

NONLINEAR OPTIMIZATION BY THE SEQUENTIAL UNCONSTRAINED MINIMIZATION
TECHNIQUE USING CONJUGATE GRADIENT METHODS

by 6408

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To

Sudha

My constant source of inspiration.

CHAPTER 1

INTRODUCTION

Analyses of industrial systems frequently involve optimization of nonlinear objective functions subject to linear or nonlinear constraints. One of the powerful techniques developed for solving such optimization problems is the sequential unconstrained minimization technique (SUMT). This method was first proposed by Carroll [1] in 1959. It was further developed by Fiacco and McCormick [6,7,8,9,10] who provided the theoretical basis for this technique. They extended the method further to handle general nonlinear nonconvex problems and provided the first computer program to utilize this technique.

The principle of the sequential unconstrained minimization technique is to transform a constrained optimization problem into a sequence of unconstrained optimization problems. The reason for this transformation is that many of the well known unconstrained optimization techniques can be used to solve constrained problems without having to develop a new one to solve such constrained problems.

The SUMT algorithm proposed by Fiacco and McCormick [6,7] uses a second order gradient method as the unconstrained minimization technique. This method presents some difficulties particularly when the objective function and the constraints are highly nonlinear. The problem lies in the difficulty of correctly taking both the first and second partial derivatives. Therefore a technique which does not require any derivatives to be evaluated would overcome these difficulties. Lai [19] developed

one such technique. The unconstrained minimization technique used by Lai was the Hooke and Jeeves pattern search [15, 16a]. This method does not require any derivatives to be evaluated, and has been demonstrated that it can solve problems of a practical nature. This method appears to be faster than the RAC SUMT computer program with regard to computing time. The disadvantage of this method is that during the process of minimization, constraints are violated frequently. There is a procedure included to pull the optimization trajectory towards the feasible region. However it is not easy to determine whether this pulling back has proceeded far enough to reach the optimum. The program requires a tolerance on constraint violation to be supplied. This is also not easy to determine.

In view of the above disadvantages it was considered useful to determine whether a minimization technique using only the first derivatives could overcome these difficulties. The conjugate gradient methods were the obvious choice since they possess the property of quadratic convergence while using only the first derivatives. Another reason for this choice was that IBM scientific subroutine packages implementing the conjugate gradient methods of Fletcher and Powell [11] and Fletcher and Reeves [12] were available.

This report presents the results of implementing SUMT by the use of conjugate gradient techniques of Fletcher and Powell and Fletcher and Reeves. Extensive use has been made of the various formulations of P functions suggested by Fiacco and McCormick [9] in this report. Suggestions are also included to indicate the type of the P function that is suitable for a particular type of problem. The conjugate

gradient methods are discrete, large step search methods. Therefore the path of optimization, particularly near a constraint, frequently goes out of the feasible region. Additional penalties were used to pull the path of optimization back into the feasible region. Details of this approach are given in chapter 5.

This newly developed approach has been used to solve two dimensional production and inventory problems with inequality as well as equality constraints, and the five and ten stage paint factory production and personnel scheduling problem of Holt et al. [14]. The above problems have also been solved by Lai [19] and Hsu et al [16]. A comparison of the computing times required on an IBM 360-50 WATFOR compiler are made. The new method has also been applied to three of the test problems suggested by Colville [2].

The experience gained in applying the present approach indicates that the method is successful in solving problems of a varied nature, provided that the first partial derivatives of the objective function and the constraints can be provided. The other conclusions are:

1. There is no generalized P function which can be used for all problems. Three types of P functions are suggested in the method, and some experimentation is involved in determining the appropriate P function.
2. The stopping criterion for the unconstrained minimization technique (EPS) has to be determined by trial and error, though a probable range of values have been suggested.

CHAPTER 2

SEQUENTIAL UNCONSTRAINED MINIMIZATION TECHNIQUE

2.1 INTRODUCTION

The general nonlinear programming problem is to determine x to
 minimize $f(x)$

subject to

$$g_i(x) \geq 0, \quad i = 1, 2, \dots, \ell,$$

$$h_i(x) = 0, \quad i = \ell+1, \dots, m.$$

The function f , $\{g_i\}$ and $\{h_i\}$ can take any form of nonlinearity.

A number of techniques have been developed to solve the above problem and the sequential unconstrained minimization technique (SUMT) is one of the powerful techniques among them. There are three classes of such sequential techniques [9]:

1. Interior point minimization techniques.
2. Exterior point minimization techniques.
3. Mixed interior-point and exterior-point techniques.

2.2 INTERIOR POINT TECHNIQUES

The interior point technique is applicable when the problem is of the form:

$$\text{minimize } f(x)$$

subject to

$$g_i(x) \geq 0, \quad i = 1, 2, \dots, m,$$

where x is an n -dimensional column vector $(x_1, x_2, \dots, x_n)^T$. The

superscript T denotes transposition. If the variables are required to be non-negative, such constraints are included in the g_i 's.

To solve this problem the following algorithm is used [6,7,8,9,10].

Define the function (called the P function)

$$P(x, r_k) = f(x) + S(r_k) \cdot I(x)$$

which satisfies the following requirements.

1. $S(r_k)$ is a scalar valued function of r_k such that

(a) if $r_1 > r_2 > 0$, then $S(r_1) > S(r_2) > 0$, and

(b) if $\lim_{k \rightarrow \infty} r_k = 0$, then $\lim_{k \rightarrow \infty} S(r_k) = 0$.

2. $I(x)$ is a scalar valued function of $g_i(x)$, $i = 1, \dots, m$, such that

(a) $I(x)$ is continuous in the region

$$R^0 = \{x | g_i(x) > 0, \quad i = 1, \dots, m\}$$

(b) if $\{x^k\}$ is an infinite sequence of feasible points converging to x^* (the local optimum) such that at least one $g_i(x) = 0$, then

$$\lim_{k \rightarrow \infty} I(x^k) = \infty$$

Some examples of the I-function are [9]:

$$(i) \quad I(x) = \sum_i \frac{1}{g_i(x)}, \quad (ii) \quad I(x) = - \sum_i \ln g_i(x)$$

3. The functions $f(x)$, $g_1(x)$, $\dots$, $g_m(x)$ are twice continuously differentiable.

4. The function $f(x)$ is convex.
5. The functions $g_1(x), \dots, g_m(x)$ are concave.
6. For every finite k , $\{x | f(x) \leq k; x \in R\}$ is a bounded set, where $R = \{x | g_i(x) \geq 0, i = 1, \dots, m\}$.
7. $R^0 = \{x | g_i(x) > 0, i = 1, 2, \dots, m\}$ is non-empty. This condition indicates that at least one point must exist within the interior of the feasible region.
8. The function $P(x, r_k) = f(x) + S(r_k) \cdot I(x)$ is, for each $S(r_k) > 0$, strictly convex for $x \in R^0$. This also indicates that either $f(x)$ is strictly convex or one of $g_1, \dots, g_m$ is strictly concave.

From practical experience it is found that certain problems can be solved even when all the above conditions are not satisfied. The four conditions which must be satisfied to obtain any useful results are conditions (1), (2), (6) and (7). Conditions (1) and (2) imply that the sequential minimization of the P function will lead to the minimization of $f(x)$. Condition (6) eliminates problems having local minima at infinite points. Condition (7) eliminates problems with equality constraints.

The characteristics of the P function are:

1. $\lim_{k \rightarrow \infty} S(r_k) \cdot I(x) = 0$
2. $\lim_{k \rightarrow \infty} f[x(r_k)] = u^*$
3. $\lim_{k \rightarrow \infty} P[x(r_k), r_k] = u^*$

4. $\{f[x(r_k)]\}$ is a monotonically decreasing sequence.
5. $\{I(x)\}$ is a monotonically increasing sequence.

The proofs of these characteristics are presented in detail by Fiacco and McCormick [6,7,8,9].

2.2.1 Intuitive Basis of the P function

The term $S(r) \cdot I(x)$ is a penalty factor included in the P function along with the objective function. This is added in order to maintain the trajectory of the optimization path within the feasible region in the process of minimizing the objective function. This can be observed by the fact that, as the value of x approaches one of the boundaries, $I(x)$ will approach infinity. Therefore the value of x will tend to remain in the feasible region.

The basic idea behind this method of formulating a P function is to transform the original constrained problem into a sequence of unconstrained minimization problems. This transformation facilitates the use of many available methods for minimizing an unconstrained function, without developing a new method to solve constrained problems. Another advantage of the P function is that it avoids movement along the boundary of the feasible region if one is met during the process of minimization. Such a motion is quite complex particularly when the constraints are nonlinear. The P function combines the objective function and the constraints in such a manner that it is not necessary to move along the boundary.

2.2.2 Computational Procedure

The procedure for interior point SUMT is given below [6,7,8,9]:

1. Select initial values for $S(r_0)$ arbitrarily or by using the formula given in reference [7].
2. Select a feasible starting point

$$x^0 = (x_1^0, x_2^0, \dots, x_n^0)$$

3. Define the P-function

$$P(x, r_k) = f(x) + S(r_k) \cdot I(x)$$

4. Minimize the P function for the current value of r_k using any unconstrained minimization technique.
5. Check if a suitable stopping criterion is satisfied.
Otherwise go to step 6.
6. Set $k = k + 1$, and $r_{k+1} = r_k/C$, $C > 1$. Go to step 3.

2.3. EXTERIOR POINT TECHNIQUES

Exterior point techniques are used when the initial starting point is in the infeasible region, that is, at least one $g_i < 0$, and the optimization path approaches the solution to the problem from the infeasible region.

To solve this problem the P function is defined as follows:

[6,7,8,9,10]

$$P(x, t_k) = f(x) + A(t_k) \cdot B(x).$$

The requirements of the P function are

1. $A(t_k)$ is a scalar valued function of t_k such that

(a) if $0 < t_1 < t_2$, then $0 < A(t_1) < A(t_2)$

(b) if $\lim_{k \rightarrow \infty} t_k = \infty$, then $\lim_{k \rightarrow \infty} A(t_k) = \infty$

2. $B(x)$ is a function of $g_i(x)$ such that

(i) $B(x) = 0$, if $g_i(x) \geq 0$, $i = 1, \dots, m$

(ii) $B(x) > 0$, otherwise.

Some examples of the B function are [9]

$$(i) \quad B(x) = \sum_i g_i^2(x),$$

$$(ii) \quad B(x) = \left\{ \frac{\sum_i g_i - \sum_i |g_i|}{2} \right\}^2$$

The other requirements are the same as those for the interior point P function.

The characteristics of the P function are:

$$1. \quad \lim_{k \rightarrow \infty} A(t_k) \cdot B(x) = 0$$

$$2. \quad \lim_{k \rightarrow \infty} f(x(t_k)) = u^*$$

$$3. \quad \lim_{k \rightarrow \infty} P[x(t_k), t_k] = u^*$$

4. $\{f[x(t_k)]\}$ is a monotonically decreasing sequence

5. $\{B(x)\}$ is a monotonically decreasing sequence.

The proofs of these characteristics are presented in detail in Fiacco and McCormick [6,7,8,9,10].

2.3.1. Intuitive Basis of the P function

The term $A(t) \cdot B(x)$ is a penalty term included in the P function. If the value of x moves too far away from the feasible region, the penalty $A(t) \cdot B(x)$ becomes quite large particularly when t_k is large. Therefore, as $k \rightarrow \infty$, $t_k \rightarrow \infty$ and the penalty function will force the path of minimization towards the feasible region.

2.3.2. Computational Procedure

The procedure for exterior point SUMT is as follows [8,9,10]:

1. Select initial values for $A(t_0)$ arbitrarily or by using the formula given in reference [7].
2. Select a starting point $x^0 = (x_1^0, x_2^0, \dots, x_n^0)$ arbitrarily. This need not be in the feasible region.
3. Define the P function

$$P(x, t_k) = f(x) + A(t_k) \cdot B(x).$$

4. Minimize the P function for the current value of t_k using any unconstrained minimization technique.
5. Check if a suitable stopping criterion is satisfied; otherwise go to step 6.
6. Set $k = k + 1$, and $A(t_{k+1}) = A(t_k) \cdot C$, $C > 1$. Go to step 3.

2.4. MIXED INTERIOR POINT - EXTERIOR POINT TECHNIQUES

Some problems, for example, those with equality constraints have no feasible region. Therefore interior point methods cannot be applied

to such problems. Other problems, where strict satisfaction of a portion of the constraints is required at all times while the remaining need be satisfied only as the solution is approached, require a different approach. The mixed interior point - exterior point technique is designed to handle problems of the above type [9].

The problem can be defined as:

$$\text{minimize } f(x)$$

subject to

$$g_i(x) \geq 0, \quad i = 1, \dots, \ell,$$

$$h_j(x) = 0, \quad j = \ell+1, \dots, m.$$

some of the $g_i(x) \geq 0$, $i = 1, \dots, q$, need to be strictly satisfied at all times.

To solve this problem the following algorithm is used [8,9,10].

Define the P function as

$$P(x, r_k, t_k) = f(x) + S(r_k) \cdot I(x) + A(t_k) \cdot B(x)$$

The conditions imposed on the P function are:

1. $S(r_k)$ is a scalar-valued function of r_k such that

$$(a) \quad \text{if } r_1 > r_2 > 0, \text{ then } S(r_1) > S(r_2) > 0$$

$$(b) \quad \text{if } \lim_{k \rightarrow \infty} r_k = 0, \text{ then } \lim_{k \rightarrow \infty} S(r_k) = 0.$$

2. $I(x)$ is a scalar valued function of $g_i(x)$, $i = 1, \dots, q$, such that

(a) $I(x)$ is continuous in the region

$$R^0 = \{x \mid g_i(x) > 0, \quad i = 1, \dots, m\}$$

(b) If $\{x^k\}$ is an infinite sequence of feasible points converging to x^* (the local optimum) such that at least one $g_i(x) = 0$, then

$$\lim_{k \rightarrow \infty} I(x^k) = \infty.$$

3. $A(t_k)$ is a scalar valued function of t_k such that

(a) if $0 < t_1 < t_2$ then $0 < A(t_1) < A(t_2)$

(b) if $\lim_{k \rightarrow \infty} t_k = \infty$ then $\lim_{k \rightarrow \infty} A(t_k) = \infty$.

4. $B(x)$ is a function of $g_i(x)$ and $h_j(x)$ such that

(i) $B(x) = 0$, if $g_i(x) \geq 0$, $j = q+1, \dots, \ell$,

and $h_j(x) = 0$, $j = \ell+1, \dots, m$,

(ii) $B(x) > 0$, otherwise.

The other requirements are the same as those for the interior-point P function.

$$1. \quad \lim_{k \rightarrow \infty} S(r_k) \cdot I(x) = 0$$

$$2. \quad \lim_{k \rightarrow \infty} A(t_k) \cdot B(x) = 0$$

$$3. \quad \lim_{k \rightarrow \infty} f(x, r_k, t_k) = u^*$$

4. $\lim_{k \rightarrow \infty} P(x, r_k, t_k) = u^*$
5. $\{f(x, r_k, t_k)\}$ is a monotonically decreasing sequence.
6. $\{I(x)\}$ is a monotonically increasing sequence.
7. $\{B(x)\}$ is a monotonically decreasing sequence.

The proofs of these characteristics are given in Fiacco and McCormick [7,8,9,10].

2.4.1. Computational procedure

The computational procedure for the mixed interior-exterior point SUMT is given below [6,7,8,9,10].

1. Select initial values for $S(r_0)$ and $A(t_0)$ arbitrarily or by using the formula given in reference [7].
2. Select a starting point $x^0 = (x_1^0, x_2^0, \dots, x_n^0)$ arbitrarily.
3. Define the P function

$$P(x, r_k, t_k) = f(x) + S(r_k) \cdot I(x) + A(t_k) \cdot B(x)$$

4. Minimize the P function for the current value of r_k and t_k using any unconstrained minimization technique.
5. Check if a suitable stopping criterion is satisfied.
Otherwise go to step 6.
6. Set $k = k + 1$, $S(r_{k+1}) = S(r_k)/C$, and $A(t_{k+1}) = A(t_k) \cdot C$,
 $C > 1$, go to step 3.

2.5 AVAILABLE COMPUTER PROGRAMS

2.5.1. RAC SUMT Computer Program

A computer program entitled "RAC Computer Program Implementing the Sequential Unconstrained Minimization Technique for Non-linear Programming" is available for solving the mathematical programming problems discussed. Its SHARE number is 3189 [20]. The program is written in FORTRAN IV.

The computational procedure used in the computer program is summarized below [6,7,8]:

1. Select an initial value of r_0 arbitrarily or use the formula for selecting r_0 .
2. Select a feasible starting point $x^0 = (x_1^0, \dots, x_n^0)$. If the feasible point cannot be easily obtained, select x^0 arbitrarily, the computer program will minimize the following P function and obtain a feasible point.

$$P(x, r_k) = -g_s(x) + r_k \sum_{t \in T} \frac{1}{g_t(x)}$$

where $g_s(x) < 0$ and $T = \{t | g_t(x^0) > 0\}$. Note that the constraint $g_s(x) \geq 0$ is violated.

3. Minimize the P function for the current value of r_k by using the second-order gradient method.
4. Verify whether a stopping criterion such as

$$\frac{f[x, r_k]}{G[x, r_k]} - 1 \leq \epsilon$$

is satisfied. If it is satisfied, the solution is optimal; otherwise go to step 5. The dual function, $G[x, r_k]$, is defined as [6]

$$G(x, r_k) = f(x, r_k) - r_k \sum_{i=1}^m \frac{1}{g_i(x, r_k)} .$$

5. Set $k = k + 1$, $r_{k+1} = r_k/C$, $C > 1$. Repeat the iteration from step 3.

The procedure described above must satisfy two stopping criteria before any meaningful optimal solution is obtained. The stopping criterion used for terminating the minimization process for P function in step 3 may be one of the following:

$$A. \left| \nabla_x P^T(x, r) \right| \left| \frac{\partial^2 P(x, r)}{\partial x_i \partial x_j} \right|^{-1} \left| \nabla_x P(x, r) \right| < \epsilon'$$

or

$$B. \left| \nabla_x P^T(x, r) \right| \left| \frac{\partial^2 P(x, r)}{\partial x_i \partial x_j} \right|^{-1} \left| \nabla_x P(x, r) \right| < \frac{P(x, r_{k-1}) - P(x, r_k)}{5}$$

or

$$C. \left| \nabla_x P(x, r) \right| < \epsilon' .$$

The second stopping criterion shown in step 4 is used to stop the search when the overall minimum for the problem is found.

2.5.2. Computer Program using SUMT and Hooke and Jeeves Pattern Search

A computer program which uses the method of SUMT, but does not

require any derivatives to be calculated, has been prepared by K.C. Lai [19]. This program makes use of the Hooke and Jeeves pattern search technique [15, 16a] as the unconstrained minimization technique. This method has the advantage that no derivatives for the objective function or the constraints need to be provided. The program is written in FORTRAN IV.

The computational procedure for using SUMT with the Hooke and Jeeves pattern search is as follows [19]:

1. Select an initial value of r_0 , an initial tolerance limit of violation to the constraints, B^0 , and the initial step-sizes needed in the search technique, d^0 .
2. Select a feasible starting point $x^0 = (x_1^0, \dots, x_n^0)$. If a feasible starting point is not easily available, the computer program will obtain one by minimizing the following function:

$$TGH = \left(\sum_{t \in T} g_t^2(x^0) + \sum_{s \in R} h_s^2(x^0) \right)^{1/2}$$

where

$$T = \{t | g_t(x^0) < 0\} \quad \text{and} \quad R = \{s | h_s(x^0) \neq 0\}$$

3. Define the P function

$$P(x, r_k) = f(x) + r_k \sum_i \frac{1}{g_i(x)} + r_k^{-1/2} \sum_j h_j^2(x)$$

where $g_i(x) \geq 0$, $i = 1, 2, \dots, m$, are the inequality constraints and $h_j(x) = 0$, $j = 1, 2, \dots, \ell$, are equality constraints.

4. Minimize the P-function by the Hooke and Jeeves pattern search method. Check after every move if the move goes out of the feasible region. If it does, go to step 5. Otherwise after the minimum of the P function is obtained go to step 6.
5. Move back to the near-feasible region and go to step 4. The near feasible region is that region where all the points satisfy the condition

$$B - TGH > 0$$

where B is the tolerance on constraint violation which is sequentially reduced after every violation during the iteration.

6. Check if the optimum $\bar{x}$, obtained in step 4 is feasible. If feasible, go to step 8. Otherwise go to step 7.
7. Move the optimum $\bar{x}$, into a near feasible region and go to step 8.
8. Check whether a stopping criterion defined as

$$\left| \left| \frac{f(x)}{G(x, r_k)} - 1 \right| \right| < \epsilon$$

is satisfied. If not satisfied go to step 9. If satisfied the solution is optimal. The dual function $G(x, r_k)$ is given by

$$G(x, r_k) = f(x) - r_k \sum_{i=1}^m \frac{1}{g_i(x)} + r_k^{-1/2} \sum_{j=1}^l h_j^2(x)$$

9. Set $k = k+1$, $r_{k+1} = r_k/C$, $C > 1$, $d_{k+1} = d^0/(k+1)$ and go to step 3.

The two stopping criteria used in this method are

- (i) Limiting stepsize for the Hooke and Jeeves pattern search,
- (ii) Overall stopping criterion defined in step 8.

CHAPTER 3

CONJUGATE GRADIENT METHODS

3.1. INTRODUCTION

Analyses of industrial systems frequently lead to optimization of a system of equations which are usually nonlinear. Analytical solutions for these equations are difficult and sometimes extremely cumbersome. This has led to the development of direct methods of optimization or a class of techniques known as search techniques. Some of them, like the Hooke and Jeeves pattern search [15, 16a] and the simplex pattern search [4,21], are based on an empirical approach. Gradient techniques on the other hand have a classical origin, and make use of the fact that if iteration is carried out in the direction of the gradient of $f(x)$, the path taken is that of the maximum rate of increase in $f(x)$ [23]. It is well known, however, that such a method has a very slow rate of convergence in the neighborhood of the optimum [23]. Hestenes and Stiefel [13] derived a method to increase the rate of convergence of the gradient method. This method, called the conjugate gradient method, makes use of the cumulated knowledge of the behaviour of the function to derive new directions of search at the end of each iteration. This technique was further developed by Davidon [3] and modified by Fletcher, Powell and Reeves [11,12]. The algorithms of Fletcher and Powell, and Fletcher and Reeves are presented here. This presentation follows closely that presented by Hwang, Fan and Pereira [17].

Conjugate gradient methods have the property of quadratic convergence, that is, if a quadratic function of n variables is being optimized, the optimum is reached in not more than n iterations.

3.2. DESCRIPTION OF THE METHOD

Before proceeding to the mechanisms of the method it is necessary to define conjugate directions.

"Two directions $\underline{u}$ and $\underline{v}$ are said to be conjugate with respect to the positive definite matrix A if $\underline{u}^T A \underline{v} = 0$ ".

The conjugate gradient method is based on the assumption that in the vicinity of a minimum, a function can be approximated by a positive definite quadratic form. The following theorem is relevant to the understanding of all conjugate gradient techniques:

Theorem: If $\xi_1, \xi_2, \dots, \xi_r$ are a set of vectors mutually conjugate with respect to the positive definite matrix A , then the minimum of the quadratic form

$$F = a + \underline{b}^T \underline{x} + 1/2 \underline{x}^T A \underline{x}$$

can be found from an arbitrary starting point $\underline{x}^{(0)}$ by a finite descent computation in which each of the vectors $\underline{\xi}_i$ ($i = 1, \dots, r$) is used as a descent direction only once. The order in which the $\underline{\xi}_i$ are used is immaterial.

The proof of this theorem is given in [18]. This theorem means that if an r -dimensional objective function of the quadratic form is being minimized, the minimum can be located starting from an arbitrary starting point $\underline{x}^0$ by making r successive one-dimensional moves in the

r conjugate directions, $\underline{\varepsilon}_i$ ($i = 1, \dots, r$), and using each direction only once. If the objective function is not quadratic, the function is approximately quadratic as the iteration approaches the minimum and therefore convergence to the solution is guaranteed.

A truncated Taylor's series expansion of the objective function $F(\underline{x})$ about an arbitrary point $\underline{x}_0 = (a_1, a_2, \dots, a_r)$ yields

$$\begin{aligned} F(\underline{x}) = & F(\underline{x}_0) + \underline{g}(\underline{x}_0) (\underline{x} - \underline{x}_0) \\ & + 1/2 (\underline{x} - \underline{x}_0)^T G(\underline{x}_0) (\underline{x} - \underline{x}_0) \end{aligned} \quad (1)$$

where

$F(\underline{x}_0)$ = objective function evaluated at $\underline{x}_0$

$\underline{g}(\underline{x}_0)$ = gradient vector evaluated at $\underline{x}_0$

$G(\underline{x}_0)$ = matrix of second order partial derivatives
(Hessian Matrix) evaluated at $\underline{x}_0$.

If $\underline{x}_0$ is at $\underline{x}_{\min}$, the point at which the minimum of the function F is located, then from equation (1)

$$F(\underline{x}) = F(\underline{x}_{\min}) + 1/2 (\underline{x} - \underline{x}_{\min})^T G(\underline{x}_{\min}) (\underline{x} - \underline{x}_{\min}) \quad (2)$$

as

$$\underline{g}(\underline{x}_{\min}) = 0.$$

Therefore from equation (2), the gradient near the optimum can be approximated as

$$\underline{g}(\underline{x}) \cong G(\underline{x}_{\min}) (\underline{x} - \underline{x}_{\min}). \quad (3)$$

3.3 THE METHOD OF FLETCHER AND POWELL [11]

If the matrix G in equation (3) is positive definite, then

$$(\underline{x} - \underline{x}_{\min}) = G^{-1}(\underline{x}_{\min}) \underline{g}(\underline{x}) \quad (4)$$

where $G^{-1}(\underline{x}_{\min})$ is the inverse of $G(\underline{x}_{\min})$. If $G^{-1}(\underline{x}_{\min})$ is available,

then $\underline{x}_{\min}$ can be computed from equation (4).

The method of Fletcher and Powell uses an indirect method to evaluate the matrix G^{-1} . Initially any positive definite symmetric matrix H is selected and this matrix is improved after each iteration by using the information obtained while proceeding along the G -conjugate directions.

If the initial matrix selected (H^0) is the identity matrix I , each iteration generates a symmetric matrix $H^{(i)}$, which tends to approach G^{-1} .

The algorithm of Fletcher and Powell is summarized below [17]:

(1) Compute $\underline{\xi}_i = -H_{i-1} \underline{g}_i$

(2) Compute λ_i to minimize

$$q_i(\lambda) = f(\underline{x}^{(i)} + \lambda_i \underline{\xi}_i)$$

(3) Set $\underline{x}^{i+1} = \underline{x}^i + \lambda_i \underline{\xi}_i$

$$\text{and } \underline{y}_i = \underline{g}_{i+1} - \underline{g}_i$$

(4) Compute

$$H_i = H_{i-1} + \frac{\lambda_i \xi_i \xi_i^T}{g_i^T H_{i-1} g_i} - \frac{H_{i-1} y_i y_i^T H_{i-1}}{y_i^T H_{i-1} y_i}$$

(5) Set $i = i+1$ and go to step (1).

In the first step of the iteration, the identity matrix I is usually selected as the initial H matrix. The value of λ in step (2) is obtained with the help of any one-dimensional search technique. Both the methods of Fletcher and Powell, and Fletcher and Reeves use a linear search procedure developed by Davidon [3].

The selection of a suitable stopping criterion is very important, particularly when the function being minimized is not quadratic. If the square of any gradient, g_i^2 , is zero, the iteration should be terminated. However since this is unlikely to occur in actual practice due to rounding off errors, the value of g_i^2 is compared with some preassigned value (EPS). A second criterion is to compare the improvement in the function value at the end of each iteration with a preassigned tolerance. If these criteria are not satisfied, iteration is continued until a preassigned number (LIMIT) of iterations is carried out.

3.4 METHOD OF FLETCHER AND REEVES [12]

In this method, successive conjugate directions are generated such that ξ_{i+1} is a linear combination of $-g_{i+1}$ and $\xi_1, \xi_i, \dots, \xi_i$, as follows:

$$\xi_{i+1} = -g_{i+1} + \lambda_i \xi_i$$

where

$$\lambda_i = \frac{\frac{g_{i+1}^2}{2}}{g_i^2}$$

The algorithm of Fletcher and Reeves may be summarized as follows:

(1) Select an arbitrary starting point $\underline{x}^{(1)}$

(2) Compute $\underline{g}_1 = \nabla F(\underline{x}^{(1)})$ and set $\underline{\xi}_1 = -\underline{g}_1$

(3) For $i = 2, 3, \dots, r+2$

(a) Set $\underline{x}^{(i)} = \underline{x}^{(i-1)} + \lambda_{i-1} \underline{\xi}_{i-1}$, where

$$\lambda_{i-1} \text{ minimize } q_{i-1}(\lambda) = F(\underline{x}^{(i-1)} + \lambda_{i-1} \underline{\xi}_{i-1}).$$

(b) Compute $\underline{g}_i = \nabla F(\underline{x}^{(i)})$

(c) If $i = r+2$ go to step 4. Otherwise set

$$\underline{\xi}_i = -\underline{g}_i + \frac{\underline{g}_i^T \underline{g}_i}{\underline{g}_{i-1}^T \underline{g}_{i-1}}$$

and return to a).

4. Replace $\underline{x}^{(1)}$ by $\underline{x}^{(r+2)}$ and restart the iteration.

The one dimensional search to locate the minimum of $q_{i-1}^{(\lambda)}$ in step 3(a) and the stopping criteria used are the same as those for the method of Fletcher and Powell.

3.5 AVAILABLE COMPUTER PROGRAMS

There are two IBM scientific subroutine packages entitled FMFP and FMCG available for implementing, respectively, the methods of Fletcher and Powell and that of Fletcher and Reeves. These are written in FORTRAN IV. Double precision versions of the same programs entitled DFMFP and DFMCG are also available. Each of these programs requires an external subprogram to supply the value of the objective function and the partial derivatives of the objective function at any given point. The stopping criteria needed to be supplied are EPS, the tolerance on function or gradient changes and LIMIT, the maximum number of iterations. In addition an estimate of the minimum value of the objective function, EST, should also be supplied. The program gives the value of IER in its output, which is an index indicating convergence. IER = 0 means that convergence is obtained, IER = 1, no convergence is obtained within the specified number of iterations, IER = -1, there are errors in the calculation of gradients, and IER = 2, the function has no minimum.

Some modifications have been made in this work to these subroutines so that intermediate results during the process of minimization could be printed out. The descriptive computational method for the conjugate gradient technique is presented in Fig. 1.

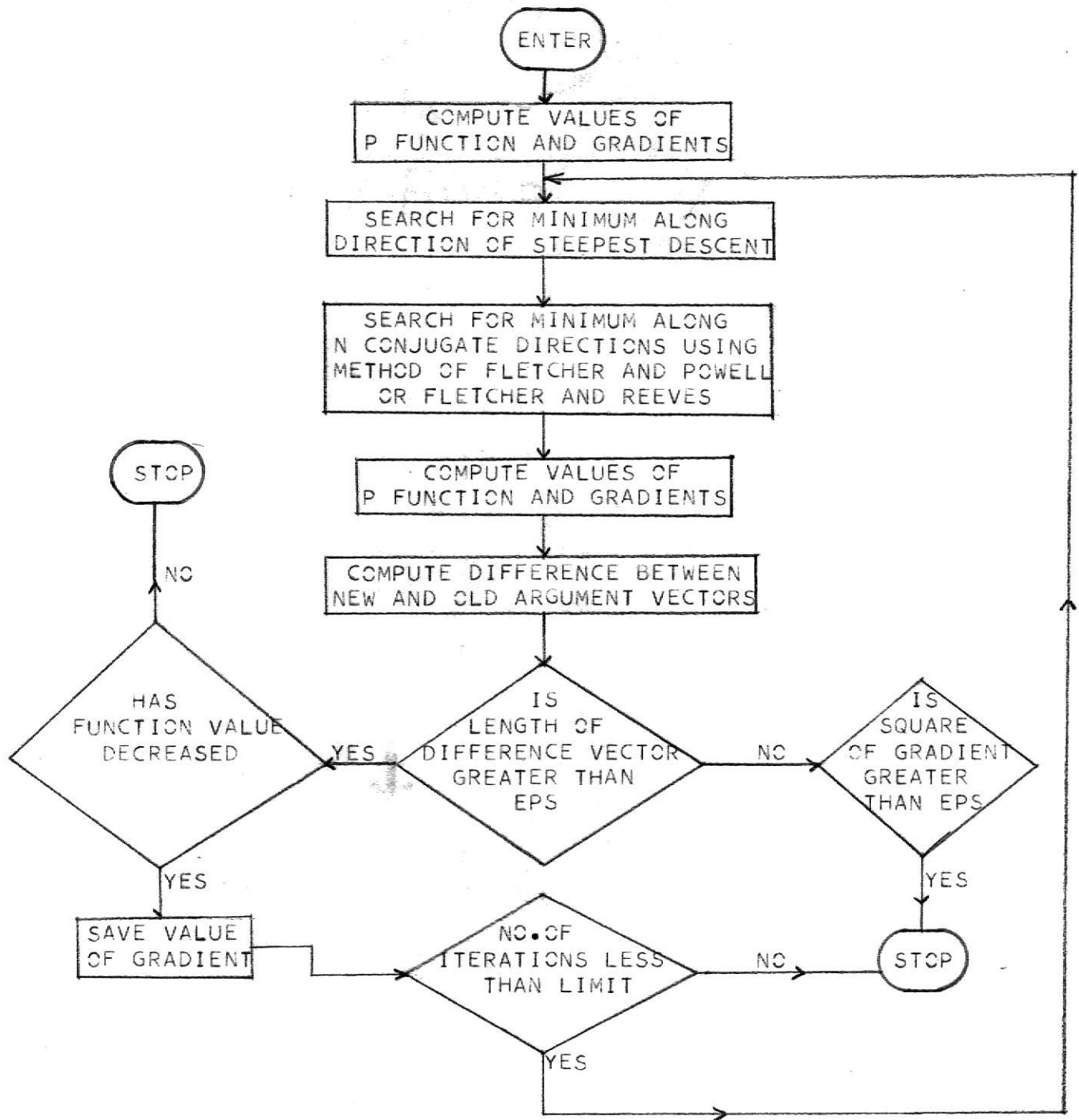


FIG. 1 DESCRIPTIVE FLOW DIAGRAM OF CONJUGATE GRADIENT METHODS.

CHAPTER 4

IMPLEMENTATION OF SEQUENTIAL UNCONSTRAINED MINIMIZATION TECHNIQUE BY
CONJUGATE GRADIENT METHODS

4.1 INTRODUCTION

Sequential unconstrained minimization technique (SUMT) is a simple and powerful technique for solving general nonlinear programming problems. The basic scheme of this technique is to transform a constrained optimization problem into a sequence of unconstrained optimization problems and to solve these with the aid of any one of the available techniques of unconstrained optimization. The computational procedure described here uses the conjugate gradient methods of Fletcher and Powell [11] and Fletcher and Reeves [12].

4.2 COMPUTATIONAL PROCEDURE

The flow-chart for the computational procedure of SUMT with the conjugate gradient method is shown in Fig. 2. The procedure is summarized below:

1. Select a starting point $x^0 = (x_1^0, x_2^0, \dots, x_n^0)$, initial values of the penalty functions $S(r_0)$, stagewise and overall stopping criterion (EPS, θ) , limit on the number of iterations at each stage (LIMIT), and an estimate of the minimum of the objective function (EST).

2. If the initial starting point is infeasible, select a feasible starting point by minimizing the total weight of violation, TGH. TGH is defined by [5]:

$$TGH = \left(\sum_{t \in T} g_t^2(x^0) + \sum_{s \in S} h_s^2(x^0) \right)^{1/2}$$

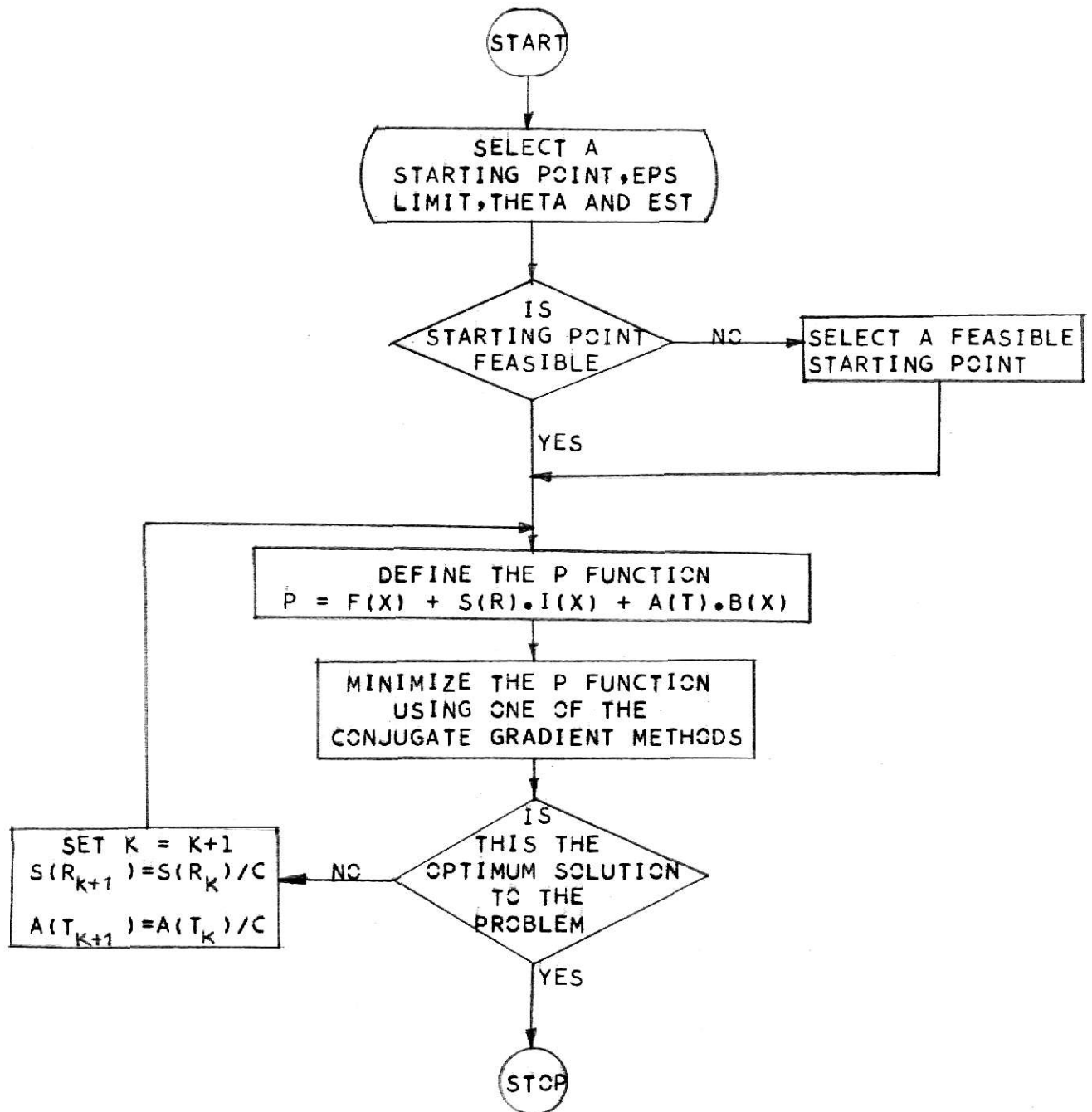


FIG.2 DESCRIPTIVE FLOW DIAGRAM OF SUMT WITH CONJUGATE GRADIENT TECHNIQUES.

where $T = \{t | g_t(x^0) < 0\}$ and $S = \{s | h_s(x^0) \neq 0\}$.

3. Define the P function as [8,9,10]

$$P(x, r, t) = f(x) + S(r) \cdot I(x) + A(t) \cdot B(x)$$

4. Minimize the P function by using either the method of Fletcher and Powell or the method of Fletcher and Reeves.

5. Check if a stopping criterion such as

$$\left| \frac{f(x)}{G(x, r, t)} - 1 \right| < \theta$$

is satisfied. Otherwise go to step 8. The dual function $G(x, r, t)$ is defined as

$$G(x, r, t) = f(x) - S(r) \cdot I(x) + A(t) \cdot B(x)$$

6. Set $k = k+1$, $S(r_{k+1}) = S(r_k)/C$, and

$$A(t_{k+1}) = A(t_k) \cdot C, \text{ where } C > 1,$$

and go to step 3.

4.3 PROCEDURE FOR SELECTING A FEASIBLE STARTING POINT

The procedure for selecting a feasible starting point when the input initial point is out of the feasible region bounded by inequality constraints, $g_i(x) \geq 0$ for $i = 1, 2, \dots, m$, utilizes the conjugate gradient technique to minimize the weight of violation of constraints.

The procedure is as follows:

1. Start at the input initial point x^0 , and compute the weight of violation

$$\text{TGH} = \left\{ \sum_{t \in T} g_t^2(x^0) + \sum_{s \in S} h_s^2(x^0) \right\}^{1/2}$$

where $T = \{t | g_t(x^0) < 0\}$ and $S = \{s | h_s(x^0) \neq 0\}$.

2. Minimize TGH using either the method of Fletcher and Powell or the method of Fletcher and Reeves.
3. Enter into the SUMT routine.

4.4. PROCEDURE FOR SELECTING AN INITIAL VALUE OF $S(r_0)$

When a suitable initial value of $S(r_0)$ is not known, the computer program calculates an initial value using the following formula.

$$S(r_0) = \frac{f(x^0)}{\sum_i \frac{1}{g_i(x^0)}}$$

If the value of $S(r_0)$ obtained is less than 1.0, $S(r_0)$ is set equal to 1.0 [7].

4.5 PROCEDURE FOR SELECTING $S(r)$, $I(x)$, $A(t)$, AND $B(x)$

The procedure for selecting the functions $S(r)$, $I(x)$, $A(t)$ and $B(x)$ depends upon the type of problem. The following suggestions are offered on the basis of experience gained in solving the seven test problems detailed in Chapter 5 and the suggestions of Fiacco and McCormick [9], Lai [19], and Hsu et al. [16].

I. If the problem consists only of inequality constraints the following penalty functions may be used:

$$S(r) = r$$

$$I(x) = \sum_i \frac{1}{g_i(x)}, \quad g_i(x) > 0$$

$$A(t) = \text{CONST}$$

$$B(x) = - \sum_j \frac{1}{g_j(x)}, \quad g_j(x) < 0.$$

The value of CONST as determined below is found to yield best results. (See Chapter 5 for discussion on the selection of CONST).

$$\begin{aligned} \text{CONST} &= |f(x)| && \text{if } |f(x)| > 10^3 \\ &= 10^3 && \text{if } |f(x)| < 10^3 \\ &= r^{-0.5} && \text{if } r^{-0.5} > |f(x)| \end{aligned}$$

The P function, therefore, will be

$$P(x, r) = f(x) + r_k \sum_i \frac{1}{g_i(x)} - \text{CONST} \sum_j \frac{1}{g_j(x)}$$

where

$$g_i(x) > 0 \quad \text{and} \quad g_j(x) < 0.$$

II. If the problem consists of inequality constraints only and the optimization process is initiated at an infeasible starting point and if the optimization path lies in the infeasible region; the following penalty functions have been found useful:

$$S(r) = r_k$$

$$I(x) = - \sum_i \ln g_i(x), \quad g_i(x) > 0$$

$$A(t) = \frac{1}{r_k}$$

$$B(x) = \sum_j g_j^2(x), \quad g_j(x) < 0$$

III. If the problem involves equality and inequality constraints, the following penalty functions may be used [16]

$$S(r) = r_k$$

$$I(x) = \sum_i \frac{1}{g_i(x)}, \quad g_i(x) > 0$$

$$A_1(t) = \text{CONST}$$

$$B_1(x) = - \sum_j \frac{1}{g_j(x)}, \quad g_j(x) > 0$$

$$A_2(t) = r_k^{-0.5}$$

$$B_2(x) = \sum_s h_s^2(x), \quad h_s(x) \neq 0, \quad s = \ell + 1, \dots, m.$$

The P function will be

$$\begin{aligned} P(x, r_k) = & f(x) + r_k \sum_i \frac{1}{g_i(x)} - \text{CONST} \sum_j \frac{1}{g_j(x)} \\ & + r_k^{-0.5} \sum_s h_s^2(x) \end{aligned}$$

where

where

$$\begin{aligned} \text{CONST} &= |f(x)| && \text{if } |f(x)| > 10^3 \\ &= 10^3 && \text{if } |f(x)| < 10^3 \\ &= r_k^{-0.5} && \text{if } r_k^{-0.5} > |f(x)| \end{aligned}$$

IV. If the problem involves inequality constraints only with non-negativity constraints on the variables, and the path of optimization tends to move into the infeasible region, the following I - function has been found to yield the optimal result. [see discussion on test problem No. 5].

$$I(x) = - \sum_{i=1}^n \ln x_i \quad x_i > 0.$$

The P function in this case will be

$$\begin{aligned} P(x, r_k) &= f(x) - r_k \sum_{i=1}^n \ln x_i + r_k \sum_j \frac{1}{g_j(x)} \\ &\quad - \text{CONST} \sum_s \frac{1}{g_s(x)} \end{aligned}$$

where

$$x_i > 0$$

$$g_j(x) > 0$$

and

$$g_s(x) < 0.$$

The computer program provides an option NSEL which can be used to select any one of the P functions listed above.

4.6 INPUT PARAMETERS FOR THE UNCONSTRAINED MINIMIZATION TECHNIQUE

A conjugate gradient method which performs the unconstrained minimization requires the following input parameters:

EST = Estimate of the minimum value of the objective function

EPS = Stopping criteria for minimization

LIMIT = Maximum number of iterations to be carried out in each stage.

The following suggestions may be useful in selecting these parameters:

1. A value of EST which is slightly smaller than the actual minimum value of the objective function is found to reduce the number of iterations as well as the computer time in solving problems.
2. The following values of EPS have been found to be useful.

$$\begin{aligned}
 \text{EPS} &= 0.4, & |f(x)| &> 10^5 \\
 &= 0.1, & 1000 &< |f(x)| < 10^5 \\
 &= 0.01, & 100 &< |f(x)| < 1000 \\
 &= 0.001, & 0 &< |f(x)| < 100.
 \end{aligned}$$

It is interesting to note that the optimum value obtained and the speed of convergence are entirely dependent upon the value of EPS selected. If too large a value is selected the method may lead to a false optimum. If too small a value is selected large amounts of computer time may be used up in the initial stages of iteration.

3. A suitable range of values for LIMIT has been found to be

$$2N \leq \text{LIMIT} \leq 3N$$

where N = number of decision variables in the problem.

4.7 GUIDE FOR USING THE COMPUTER PROGRAM IMPLEMENTING SUMIT BY CONJUGATE GRADIENTS

The computer program requires two forms of input. The first is in the form of statements and the second is in the form of data cards. The objective function, the constraints, and the first partial derivatives of the objective function and the constraints should be included in the subroutine INPUT. The computer program is designed for 50 variables and 20 each of inequality and equality constraints.

The second form of input through data cards are as follows:

DATA CARD NO. 1 - This is used to name the problem being solved. The name can be 80 columns long.

DATA CARD NO. 2 - This is used to provide the number of variables and constraints pertaining to the problem as well as information regarding the selection of initial value of $S(r_0)$, printing of intermediate results, limit on the number of iterations in each stage, inclusion of non-negativity constraints, type of P function being selected and the method of unconstrained minimization selected.

The following suggestions are offered in selecting the above options:

(i) NR is the option used to select $S(r_0)$. If the user can supply an initial value, NR should be zero. If $S(r_0)$ is not known and has to be computed NR should be 1.

(ii) INTER is the option used for printing intermediate results. If details of iteration within a stage are needed INTER should be 1.

(iii) LIMIT is the maximum permissible number of iterations at each stage. A suggested range for LIMIT is $2N \leq \text{LIMIT} \leq 3N$, where N is the number of variables.

(iv) NEGAT is the option used to include non-negativity constraints. NEGAT = 1 results in non-negativity constraints being included. It is suggested that NEGAT be set to zero unless the value of x is likely to become negative during the minimization process.

(v) NSEL is the option used to select the type of P function to be used. It is suggested that NSEL be set to zero to start with. If this does not yield the minimum, then NSEL = 1 or 2 can be tried. When NSEL is set equal to zero, the following P function is used.

$$P(x, r_k) = f(x) + r_k \sum_{s \in S} \frac{1}{g_s(x)} - \text{CONST} \sum_{t \in T} \frac{1}{g_t(x)} \\ + r_k^{-1/2} \sum h_j^2(x)$$

where

$$S = \{s | g_s(x) > 0\} \quad T = \{t | g_t(x) < 0\}$$

and $h_j(x)$, $j = 1, \dots, m$, are equality constraints. This P function is suitable for problems where a feasible starting point is known.

When a value of NSEL = 1 is used, the following P function is selected.

$$P(x, r_k) = f(x) - r_k \sum_{s \in S} \ln g_s(x) + \frac{1}{r_k} \sum_{t \in T} g_t^2(x)$$

where

$$S = \{s | g_s(x) > 0\} \quad \text{and} \quad T = \{t | g_t(x) < 0\}$$

This P function is suitable when the initial starting point is infeasible, the path of optimization frequently moves into the infeasible region and the variables remain positive throughout.

If NSEL is set equal to 2, the P function used is

$$P(x, r_k) = f(x) - r_k \sum_{s \in S} \ln x_s + \frac{1}{r_k} \sum_{t \in T} g_t^2(x)$$

where

$$S = \{s | x_s > 0\} \quad \text{and} \quad T = \{t | g_t(x) < 0\}.$$

This P function is suitable when the initial starting point is infeasible, the path of optimization frequently moves into the infeasible region and the variables tend to take negative values thus violating non-negativity constraints.

(vi) METHOD is the option used to select the unconstrained minimization technique to be used. METHOD = 0 indicates the method of Fletcher and Reeves and METHOD = 1 the method of Fletcher and Powell. It is suggested that METHOD be set to zero, so that the method of Fletcher and Reeves is used first.

DATA CARD NO. 3 - This is used to provide the initial value of $S(r_0)$, ratio and the stopping criteria. If a value of $S(r_0)$ is known, it can be supplied here. Otherwise this may be set to zero and NR set equal to 1. The ratio for reducing r_k at each stage is to be supplied and a suggested value is $RATIO = 4$. EPS is the stagewise stopping criteria and a suggested range of values is

EPS = 0.4	if	$ f(x) > 10^5$
= 0.1	if	$10^3 < f(x) < 10^5$
= 0.01	if	$100 < f(x) < 10^3$
= 0.001	if	$0 < f(x) < 100.$

Depending upon the problem, however, other values of EPS may also be tried. The overall stopping criteria THETA may be set to 10^{-5} . EST is an estimate of the minimum value of the objective function. An overestimated value is found useful when the method of Fletcher and Reeves is used, whereas an underestimated value is useful for the method of Fletcher and Powell.

DATA CARD NO. 4 - This provides the initial starting value of x. These are one per card with the variable J being used to identify the sequence in which they are placed.

A detailed computer program which implements the sequential unconstrained minimization technique by conjugate gradient methods is presented in Appendix.

CHAPTER 5

APPLICATIONS

The following test problems were solved by the computer program using the method of SUMT with conjugate gradients:

1. Two-period production and inventory control problem with two decision variables and four inequality constraints.
2. Two-period production and inventory control problem with two decision variables, four inequality constraints and one equality constraint.
3. Five-stage production and inventory control problem with 10 decision variables and 10 inequality constraints.
4. Ten-stage production and inventory control problem with 20 decision variables and 20 inequality constraints.
5. Test Problem from Shell Development Company with 5 decision variables and 15 inequality constraints.
6. Test Problem from Proctor and Gamble Company with 5 decision variables, 6 non-linear constraints and 10 bounds on the variables.
7. Test Problem from Westinghouse Research Laboratory with 4 decision variables and 8 constraints.

5.1. PROBLEM NO. 1

This is a two-dimensional production scheduling problem with four inequality constraints [16]. The problem is to minimize the sum of the production cost and inventory cost subject to the constraints of the nonnegative inventory and the maximum capacity of the machine which

produces the required items. The demand of each period is known and must be satisfied.

The costs for changing the production level and for carrying inventory are given by

$$C(x_i - x_{i-1})^2 = \text{cost due to the change in the production level from the } (i-1)^{\text{th}} \text{ period to the } i^{\text{th}} \text{ period}$$

and

$$D(E - I_i)^2 = \text{inventory cost at the } i^{\text{th}} \text{ period}$$

where C , D , and E are positive constants. x_i and I_i are, respectively, the production level and inventory level at the i^{th} period.

The problem is to find $\underline{x}^* = (x_1^*, x_2^*, \dots, x_n^*)$ which minimizes

$$f(x) = \sum_{i=1}^n [C(x_i - x_{i-1})^2 + D(E - I_i)^2]$$

subject to

$$I_i = I_{i-1} + x_i - Q_i \geq 0, \quad i = 1, 2, \dots, n$$

and

$$0 \leq x_i \leq M, \quad i = 1, 2, \dots, n$$

where M is the maximum production capacity and Q_i represents the sales at the i^{th} period. x_0 and I_0 are the production level and inventory level at the initial period respectively.

For a two period production and inventory system the problem can be stated as:

Minimize

$$f(x) = C(x_1 - x_0)^2 + D(E - I_1)^2 \\ + C(x_2 - x_1)^2 + D(E - I_2)^2$$

subject to

$$g_1(x) = I_1 = I_0 + x_1 - Q_1 \geq 0$$

$$g_2(x) = I_2 = I_1 + x_2 - Q_2 \geq 0$$

$$g_3(x) = M - x_1 \geq 0$$

$$g_4(x) = M - x_2 \geq 0.$$

The values of the constants $C, D, E, M, x_0, I_0, Q_1$ and Q_2 are

$$C = 100, \quad D = 20, \quad E = 10, \quad M = 30, \\ x_0 = 15, \quad I_0 = 12, \quad Q_1 = 30, \quad Q_2 = 10.$$

5.1.1 Approach Used

This problem was first approached as an interior point SUMT problem.

The P function was defined as

$$P(x, r_k) = f(x) + r_k \sum_{i=1}^4 \frac{1}{g_i(x)} \\ = 100(x_1 - 15)^2 + 20(28 - x_1)^2 + 100(x_2 - x_1)^2 + 20(38 - x_1 - x_2)^2 \\ + r_k \left(\frac{1}{x_1 - 18} + \frac{1}{x_1 + x_2 - 28} + \frac{1}{30 - x_1} + \frac{1}{30 - x_2} \right)$$

where r_k corresponds to $S(r_k)$ and $\sum_{i=1}^4 \frac{1}{g_i(x)}$ corresponds to $I(x)$.

The computer program for SUMT with conjugate gradients was used to solve the problem. This resulted in one of the constraints, $g_1(x)$ being violated and the solution converged to the unconstrained optimum of $x^{**} = (x_1^{**}, x_2^{**}) = (17.82, 18.21)$.

The reason for this violation can be explained as follows:

According to the intuitive concept of the P function, the penalty term $\sum \frac{1}{g_i(x)}$ tends to approach infinity as the constraint is approached from inside the feasible region. This is considered sufficient to prevent any constraint being violated. However, in actual practice this is not necessarily true. This is due to the fact that all iterative methods of unconstrained optimization move towards the optimal solution in discrete stepsizes. This means that at one stage during the iterative procedure the point may be inside the feasible region. This may move into the infeasible region when the next point is obtained using the calculated stepsize. In this problem, for example, the point before the constraint was violated was $\underline{x} = (x_1, x_2) = (18.737, 18.881)$. The next point obtained by the method of Fletcher and Reeves was $(17.792, 18.175)$. This resulted in constraint $g_1(x)$ being violated. Once the trajectory moves into the infeasible region, the term $r_k \sum \frac{1}{g_i(x)}$ ceases to be a penalty term and the P function decreases in value as the trajectory moves into the infeasible region. This results in the optimal trajectory moving towards a point which is the optimum of the unconstrained objective function.

Therefore it becomes necessary to impose additional penalty factors on the P function so that the P function increases in value considerably whenever a constraint is violated. The following modifications to the

P function, which impose extra penalties for constraint violation, were tested on this problem:

$$1. \quad P(x, r_k) = f(x) + r_k \sum_{s \in S} \frac{1}{g_s(x)} - r_k \sum_{t \in T} \frac{1}{g_t(x)} |f(x)| \quad (1)$$

where $S = \{s | g_s(x) > 0\}$ and $T = \{t | g_t(x) < 0\}$.

$$2. \quad P(x, r_k) = f(x) + r_k \sum_{s \in S} \frac{1}{g_s(x)} + r_k^{-1/2} \sum_{t \in T} g_t^2(x) \quad (2)$$

where $S = \{s | g_s(x) > 0\}$ and $T = \{t | g_t(x) < 0\}$.

Both the above modifications of the P functions did not prevent the constraints being violated. However, they were able to bring the trajectory of the optimal path back into the feasible region. The P function defined by equation (2) required considerably more computer time and more number of iterations than the P function defined by equation (1) to converge to the optimum. The P function defined by equation (1) was suitable as long as the value of $r_k \cdot |f(x)|$ was greater than 1. If this value became smaller than 1, the penalty imposed by this term was not adequate and as such the trajectory of optimization moved out into the infeasible region. Therefore the P function was further modified as

$$P(x, r_k) = f(x) + r_k \sum_{s \in S} \frac{1}{g_s(x)} - \text{CONST} \sum_{t \in T} \frac{1}{g_t(x)}$$

where

$$S = \{s | g_s(x) > 0\}, \quad T = \{t | g_t(x) < 0\}$$

and

$$\begin{aligned} \text{CONST} &= |f(x)| && \text{if } |f(x)| > r_k^{-0.5} \\ &= r_k^{-0.5} && \text{otherwise.} \end{aligned}$$

This modification to the P function enabled the program to handle constraint violations for different values of r_k and to converge to an optimum solution. With this modification this problem is an example of the application of the mixed interior point-exterior point SUMT.

The P function that was finally adopted for this problem was

$$P(x, r_k) = f(x) + r_k \sum_{s \in S} \frac{1}{g_s(x)} - \text{CONST} \sum_{t \in T} \frac{1}{g_t(x)} \quad (3)$$

where

$$S = \{s | g_s(x) > 0, s = 1, 2, 3, 4\},$$

$$T = \{t | g_t(x) < 0, t = 1, 2, 3, 4\}$$

and

$$\begin{aligned} \text{CONST} &= |f(x)| && \text{if } |f(x)| > r_k^{-1/2} \\ &= r_k^{-1/2} && \text{otherwise.} \end{aligned}$$

The terms r_k , $\sum \frac{1}{g_s(x)}$, CONST and $-\sum \frac{1}{g_t(x)}$ in the above P function

correspond, respectively, to $S(r)$, $I(x)$, $A(t)$ and $B(x)$ in the mixed point SUMT algorithm.

5.1.2. Infeasible Starting Point

The problem was attempted with the P function given by equation (3), but by starting from an infeasible starting point $x^0 = (x_1^0, x_2^0) = (5, 10)$. It can be seen that at this point constraints $g_1(x)$ and $g_2(x)$ are violated. Using the method of Fletcher and Powell the optimal solution $x^* = (x_1^*, x_2^*) = (18.002, 18.331)$ was obtained, but the method of Fletcher and Reeves failed to yield the optimal solution. Therefore, a modified P function as suggested in reference [9] and shown below was tried:

$$P(x, r_k) = f(x) - r_k \sum_{s \in S} \ln g_s(x) + \frac{1}{r_k} \sum_{t \in T} g_t^2(x) \quad (4)$$

where $S = \{s | g_s(x) > 0\}$, and $T = \{t | g_t(x) < 0\}$.

With this P function the optimal solution was obtained with a considerably lesser number of iterations and computing time as shown below.

	Computing Time	No. of Iterations
P function, equation (3)	79.4 sec.	120
P function, equation (4)	60.9 sec.	53

The same P function was tried for this problem with a feasible starting point of $x^0 = (x_1^0, x_2^0) = (25, 29)$. However, this resulted in constraints being violated and the optimum of the unconstrained problem being reached. Therefore the P function defined by equation (3) should be considered suitable when the initial starting point is feasible. If the initial starting point is infeasible, the P function of equation (4) should be used.

5.1.3 Results and Discussions

The optimal results obtained by the present approach are presented in Tables 1, 2, 2A, 3, and 4 . Table 1 presents the optimal results and the details of convergence towards the optimal result from a feasible starting point by the method of Fletcher and Powell. Tables 2 and 2A present the optimal results and details of convergence starting from an infeasible starting point using the method of Fletcher and Powell and with the P function defined by equation (3) and equation (4) respectively. Table 3 presents the optimal results and details of convergence starting from a feasible starting point and using the method of Fletcher and Reeves. Table 4 presents the optimal results and details of convergence when the initial starting point was infeasible using the method of Fletcher and Reeves.

Four formulations of P functions were tried in solving this problem. The following P functions were found suitable.

1. When the starting point is feasible,

$$P(x, r_k) = f(x) + r_k \sum_{s \in S} \frac{1}{g_s(x)} - \text{CONST} \sum_{t \in T} \frac{1}{g_t(x)}$$

where

$$S = \{s | g_s(x) > 0\}, \quad T = \{t | g_t(x) < 0\}$$

and

$$\begin{aligned} \text{CONST} &= |f(x)| && \text{if } |f(x)| > r_k^{-1/2} \\ &= r_k^{-1/2} && \text{otherwise.} \end{aligned}$$

TABLE 1 TEST PROBLEM NO.1
METHOD OF FLETCHER AND POWELL

STAGE NUMBER	VALUE OF R	NO.OF ITERATIONS	X(1)	X(2)	F(X)	P(X)
0	0.00E 00	0	25.000	25.000	16900.0	0.0
1	0.30E 04	8	20.668	18.168	4927.0	7214.0
2	0.75E 03	9	20.651	18.174	4901.0	5474.0
3	0.19E 03	9	20.611	18.188	4840.0	1984.0
4	0.47E 02	8	20.488	18.190	4678.0	4715.0
5	0.12E 02	9	20.009	18.107	4148.0	4158.0
6	0.29E 01	9	18.681	18.139	3149.0	3154.0
7	0.73E 00	8	18.075	18.399	2973.0	2983.0
8	0.18E 00	8	18.066	18.426	2972.0	2975.0
9	0.46E-01	8	18.049	18.320	2970.7	2971.6
10	0.11E-01	9	18.036	18.308	2969.6	2969.9
11	0.29E-02	9	18.010	18.336	2967.4	2967.6
12	0.72E-03	8	18.003	18.338	2966.9	2967.1

TABLE 2 TEST PROBLEM NO.1
METHOD OF FLETCHER AND POWELL, INFEASIBLE STARTING POINT

STAGE NUMBER	VALUE CF R	NO.OF ITERATIONS	X(1)	X(2)	F(X)	P(X)
0	0.00E 00	0	5.000	10.000	33660.0	0.0
1	0.30E 04	8	20.959	21.706	5033.0	6945.0
2	0.75E 03	8	20.959	21.705	5033.0	5511.0
3	0.19E 03	9	20.157	20.526	4047.0	4188.0
4	0.47E 02	8	19.802	20.184	3743.3	3783.0
5	0.12E 02	8	18.266	18.511	2998.0	3045.0
6	0.29E 01	5	18.154	18.436	2981.0	3001.0
7	0.73E 00	8	18.101	18.461	2976.0	2983.0
8	0.18E 00	9	18.101	18.460	2976.0	2978.0
9	0.46E-01	8	18.098	18.421	2975.0	2976.0
10	0.11E-01	8	18.079	18.367	2973.1	2973.3
11	0.29E-02	9	18.007	18.164	2970.8	2971.2
12	0.72E-03	8	18.014	18.332	2967.6	2967.7
13	0.18E-03	8	18.014	18.331	2967.6	2967.7
14	0.45E-04	8	18.011	18.331	2967.4	2967.5
15	0.11E-04	8	18.002	18.331	2966.8	2966.8

TABLE 2A TEST PROBLEM NO.1
METHOD OF FLETCHER AND POWELL, INFEASIBLE STARTING POINT

STAGE NUMBER	VALUE OF R	NO.OF ITERATIONS	X(1)	X(2)	F(X)	P(X)
0	0.00E 00	0	5.000	10.000	33660.0	0.0
1	0.18E 00	8	17.327	18.219	2960.7	2959.6
2	0.46E-01	5	17.840	18.227	2960.8	2961.0
3	0.12E-01	6	17.878	18.252	2961.3	2962.5
4	0.29E-02	6	17.938	18.292	2963.2	2964.6
5	0.72E-03	9	17.979	18.319	2965.3	2966.0
6	0.18E-03	7	17.994	18.330	2966.3	2966.5
7	0.45E-04	5	17.999	18.332	2966.6	2966.7
8	0.11E-04	8	18.000	18.332	2966.6	2966.6

TABLE 3 TEST PROBLEM NO.1
METHOD OF FLETCHER AND REEVES

STAGE NUMBER	VALUE OF R	NO. OF ITERATIONS	X(1)	X(2)	F(X)	P(X)
0	0.00E 00	0	25.000	29.000	16900.0	0.0
1	0.30E 04	5	19.937	19.601	3796.0	6192.0
2	0.75E 03	5	19.204	19.134	3317.0	4151.0
3	0.19E 03	7	18.740	18.828	3118.0	3425.0
4	0.47E 02	8	18.448	18.639	3034.0	3152.0
5	0.12E 02	5	18.266	18.515	2998.0	3045.0
6	0.29E 01	7	18.154	18.436	2981.0	3001.0
7	0.73E 00	7	18.086	18.390	2974.0	2983.0
8	0.18E 00	11	18.047	18.364	2970.2	2974.2
9	0.46E-01	8	18.025	18.350	2968.4	2970.3
10	0.11E-01	17	18.013	18.341	2967.5	2968.5
11	0.29E-02	4	18.006	18.341	2967.1	2967.6
12	0.72E-03	9	18.003	18.337	2966.9	2967.1

TABLE 4 TEST PROBLEM NO.1
METHOD OF FLETCHER AND REEVES, INFEASIBLE STARTING POINT

STAGE	VALUE	NO.OF	X(1)	X(2)	F(X)	P(X)
NUMBER	OF R	ITERATIONS				
0	0.00E+00	0	5.000	10.000	33660.0	0.0
1	0.18E+00	5	17.825	18.213	2960.7	2959.6
2	0.46E+01	7	17.840	18.227	2960.8	2961.0
3	0.11E+01	4	17.878	18.251	2961.3	2962.5
4	0.29E+02	9	17.937	18.288	2963.2	2964.6
5	0.72E+03	4	17.979	18.320	2965.4	2966.0
6	0.18E+03	4	17.994	18.321	2966.3	2966.5
7	0.45E+04	4	17.999	18.322	2966.6	2966.7
8	0.11E+04	4	18.000	18.322	2966.6	2966.6

2. When the starting point is infeasible,

$$P(x, r_k) = f(x) - r_k \sum_{s \in S} \ln g_s(x) + \frac{1}{r_k} \sum_{t \in T} g_t^2(x)$$

where

$$S = \{s | g_s(x) > 0\} \quad \text{and} \quad T = \{t | g_t(x) < 0\}.$$

An initial value of $r_0 = 3000$, which was found suitable for this problem when it was solved by Hsu, et. al [16], was used. The stopping criteria used were $EPS = 0.1$ and $THETA = 10^{-4}$ while the value of EST used was 2000 for the method of Fletcher and Powell and 3000 for the method of Fletcher and Reeves.

The computing time required for this problem on an IBM 360-50 WATFOR compiler were (i) 53.1 and 56.5 seconds by the method of Fletcher and Powell starting from a feasible starting point and an infeasible starting point respectively, (ii) 59.7 and 52.5 seconds by the method of Fletcher and Reeves starting from a feasible starting point and an infeasible starting point respectively. The computing times required by Lai [19] and Hsu [16] on the same compiler for this problem are 36.0 and 100.8 seconds respectively.

5.2 PROBLEM NO. 2

This problem is the same as Problem No. 1 except that an equality constraint is added. The new constraint is a requirement that the production level in the first period be exactly five units larger than that in the second period.

The problem then is:

Minimize

$$f(x) = 100(x_1 - 15)^2 + 20(28 - x_1)^2 + 100(x_2 - x_1)^2 \\ + 20(38 - x_1 - x_2)^2$$

subject to

$$g_1(x) = x_1 - 18 \geq 0$$

$$g_2(x) = x_1 + x_2 - 28 \geq 0$$

$$g_3(x) = 30 - x_1 \geq 0$$

$$g_4(x) = 30 - x_2 \geq 0$$

$$h_1(x) = x_1 - x_2 - 5 = 0$$

5.2.1. Approach Used

This problem is similar to Problem No. 1 except for an additional equality constraint. Therefore a P function similar to that of problem 1 with suitable modifications to handle equality constraints as suggested in reference [16] was used.

The P function is

$$P(x, r_k) = f(x) + r_k \sum_{s \in S} \frac{1}{g_s(x)} - \text{CONST} \sum_{t \in T} \frac{1}{g_t(x)} \\ + r_k^{-1/2} \sum_{j=1} h_j^2(x)$$

where $S = \{s | g_s(x) > 0\}$, $T = \{t | g_t(x) < 0\}$ and $h_1(x)$ is the equality constraint and

$$\begin{aligned} \text{CONST} &= |f(x)| & \text{if } |f(x)| > r^{-0.5} \\ &= r^{-0.5} & \text{otherwise.} \end{aligned}$$

The problem was first solved by the method of Fletcher and Reeves from a starting point of $\underline{x}^0 = (x_1^0, x_2^0) = (25, 29)$. At this point $h_1(x)$ has a value of -9.0. A value of $r_0 = 3000$, $\text{EPS} = 0.1$, $\text{EST} = 3000$ and overall stopping criterion of 10^{-5} was used. The optimum value of objective function $f(x) = 6217.4$ was obtained which compares well with the value of $f(x) = 6210.0$ obtained by Lai [19].

The same problem was then solved using the method of Fletcher and Powell. The input parameters used were, $r_0 = 3000$, $\text{EPS} = 0.1$, $\text{EST} = 2000$ and overall stopping criterion = 10^{-5} . The optimum value obtained was $f(x) = 6217.4$.

The problem was next attempted from an infeasible starting point of $\underline{x}^0 = (x_1^0, x_2^0) = (5, 10)$. The violated constraints at this point are $g_1(x)$, $g_2(x)$ and $h_1(x)$. Two approaches were used to solve this problem.

1. The first approach was to solve this as an exterior point SUMT problem using the following P function.

$$P(x, r_k) = f(x) + r_k \sum_{s \in S} \frac{1}{g_s(x)} - \text{CONST} \sum_{t \in T} \frac{1}{g_t(x)} + r_k^{-1/2} h_1^2(x)$$

where $S = \{s | g_s(x) > 0\}$, $T = \{t | g_t(x) < 0\}$ and $h_1(x)$ is the inequality constraint.

The solution of $f(x) = 6217.4$ was obtained by the method of Fletcher and Powell, and that of $f(x) = 6217.0$ by the method of Fletcher and Reeves.

2. The second approach attempted was to use the following P function, which was successfully used in Problem No. 1:

$$P(x, r_k) = f(x) - r_k \sum_{s \in S} \ln g_s(x) + \frac{1}{r_k} \sum_{t \in T} g_t^2(x) + r_k^{-1/2} h_1^2(x)$$

where $S = \{s | g_s(x) > 0\}$, $T = \{t | g_t(x) < 0\}$ and $h_1(x)$ is the equality constraint. However, with the above P function the iterations converged to a false optimum of $f(x) = 6428$ and $\underline{x}^* = (x_1^*, x_2^*) = (19.936, 14.940)$ with $h_1(x) = -0.004$.

5.2.2. Results and Discussion

The optimal results obtained by using the new method are presented in Tables 5, 6, 7 and 8. Tables 5 and 6 represent the optimal results and details of convergence towards the optimal result using the method of Fletcher and Powell and starting from a feasible starting point and an infeasible starting point respectively. Tables 7 and 8 represent the optimal results and details of convergence using the method of Fletcher and Reeves and starting from a feasible starting point and an infeasible starting point respectively.

Two formulations of P functions were examined when solving this problem. The P function which was successful in yielding the optimum was

TABLE 5 TEST PROBLEM NC.2
METHOD OF FLETCHER AND POWELL

STAGE NUMBER	VALUE OF R	NO. OF ITERATIONS	X(1)	X(2)	F(X)	P(X)	FH(1)
0	0.00E+00	0	25.000	29.000	16900.0	0.0	-9.000
1	0.51E+02	8	24.688	24.863	12277.0	12315.0	-5.200
2	0.13E+02	8	22.190	19.226	6956.0	6965.0	-2.040
3	0.32E+01	8	20.669	18.866	4661.0	4669.0	-3.200
4	0.80E+00	8	18.063	18.352	2472.0	3016.0	-5.300
5	0.20E+00	8	18.066	18.291	2973.0	3037.0	-5.200
6	0.50E-01	9	18.057	18.157	2977.0	3094.0	-5.100
7	0.12E-01	8	18.043	17.916	2994.0	3206.0	-4.800
8	0.31E-02	9	18.029	17.708	3019.0	3410.0	-4.700
9	0.78E-03	8	18.083	17.171	3151.0	3749.0	-4.100
10	0.20E-03	3	18.242	16.533	3455.0	4230.0	-3.300
11	0.49E-04	3	18.426	15.795	3985.0	4787.0	-2.400
12	0.12E-04	3	18.596	15.114	4643.0	5301.0	-1.500
13	0.31E-05	3	18.723	14.607	5239.0	5685.0	-0.880
14	0.76E-06	3	18.804	14.285	5663.0	5928.0	-0.480
15	0.19E-06	7	18.850	14.101	5921.0	6066.0	-0.250
16	0.48E-07	5	18.874	14.003	6064.0	6140.0	-0.130
17	0.12E-07	7	18.887	13.952	6140.0	6179.0	-0.070
18	0.30E-08	5	18.893	13.926	6178.0	6198.0	-0.030
19	0.75E-09	8	18.897	13.913	6198.0	6208.0	-0.020
20	0.19E-09	9	18.898	13.907	6208.0	6213.0	-0.008
21	0.47E-10	9	18.900	13.905	6213.0	6216.0	-0.004
22	0.12E-10	8	18.900	13.902	6216.0	6217.0	-0.002
23	0.29E-11	9	18.900	13.901	6216.7	6217.4	-0.001
24	0.73E-12	8	18.900	13.900	6217.4	6217.8	-0.001

TABLE 6 TEST PROBLEM NO.2
METHOD OF FLETCHER AND POWELL, INFEASIBLE STARTING POINT

STAGE NUMBER	VALUE OF R	NO. OF ITERATIONS	X(1)	X(2)	F(X)	P(X)	FH(1)
0	0.00E+00	0	5.000	10.000	33660.0	0.0	-10.000
1	0.51E+02	8	11.843	10.647	11172.0	208247.0	-3.800
2	0.13E+02	8	11.975	10.716	10897.0	60414.0	-3.700
3	0.32E+01	8	19.184	17.964	3468.0	3480.0	-3.800
4	0.80E+00	9	18.091	18.343	2975.0	3015.0	-5.300
5	0.20E+00	8	18.089	18.314	2975.0	3038.0	-5.200
6	0.50E+01	9	18.065	18.118	2980.0	3095.0	-5.100
7	0.12E+01	9	18.046	17.934	2992.0	3206.0	-4.900
8	0.30E+02	9	18.047	17.703	3023.0	3410.0	-4.700
9	0.78E+03	5	18.083	17.170	3152.0	3749.0	-4.100
10	0.20E+03	6	18.242	16.533	3455.0	4230.0	-3.300
11	0.50E+04	3	18.426	15.795	3985.0	4787.0	-2.400
12	0.12E+04	3	18.597	15.114	4643.0	5301.0	-1.500
13	0.31E+05	3	18.723	14.607	5239.0	5685.0	-0.900
14	0.76E+06	3	18.804	14.285	5663.0	5928.0	-0.500
15	0.19E+06	7	18.850	14.101	5921.0	6066.0	-0.250
16	0.48E+07	5	18.874	14.003	6064.0	6140.0	-0.130
17	0.12E+07	7	18.887	13.952	6140.0	6179.0	-0.070
18	0.30E+08	5	18.893	13.926	6178.0	6198.0	-0.030
19	0.75E+09	8	18.897	13.913	6198.0	6208.0	-0.020
20	0.19E+09	9	18.898	13.907	6208.0	6213.0	-0.008
21	0.47E+10	9	18.900	13.905	6213.0	6216.0	-0.004
22	0.12E+10	8	18.900	13.902	6215.0	6217.0	-0.002
23	0.30E+11	9	18.900	13.901	6216.7	6217.4	-0.001
24	0.70E+12	8	18.900	13.900	6217.4	6217.7	-0.001

TABLE 7 TEST PROBLEM NO.2
METHOD OF FLETCHER AND REEVES

STAGE NUMBER	VALUE OF R	NO.OF ITERATIONS	X(1)	X(2)	F(X)	P(X)	FH(1)
0	0.00E-00	0	25.000	29.000	16900.0	0.0	-9.000
1	0.51E-02	5	18.644	18.994	3093.0	3197.0	-5.350
2	0.13E-02	4	18.643	18.994	3093.0	3126.0	-5.350
3	0.32E-01	4	18.641	18.993	3092.7	3115.0	-5.350
4	0.80E-00	4	18.634	18.992	3090.7	3124.0	-5.360
5	0.20E-00	4	18.614	18.985	3085.1	3150.0	-5.370
6	0.50E-01	4	18.586	18.970	3077.2	3207.0	-5.380
7	0.12E-01	5	18.059	18.009	2987.1	3206.0	-4.950
8	0.31E-02	4	17.970	17.623	3022.1	3409.0	-4.650
9	0.78E-03	4	18.085	17.174	3150.9	3749.0	-4.090
10	0.20E-03	3	18.242	16.533	3455.2	4230.0	-3.290
11	0.49E-04	3	18.426	15.795	3984.9	4787.0	-2.370
12	0.12E-04	3	18.597	15.114	4642.6	5301.0	-1.520
13	0.31E-05	3	18.723	14.606	5238.5	5685.0	-0.880
14	0.76E-06	4	18.804	14.285	5662.9	5927.0	-0.480
15	0.19E-06	4	18.850	14.101	5921.0	6066.0	-0.250
16	0.48E-07	4	18.874	14.003	6064.2	6140.0	-0.130
17	0.12E-07	4	18.877	13.952	6139.7	6179.0	-0.065
18	0.30E-08	4	18.895	13.927	6178.5	6198.0	-0.033
19	0.75E-09	4	18.903	13.919	6198.1	6208.0	-0.016
20	0.19E-09	4	18.898	13.907	6208.0	6213.0	0.000
21	0.47E-10	4	18.900	13.904	6213.0	6215.5	-0.004
22	0.12E-10	4	18.901	13.904	6215.5	6216.7	-0.002
23	0.30E-11	4	18.902	13.903	6216.7	6217.3	-0.001
24	0.73E-12	4	18.902	13.903	6217.4	6217.6	-0.001

TABLE 8 TEST PROBLEM NO.2
METHOD OF FLETCHER AND REEVES, INFEASIBLE STARTING POINT

STAGE NUMBER	VALUE OF R	NO.OF ITERATIONS	X(1)	X(2)	F(X)	P(X)	FH(1)
0	0.00E 00	0	5.000	10.000	33660.0	0.0	-10.000
1	0.73E 00	4	5.000	10.000	33660.0	146457.0	-10.000
2	0.18E 00	5	7.774	9.954	22096.0	57949.0	-7.200
3	0.45E-01	4	11.457	10.272	12165.0	26528.0	-3.800
4	0.11E-01	4	13.139	11.032	9033.0	14453.0	-2.900
5	0.29E-02	6	16.876	15.640	3581.0	6392.0	-3.800
6	0.72E-03	4	16.188	15.640	3581.0	4747.0	-3.800
7	0.18E-03	4	17.092	15.610	3598.0	4721.0	-3.500
8	0.44E-04	4	18.147	15.130	4288.0	1876.0	-2.000
9	0.11E-04	3	18.606	15.075	4685.0	5331.0	-1.500
10	0.28E-05	4	18.730	14.580	5272.0	5704.0	-0.900
11	0.70E-06	4	18.808	14.269	5684.0	5939.0	-0.500
12	0.17E-06	4	18.852	14.093	5933.0	6072.0	-0.200
13	0.44E-07	4	18.875	13.999	6071.0	6144.0	-0.100
14	0.11E-07	4	18.888	13.950	6143.0	6180.0	-0.060
15	0.27E-08	4	18.903	13.934	6180.0	6199.0	-0.030
16	0.68E-09	4	18.906	13.922	6199.0	6208.0	-0.020
17	0.17E-09	4	18.910	13.918	6208.0	6213.0	-0.008
18	0.43E-10	4	18.912	13.916	6213.0	6216.0	-0.004
19	0.11E-10	4	18.913	13.915	6216.0	6217.0	-0.002
20	0.27E-11	4	18.913	13.914	6217.0	6217.0	-0.001

$$P(x, r_k) = f(x) + r_k \sum_{s \in S} \frac{1}{g_s(x)} - \text{CONST} \sum_{t \in T} \frac{1}{g_t(x)} \\ + r_k^{-1/2} h_1^2(x)$$

where $S = \{s | g_s(x) > 0\}$, $T = \{t | g_t(x) < 0\}$ and $h_1(x)$ is the inequality constraint.

The parameters and stopping criterion used for this problem are $r_0 = 3000$, $\text{EST} = 2000$ for the method of Fletcher and Powell and $\text{EST} = 3000$ for the method of Fletcher and Reeves, $\text{EPS} = 0.1$ and $\text{THETA} = 10^{-5}$.

The optimum value of $f(x)$ obtained was 6217.0 which compares favorably with $f(x) = 6210.1$ obtained by Lai [19]. The computing time required on an IBM 360-50 WATFOR compiler were (i) 101.6 and 104.5 seconds using the method of Fletcher and Powell and initial starting points of $\underline{x}^0 = (25, 29)$ and $\underline{x}^0 = (5, 10)$ respectively, (ii) 90.0 and 105.8 seconds using the method of Fletcher and Reeves and initial starting points of (25, 29) and (5, 10) respectively.

5.3 PROBLEM NO. 3 AND 4

This problem is the well-known multi-stage personnel and production scheduling problem of Holt, Modigliani, Muth and Simon [14]. The problem is to find the optimal production level and work force level which minimize the overall operation cost in a paint factory. The subcosts which make up the overall operation cost are (i) the cost of pay-roll, (ii) the cost of hiring and laying off, (iii) the cost of overtime and (iv) the inventory cost. The schematic diagram of the system is shown in Fig. 3. The problem is to minimize the sum of all four different costs

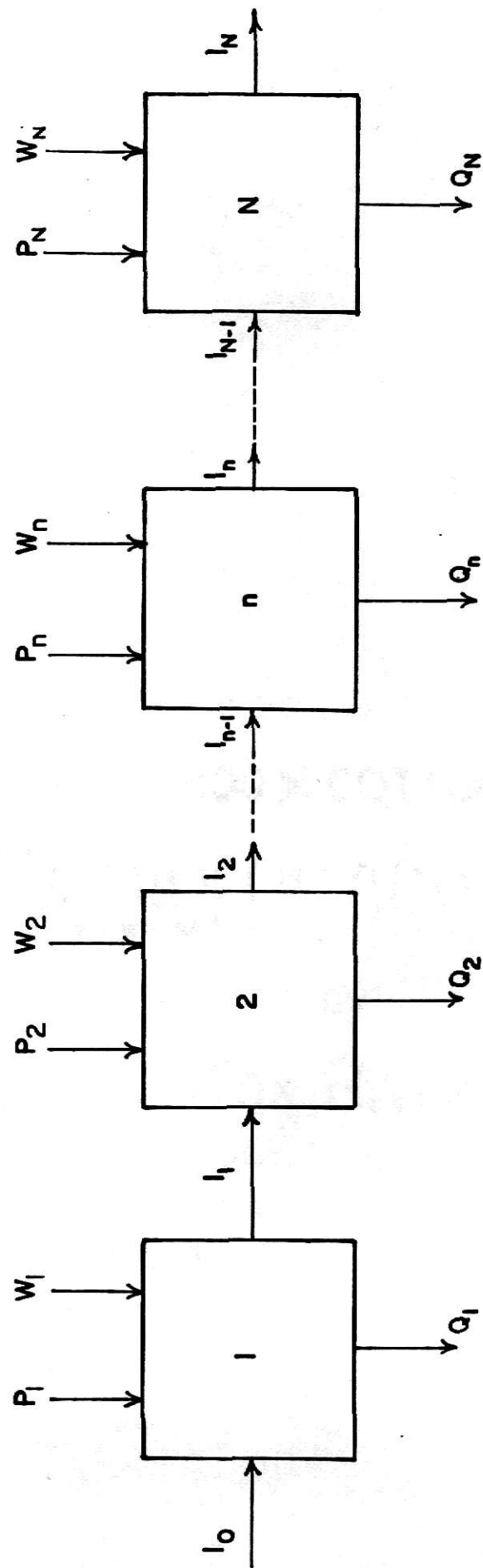


Fig.3 Block diagram for personnel and production scheduling.

over a planning period subject to the constraints of the non-negative inventory and non-negative overtime cost. The demand for each period is known and must be satisfied.

Let

n = a month in the planning horizon

N = duration, in months

P_n = production rate in the n^{th} month

W_n = work force level in the n^{th} month

Q_n = demand in the n^{th} month

I_n = inventory level at the end of the n^{th} month

The inventory level at the end of each month is represented by the recursive relationship among current demand, current production and inventory level of the preceding month.

$$I_n = I_{n-1} + P_n - Q_n, \quad n = 1, 2, \dots, N.$$

The model considers the following monthly operation costs

1. Regular payroll cost = $340.0 W_n$
2. Hiring and lay off cost = $64.3 (W_n - W_{n-1})^2$
3. Overtime cost = $0.2 (P_n - 5.67 W_n)^2 + 51.2 P_n - 281.0 W_n$
4. Inventory cost = $0.0825 (I_n - 320.0)^2$

The system can then be represented by the following model.

Minimize

$$f(P_1, P_2, \dots, P_N; W_1, W_2, \dots, W_N) = \sum_{n=1}^N S_n$$

subject to

$$I_n = I_{n-1} + P_n - Q_n \geq 0, \quad n = 1, 2, \dots, N-1$$

$$I_N = I_{N-1} + P_N - Q_N \geq I_f$$

and

$$0.2 (P_n - 5.67 W_n)^2 + 51.2 P_n - 281.0 W_n \geq 0, \quad n = 1, 2, \dots, N$$

where

$$\begin{aligned} S_n = & [340.0 W_n] + [64.3 (W_n - W_{n-1})^2] \\ & + [0.2 (P_n - 5.67 W_n)^2 + 51.2 P_n - 281.0 W_n] \\ & + [0.0825 (I_n - 320.0)^2] \end{aligned}$$

The reason for considering the non-negative overtime cost is due to the characteristics of its mathematical formula:

$$\text{overtime cost} = 0.2 (P_n - 5.67 W_n)^2 + 51.2 P_n - 281.0 W_n$$

The equation is in quadratic form and can have negative cost contours, as shown in Fig. 4 [16]. Taubert [22] found that minimizing total costs over the planning period by selecting a certain W_n and P_n combination contributed a negative overtime cost to the objective function. Since such a negative cost is illogical in the context of the paint factory problem, the constraint of non-negative overtime cost should be imposed.

Problem No. 3 considers five-months of the above problem. The numerical data that were used are as follows:

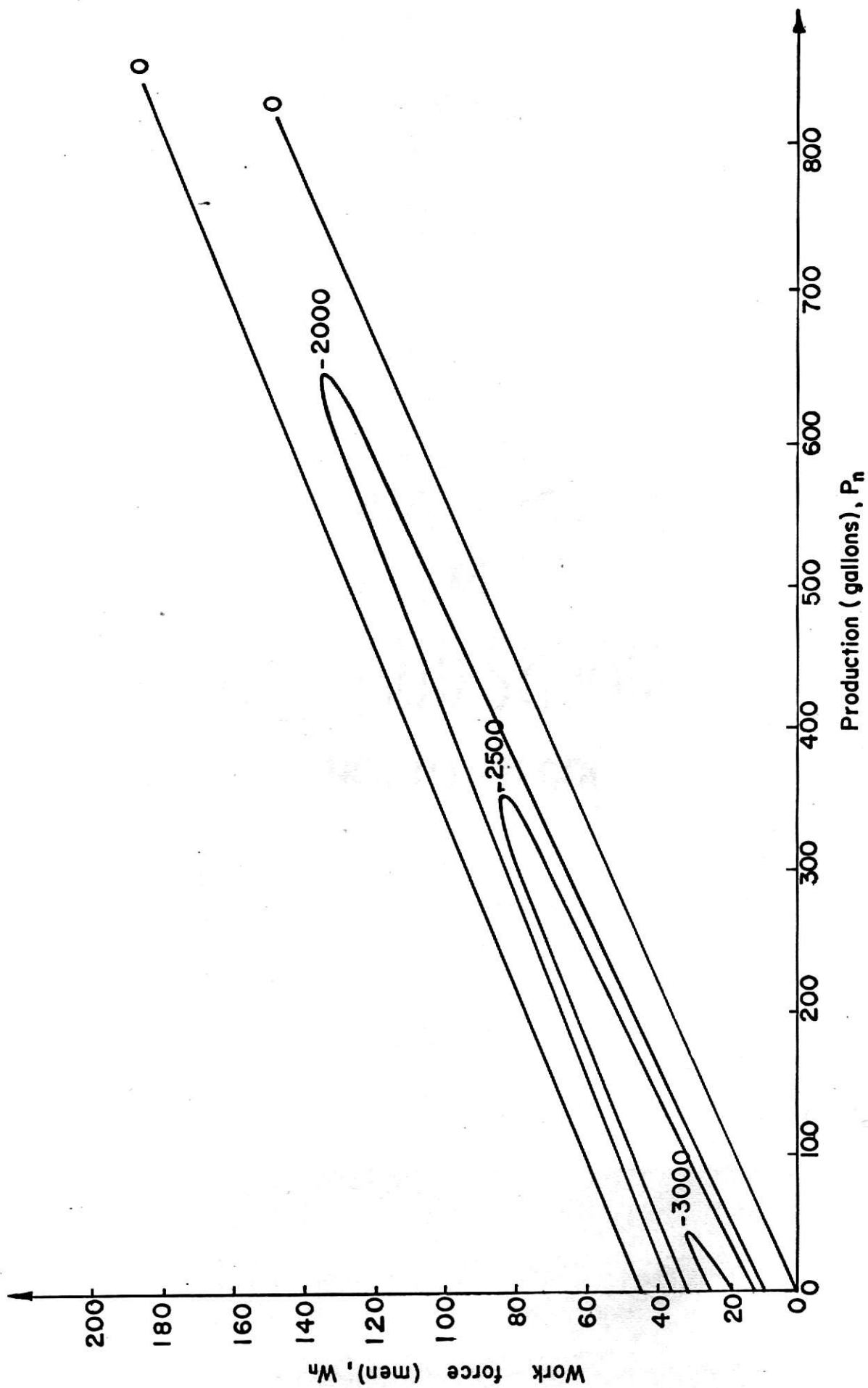


Fig.4. Negative cost contours of the overtime cost equation [18].

$$\text{Demand: } Q_1 = 430$$

$$Q_2 = 447$$

$$Q_3 = 440$$

$$Q_4 = 316$$

$$Q_5 = 397$$

$$\text{Initial inventory, } I_0 = 263$$

$$\text{Inventory for last month, } I_f = 263$$

$$\text{Initial work force level, } W_0 = 81$$

The starting point was chosen arbitrarily at

$$\begin{aligned} \underline{x}^0 &= [P_1^0, W_1^0; P_2^0, W_2^0; P_3^0, W_3^0; P_4^0, W_4^0; P_5^0, W_5^0] \\ &= [500, 90; 500, 90; 500, 90; 500, 90; 500, 90] \end{aligned}$$

5.3.1 Approach used

An attempt was made to solve this problem by using an interior-point SUMT P function defined by

$$P(x, r_k) = f(x) + r_k \sum_{i=1}^{10} \frac{1}{g_i(x)}$$

and using the method of Fletcher and Reeves. This, however, resulted in the trajectory of optimization moving out of the feasible region. The reasons for this are explained in Problem No. 1. Therefore the following P function which was adequate in solving test Problem No. 1 was next tried:

$$P(x, r_k) = f(x) + r_k \sum_{s \in S} \frac{1}{g_s(x)} - \text{CONST} \sum_{t \in T} \frac{1}{g_t(x)}$$

where

$$S = \{s | g_s(x) > 0\}, \quad T = \{t | g_t(x) < 0\}$$

and

$$\begin{aligned} \text{CONST} &= |f(x)| && \text{if } |f(x)| > r^{-0.5} \\ &= r^{-0.5} && \text{otherwise} \end{aligned}$$

An initial value of $r_0 = 3000$, which has earlier been used by Hsu et al. [16] to solve this problem was used. The overall stopping criteria used was 10^{-5} . A value for the stagewise stopping criterion EPS was not immediately apparent. Since the order of magnitude of the minimum of the objective function was forty times that of the minimum of test Problem No. 1 a value of EPS = 5.0 was first tried. This resulted in the iterations terminating at a false optimum as shown in table 10A. The optimum value of $f(x) = 138,291$ obtained was quite different from the optimum $f(x) = 129,096$ obtained by Hsu et al. Next a value of EPS = 1.0 was tried and again the optimum value obtained was $f(x) = 138,291$. Finally a value of EPS = 0.1, yielded the optimum value of $f(x) = 129,137$, which compares favourably with the solution obtained by Hsu [16]. A value of EST = 200,000, which is an estimate of the minimum value of the objective function was used throughout.

While this program was being solved a run was made in which the overall stopping criterion was accidentally made equal to 0.0. This resulted in the problem being executed for 10 minutes which was the

maximum time specified for this run. This, however, produced a new constrained optimum value of $f(x) = 127,551$. Compared to the earlier optimum value of $f(x) = 129,137$, this required 94 more iterations. The actual value of the stopping criteria at this point was 10^{-24} .

The problem was then attempted using the method of Fletcher and Powell and values of $EPS = 0.1$, and $EST = 200,000$. This resulted in the trajectory of optimization going out of the feasible region. An analysis of the IBM subroutine FMFP (which was used in the computer program) indicated that better results would be obtained if a value of EST which is less than the actual minimum of the objective function were used. Therefore a value of $EST = 100,000$ was tried next. This resulted in an optimum value of $f(x) = 129,546$ which is about 0.4 percent away from the optimal solution presented by Hsu. The method of Fletcher and Powell was found to be quite sensitive to changes in EST .

5.3.2 Results and Discussion

The optimal results obtained by the present approach are presented in Tables 9, 10 and 10A.

Table 9 represents the optimal result and details of convergence using the method of Fletcher and Powell. Tables 10 and 10A represent the optimal result and rate of convergence using the method of Fletcher and Powell and with values of $EPS = 0.1$ and 5.0 respectively.

Two formulations of P functions were tested on this problem and the P function which was successful in yielding the optimal result was

$$P(x, r_k) = f(x) + r_k \sum_{s \in S} \frac{1}{g_s(x)} - \text{CONST} \sum_{t \in T} \frac{1}{g_t(x)}$$

TABLE 9 TEST PROBLEM NO.3
METHOD OF FLETCHER AND POWELL

STAGE NUMBER	VALUE OF R	NO.OF ITERATIONS	P(1) (W(1))	P(2) (W(2))	P(3) (W(3))	P(4) (W(4))	P(5) (W(5))	F(X)	P(X)
0	0.00E 00	0	500.0 (90.0)	500.0 (90.0)	500.0 (90.0)	500.0 (90.0)	500.0 (90.0)	183548.	C.
1	0.30E 04	103	460.0 (77.0)	435.8 (74.3)	416.9 (71.7)	382.8 (68.5)	387.4 (69.1)	131189.	131309.
2	0.75E 03	162	459.9 (77.0)	435.8 (74.3)	416.8 (71.7)	382.8 (68.5)	387.3 (69.0)	131176.	131205.
3	0.19E 03	113	458.5 (76.7)	433.8 (73.5)	413.4 (71.3)	378.9 (68.9)	381.7 (69.4)	130457.	130471.
4	0.47E 02	182	456.8 (77.0)	431.3 (74.3)	408.8 (71.6)	377.0 (68.7)	373.2 (67.8)	129625.	129630.
5	0.12E 02	35	456.5 (77.0)	430.9 (74.3)	408.2 (71.5)	376.4 (68.5)	372.1 (67.6)	129546.	129547.

TABLE 10 TEST PROBLEM NO.3
METHOD OF FLETCHER AND REEVES

STAGE NUMBER	VALUE OF R	NO.OF ITERATIONS	P(1) (W(1))	P(2) (W(2))	P(3) (W(3))	P(4) (W(4))	P(5) (W(5))	F(X)	P(X)
0	0.00E 00	0	500.0 (90.0)	500.0 (90.0)	500.0 (90.0)	500.0 (90.0)	500.0 (90.0)	183548.	C.
1	0.30E 04	27	415.2 (75.6)	415.3 (74.1)	418.9 (73.2)	426.0 (72.4)	445.7 (72.3)	135434.	135605.
2	0.75E 03	79	453.3 (77.5)	428.3 (74.3)	401.1 (71.4)	380.1 (69.1)	372.9 (67.9)	129291.	129466.
3	0.19E 03	28	453.5 (77.6)	428.1 (74.4)	400.5 (71.4)	379.1 (69.1)	372.0 (67.8)	129220.	129296.
4	0.47E 02	20	453.3 (77.5)	427.9 (74.3)	400.1 (71.5)	378.6 (69.1)	371.5 (67.8)	129177.	129219.
5	0.12E 02	36	453.6 (77.6)	428.0 (74.3)	399.9 (71.4)	378.2 (69.0)	371.1 (67.7)	129156.	129174.
6	0.29E 01	24	453.6 (77.5)	428.0 (74.3)	399.8 (71.4)	378.0 (69.0)	370.9 (67.7)	129146.	129155.
7	0.73E 00	10	453.6 (77.5)	428.0 (74.3)	399.7 (71.4)	378.0 (69.0)	370.9 (67.7)	129141.	129147.
8	0.18E 00	25	453.6 (77.5)	427.9 (74.3)	399.7 (71.4)	378.0 (69.0)	370.9 (67.7)	129137.	129141.
9	0.43E-10	94	460.9 (77.3)	430.0 (73.8)	393.7 (70.4)	342.5 (67.7)	310.3 (66.1)	127551.	127551.

TABLE 10A TEST PROBLEM NO.3
METHOD OF FLETCHER AND REEVES

EPS = 5.0

STAGE NUMBER	VALUE CF R	NO.OF ITERATIONS	P(1) (W(1))	P(2) (W(2))	P(3) (W(3))	P(4) (W(4))	P(5) (W(5))	F(X)	P(X)
0	0.00E 00	0	500.0 (90.0)	500.0 (90.0)	500.0 (90.0)	500.0 (90.0)	500.0 (90.0)	183548.	0.
1	0.30E 04	13	421.3 (76.8)	422.2 (75.8)	425.8 (71.6)	432.6 (73.1)	450.9 (69.5)	138276.	138540.
2	0.75E 03	13	421.3 (76.9)	422.2 (75.8)	425.8 (71.6)	432.5 (73.1)	450.9 (69.5)	138276.	138342.
3	0.19E 03	13	421.0 (76.9)	422.2 (75.8)	425.8 (71.6)	432.5 (73.1)	450.9 (69.5)	138274.	138291.

where

$$S = \{s | g_s(x) > 0\} \quad \text{and} \quad T = \{t | g_t(x) < 0\}.$$

The problem was attempted with different values of EPS. Table 10A indicates that the false optimum was obtained with $EPS = 5.0$. A value of $EPS = 0.1$ was finally found suitable. An overall stopping criteria of $THETA = 10^{-5}$ was used. The parameters used were $r_0 = 3000$ and $EST = 100,000$ for the method of Fletcher and Powell and $EST = 200,000$ for the method of Fletcher and Reeves.

The method of Fletcher and Reeves yielded a new optimum of $f(x) = 127,551$ compared to $f(x) = 129,096$ obtained by Hsu [16], whereas the method of Fletcher and Powell yielded $f(x) = 129,546$.

The computing time required on an IBM 360-50 WATFOR compiler for this problem were 154.7 and 227.8 seconds using the methods of Fletcher and Powell and Fletcher and Reeves respectively. The times used by Hsu [16] on the same compiler was 208.8 seconds.

5.3.3. Test Problem No. 4

Problem No. 4 considers ten stages of the multistage personnel and production scheduling problem. The numerical data that were used are as follows:

Demand

$$Q_1 = 430 \quad Q_6 = 375$$

$$Q_2 = 447 \quad Q_7 = 292$$

$$Q_3 = 440 \quad Q_8 = 458$$

$$Q_4 = 316 \quad Q_9 = 400$$

$$Q_5 = 397 \quad Q_{10} = 350$$

Initial inventory, $I_0 = 263$.

Inventory for last month, $I_f = 263$.

Initial work force level, $W_0 = 81$.

The starting point chosen is

$$\begin{aligned} x^0 &= [P_1^0, W_1^0; P_2^0, W_2^0; P_3^0, W_3^0; P_4^0, W_4^0; P_5^0, W_5^0; \\ &\quad P_6^0, W_6^0; P_7^0, W_7^0; P_8^0, W_8^0; P_9^0, W_9^0; P_{10}^0, W_{10}^0] \\ &= [500, 90; 500, 90; 500, 90; 500, 90; 500, 90; \\ &\quad 500, 90; 500, 90; 500, 90; 500, 90; 500, 90] \end{aligned}$$

The problem can be stated as

Minimize

$$f(P_1, P_2, \dots, P_{10}; W_1, W_2, \dots, W_{10}) = \sum_{n=1}^{10} S_n$$

subject to

$$g_n(x) = I_n = I_{n-1} + P_n - Q_n \geq 0, \quad n = 1, 2, \dots, 9$$

$$g_{10}(x) = I_9 + P_{10} - Q_{10} \geq 263$$

and

$$g_{(n+10)}(x) = 0.2 (P_n - 5.67 W_n)^2 + 51.2 P_n - 281.0 W_n \geq 0,$$

$$n = 1, 2, \dots, 10$$

where

$$S_n = [340.0 W_n] + [64.3 (W_n - W_{n-1})^2 + 51.2 P_n - 281.0 W_n] \\ + [0.0825 (I_n - 320)^2]$$

5.3.4. Approach used

The approach used for solving this problem is similar to that used for test Problem No. 3. The P function for this problem was formulated as:

$$P(x, r_k) = f(x) = r_k \sum_{s \in S} \frac{1}{g_s(x)} - \text{CONST} \sum_{t \in T} \frac{1}{g_t(x)}$$

where

$$S = \{s | g_s(x) > 0\}, \quad T = \{t | g_t(x) < 0\}$$

$$\text{CONST} = |f(x)| \quad \text{if } |f(x)| > r_k^{-0.5} \\ = r_k^{-0.5} \quad \text{otherwise.}$$

An initial value of $r_0 = 3000$ was used. The problem was first attempted using the method of Fletcher and Reeves with $\text{EPS} = 0.1$, $\text{EST} = 200,000$ and overall stopping criteria of 10^{-5} . However the method failed to converge to a solution even after 15 minutes of computer time. Then the trials were repeated with values of EPS equal to 0.2, 0.4, and 1.0. The optimal solution of $f(x) = 244,869$ was obtained with $\text{EPS} = 0.4$. A value of $\text{EPS} = 0.2$ resulted in failure to converge to a solution after 15 minutes of computer time, whereas a value of $\text{EPS} = 1.0$ resulted in iteration terminating at a false optimum of $f(x) = 265,150$.

Using the same P function as above, the problem was then attempted by the method of Fletcher and Powell. Values of $\text{EPS} = 0.4$, $\text{EST} = 200,000$ and $r_0 = 3000$ were used. The method failed to converge to a solution even after 25 minutes of computer time was used. In an effort to determine the cause for this failure the following alternatives were tried.

(i) use of values of $\text{EPS} = 0.1, 1.0$ and 5.0 , (ii) use of $\text{EST} = 244,869$, the minimum found by the method of Fletcher and Reeves, (iii) use of a value of $\text{LIMIT} = 100$ as compared to 80 used earlier. However even after all these trials the method failed to converge to a solution.

5.3.5 Results and Discussion

The optimal result obtained by using the method of Fletcher and Reeves and the details of convergence are summarized in Table 11.

The following P function was tried and found successful for this problem:

$$P(x, r_k) = f(x) + r_k \sum_{s \in S} \frac{1}{g_s(x)} - \text{CONST} \sum_{t \in T} \frac{1}{g_t(x)}$$

where

$$S = \{s | g_s(x) > 0\}, \quad T = \{t | g_t(x) < 0\}$$

and

$$\begin{aligned} \text{CONST} &= |f(x)| && \text{if } |f(x)| > r_k^{-0.5} \\ &= r_k^{-0.5} && \text{otherwise.} \end{aligned}$$

TABLE 11 TEST PROBLEM NO.4
METHOD OF FLETCHER AND REEVES

STAGE NO.	VALUE OF R	NO. OF ITERN	P(1) (W(1))	P(2) (W(2))	P(3) (W(3))	P(4) (W(4))	P(5) (W(5))	P(6) (W(6))	P(7) (W(7))	P(8) (W(8))	P(9) (W(9))	P(10) (W(10))	F(X)	P(X)
C	0.00E 00	0	500.0 (90.0)	500.0 (90.0)	500.0 (90.0)	500.0 (90.0)	500.0 (90.0)	500.0 (90.0)	500.0 (90.0)	500.0 (90.0)	500.0 (90.0)	500.0 (90.0)	613167.	0.
1	0.30E 04	89	458.1 (77.3)	430.4 (73.9)	409.6 (70.5)	387.2 (67.9)	378.6 (65.5)	368.9 (63.1)	362.7 (61.6)	372.2 (60.1)	375.1 (58.5)	379.7 (55.7)	247161.	247447.
2	0.75E 03	18	464.5 (77.5)	446.4 (74.1)	415.4 (70.5)	391.3 (68.3)	375.0 (66.4)	362.5 (64.9)	353.8 (63.5)	364.0 (62.6)	365.3 (62.1)	367.9 (61.9)	244699.	244869.

The parameters used for this problem were $r_0 = 3000$, $EST = 200,000$, $EPS = 0.1$ and $THETA = 10^{-5}$. The method of Fletcher and Reeves yielded an optimal value of $f(x) = 244,699$ while the method of Fletcher and Powell failed to converge to a solution. The computing time used on an IBM 360-50 WATFOR compiler was 480.0 seconds compared to 480.0 and 900.7 used by Lai [19] and Hsu [16].

5.4 PROBLEMS 5, 6 AND 7

Colville [2] made a comparative study of nonlinear programming codes to test the relative efficiency of various available minimization techniques. For comparison he selected seven test problems. These problems were solved by several investigators using various nonlinear programming techniques. Since each investigator solved the problems on a computer available to him and since the computing capability of each installation was different, a scheme which would standardize the computing times for solving each problem was evolved. This consisted of determining the time required for inverting a given matrix by using a standard computer program. Then the time taken for solving each problem was divided by this time to obtain the standard time. The standard time required for each technique was used as the measure of efficiency of the technique.

Three of the test problems suggested by Colville were solved by the present method and are presented here. The computing time required for these problems are given in terms of time taken on a IBM 360-50 WATFOR compiler. The standard time suggested by Colville could not be obtained as the standard matrix inversion computer program was not available.

5.4.1 Test Problem No. 5

This is a problem from the Shell Development Company. The objective function is cubic and has 5 independent variables. There are ten linear constraints and the variables are required to be non-negative. The problem is stated as follows:

Minimize

$$f(x) = \sum_{j=1}^5 e_j x_j + \sum_{j=1}^5 \sum_{i=1}^5 c_{ij} x_i x_j + \sum_{j=1}^5 d_j x_j^3$$

subject to

$$g_i(x) = \sum_{j=1}^5 a_{ij} x_j \geq b_i, \quad i = 1, 2, \dots, 10$$

and

$$g_j(x) = x_j \geq 0, \quad j = 1, 2, \dots, 5.$$

The values of the constants used are

1. 'a' - values

i	j	<u>a_{ij}</u>				
		1	2	3	4	5
1		-16	2	0	1	0
2		0	-2	0	0.5	2
3		-3.5	0	2	0	0
4		0	-2	0	-4	-1
5		0	-9	-2	1	-2.8
6		2	0	-4	0	0
7		-1	-1	-1	-1	-1
8		-1	-2	-3	-2	-1
9		1	2	3	4	5
10		1	1	1	1	1

2. 'b'-values

	<u>b_i</u>									
$i \rightarrow$	1	2	3	4	5	6	7	8	9	10
	-40	-2	-0.25	-4	-4	-1	-40	-60	5	1

3. 'c'-values

		<u>c_{ij}</u>				
	j	1	2	3	4	5
i						
1		30	-20	-10	32	-10
2		-20	39	-6	-31	32
3		-10	-6	10	-6	-10
4		32	-31	-6	39	-20
5		-10	32	-10	-20	30

4. 'd'-values

	<u>d_j</u>				
j	1	2	3	4	5
	4	8	10	6	2

5. 'e'-values

	$\underline{e_j}$				
$j \rightarrow$	1	2	3	4	5
	-15	-27	-36	-18	-12

The feasible starting point chosen was

$$x^0 = [x_1^0, x_2^0, x_3^0, x_4^0, x_5^0].$$

$$= [0, 0, 0, 0, 1]$$

The initial value of r_k was chosen by using the formula

$$r_0 = \left(\frac{f(x^0)}{\sum_i \frac{1}{g_i(x^0)}} \right)$$

Since this yielded a value of $r_0 < 1.0$, a value of $r_0 = 1.0$ was selected [7].

5.4.1.1. Approach Used.

This problem was first attempted using the following P function which was adequate in solving Problems 1, 3 and 4.

$$P(x, r_k) = f(x) + r_k \sum_{s \in S} \frac{1}{g_s(x)} - \text{CONST} \sum_{t \in T} \frac{1}{g_t(x)}$$

where

$$S = \{s | g_s(x) > 0\} \quad T = \{t | g_t(x) < 0\}$$

and

$$\begin{aligned} \text{CONST} &= |f(x)| && \text{if } |f(x)| > r^{-0.5} \\ &= r^{-0.5} && \text{otherwise.} \end{aligned}$$

The value of $r_0 = 1.0$, $\text{EPS} = 0.01$ and $\text{EST} = -100$ were used. The value of EPS and EST were selected based upon the order of the minimum value of the objective function of -32.34 . The method, however, failed to converge to an optimal solution. The solution obtained was $f(x) = -17.13$. The problem was also solved with $\text{EPS} = 0.001$ without any further improvement in the minimum.

The following P function suggested in reference [9] was next tried.

$$P(x, r_k) = f(x) - r_k \sum_{s \in S} \ln g_s(x) + \frac{1}{r_k} \sum_{t \in T} g_t^2(x)$$

where

$$S = \{s | g_s(x) > 0\} \quad \text{and} \quad T = \{t | g_t(x) < 0\}.$$

With this P function, the optimization path moved into the infeasible region and as such the optimal solution was not reached. The P function was again modified as follows:

$$P(x, r_k) = f(x) - r_k \sum_{s \in S} \ln x_s + \frac{1}{r_k} \sum_{t \in T} g_t^2(x)$$

where

$$S = \{s | x_s > 0\}, \quad \text{and} \quad T = \{t | g_t(x) < 0\}.$$

This P function is different from the one suggested in reference [9] in that it does not include the constraints which are satisfied. The problem was solved with $EPS = 0.01$, and $EST = -100$. The optimal value of $f(x) = -32.3$ compared well with the value of $f(x) = -32.34$ shown by Colville.

5.4.1.2 Results and Discussion

The optimal results obtained by the new method are presented in Tables 12 and 13. Table 12 presents the optimal solution and details of convergence using the method of Fletcher and Powell. Table 13 presents the optimal solution and details of convergence using the method of Fletcher and Reeves.

Three P functions were tried while solving the problem and the P function that was finally used was

$$P(x, r_k) = f(x) - r_k \sum_{s \in S} \ln x_s + \frac{1}{r_k} \sum_{t \in T} g_t^2(x)$$

where

$$S = \{s | x_s > 0\} \quad \text{and} \quad T = \{t | g_t(x) < 0\}.$$

The parameters used were $r_0 = 1.0$, $EST = -100$, $EPS = 0.01$ and $THETA = 10^{-5}$. The optimum $f(x) = -32.3$ compares favorably with $f(x) = -32.34$ shown by Colville.

The computing time on an IBM 360-50 WATFOR compiler for this problem was 351.0 and 294.9 seconds using the methods of Fletcher and Powell and Fletcher and Reeves respectively.

TABLE 12 TEST PROBLEM NO.5
METHOD OF FLETCHER AND POWELL

STAGE NUMBER	VALUE OF R	NO. OF ITERATIONS	X(1)	X(2)	X(3)	X(4)	X(5)	F(X)	P(X)
0	0.00E+00	0	0.000	0.000	0.000	0.000	1.000	20.00	0.00
1	0.10E+01	20	0.534	0.323	0.955	0.476	0.511	-50.80	-41.40
2	0.25E+00	18	0.533	0.242	0.728	0.339	0.414	-42.30	-37.40
3	0.63E-01	20	0.413	0.280	0.540	0.378	0.316	-36.80	-34.30
4	0.16E-01	20	0.337	0.301	0.444	0.396	0.294	-33.70	-32.80
5	0.39E-02	20	0.320	0.304	0.423	0.406	0.277	-32.90	-32.00
6	0.98E-03	9	0.294	0.311	0.401	0.405	0.277	-32.19	-32.16

TABLE 13 TEST PROBLEM NO.5
METHOD OF FLETCHER AND REEVES

STAGE NUMBER	VALUE OF R	NO.OF ITERATIONS	X(1)	X(2)	X(3)	X(4)	X(5)	F(X)	P(X)
0	0.00E-00	0	0.000	0.000	0.000	0.000	1.000	20.00	0.00
1	0.10E-01	16	0.535	0.325	0.955	0.478	0.509	-50.80	-41.40
2	0.25E-00	19	0.533	0.242	0.728	0.339	0.414	-42.30	-37.40
3	0.63E-01	32	0.408	0.285	0.532	0.372	0.302	-36.50	-34.30
4	0.16E-01	32	0.333	0.315	0.437	0.417	0.260	-33.60	-32.90
5	0.39E-02	7	0.307	0.318	0.409	0.416	0.260	-32.60	-32.40
6	0.98E-03	7	0.301	0.320	0.402	0.416	0.260	-32.40	-32.30
7	0.24E-03	7	0.301	0.321	0.401	0.416	0.260	-32.30	-32.30

5.4.2 Test problem No. 6

This is a problem contributed by W. C. Mylander of Research Analysis Corporation and is a process design problem from Proctor and Gamble Company. The objective function is quadratic and has 5 independent variables. There are six quadratic constraints and ten bounds on the variables. The problem is stated as follows:

Minimize

$$f(x) = 5.3578547 x_3^2 + 0.8356891 x_1 x_5 \\ + 37.293239 x_1 - 40792.141$$

subject to

$$g_1(x) = 85.33407 + 0.0056858 x_2 x_5 \\ + 0.0006262 x_1 x_4 - 0.0022053 x_3 x_5 \geq 0.$$

$$g_2(x) = 92 - g_1(x) \geq 0$$

$$g_3(x) = 80.51249 + 0.0071317 x_2 x_5 + 0.0029955 x_1 x_2 - 90 \geq 0$$

$$g_4(x) = 110 - g_3(x) \geq 0$$

$$g_5(x) = 9.300961 + 0.0047026 x_3 x_5 \\ + 0.0012547 x_1 x_3 + 0.0019085 x_3 x_4 - 20 \geq 0$$

$$g_6(x) = 25 - g_5(x) \geq 0$$

$$g_7(x) = x_1 - 78 \geq 0$$

$$g_8(x) = 102 - x_1 \geq 0$$

$$g_9(x) = x_2 - 33 \geq 0$$

$$g_{10}(x) = 45 - x_2 \geq 0$$

$$g_{11}(x) = x_3 - 27 \geq 0$$

$$g_{12}(x) = 45 - x_3 \geq 0$$

$$g_{13}(x) = x_4 - 27 \geq 0$$

$$g_{14}(x) = 45 - x_4 \geq 0$$

$$g_{15}(x) = x_5 - 27 \geq 0$$

$$g_{16}(x) = 45 - x_5 \geq 0$$

The feasible starting point suggested by Colville is

$$\begin{aligned} x^0 &= [x_1^0, x_2^0, x_3^0, x_4^0, x_5^0] \\ &= [78.62, 33.44, 31.07, 44.10, 35.22] \end{aligned}$$

5.4.2.1 Approach Used

The initial value of r_0 was chosen by the computer program. The P function for this problem was defined as

$$P(x, r_k) = f(x) - r_k \sum_{s \in S} \ln g_s(x) + \frac{1}{r_k} \sum_{t \in T} g_t^2(x)$$

where

$$S = \{s | g_s(x) > 0\} \quad \text{and} \quad T = \{t | g_t(x) < 0\}$$

The stopping criterion used were $EPS = 0.1$, $EST = -30,000$ and the overall stopping criterion of 10^{-5} . The optimal value of $f(x) = -30,663$ compares favorably with the value of $f(x) = -30665.5$ given by Colville. The success of the above formulation of the P function for this problem and the failure of the same for problem 5 suggests that this P function is suitable where the values of the x_i are not likely to become negative.

5.4.2.2 Results and Discussions

The optimal results obtained by using the new method are presented in Tables 14 and 15. Table 14 presents the details of convergence and the optimal result using the method of Fletcher and Powell. Table 15 represents the details of convergence and the optimal result using the method of Fletcher and Reeves.

The following P function was tried and was successful in solving the problem.

$$P(x, r_k) = f(x) - r_k \sum_{s \in S} \ln g_s(x) + \frac{1}{r_k} \sum_{t \in T} g_t^2(x)$$

where

$$S = \{s | g_s(x) > 0\} \quad \text{and} \quad T = \{t | g_t(x) < 0\}.$$

The parameters used were $r_0 = 1.0$, $EST = -30000$, $EPS = 0.1$ and $THETA = 10^{-5}$. The optimum value of $f(x)$ obtained was $-30,663$ which compares favorably with $f(x) = -30665.5$ given by Colville.

TABLE 14. TEST PROBLEM NO.6
METHOD OF FLETCHER AND POWELL

STAGE NUMBER	VALUE OF μ	NO. OF ITERATIONS	X(1)	X(2)	X(3)	X(4)	X(5)	F(X)	P(X)
0	0.00E+00	0	78.62	33.44	31.07	44.18	35.22	-30374.	0.
1	0.10E+01	20	58.26	41.44	45.09	41.55	32.93	-38350.	-36962.
2	0.25E+00	20	71.35	40.06	12.38	45.17	19.89	-36125.	-34712.
3	0.63E+01	20	76.32	39.89	21.21	45.16	25.45	-33913.	-33006.
4	0.16E+01	20	79.51	38.41	25.72	45.29	27.64	-32445.	-31916.
5	0.39E+02	20	77.90	34.34	26.92	44.48	35.40	-31700.	-31274.
6	0.98E+03	20	77.97	33.20	29.33	45.01	36.71	-30883.	-30812.
7	0.24E+03	20	78.48	33.00	29.74	44.88	36.86	-30710.	-30684.
8	0.61E+04	20	78.42	33.00	29.86	44.91	36.85	-30667.	-30651.

TABLE 15 TEST PROBLEM NO.6
METHOD OF FLETCHER AND REEVES

STAGE NUMBER	VALUE CF R	NO. OF ITERATIONS	X(1)	X(2)	X(3)	X(4)	X(5)	F(X)	P(X)
0	0.00E-00	0	78.62	33.44	31.07	44.18	35.22	-30374.	0.
1	0.10E-01	18	58.13	41.32	45.06	41.78	32.41	-38158.	-36962.
2	0.25E-00	9	71.35	41.28	12.39	45.18	49.95	-36119.	-34712.
3	0.63E-01	8	76.28	39.70	21.20	45.17	25.43	-33918.	-33006.
4	0.16E-01	8	77.62	39.34	25.70	45.13	26.82	-32619.	-32020.
5	0.39E-02	22	77.90	32.98	26.88	45.06	35.82	-31683.	-31284.
6	0.98E-03	11	77.98	32.98	28.85	45.02	36.89	-31021.	-30851.
7	0.24E-03	7	78.00	32.99	29.64	44.85	37.06	-30760.	-30711.
8	0.61E-04	8	78.00	33.00	29.87	44.85	36.96	-30693.	-30671.
9	0.15E-04	7	78.00	33.00	29.99	44.82	36.86	-30663.	-30661.

The computing time required to solve this problem on an IBM 360-50 WATFOR compiler was 600.0 and 525.8 seconds using the method of Fletcher and Powell and Fletcher and Reeves respectively.

5.4.3 Test Problem No. 7

This is a test program from Westinghouse Research Laboratory. The objective function has four independent variables and there are 4 bounds on the variables.

The problem can be stated as:

Minimize

$$\begin{aligned} f(x) = & 100 (x_2 - x_1^2)^2 + (1 - x_1)^2 + 90 (x_4 - x_3^2)^2 \\ & + (1 - x_3)^2 + 10.1 \{(x_2 - 1)^2 + (x_4 - 1)^2\} \\ & + 19.8 (x_2 - 1) (x_4 - 1) \end{aligned}$$

subject to

$$g_1(x) = x_1 + 10 \geq 0$$

$$g_2(x) = 10 - x_1 \geq 0$$

$$g_3(x) = x_2 + 10 \geq 0$$

$$g_4(x) = 10 - x_2 \geq 0$$

$$g_5(x) = x_3 + 10 \geq 0$$

$$g_6(x) = 10 - x_3 \geq 0$$

$$g_7(x) = x_4 + 10 \geq 0$$

$$g_8(x) = 10 - x_4 \geq 0.$$

The feasible starting point suggested by Colville is

$$\begin{aligned} x^0 &= [x_1^0, x_2^0, x_3^0, x_4^0] \\ &= [-3, -1, -3, -1]. \end{aligned}$$

5.4.3.1. Approach Used

The initial value of r_0 was selected by the computer program and $r_0 = 22,750$. Since the value of $f(x)$ at the starting point is 19,192, a value of $EPS = 0.1$ was selected. An overall stopping criterion of 10^{-5} and $EST = 0.0$ were used.

The mixed-point SUMT P function was first attempted. The P function is

$$P(x, r_k) = f(x) + r_k \sum_{s \in S} \frac{1}{g_s(x)} + r_k^{-0.5} \sum_{t \in T} g_t^2(x)$$

where $S = \{s | g_s(x) > 0\}$ and $T = \{t | g_t(x) < 0\}$.

This P function proved to be adequate and the optimal value of $f(x) = 0.3 \times 10^{-5}$ was obtained by both the methods of Fletcher and Powell, and Fletcher and Reeves.

5.4.3.2 Results and Discussion

The optimal results obtained by using the new method are presented in Tables 16 and 17. Table 16 represents the optimal result and the details of convergence using the method of Fletcher and Powell. Table 17 shows the optimal result and the details of convergence using the method of Fletcher and Reeves.

TABLE 16. TEST PROBLEM NO.7
METHOD OF FLETCHER AND POWELL

STAGE NUMBER	VALUE OF R	NO. OF ITERATIONS	X(1)	X(2)	X(3)	X(4)	F(X)	P(X)
0	C.00	0	-3.000	-1.000	-3.000	-1.000	0.19E 05	0.00E 00
1	22750.00	20	C.043	0.122	C.040	0.130	0.35E 02	0.18E 05
2	5687.50	20	0.638	0.461	0.633	0.463	0.13E 02	0.46E 04
3	1422.00	12	0.894	0.813	C.893	0.814	0.14E 01	0.11E 04
4	355.50	11	0.972	0.948	0.972	0.948	C.11E 00	0.29E 03
5	88.90	8	C.993	0.987	C.993	0.987	0.70E-02	0.72E 02
6	22.20	6	0.998	0.997	0.998	0.997	0.4CE-03	0.18E 02
7	5.60	4	0.999	C.999	C.999	0.999	0.30E-04	0.45E 01
8	1.40	4	1.000	1.000	1.000	1.000	0.00E 00	0.11E 01
9	C.35	4	1.000	1.000	1.000	1.000	0.00E 00	0.28E 00

TABLE 17 TEST PROBLEM NO.7
METHOD OF FLETCHER AND REEVES

STAGE NUMBER	VALUE CF R	NO.OF ITERATIONS	X(1)	X(2)	X(3)	X(4)	F(X)	P(X)
0	0.00	0	-3.000	-1.000	-3.000	-1.000	0.19E 05	0.00E 00
1	22750.00	6	0.048	0.125	0.052	0.131	0.35E 02	0.18E 05
2	5687.50	9	0.630	0.447	0.644	0.478	0.12E 02	0.46E 04
3	1421.90	7	0.889	0.805	0.898	0.821	0.15E 01	0.11E 04
4	355.50	7	0.968	0.941	0.976	0.956	0.11E 00	0.29E 03
5	88.90	6	0.989	0.980	0.996	0.993	0.70E-02	0.72E 02
6	17.95	5	0.995	0.990	1.001	1.003	0.50E-03	0.18E 02
7	5.55	6	0.997	0.993	1.002	1.005	0.80E-04	0.56E 01
8	1.39	7	0.997	0.995	1.002	1.004	0.30E-04	0.14E 01
9	0.35	7	0.999	0.997	1.001	1.002	0.10E-04	0.28E 00
10	0.09	7	0.999	0.998	1.001	1.002	0.40E-05	0.70E-01
11	0.00	7	0.999	0.998	1.000	1.002	0.30E-05	0.20E-01

The following P function was first tried and yielded the optimal value

$$P(x, r_k) = f(x) + r_k \sum_{i=1}^8 \frac{1}{g_i(x)}$$

The parameters used were $r_0 = 22750$, $EST = 0.0$, $EPS = 0.1$ and $THETA = 10^{-5}$. The optimal value obtained was $f(x) = 0.3 \times 10^{-5}$ which compares well with $f(x) = 0.0$ given by Colville. The computing time taken on an IBM 360-50 WATFOR compiler were 197.0 and 101.0 seconds using the methods of Fletcher and Powell and Fletcher and Reeves respectively.

5.5 CONCLUSIONS

From the results presented in this chapter the following conclusions can be drawn:

1. There is no general formulation of a P function which is suitable for all problems. The different types of P functions as suggested by Fiacco and McCormick [9] were tried while solving the test problems and the following approach has been found satisfactory.

(i) If a problem involves equality and inequality constraints and a feasible starting point is known, the following P function should be used.

$$P(x, r_k) = f(x) + r_k \sum_{s \in S} \frac{1}{g_s(x)} - \text{CONST} \sum_{t \in T} \frac{1}{g_t(x)} \\ + r_k^{-0.5} \sum h_j^2(x)$$

where

$$S = \{s | g_s(x) > 0\}, \quad T = \{t | g_t(x) < 0\}$$

and $h_j(x)$ are the inequality constraints.

(ii) If the problem involves inequality constraints only and the starting point is infeasible, or the path of optimization lies in the infeasible region, the following P function should be used.

$$P(x, r_k) = f(x) - r_k \sum_{s \in S} \ln g_s(x) + \frac{1}{r_k} \sum_{t \in T} g_t^2(x)$$

where

$$S = \{s | g_s(x) > 0\} \quad \text{and} \quad T = \{t | g_t(x) < 0\}$$

(iii) If a problem involves inequality constraints only with non-negativity constraints on the variables and the variables tend to have negative values during the process of optimization, the following P function should be used.

$$P(x, r_k) = f(x) - r_k \sum_{s \in S} \ln x_s - \text{CONST} \sum_{t \in T} \frac{1}{g_t(x)}$$

where

$$S = \{s | x_s > 0\} \quad \text{and} \quad T = \{t | g_t(x) < 0\}$$

2. The optimal results obtained are entirely dependent upon the value of the stagewise stopping criterion (EPS) used. This value changes in proportion to the absolute value of the objective function at the beginning of the iterative search as well as the minimum value. Different values of EPS were tried for each problem before deciding upon a suitable value for the problem. Based on this experience the following values of EPS are suggested.

$$\begin{aligned}
 \text{EPS} &= 0.4 && \text{if } |f(x)| > 10^5 \\
 &= 0.1 && \text{if } 10^3 < |f(x)| < 10^5 \\
 &= 0.01 && \text{if } 100 < |f(x)| < 10^3 \\
 &= 0.001 && \text{if } 0 < |f(x)| < 100
 \end{aligned}$$

However depending upon the problem being solved some experimentation is needed to determine the value of EPS suitable to the problem.

3. The procedure used for selecting the initial value of $S(r_0)$ is based on the principle that the penalty and the objective function be of the same order at the starting point. This procedure works well with the test problems solved and seems to be adequate.

4. The new approach appears to be able to handle various problems where the functions and the constraints are differentiable.

5. The new approach required more computing time than the methods of Lai and RAC program. The optimal solution obtained by the new method was close to the ones obtained by Lai and Hsu. The difference in the solution was larger as the size of the problem became larger. For example in the two dimensional test problem, the solutions obtained were the same, whereas in the 20 dimensional test problem the difference was 0.2 percent. This difference can be reduced if a value of EPS which is correctly suited is determined and used.

6. The method of Fletcher and Reeves appeared to be better than the method of Fletcher and Powell both in the quality of the solution and the computing time required to solve a problem. This was more noticeable when the size of the problem became larger.

5.6 SUGGESTIONS FOR FUTURE WORK IN THIS FIELD.

The method of SUMT with conjugate gradients provides a useful technique for problems with differentiable objective function and constraints.

However additional experimentation is needed to determine the following:

1. The suitable types of P functions for different kinds of practical problems. The 7 test problems suggested by Colville [2] may be used for this purpose.
2. Suitable values of EPS and EST to yield the optimum a given problem with minimum computing time. This is necessary since the optimum value obtained and the computing time required are dependent upon these values.
3. Powell's method of conjugate gradients, which does not need any derivatives, may be tried as one of the minimization techniques.

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APPENDIX

COMPUTER PROGRAM FOR IMPLEMENTING SUMT BY CONJUGATE

GRADIENT TECHNIQUES

The computational procedure is presented in the flow charts shown in Figs. A-1 through A-4; the FORTRAN program symbols, their explanation and the corresponding mathematical notations are summarized in Table 18. The computer program listing follows this table.

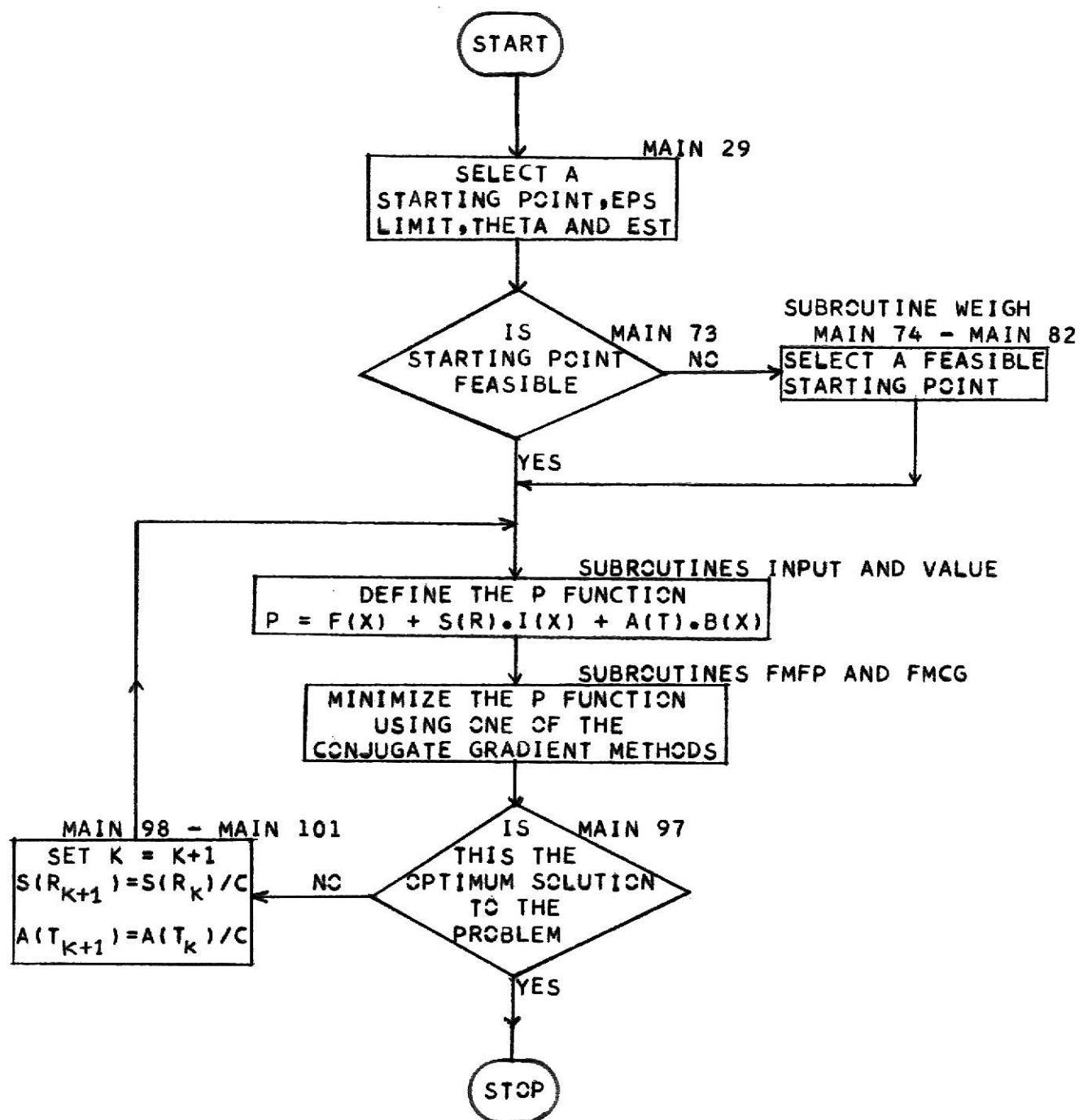


FIG.A-1 DESCRIPTIVE FLOW DIAGRAM OF SUMT WITH CONJUGATE GRADIENT TECHNIQUES

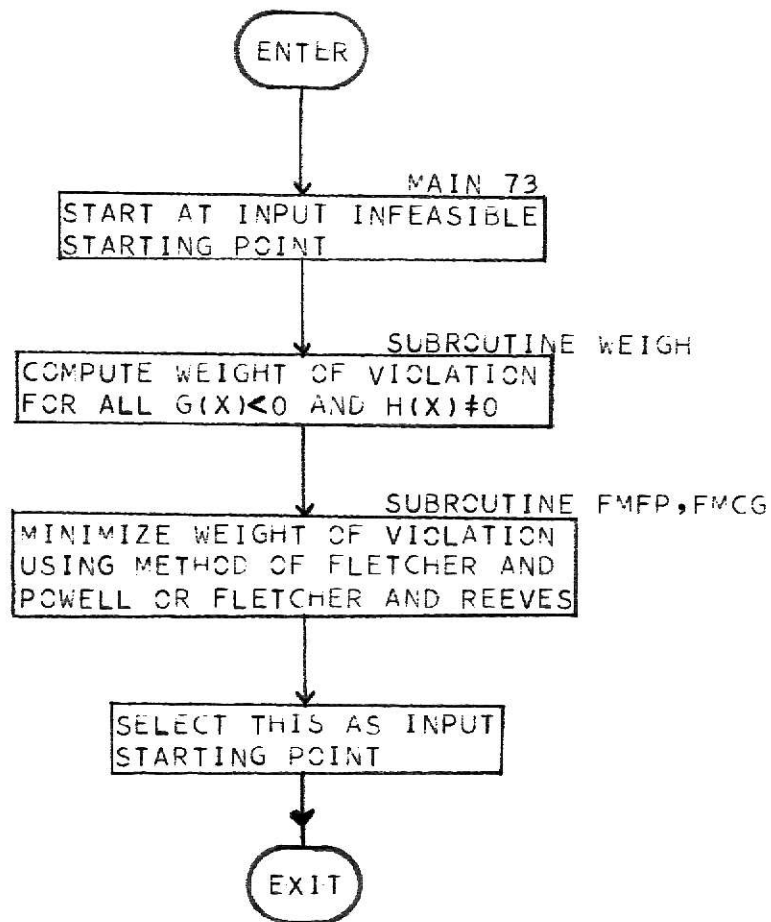


FIG. A-2 DESCRIPTIVE FLOW DIAGRAM FOR SELECTING A FEASIBLE STARTING POINT (SUBROUTINE WEIGH)

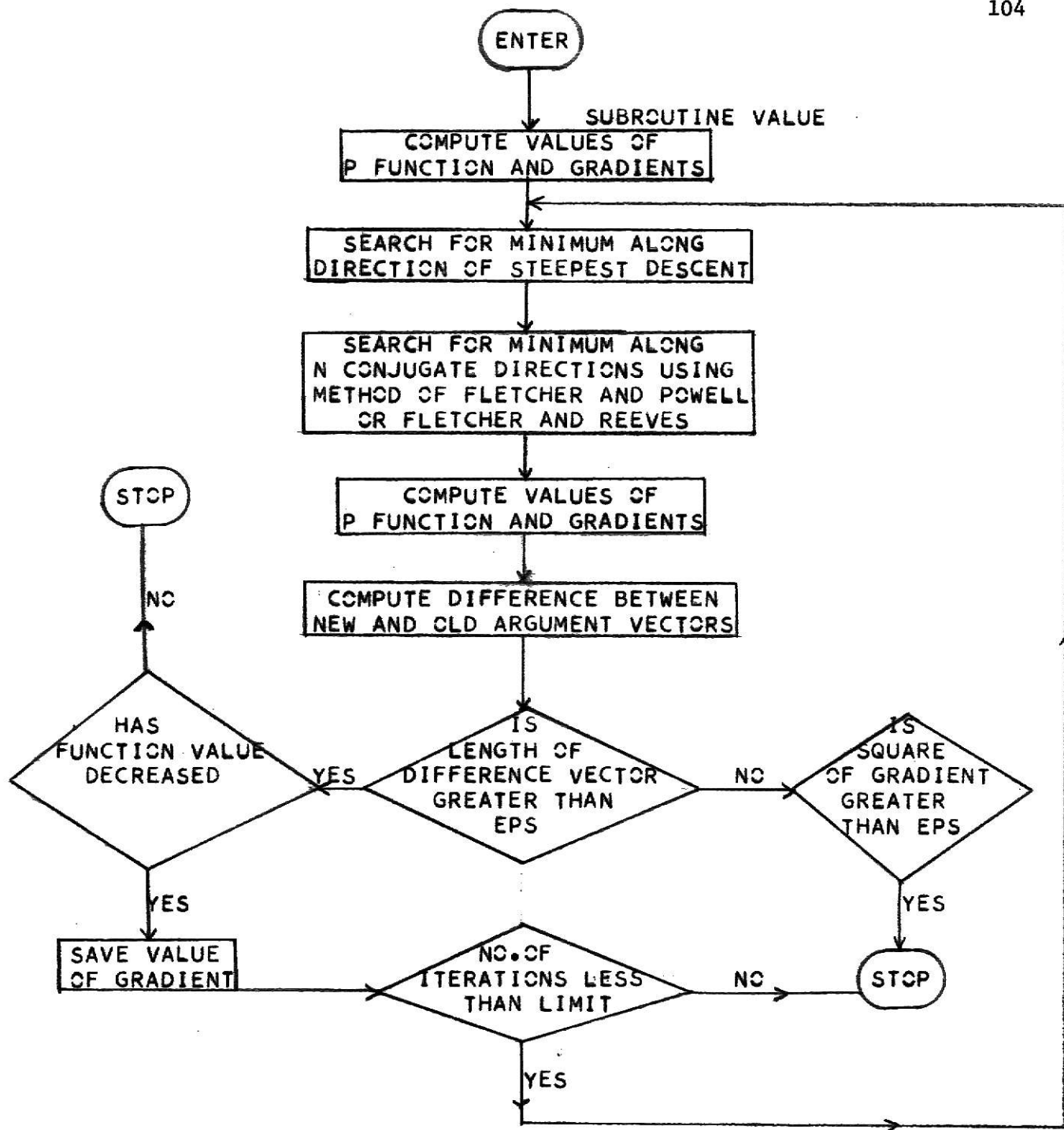


FIG.A-3 DESCRIPTIVE FLOW DIAGRAM OF CONJUGATE GRADIENT METHODS.(SUBROUTINES FMFP AND FMCG)

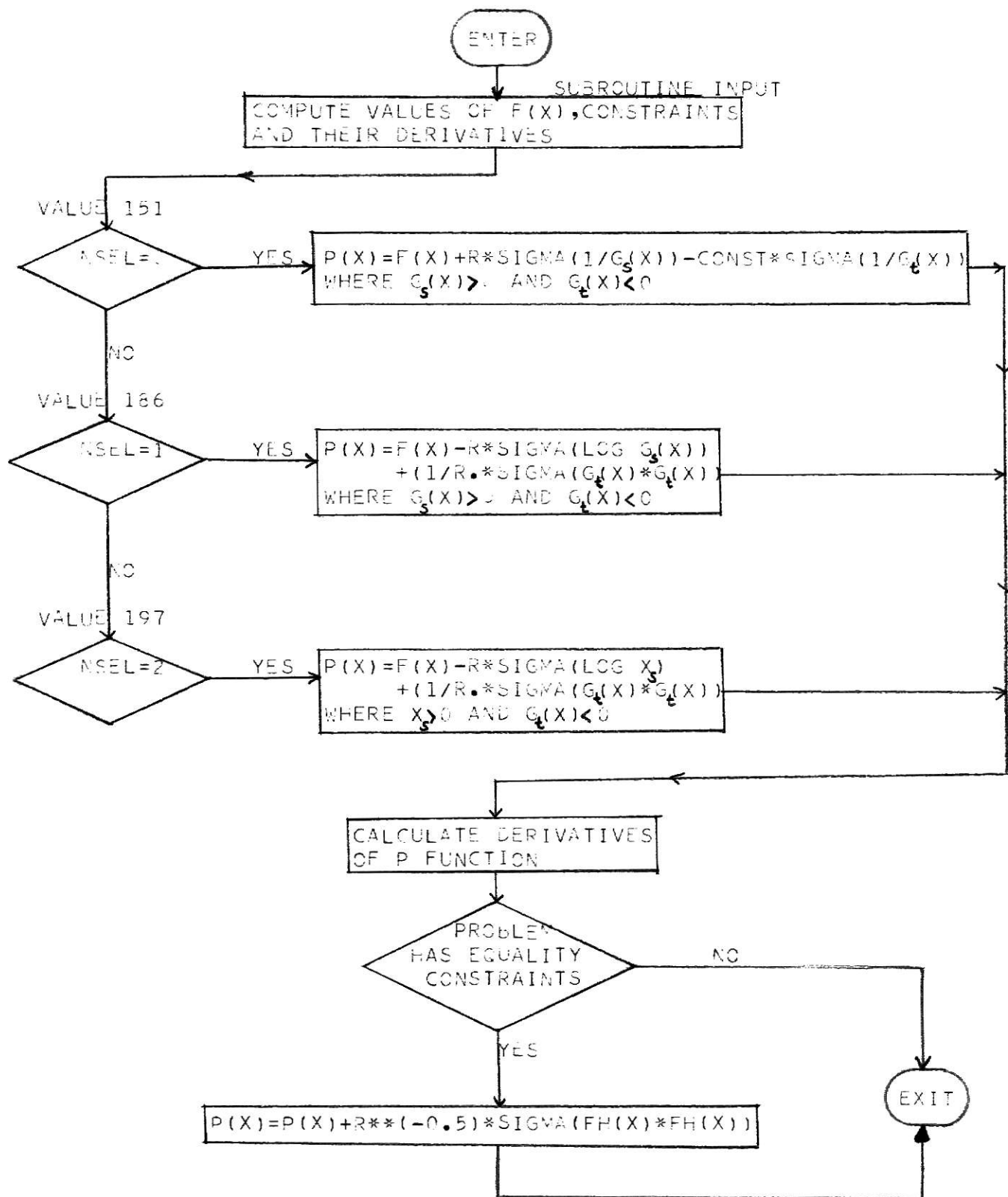


FIG. A-4 DESCRIPTIVE FLOW DIAGRAM OF SUBROUTINE VALUE

Table 18 Program Symbols and Explanation

Program Symbols	Explanation	Mathematical Symbol
A(I) } B(I) }	Indices to determine which of the inequality constraints are violated	
CONST	Weighting factor for violated constraints	
DG(I,J)	Partial derivative of inequality constraint $g_j(x)$ with respect to variable x_i	$\frac{\partial g_j(x)}{\partial x_i}$
DH(I,J)	Partial derivative of inequality constraint $h_j(x)$ with respect to variable x_i	$\frac{\partial h_j(x)}{\partial x_i}$
EPS	Stagewise stopping criterion	ϵ
EST	Estimate of the minimum value of the objective function	
F	Objective function	$f(x)$
FG(I)	Inequality constraint i	$g_i(x)$
FH(J)	Equality constraint j	$h_j(x)$
GRAD(I)	Partial derivative of the objective function with respect to variable x_i	$\frac{\partial f}{\partial x_i}$
INTER	Option for printing intermediate results	
KOUNT	Number of iterations in each stage	
LOST	Index to determine whether any constraint is violated	
LIMIT	Limit for number of iterations in each stage	
METHOD	Option for selecting the unconstrained minimization method	
MG	Number of inequality constraints	
MH	Number of equality constraints	
N	Number of decision variables	n
NEGAT	Option for including non-negativity constraints	
NR	Option for selecting initial value of r_0	

Table 18 Program Symbols and Explanation (continued)

Program Symbols	Explanation	Mathematical Symbol
NSEL	Option for selecting the type of P function	
NSTAGE	Number of stages	
P	P function	$P(x)$
PENAT	Sum of the penalty functions	$\sum_i \frac{1}{g_i(x)}$
R	Penalty coefficient	r_k
RATIO	Ratio for decreasing r_k at each stage	C
TGH	Weight of violation at infeasible starting point	$(\sum_k g_k^2 + \sum_s h_s^2)^{1/2}$
THETA	Overall stopping criterion	
X	Decision variable	x
YSTOP	Calculated value of overall stopping criterion	$\left \left \frac{f - r_k \sum_i \frac{1}{g_i} + r_k^{-1/2} \sum_j h_j^2}{\sum_i \frac{1}{g_i} + r_k^{-1/2} \sum_j h_j^2} \right ^{-1} \right $

```

$JOB          SVG,RUN=CHECK,KP=29,TIME=2,PAGES=100
1  CCMPCH /DATA/ X(50),FG(50),FH(50),A(50),B(50)
2  CCMPCH /DERIV/ DG(20,20),DH(20,20)
3  CCMPCH /PARAM/ R,N,MG,MH,LOST,KOUNT,INTER,NEGAT,NFUNCT,NSEL
4  DIMENSION AG(50),AH(500),GRAL(50),PROB(25)
5  EXTERNAL WEIGH
6  EXTERNAL VALUE

C
C  THIS IS A PROGRAM FOR OPTIMIZATION BY THE METHOD OF SEQUENTIAL
C  UNCONSTRAINED MINIMIZATION TECHNIQUE (SUMT).IT USES THE PENALTY
C  FUNCTION APPROACH OF SUMT,COMBINED WITH THE UNCONSTRAINED
C  MINIMIZATION TECHNIQUES OF FLETCHER & POWELL AND FLETCHER & REEVES.
C  THIS PROGRAM IS DESIGNED TO HANDLE 50 VARIABLES AND 20 EACH OF
C  EQUALITY CONSTRAINTS AND INEQUALITY CCNSTRANTS
C  IF A LARGER PROBLEM IS TO BE SOLVED THE COMMON AREA DENCTED BY
C  'DATA'NEEDS TO BE CORRESPONDINGLY CHANGED
C
C  *** DATA TO BE FURNISHED BY THE USER ***
C
C  N = NUMBER OF DECISION VARIABLES
C  MG = NUMBER OF INEQUALITY CCNSTRANTS
C  MH = NUMBER OF  EQUALITY CCNSTRANTS
C  X = DECISION VARIABLE
C  F = OBJECTIVE FUNCTION
C  GRAD(I) = DERIVATIVE OF THE OBJECTIVE FUNCTION WITH RESPECT TO
C           VARIABLE 'I'
C  FG(I) = INEQUALITY CONSTRAINT 'I',OF THE FORM FG(I)>0
C  FH(I) =  EQUALITY CONSTRAINT 'I',OF THE FORM FH(I) = 0
C  DG(I,J) = DERIVATIVE OF INEQUALITY CONSTRAINT 'J' WITH RESPECT TO
C           VARIABLE 'I'
C  DH(I,J) = DERIVATIVE OF EQUALITY CONSTRAINT 'J' WITH RESPECT TO
C           VARIABLE 'I'
C  R = PENALTY CO-EFFICIENTS
C  RATIO = RATIO FOR DECREASING 'R' AT EACH STAGE
C  THETA = CVERALL STOPPING CRITERION
C  EPS = STAGE STOPPING CRITERION
C  EST = INITIAL ESTIMATE OF OBJECTIVE FUNCTION
C  LIMIT = LIMIT FOR NUMBER OF ITERATIONS IN EACH STAGE
C
C  **** OPTIONS TO BE PROVIDED BY THE USER ****
C
C  NR = OPTION FOR SELECTING 'R'
C       = 0, 'R' VALUE SUPPLIED BY USER
C       = 1, 'R' VALUE TO BE COMPUTEC
C  INTER = OPTION FOR PRINTING INTERMEDIATE STAGE-WISE RESULTS
C          = 0,DO NOT PRINT INTERMECIATE VALUES
C          = 1,PRINT INTERMEDIATE VALUES
C  NEGAT = OPTION FOR NCN-NEGATIVITY CONSTRAINTS
C          = 0,DO NOT INCLUDE NCN-NEGATIVITY CCNSTRANTS
C          = 1,INCLUDE THESE CONSTRAINTS
C  METHOD = OPTION FOR SELECTING THE UNCCNSTRAINED MINIMIZATION TECHNIQUE
C          = 0,METHOD OF FLETCHER AND REEVES
C          = 1,METHOD OF FLETCHER AND POWELL
C  NSEL = OPTION FOR SELECTING THE TYPE OF 'P' FUNCTION
C          =0,  $P(X) = F(X) + R*(SIGMA(1/FG(I)))$ 
C          =1,  $P(X) = F(X) - R*(SIGMA(ALOG(FG(I)) + (1/R)*SIGMA(FG(J)**2)$ 
C              WHERE  $FG(I) > 0$  AND  $FG(J)$  IS LESS THAN 0
C          = 2,  $P(X) = F(X) - R*(SIGMA(ALOG(X(I)) + (1/R)*SIGMA(FG(J)**2)$ 
C              WHERE  $X(I) > 0$  AND  $FG(J)$  IS LESS THAN 0
C

```

```

C   *** OTHER VARIABLES USED IN THIS PROGRAM ***
C
C   KOUNT = NUMBER OF ITERATIONS IN EACH STAGE
C   NSTAGE = NUMBER OF STAGES
C   P = 'P' FUNCTION
C   YSTOP = CALCULATED OVERALL STOPPING CRITERION
C
7   1000 FORMAT (16I5)
8   1001 FORMAT (5E15.4)
9   1002 FORMAT (15,E15.4)
10  1003 FORMAT (////,7X,' NUMBER OF VARIABLES = ',I5//7X,' NUMBER OF INEQU
    1ALITY CONSTRAINTS = ',I5//7X,' NUMBER OF EQUALITY CONSTRAINTS = ',
    2I5,' NOT INCLUDING NON-NEGATIVITIES ')
11  1004 FORMAT (7X,' F(X) = ',E14.6,' P(X) = ',E14.6,' R = ',E14.6,' RATIO
    1 = ',F2.1,' THETA = ',E14.6,' EPS = ',E14.6//7X,' LIMIT = ',I4,'
    2 EST = ',E14.6)
12  1005 FORMAT (/,10X,' X(',I3,') = ',E14.6)
13  1006 FORMAT (/,10X,' CONSTRAINTS ',/14X,5('FG(',I3,') = ',E13.6,2X)))
14  1007 FORMAT (/,10X,' FH(',I3,') = ',E14.6)
15  1008 FORMAT (///,7X,' SELECTED FEASIBLE STARTING POINT ')
16  1009 FORMAT (///,7X,' STAGE NUMBER = ',I5,' F(X) = ',E14.6,' P(X) = ',E1
    14.6,' R = ',E14.6//7X,' ITERATIONS IN THIS STAGE = ',I5,' YSTOP =
    2',E14.6)
17  1010 FORMAT (///,7X,' THE ABOVE RESULTS ARE THE FINAL OPTIMUM')
18  1020 FORMAT (3X,100(1H*))
19  1021 FORMAT ('1',10X,' *** INITIAL VALUES *** ')
20  1022 FORMAT (20A4)
21  1023 FORMAT (///,20X,20A4)
22  1024 FORMAT (////////,7X,'***** SEQUENTIAL UNCONSTRAINED MINIMIZATION TEC
    1HNIQUE WITH CONJUGATE GRADIENTS *****')
23  1025 FORMAT (10X,' *** NUMBER OF FUNCTION EVALUATIONS = ',I4)
24  1026 FORMAT (///,20X,' METHOD OF FLETCHER AND REEVES ')
25  1027 FORMAT (///,20X,' METHOD OF FLETCHER AND POWELL ')
26  PRINT 1024
C
C   INPUT DATA CARD NUMBER 1 - PROBLEM NAME. USE COLUMN 1 - 80. THE NAME
C   CAN BE ANY ALPHA-NUMERIC OR SPECIAL CHARACTER 80 COLUMNS LONG
C
27  READ 1022,(PROB(I),I = 1,20)
28  PRINT 1023,(PROB(I),I = 1,20)
C
C   INPUT DATA CARD NUMBER 2 - PARAMETERS
C
29  READ (1,1000) N,MG,MH,NR,INTER,LIMIT,NEGAT,NSEL,METHOD
30  IF (METHOD) 130,131,130
31  131 PRINT 1026
32  GO TO 132
33  130 PRINT 1027
34  132 CONTINUE
C
C   INPUT DATA CARD NUMBER 3 - INITIAL VALUE OF R,RATIO,STOPPING CRITERIA
C
35  READ(1,1001) R,RATIO,EPS,THETA,EST
36  WRITE (3,1003) N,MG,MH
C
C   INPUT DATA CARD NUMBER 4 - INITIAL GUESS OF THE X(I)'S
C
37  READ(1,1002) (J,X(I),I = 1,N)
38  PENAL = 0.0
39  LOST = 0

```

```

C
C   INCLUDE NON-NEGATIVITIES IF NEEDED
C
40      N1 = MG + 1
41      N2 = MG + N
42      IF (NEGAT) 4,5,4
43      4 DO 6 I = N1,N2
44          J = I - MG
45          FG(I) = X(J)
46      6 CONTINUE
47      MG = MG + N
48      5 CONTINUE
49      DO 3 J = 1, MG
50          A(J) = 0.0
51          B(J) = 1.0
52      3 DO 3 I = 1, N
53          DG(I,J) = 0.0
54          DH(I,J) = 0.0
55      3 CONTINUE
56      CALL INPUT (F,GRAD)

C
C   CALCULATE 'R' IF NECESSARY
C
57      IF (NR) 15,20,15
58      15 DO 10 I = 1, MG
59          IF (FG(I).LT.0.0001) GO TO 10
60          PENAT = PENAT + (1.0/FG(I))
61      10 CONTINUE
62          R = F/PENAT
63          R = ABS(R)
64          IF (R.LT.1.0) R = 1.0
65      20 WRITE (3,1021)
66          CALL VALUE (F,P,GRAD)
67          WRITE (3,1004) F,P,R,RATIC,THETA,EPS,LIMIT,EST
68          WRITE (3,1005) (I,X(I),I = 1,N)
69          WRITE (3,1006) (I,FG(I),I = 1, MG)
70          IF (MH) 1,2,1
71          1 WRITE (3,1007) (I,FH(I),I = 1,MH)
72          2 NSTAGE = 1

C   IF THE PROBLEM IS TO BE SOLVED FROM AN INFEASIBLE STARTING POINT
C   WITHOUT GOING INTO A FEASIBLE POINT ROUTINE, REMOVE THE 'C' FROM
C   COLUMN 1 OF THE FOLLOWING CARD
C
73      LOST = 0
74      IF (LCST.NE.2) GO TO 50
75      IF (METHOD.EQ.1) GO TO 70
76      CALL FMCG (WEIGH,F,AG,EST,EPS,LIMIT,IER,AH)
77      GO TO 71
78      70 CALL FMFP (WEIGH,F,AG,EST,EPS,LIMIT,IER,AH)
79      71 CALL VALUE (F,P,GRAD)
80      WRITE (3,1008)
81      WRITE (3,1004) F,P,R,RATIC,THETA,EPS
82      WRITE (3,1005) (I,X(I),I = 1,N)
83      WRITE (3,1006) (I,FG(I),I = 1, MG)
84      50 IF (METHOD.EQ.1) GO TO 72
85      CALL FMCG (VALUE,F,AG,EST,EPS,LIMIT,IER,AH)
86      GO TO 73
87      72 CALL FMFP (VALUE,F,AG,EST,EPS,LIMIT,IER,AH)
88      73 CALL VALUE (F,P,GRAD)

C
C   CALCULATE STOPPING CRITERION

```

```

      C
88      YSTOP = ABS(F/(2.0*F - P))
89      YSTOP = ABS(YSTOP - 1.0)
90      PRINT 1020
91      WRITE (3,1009) NSTAGE,F,P,R,KOUNT,YSTOP
92      PRINT 1025,NFUNCT
93      WRITE (3,1005) (I,X(I),I = 1,N)
94      WRITE (3,1006) (I,FG(I),I = 1,MG)
95      IF (MH) 60,61,60
96      60 WRITE (3,1007) (I,FH(I),I = 1,MH)
      C
      C CHECK IF MINIMUM IS REACHED
      C
97      61 IF (YSTOP - THETA) 110,110,100
98      100 R = R/RATIO
99      NSTAGE = NSTAGE + 1
100     PRINT 1020
      C
      C REPEAT STAGE-WISE SEARCH
      C
101     GC TC 50
102     110 WRITE (3,1010)
103     STOP
104     END

```

```

105      SUBROUTINE INPUT (F,GRAD)
106      COMMON /DATA/ X(50),FG(50),FH(50),A(50),B(50)
107      COMMON /DERIV/ DG(20,20),DH(20,20)
108      COMMON /PARAM/ R,N,MG,MH,LOST,KOUNT,INTER,NEGAT,NFUNCT,NSEL
109      DIMENSION GRAD(1)

C
C      THIS IS A USER SUPPLIED SUBROUTINE.THIS SUBROUTINE COMPUTES THE
C      OBJECTIVE FUNCTION,THE CONSTRAINTS AND THE DERIVATIVES OF THESE
C      FUNCTIONS.THE FUNCTIONS MUST BE OF THE FORM :
C      F = OBJECTIVE FUNCTION EXPRESSED AS A FUNCTION OF 'X'.
C      FG(I) = INEQUALITY CONSTRAINT OF THE FORM FG(I)>0.0
C      FH(I) = EQUALITY CONSTRAINT OF THE FORM FH(I)=0.0
C      GRAD(I) = DERIVATIVE OF THE OBJECTIVE FUNCTION WITH RESPECT TO
C      VARIABLE X(I)
C      DG(I,J) = DERIVATIVE OF INEQUALITY CONSTRAINT 'J' WITH
C      RESPECT TO VARIABLE X(I)
C      CH(I,J) = DERIVATIVE OF EQUALITY CONSTRAINT 'J' WITH RESPECT
C      TO VARIABLE X(I)
C
110      DO 1 I = 1,MG
111      1 FG(I) = 0.0
112      N1 = MG + 1 - N
113      IF (NEGAT) 4,5,4
114      4 DO 6 I = N1,MG
115      J = I - MG + N
116      FG(I) = X(J)
117      6 CONTINUE
118      5 CONTINUE

C
C      INTRODUCE THE REQUIRED CARDS BETWEEN THE CARDS MARKED ***.
C
C      *****
C      THE FOLLOWING IS AN EXAMPLE OF THE INPUT DATA FOR TEST PROBLEM NO.2
119      DG(1,1) = 1.0
120      DG(1,2) = 1.0
121      DG(1,3) = -1.0
122      DG(1,5) = 1.0
123      DG(2,2) = 1.0
124      DG(2,4) = -1.0
125      DG(2,6) = 1.0
126      DH(1,1) = 1.0
127      DH(2,1) = - 1.0
128      FH(1) = X(1) - X(2) - 5.0
129      FG(1) = X(1) - 18.0
130      FG(2) = X(1) + X(2) - 28.0
131      FG(3) = 30.0 - X(1)
132      FG(4) = 30.0 - X(2)
133      F = 100.0*(X(1) - 15.0)**2 + 20.0*(28.0 - X(1))**2 + 100.0*(X(2)
134      - X(1))**2 + 20.0*(38.0 - X(1) - X(2))**2
135      GRAD (1) = 200.0*( X(1) - 15.0 ) - 40.0*( 28.0 - X(1) ) - 200.0* (
136      1X(2) - X(1)) - 40.0*(38.0 - X(1) - X(2))
137      GRAD(2) = 200.0*( X(2) - X(1)) - 40.0*( 38.0 - X(1) - X(2))

C
C      *****
C
136      RETURN
137      END

```



```

128     SUBROUTINE VALUE (F,P,GRAD)
129     COMMON /DATA/ X(50),FG(50),FF(50),A(50),B(50)
140     COMMON /DERIV/ DG(20,20),DF(20,20)
141     COMMON /PARAM/ R,N,MG,PH,LOST,KOUNT,INTER,NEGAT,AFUNCT,NSEL
142     DIMENSION GRAD(50),SAVE(50)
143     CALL INPUT (F,GRAD)

C
C     THIS SUBROUTINE CALCULATES THE 'P' FUNCTION AND THE DERIVATIVES OF
C     THE 'P' FUNCTION USING THE DATA SUPPLIED BY THE USER IN THE
C     SUBROUTINE 'INPUT'.
C

144     LOST = 0
145     CONST = ABS(F)
146     IF (CCNST.LT.1.0E03) CCNST = 1.0E03
147     PENA = 1.0/(R**0.5)
148     IF (PENA.GT.CONST) CONST = PENA
149     DO 30 I = 1,MG
150     30 SAVE(I) = FG(I)
151     IF (NSEL) 60,70,60
152     70 DO 1 I = 1,MG

C
C     THIS 'P' FUNCTION IS USED WHEN NSEL=0
C      $P(X) = F(X) + R * (\text{SIGMA}(1/FG(I)))$ 
C

153     A(I) = 0.0
154     B(I) = 1.0
155     IF (FG(I)) 3,2,1
156     3 A(I) = 1.0
157     B(I) = 0.0
158     FG(I) = -FG(I)/CONST
159     LCST = 2
160     GO TO 1
161     2 FG(I) = 0.1E-48
162     1 CONTINUE
163     P = F
164     IF (MG) 10,11,10
165     10 DO 12 I = 1,MG
166     P = P + R/FG(I)
167     12 CONTINUE
168     DO 15 I = 1,N
169     DO 15 J = 1,MG
170     IF (FG(J).EQ.0.0) GO TO 15
171     GRAD(I) = GRAD(I) - R*DG(I,J)/(FG(J)*FG(J))
172     15 CONTINUE
173     DO 31 I = 1,MG
174     31 FG(I) = SAVE(I)
175     GO TO 11
176     60 DO 50 I = 1,MG
177     B(I) = 0.0
178     IF (FG(I)) 51,50,50
179     51 LCST = 2
180     B(I) = 1.0
181     50 CONTINUE
182     P = F
183     DO 40 I = 1,MG
184     P = P + B(I)*FG(I)*FG(I)/R
185     40 CONTINUE
186     GO TO (71,72),NSEL

C
C     THIS 'P' FUNCTION IS USED WHEN NSEL=1

```

```

C          P(X) = F(X) - R*SIGMA(ALOG(FG(I))) + (1/R)*SIGMA(FG(J)**2)
C          WHERE FG(I) >0 AND FG(J) IS LESS THAN 0
C
147      71 DO 41 I = 1,MG
148          IF (FG(I).LE.0.0) GO TO 41
149          P = P - R*ALOG(FG(I))
150      41 CONTINUE
151          DO 42 I = 1,N
152              DO 42 J = 1,MG
153                  IF (FG(J).LE.0.0) GO TO 42
154                  GRAD(I) = GRAD(I) - R*DG(I,J)/(FG(J))
155      42 CONTINUE
156          GO TO 73
C
C      THIS 'P' FUNCTION IS USED WHEN NSEL=2
C          P(X) = F(X) - R*SIGMA(ALOG(X(I))) + (1/R)*SIGMA(FG(J)**2)
C          WHERE X(I) >0 AND FG(J) IS LESS THAN 0
C
157      72 DO 44 I = 1,N
158          IF (X(I).LE.0.0) GO TO 44
159          P = P - R*ALOG(X(I))
200      44 CONTINUE
201          DO 45 I = 1,N
202              IF (X(I).LE.0.0) GO TO 45
203              GRAD(I) = GRAD(I) - R/X(I)
204      45 CONTINUE
205      73 DO 43 I = 1,N
206          DO 43 J = 1,MG
207              GRAD(I) = GRAD(I) + B(J)*2.0*DG(I,J)*FG(J)/R
208      43 CONTINUE
209      11 IF (MF) 20,21,20
210      20 DO 22 I = 1,MH
211          P = P + R**(-0.5)*FH(I)*FH(I)
212      22 CONTINUE
213          DO 25 I = 1,N
214              DO 25 J = 1,MH
215                  GRAD(I) = GRAD(I) + 2.0*(R**(-0.5))*FH(J)*CH(I,J)
216      25 CONTINUE
217      21 RETURN
218      END

```

```

219      SUBROUTINE WEIGH (P,TGH,GRAD)
220      COMMON /DATA/ X(50),FG(50),FH(50),A(50),B(50)
221      COMMON /DERIV/ DG(20,20),CH(20,20)
222      COMMON /PARAM/ R,N,MG,MH,LOST,KOUNT,INTER,NEGAT,NFUNCT,NSEL
223      DIMENSION GRAD(50)
224      DIMENSION SUM(50),SSUM(50)
C      THIS SUBROUTINE IS USED WHEN THE INITIAL GUESS OF THE X(I)'S ARE
C      IN THE INFEASIBLE REGION.IT SELECTS THOSE VALUES OF X(I) THAT
C      MINIMIZE :
C          TGH = (FG(I)**2 + FH(I)**2)**0.5
C          WHERE :
C          FG(I) IS LESS THAN ZERO,AND
C          FH(I) NOT EQUAL TO ZERO.
C
225      P = C.C
226      TGH = C.0
227      DO 100 I = 1,N
228          SUM(I) = 0.0
229          SSUM(I) = 0.0
230      100 CONTINUE
231      CALL INPUT (F,GRAD)
232      IF (MG) 1,2,1
233      1 DO 10 I = 1,MG
234          A(I) = 0.0
235          IF (FG(I)) 5,10,10
236      5 TGH = TGH + FG (I)*FG(I)
237          A(I) = 1.0
238      10 CONTINUE
239      2 IF (MH) 15,20,15
240      15 DO 25 I = 1,MH
241          IF (FH(I)) 21,25,21
242      21 TGH = TGH + FH(I)*FH(I)
243      25 CONTINUE
244      20 TGH = TGH**(0.5)
245      DO 50 I = 1,N
246          IF(MH) 70,80,70
247      70 DO 30 J = 1,MH
248          30 SUM(I) = SUM(I) + 2.0*FH(J)*CH(I,J)
249      80 DO 40 K = 1,MG
250          40 SSUM(I) = SSUM(I) + 2.0*FG(K)*DG(I,K)*A(K)
251      50 CONTINUE
252      DO 60 I = 1,N
253          IF(TGH) 61,62,61
254      61 GRAD(I) = 0.5*(TGH**(-0.5))*(SUM(I) + SSUM(I))
255          GO TO 60
256      62 GRAD(I) = 0.0
257      60 CONTINUE
258      DO 26 I = 1,MG
259          A(I) = 0.0
260      26 CONTINUE
261      RETURN
262      END

```

```

263 SUBROUTINE FMCG(VALUE,F,G,EST,EPS,LIMIT,IER,H)
264 COMMON /DATA/ X(50),FG(50),FF(50),A(50),B(50)
265 COMMON /PARAM/ R,N,MG,PH,LOST,KCUNT,INTER,NEGAT,NFUNCT,NSEL
266 DIMENSION G(50),H(1)
C
C ..... FMCG 10
C FMCG 20
C FMCG 30
C SUBROUTINE FMCG FMCG 40
C FMCG 50
C PURPOSE FMCG 60
C TO FIND A LOCAL MINIMUM OF A FUNCTION OF SEVERAL VARIABLES FMCG 70
C BY THE METHOD OF CONJUGATE GRADIENTS FMCG 80
C FMCG 90
C USAGE FMCG 100
C CALL FMCG(FUNCT,N,X,F,G,EST,EPS,LIMIT,IER,H) FMCG 120
C DESCRIPTION OF PARAMETERS FMCG 130
C FUNCT - USER-WRITTEN SUBROUTINE CONCERNING THE FUNCTION TO FMCG 140
C BE MINIMIZED. IT MUST BE OF THE FORM FMCG 150
C SUBROUTINE FUNCT(IN,ARG,VAL,GRAD) FMCG 160
C AND MUST SERVE THE FOLLOWING PURPOSE FMCG 170
C FOR EACH N-DIMENSIONAL ARGUMENT VECTOR ARG, FMCG 180
C FUNCTION VALUE AND GRADIENT VECTOR MUST BE COMPUTED FMCG 190
C AND, ON RETURN, STORED IN VAL AND GRAD RESPECTIVELY FMCG 200
C N - NUMBER OF VARIABLES FMCG 210
C X - VECTOR OF DIMENSION N CONTAINING THE INITIAL FMCG 220
C ARGUMENT WHERE THE ITERATION STARTS. ON RETURN, FMCG 230
C X HOLDS THE ARGUMENT CORRESPONDING TO THE FMCG 240
C COMPUTED MINIMUM FUNCTION VALUE FMCG 250
C F - SINGLE VARIABLE CONTAINING THE MINIMUM FUNCTION FMCG 260
C VALUE ON RETURN, I.E. F=F(X). FMCG 270
C G - VECTOR OF DIMENSION N CONTAINING THE GRADIENT FMCG 280
C VECTOR CORRESPONDING TO THE MINIMUM ON RETURN, FMCG 290
C I.E. G=G(X). FMCG 300
C EST - IS AN ESTIMATE OF THE MINIMUM FUNCTION VALUE. FMCG 310
C EPS - TESTVALUE REPRESENTING THE EXPECTED ABSOLUTE ERROR. FMCG 320
C A REASONABLE CHOICE IS 10**(-6), I.E. FMCG 330
C SOMEWHAT GREATER THAN 10**(-C), WHERE C IS THE FMCG 340
C NUMBER OF SIGNIFICANT DIGITS IN FLOATING POINT FMCG 350
C REPRESENTATION. FMCG 360
C LIMIT - MAXIMUM NUMBER OF ITERATIONS. FMCG 370
C IER - ERROR PARAMETER FMCG 380
C IER = 0 MEANS CONVERGENCE WAS OBTAINED FMCG 390
C IER = 1 MEANS NO CONVERGENCE IN LIMIT ITERATIONS FMCG 400
C IER = -1 MEANS ERRORS IN GRADIENT CALCULATION FMCG 410
C IER = 2 MEANS LINEAR SEARCH TECHNIQUE INDICATES FMCG 420
C IT IS LIKELY THAT THERE EXISTS NO MINIMUM. FMCG 430
C F - WORKING STORAGE OF DIMENSION 2*N. FMCG 440
C FMCG 450
C FMCG 460
C REMARKS FMCG 460
C I) THE SUBROUTINE NAME REPLACING THE DUMMY ARGUMENT FUNCT FMCG 470
C MUST BE DECLARED AS EXTERNAL IN THE CALLING PROGRAM. FMCG 480
C II) IER IS SET TO 2 IF, STEPPING ON ONE OF THE COMPUTED FMCG 490
C DIRECTIONS, THE FUNCTION WILL NEVER INCREASE WITHIN FMCG 500
C A TOLERABLE RANGE OF ARGUMENT. FMCG 510
C IER = 2 MAY OCCUR ALSO IF THE INTERVAL WHERE F FMCG 520
C INCREASES IS SMALL AND THE INITIAL ARGUMENT WAS FMCG 530
C RELATIVELY FAR AWAY FROM THE MINIMUM SUCH THAT THE FMCG 540
C MINIMUM WAS OVERLEAPED. THIS IS DUE TO THE SEARCH FMCG 550
C TECHNIQUE WHICH DOUBLES THE STEPSIZE UNTIL A POINT FMCG 560

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	IS FOUND WHERE THE FUNCTION INCREASES.	FMCG 570
		FMCG 580
	SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED	FMCG 590
	FUNCT	FMCG 600
		FMCG 610
	THE METHOD IS DESCRIBED IN THE FOLLOWING ARTICLE	FMCG 630
	METHOD	FMCG 620
	R.FLETCHER AND C.M.REEVES, FUNCTION MINIMIZATION BY	FMCG 640
	CONJUGATE GRADIENTS,	FMCG 650
	COMPUTER JOURNAL VOL.7, ISS.2, 196 , PP.149-154.	FMCG 660
		FMCG 670
		FMCG 680
		FMCG 690
267	NFUNCT = 0	
268	IF (INTER) 401,400,401	
269	401 WRITE (3,1000)	
270	400 CALL VALUE (P,F,G)	
271	NFUNCT = NFUNCT + 1	
		FMCG 740
		FMCG 750
	COMPUTE FUNCTION VALUE AND GRADIENT VECTOR FOR INITIAL ARGUMENT	FMCG 760
		FMCG 780
	RESET ITERATION COUNTER	FMCG 790
272	KCOUNT = 0	FMCG 800
273	IER = 0	FMCG 810
274	N1=N+1	FMCG 820
		FMCG 830
	START ITERATION CYCLE FOR EVERY N+1 ITERATIONS	FMCG 840
275	1 DO 43 II=1,N1	FMCG 850
		FMCG 860
	STEP ITERATION COUNTER AND SAVE FUNCTION VALUE	FMCG 870
276	KCOUNT=KCOUNT+1	FMCG 880
277	CLDF = F	FMCG 890
		FMCG 900
	COMPUTE SQUARE OF GRADIENT AND TERMINATE IF ZERO	FMCG 910
278	GNRM=C.	FMCG 920
279	DO 2 J=1,N	FMCG 930
280	2 GNRM=GNRM+G(J)*G(J)	FMCG 940
281	IF (GNRM) 146,46,3	FMCG 950
		FMCG 960
	EACH TIME THE ITERATION LOOP IS EXECUTED , THE FIRST STEP WILL	FMCG 970
	BE IN DIRECTION OF STEEPEST DESCENT	FMCG 980
282	3 IF (II-1) 4,4,6	FMCG 990
283	4 DO 5 J=1,N	FMCG1000
284	5 H(J)=-G(J)	FMCG1010
285	GO TO 6	FMCG1020
		FMCG1030
	FURTHER DIRECTION VECTORS H WILL BE CHOSEN CORRESPONDING	FMCG1040
	TO THE CONJUGATE GRADIENT METHOD	FMCG1050
286	6 A*BDA = GNRM/OLDG	FMCG1060
287	DO 7 J=1,N	FMCG1070
288	7 H(J)=A*BDA*H(J)-G(J)	FMCG1080
		FMCG1090
	COMPUTE TESTVALUE FOR DIRECTIONAL VECTOR AND DIRECTIONAL	FMCG1100
	DERIVATIVE	FMCG1110
289	8 DY=C.	FMCG1120
290	HARM=C.	FMCG1130
291	DO 9 J=1,N	FMCG1140
292	K=J+N	FMCG1150
		FMCG1160

253	C	SAVE ARGUMENT VECTOR	FMCG1170
254	C	H(K)=X(IJ)	FMCG118
255	C	HNRM=HNRM+ABS(H(IJ))	FMCG1190
	9	DY=DY+H(IJ)*G(IJ)	FMCG1200
	C		FMCG1210
	C	CHECK WHETHER FUNCTION WILL DECREASE STEPPING ALONG 'H' AND	FMCG1220
	C	SKIP LINEAR SEARCH ROUTINE IF NOT	FMCG1230
256	C	IF(DY)10,42,42	FMCG1240
	C		FMCG1250
	C	COMPUTE SCALE FACTOR USED IN LINEAR SEARCH SUBROUTINE	FMCG1260
257	10	SNRM=1./HNRM	FMCG1270
	C		FMCG1280
	C	SEARCH MINIMUM ALONG DIRECTION H	FMCG1290
	C		FMCG1300
	C	SEARCH ALONG H FOR POSITIVE DIRECTIONAL DERIVATIVE	FMCG1310
258		FY=F	FMCG1320
259		ALFA=2.*(EST-F)/DY	FMCG1330
300		AMBCA=SNRM	FMCG1340
	C		FMCG1350
	C	USE ESTIMATE FOR STEPSIZE ONLY IF IT IS POSITIVE AND LESS THAN	FMCG1360
	C	SNRM. OTHERWISE TAKE SNRM AS STEPSIZE	FMCG1370
301		IF(ALFA)13,13,11	FMCG1380
302	11	IF(ALFA-AMBCA)12,13,13	FMCG1390
303	12	AMBCA=ALFA	FMCG1400
304	13	ALFA=C.	FMCG1410
	C		FMCG1420
	C	SAVE FUNCTION AND DERIVATIVE VALUES FOR OLD ARGUMENT	FMCG1430
305	14	FX=FY	FMCG1440
306		DX=DY	FMCG1450
	C		FMCG1460
	C	STEP ARGUMENT ALONG H	FMCG1470
307	CC	15 I=1,N	FMCG1480
308	15	X(I)=X(I)+AMBCA*H(I)	FMCG1490
	C		FMCG1500
	C	COMPUTE FUNCTION VALUE AND GRADIENT FOR NEW ARGUMENT	FMCG1510
309		CALL VALUE (P,F,G)	
310		NFUNCT = NFUNCT + 1	
311		FY=F	FMCG1530
	C		FMCG1540
	C	COMPUTE DIRECTIONAL DERIVATIVE DY FOR NEW ARGUMENT. TERMINATE	FMCG1550
	C	SEARCH, IF DY POSITIVE. IF DY IS ZERO THE MINIMUM IS FOUND	FMCG1560
312		DY=C.	FMCG1570
313	CC	16 I=1,N	FMCG1580
314	16	DY=DY+G(I)*H(I)	FMCG1590
315		IF(DY)17,38,20	FMCG1600
	C		FMCG1610
	C	TERMINATE SEARCH ALSO IF THE FUNCTION VALUE INDICATES THAT	F43 16
	C	A MINIMUM HAS BEEN PASSED	FMCG1630
316	17	IF(FY-FX)18,20,20	FMCG1640
	C		FMCG1650
	C	REPEAT SEARCH AND DOUBLE STEPSIZE FOR FURTHER SEARCHES	FMCG1660
317	18	AMBCA=AMBCA*ALFA	FMCG1670
318		ALFA=AMBCA	FMCG1680
	C		FMCG1690
	C	TERMINATE IF THE CHANGE IN ARGUMENT GETS VERY LARGE	FMCG1700
319		IF(HNRM*AMBCA-1.E10)14,14,19	FMCG1710
	C		FMCG1720
	C	LINEAR SEARCH TECHNIQUE INDICATES THAT NO MINIMUM EXISTS	FMCG1730
320	19	IER=2	FMCG1740
	C		FMCG1741

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C      RESTORE OLD VALUES OF FUNCTION AND ARGUMENTS
321      F=OLCF
322      DC 100 J=1,N
323      G(J)=F(J)
324      K=N+J
325      100 X(J)=F(K)
326      PRINT 1006,IER
327      RETURN
C      END OF SEARCH LOCP
C
C      INTERPOLATE CUBICALLY IN THE INTERVAL DEFINED BY THE SEARCH
C      ABOVE AND COMPUTE THE ARGUMENT X FOR WHICH THE INTERPOLATION
C      POLYNOMIAL IS MINIMIZED
C
328      20 T = C.C
329      21 IF (AMECA) 22,38,22
330      22 Z=3.*(FX-FY)/AMBCA+CX+CY
331      ALFA = AMAX1(ABS(Z),ABS(CX),ABS(CY))
332      DALFA=Z/ALFA
333      CALFA=CALFA+DALFA-CX/ALFA*CY/ALFA
334      IF(CALFA)23,27,27
C
C      RESTORE OLD VALUES OF FUNCTION AND ARGUMENTS
335      23 DC 24 J=1,N
336      K=N+J
337      24 X(J)=F(K)
338      CALL VALUE (P,F,G)
339      NFUNC = NFUNC + 1
C
C      TEST FOR REPEATED FAILURE OF ITERATION
340      25 IF(IER)47,26,47
341      26 IER=-1
342      GOTO 1
343      27 W=ALFA*SQRT(DALFA)
344      ALFA=CY-DX+W+W
345      IF (ALFA) 270,271,270
346      270 ALFA=(CY-Z+W)/ALFA
347      GO TO 272
348      271 ALFA=(Z+DY-W)/(Z+DX+Z+DY)
349      272 ALFA=ALFA*AMBCA
350      DC 28 I=1,N
351      28 X(I)=X(I)+(1-ALFA)*H(I)
C
C      TERMINATE, IF THE VALUE OF THE ACTUAL FUNCTION AT X IS LESS
C      THAN THE FUNCTION VALUES AT THE INTERVAL ENDS. OTHERWISE REDUCE
C      THE INTERVAL BY CHOOSING ONE END-POINT EQUAL TO X AND REPEAT
C      THE INTERPOLATION. WHICH END-POINT IS CHOSEN DEPENDS ON THE
C      VALUE OF THE FUNCTION AND ITS GRADIENT AT X
C
352      CALL VALUE (P,F,G)
353      NFUNC = NFUNC + 1
354      IF(F-FX)29,29,30
355      29 IF(F-FY)38,38,30
C
C      COMPUTE DIRECTIONAL DERIVATIVE
356      30 DALFA=C.
357      DC 31 I=1,N
358      31 DALFA=CALFA+G(I)*H(I)
359      IF(CALFA)32,35,35
360      32 IF(F-FX)34,33,35

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FMCG1742

FMCG1743

FMCG1744

FMCG1745

FMCG1746

FMCG1747

FMCG1750

FMCG1760

FMCG1770

FMCG1780

FMCG1790

FMCG1800

FMCG1810

FMCG1820

FMCG1830

FMCG1840

FMCG1860

FMCG1870

FMCG1880

FMCG1890

FMCG1900

FMCG1910

FMCG1920

FMCG1930

FMCG1950

FMCG1960

FMCG1970

FMCG1980

FMCG1990

FMCG2000

FMCG2010

FMCG2011

FMCG2012

FMCG2013

FMCG2014

FMCG2015

FMCG2020

FMCG2030

FMCG2040

FMCG2050

FMCG2060

FMCG2070

FMCG2080

FMCG2090

FMCG2100

FMCG2120

FMCG2130

FMCG2140

FMCG2150

FMCG2160

FMCG2170

FMCG2180

FMCG2190

FMCG2200

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361      33 IF(CX-CALFA)34,38,34      FMCG2210
362      34 FX=F                      FMCG2220
363          DX=CALFA                 FMCG2230
364          T=ALFA                   FMCG2240
365          AMBDA=ALFA               FMCG2250
366          GC TC 21                 FMCG2260
367      35 IF(FY-F)37,36,37          FMCG2270
368      36 IF(CY-CALFA)37,38,37      FMCG2280
369      37 FY=F                      FMCG2290
370          CY=CALFA                 FMCG2300
371          AMBDA=AMBDA-ALFA         FMCG2310
372      GC TC 20                     FMCG2320
C                                     FMCG2330
C      TERMINATE, IF FUNCTION HAS NOT DECREASED DURING LAST ITERATION FMCG2340
C      OTHERWISE SAVE GRADIENT NORM FMCG2350
373      38 IF(CLOG-F+EPS) 19,25,39   FMCT2360
374      39 CLOG=GNRM                 FMCG2370
C                                     FMCG2380
C      COMPUTE DIFFERENCE OF NEW AND OLD ARGUMENT VECTOR. FMCG2390
C      T=0.                          FMCG2400
375          CC 4C J=1,N              FMCG2410
376          K=J+N                    FMCG2420
377          F(K)=X(J)-F(K)           FMCG2430
378          4C T=T+ABS(F(K))          FMCG2440
379          IF (INTER) 300,301,300
380      300 WRITE (3,10C1) KCUNT,P,F
381          WRITE (3,10C2) (I,X(I),I = 1,N)
382          WRITE (3,10C3) (I,FG(I),I = 1,MG)
383          IF (MH) 302,303,302
384      302 WRITE (3,10C4) (I,FH(I),I = 1,MH)
385      303 WRITE (3,10C5) (I,G(I),I = 1,N)
386      1000 FORMAT (//,7X,' *** INTERMEDIATE RESULTS *** ')
387      1001 FORMAT (//,7X,' ITERATION NUMBER = ',I5,' F(X) = ',E14.6,' P(X) = ',E14.6)
388      1002 FORMAT (10X,' X(',I3,') = ',E14.6)
389      1003 FORMAT (//,10X,' CONSTRAINTS '/(4X,5('FG(',I3,') = ',E13.6,2X)))
390      1004 FORMAT (10X,' FH(',I3,') = ',E14.6)
391      1005 FORMAT (//,10X,' GRADIENTS '/(4X,5('X(',I3,') = ',E13.6,2X)))
392      1006 FORMAT (7X,'**** THE VALUE OF IER IS ',I4)
C                                     FMCG2450
C      TEST LENGTH OF DIFFERENCE VECTOR IF AT LEAST N+1 ITERATIONS FMCG2460
C      HAVE BEEN EXECUTED. TERMINATE, IF LENGTH IS LESS THAN EPS FMCG2470
394      301 IF (KCLNT-N1) 42,41,41
395      41 IF (T-EPS) 45,45,42
396      42 IF (KCLNT-LIMIT) 43,44,44
397      43 IER=C
C      END OF ITERATION CYCLE
C
C      START NEXT ITERATION CYCLE
398      GC TC 1
C
C      NO CONVERGENCE AFTER LIMIT ITERATIONS
399      44 IER=1
400      IF (GNRM-EPS)46,46,47
C
C      TEST FOR SUFFICIENTLY SMALL GRADIENT
401      45 IF ((GNRM - EPS).LE.C.) GO TC 46
402      50 IF (IER.NE.0) GC TC 47
403      51 IER = - 1
404      GC TC 1

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405      46 IER=C
406      47 PRINT 1006,IER
407          RETLRA
408          END
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FMCG2650

FMCG2670

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405 SUBROUTINE FMFP(VALUE,F,G,EST,EPS,LIMIT,IER,H)
410 COMMON /DATA/ X(50),FG(50),F(50),A(50),B(50)
411 COMMON /PARAM/ R,N,MG,MH,LCS1,KCUNT,INTER,NEGAT,NFUNCT,NSEL
412 DIMENSION G(50),H(1)

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..... FMFP 10

SUBROUTINE FMFP FMFP 20

PURPOSE FMFP 30

TO FIND A LOCAL MINIMUM OF A FUNCTION OF SEVERAL VARIABLES FMFP 40

BY THE METHOD OF FLETCHER AND POWELL FMFP 50

USAGE FMFP 60

CALL FMFP(FUNCT,N,X,F,G,EST,EPS,LIMIT,IER,H) FMFP 70

DESCRIPTION OF PARAMETERS FMFP 80

FUNCT - USER-WRITTEN SUBROUTINE CONCERNING THE FUNCTION TO FMFP 90

BE MINIMIZED. IT MUST BE OF THE FORM FMFP 100

SUBROUTINE FUNCT(N,ARG,VAL,GRAD) FMFP 110

AND MUST SERVE THE FOLLOWING PURPOSE FMFP 120

FOR EACH N-DIMENSIONAL ARGUMENT VECTOR ARG, FMFP 130

FUNCTION VALUE AND GRADIENT VECTOR MUST BE COMPUTED FMFP 140

AND, ON RETURN, STORED IN VAL AND GRAD RESPECTIVELY FMFP 150

N - NUMBER OF VARIABLES FMFP 160

X - VECTOR OF DIMENSION N CONTAINING THE INITIAL FMFP 170

ARGUMENT WHERE THE ITERATION STARTS. ON RETURN, FMFP 180

X HOLDS THE ARGUMENT CORRESPONDING TO THE FMFP 190

COMPUTED MINIMUM FUNCTION VALUE FMFP 200

F - SINGLE VARIABLE CONTAINING THE MINIMUM FUNCTION FMFP 210

VALUE ON RETURN, I.E. $F=F(X)$. FMFP 220

G - VECTOR OF DIMENSION N CONTAINING THE GRADIENT FMFP 230

VECTOR CORRESPONDING TO THE MINIMUM ON RETURN, FMFP 240

I.E. $G=G(X)$. FMFP 250

EST - IS AN ESTIMATE OF THE MINIMUM FUNCTION VALUE. FMFP 260

EPS - TESTVALUE REPRESENTING THE EXPECTED ABSOLUTE ERROR. FMFP 270

A REASONABLE CHOICE IS $10^{-(1-6)}$, I.E. FMFP 280

SOMEWHAT GREATER THAN $10^{-(1-D)}$, WHERE D IS THE FMFP 290

NUMBER OF SIGNIFICANT DIGITS IN FLOATING POINT FMFP 300

REPRESENTATION. FMFP 310

LIMIT - MAXIMUM NUMBER OF ITERATIONS. FMFP 320

IER - ERROR PARAMETER FMFP 330

IER = 0 MEANS CONVERGENCE WAS OBTAINED FMFP 340

IER = 1 MEANS NO CONVERGENCE IN LIMIT ITERATIONS FMFP 350

IER = -1 MEANS ERRORS IN GRADIENT CALCULATION FMFP 360

IER = 2 MEANS LINEAR SEARCH TECHNIQUE INDICATES FMFP 370

IT IS LIKELY THAT THERE EXISTS NO MINIMUM. FMFP 380

F - WORKING STORAGE OF DIMENSION $N*(N+1)/2$. FMFP 390

REMARKS FMFP 400

I) THE SUBROUTINE NAME REPLACING THE DUMMY ARGUMENT FUNCT FMFP 410

MUST BE DECLARED AS EXTERNAL IN THE CALLING PROGRAM. FMFP 420

III) IER IS SET TO 2 IF, STEPPING IN ONE OF THE COMPUTED FMFP 430

DIRECTIONS, THE FUNCTION WILL NEVER INCREASE WITHIN FMFP 440

A TOLERABLE RANGE OF ARGUMENT. FMFP 450

IER = 2 MAY OCCUR ALSO IF THE INTERVAL WHERE F FMFP 460

INCREASES IS SMALL AND THE INITIAL ARGUMENT WAS FMFP 470

RELATIVELY FAR AWAY FROM THE MINIMUM SUCH THAT THE FMFP 480

MINIMUM WAS OVERLEAPED. THIS IS DUE TO THE SEARCH FMFP 490

TECHNIQUE WHICH DOUBLES THE STEPSIZE UNTIL A POINT FMFP 500

FMFP 510

FMFP 520

FMFP 530

FMFP 540

FMFP 550

FMFP 560

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C          IS FOUND WHERE THE FUNCTION INCREASES. FMFP 570
C          SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED FMFP 580
C          FUNCT FMFP 590
C          METHOD FMFP 600
C          THE METHOD IS DESCRIBED IN THE FOLLOWING ARTICLE FMFP 610
C          R. FLETCHER AND M.J.D. POWELL, A RAPID DESCENT METHOD FOR FMFP 620
C          MINIMIZATION, FMFP 630
C          COMPUTER JOURNAL VOL.6, ISS. 2, 1963, PP.163-168. FMFP 640
C          FMFP 650
C          FMFP 660
C          FMFP 670
C          FMFP 680
C          FMFP 690
C          FMFP 700
C          FMFP 710
C          FMFP 720
C          FMFP 730
C          FMFP 740
C          FMFP 750
C          COMPUTE FUNCTION VALUE AND GRADIENT VECTOR FOR INITIAL ARGUMENT
413 NFUNCT = 0
414 IF (INTER) 401,400,401
415 401 WRITE (3,1000)
416 400 CALL VALUE (P,F,G)
417 NFUNCT = NFUNCT + 1
C
C          RESET ITERATION COUNTER AND GENERATE IDENTITY MATRIX FMFP 770
C          FMFP 780
418 IER=C FMFP 790
419 KCUNT=C FMFP 800
420 N2=N+N FMFP 810
421 N3=N2+N FMFP 820
422 N31=N3+1 FMFP 830
423 1 K=N31 FMFP 840
424 DO 4 J=1,N FMFP 850
425 H(K)=1. FMFP 860
426 NJ=N-J FMFP 870
427 IF(NJ)5,5,2 FMFP 880
428 2 DO 3 L=1,NJ FMFP 890
429 KL=K+L FMFP 900
430 3 H(KL)=C. FMFP 910
431 4 K=KL+1 FMFP 920
C FMFP 930
C          START ITERATION LOOP FMFP 940
432 5 KCUNT=KCUNT +1 FMFP 950
C FMFP 960
C          SAVE FUNCTION VALUE, ARGUMENT VECTOR AND GRADIENT VECTOR FMFP 970
433 OLDF=F FMFP 980
434 DO 9 J=1,N FMFP 990
435 K=N+J FMFP1000
436 H(K)=G(J) FMFP1010
437 K=K+N FMFP1020
438 H(K)=X(J) FMFP1030
C FMFP1040
C          DETERMINE DIRECTION VECTOR H FMFP1050
439 K=J+N3 FMFP1060
440 T=0. FMFP1070
441 DO 8 L=1,N FMFP1080
442 T=T-G(L)*H(K) FMFP1090
443 IF(L-J)6,7,7 FMFP1100
444 6 K=K+N-L FMFP1110
445 GO TO 8 FMFP1120
446 7 K=K+1 FMFP1130
447 8 CONTINUE FMFP1140

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448      9 H(J)=T                                FMFP1150
      C                                          FMFP1160
      C      CHECK WHETHER FUNCTION WILL DECREASE STEPPING ALONG H.    FMFP1170
449      CY=C.                                    FMFP1180
450      HNRN=C.                                  FMFP1190
451      GNRN=C.                                  FMFP1200
      C                                          FMFP1210
      C      CALCULATE DIRECTIONAL DERIVATIVE AND TESTVALUES FOR DIRECTION FMFP1220
      C      VECTOR H AND GRADIENT VECTOR G.                            FMFP1230
452      DO 10 J=1,N                             FMFP1240
453      HNRN=HNRN+ABS(H(J))                      FMFP1250
454      GNRN=GNRN+ABS(G(J))                      FMFP1260
455      10 CY=CY+H(J)*G(J)                      FMFP1270
      C                                          FMFP1280
      C      REPEAT SEARCH IN DIRECTION OF STEEPEST DESCENT IF DIRECTIONAL FMFP1290
      C      DERIVATIVE APPEARS TO BE POSITIVE OR ZERO.                FMFP1300
456      IF(CY)11,51,51                          FMFP1310
      C                                          FMFP1320
      C      REPEAT SEARCH IN DIRECTION OF STEEPEST DESCENT IF DIRECTION FMFP1330
      C      VECTOR H IS SMALL COMPARED TO GRADIENT VECTOR G.          FMFP1340
457      11 IF(HNRN/GNRN-EPS)51,51,12             FMFP1350
      C                                          FMFP1360
      C      SEARCH MINIMUM ALONG DIRECTION H                          FMFP1370
      C                                          FMFP1380
      C      SEARCH ALONG H FOR POSITIVE DIRECTIONAL DERIVATIVE        FMFP1390
458      12 FY=F                                  FMFP1400
459      ALFA=2.*(EST-F)/CY                        FMFP1410
460      AMBDA=1.                                  FMFP1420
      C                                          FMFP1430
      C      USE ESTIMATE FOR STEPSIZE ONLY IF IT IS POSITIVE AND LESS THAN FMFP1440
      C      1. OTHERWISE TAKE 1. AS STEPSIZE                          FMFP1450
461      IF(ALFA)15,15,13                         FMFP1460
462      13 IF(ALFA-AMBDA)14,15,15                FMFP1470
463      14 AMBDA=ALFA                             FMFP1480
464      15 ALFA=C.                                FMFP1490
      C                                          FMFP1500
      C      SAVE FUNCTION AND DERIVATIVE VALUES FOR OLD ARGUMENT    FMFP1510
465      16 FX=FY                                  FMFP1520
466      DX=CY                                      FMFP1530
      C                                          FMFP1540
      C      STEP ARGUMENT ALONG H                                      FMFP1550
467      DO 17 I=1,N                             FMFP1560
468      17 X(I)=X(I)+AMBDA*H(I)                  FMFP1570
      C                                          FMFP1580
      C      COMPUTE FUNCTION VALUE AND GRADIENT FOR NEW ARGUMENT      FMFP1590
469      CALL VALUE (P,F,G)
470      NFUNCT = NFUNCT + 1
471      FY=F                                FMFP1610
      C                                          FMFP1620
      C      COMPUTE DIRECTIONAL DERIVATIVE CY FOR NEW ARGUMENT. TERMINATE FMFP1630
      C      SEARCH, IF CY IS POSITIVE. IF CY IS ZERO THE MINIMUM IS FOUND FMFP1640
472      CY=C.                                    FMFP1650
473      DO 18 I=1,N                             FMFP1660
474      18 CY=CY+G(I)*H(I)                      FMFP1670
475      IF(CY)19,36,22                          FMFP1680
      C                                          FMFP1690
      C      TERMINATE SEARCH ALSO IF THE FUNCTION VALUE INDICATES THAT FMFP1700
      C      A MINIMUM HAS BEEN PASSED                                FMFP1710
476      19 IF(FY-FX)20,22,22                    FMFP1720
      C                                          FMFP1730

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C      REPEAT SEARCH AND DOUBLE STEPSIZE FOR FURTHER SEARCHES      FMFP1740
477 2C AMBCA=AMBDA*ALFA      FMFP1750
478      ALFA=AMBDA      FMFP1760
C      END OF SEARCH LOOP      FMFP1770
C      FMFP1780
C      TERMINATE IF THE CHANGE IN ARGUMENT GETS VERY LARGE      FMFP1790
479      IF(AMB*AMBDA-1.E10)16,16,21      FMFP1800
C      FMFP1810
C      LINEAR SEARCH TECHNIQUE INDICATES THAT NO MINIMUM EXISTS      FMFP1820
480 21 IER=2      FMFP1830
481      PRINT 1006,IER
482      RETURN
C      FMFP1840
C      INTERPOLATE CUBICALLY IN THE INTERVAL DEFINED BY THE SEARCH      FMFP1850
C      ABOVE AND COMPUTE THE ARGUMENT X FOR WHICH THE INTERPOLATION      FMFP1860
C      POLYNOMIAL IS MINIMIZED      FMFP1870
C      FMFP1880
483 22 T=0.      FMFP1890
484 23 IF(AMBCA)24,36,24      FMFP1900
485 24 Z=3.*(FX-FY)/AMBCA+DX+CY      FMFP1910
486      ALFA=AMAX1(ABS(Z),ABS(CX),ABS(CY))      FMFP1920
487      DALFA=Z/ALFA      FMFP1930
488      DALFA=DALFA*DALFA-CX/ALFA+DY/ALFA      FMFP1940
489      IF(DALFA)51,25,25      FMFP1950
490 25 W=ALFA*SQRT(DALFA)      FMFP1960
491      ALFA=CY-DX+W*W      FMFP1970
492      IF (ALFA) 250,251,250      FMFP1971
493 250 ALFA=(CY-Z+W)/ALFA      FMFP1972
494      GO TO 252      FMFP1973
495 251 ALFA=(Z+DY-W)/((Z+CX+Z+DY)      FMFP1974
496 252 ALFA=ALFA*AMBCA      FMFP1975
497      DO 26 I=1,N      FMFP1980
498 26 X(I)=X(I)+(1-ALFA)*H(I)      FMFP1990
C      FMFP2000
C      TERMINATE, IF THE VALUE OF THE ACTUAL FUNCTION AT X IS LESS      FMFP2010
C      THAN THE FUNCTION VALUES AT THE INTERVAL ENDS. OTHERWISE REDUCE      FMFP2020
C      THE INTERVAL BY CHOOSING ONE END-POINT EQUAL TO X AND REPEAT      FMFP2030
C      THE INTERPOLATION. WHICH END-POINT IS CHOSEN DEPENDS ON THE      FMFP2040
C      VALUE OF THE FUNCTION AND ITS GRADIENT AT X      FMFP2050
C      FMFP2060
499      CALL VALUE (P,F,G)
500      NFUNCT = NFUNCT + 1
501      IF(F-FX)27,27,28      FMFP2080
502 27 IF(F-FY)36,36,28      FMFP2090
503 28 DALFA=C.      FMFP2100
504      DO 29 I=1,N      FMFP2110
505 29 DALFA=DALFA+G(I)*H(I)      FMFP2120
506      IF(DALFA)30,33,33      FMFP2130
507 30 IF(F-FX)32,31,33      FMFP2140
508 31 IF(CX-DALFA)32,36,32      FMFP2150
509 32 FX=F      FMFP2160
510      DX=DALFA      FMFP2170
511      T=ALFA      FMFP2180
512      AMBCA=ALFA      FMFP2190
513      GO TO 23      FMFP2200
514 33 IF(FY-F)35,34,35      FMFP2210
515 34 IF(CY-DALFA)35,36,35      FMFP2220
516 35 FY=F      FMFP2230
517      CY=DALFA      FMFP2240
518      AMBCA=AMBDA-ALFA      FMFP2250
519      GO TO 22      FMFP2260

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C
C      TERMINATE, IF FUNCTION HAS NOT DECREASED DURING LAST ITERATION
520 36 IF(1CLCF-F+EPS) 51,38,38
C
C      COMPUTE DIFFERENCE VECTORS OF ARGUMENT AND GRADIENT FROM
C      TWO CONSECUTIVE ITERATIONS
521 38 DO 37 J=1,N
522     K=N+J
523     F(K)=G(J)-F(K)
524     K=N+K
525 37 H(K)=X(J)-H(K)
526     IF (INTER) 300,301,300
527 300 WRITE (3,1001) KCUNT,P,F
528     WRITE (3,1002) (I,X(I),I = 1,N)
529     WRITE (3,1003) (I,FG(I),I = 1,PG)
530     IF (MH) 302,303,302
531 302 WRITE (3,1004) (I,FH(I),I = 1,MH)
532 303 WRITE (3,1005) (I,G(I),I = 1,N)
533 1000 FORMAT (//,7X,' *** INTERMEDIATE RESULTS *** ')
534 1001 FORMAT (//,7X,' ITERATION NUMBER = ',15,' F(X) = ',E14.6,' P(X) = ',E14.6)
535 1002 FORMAT (10X,' X(',13,') = ',E14.6)
536 1003 FORMAT (//,10X,' CONSTRAINTS '/(4X,5('FG(',13,') = ',E13.6,2X)))
537 1004 FORMAT (10X,' FH(',13,') = ',E14.6)
538 1005 FORMAT (//,10X,' GRADIENTS '/(4X,5('X(',13,') = ',E13.6,2X)))
539 1006 FORMAT (7X,'**** THE VALUE OF IER IS ',14)
C
C      TEST LENGTH OF ARGUMENT DIFFERENCE VECTOR AND DIRECTION VECTOR
C      IF AT LEAST N ITERATIONS HAVE BEEN EXECUTED. TERMINATE, IF
C      BOTH ARE LESS THAN EPS
540 301 IER=C
541     IF(KCLNT-N)42,39,39
542 39 I=0.
543     Z=0.
544     DO 40 J=1,N
545     K=N+J
546     H=H(K)
547     K=K+N
548     T=T+ABS(H(K))
549 40 Z=Z+H+H(K)
550     IF(I-NRP-EPS)41,41,42
551 41 IF(T-EPS)56,56,42
C
C      TERMINATE, IF NUMBER OF ITERATIONS WOULD EXCEED LIMIT
552 42 IF(KCLNT-LIMIT)43,50,50
C
C      PREPARE UPDATING OF MATRIX H
553 43 ALFA=C.
554     DO 47 J=1,N
555     K=J+N3
556     H=0.
557     DO 46 L=1,N
558     KL=N+L
559     H=H+H(KL)*H(K)
560     IF(L-J)44,45,45
561 44 K=K+N-L
562     GO TO 46
563 45 K=K+1
564 46 CONTINUE
565     K=N+J

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FMFP2270
 FMFP2280
 FMFP2290
 FMFP2300
 FMFP2310
 FMFP2320
 FMFP2330
 FMFP2340
 FMFP2350
 FMFP2360
 FMFP2370

 FMFP2380
 FMFP2390
 FMFP2400
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 FMFP2660
 FMFP2670
 FMFP2680
 FMFP2690
 FMFP2700
 FMFP2710

566	ALFA=ALFA+H*H(K)	FMFP2720
567	47 F(IJ)=H	FMFP2730
	C	FMFP2740
	C REPEAT SEARCH IN DIRECTION OF STEEPEST DESCENT IF RESULTS	FMFP2750
	C ARE NOT SATISFACTORY	FMFP2760
568	IF(Z*ALFA)48,1,48	FMFP2770
	C	FMFP2780
	C UPDATE MATRIX H	FMFP2790
569	48 K=N31	FMFP2800
570	CC 49 L=1,N	FMFP2810
571	KL=N2+L	FMFP2820
572	CC 49 J=L,N	FMFP2830
573	NJ=N2+J	FMFP2840
574	H(K)=F(K)+H(KL)*H(NJ)/Z-H(L)*H(J)/ALFA	FMFP2850
575	49 K=K+1	FMFP2860
576	GC TC 5	FMFP2870
	C END OF ITERATION LOOP	FMFP2880
	C	FMFP2890
	C NO CONVERGENCE AFTER LIMIT ITERATIONS	FMFP2900
577	50 IER=1	FMFP2910
578	PRINT 1006,IER	
579	RETURN	FMFP2920
	C	FMFP2930
	C RESTORE OLD VALUES OF FUNCTION AND ARGUMENTS	FMFP2940
580	51 CC 52 J=1,N	FMFP2950
581	K=N2+J	FMFP2960
582	52 X(J)=F(K)	FMFP2970
583	CALL VALUE (P,F,G)	
584	NFUNCT = NFUNCT + 1	
585	IF(GAMP-EPS)55,55,53	FMFP3020
	C	FMFP3030
	C TEST FOR REPEATED FAILURE OF ITERATION	FMFP3040
586	53 IF(IER)56,54,54	FMFP3050
587	54 IER=-1	FMFP3060
588	GCTC 1	FMFP3070
589	55 IER=C	FMFP3080
590	56 PRINT,IER	
591	RETURN	
592	END	FMFP3100

NONLINEAR OPTIMIZATION BY THE SEQUENTIAL UNCONSTRAINED MINIMIZATION
TECHNIQUE USING CONJUGATE GRADIENT METHODS

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AN ABSTRACT OF A MASTER'S REPORT

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1971

ABSTRACT

The purpose of this report is to present the implementation of sequential unconstrained minimization technique (SUMT) by the method of conjugate gradients. The need for developing this approach arose due to the difficulties in the existing computer programs implementing SUMT. The RAC computer program requires the user to provide the first and second derivatives of the objective function and constraints functions. This is difficult when both are highly nonlinear. In the computer program of Lai which uses Hooke and Jeeves pattern search technique, the path of optimization frequently moves into the infeasible region and therefore a tolerance on constraint violation needs to be specified. This is not easy to determine.

This report contains the computer program to implement SUMT using conjugate gradient methods of Fletcher and Powell and Fletcher and Reeves. These methods possess the property of quadratic convergence. The computer program has been used for solving seven test problems. It was noticed that there was no single unconstrained minimization function, P-function, which was suitable for all types of problems. Six different kinds of P-functions as suggested by Fiacco and McCormick were tried. Three types of P-functions were found suitable for the test problems and have been included in the computer program. The optimal results obtained were found to be dependant on the stagewise stopping criterion and a suitable range of values have been suggested.

The following conclusions are drawn:

1. The new approach is capable of handling a variety of problems provided the objective function and the constraints are differentiable.
2. There is no single P function suitable for all problems.
3. The optimal solution is dependant on the stagewise stopping criterion.