

Non-k-equal configuration and immersion spaces

by

Keely Grossnickle

B.S., Emporia State University 2013

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AN ABSTRACT OF A DISSERTATION

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DOCTOR OF PHILOSOPHY

Department of Mathematics  
College of Arts and Sciences

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# Abstract

Let  $\mathcal{M}_d(n)$  be the configuration space of  $n$  distinct, labeled points in  $\mathbb{R}^d$ . We can impose a non- $k$ -equal condition on the configurations that no  $k$  points coincide. Denote this space by  $\mathcal{M}_d^{(k)}(n)$ . One closely related space is the  $k$ th-component of the little discs operad, denoted by  $\mathcal{B}_d(n)$ , it is the configuration space of  $n$  open, labeled, distinct discs in the unit disc. Similarly, we can impose a non- $k$ -overlapping condition such that no  $k$  discs share a common point. Let this space be denoted  $\mathcal{B}_d^{(k)}(n)$ .

The homology of the little discs operad and the homology of the non- $k$ -equal configuration spaces have both been known for several decades. Dobrinskaya and Turchin gave a geometric description of  $H_*\mathcal{B}_d^{(k)}(n)$  using the operadic interpretation that is extensively used throughout the first four chapters.

The first two chapters of this dissertation give the needed background on operads, modules, and bimodules, in general, and then more details about the little discs operad. There is also information given about symmetric sequences, including the homology of the non- $k$ -overlapping discs. This leads to the third chapter where we give an explicit formula to compute the traces (or characters) of the symmetric group action on  $H_*\mathcal{M}_d^{(k)}(n)$ . This yields a generating function of these characters that is called the *Cycle Index Sum*. The fourth chapter defines the operad of overlapping discs, which is a filtered operad. The culmination of this chapter is a theorem that gives a description of an element in  $H_*\mathcal{B}_d^{(k)}(n)$  that occurs when braces are composed with braces. In the final chapter of this dissertation, we define a cosimplicial model for the limit of the Taylor tower associated to the homotopy fiber of non- $k$ -equal spaces of immersions of  $D^1 \rightarrow D^n$  over the space of all immersions  $D^1 \rightarrow D^n$ .

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Approved by:

Major Professor  
Victor Turchin

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The first two chapters of this dissertation give the needed background on operads, modules, and bimodules, in general, and then more details about the little discs operad. There is also information given about symmetric sequences, including the homology of the non- $k$ -overlapping discs. This leads to the third chapter where we give an explicit formula to compute the traces (or characters) of the symmetric group action on  $H_*\mathcal{M}_d^{(k)}(n)$ . This yields a generating function of these characters that is called the *Cycle Index Sum*. The fourth chapter defines the operad of overlapping discs, which is a filtered operad. The culmination of this chapter is a theorem that gives a description of an element in  $H_*\mathcal{B}_d^{(k)}(n)$  that occurs when braces are composed with braces. In the final chapter of this dissertation, we define a cosimplicial model for the limit of the Taylor tower associated to the homotopy fiber of non- $k$ -equal spaces of immersions of  $D^1 \rightarrow D^n$  over the space of all immersions  $D^1 \rightarrow D^n$ .

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# Dedication

To those with Imposter Syndrome

# Chapter 1

## Operads, Modules and Bimodules

### 1.1 Operads

**Definition 1.1.1.** Define an operad  $\{\mathcal{O}(n), n \geq 0\}$  to be a sequence of  $\mathbb{k}$  that has the following structure data:

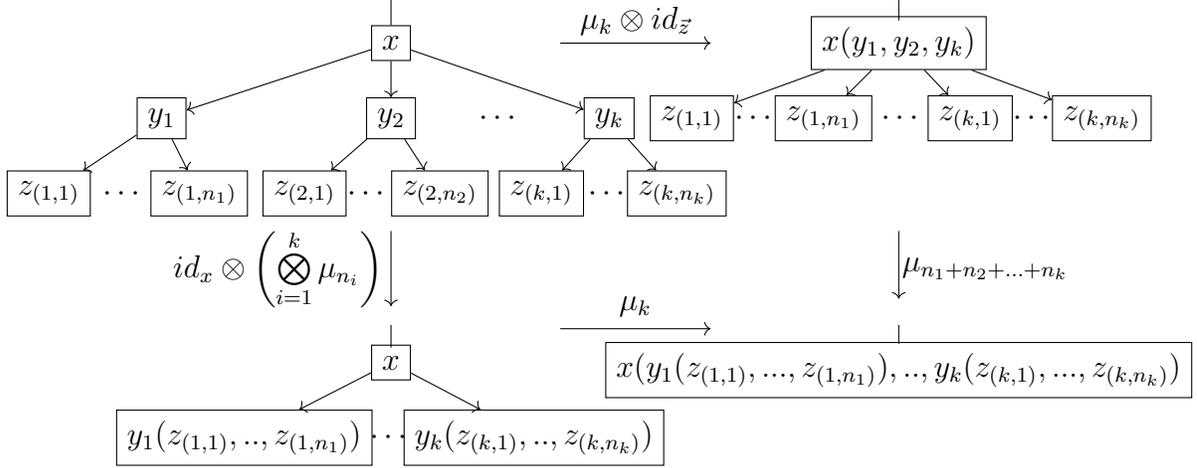
1. An action of the symmetric group  $\Sigma_n$ , for  $n \geq 0$ .
2. Composition maps  $\mu_{k(m_1, m_2, \dots, m_k)} : \mathcal{O}(k) \otimes (\mathcal{O}(m_1) \otimes \mathcal{O}(m_2) \otimes \dots \otimes \mathcal{O}(m_k)) \rightarrow \mathcal{O}(m_1 + m_2 + \dots + m_k)$ , where  $k, m_1, m_2, \dots, m_k \geq 0$ .

If the  $m_i$ 's are understood, then we will abbreviate  $\mu_{k(m_1, m_2, \dots, m_k)}$  by  $\mu_k$ . For  $x \in \mathcal{O}(k)$  and  $y_i \in \mathcal{O}(m_i)$ , we will write  $\mu_k(x, y_1, \dots, y_k) = x(y_1, \dots, y_k)$ .

3. An element  $id \in \mathcal{O}(1)$  that will be a unital element or identity.

An operad  $\mathcal{O}(n)$  is also required to satisfy associative, unital and a compatibility with the symmetric group action axioms:

1. The composition maps will satisfy the following commutative square diagram:



2. An operad satisfies the unital property:  $x(id, \dots, id) = x = id(x)$ .

3. An operad satisfies two symmetric properties. First,

$$x(y_1 \cdot \sigma_1, y_2 \cdot \sigma_2, \dots, y_k \cdot \sigma_k) = x(y_1, y_2, \dots, y_k) \cdot \vec{\sigma},$$

where  $\vec{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_k) \in \Sigma_{n_1 + \dots + n_k}$ . Here the inputs of the  $y_i$ 's are permuted.

Second, a block permutation action where the  $y_i$ 's are permuted and their corresponding  $z_{i,j}$ 's move with them.

$$x(y_1, y_2, \dots, y_k) \cdot \sigma(\vec{n}) = (x \cdot \sigma)(y_{\sigma_1^{-1}}, y_{\sigma_2^{-1}}, \dots, y_{\sigma_k^{-1}})$$

where  $\sigma(\vec{n}) = (\sigma_1, \sigma_2, \dots, \sigma_k) \in \Sigma_{n_1 + \dots + n_k}$  is the permutation of blocks obtained from  $\sigma \in \Sigma_k$

## 1.2 Examples

1. For any vector space  $A$ , let  $\mathcal{E}nd_A$  be the operad of endomorphisms on  $A$  where  $\mathcal{E}nd_A(k) = \text{Hom}(A^{\otimes k}, A)$ . The composition maps on  $\mathcal{E}nd_A(k)$  are

$$x(y_1, \dots, y_k) = x \circ (y_1 \otimes y_2 \otimes \dots \otimes y_k) \in \text{Hom}(A^{\otimes(n_1 + \dots + n_k)}, A).$$

where  $A^{\otimes(n_1+\dots+n_k)} = A^{\otimes n_1} \otimes \dots \otimes A^{\otimes n_k}$ .

$$y_1 \otimes y_2 \otimes \dots \otimes y_k : A^{\otimes n_1} \otimes \dots \otimes A^{\otimes n_k} \rightarrow A \otimes A \otimes \dots \otimes A = A^k,$$

and  $x : A^k \rightarrow A$ .

The identity,  $id \in \mathcal{E}nd_A(1) = Hom(A, A)$  and the identity map is  $id : A \rightarrow A$ . The  $\Sigma_n$  action of  $\mathcal{E}nd_A(n) = Hom(A^{\otimes n}, A)$  permutes the factors in  $A^{\otimes n}$ .

**Definition 1.2.1.** For an operad  $\mathcal{O}$ , a vector space  $A$  is an  $\mathcal{O}$ -algebra if one is given a map of operads  $f : \mathcal{O} \rightarrow \mathcal{E}nd_A$ .

An operad  $\mathcal{O}$  describes all natural multi-linear  $n$ -arity operations of a given algebra structure. We will look at classical algebra structures such as commutative, associative, and Lie algebras as well as Poisson algebras and see how they arise from looking at such maps in small arities of the corresponding operads.

2. The commutative unital operad  $\mathcal{C}om$ :

$\mathcal{C}om(n) = \mathbb{k} = \mathbb{k}\langle x_1 \cdot x_2 \cdots x_n \rangle$  for all  $n \geq 0$ , the one dimensional vector space of monomials in  $n$  commutative variables. The identity element is  $x_1 \in \mathcal{C}om(1)$ . The symmetric group actions of  $\Sigma_n$  is trivial. All compositions are induced by the  $\mathbb{k} \otimes \mathbb{k} \simeq \mathbb{k}$ , using the usual multiplication. A map  $\mathcal{C}om \rightarrow \mathcal{E}nd_A$  gives the structure of a commutative algebra on  $A$ . Pictorially, the symmetric group action can be described as tree diagrams. This is shown below in the case that  $n = 2$ :

$$\begin{array}{c} | \\ \boxed{(x_1 \cdot x_2)} \\ / \quad \backslash \\ 2 \quad 1 \end{array} = \begin{array}{c} | \\ \boxed{(x_2 \cdot x_1)} \\ / \quad \backslash \\ 1 \quad 2 \end{array} = \begin{array}{c} | \\ \boxed{(x_1 \cdot x_2)} \\ / \quad \backslash \\ 1 \quad 2 \end{array}$$

Now let us look at a few of the small arities of  $\mathcal{C}om$ :

$$\mathcal{C}om(0) = \mathbb{k} = \mathbb{k}\langle 1 \rangle.$$

$Com(1) = \mathbb{k}$  since the only map from  $A$  to  $A$  is the identity map  $id : A \rightarrow A$  mapping  $x_1 \mapsto x_1$ , including scalar multiples of the map. Thus  $Com(1) = \mathbb{k}\langle x_1 \rangle = \mathbb{k}$ .

For  $Com(2)$  we look at the possible maps  $A^{\otimes 2} \rightarrow A$  that come from a commutative algebra structure. Given two elements  $x_1, x_2 \in A$  the only possible operation is commutative multiplication. This implies all maps of  $A^{\otimes 2} \rightarrow A$  are equivalent to scalar multiples of the map  $(x_1, x_2) \mapsto x_1 \cdot x_2$ . Therefore  $Com(2) = \mathbb{k}\langle x_1 \cdot x_2 \rangle = \mathbb{k}$ .

For  $Com(3)$  we look at the possible maps  $A^{\otimes 3} \rightarrow A$  that arise from a commutative algebra structure. Given three elements  $x_1, x_2, x_3 \in A$  the only possible operation is multiplication. Since we are in a commutative setting, all maps of  $A^{\otimes 3} \rightarrow A$  are equivalent to scalar multiples of the map  $(x_1, x_2, x_3) \mapsto x_1 \cdot x_2 \cdot x_3$ . Therefore  $Com(3) = \mathbb{k}\langle x_1 \cdot x_2 \cdot x_3 \rangle = \mathbb{k}$ .

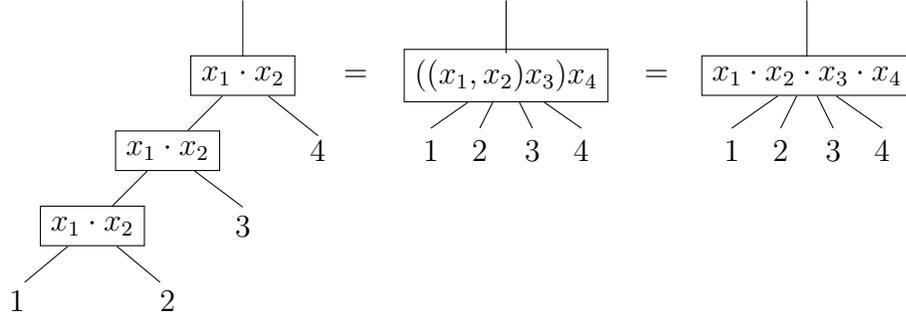
There are two versions of a commutative algebra structure, either unital or non-unital. Above we have described the unital case. The non-unital version is governed by how  $Com(0)$  is defined. We denote the non-unital commutative operad by  $Com_{>0}$  and we define  $Com_{>0}(0) = 0$ .

### 3. The associative unital operad $\mathcal{A}ssoc$ :

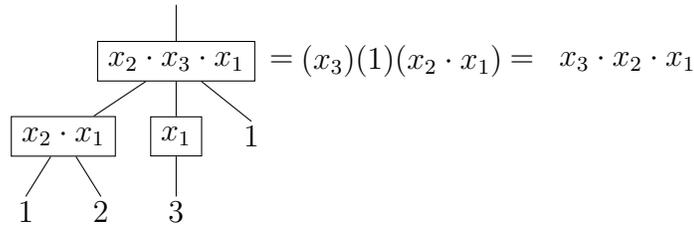
$\mathcal{A}ssoc(n)$  is the subspace of a free associative algebra on  $n$  generators,  $x_1, \dots, x_n$ , spanned by monomials where every generator is used exactly once. The symmetric group action of  $\Sigma_n$  on  $\mathcal{A}ssoc(n)$  is renumeration of the generators. Thus  $\mathcal{A}ssoc(n) = \mathbb{k}^{n!} = \mathbb{k}[\Sigma_n]$ . The identity element is  $x_1 \in \mathcal{A}ssoc(1)$ . Since  $\mathcal{A}ssoc$  is unital,  $\mathcal{A}ssoc(0) = \mathbb{k}\langle 1 \rangle$ .

As with commutative algebras, there are two versions of an associative algebra, either unital or non-unital. The non-unital version is governed by how  $\mathcal{A}ssoc(0)$  is defined. We will denote the non-unital version by  $\mathcal{A}ssoc_{>0}$  and we define  $\mathcal{A}ssoc_{>0}(0) = 0$ .

Pictorially, we can represent the composition maps of  $\mathcal{A}ssoc$  as a tree diagram. Example of such diagram for  $\mathcal{A}ssoc(4)$  is shown below.



An example of composition in  $\mathcal{A}socc(3)$  is shown below:



4. The Lie operad  $\mathcal{L}ie$ :

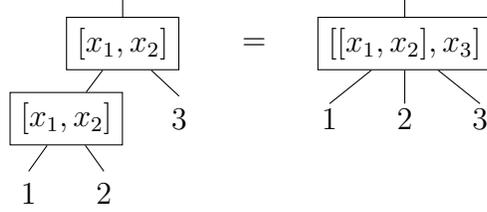
Recall that a Lie algebra is a vector space  $\mathfrak{g}$  with a Lie bracket  $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ . This bracket is a bilinear operation with an anti-symmetric relation,  $[x, y] = -[y, x]$ , and obeys the Jacobi relation:  $[[x, y], z] + [[z, x], y] + [[y, z], x] = 0$ .

Now we define  $\mathcal{L}ie(n)$  as the subspace of a free Lie algebra on  $n$  generators  $x_1, \dots, x_n$ , spanned by the Lie bracket where every generator is used exactly once. The identity element is  $x_1 \in \mathcal{L}ie(1)$ . The symmetric group action of  $\Sigma_n$  on  $\mathcal{L}ie(n)$  is renumeration of the generators.

It is a known fact that  $\mathcal{L}ie(n) = \mathbb{k}^{\binom{n-1}{1}} = \mathbb{k}\langle [\dots[[x_1, x_{\sigma(2)}], x_{\sigma(3)}], \dots, x_{\sigma(n)}] \rangle$ . The first generator,  $x_1$ , can always be moved to the front by the anti-symmetry relation and the bracket can be made linear by the Jacobi relation. We say that a Lie bracket is linear if it is of the form  $[\dots[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}], \dots, x_{\sigma(n)}]$ .

*Remark:* We have an inclusion of operads:  $\mathcal{L}ie \hookrightarrow \mathcal{A}ssoc$ . The Lie bracket can be interpreted as  $[x_1, x_2] = x_1 \cdot x_2 - x_2 \cdot x_1$  in  $\mathcal{A}ssoc$ .

Pictorially, we can represent composition maps in  $\mathcal{L}ie(3)$  as follows:



Now we look at a few small arities of  $\mathcal{L}ie$ :

$\mathcal{L}ie(0) = 0$  as there is no natural map  $\mathbb{k} \rightarrow \mathfrak{g}$  that arises from the Lie algebra structure.

$\mathcal{L}ie(1) = \mathbb{k} = \mathbb{k}\langle x_1 \rangle$  as the only map from  $\mathfrak{g} \rightarrow \mathfrak{g}$  is the identity map and its scalar multiples.

For  $\mathcal{L}ie(2)$  we look at the possible maps  $\mathfrak{g}^{\otimes 2} \rightarrow \mathfrak{g}$  that come from a Lie algebra structure. Given two elements  $x_1, x_2 \in \mathfrak{g}$ , the only operation we can apply is the Lie bracket. By the anti-symmetry property, all maps of  $\mathfrak{g}^{\otimes 2} \rightarrow \mathfrak{g}$  are equivalent to scalar multiples of the map  $(x_1, x_2) \mapsto [x_1, x_2]$ . Thus  $\mathcal{L}ie(2) = \mathbb{k} = \mathbb{k}\langle [x_1, x_2] \rangle$ .

For  $\mathcal{L}ie(3)$  we look at the possible maps  $\mathfrak{g}^{\otimes 3} \rightarrow \mathfrak{g}$  that arise from a Lie algebra structure. Given three elements  $x_1, x_2, x_3 \in \mathfrak{g}$ , the only operation we can apply is the Lie bracket iteratively. Up to the Jacobi relation, all maps of  $\mathfrak{g}^{\otimes 3} \rightarrow \mathfrak{g}$  are equivalent to either scalar multiples of  $(x_1, x_2, x_3) \mapsto [[x_1, x_2], x_3]$  or scalar multiples of  $(x_1, x_2, x_3) \mapsto [[x_1, x_3], x_2]$ . Thus  $\mathcal{L}ie(3) = \mathbb{k}^2 = \mathbb{k}\langle [[x_1, x_2], x_3], [[x_1, x_3], x_2] \rangle$ .

5. The unital Poisson operad  $\mathcal{P}ois$ :

Recall the notion of a Poisson algebra which is both a commutative algebra and a Lie algebra so it has a commutative multiplication and a Lie bracket. The two operations are related by the Leibniz relation:  $[a, bc] = [a, b]c + b[a, c]$ .

$\mathcal{P}ois(n)$  is the subspace of a free Poisson algebra on  $n$  generators  $x_1, \dots, x_n$ , spanned by products of iterated brackets where every generator is used exactly once. It is a known fact that  $\mathcal{P}ois(n) = \mathbb{k}^n$ . The identity element is  $x_1 \in \mathcal{P}ois(1)$ . The symmetric group action of  $\Sigma_n$  on  $\mathcal{P}ois(n)$  is renumeration of the generators.

Next we look at small arities of  $\mathcal{P}ois$ :

$$\mathcal{Pois}(0) = \mathbb{k} = \mathbb{k}\langle 1 \rangle$$

$\mathcal{Pois}(1) = \mathbb{k}\langle x_1 \rangle$  as there is only the identity map from  $A \rightarrow A$  and its scalar multiples.

For  $\mathcal{Pois}(2)$  we look at the possible maps  $A^{\otimes 2} \rightarrow A$  that arise from a Poisson algebra structure. Given two elements  $x_1, x_2 \in A$ , there are two possible operations, either commutative multiplication or the Lie bracket. The commutative multiplication gives a map  $(x_1, x_2) \mapsto x_1 \cdot x_2$  with scalar multiples. The Lie bracket, up to the anti-symmetry property and scalar multiples, gives the map  $(x_1, x_2) \mapsto [x_1, x_2]$ . Therefore  $\mathcal{Pois}(2) = \mathbb{k}^2 = \mathbb{k}\langle x_1 \cdot x_2, [x_1, x_2] \rangle$ .

For  $\mathcal{Pois}(3)$  we look at the possible maps  $A^{\otimes 3} \rightarrow A$  that arise from a Poisson algebra structure. Given three elements  $x_1, x_2, x_3 \in A$ , there are two possible operations, either commutative multiplication or Lie bracket. Up to the Jacobi, anti-symmetry and Leibniz relations and commutativity, there are six possible maps and their scalar multiples:  $(x_1, x_2, x_3) \mapsto x_1 \cdot x_2 \cdot x_3$ ,  $(x_1, x_2, x_3) \mapsto [x_1, x_2] \cdot x_3$ ,  $(x_1, x_2, x_3) \mapsto [x_1, x_3] \cdot x_2$ ,  $(x_1, x_2, x_3) \mapsto [x_2, x_3] \cdot x_1$ ,  $(x_1, x_2, x_3) \mapsto [[x_1, x_2], x_3]$ , and  $(x_1, x_2, x_3) \mapsto [[x_1, x_3], x_2]$ . So  $\mathcal{Pois}(3) = \mathbb{k}^6 = \mathbb{k}\langle x_1 \cdot x_2 \cdot x_3, [x_1, x_2] \cdot x_3, [x_1, x_3] \cdot x_2, [x_2, x_3] \cdot x_1, [[x_1, x_2], x_3], [[x_1, x_3], x_2] \rangle$ .

There are two versions of a Poisson algebra, either unital or non-unital. The unital version is described above. The non-unital version is governed by how  $\mathcal{Pois}(0)$  is defined. We will denote the non-unital version by  $\mathcal{Pois}_{>0}$  and define  $\mathcal{Pois}_{>0}(0) = 0$ .

## 6. Graded unital Poisson Operad:

There is also a graded version of  $\mathcal{Pois}(n)$ , which we denote  $\mathcal{Pois}_{d-1}(n)$ , where the Poisson bracket has a grading of degree  $(d-1)$ . This is an example of an operad in the category of graded vector spaces. We will use this graded version of  $\mathcal{Pois}(n)$  below in Theorem 2.1.1. The grading of the bracket does not affect the identity or the identity axiom. However the structure data for the composition maps and the symmetric group action as well as the associativity and symmetry axioms are affected by the grading with the appearance of a sign that depends on the grading of the bracket. In example,  $[x_1, x_2] = (-1)^{d-1}[x_2, x_1]$ . The grading corresponds to the central symmetry of  $S^{d-1}$ . The algebra over  $\mathcal{Pois}_{d-1}$  is a

$(d - 1)$ -graded Poisson algebra.

### 1.3 Modules over an Operad

**Definition 1.3.1.** A left module over an operad  $\mathcal{O}$  is a sequence of vector spaces  $M = \{M(n), n \geq 0\}$ , with a  $\Sigma_n$ -action, together with maps

$$\lambda_{k(m_1, m_2, \dots, m_k)} = \mathcal{O}(k) \otimes (M(m_1) \otimes M(m_2) \otimes \dots \otimes M(m_k)) \rightarrow M(m_1 + \dots + m_k)$$

for any  $k, m_1, \dots, m_k \geq 0$ , satisfying similar associative, unital and symmetric group action axioms as in Definition 1.1.1. That is, there is a similar commutative square to the one in Axiom 1, Definition 1.1.1 where the  $x$  and  $y_i$ 's are in  $\mathcal{O}$  and the  $z_{i,j}$ 's are in  $M$ . The condition  $id(x) = x$ , for  $x \in M$ ,  $id \in \mathcal{O}(1)$  is satisfied. There is also similar symmetric group action conditions to Axiom 3 in Definition 1.1.1 where  $x$  is in  $\mathcal{O}$  and the  $y_i$ 's are in  $M$ .

**Definition 1.3.2.** Similarly, a right module over an operad  $\mathcal{O}$  is a sequence of vector spaces,  $M = \{M(n), n \geq 0\}$ , with a  $\Sigma_n$ -action, together with maps:

$$\rho_{k(m_1, \dots, m_k)} = M(k) \otimes \mathcal{O}(m_1) \otimes \dots \otimes \mathcal{O}(m_k) \rightarrow M(m_1 + \dots + m_k)$$

for any  $k, m_1, \dots, m_k \geq 0$ . Just as will the left module over  $\mathcal{O}(k)$ , there are analogous axioms to Axioms 1-3 in Definition 1.1.1. For the associative axiom, there is similar commutative square where  $x \in M$  and the  $y_i$ 's and the  $z_{i,j}$ 's are in  $\mathcal{O}$ . The condition  $x(id, \dots, id) = x$ , for  $x \in M$ ,  $id \in \mathcal{O}(1)$  is satisfied. For the symmetric group action axiom, there are similar conditions where  $x$  is in  $M$  and the  $y_i$ 's are in  $\mathcal{O}$ .

**Definition 1.3.3.** Define a sequence of vector spaces  $M = \{M(n), n \geq 0\}$  with a  $\Sigma_n$ -action, to be a bimodule over an operad  $\mathcal{O}$  if it is both a left and a right module over  $\mathcal{O}$ . There is also a compatibility condition that can be expressed as a commutative square, as in Axiom 1 of Definition 1.1.1, where the  $x$  and  $z_{i,j}$ 's are in  $\mathcal{O}$  and the  $y_i$ 's are in  $M$ .

## 1.4 Categorical Definitions of Operads and Modules

**Definition 1.4.1.** Let  $\mathcal{C}^\Sigma$  denote the category of symmetric sequences in  $\mathcal{C}$ . Objects in  $\mathcal{C}^\Sigma$  are sequences  $X = \{X(n), n \geq 0\}$  of objects  $X(n)$ ,  $n \geq 0$ , each endowed with a  $\Sigma_n$ -action. A morphism  $f : X \rightarrow Y$  is a collection of  $\Sigma_n$ -equivariant maps  $f_n : X(n) \rightarrow Y(n)$ ,  $n \geq 0$ .

Operads can also be defined categorically. In order to do so, we define two monoidal structure on the category of symmetric sequences. Let  $\mathcal{C}$  be a symmetric monoidal category with monoidal product  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ , and unit object  $I$ . Assume also the category  $\mathcal{C}$  has limits and colimits where colimits are preserved by the monoidal structure. In particular,  $\mathcal{C}$  has an initial element  $\emptyset$ .

The category  $\mathcal{C}$  has two monoidal structures. For two objects  $M$  and  $N$  in  $\mathcal{C}$

$$(M \otimes N)(k) = \bigsqcup_{k_1+k_2+\dots+k_r=k} \text{Ind}_{\Sigma_{k_1} \times \Sigma_{k_2}}^{\Sigma_k} M(k_1) \otimes N(k_2)$$

The identity is given by:

$$1(n) = \begin{cases} I, & n = 0; \\ \emptyset & n \neq 0. \end{cases}$$

**Definition 1.4.2.** [17, Section 2.2.2] For two symmetric sequences  $M$  and  $N$  in a symmetric monoidal category  $\mathcal{C}$ , we can define their composition sequence:

$$M \circ N = \bigsqcup_{r=0}^{\infty} (M(r) \otimes N^{\otimes r})_{\Sigma_r} \quad (1.4.1)$$

where  $N^{\otimes r}$  is the  $r$ -th symmetric sequence given by explicitly:

$$N^{\otimes r}(k) = \bigsqcup_{k_1+k_2+\dots+k_r=k} \text{Ind}_{\Sigma_{k_1} \times \Sigma_{k_2} \times \dots \times \Sigma_{k_r}}^{\Sigma_k} N(k_1) \otimes N(k_2) \dots \otimes N(k_r) \quad (1.4.2)$$

and the  $\Sigma_r$  denotes quotienting by the symmetric action of  $\Sigma_r$ .

The operation (1.4.1) endows  $\mathcal{C}$  with monoidal structure. We will denote this monoidal

category by  $(\mathcal{C}^\Sigma, \circ, \mathbb{1})$  where  $\circ$  is the monoidal product as in (1.4.1), and  $\mathbb{1}$  is the unit defined as

$$\mathbb{1}(n) = \begin{cases} I, & n = 1; \\ \emptyset & n \neq 1. \end{cases}$$

**Definition 1.4.3.** *An operad  $\mathcal{O}$  is a monoid in  $(\mathcal{C}^\Sigma, \circ, \mathbb{1})$ .*

**Definition 1.4.4.** [26, Section 5.2.1] *Recall that the structure of a monoid is given by the following data:*

1. *There is a unit map  $\eta : \mathbb{1} \rightarrow \mathcal{O}(1)$ .*
2. *Compositions maps  $\mathcal{O} \circ \mathcal{O} \rightarrow \mathcal{O}$ .*

*As well as satisfying the following associativity and unitary axioms:*

1. *It satisfies the following commutative diagram for associativity:*

$$\begin{array}{ccc} (\mathcal{O} \circ \mathcal{O}) \circ \mathcal{O} & \xrightarrow{\cong} & \mathcal{O} \circ (\mathcal{O} \circ \mathcal{O}) \\ \mu \circ id \downarrow & & \downarrow id \circ \mu \\ \mathcal{O} \circ \mathcal{O} & & \mathcal{O} \circ \mathcal{O} \\ \cong \searrow & & \swarrow \cong \\ & \mathcal{O} & \end{array}$$

2. *It satisfies the following commutative diagram for the unitary axiom:*

$$\begin{array}{ccccc} \mathbb{1} \circ \mathcal{O} & \xrightarrow{\eta \circ id} & \mathcal{O} \circ \mathcal{O} & \xleftarrow{id \circ \eta} & \mathcal{O} \circ \mathbb{1} \\ \cong \searrow & & \downarrow \mu & & \swarrow \cong \\ & & \mathcal{O} & & \end{array}$$

The composition map of  $\mathcal{O} \in (\mathcal{C}^\Sigma, \circ, \mathbb{1})$  encodes the composition map  $\mu_{k(m_1, \dots, m_k)}$  given in Definition 1.1.1. Similarly, the identity map  $id \in \mathcal{O}(1)$  from Definition 1.1.1 is encoded by the map  $\eta : \mathbb{1} \rightarrow \mathcal{O}$ .

**Definition 1.4.5.** [16, Section 2.1.6] *Let  $(M, \mu, \eta)$  be a monoid in any category. Define  $X$  to be a left module over  $M$  if it satisfies the following relations for the composition morphism  $\lambda_x : M \circ X \rightarrow X$  so that the following diagrams commute:*

$$\begin{array}{ccc}
M \circ M \circ X & \xrightarrow{id_M \circ \lambda_x} & M \circ X \\
\mu \circ id_X \downarrow & & \downarrow \lambda_X \\
M \circ X & \xrightarrow{\lambda_x} & X
\end{array}$$

and

$$\begin{array}{ccc}
I \circ X & \xrightarrow{\eta \circ id_X} & M \circ X \\
& \searrow \simeq & \downarrow \lambda_X \\
& & X
\end{array}$$

**Definition 1.4.6.** [16, Section 2.1.5] Let  $(M, \mu, \eta)$  be a monoid in any category. Define  $X$  to be a right module over  $M$  if it satisfies the following relations for the composition morphism  $\rho_X : X \circ M \rightarrow X$  so that the following diagrams commute:

$$\begin{array}{ccc}
X \circ M \circ M & \xrightarrow{id_X \circ \mu} & X \circ M \\
\rho_X \circ id_M \downarrow & & \downarrow \rho_X \\
X \circ M & \xrightarrow{\rho_X} & X
\end{array}$$

and

$$\begin{array}{ccc}
X \circ I & \xrightarrow{id_X \circ \eta} & X \circ M \\
& \searrow \simeq & \downarrow \rho_X \\
& & X
\end{array}$$

**Definition 1.4.7.** Define  $X$  to be a bimodule of  $(M, \mu, \eta)$ , a monoid in any category, if it is both a left and a right module of  $M$  and satisfies the following compatibility between left and right actions:

$$\begin{array}{ccc}
M \circ X \circ M & \xrightarrow{id_M \circ \rho_X} & M \circ X \\
\lambda_X \circ id_M \downarrow & & \downarrow \lambda_X \\
X \circ M & \xrightarrow{\rho_X} & X
\end{array}$$

**Definition 1.4.8.** For an operad  $\mathcal{O} \in (\mathcal{C}^\Sigma, \circ, \mathbb{1})$ , a left module  $L$  (respectively, right module  $R$ ) over  $\mathcal{O}$  is a  $\Sigma$ -sequence in  $(\mathcal{C}^\Sigma, \circ, \mathbb{1})$  endowed with a left action  $\lambda_L : \mathcal{O} \circ L \rightarrow L$  (respectively, a right action  $\rho_R : R \circ \mathcal{O} \rightarrow R$ ) from Definition 1.4.5 (respectively, Definition 1.4.6). Then define  $X$  to be a bimodule of  $\mathcal{O}$  to be a  $\Sigma$ -sequence in  $(\mathcal{C}^\Sigma, \circ, \mathbb{1})$  that is both a left and right module over  $\mathcal{O}$  and satisfies the compatibility between left and right actions in Definition 1.4.7.

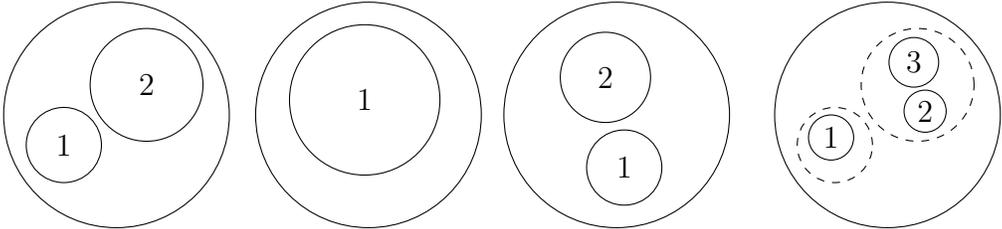
# Chapter 2

## Operads and Bimodules of Little Discs

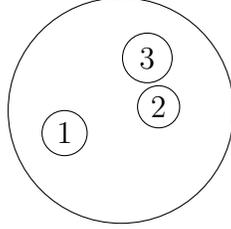
### 2.1 The Little $d$ -discs Operad $\mathcal{B}_d$

This is an example of a topological operad which will be essential to this paper. Topological operads are defined in an analogous way as operads of (graded) vector spaces where  $\{\mathcal{O}(n), n \geq 0\}$  is now a sequence of topological spaces with the same set of structure data as given in Definition 1.1.1 with the exception one uses  $\times$ , the product of spaces, in place of  $\otimes$ . Define  $\mathcal{B}_d(k)$  to be the configuration space of  $k$  disjoint, open, labeled discs in the unit disc:  $\mathcal{B}_d(k) = sEmb\left(\bigsqcup_k D^d, D^d\right)$ , the space of special embeddings,  $\bigsqcup_k D^d \mapsto D^d$  given on each component by translation and rescaling only. Composition maps in  $\mathcal{B}_d(k)$  can be viewed similarly to composition maps in  $\mathcal{E}nd_A(k)$ . Geometrically, the composition maps in  $\mathcal{B}_d(k)$  are given by insertion of discs. The identity is the configuration of one disc that encompasses the entire unit disc. The symmetric group action is renumeration of the discs.

An example of a composition map  $\mu_2 : \mathcal{B}_d(2) \times (\mathcal{B}_d(1) \times \mathcal{B}_d(2)) \rightarrow \mathcal{B}_d(3)$  would be as follows:



The dotted circles do not actually appear in  $\mathcal{B}_d(3)$  but are shown to help the reader visualize how the second and third disc compose into the first disc. The actual result of  $\mu_2$  in  $\mathcal{B}_d(3)$  is:



**Theorem 2.1.1** (F. Cohen, [11]). *The homology operad  $H_*\mathcal{B}_d$  is the operad  $\mathcal{A}ssoc$  of associative unital algebras in  $d = 1$  and the operad  $\mathcal{P}ois_{d-1}$  of graded unital Poisson algebras with bracket of degree  $(d - 1)$  in the case  $d \geq 2$ .*

We will not prove this theorem but will show that it is true for some small arities. Note that the space is torsion free so we use integral homology. We can define operads in the category of  $\mathbb{Z}$  – modules rather than vector spaces with no changes to the definition. Therefore, we use  $\mathbb{Z}$  instead of  $\mathbb{k}$  below.

**Case  $d=1$ :** It is easy to see  $\mathcal{B}_1(n)$  has  $n!$  contractible components. Then  $H_*\mathcal{B}_1(n) = H_0\mathcal{B}_1(n) = \mathbb{Z}^{n!}$ . As seen above  $\mathcal{A}ssoc(n) = \mathbb{Z}^{n!} = \mathbb{Z}\langle x_{\sigma_1} \cdot x_{\sigma_2} \cdot \dots \cdot x_{\sigma_n}, \sigma \in \Sigma_n \rangle$ .

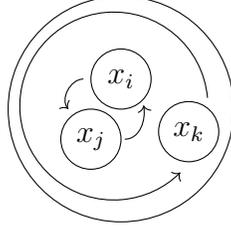
**Case  $d \geq 2$ :** We will look at small arities of  $H_*\mathcal{B}_d(n)$  to see that they agree with the (graded) Poisson operad of the same arity above. Since  $\mathcal{B}_d(0) = *$ ,  $H_0\mathcal{B}_d(0) = \mathbb{Z}$  and  $H_{>0}\mathcal{B}_d(0) = 0$ . Thus  $H_*\mathcal{B}_d(0) = \mathbb{Z}\langle 1 \rangle$ .

Similarly  $\mathcal{B}_d(1) \cong *$ ,  $H_0\mathcal{B}_d(1) = \mathbb{Z}$  and  $H_{>0}\mathcal{B}_d(1) = 0$ . Thus  $H_*\mathcal{B}_d(1) = \mathbb{Z}\langle x_1 \rangle$ .

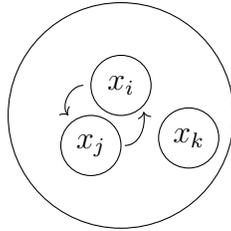
For  $d \geq 2$ ,  $\mathcal{B}_d(2) = S^{d-1}$ , so  $H_*\mathcal{B}_d(2) = H_*S^{d-1}$ . One has  $H_*S^{d-1} = \mathbb{Z}$  when  $*$  = 0 or  $d-1$ , and  $H_*S^{d-1} = 0$  otherwise. Hence  $H_*S^{d-1} = \langle x_1 \cdot x_2, [x_1, x_2] \rangle$ , where  $x_1 \cdot x_2$  generates  $H_0S^{d-1}$  and  $[x_1, x_2]$  generates  $H_{d-1}S^{d-1}$ . This matches with what we have above when describing the first few arities of  $\mathcal{P}ois(n)$ .

These generators in  $H_*\mathcal{B}_d(n)$  have nice geometric interpretations by thinking about the  $x_i$ 's as planets and the brackets giving the orbits relating the planets to one another. Looking at the generators of  $\mathcal{P}ois(3)=H_*\mathcal{B}_d(3)$ , the generator  $x_1 \cdot x_2 \cdot x_3$  is the only one where the  $x_i$ 's do not interact with one another. Geometrically this generator is just a point. For

generators of the form  $[[x_i, x_j], x_k]$ , here  $x_i$  and  $x_j$  orbit closely around each other like a binary star system and  $x_k$  orbits around the pair of them. Geometrically this generator corresponds to a map  $S^{d-1} \times S^{d-1} \rightarrow \mathcal{B}_d(3)$ .



For the generators of the form  $[x_i, x_j] \cdot x_k$ , again  $x_i$  and  $x_j$  orbit closely around each other. However, unlike the previous generator,  $x_k$  does not interact with the other two generators. Geometrically this generator corresponds to the map  $S^{d-1} \rightarrow \mathcal{B}_d(3)$ .



## 2.2 Bimodule of Overlapping Discs $\mathcal{B}_d^{(k)}$

Below we describe a bimodule over  $\mathcal{B}_d$ , the little  $d$ -discs operad, but first recall the configuration space.

**Definition 2.2.1.** 1. Let  $\mathcal{M}_d(n)$  be the configuration space of  $n$  distinct, labeled points in  $\mathbb{R}^d$ , equivalently:

$$\mathcal{M}_d(n) = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^d \mid x_i \neq x_j \text{ if } i \neq j\}.$$

2. We can impose a non- $k$ -equal condition on the configurations of  $\{x_1, x_2, \dots, x_n\}$ : no  $k$  points coincide. We will denote the configuration space of  $n$  labeled points with the non- $k$ -equal condition by  $\mathcal{M}_d^{(k)}(n)$ .

Notice that  $\mathcal{M}_d(n)$  is homotopy equivalent to  $\mathcal{B}_d(n)$  by contracting each disc of  $\mathcal{B}_d(n)$  to its center.

Similarly we can impose a *non- $k$ -overlapping* condition on the configurations of discs where no  $k$  discs share a common point. We will denote this space as  $\mathcal{B}_d^{(k)}(n)$ . One can view  $\mathcal{B}_d^{(k)}(n)$  as a space  $sImm^k\left(\bigsqcup_n D^d, D^d\right)$ , the space of special immersions,  $\bigsqcup_n D^d \mapsto D^d$  given on each component by translation and rescaling only.

The sequence  $\mathcal{B}_d^{(k)}$  is a bimodule over  $\mathcal{B}_d$ . The right action

$$\mathcal{B}_d^{(k)}(n) \times \mathcal{B}_d(m_1) \times \mathcal{B}_d(m_2) \times \dots \times \mathcal{B}_d(m_n) \rightarrow \mathcal{B}_d^{(k)}(m_1 + m_2 + \dots + m_n) \quad (2.2.1)$$

and the left action

$$\mathcal{B}_d(n) \times \mathcal{B}_d^{(k)}(m_1) \times \mathcal{B}_d^{(k)}(m_2) \times \dots \times \mathcal{B}_d^{(k)}(m_n) \rightarrow \mathcal{B}_d^{(k)}(m_1 + m_2 + \dots + m_n) \quad (2.2.2)$$

are given by composition of disc maps.

The composition maps of  $\mathcal{B}_d^{(k)}$  are well-defined as resulting configuration in  $\mathcal{B}_d^{(k)}(m_1 + m_2 + \dots + m_k)$  will satisfy the non- $k$ -overlapping condition.

In [12], the authors give a geometrical description of  $H_*\mathcal{B}_d^{(k)}$  as a pointed left module under  $H_*\mathcal{B}_d$  in Theorem 3.4 and Theorem 3.6 of their paper. We will include these theorems below for completeness. However, first we will need three more definitions.

**Definition 2.2.2.** *Define  $M$  to be a left module (respectively, bimodule) under an operad  $\mathcal{O}$  if it is a left module (respectively, bimodule) over  $\mathcal{O}$  and is endowed with a map of left modules (respectively, bimodules)  $\mathcal{O} \rightarrow M$ .*

**Definition 2.2.3.** *An operad  $\mathcal{O}$  in graded  $\mathbb{Z}$ -modules is called augmented if it is endowed with a surjective map of operads  $\mathcal{O} \rightarrow \text{Com}$ .*

**Definition 2.2.4.** *Define  $M$  to be a pointed left module (respectively, bimodule) under an augmented operad  $\mathcal{O}$  if  $M$  is a left module (respectively, bimodule) under  $\mathcal{O}$ , the structure map  $\mathcal{O} \rightarrow M$  factors through  $\text{Com}$ , and the map  $\text{Com} \rightarrow M$  is an inclusion.*

**Theorem 2.2.5** (N. Dobrinskaya and V. Turchin, [12]). *For  $k \geq 3$ , the pointed left module  $H_*\mathcal{B}_d^{(k)}$  under  $H_*\mathcal{B}_d$  is generated by a single element  $\{x_1, \dots, x_k\} \in H_{(k-1)d-1}\mathcal{B}_d^{(k)}(k)$  which is symmetric or skew symmetric depending on the parity of  $d$ :*

$$\{x_{\sigma_1} \dots x_{\sigma_k}\} = (-1)^{|\sigma|d} \{x_1, \dots, x_k\}, \quad \sigma \in \Sigma_k. \quad (2.2.3)$$

*The only relation of the left action is the generalized Jacobi:*

$$\sum_{i=1}^{k+1} (-1)^{(i-1)d} [x_i, \{x_1, \dots, \hat{x}_i, \dots, x_{k+1}\}] = 0. \quad (2.2.4)$$

**Theorem 2.2.6** (N. Dobrinskaya and V. Turchin, [12]). *For  $k \geq 3$ , the pointed bimodule  $H_*\mathcal{B}_d^{(k)}$  under  $H_*\mathcal{B}_d$  is generated by a single element  $\{x_1, \dots, x_k\} \in H_{(k-1)d-1}\mathcal{B}_d^{(k)}(k)$  satisfying the symmetry (2.2.3), generalized Jacobi (2.2.4), and Leibniz relations with respect to the right action:*

$$\{x_1, \dots, x_{k-1}, x_k \cdot x_{k+1}\} = x_k \cdot \{x_1, \dots, x_{k-1}, x_{k+1}\} + \{x_1, \dots, x_k\} \cdot x_{k+1}; \quad (2.2.5)$$

$$\{x_1, \dots, x_{k-1}, [x_k, x_{k+1}]\} = (-1)^d [\{x_1, \dots, x_{k-1}, x_{k+1}\}, x_k] + [\{x_1, \dots, x_k\}, x_{k+1}]. \quad (2.2.6)$$

As a pointed left module (respectively, bimodule)  $H_*\mathcal{B}_d^{(k)}(k)$  has only one generator, given by  $\{x_1, \dots, x_k\}$ . Below we give a geometrical description of  $\{x_1, \dots, x_k\}$ . We know

$$\mathcal{B}_d^{(k)}(k) \simeq \mathcal{M}_d^{(k)}(k) = \{(x_1, \dots, x_k) \in (\mathbb{R}^d)^k \mid \text{exclude } x_1 = x_2 = \dots = x_k\} = \mathbb{R}^{dk} - \mathbb{R}^d \simeq S^{(k-1)d-1}.$$

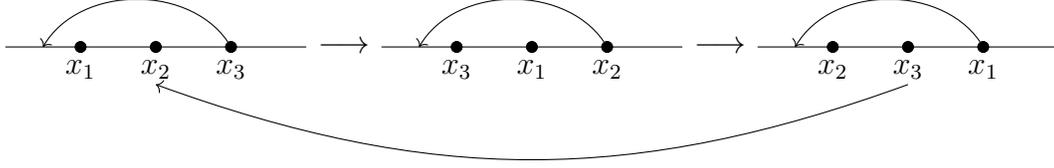
This implies that the element  $\{x_1, \dots, x_k\} \in H_{(k-1)d-1}\mathcal{B}_d^{(k)}(k)$  can geometrically be realized as a  $[(k-1)d-1]$ -sphere:

$$|x_1|^2 + |x_2|^2 + \dots + |x_k|^2 = \varepsilon^2 \quad \sum_{i=1}^k x_i = 0 \quad (2.2.7)$$

Each of the  $x_i$ 's represents the center of the  $i$ th disc or the  $i$ th point in the configuration.

**Example:** Let  $d = 1$ ,  $k = 3$  then  $\{x_1, x_2, x_3\} \in H_{(3-1)1-1}\mathcal{B}_1^{(3)}(3) = H_1\mathcal{B}_1^{(3)}(3) = H_1S^1$ .

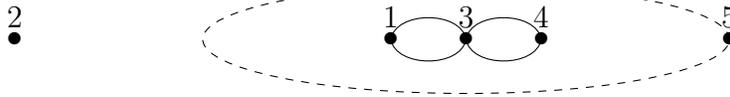
Geometrically we can see this as follows:



The above example is for the case  $d = 1$ . Theorem 2.2.5 and Theorem 2.2.6 still hold for  $d = 1$ . However the bracket  $[x_1, x_2]$  should be understood as  $x_1x_2 - x_2x_1$  so the only operation is multiplication, which implies that the underlying operad is  $\mathcal{A}ssoc$ , the associative operad. In this case (2.2.5) implies (2.2.6). The relation (2.2.4) is instead equivalently written as

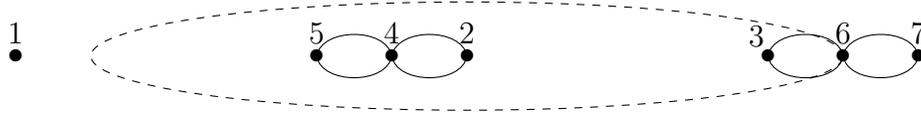
$$\sum_{i=1}^{k+1} (x_i \cdot \{x_1, \dots, \hat{x}_i, \dots, x_{k+1}\} - \{x_1, \dots, \hat{x}_i, \dots, x_{k+1}\} \cdot x_i).$$

**Example:** Let us look at an example of an element in  $H_{(k-1)d-1}\mathcal{B}_d^{(k)}(n)$  for  $k = 3$ ,  $n = 5$ . Let us examine the element  $x_2 \cdot [\{x_1, x_3, x_4\}, x_5]$ . Here  $x_1, x_3, x_4$  orbit closely around each other making a spherical class as in the previous example. The disc  $x_5$  orbits around  $x_1, x_3, x_4$ . Lastly,  $x_2$  does not interact with the other discs and stays still far away from the other points.



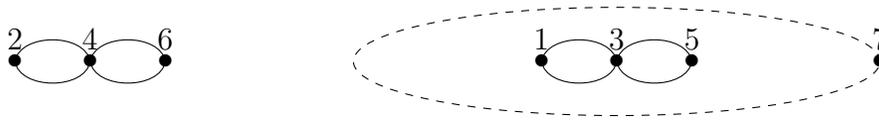
Elements in  $H_{(k-1)d-1}\mathcal{B}_d^{(k)}(n)$ ,  $d \geq 2$  can be thought as products of iterated brackets. Formally, one can have a bracket,  $[\cdot, \cdot]$ , or multiplication inside of a brace, however by the two Leibniz relations (2.2.5) and (2.2.6), they can be pulled outside of the brace. This means the left action on the two generators span all of the homology and nothing new comes from the right action. All braces must be of the same length and braces cannot be inside of other braces. As seen in the example above, singletons are allowed in the products of brackets. One must have at least one brace inside a bracket, that is,  $[[x_1, x_3], x_2]$  is zero as an element. It is possible to have two or more braces inside a bracket, for example  $[\{x_1, x_3, x_5\}, \{x_2, x_4, x_6\}]$ .

Other examples of possible forms of elements in  $H_*\mathcal{B}_d^{(k)}(n)$  for  $k = 3$ ,  $n = 7$  are  $x_1 \cdot [\{x_5, x_4, x_2\}, \{x_3, x_6, x_7\}]$  and  $\{x_2, x_4, x_6\} \cdot [\{x_1, x_3, x_5\}, x_7]$ . The degree of these elements is  $2[(k-1)d-1] + (d-1) = 2(kd-d-1) + d-1 = 2kd-2d-2+d-1 = 2kd-d-3 = (2k-1)d-3$ . Initially, we have  $2[(k-1)d-1]$  since we have two braces and then add  $d-1$  for the single bracket. Both of the elements can be seen as a map:  $S^{(k-1)d-1} \times S^{(k-1)d-1} \times S^{d-1} \rightarrow \mathcal{B}_d^{(3)}(7)$ . The element  $x_1 \cdot [\{x_5, x_4, x_2\}, \{x_3, x_6, x_7\}]$  can be seen geometrically as:



Here  $x_3, x_6, x_7$  are close together and orbit around each other and  $x_5, x_4, x_2$  are also closely orbiting around one another, as shown by the solid ellipses. The generators  $x_3, x_6, x_7$  also orbit around  $x_5, x_4, x_2$ , as represented by the dotted ellipse. Finally,  $x_1$  stays far away from all the others and stays still.

Geometrically,  $\{x_2, x_4, x_6\} \cdot [\{x_1, x_3, x_5\}, x_7]$  can be seen as:



As just left modules (respectively, bimodules),  $H_*\mathcal{B}_d^{(k)}(n)$  has two generators:  $x_1 \in H_0\mathcal{B}_d^{(k)}(1)$  and  $\{x_1, x_2, \dots, x_k\} \in H_{(k-1)d-1}\mathcal{B}_d^{(k)}(n)$ . There is an additional left action relation given by the following diagram:

$$\mathcal{L}ie(2) \ni \begin{array}{c} \boxed{[x_1, x_2]} \\ \swarrow \quad \searrow \\ \boxed{x_1} \quad \boxed{x_1} \\ \downarrow \quad \downarrow \\ 1 \quad 2 \end{array} = 0$$

generators

There is also an additional right action relation:

$$\begin{array}{ccc}
\text{generator} & \begin{array}{c} | \\ \boxed{x_1} \\ | \\ \boxed{[x_1, x_2]} \\ / \quad \backslash \\ 1 \quad 2 \end{array} & = 0 \\
\mathcal{L}ie(2) \ni & & 
\end{array}$$

In other words, the action of  $x_1$  by  $\mathcal{P}ois_{d-1}$  only gives  $\mathcal{C}om$ .

Finally, there is also a relation between the left and right action relations given by the following diagram:

$$\begin{array}{ccc}
\text{generator} & \begin{array}{c} | \\ \boxed{x_1} \\ | \\ \boxed{x_1 \cdot x_2} \\ / \quad \backslash \\ 1 \quad 2 \end{array} & \in \mathcal{C}om(2) \\
\mathcal{C}om(2) \ni & = & \begin{array}{c} | \\ \boxed{x_1 \cdot x_2} \\ / \quad \backslash \\ \boxed{x_1} \quad \boxed{x_1} \\ | \quad | \\ 1 \quad 2 \end{array} \text{ generators}
\end{array}$$

## 2.3 Homotopy Equivalence $\mathcal{B}_d^{(k)}(n) \simeq \mathcal{M}_d^{(k)}(n)$

Recall part 2 of Definition 2.2.1 of non- $k$ -equal configuration spaces. For completion we will prove the following proposition.

**Proposition 2.3.1.** *The spaces  $\mathcal{B}_d^{(k)}(n)$  and  $\mathcal{M}_d^{(k)}(n)$  are homotopy equivalent.*

*Proof.* Define  $\mathring{\mathcal{M}}_d^{(k)}(n) := \mathcal{M}_d^{(k)}(n) \cap (\mathring{D}^d)^{\times n}$ , where  $\mathring{D}^d$  is the open unit ball in  $\mathbb{R}^d$ . Then the inclusion  $\mathring{\mathcal{M}}_d^{(k)}(n) \hookrightarrow \mathcal{M}_d^{(k)}(n)$  is homotopy equivalent since  $\mathring{D}^d \subset \mathbb{R}^d$  is isotopic to  $\mathbb{R}^d$ .

There is a natural map  $f : \mathcal{B}_d^{(k)}(n) \rightarrow \mathcal{M}_d^{(k)}(n)$  that factors through  $\mathring{\mathcal{M}}_d^{(k)}(n)$  and is given by taking the centers of the discs and forgetting their radii. Next, we will construct a map  $g : \mathring{\mathcal{M}}_d^{(k)}(n) \rightarrow \mathcal{B}_d^{(k)}(n)$  that is the homotopy inverse of  $f$ , where  $\mathring{f} : \mathcal{B}_d^{(k)}(n) \rightarrow \mathring{\mathcal{M}}_d^{(k)}(n)$ . Let  $(x_1, \dots, x_n)$  be the  $n$  points in a configuration  $P$  in  $\mathring{\mathcal{M}}_d^{(k)}(n)$ . We will use  $\vec{i}$  to denote  $(i_1, \dots, i_k)$ . Next, define  $\varepsilon(x_1, \dots, x_n) = \frac{1}{\sqrt{k}} \min_{1 \leq i_1 < \dots < i_k \leq n} \sqrt{\sum_{j=1}^k |x_{i_j} - \bar{x}_{\vec{i}}|^2}$ , where  $\bar{x}_{\vec{i}} = \frac{1}{k}(x_{i_1} + \dots + x_{i_k})$ . Finally, define  $r(x_1, \dots, x_n) = \min(\varepsilon(x_1, \dots, x_n), 1 - |x_1|, \dots, 1 - |x_n|)$ . This is a continuous, strictly positive map  $r : \mathring{\mathcal{M}}_d^{(k)}(n) \rightarrow \mathbb{R}_{>0}$  that gives each disc  $x_i$  the same radius  $r$ . The  $(1 - |x_i|)$ 's guarantee that all the discs are completely contained inside

of the unit disc. The  $\varepsilon(x_1, \dots, x_n)$  guarantees that no  $k$  discs overlap. To see this, assume by contradiction that  $k$  discs overlap, which implies that there is some  $y$  in  $\mathring{D}^d$  such that  $d(x_j, y) < r$  for  $j = i_1, \dots, i_k$ , where  $d(x_j, y)$  is the usual distance function for  $\mathbb{R}^d$ . This implies that  $\sum_{j=1}^k |x_{i_j} - y|^2 < kr^2$ . This gives the following string of inequalities:

$$kr^2 > \sum_{j=1}^k |x_{i_j} - y|^2 \geq \sum_{j=1}^k |x_{i_j} - \bar{x}_{\rightarrow i}|^2 \geq k\varepsilon^2$$

This implies that  $\varepsilon^2 < r^2$ . However by the definition of  $r$ ,  $r \leq \varepsilon$ , thus giving a contradiction and proving that no  $k$  discs overlap.

One has that  $\mathring{f} \circ g = id$ . Now we will prove  $g \circ \mathring{f} \simeq id$ . Let  $P$  be a point (i.e. a configuration of discs) in  $\mathcal{B}_d^{(k)}(n)$ . For  $i = 1, \dots, n$ , let  $(x_i, r_i)$  be the  $i$ -th disc where  $x_i$  is the center of the disc and  $r_i$  is the radius of the disc. Let  $(x_i, r)$  denote the  $i$ -th disc of  $(g \circ \mathring{f})(P)$ . Define  $H : \mathcal{B}_d^{(k)}(n) \times [0, 1] \rightarrow \mathcal{B}_d^{(k)}(n)$  as follows: subdivide  $[0, 1]$  into 2 subintervals. On the first subinterval, shrink continuously all radii  $r_i > r$  to  $r$ . We want to do this first to avoid any issues with the number of overlaps. Then on the second subinterval, continuously enlarge all radii  $r_i < r$  to  $r$ . This homotopy  $H$  proves that the identity map is homotopic to the composition  $g \circ \mathring{f}$ .  $\square$

## 2.4 Homology of Overlapping Discs as a Symmetric Sequence

Björner and Welker in [6] first computed the homology of  $\mathcal{M}_d^{(k)}(n)$  for  $k \geq 3$ . Sundaram and Wachs in [30] later computed the symmetric group action on the homology of the intersection lattice corresponding to  $\mathcal{M}_d^{(k)}(n)$ ; their computations imply the following isomorphism of symmetric sequences:

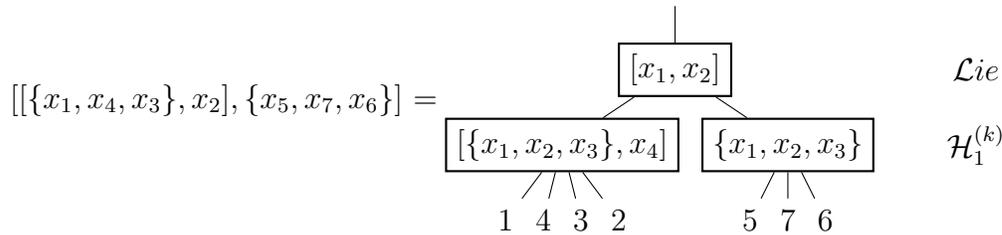
**Theorem 2.4.1.** *There is a natural isomorphism of symmetric sequences*

$$H_*\mathcal{B}_d^{(k)} \simeq \mathcal{C}om \circ \left( \mathbb{1} \oplus (\mathcal{L}ie \circ \mathcal{H}_1^{(k)})\{d-1\} \right). \quad (2.4.1)$$

The isomorphism (2.4.1) holds integrally for  $d \geq 2$ ,  $k \geq 3$  and rationally for  $d \geq 1$ ,  $k \geq 2$ .

Explicitly, with an independent proof, this formula appears in [12, Theorem 10.3]. In the above isomorphism (2.4.1),  $\circ$  is the graded composition product for symmetric sequences as defined in (1.4.1).  $\mathcal{C}om$  and  $\mathcal{L}ie$  are the underlying symmetric sequences of the commutative and Lie operads and  $\mathcal{H}_1^{(k)}$  is the symmetric sequence of hook representations which we describe below in Section 3.3. The notation  $\{d-1\}$  is the operadic degree  $(d-1)$  suspension of symmetric sequences. The symmetric sequence  $\mathbb{1}$  is the unit with respect to the composition product. It is a one dimensional space concentrated in arity 1. In section 3.4, we give an explicit formula for the cycle index sum of the symmetric sequences obtained in (2.4.1).

The space  $\mathcal{H}_1^{(k)}(n)$  is the natural subspace of  $H_{k-2}\mathcal{M}_1^{(k)}(n)$ , spanned by iterated brackets having exactly one brace. That is, for  $k = 3$ ,  $n = 4$ , elements are of the form  $[\{x_1, x_4, x_3\}, x_2]$ . Composing  $\mathcal{L}ie$  and  $\mathcal{H}_1^{(k)}$  implies that elements can have any positive number of braces. For example, for  $k = 3$ ,  $n = 7$ ,  $[[\{x_1, x_4, x_3\}, x_2], \{x_5, x_7, x_6\}]$  is a possible element. This can be represented as the following tree diagram:



As another example is  $[x_1, [\{x_2, x_3, x_4\}, \{x_5, x_6, x_7\}]] \in H_*\mathcal{B}_d^{(3)}(7)$ . By applying the Jacobi relation, one gets  $[x_1, [\{x_2, x_3, x_4\}, \{x_5, x_6, x_7\}]] = [\{x_5, x_6, x_7\}, [\{x_2, x_3, x_4\}, x_1]] + [[\{x_5, x_6, x_7\}, x_1], \{x_2, x_3, x_4\}]$ . Both of the summands are in  $\mathcal{L}ie \circ \mathcal{H}_1^{(k)}$  and thus so is  $[x_1, [\{x_2, x_3, x_4\}, \{x_5, x_6, x_7\}]]$ .

The  $(d-1)$  suspension distributes over the composition so  $(\mathcal{L}ie \circ \mathcal{H}_1^{(k)})\{d-1\} = \mathcal{L}ie\{d-1\} \circ \mathcal{H}_1^{(k)}\{d-1\}$  where  $\mathcal{H}_1^{(k)}\{d-1\}(n) = \mathcal{H}_d^{(k)}(n)$  as a subspace of  $H_{(k-2)+(n-1)(d-1)}\mathcal{B}_d^{(k)}(n)$ . The



be moved to the front of the iterated brackets by the anti-symmetry property and the Lie bracket can always be made linear by the Jacobi relation.

Now let us look at  $\mathcal{H}_1^{(k)}(n)$ . First, using the anti-symmetric property of the Lie bracket and the fact that if an element contains  $[x_i, x_j]$  then the element is 0, all elements of  $\mathcal{H}_1^{(k)}(n)$  are either equal, up to a sign, to an element of the form  $[\dots[\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}, x_{j_1}], \dots, x_{j_{n-k}}]$ , where the  $x_i$ 's are in any order and  $x_j$ 's are in any order, or the element contains  $[x_i, x_j]$  and therefore is 0. Then the  $x_i$ 's inside the brace can be ordered  $i_1 < i_2 < \dots < i_k$  by the skew symmetric property (2.2.3) and the  $x_j$ 's can also be ordered  $j_1 < j_2 < \dots < j_{n-k}$ . Changing the order of the  $x_j$ 's gives just a sign change. This can be seen in the following example for  $k = 3, n = 5$ : By the usual Jacobi relation  $[[\{x_1, x_2, x_5\}, x_4], x_3] = (-1)^{d-1}[[\{x_1, x_2, x_5\}, x_3], x_4] + [\{x_1, x_2, x_5\}, [x_3, x_4]]$ . The second summand of the right hand side is 0, thus  $[[\{x_1, x_2, x_5\}, x_4], x_3] = (-1)^{d-1}[[\{x_1, x_2, x_5\}, x_3], x_4]$ . The group  $\Sigma_k$  acts on the  $i_m$ 's and the group  $\Sigma_{n-k}$  acts on the  $j_m$ 's. If  $d$  is even, then  $\Sigma_k$  acts trivially on the  $i_m$ 's and  $\Sigma_{n-k}$  acts by the sign representation on the  $j_m$ 's. Conversely if  $d$  is odd, then  $\Sigma_k$  acts on the  $i_m$ 's by the sign representation and  $\Sigma_{n-k}$  acts trivially on the  $j_m$ 's. Finally, using the generalized Jacobi (2.2.4), we can make  $i_1 = 1$ . Therefore any element of the form  $[\dots[\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}, x_{j_1}], \dots, x_{j_{n-k}}]$  is a linear combination of elements from the basis  $[\dots[\{x_1, x_{i_2}, \dots, x_{i_k}\}, x_{j_1}], \dots, x_{j_{n-k}}]$ .

Next we give small arities of the basis of  $\mathcal{H}_1^{(k)}$ : For  $\mathcal{H}_1^{(k)}(k)$ , that is,  $n = k$ , the dimension is 1 and the only element is  $\{x_1, \dots, x_k\}$ .

For  $\mathcal{H}_1^{(k)}(k+1)$ ,  $n - k = 1$ , the dimension of this space is  $\binom{k+1-1}{k-1} = \binom{k}{k-1} = k$ . The basis of  $\mathcal{H}_1^{(k)}(k+1)$  is

$$[\{x_1, \dots, \hat{x}_i, \dots, x_{k+1}\}, x_i], \quad i > 1.$$

For  $\mathcal{H}_1^{(k)}(k+2)$ ,  $n - k = 2$ , the dimension of this space is  $\binom{k+2-1}{k-1} = \binom{k+1}{k-1} = \frac{(k+1)k}{2}$ . The basis of  $\mathcal{H}_1^{(k)}(k+2)$  is

$$[[\{x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{k+2}\}, x_i], x_j], \quad 1 < i < j.$$



# Chapter 3

## Cycle Index Sum of Non- $k$ -equal Configurations

The main result of this section is an explicit formula for the trace of the symmetric group action on the homology of overlapping discs. The formula is obtained from (2.4.1) and given in Section 3.4 and then proved in Section 3.5. This result appeared in publication [24] by the Ph.D candidate and V. Turchin.

### 3.1 Cycle Index Sum: Definitions and Formulas

We will use the variable  $q$  for the formal variable responsible for the homological degree. For  $\sigma \in \Sigma_n$  we will denote the number of its cycles of length  $j$  by  $i_j(\sigma)$ . Let  $\rho : \Sigma_n \rightarrow GL(V)$  be a representation of the symmetric group  $\Sigma_n$ , where  $V$  is a graded vector space, and let  $(p_1, p_2, p_3, \dots)$  be a family of infinite commuting variables. Then the cycle index sum of  $\rho$ , denoted  $Z_V(q; p_1, p_2, p_3, \dots)$ , is defined by

$$Z_V(q; p_1, p_2, p_3, \dots) = \frac{1}{|\Sigma_n|} \sum_{\sigma \in \Sigma_n} \text{tr}(\rho(\sigma)) \prod_j p_j^{i_j(\sigma)}, \quad (3.1.1)$$

where  $tr(\rho(\sigma))$  is the graded trace that is a polynomial of  $q$  obtained as the generating function of traces on each component. We also define an auxiliary *cardinality degree* given by  $p_i$ 's where each  $p_i$  is said to have cardinality degree  $i$ .

Let  $V$  be a  $\Sigma_k$ -module and  $W$  be a  $\Sigma_n$ -module. Then from [4, 5, 27, section 6.1; section 3.1, proposition 8, part c; 7.3, respectively],

$$Z_{\text{Ind}_{\Sigma_k \times \Sigma_n}^{\Sigma_{k+n}}(V \otimes W)} = Z_V \cdot Z_W. \quad (3.1.2)$$

For a symmetric sequence  $M(\bullet) = \{M(n), n \geq 0\}$ , one defines its cycle index sum as

$$Z_M(q; p_1, p_2, p_3, \dots) = \sum_{n=0}^{\infty} Z_{M(n)}(q; p_1, p_2, p_3, \dots). \quad (3.1.3)$$

We will need the following formulas later in this section. The cycle index sum of the composition of two symmetric sequences is given by the graded plethysm formula:

$$Z_{M \circ N} = Z_M * Z_N = Z_M(q; p_i \mapsto p_i * Z_N), \quad (3.1.4)$$

where

$$p_i * Z_N = Z_N(q \mapsto (-1)^{i-1} q^i; p_j \mapsto p_{ij}). \quad (3.1.5)$$

The usual plethysm without the grading can be found in [4, 5, 27, equation 3.25, section 3.8; definition 3, section 1.4; equation 8.1-8.2, section 8, respectively]. For the graded case, it is done when  $q = -1$  in [20, section 7.20]. Unfortunately, the graded version of this formula doesn't seem to appear in the literature, though it is known to experts [10, 28]. To prove our formula, we notice that the sign convention is correct and holds when  $q = -1$  and the  $q$ -grading contribution is correct by the same argument as in [13, Section 3.5, definition 3].

To recall the operadic suspension  $\mathcal{M}\{1\}$  of the symmetric sequence  $\mathcal{M}$  is defined as

$$\mathcal{M}\{1\}(n) = s^{n-1} \mathcal{M}(n) \otimes V_{(1^n)},$$

where  $s^{n-1}$  is the degree  $(n-1)$  suspension and  $V_{(1^n)}$  is the sign representation. One can easily see that

$$Z_{\mathcal{M}\{1\}} = \frac{1}{q} Z_{\mathcal{M}}(q; p_i \mapsto (-1)^{i-1} q^i p_i).$$

We will use the formula for the  $\{d-1\}$  operadic suspension, which is an easy formula to obtain from the above:

$$Z_{\mathcal{M}\{d-1\}} = (q)^{1-d} Z_{\mathcal{M}}(q; p_i \mapsto (-1)^{(i-1)(d-1)} q^{i(d-1)} p_i). \quad (3.1.6)$$

Lastly, we will need the cycle index sums of  $Com$  and  $\mathcal{L}ie$ . From [13, 20], the cycle index sum for  $Com$  is

$$Z_{Com} = \exp\left(\sum_{i=1}^{\infty} \frac{p_i}{i}\right); \quad (3.1.7)$$

and from [7, 13, 20], the cycle index sum for  $\mathcal{L}ie$  is

$$Z_{\mathcal{L}ie} = \sum_{i=1}^{\infty} \frac{-\mu(i) \ln(1-p_i)}{i}, \quad (3.1.8)$$

where, here and throughout this paper,  $\mu(i)$  is the usual Möbius function.

We will also use the notation  $V_{\lambda}$  to denote the irreducible  $\Sigma_n$ -representation corresponding to the partition  $\lambda$ , see [19].

## 3.2 Cycle Index Sum for $H_*\mathcal{B}_d$

From [11],  $H_*\mathcal{B}_d = \mathcal{P}ois_{d-1} = Com \circ \mathcal{L}ie\{d-1\}$ . Below we give an independent proof of the following well-known result.

**Theorem 3.2.1.** [25, Lehrer] For  $d \geq 1$ ,

$$Z_{H_*\mathcal{B}_d} = Z_{Com \circ \mathcal{L}ie\{d-1\}}(q; p_1, p_2, p_3, \dots) = \prod_{m=1}^{\infty} \left(1 + (-1)^d (-q)^{m(d-1)} p_m\right)^{(-1)^d E_m\left(\frac{1}{(-q)^{d-1}}\right)} \quad (3.2.1)$$

where  $E_m(y) = \frac{1}{m} \sum_{i|m} \mu(i) y^{\frac{m}{i}}$ .

*Proof.* To show this, first we apply the  $\{d-1\}$ -suspension (3.1.6) to  $\mathcal{L}ie$  (3.1.8) to get:

$$\begin{aligned} Z_{\mathcal{L}ie\{d-1\}}(q; p_1, p_2, p_3, \dots) &= q^{1-d} \sum_{i=1}^{\infty} \frac{-\mu(i)}{i} \ln(1 - (-1)^{(i-1)(d-1)}(q)^{i(d-1)}p_i). \\ &= q^{1-d} \sum_{i=1}^{\infty} \frac{-\mu(i)}{i} \ln(1 + (-1)^d(-q)^{i(d-1)}p_i). \end{aligned}$$

Next use (3.1.4) and(3.1.5) to compute the graded composition of  $\mathcal{C}om$  and the above equation:

$$\begin{aligned} Z_{\mathcal{C}om \circ \mathcal{L}ie\{d-1\}}(q; p_1, p_2, p_3, \dots) &= \\ &= \exp \left( \sum_{j=1}^{\infty} \frac{1}{j} \left( (-1)^{d-1}(-q)^{j(1-d)} \sum_{i=1}^{\infty} \frac{-\mu(i)}{i} \ln \left( 1 + (-1)^d(-q)^{ij(d-1)}p_{ij} \right) \right) \right). \end{aligned}$$

Let  $m = ij$

$$\begin{aligned} &= \exp \left( \sum_{m=1}^{\infty} \frac{1}{m} \left( (-1)^d \sum_{i|m} (-q)^{\frac{m}{i}(1-d)} \mu(i) \ln(1 + (-1)^d(-q)^{m(d-1)}p_m) \right) \right) \\ &= \exp \left( \sum_{m=1}^{\infty} \ln(1 + (-1)^d(-q)^{m(d-1)}p_m) \right)^{\frac{(-1)^d}{m} \sum_{i|m} (-q)^{\frac{m}{i}(1-d)}} \end{aligned}$$

Using the fact that exponential functions and logarithm functions are inverses to one another and  $E_m(y) = \frac{1}{m} \sum_{i|m} \mu(i)y^{\frac{m}{i}}$  that we get the result. □

### 3.3 Cycle Index Sum for $\mathcal{H}_1^{(k)}$

By [30],

$$\mathcal{H}_1^{(k)}(n) \cong_{\mathbb{Q}} \begin{cases} 0, & n < k; \\ s^{k-2}V_{(n-k+1, 1^{k-1})} & n \geq k, \end{cases}$$

where  $V_{(n-k+1, 1^{k-1})}$  is the hook representation corresponding to the partition  $\lambda = (n - k + 1, 1^{k-1})$  and  $s^{k-2}$  is the  $(k - 2)$ -suspension. The space  $\mathcal{H}_1^{(k)}(n)$  is some natural subspace of  $H_{k-2}\mathcal{M}_1^{(k)}(n)$ , in turn it lies in degree  $k - 2$ , see [12].

**Proposition 3.3.1.** *For  $k \geq 2$ ,*

$$Z_{\mathcal{H}_1^{(k)}}(q; p_1, p_2, p_3, \dots) = (-q)^{k-2} - (-q)^{k-2} \left( \exp \left( - \sum_{i=1}^{\infty} \frac{p_i}{i} \right) \right)_{\leq k-1} \left( \exp \left( \sum_{i=1}^{\infty} \frac{p_i}{i} \right) \right), \quad (3.3.1)$$

where  $\leq k - 1$  is the truncation with respect to the cardinality degree.

We will prove this proposition with the following well-known facts and lemmas. First, let  $W_n = \mathbb{Q}[\underline{n}]$ , where  $\underline{n} = \{1, 2, \dots, n\}$ , be the canonical  $n$ -dimensional representation of  $\Sigma_n$ . Then  $W_n$  can be decomposed in the following way:  $W_n = V_{(n-1, 1)} \oplus V_{(n)}$  where  $V_{(n-1, 1)}$  is the  $(n - 1)$  dimensional representation and  $V_{(n)} = \mathbb{Q}$  is the one-dimensional trivial representation.

**Lemma 3.3.2.** *For  $n \geq k \geq 0$ ,  $\wedge^k W_n = \text{Ind}_{\Sigma_k \times \Sigma_{n-k}}^{\Sigma_n} V_{(1^k)} \otimes V_{(n-k)}$ .*

*Proof.* First recall that  $V_{(1^k)}$  is the sign representation of  $\Sigma_k$  and that  $V_{(n-k)}$  is the trivial representation of  $\Sigma_{n-k}$ . Also, note that  $\wedge^k W$  and  $\text{Ind}_{\Sigma_k \times \Sigma_{n-k}}^{\Sigma_n} V_{(1^k)} \otimes V_{(n-k)}$  have the same dimension, namely  $\binom{n}{k}$ . To start, let  $e_1, e_2, \dots, e_n$  be the usual basis of  $W_n$ . Now examine how  $\Sigma_n$  acts on a vector  $e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k} \in W_n$ . For  $\sigma \in \Sigma_n$ ,  $\sigma(e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}) = e_{\sigma(i_1)} \wedge e_{\sigma(i_2)} \wedge \dots \wedge e_{\sigma(i_k)}$ . By definition,  $\text{Ind}_{\Sigma_k \times \Sigma_{n-k}}^{\Sigma_n} V_{(1^k)} \otimes V_{(n-k)} = \mathbb{Q}[\Sigma_n] \otimes_{\mathbb{Q}[\Sigma_k \times \Sigma_{n-k}]} V_{(1^k)} \otimes V_{(n-k)}$ . Define

$$I_{(k, n-k)} : \mathbb{Q}[\Sigma_n] \otimes_{\mathbb{Q}[\Sigma_k \times \Sigma_{n-k}]} V_{(1^k)} \otimes V_{(n-k)} \rightarrow \wedge^k W_n$$

by  $I_{(k, n-k)}(\sigma \otimes \mathbb{1}) \mapsto \sigma(e_1 \wedge \dots \wedge e_k) = e_{\sigma(1)} \wedge e_{\sigma(2)} \wedge \dots \wedge e_{\sigma(k)}$ . We claim this is the desired isomorphism. First, we will show that it is well defined. Let  $(\alpha, \beta) \in \Sigma_k \times \Sigma_{n-k}$ . Then

$I_{(k,n-k)}(\sigma \cdot (\alpha, \beta) \otimes \mathbb{1}) = e_{\sigma(\alpha(1))} \wedge e_{\sigma(\alpha(2))} \wedge \dots \wedge e_{\sigma(\alpha(k))} = (-1)^{|\alpha|} e_{\sigma(1)} \wedge e_{\sigma(2)} \wedge \dots \wedge e_{\sigma(k)} = (-1)^{|\alpha|} \sigma \otimes \mathbb{1}$ . On the other hand,  $I(\sigma \otimes (\alpha, \beta) \cdot \mathbb{1}) = \sigma \otimes (-1)^{|\alpha|} \mathbb{1} = (-1)^{|\alpha|} \sigma \otimes \mathbb{1}$ . Therefore  $I$  is well defined. As previously mentioned, these two spaces have the same dimension and by construction  $I_{(k,n-k)}$  is surjective and therefore  $I_{(k,n-k)}$  is bijective.  $\square$

**Lemma 3.3.3.** *For  $n > k$ , one has an isomorphism of  $\Sigma_n$ -modules:  $V_{(n-k,1^k)} = \wedge^k V_{(n-1,1)}$ .*

This lemma is a standard exercise in representation theory [19, Exercise 4.6].

**Corollary 3.3.4.** *One has an isomorphism of  $\Sigma_n$ -modules:  $\wedge^k W_n = \wedge^k V_{(n-1,1)} \oplus \wedge^{k-1} V_{(n-1,1)}$ .*

*Proof.*  $\wedge^k W_n = \wedge^k (V_{(n-1,1)} \oplus V_{(n)}) = \wedge^k (V_{(n-1,1)}) \oplus \wedge^{k-1} (V_{(n-1,1)}) \otimes V_{(n)}$ , where  $V_{(n)}$  is just the trivial representation and thus we have our desired isomorphism.  $\square$

**Corollary 3.3.5.** *One has an isomorphism of virtual  $\Sigma_n$ -modules:  $\wedge^k V_{(n-1,1)} = \sum_{i=0}^k (-1)^i \wedge^{k-i} W_n$*

*Proof.*  $\wedge^k V_{(n-1,1)} = \wedge^k W_n - \wedge^{k-1} V_{(n-1,1)}$  by Corollary 3.3.4. We apply the same corollary to  $\wedge^{k-1} V_{(n-1,1)}$  again and we have  $\wedge^k V_{(n-1,1)} = \wedge^k W_n - \wedge^{k-1} V_{(n-1,1)} = \wedge^k W_n - \wedge^{k-1} W_n + \wedge^{k-2} V_{(n-1,1)}$ . We can apply Corollary 3.3.4 iteratively to obtain the desired isomorphism.  $\square$

Now we are ready to prove Proposition 3.3.1.

*Proof of Proposition 3.3.1.* Let

$$\mathcal{H}(n) = \begin{cases} 0, & n < k; \\ V_{(n-k+1,1^{k-1})}, & \text{otherwise.} \end{cases}$$

In order to prove Proposition 3.3.1, it is sufficient to show that

$$Z_{\mathcal{H}}(p_1, p_2, p_3, \dots) = (-1)^{k-2} - (-1)^{k-2} \left( \exp \left( - \sum_{i=1}^{\infty} \frac{p_i}{i} \right) \right)_{\leq k-1} \left( \exp \left( \sum_{i=1}^{\infty} \frac{p_i}{i} \right) \right). \quad (3.3.2)$$

For  $n \geq k$ , one has

$$\begin{aligned}
\mathcal{H}(n) &= V_{(n-k+1, 1^{k-1})} && \text{(by definition)} \\
&= \wedge^{k-1} V_{(n-1, 1)} && \text{(by Lemma 3.3.3)} \\
&= \sum_{i=0}^{k-1} (-1)^i \wedge^{k-1-i} W_n && \text{(by Corollary 3.3.5)} \\
&= \sum_{i=0}^{k-1} (-1)^i \text{Ind}_{\Sigma_{k-1-i} \times \Sigma_{n-k+1+i}}^{\Sigma_n} V_{(1^{k-1-i})} \otimes V_{(n-k+1+i)} && \text{(by Lemma 3.3.2)} \\
&= (-1)^{k-1} \sum_{j=0}^{k-1} (-1)^j \text{Ind}_{\Sigma_j \times \Sigma_{n-j}}^{\Sigma_n} V_{(1^j)} \otimes V_{(n-j)}. && \text{(by taking } j = k - i - 1)
\end{aligned}$$

Next we apply (3.1.2).

$$Z_{\mathcal{H}(n)} = Z_{V_{(n-k+1, 1^{k-1})}} = (-1)^{k-1} \sum_{j=0}^{k-1} (-1)^j Z_{V_{(1^j)}} \cdot Z_{V_{(n-j)}}. \quad (3.3.3)$$

Thus,

$$Z_{\mathcal{H}} = (-1)^{k-1} \sum_{n \geq k} \sum_{j=0}^{k-1} (-1)^j Z_{V_{(1^j)}} \cdot Z_{V_{(n-j)}}. \quad (3.3.4)$$

Note that  $Z_{V_{(1^j)}}$  is the cycle index sum for the sign representation and  $Z_{V_{(n-j)}}$  is the cycle index sum for the trivial representation. We claim that (3.3.4) is equal to (3.3.2). We will prove this claim in two cases: when cardinality  $n < k$  and when cardinality  $n \geq k$ .

We will first do the case when  $n < k$ . Clearly (3.3.4) is equal to 0 when  $n < k$  as the sum starts when  $n \geq k$  and thus has no terms. When  $n < k$ , (3.3.2) is also 0 since the exponentials are inverses to one another:

$$\begin{aligned}
(-1)^{k-2} - (-1)^{k-2} \left( \exp \left( - \sum_{i=1}^{\infty} \frac{p_i}{i} \right) \right)_{\leq k-1} \left( \exp \left( \sum_{i=1}^{\infty} \frac{p_i}{i} \right) \right)_{\leq k-1} &= 0 \\
(-1)^{k-2} - (-1)^{k-2} \left( \exp \left( - \sum_{i=1}^{\infty} \frac{p_i}{i} \right) \right) \left( \exp \left( \sum_{i=1}^{\infty} \frac{p_i}{i} \right) \right) &= 0.
\end{aligned}$$

Now we look at the case when the cardinality degree  $n \geq k$ . It follows from (3.1.7) that

$$\sum_{n=0}^{\infty} Z_{V_{(1^n)}} = \exp \left( \sum_{i=0}^{\infty} \frac{(-1)^{i-1} p_i}{i} \right).$$

By replacing  $p_i \mapsto (-1)^i p_i$ , we get

$$\sum_{n=0}^{\infty} (-1)^n Z_{V_{(1^n)}} = \exp \left( \sum_{i=0}^{\infty} \frac{-p_i}{i} \right).$$

Then,

$$\sum_{n=0}^{k-1} (-1)^n Z_{V_{(1^n)}} = \left( \exp \left( \sum_{i=0}^{\infty} \frac{-p_i}{i} \right) \right)_{\leq k-1}.$$

We also know that

$$\sum_{n=0}^{\infty} Z_{V_{(n)}} = \exp \left( \sum_{i=0}^{\infty} \frac{p_i}{i} \right).$$

From these formulas, one can easily see that in cardinality  $n \geq k$ , (3.3.2) and (3.3.4) are both equal to (3.3.3). Thus in arity  $n$ , (3.3.2) is equal to (3.3.4), completing the proof.  $\square$

### 3.4 Cycle Index Sum of $H_* \mathcal{B}_d^{(k)}$

The cycle index sum of the symmetric group action on  $H_* \mathcal{B}_d^{(2)}$ , the usual configuration space, was computed in [2, 25] to be, also see Theorem 3.2.1:

$$Z_{H_* \mathcal{B}_d^{(2)}} = \prod_{m=1}^{\infty} \left( 1 + (-1)^d (-q)^{m(d-1)} p_m \right)^{(-1)^d E_m \left( \frac{1}{(-q)^{d-1}} \right)}. \quad (3.4.1)$$

From (2.4.1), in [12] the exponential generating function of Poncaré polynomials for the

sequence  $H_*\mathcal{B}_d^{(k)}$  is computed to be:

$$F_{H_*\mathcal{B}_d^{(k)}}(x) = \sum_{n=0}^{\infty} P_{H_*\mathcal{B}_d^{(k)}(n)}(q) \frac{x^n}{n!} = e^x \left( 1 - (-q)^{k-2} + (-q)^{k-2} \left( \sum_{j=0}^{k-1} \frac{(-q^{d-1}x)^j}{j!} \right) e^{q^{d-1}x} \right)^{-\frac{1}{q^{d-1}}}. \quad (3.4.2)$$

The main result of this section describes the cycle index sum of the symmetric sequence  $H_*\mathcal{B}_d^{(k)}$  obtained from the isomorphism (2.4.1).

**Theorem 3.4.1.** *For  $k \geq 2$ ,  $d \geq 1$*

$$Z_{H_*\mathcal{B}_d^{(k)}}(q; p_1, p_2, p_3, \dots) = e^{(\sum_{i=1}^{\infty} \frac{p_i}{i})} \prod_{m=1}^{\infty} \left( 1 - (-q)^{m(k-2)} + (-q)^{m(k-2)} \left( e^{-\sum_{j=1}^{\infty} \frac{(-1)^{d-1}(-q)^{mj(d-1)} p_{mj}}{j}} \right)_{\leq m(k-1)} \left( e^{\sum_{j=1}^{\infty} \frac{(-1)^{d-1}(-q)^{mj(d-1)} p_{mj}}{j}} \right) \right)^{(-1)^d E_m \left( \frac{1}{(-q)^{d-1}} \right)}, \quad (3.4.3)$$

where  $\leq m(k-1)$  denotes the truncation with respect to the cardinality degree ( $|p_i| = i$ ) and  $E_m(y) = \frac{1}{m} \sum_{i|m} \mu(i) y^{\frac{m}{i}}$ , where  $\mu(i)$  is the usual Möbius function.

Most of the computations are straightforward. The main difficult part is computing the cycle index sum for  $\mathcal{H}_1^{(k)}$ , which is done above in Section 3.3.

We also establish a refinement of Theorem 3.4.1. The homology groups of  $H_*\mathcal{B}_d^{(k)}$  can be described as linear combinations of products of iterated brackets as seen in Section 2.2. The number of braces and the number of brackets represent two additional gradings on the space. The cycle index sum of  $H_*\mathcal{B}_d^{(k)}$  can be adjusted with the use of two additional variables to account for these two additional gradings. See Section 3.6.

It is easy to see that from (3.4.3) one can recover (3.4.1) by setting  $k = 2$ .

*Proof.* Begin with (3.4.3) and set  $k = 2$ :

$$\begin{aligned}
Z_{H^* \mathcal{B}_d^{(2)}}(q; p_1, p_2, p_3, \dots) &= \exp\left(\sum_{l=1}^{\infty} \frac{p_l}{l}\right) \prod_{m=1}^{\infty} \left(1 - (-q)^0 + (-q)^0\right. \\
&\exp\left(-\sum_{j=1}^{\infty} \frac{(-1)^{d-1}(-q)^{mj(d-1)} p_{mj}}{j}\right) \left.\exp\left(\sum_{j=1}^{\infty} \frac{(-1)^{d-1}(-q)^{mj(d-1)} p_{mj}}{j}\right)\right)^{(-1)^d E_m\left(\frac{1}{(-q)^{d-1}}\right)} \\
&= \exp\left(\sum_{l=1}^{\infty} \frac{p_l}{l}\right) \prod_{m=1}^{\infty} \left(1 - 1 + 1 \exp(-(-1)^{1-d}(-q)^{m(d-1)} p_m)\right)_{\leq m} \\
&\exp\left(\sum_{j=1}^{\infty} \frac{(-1)^{1-d}(-q)^{mj(d-1)} p_{mj}}{j}\right)^{(-1)^d E_m\left(\frac{1}{(-q)^{d-1}}\right)} \\
&= \exp\left(\sum_{l=1}^{\infty} \frac{p_l}{l}\right) \prod_{m=1}^{\infty} \left(1 - (-1)^{1-d}(-q)^{m(d-1)} p_m\right)^{(-1)^d E_m\left(\frac{1}{(-q)^{d-1}}\right)} \\
&\prod_{m=1}^{\infty} \prod_{j=1}^{\infty} \left(\exp\left(\frac{(-1)^d E_m\left((-q)^{1-d}\right) (-1)^{1-d}(-q)^{mj(d-1)} p_{mj}}{j}\right)\right) \\
&= \exp\left(\sum_{l=1}^{\infty} \frac{p_l}{l}\right) \prod_{m=1}^{\infty} \left(1 - (-1)^{1-d}(-q)^{m(d-1)} p_m\right)^{(-1)^d E_m\left(\frac{1}{(-q)^{d-1}}\right)} \\
&\prod_{m=1}^{\infty} \prod_{j=1}^{\infty} \exp\left(\frac{1}{j} (-1)^d E_m\left((-q)^{1-d}\right) (-1)^{1-d}(-q)^{mj(d-1)} p_{mj}\right) \quad (3.4.4)
\end{aligned}$$

Notice that in (3.4.4) we have the formula we want as the first factor  $\exp\left(\sum_{l=1}^{\infty} \frac{p_l}{l}\right)$  cancels out with the factor on the second line:

$$\prod_{m=1}^{\infty} \left(1 - (-1)^{1-d}(-q)^{m(d-1)} p_m\right)^{(-1)^d E_m\left(\frac{1}{(-q)^{d-1}}\right)}$$

So let us look at the factor from the second line in (3.4.4):

$$\prod_{m=1}^{\infty} \prod_{j=1}^{\infty} \exp\left(\frac{1}{j}(-1)^d E_m\left((-q)^{1-d}\right) (-1)^{1-d} (-q)^{mj(d-1)} p_{mj}\right) \quad (3.4.5)$$

Let  $l = mj$  in (3.4.5):

$$\begin{aligned} & \prod_{l=1}^{\infty} \prod_{m|l} \exp\left(\frac{m}{l}(-1)^d E_m\left((-q)^{1-d}\right) (-1)^{1-d} (-q)^{l(d-1)} p_l\right) \\ &= \prod_{l=1}^{\infty} \exp\left(\frac{pl}{l}(-1)(-q)^{l(d-1)} \sum_{m|l} m E_m\left((-q)^{1-d}\right)\right) \end{aligned}$$

Recall  $E_m(y) = \frac{1}{m} \sum_{i|m} \mu(i) y^{\frac{m}{i}}$ .

$$\begin{aligned} &= \prod_{l=1}^{\infty} \exp\left(\frac{-pl}{l}(-q)^{l(d-1)} \sum_{m|l} \sum_{k|m} \mu(k) \left((-q)^{1-d}\right)^{m/k}\right) \\ &= \exp\left(\sum_{l=1}^{\infty} \frac{-pl}{l}(-q)^{l(d-1)} \sum_{m|l} \sum_{k|m} \mu(k) \left((-q)^{1-d}\right)^{m/k}\right) \quad (3.4.6) \end{aligned}$$

Let us examine

$$\sum_{m|l} \sum_{k|m} \mu(k) \left((-q)^{1-d}\right)^{m/k}$$

In the space of  $\mathbb{R}$ -valued functions on  $\mathbb{N}$  (excluding 0), let  $*$  denote the Dirichlet convolution operator where

$$(f * g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right).$$

Let  $\mathbb{1}$  be the constant function where  $\mathbb{1}(n) = 1$  for all  $n$  and let  $\mu$  be the usual Möbius function. It is a known fact that  $\mathbb{1} * \mu = \varepsilon$ , where  $\varepsilon$  is the multiplicative identity (or unit) function, that is,

$$\varepsilon = \begin{cases} 1 & \text{if } n = 1; \\ 0 & \text{if } n > 1, \end{cases}$$

and  $(\varepsilon * g) = (g * \varepsilon) = g$  for any function  $g$ . Then  $\sum_{m|l} \sum_{k|m} \mu(k) ((-q)^{1-d})^{m/k} = (\mathbb{1} * \mu * f)(l)$  where  $f(l) = ((-q)^{1-d})^l$ . Using the above fact,  $(\mathbb{1} * \mu * f)(l) = (\varepsilon * f)(l) = f(l)$ , which implies  $\sum_{m|l} \sum_{k|m} \mu(k) ((-q)^{1-d})^{m/k} = f(l) = ((-q)^{1-d})^l$ .

Substituting this into (3.4.6), we have

$$\exp\left(\sum_{l=1}^{\infty} \frac{-pl}{l} (-q)^{l(d-1)} \cdot ((-q)^{1-d})^l\right) = \exp\left(\sum_{l=1}^{\infty} \frac{-pl}{l}\right) \quad (3.4.7)$$

Now substitute (3.4.7) back into (3.4.4):

$$\exp\left(\sum_{l=1}^{\infty} \frac{pl}{l}\right) \prod_{m=1}^{\infty} \left(1 - (-1)^{1-d} (-q)^{m(d-1)} p_m\right)^{(-1)^d E_m\left(\frac{1}{(-q)^{d-1}}\right)} \exp\left(\sum_{l=1}^{\infty} \frac{-pl}{l}\right) \quad (3.4.8)$$

The first term in the product

$$\exp\left(\sum_{l=1}^{\infty} \frac{pl}{l}\right)$$

and the last term in the product

$$\exp\left(\sum_{l=1}^{\infty} \frac{-pl}{l}\right)$$

are inverses of each other, completely the proof.  $\square$

Similarly, (3.4.2) can be recovered from (3.4.3) by setting  $p_1 = x$  and  $p_i = 0$  for  $i \geq 2$ .

*Proof.* If we want to set  $p_1 = x$  and  $p_i = 0$  for  $i \geq 2$  in (3.4.3), this is only possible if  $l = m = j = 1$ . This yields

$$\begin{aligned} & e^x \left(1 - (-q)^{k-2} + (-q)^{k-2} \exp((-1)^{d-1} (-q)^{d-1} x)_{\leq k-1} \exp((-1)^{d-1} (-q)^{d-1} x)\right)^{(-1)^d (-q)^{1-d}} \\ & = e^x \left(1 - (-q)^{k-2} + (-q)^{k-2} \exp(-q^{d-1} x)_{\leq (k-1)} \exp((-1)^{d-1} (-q)^{d-1} x)\right)^{\frac{-1}{-q^{d-1}}} \end{aligned}$$

Recall  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ . Then

$$\exp(-q^{d-1}x)_{\leq(k-1)} = \sum_{j=0}^{k-1} \frac{(-q^{d-1}x)^j}{j!}.$$

This gives us the result we desire:

$$= e^x \left( 1 - (-q)^{k-2} + (-q)^{k-2} \left( \sum_{j=0}^{k-1} \frac{(-q^{d-1}x)^j}{j!} \right) e^{q^{d-1}x} \right)^{-\frac{1}{q^{d-1}}}.$$

□

### 3.5 Proof of Theorem 3.4.1

First we compute the plethysm of  $\mathcal{L}ie$  and  $\mathcal{H}_1^{(k)}$  using (3.1.4) and (3.1.5).

$$\begin{aligned} & Z_{\mathcal{L}ie \circ \mathcal{H}_1^{(k)}}(q; p_1, p_2, p_3, \dots) = \\ & \sum_{i=1}^{\infty} \frac{-\mu(i)}{i} \ln \left( 1 - ((-1)^i q^i)^{k-2} + ((-1)^i q^i)^{k-2} \left[ \exp \left( - \sum_{j=1}^{\infty} \frac{p_{ij}}{j} \right) \right]_{\leq i(k-1)} \left[ \exp \left( \sum_{j=1}^{\infty} \frac{p_{ij}}{j} \right) \right] \right) \\ & = \sum_{i=1}^{\infty} \frac{-\mu(i)}{i} \ln \left( 1 - (-q)^{i(k-2)} + (-q)^{i(k-2)} \left[ \exp \left( - \sum_{j=1}^{\infty} \frac{p_{ij}}{j} \right) \right]_{\leq i(k-1)} \left[ \exp \left( \sum_{j=1}^{\infty} \frac{p_{ij}}{j} \right) \right] \right). \end{aligned}$$

Next we use (3.1.6) to compute the  $\{d-1\}$  suspension of the above equation:

$$\begin{aligned} & Z_{(\mathcal{L}ie \circ \mathcal{H}_1^{(k)})\{d-1\}}(q; p_1, p_2, p_3, \dots) = \\ & q^{1-d} \sum_{i=1}^{\infty} \frac{-\mu(i)}{i} \ln \left( 1 - (-q)^{i(k-2)} + (-q)^{i(k-2)} \times \right. \\ & \left. \times \left[ \exp \left( - \sum_{i=1}^{\infty} \frac{(-1)^{(ij-1)(d-1)} q^{ij(d-1)} p_{ij}}{j} \right) \right]_{\leq i(k-1)} \left[ \exp \left( \sum_{i=1}^{\infty} \frac{(-1)^{(ij-1)(d-1)} q^{ij(d-1)} p_{ij}}{j} \right) \right] \right) \end{aligned}$$

$$\begin{aligned}
&= q^{1-d} \sum_{i=1}^{\infty} \frac{-\mu(i)}{i} \ln \left( 1 - (-q)^{i(k-2)} + \right. \\
&\quad \left. + (-q)^{i(k-2)} \left[ e^{-\sum_{j=1}^{\infty} \frac{(-1)^{d-1} (-q)^{ij(d-1)} p_{ij}}{j}} \right]_{\leq i(k-1)} \left[ e^{\sum_{j=1}^{\infty} \frac{(-1)^{d-1} (-q)^{ij(d-1)} p_{ij}}{j}} \right] \right).
\end{aligned}$$

Now we will simply add the  $\mathbb{1}$ , the trivial representation of  $\Sigma_1$ , to the equation above:

$$\begin{aligned}
Z_{\mathbb{1} \oplus (\mathcal{L}ie \circ \mathcal{H}_1^{(k)}) \{d-1\}}(q; p_1, p_2, p_3, \dots) = \\
p_1 + q^{1-d} \sum_{i=1}^{\infty} \frac{-\mu(i)}{i} \ln \left( 1 - (-q)^{i(k-2)} + \right. \\
\left. + (-q)^{i(k-2)} \left[ e^{-\sum_{i=1}^{\infty} \frac{(-1)^{d-1} (-q)^{ij(d-1)} p_{ij}}{j}} \right]_{\leq i(k-1)} \left[ e^{\sum_{i=1}^{\infty} \frac{(-1)^{d-1} (-q)^{ij(d-1)} p_{ij}}{j}} \right] \right).
\end{aligned}$$

Finally, we again use (3.1.4) and (3.1.5) to compute the graded composition product of  $Com$  with the above equation to get an explicit formula.

$$\begin{aligned}
Z_{Com \circ (\mathbb{1} \oplus (\mathcal{L}ie \circ \mathcal{H}_1^{(k)}) \{d-1\})}(q; p_1, p_2, p_3, \dots) = \\
\exp \left( \sum_{l=1}^{\infty} \frac{1}{l} \left[ p_l + \left( (-1)^{l-1} q^l \right)^{1-d} \sum_{i=1}^{\infty} \frac{-\mu(i)}{i} \ln \left( 1 - \left( (-1)^l q^l \right)^{i(k-2)} + \left( (-1)^l q^l \right)^{i(k-2)} \times \right. \right. \\
\left. \left. \times \exp \left( - \sum_{j=1}^{\infty} \frac{(-1)^{1-d} \left( (-1)^l q^l \right)^{ij(d-1)} p_{ijl}}{j} \right)_{\leq il(k-1)} \times \right. \right. \\
\left. \left. \times \exp \left( \sum_{j=1}^{\infty} \frac{(-1)^{1-d} \left( (-1)^l q^l \right)^{ij(d-1)} p_{ijl}}{j} \right) \right) \right] \right) \\
= \exp \left( \sum_{l=1}^{\infty} \frac{1}{l} \left[ p_l + (-1)^{(l-1)(1-d)} q^{l(1-d)} \sum_{i=1}^{\infty} \frac{-\mu(i)}{i} \ln \left( 1 - (-1)^{li(k-2)} q^{li(k-2)} + \right. \right. \\
\left. \left. + (-1)^{li(k-2)} q^{li(k-2)} \exp \left( - \sum_{j=1}^{\infty} \frac{(-1)^{1-d} (-1)^{lij(d-1)} (q)^{lij(d-1)} p_{ijl}}{j} \right)_{\leq il(k-1)} \times \right. \right. \\
\left. \left. \times \exp \left( \sum_{j=1}^{\infty} \frac{(-1)^{1-d} (-1)^{lij(d-1)} (q)^{lij(d-1)} p_{ijl}}{j} \right) \right) \right] \right)
\end{aligned}$$

$$\begin{aligned}
&= \exp \left( \sum_{l=1}^{\infty} \frac{1}{l} \left[ p_l + (-1)^{(l-1)(1-d)} q^{l(1-d)} \sum_{i=1}^{\infty} \frac{-\mu(i)}{i} \ln \left( 1 - (-q)^{li(k-2)} + (-q)^{li(k-2)} \right. \right. \right. \\
&\quad \left. \left. \exp \left( - \sum_{j=1}^{\infty} \frac{(-1)^{1-d} (-q)^{lij(d-1)} p_{ijl}}{j} \right) \right]_{\leq il(k-1)} \exp \left( \sum_{j=1}^{\infty} \frac{(-1)^{1-d} (-q)^{lij(d-1)} p_{ijl}}{j} \right) \right) \Bigg) \\
&= \exp \left( \sum_{l=1}^{\infty} \frac{p_l}{l} \right) \exp \left( \sum_{l=1}^{\infty} (-1)^{d-1} (-q)^{l(1-d)} \sum_{i=1}^{\infty} \frac{\mu(i)}{li} \ln \left[ 1 - (-q)^{li(k-2)} + (-q)^{li(k-2)} \times \right. \right. \\
&\quad \left. \left. \times \exp \left( - \sum_{j=1}^{\infty} \frac{(-1)^{1-d} (-q)^{lij(d-1)} p_{ijl}}{j} \right) \right]_{\leq il(k-1)} \exp \left( \sum_{j=1}^{\infty} \frac{(-1)^{1-d} (-q)^{lij(d-1)} p_{ijl}}{j} \right) \right) \Bigg)
\end{aligned}$$

Let  $m = il$ , then

$$\begin{aligned}
&Z_{\text{Com}(\mathbb{1} \oplus (\mathcal{L}ie \circ \mathcal{H}_1^{(k)})_{\{d-1\}})}(q; p_1, p_2, p_3, \dots) = \\
&\quad \exp \left( \sum_{l=1}^{\infty} \frac{p_l}{l} \right) \exp \left( \sum_{m=1}^{\infty} \frac{(-1)^{d-1}}{m} \sum_{i|m} q^{\frac{m}{i}(d-1)} \mu(i) \ln \left[ 1 - (-q)^{m(k-2)} + (-q)^{m(k-2)} \times \right. \right. \\
&\quad \left. \left. \times \exp \left( - \sum_{j=1}^{\infty} \frac{(-1)^{1-d} (-q)^{mj(d-1)} p_{mj}}{j} \right) \right]_{\leq m(k-1)} \exp \left( \sum_{j=1}^{\infty} \frac{(-1)^{1-d} (-q)^{mj(d-1)} p_{mj}}{j} \right) \right) \Bigg) \\
&= \exp \left( \sum_{l=1}^{\infty} \frac{p_l}{l} \right) \prod_{m=1}^{\infty} \left( 1 - (-q)^{m(k-2)} + (-q)^{m(k-2)} \times \right. \\
&\quad \left. \times \exp \left[ - \sum_{j=1}^{\infty} \frac{(-1)^{1-d} (-q)^{mj(d-1)} p_{mj}}{j} \right]_{\leq m(k-1)} \times \right. \\
&\quad \left. \times \exp \left[ \sum_{j=1}^{\infty} \frac{(-1)^{1-d} (-q)^{mj(d-1)} p_{mj}}{j} \right] \right)^{\frac{(-1)^{d-1}}{m} \sum_{i|m} q^{\frac{m}{i}(d-1)} \mu(i)}
\end{aligned}$$

Recall that  $E_m(y) = \frac{1}{m} \sum_{i|m} (\mu(i)y^{\frac{m}{i}})$ .

$$\begin{aligned}
&= \exp\left(\sum_{l=1}^{\infty} \frac{p_l}{l}\right) \prod_{m=1}^{\infty} \left(1 - (-q)^{m(k-2)} + (-q)^{m(k-2)} \times \right. \\
&\quad \times \exp\left[-\sum_{j=1}^{\infty} \frac{(-1)^{1-d}(-q)^{mj(d-1)}p_{mj}}{j}\right]_{\leq m(k-1)} \times \\
&\quad \left. \times \exp\left[\sum_{j=1}^{\infty} \frac{(-1)^{1-d}(-q)^{mj(d-1)}p_{mj}}{j}\right]\right)^{(-1)^d E_m\left(\frac{1}{(-q)^{d-1}}\right)}
\end{aligned}$$

### 3.6 Refinement

Recall from Section 2.2 that elements in  $H_*\mathcal{B}_d^{(k)}(n)$  are described as linear combinations of certain products of iterated brackets, where there are two types of brackets, long and short, or braces and brackets respectively.

The number of long and short brackets are additional gradings that we consider on  $H_*\mathcal{B}_d^{(k)}(n)$ . We add the variable  $u$  to be responsible for the number of short brackets grading and the variable  $w$  to be responsible for the number of long brackets grading in the graded trace used for the cycle index sum (3.1.1). The sequence  $Com$  does not contribute to these additional gradings and thus (3.1.7) remains unchanged in the refinement. The graded suspension does not interact with the long and short brackets and thus (3.1.6) also remains unchanged. However, there are short brackets in  $\mathcal{L}ie$ . In cardinality  $k$ , there are always  $k - 1$  (short) brackets, which is why we divide by  $u$  and replace  $p_i$  by  $u^i p_i$  in the formula below. By abuse of notation, we also denote by  $Z_{\mathcal{L}ie}$  the cycle index sum of  $\mathcal{L}ie$  with this refinement:

$$Z_{\mathcal{L}ie}(u, q; p_1, p_2, p_3, \dots) = \sum_{i=1}^{\infty} \frac{-\mu(i)}{u} \frac{\ln(1 - u^i p_i)}{i}. \quad (3.6.1)$$

Recall from Section 2.5 the space  $\mathcal{H}_1^{(k)}(n)$  is a subspace of  $H_{k-2}\mathcal{B}_1^{(k)}(n)$ , defined as a subspace spanned by iterated brackets that have exactly one long bracket [12]. This explains why we multiply by  $w$  in the formula below. The iterated brackets of  $\mathcal{H}_1^{(k)}(n)$  have exactly

$n - k$  short brackets, which explains in the formula below why we divide by  $u^k$  and replace  $p_i$  by  $u^i p_i$  in the refinement. Similarly we abuse notation to denote the cycle index sum of  $\mathcal{H}_1^{(k)}(n)$  with the refinement as before by  $Z_{\mathcal{H}_1^{(k)}}$ .

$$Z_{\mathcal{H}_1^{(k)}}(u, w, q; p_1, p_2, p_3, \dots) = \frac{w}{u^k} \left( (-q)^{k-2} - (-q)^{k-2} \left( \exp \left( - \sum_{i=1}^{\infty} \frac{u^i p_i}{i} \right) \right)_{\leq k-1} \left( \exp \left( \sum_{i=1}^{\infty} \frac{u^i p_i}{i} \right) \right) \right). \quad (3.6.2)$$

The plethysm also affects the long and short brackets and is now defined as:

$$Z_{M \circ N} = Z_M * Z_N = Z_M(u; w; q; p_i \mapsto p_i * Z_N), \quad (3.6.3)$$

where

$$p_i * Z_N = Z_N(u \mapsto u^i; w \mapsto w^i; q \mapsto (-1)^{i-1} q^i; p_j \mapsto p_{ij}). \quad (3.6.4)$$

**Theorem 3.6.1.** *For  $k \geq 3$  and  $d \geq 2$ ,*

$$Z_{H_* \mathcal{M}_d^{(k)}}(u, w, q; p_1, p_2, p_3, \dots) = e^{(\sum_{l=1}^{\infty} \frac{p_l}{l})} \prod_{m=1}^{\infty} \left( 1 - \frac{(-q)^{m(k-2)} w^m}{u^{m(k-1)}} \left[ 1 - \left( e^{-\sum_{j=1}^{\infty} \frac{(-1)^{d-1} (-q)^{mj(d-1)} u^{mj} p_{mj}}}{j} \right)_{\leq m(k-1)} \left( e^{\sum_{j=1}^{\infty} \frac{(-1)^{d-1} (-q)^{mj(d-1)} u^{mj} p_{mj}}}{j} \right) \right] \right)^{(-1)^d E_m \left( \frac{1}{(-q)^{d-1} u} \right)}. \quad (3.6.5)$$

*Proof.* The proof for this theorem follows the same steps as the proof for 3.4.1.

First we compose the refinements of  $\mathcal{L}ie$  (3.6.1) and  $\mathcal{H}_1^{(k)}$  (3.6.2) now using (3.6.3) and

(3.6.4):

$$Z_{\mathcal{L}ie\mathcal{H}_1^{(k)}}(u, w, q; p_1, p_2, p_3 \dots) = \sum_{i=1}^{\infty} \frac{-\mu(i)}{u \cdot i} \ln \left( 1 - \frac{w^i}{u^{i(k-1)}} \left( (-q)^{i(k-2)} - (-q)^{i(k-2)} \exp \left( - \sum_{j=1}^{\infty} \frac{u^{ij} p_{ij}}{j} \right)_{\leq i(k-1)} \exp \left( \sum_{j=1}^{\infty} \frac{u^{ij} p_{ij}}{j} \right) \right) \right).$$

Next compute the  $\{d-1\}$ -suspension still using (3.1.6):

$$Z_{(\mathcal{L}ie\mathcal{H}_1^{(k)})\{d-1\}}(u, w, q; p_1, p_2, p_3 \dots) = (q)^{1-d} \sum_{i=1}^{\infty} \frac{-\mu(i)}{u \cdot i} \ln \left( 1 - \frac{w^i}{u^{i(k-1)}} \left( (-q)^{i(k-2)} - (-q)^{i(k-2)} \times \exp \left( - \sum_{j=1}^{\infty} \frac{u^{ij} (-1)^{1-d} (-q)^{ij(d-1)} p_{ij}}{j} \right)_{\leq i(k-1)} \exp \left( \sum_{j=1}^{\infty} \frac{u^{ij} (-1)^{1-d} (-q)^{ij(d-1)} p_{ij}}{j} \right) \right) \right).$$

Now add  $\mathbb{1}$  to the above equation:

$$Z_{\mathbb{1} \oplus (\mathcal{L}ie\mathcal{H}_1^{(k)})\{d-1\}}(u, w, q; p_1, p_2, p_3 \dots) = p_1 + (q)^{1-d} \sum_{i=1}^{\infty} \frac{-\mu(i)}{u \cdot i} \ln \left( 1 - \frac{w^i}{u^{i(k-1)}} \left( (-q)^{i(k-2)} - (-q)^{i(k-2)} \times \exp \left( - \sum_{j=1}^{\infty} \frac{u^{ij} (-1)^{1-d} (-q)^{ij(d-1)} p_{ij}}{j} \right)_{\leq i(k-1)} \exp \left( \sum_{j=1}^{\infty} \frac{u^{ij} (-1)^{1-d} (-q)^{ij(d-1)} p_{ij}}{j} \right) \right) \right).$$

Finally, use (3.6.3) and (3.6.4) again to compute the graded composition with  $\mathcal{C}om$  with

the above equation:

$$\exp\left(\sum_{l=1}^{\infty} \frac{p_l}{l}\right) \exp\left(\sum_{l=1}^{\infty} \frac{-\mu(i)}{u^l \cdot l \cdot i} (-1)^d (-q)^{l(1-d)} \ln\left(1 - \frac{w^{li}}{u^{li(k-1)}} \left[(-q)^{li(k-2)} - (-q)^{li(k-2)} \times \right.\right.\right. \\ \left.\left.\left. \times \exp\left(-\sum_{j=1}^{\infty} \frac{u^{ijl} (-1)^{1-d} (-q)^{ijl(d-1)} p_{ijl}}{j}\right)_{\leq i(k-1)} \times \right.\right.\right. \\ \left.\left.\left. \times \exp\left(\sum_{j=1}^{\infty} \frac{u^{ijl} (-1)^{1-d} (-q)^{ijl(d-1)} p_{ijl}}{j}\right)_{\leq i(k-1)}\right]\right)\right)$$

Let  $m = li$  and recall  $E_m(y) = \frac{1}{m} \sum_{i|m} (\mu(i) y^{\frac{m}{i}})$ . This, along with the fact that the exponential function and natural logarithms are inverses, gives the desired result (3.6.5).  $\square$

### 3.7 First Approach

When first attempting to find (and verify in small arities) a general formula for the cycle index sum of  $\mathcal{H}_1^{(k)}$ , we began by using the Frobenius Formula for Characters of Symmetric Groups.

We will use the following notation: Let  $\mu$  be a partition of  $n$  and  $c_\mu$  be the corresponding conjugacy class in  $\Sigma_n$ . We can write  $\mu = 1^{i_1} \cdot 2^{i_2} \cdot 3^{i_3} \cdot \dots$ , where  $i_j$  is the number of cycles of length  $j$ . For convenience we can also write  $c_\mu = (i_1, i_2, i_3, \dots)$  and will denote the order of the conjugacy class by  $|c_\mu|$ . For another partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$  of  $n$ , where  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$ , one has the corresponding irreducible representation  $V_\lambda$ . Define  $\ell_j = \lambda_j + m - j$ . Next we introduce variables  $x_1, x_2, \dots, x_m$ , where  $m$  is the number of rows in the Young diagram corresponding to the partition  $\lambda$ . The  $j$ th-power sum in variables  $x_1, x_2, \dots, x_m$  is given by  $P_j(x) = x_1^j + x_2^j + \dots + x_m^j$ . The Vandermonde determinant is denoted  $\Delta(x) = \prod_{i < j} (x_i - x_j)$ .

The Frobenius Formula for Characters of  $V_\lambda$  is given by:

$$tr_\lambda(c_\mu) = \left[ \Delta(x) \prod_j P_j(x)^{i_j} \right]_{\ell_1, \ell_2, \dots, \ell_m}, \quad (3.7.1)$$

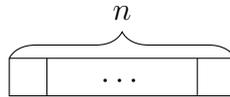
that is, we are looking for the coefficient of  $x_1^{\ell_1} \cdot x_2^{\ell_2} \cdot \dots \cdot x_m^{\ell_m}$  in  $[\Delta(x) \prod_j P_j(x)^{i_j}]$ . Note, in the literature usually the letter  $\mathcal{X}$  is used to denote characters in Representation theory. To better match our notation for the cycle index sum formula, we instead use the trace notation. If the  $\lambda$  is understood, we can instead denote the left hand side simply by  $tr(c_\mu)$ . The Vandermonde determinant  $\Delta(x)$  and the coefficient that we are looking for do not change within each conjugacy class  $c_\mu$ . The only term in the formula (3.7.1) that depends on the conjugacy class is  $\prod_j P_j(x)^{i_j}$ .

Lastly, we introduce an infinite family of commuting variables  $p_1, p_2, p_3, \dots$  where the cardinality degree of each  $p_i$  is  $i$ . Then the Cycle Index Sum of  $V_\lambda$  is given by:

$$Z_{V_\lambda} = \frac{1}{n!} \sum_{\mu \vdash n} tr_\lambda(\mu) \cdot |c_\mu| \cdot \prod_j p_j^{i_j(\mu)}. \quad (3.7.2)$$

This formula also appears as (3.1.1) in Section 3.1 with slightly different notation. Recall that  $i_j(\mu)$  denotes the number of cycles of  $\mu$  that have length  $j$ , which means the  $i_j(\mu)$ 's are equivalent to the  $i_j$ 's when we write the conjugacy class as  $c_\mu = (i_1, i_2, i_3, \dots)$ .

To begin, note that when  $\lambda = (n)$ , the corresponding representation  $V_{(n)}$  is just the trivial representation on  $\Sigma_n$ . Note when  $n = 0$  in this case, the cycle index sum for  $Z_{V_{(0)}} = 1$ . The partition  $\lambda = (n)$  corresponds to the following Young diagram:



It is clear then that  $tr(c_\mu) = 1$ , however we will verify this with the Frobenius formula. As stated above,  $\lambda = (n)$  corresponds to the representation of  $\Sigma_n$  and  $\ell_1 = n$ . That implies that we are looking for the coefficient to  $x_1^n$ . The Vandermonde determinant is 1. The power

sum is  $\prod_j P_j(x)^{i_j} = (x_1)^n$ . Finally,

$$\begin{aligned}
tr(c_\mu) &= [(1) \prod_j P_j(x)^{i_j}]_{x_1^n} \\
&= [\prod_j P_j(x)^{i_j}]_{x_1^n} \\
&= [x_1^{\sum j \cdot i_j}]_{x_1^n} \\
&= [x_1^n]_{x_1^n} \\
&= 1.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{n=1}^{\infty} Z_{V(n)} &= \frac{1}{n!} \sum_{\mu \vdash n+1} \left( [(1) \prod_{j \geq 1} P_j(x)^{i_j}]_{x_1^n} \times \frac{n!}{1^{i_1} i_1! \cdot 2^{i_2} i_2! \cdot 3^{i_3} i_3! \cdot \dots} \prod_j p_j^{i_j} \right) \\
&= \sum_{i_j} \left( \prod_{j \geq 1} \frac{p_j^{i_j}}{(j)^{i_j} (i_j)!} \right) \\
&= \prod_{j \geq 1} \sum_i \frac{\left(\frac{p_j}{j}\right)^i}{i!} \\
&= \prod_{j \geq 1} \exp\left(\frac{p_j}{j}\right) \\
&= \exp\left(\sum_{j \geq 1} \frac{p_j}{j}\right)
\end{aligned}$$

Next, we look at hooks of the form  $(n,1)$ , with  $n \geq 1$ . This corresponds to Young diagrams with  $n$  boxes in the first row and one box in the second row. In this case, we will only need two variables  $x_1$  and  $x_2$ . Hence we will be looking for the coefficient to  $x_1^{\ell_1} x_2^{\ell_2}$ , even further  $\ell_1 = n + 2 - 1 = n + 1$  and  $\ell_2 = 1 + 2 - 2 = 1$ . The Vandermonde Determinant will always be  $(x_1 - x_2)$  for  $n \geq 1$ .

We looked at small arities of  $n$ . First we examine the hook diagram corresponding to the

partition  $\lambda = (1, 1)$ , that is the Young diagram is as follows:



This partition corresponds to the 1-dimensional sign representation. We know that the characters are 1 and -1, corresponding to the identity conjugacy class and the conjugacy represented by a transposition, respectively. We will confirm this below with the Frobenius formula.

Since  $\lambda = (1, 1)$ , then  $\ell_1 = 1 + 1 = 2$  and  $\ell_2 = 1$  and we are looking for the coefficient to  $x_1^2 x_2$ . This  $\lambda$  corresponds to representations of  $\Sigma_2$ . There are two conjugacy classes of  $\Sigma_2$ : the identity conjugacy class, which we denote (id) and the other conjugacy class we denote by (12). Note that  $|(id)| = 1 = |(12)|$ .

We can equivalently write  $(id) = (1^2)$ , so  $i_1 = 2$ . Then the power sum for this conjugacy class is given by  $\prod_j P_j(x)^{i_j} = (x_1 + x_2)^2$ . Combining this with what we already found,

$$tr((id)) = [(x_1 - x_2)(x_1 + x_2)^2]_{x_1^2 x_2} = 1.$$

We can equivalently write  $(12) = (1^0 2^1) = (2^1)$ , so  $i_1 = 0$  and  $i_2 = 1$ . Then for this conjugacy class  $\prod_j P_j(x)^{i_j} = (x_1^2 + x_2^2)$ . Now we can calculate

$$tr((12)) = [(x_1 - x_2)(x_1^2 + x_2^2)]_{x_1^2 x_2} = -1.$$

Now that the traces have been found we can calculate the cycle index sum of  $V_{(1,1)}$ :

$$Z_{V_{(1,1)}} = \frac{1}{2!} ((1)(1)p_1^2 + (-1)(1)p_2^1) = \frac{1}{2} (p_1^2 - p_2).$$

The two previous examples are known without using the Frobenius formula. Now we will look at two examples where the Frobenius formula is needed.

The Young diagram corresponding to  $\lambda = (2, 1)$  is:



Since  $\lambda = (2, 1)$ ,  $\ell_1 = 2 + 1 = 3$  and  $\ell_2 = 1$ . Hence we are looking for the coefficient of  $x_1^3 x_2$ . This  $\lambda$  corresponds to representations of  $\Sigma_3$ . There are three conjugacy classes of  $\Sigma_3$ : again we denote the conjugacy class corresponding to the identity in  $\Sigma_3$  by (id) and  $|(id)| = 1$ . The conjugacy with one transposition we will denote by (12)(3) and  $|(12)(3)| = 3$ . The last conjugacy we denote by (123) and  $|(123)| = 2$ .

We can equivalently write (id) = (1<sup>3</sup>) and hence  $i_1 = 3$ . Then for this conjugacy class  $\prod_j P_j(x)^{i_j} = (x_1 + x_2)^3$ . Therefore:

$$tr(id) = [(x_1 - x_2)(x_1 + x_2)^3]_{x_1^3 x_2} = 2.$$

The conjugacy class (12)(3) can equivalently be written as (1<sup>1</sup>2<sup>1</sup>), so  $i_1 = 1 = i_2$ . Then for this conjugacy class  $\prod_j P_j(x)^{i_j} = (x_1 + x_2)(x_1^2 x_2^2)$ . Therefore:

$$tr((12)(3)) = [(x_1 - x_2)(x_1 + x_2)(x_1^2 x_2^2)]_{x_1^3 x_2} = 0.$$

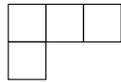
We can equivalently write the conjugacy class (123) as (1<sup>0</sup>2<sup>0</sup>3<sup>1</sup>) = (3<sup>1</sup>), so  $i_1 = 0$ ,  $i_2 = 0$  and  $i_3 = 1$ . Then for this conjugacy class  $\prod_j P_j(x)^{i_j} = (x_1^3 + x_2^3)$ . Therefore:

$$tr((123)) = [(x_1 - x_2)(x_1^3 + x_2^3)]_{x_1^3 x_2} = -1.$$

The cycle index sum for  $V_{(2,1)}$  is given by

$$Z_{V_{(2,1)}} = \frac{1}{3!} ((1)(2)p_1^3 + (3)(0)p_1 p_2 + (2)(-1)p_3) = \frac{1}{6}(2p_1^3 - 2p_3) = \frac{p_1^3 - p_3}{3}.$$

The partition  $\lambda = (3, 1)$  given by the Young diagram



This  $\lambda$  corresponds to representations of  $\Sigma_4$ . Since  $\lambda = (3, 1)$  then  $\ell_1 = 3 + 1 = 4$  and  $\ell_2 = 1$ . Hence we are looking for the coefficient of  $x_1^4 x_2$ . There are five conjugacy classes of  $\Sigma_4$  which we denote by (id), (12)(3)(4), (123)(4), (12)(34), and (1234). The orders of

these conjugacy classes are  $|id| = 1$ ,  $|(12)(3)(4)| = 6$ ,  $|(123)(4)| = 8$ ,  $|(12)(34)| = 3$ , and  $|(1234)| = 6$ .

For the (id) conjugacy class,  $i_1 = 4$ . This implies  $\prod_j P_j(x)^{i_j} = (x_1 + x_2)^4$ . Then

$$tr(id) = [(x_1 - x_2)(x_1 + x_2)^4]_{x_1^4 x_2} = 3.$$

For the conjugacy class of  $(12)(3)(4)$ ,  $i_1 = 2$  and  $i_2 = 1$ . This gives  $\prod_j P_j(x)^{i_j} = (x_1 + x_2)^2(x_1^2 + x_2^2)$ . Then

$$tr((12)(3)(4)) = [(x_1 - x_2)(x_1 + x_2)^2(x_1^2 + x_2^2)]_{x_1^4 x_2} = 1.$$

For the conjugacy class of  $(123)(4)$ ,  $i_1 = 1$  and  $i_3 = 1$ . This implies that  $\prod_j P_j(x)^{i_j} = (x_1 + x_2)(x_1^3 + x_2^3)$ . Then

$$tr((123)(4)) = [(x_1 - x_2)(x_1 + x_2)(x_1^3 + x_2^3)]_{x_1^4 x_2} = 0.$$

For the conjugacy class of  $(12)(34)$ ,  $i_2 = 2$ . This gives  $\prod_j P_j(x)^{i_j} = (x_1^2 + x_2^2)^2$ . Then

$$tr((12)(34)) = [(x_1 - x_2)(x_1^2 + x_2^2)^2]_{x_1^4 x_2} = -1.$$

For the conjugacy class of  $(1234)$ ,  $i_4 = 1$ . This gives that  $\prod_j P_j(x)^{i_j} = (x_1^4 + x_2^4)$ . Then

$$tr((1234)) = [(x_1 - x_2)(x_1^4 + x_2^4)]_{x_1^4 x_2} = -1.$$

Therefore the cycle index sum for the hook  $(3,1)$  is given by

$$\begin{aligned} Z_{V_{(3,1)}} &= \frac{1}{4!}((1)(3)p_1^4 + (6)(1)p_1^2 p_2 + (8)(0)p_1 p_3 + (3)(-1)p_2^2 + (6)(-1)p_4) \\ &= \frac{1}{24}(3p_1^4 + 6p_1^2 p_2 - 3p_2^2 - 6p_4) \end{aligned}$$

$$= \frac{1}{8}p_1^4 + \frac{1}{4}p_1^2p_2 - \frac{1}{8}p_2^2 - \frac{1}{4}p_4.$$

From the Frobenius formula we can write the cycle index sum for each  $V_{(n,1)}$  where  $n \geq 1$ ,  
as

$$Z_{V_{(n,1)}} = \frac{1}{n!} \sum_{\mu \vdash n+1} \left( \left[ (x_1 - x_2) \prod_j (x_1^j + x_2^j)^{i_j} \right]_{x_1^{n+1}x_2} \times \right. \\ \left. \times \frac{n!}{1^{i_1}i_1! \cdot 2^{i_2}i_2! \cdot 3^{i_3}i_3! \cdot \dots} \prod_j p_j^{i_j} \right). \quad (3.7.3)$$

where  $i_j$  is the number of cycles of length  $j$  in partition  $\mu$  of  $\Sigma_{n+1}$ .

We can rewrite (3.7.3) in the following way:

$$\sum_{n=1}^{\infty} Z_{V_{(n,1)}} = \frac{\partial}{\partial x_2} \left( \sum_{i_j} \left( (x_1 - x_2) \prod_j \frac{(x_1^j + x_2^j)^{i_j} p_j^{i_j}}{(j)^{i_j} (i_j)!} \right) - P_0(x_1, x_2)^0 p_0^0 - P_1(x_1, x_2)^1 p_1^1 \right) \Big|_{x_1=1, x_2=0}$$

The last two terms in the above equation correspond to the cardinality degree 0 and 1, respectively. Since  $V_{(n,1)}$  begins in cardinality degree 2, we want to subtract off anything in degree 0 or 1 so that they do not contribute anything.

$$= \frac{\partial}{\partial x_2} \left( (x_1 - x_2) \prod_j \sum_i \left( \frac{\left( \frac{(x_1^j + x_2^j) p_j}{j} \right)^i}{i!} \right) - 1 - (x_1 + x_2) p_1 \right) \Big|_{x_1=1, x_2=0} \\ = \frac{\partial}{\partial x_2} \left( (x_1 - x_2) \prod_j \exp \left( \frac{(x_1^j + x_2^j) p_j}{j} \right) - 1 - (x_1 + x_2) p_1 \right) \Big|_{x_1=1, x_2=0} \\ = \frac{\partial}{\partial x_2} \left( (x_1 - x_2) \exp \left( \sum_j \frac{(x_1^j + x_2^j) p_j}{j} \right) - 1 - (x_1 + x_2) p_1 \right) \Big|_{x_1=1, x_2=0}$$

Next take the derivative of the above equation with respect to  $x_2$ :

$$\begin{aligned}
&= \left[ (-1) \left( \exp \left( \sum_{i_j} \frac{(x_1^j + x_2^j) p_j}{j} \right) - 1 - (x_1 + x_2) p_1 \right) + \right. \\
&\quad \left. + (x_1 - x_2) \left( \exp \left( \sum_{i_j} \frac{(x_1^j + x_2^j) p_j}{j} \right) \left( \sum_{i_j} x_2^{j-1} p_j \right) - p_1 \right) \right] \Big|_{x_1=1, x_2=0} \\
&= \left[ (x_1 - x_2) \left( \exp \left( \sum_{i_j} \frac{(x_1^j + x_2^j) p_j}{j} \right) \left( \sum_{i_j} x_2^{j-1} p_j \right) - p_1 \right) - \right. \\
&\quad \left. - \left( \exp \left( \sum_{i_j} \frac{(x_1^j + x_2^j) p_j}{j} \right) - 1 - (x_1 + x_2) p_1 \right) \right] \Big|_{x_1=1, x_2=0} \\
&= \left[ (x_1 - x_2) \left( \exp \left( \sum_{i_j} \frac{(x_1^j + x_2^j) p_j}{j} \right) \left( \sum_{i_j} x_2^{j-1} p_j \right) - p_1 \right) - \right. \\
&\quad \left. - \exp \left( \sum_{i_j} \frac{(x_1^j + x_2^j) p_j}{j} \right) + 1 + (x_1 + x_2) p_1 \right] \Big|_{x_1=1, x_2=0}
\end{aligned}$$

Now evaluate the  $x_1 \mapsto 1$  and  $x_2 \mapsto 0$ :

$$= \exp \left( \sum_{i_j} \frac{1}{j} p_j \right) p_1 - p_1 - \exp \left( \sum_{i_j} \frac{1}{j} p_j \right) + 1 + p_1$$

Finally, we get a nicer formula for the cycle index sum of partitions  $(n,1)$  for all  $n$ :

$$(p_1 - 1) \exp \left( \sum_{i_j} \frac{1}{j} p_j \right) + 1$$

Next we tried to apply this same approach to partitions  $(n,1,1)$ , that is, hooks with  $n$  boxes in the first row and 1 box in the second and in the third row. This requires another variable,  $x_3$ . While find the cycle index sum for representations  $V_{(n,1,1)}$  for small  $n$  was

attainable, as a whole, the process was not replicable. A formula similar to (3.7.3) was found:

$$Z_{V_{(n,1,1)}} = \frac{1}{n!} \sum_{\mu \vdash n+1} \left[ (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \prod_j (x_1^j + x_2^j + x_3^j)^{i_j} \right]_{x_1^{n+2} x_2^2 x_3} \times \left. \frac{n!}{1^{i_1} i_1! \cdot 2^{i_2} i_2! \cdot 3^{i_3} i_3! \dots \prod_j p_j^{i_j}} \right|_{x_1=1, x_2=x_3=0}. \quad (3.7.4)$$

We can rewrite (3.7.4) in the following way:

$$\sum_{n=1}^{\infty} Z_{V_{(n,1,1)}} = \frac{1}{2} \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_3} (F(x_1, x_2, x_3; p_1, p_2, \dots)) \Big|_{x_1=1, x_2=x_3=0}$$

where

$$F(x_1, x_2, x_3; p_1, p_2, \dots) = \sum_{i_j} \left( (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \prod_j \frac{(x_1^j + x_2^j + x_3^j)^{i_j} p_j^{i_j}}{(j)^{i_j} (i_j)!} \right) - P_0(x_1, x_2)^0 p_0^0 - P_1(x_1, x_2)^1 p_1^1 - \frac{1}{2} (P_1(x_1, x_2, x_3)^2 p_1^2 + P_2(x_1, x_2, x_3) p_2)$$

Similarly to (n,1), we subtract off the terms of cardinality degree less than 3, which is done in the second line of the above equation. Although the process was not complete, some work was done to replicate the process above:

$$F(x_1, x_2, x_3; p_1, p_2, \dots) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \left( \prod_j \sum_i \left( \frac{\left( \frac{(x_1^j + x_2^j + x_3^j) p_j}{j} \right)^i}{i!} \right) \right) - 1 - p_1 - \frac{1}{2} ((x_1 + x_2 + x_3)^2 p_1^2 + (x_1^2 + x_2^2 + x_3^2) p_2)$$

$$= (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \left( \prod_j \exp \left( \frac{(x_1^j + x_2^j + x_3^j) p_j}{j} \right) \right) - 1 - p_1 - \frac{1}{2} ((x_1 + x_2 + x_3)^2 p_1^2 + (x_1^2 + x_2^2 + x_3^2) p_2)$$

$$(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \left( \exp \left( \sum_j \frac{(x_1^j + x_2^j)p_j}{j} \right) - 1 - p_1 - \frac{1}{2}((x_1 + x_2 + x_3)^2 p_1^2 + (x_1^2 + x_2^2 + x_3^2)p_2) \right)$$

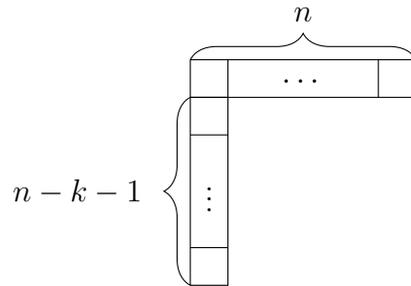
Hence,

$$\sum_{n=1}^{\infty} Z_{V(n,1,1)} = \frac{1}{2} \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_3} \left( (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \left( \exp \left( \sum_j \frac{(x_1^j + x_2^j)p_j}{j} \right) - 1 - p_1 - \frac{1}{2}((x_1 + x_2 + x_3)^2 p_1^2 + (x_1^2 + x_2^2 + x_3^2)p_2) \right) \right) \Big|_{x_1=1, x_2=x_3=0}$$

The difficulty lies in the next steps, namely taking the derivatives and mapping  $x_1 \mapsto 1$ ,  $x_2 \mapsto 0$  and  $x_3 \mapsto 0$ . An attempt was made at these next steps and although done with minor errors, the formula (3.7.5) found still proved helpful.

$$\frac{1}{2} \exp \left( \sum_{i_j} \frac{1}{j} p_j \right) (2(p_1 + 1) - p_1^2 p_2) - 1 - 2p_1 - p_1^2 + \frac{1}{2} p_2 \tag{3.7.5}$$

This difficulty would continue grow with more variables (and more derivatives) for  $(n,1,1,1)$ ,  $(n,1,1,1,1)$ ,  $(n,1,1,1,1,1)$  and so on. If these formulas had been easier to obtain, the idea would be to use them to find (and check) a more general formula for a hook of the form  $(n, 1^{n-k})$ , that is ones given by the following Young diagram for any  $0 \leq k \leq n - 1$ :



However, from (3.7.3) and (3.7.5) we were able to make a guess as to what the cycle index sum should be for  $\mathcal{H}_1^{(k)}$ , which we then verified using induced representations. See Section

3.3.

# Chapter 4

## Overlapping Discs Filtration in the Commutative Operad

The goal of this chapter is to define the operad of overlapping discs. This new operad will contain information on both the operad  $\mathcal{B}_d$  and all the bimodules  $\mathcal{B}_d^{(k)}$ . This operad is endowed with a natural filtration by the degree of the overlap.

### 4.1 Filtered Operads of Overlapping Discs

An operad  $\mathcal{O}$  is filtered if there is a filtration in each component of  $F_0\mathcal{O}(n) \subset F_1\mathcal{O}(n) \subset F_2\mathcal{O}(n) \subset \dots$  compatible with the composition maps:

$$\circ_i : F_{k_1}\mathcal{O}(n_1) \times F_{k_2}\mathcal{O}(n_2) \rightarrow F_{k_1+k_2}\mathcal{O}(n_1 + n_2 - 1). \quad (4.1.1)$$

We assume that  $id \in F_0\mathcal{O}(1)$ . Note that  $F_0\mathcal{O}$  is a suboperad of  $\mathcal{O}$ .

The filtration of the operad induces a sequence of maps in homology:

$$H_*F_0\mathcal{O}(n) \rightarrow H_*F_1\mathcal{O}(n) \rightarrow H_*F_2\mathcal{O}(n) \rightarrow \dots \quad (4.1.2)$$

and the composition maps:

$$\circ_i : H_*F_{k_1}\mathcal{O}(n_1) \otimes H_*F_{k_2}\mathcal{O}(n_2) \rightarrow H_*F_{k_1+k_2}\mathcal{O}(n_1 + n_2 - 1). \quad (4.1.3)$$

One has the inclusion  $\mathcal{B}_d^{(k)}(n) \subset \mathcal{B}_d^{(k+1)}(n)$  since the non- $k$ -overlapping condition is stricter than the non- $(k+1)$ -overlapping condition. Now define  $F_i\mathcal{O}(n) := \mathcal{B}_d^{(i+2)}(n)$ , where  $\mathcal{O}(n) := \mathcal{B}_d^{(\infty)}(n) := \bigcup_{i=2}^{\infty} \mathcal{B}_d^{(i)}(n)$ . Just as with  $\mathcal{B}_d$ , here composition (4.1.1) is inserting a configuration of  $n_2$  discs from  $\mathcal{B}_d^{(k_2)}(n_2)$  into the  $i$ -th disc of  $\mathcal{B}_d^{(k_1)}(n_1)$ . When  $k_1 = k_2 = 2$  we get the usual operadic composition in  $\mathcal{B}_d$ . Note that  $\mathcal{B}_d^{(2)}$  is the usual little discs operad  $\mathcal{B}_d$ . When  $k_1 = 2$  we get the infinitesimal version of the left action (2.2.2) and when  $k_2 = 2$  we get the infinitesimal version of the right action (2.2.1). Note that  $\mathcal{B}_d^{(\infty)}(n) = (\mathcal{B}_d(1))^n$  and therefore is contractible. Thus  $\mathcal{B}_d^{(\infty)}$  is equivalent to the commutative operad.

From [12], we already know the homology groups of  $H_*\mathcal{B}_d^{(k)}(n)$ ,  $k \geq 2$ , and how the composition maps work when either  $k_1$  or  $k_2$  is 2. Now we want to understand the maps in the sequence (4.1.2) as well as the composition maps from (4.1.3), for  $k_1, k_2 > 2$ . To understand the sequence (4.1.2) we will need the following lemma, which will also be useful when understanding the composition maps (4.1.3).

**Lemma 4.1.1.** *For all  $d \geq 1$ ,  $k \geq 2$  and  $n \geq 0$ , the inclusion  $\mathcal{B}_d^{(k)}(n) \subset \mathcal{B}_d^{(k+1)}(n)$  is null-homotopic.*

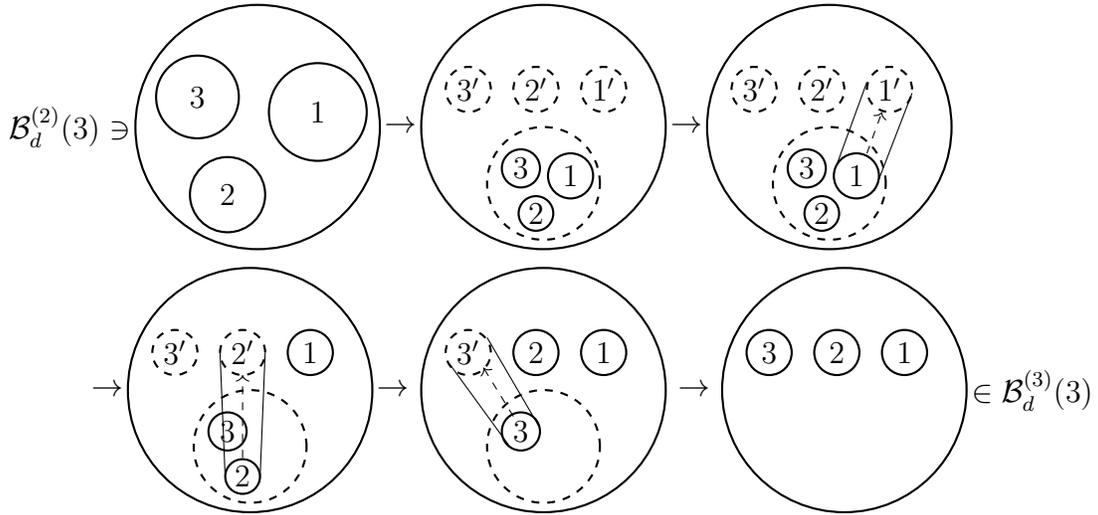
*Proof.* Define a homotopy  $H : \mathcal{B}_d^{(k)}(n) \times [0, 1] \rightarrow \mathcal{B}_d^{(k+1)}(n)$ . Subdivide  $[0, 1]$  into  $n + 1$  subintervals. Recall that a point in  $\mathcal{B}_d^{(k)}(n)$  is a configuration of  $n$ -discs in the unit disc with the condition that the intersection of any  $k$  of them is empty. Fix a point in  $\mathcal{B}_d^{(k+1)}(n)$  where all the discs, labeled  $1', 2', \dots, n'$ , are disjoint. We will call this configuration the standard position for the discs. Now take any point  $P \in \mathcal{B}_d^{(k)}(n)$ , discs labeled  $1, 2, \dots, n$ . Recall that  $P$  lies inside of a unit disc. We can smoothly rescale and translate this unit disc so that it is disjoint from the  $1', \dots, n'$  discs in the standard configuration. This homotopy is done on the first subinterval  $[0, \frac{1}{n+1}]$ .

Next, we can smoothly rescale and translate the disc labeled 1 in  $P$  to the disc labeled

1' in the standard position during the second interval of  $H$ . Then we can smoothly rescale and translate the disc labeled 2 in  $P$  to the disc labeled 2' in the standard position during the third interval of  $H$ . We can iteratively do this for all  $n$  discs in  $P$  until each disc is in the standard position in  $\mathcal{B}_d^{(k+1)}(n)$ . In the  $i$ -th interval of  $H$ , rescale and translate the disc labeled  $(i - 1)$  to the standard position, for  $i \geq 2$ .

Note when moving the discs, up to  $k$  overlaps can occur. However since the  $k$  overlaps are allowed in  $B_d^{(k+1)}(n)$ , the homotopy is well-defined.

We show the idea of the above proof below pictorially for  $\mathcal{B}_d^{(2)}(3) \subset \mathcal{B}_d^{(3)}(3)$ .



Notice that when disc 2 is moved to 2', it overlaps with disc 3. However since the movement occurs in  $B_d^{(3)}(n)$ , it does not cause any issues.

□

As an immediate consequence of Lemma 4.1.1, we get the following corollary:

**Corollary 4.1.2.** *For  $d \geq 1$ , the sequence of inclusions  $\mathcal{B}_d^{(2)} \subset \mathcal{B}_d^{(3)} \subset \mathcal{B}_d^{(4)} \subset \dots$  induces maps in the homology with each map factoring through  $\mathcal{Com}$ .*

$$\begin{array}{ccccccc}
 H_*\mathcal{B}_d^{(2)} & \longrightarrow & H_*\mathcal{B}_d^{(3)} & \longrightarrow & H_*\mathcal{B}_d^{(4)} & \longrightarrow & \dots \\
 \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \\
 \mathcal{Com} & & \mathcal{Com} & & \mathcal{Com} & & \dots
 \end{array}$$

For  $d \geq 2$  or  $k \geq 3$ , the map  $H_0\mathcal{B}_d^{(k)} \rightarrow \mathcal{Com}$  is just the projection to  $H_0\mathcal{B}_d^{(k)} = \mathcal{Com}$ . For  $d = 1, k = 2$ ,  $H_*\mathcal{B}_1^{(2)} = \mathcal{Assoc}$ . The map  $H_*\mathcal{B}_1^{(2)} \rightarrow \mathcal{Com}$  is the natural projection

*Assoc*  $\rightarrow$  *Com*. The map *Com*  $\hookrightarrow H_*\mathcal{B}_d^{(k)}$  is always the inclusion  $H_0\mathcal{B}_d^{(k)} \rightarrow H_*\mathcal{B}_d^{(k)}$ , as  $H_0\mathcal{B}_d^{(k)} = \text{Com}$  for  $k \geq 3$ .

## 4.2 Compositions in $H_*\mathcal{B}_d^{(\bullet)}$

In the previous Section 4.1 we understood the sequence of maps (4.1.2), now we want to understand the compositions maps (4.1.3). Corollary 4.1.2 tells us that the map  $H_*\mathcal{B}_d^{(k)} \rightarrow H_*\mathcal{B}_d^{(k+1)}$  can be factored through *Com*. The spherical cycle  $\{x_1, \dots, x_k\}$  is the boundary of the chain (disc)  $c(x_1, \dots, x_k) : \{x_1, \dots, x_k\} = \partial(c(x_1, \dots, x_k))$  in  $\mathcal{B}_d^{(k+1)}(k)$ . Hence  $\{x_1, \dots, x_k\} = 0$  in  $H_*\mathcal{B}_d^{(k+1)}(k)$ , by Corollary 4.1.2. The chain  $c(x_1, \dots, x_k)$  can be explicitly described as follows:

$$|x_1|^2 + |x_2|^2 + \dots + |x_k|^2 \leq \varepsilon^2 \quad \sum_{i=1}^k x_i = 0 \quad (4.2.1)$$

where  $x_i$  represents the center of the  $i$ -th disc.

### 4.2.1 Examples

We want to explicitly describe the composition maps

$$\circ_i : H_*F_{k_1}\mathcal{O}(n_1) \otimes H_*F_{k_2}\mathcal{O}(n_2) \rightarrow H_*F_{k_1+k_2}\mathcal{O}(n_1 + n_2 - 1).$$

Before describing the general case of composition, let us examine some examples of the composition maps on the level of homology.

Let  $k_1 = k_2 = 3$ ,  $n_1 = 5$ , and  $n_2 = 3$ :

$$\circ_i : H_*\mathcal{B}_d^{(3)}(5) \otimes H_*\mathcal{B}_d^{(3)}(3) \rightarrow H_*\mathcal{B}_d^{(4)}(7).$$

Let us examine the composition when  $i = 5$ :

$$[\{x_1, x_2, x_3\}, x_4] \cdot x_5 \circ_5 \{x_1, x_2, x_3\} = [\{x_1, x_2, x_3\}, x_4] \cdot \{x_5, x_6, x_7\} \in \text{Im}(H_*\mathcal{B}_d^{(3)}(7)) \subset H_*\mathcal{B}_d^{(4)}(7).$$

Here we insert the second brace in  $x_5$ . This element is in the image of  $H_*\mathcal{B}_d^{(3)}(7)$ . By the Corollary 4.1.2,  $[\{x_1, x_2, x_3\}, x_4] \cdot \{x_5, x_6, x_7\} = 0$ . Explicitly this element is the boundary of the chain  $[\{x_1, x_2, x_3\}, x_4] \cdot c(x_5, x_6, x_7)$ .

Now let us look at the composition when  $i = 4$ :

$$[\{x_1, x_2, x_3\}, x_4] \cdot x_5 \circ_4 \{x_1, x_2, x_3\} = [\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}] \cdot x_7.$$

Note that

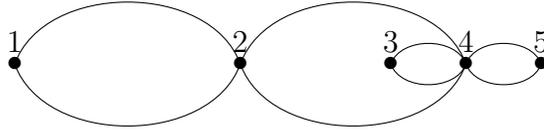
$$[\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}] \cdot x_7 = \partial([\{x_1, x_2, x_3\}, c(x_4, x_5, x_6)] \cdot x_7).$$

This element is also zero by the same argument as above.

Lastly, let us look at the composition for  $i = 3$ :

$$[\{x_1, x_2, x_3\}, x_4] \cdot x_5 \circ_3 \{x_1, x_2, x_3\} = [\{x_1, x_2, \{x_3, x_4, x_5\}\}, x_6] \cdot x_7.$$

Note that compositions  $\circ_1$  and  $\circ_2$  give similar results to  $\circ_3$ . We claim that resulting element  $[\{x_1, x_2, \{x_3, x_4, x_5\}\}, x_6] \cdot x_7 \in H_*\mathcal{B}_d^{(4)}(7)$  is non-trivial. To show this, we must understand the composition of braces, that is,  $\{x_1, x_2, \{x_3, x_4, x_5\}\}$ , which can be realized as a map  $S^{2d-1} \times S^{2d-1} \rightarrow \mathcal{B}_d^{(4)}(7)$ . We can geometrically see  $\{x_1, x_2, \{x_3, x_4, x_5\}\}$  as follows:



Here  $x_3, x_4, x_5$  orbit closely around each other while at the same time they collectively as a cluster closely orbit with  $x_1$  and  $x_2$ .

## 4.2.2 Sign Conventions

We choose our sign convention so that it agrees with that of the Dobrinskaya-Turchin paper [12]. When we describe elements in the homology of  $B_d^{(k)}$  in terms of products of iterated

brackets, cycles are realized as products of chains (usually spheres). We read the ordering of the factors (spheres) from the way the product of iterated brackets is written. We follow the rule that the spherical cycle corresponding to the brace is taken into account in the ordering by the left brace. Similarly, the spherical cycle corresponding to the bracket is taken into account by the comma. Non-spherical cycles are represented by letters (such as  $Y$  and  $Z$ ) and are taken into account in the ordering when they appear.

**Example:**

$\{x_1, \dots, x_{k-1}, [\mathbf{Y}, \mathbf{Z}]\}$  corresponds to a cycle realized by the product  $S^{(k-1)d-1} \times Y \times S^{d-1} \times Z$ . First we get  $S^{(k-1)d-1}$  due to the brace since the sign contribution is placed on the left most brace. Then we have all of the  $x_i$ 's which do not contribute to the sign. Next we have the contribution from  $Y$  and then  $S^{d-1}$ , the contribution from the bracket, recall that it is taken into account by the comma. Lastly, we have the contribution from  $Z$ .

### 4.2.3 Compositions

From the examples in Section 4.2.1 one can see that many compositions will be trivial. In fact, all non-trivial elements come from either composing braces inside of braces, or from degree 0 classes. We can categorize elements in  $H_*\mathcal{B}_d^{(k)}$  into 3 different types:

I)  $H_0\mathcal{B}_d^{(k)}(n) = \mathcal{Com}(n)$ .

II) Products with exactly one iterated bracket (the other factors singletons) that contain exactly one brace.

III) The space spanned by all the other products of iterated brackets. In particular, elements of this type will have at least two braces.

For all  $i$ ,  $I \circ_i I \neq 0$  as these are just compositions in  $\mathcal{Com}$ . Compositions of the form  $II \circ_i II \neq 0$  if and only if  $i$  is inside of the brace. We claim all the other compositions are trivial.

**Proposition 4.2.1.** *Assume both  $k_1$  and  $k_2$  are greater than 2. Then the composition maps*

$$\circ_i : H_*\mathcal{B}_d^{(k_1)}(n_1) \otimes H_*\mathcal{B}_d^{(k_2)}(n_2) \rightarrow H_*\mathcal{B}_d^{(k_1+k_2-2)}(n_1 + n_2 - 1) \quad (4.2.2)$$

are trivial restricted on  $I \circ_i II$ ,  $I \circ_i III$ ,  $II \circ_i I$ ,  $II \circ_i III$ ,  $III \circ_i I$ ,  $III \circ_i II$ ,  $III \circ_i III$  for all  $i$  and on  $II \circ_i II$  if  $i$  is outside of the brace.

*Proof.* Cases  $I \circ_i II$  and  $I \circ_i III$ : Let  $x_1 \cdots x_{n_1} \in H_0 \mathcal{B}_d^{(k_1)}(n_1)$  be of type I and  $\beta(x_1, \dots, x_{n_2}) \in H_* \mathcal{B}_d^{(k_2)}(n_2)$  be of type II or III. Compositions in these cases are as follows:

$$x_1 \cdots x_{n_1} \circ_i \beta(x_1, \dots, x_{n_2}) = x_1 \cdots x_{i-1} \cdot \beta(x_i, \dots, x_{i+n_2-1}) \cdots x_{n_1+n_2-1}.$$

The result of this composition is in  $\text{Im}(H_{>0} \mathcal{B}_d^{(k_2)}(n_1+n_2-1)) \subset H_{>0} \mathcal{B}_d^{(k_1+k_2-2)}(n_1+n_2-1)$ . Since  $k_1$  is assumed to be greater than 2, by Lemma 4.1.1, the compositions of these forms are trivial.

Cases  $II \circ_i I$  and  $III \circ_i I$ : Let  $\alpha(x_1, \dots, x_{n_1}) \in H_* \mathcal{B}_d^{(k_1)}(n_1)$  be of type II or III and  $x_1 \cdots x_{n_2} \in H_0 \mathcal{B}_d^{(k_2)}(n_2)$  be of type I. Compositions in these cases are as follows:

$$\alpha(x_1, \dots, x_{n_1}) \circ_i x_1 \cdots x_{n_2} = \alpha(x_1, \dots, x_{i-1}, x_i \cdots x_{i+n_2-1}, x_{i+n_2}, \dots, x_{n_1+n_2-1}).$$

If  $i$  is not in the brace of  $\alpha(x_1, \dots, x_{n_1})$  then the result of the composition is clearly in  $\text{Im}(H_{>0} \mathcal{B}_d^{(k_1)}(n_1+n_2-1)) \subset H_{>0} \mathcal{B}_d^{(k_1+k_2-2)}(n_1+n_2-1)$ . Since  $k_2$  is assumed to be greater than 2, by Lemma 4.1.1, the compositions of these forms are trivial.

Case  $II \circ_i II$ : Let  $\alpha(x_1, \dots, x_{n_1}) \in H_* \mathcal{B}_d^{(k_1)}(n_1)$  be an element of type II and let  $\beta(x_1, \dots, x_{n_2}) \in H_* \mathcal{B}_d^{(k_2)}(n_2)$  be another element of type II.

Compositions are as follows:

$$\alpha(x_1, \dots, x_{n_1}) \circ_i \beta(x_1, \dots, x_{n_2}) = \alpha(x_1, \dots, x_{i-1}, \beta(x_i, \dots, x_{i+n_2-1}), x_{i+n_2}, \dots, x_{n_1+n_2-1}).$$

Since  $i$  is not inside the brace, the result of the composition is in  $\text{Im}(H_{>0} \mathcal{B}_d^{(k')}(n_1+n_2-1))$ , where  $k' = \max(k_1, k_2)$ . Then  $\text{Im}(H_{>0} \mathcal{B}_d^{(k')}(n_1+n_2-1)) \subset H_{>0} \mathcal{B}_d^{(k_1+k_2-2)}(n_1+n_2-1)$ . Since  $k_1$  and  $k_2$  are assumed to be greater than 2, by Lemma 4.1.1, any composition of this form is trivial.

Cases  $III \circ_i II$  and  $III \circ_i III$ : Let  $\alpha(x_1, \dots, x_{n_1}) \in H_* \mathcal{B}_d^{(k_1)}(n_1)$  be an element of type

III and let  $\beta(x_1, \dots, x_{n_2}) \in H_*\mathcal{B}_d^{(k_2)}(n_2)$  be an element of type II or III. Compositions are as follows:

$$\alpha(x_1, \dots, x_{n_1}) \circ_i \beta(x_1, \dots, x_{n_2}) = \alpha(x_1, \dots, x_{i-1}, \beta(x_i, \dots, x_{i+n_2-1}), \dots, x_{n_1+n_2-1}).$$

Since  $\alpha(x_1, \dots, x_{n_1})$  is of type III, it has at least two braces. Therefore for any  $i$ , there is at least one brace of  $\alpha(x_1, \dots, x_{n_1})$  that is unaffected by the composition. That brace is of length  $k_1$  and is the boundary of the chain  $c(x_1, \dots, x_{k_1})$ . Therefore the composition is trivial.

**Example :** For  $k_1 = 3$ ,  $k_2 = 4$ ,  $n_1 = 6$ ,  $n_2 = 4$ , we get a map  $\circ_2 : \mathcal{B}_d^{(3)}(6) \times \mathcal{B}_d^{(4)}(4) \rightarrow \mathcal{B}_d^{(5)}(9)$ . Let  $\alpha = [\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}]$  and  $\beta = \{x_1, x_2, x_3, x_4\}$ . The composite cycle  $\alpha \circ_2 \beta$  can be realized as a product of spheres  $S^{2d-1} \times S^{3d-1} \times S^{d-1} \times S^{2d-1}$ , which is a boundary  $\partial(S^{2d-1} \times S^{3d-1} \times S^{d-1} \times D^{2d})$ :

$$\begin{aligned} [\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}] \circ_2 \{x_1, x_2, x_3, x_4\} &= [\{x_1, \{x_2, x_3, x_4, x_5\}, x_6\}, \{x_7, x_8, x_9\}] \\ &= \partial[\{x_1, \{x_2, x_3, x_4, x_5\}, x_6\}, c(x_7, x_8, x_9)] \\ &= 0. \end{aligned}$$

For the inner brace,  $\{x_2, x_3, x_4, x_5\}$ , any three of the discs 2, 3, 4, and 5 can overlap. If we replace  $\{x_2, x_3, x_4, x_5\}$  with  $Y$  in the brace  $\{x_1, \{x_2, x_3, x_4, x_5\}, x_6\}$ , we have  $\{x_1, Y, x_6\}$  and any two of the discs 1,  $Y$ , and 6 can overlap. Therefore if  $Y$  overlaps with either 1 or 6 there is at most four discs overlapping, so the result is indeed in  $\mathcal{B}_d^{(5)}(9)$ . However since  $\{x_7, x_8, x_9\}$  bounds a disc, as seen above, the result of the composition is zero in  $\mathcal{B}_d^{(5)}(9)$ .

The remaining case  $II \circ_i III$  is the most difficult and we will need the following lemma:

**Lemma 4.2.2.** *Let  $\{x_1, \dots, x_{k_1}\} \in H_{(k_1-1)d-1}\mathcal{B}_d^{(k_1)}(k_1)$  and let  $\beta(x_1, \dots, x_{n_2}) \in H_*\mathcal{B}_d^{(k_2)}(n_2)$  be an element of type III.*

*We can either write  $\beta = Y \cdot Z$  or  $\beta = [Y, Z]$  for some  $Y$  and  $Z$ . When  $\beta = Y \cdot Z$ , then*

$$\{x_1, \dots, x_{k_1-1}, (Y \cdot Z)\} = (-1)^{|Y|((k_1-1)d-1)} Y \cdot \{x_1, \dots, x_{k_1-1}, Z\} + \{x_1, \dots, x_{k_1-1}, Y\} \cdot Z, \quad (4.2.3)$$

and when  $\beta = [Y, Z]$ , then

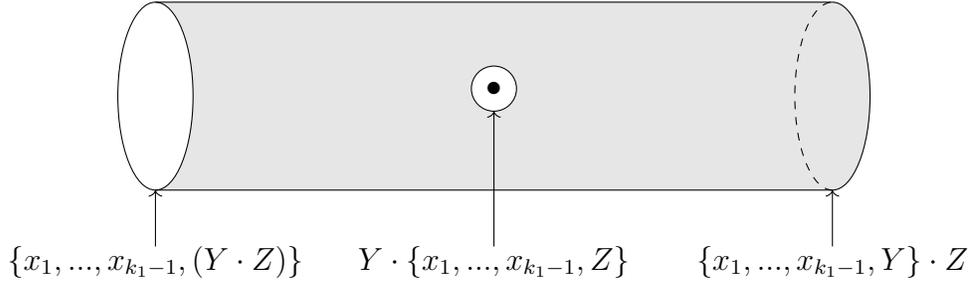
$$\{x_1, \dots, x_{k_1-1}, [Y, Z]\} = (-1)^{(|Y|+d-1)((k_1-1)d-1)}[Y, \{x_1, \dots, x_{k_1-1}, Z\}] + [\{x_1, \dots, x_{k_1-1}, Y\}, Z]. \quad (4.2.4)$$

*Proof.* The formulas (4.2.3) and (4.2.4) do not follow from (2.2.5) and (2.2.6), but are proved by a similar argument as in [12, Examples 5.2 and 5.3], which was inspired from [3]. Consider the cycle  $\{x_1, \dots, x_{k_1-1}, (Y \cdot Z)\}$ . When we pull  $Z$  far away, it forms a chain, which might have a forbidden  $(k_1 + k_2 - 1)$ -overlap. This could only happen near the plane

$$x_1 = \dots = x_{k_1-1} = Z,$$

where abusing notation,  $Z$  denotes the center of mass of points appearing in the chain  $Z$ . We remove a small tubular neighborhood of this forbidden plane and this produces the cycle  $Y \cdot \{x_1, \dots, x_{k_1-1}, Z\}$ . On the other hand when  $Z$  is far away we get the cycle  $\{x_1, \dots, x_{k_1-1}, Y\} \cdot Z$ . This proves the relation (4.2.3).

Geometrically we can see the chain  $C$  below.



Now consider the cycle  $\{x_1, \dots, x_{k_1-1}, [Y, Z]\}$ . We pull  $[Y, Z]$  together far away. This produces a chain that intersects forbidden strata. Notice  $Y$  and  $Z$  rotate around one another and thus never meet. This chain meets the plane

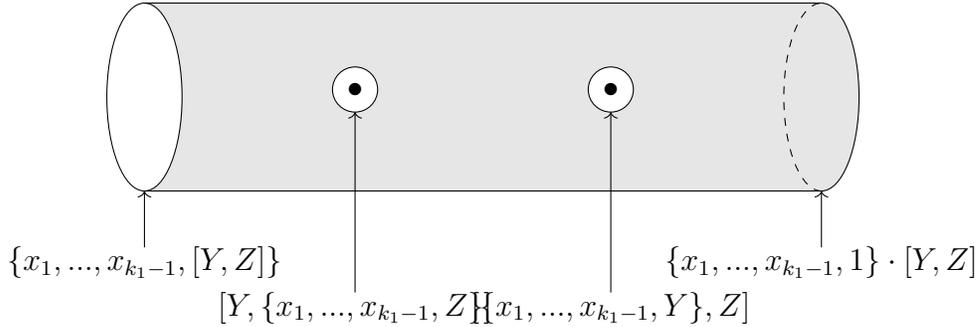
$$x_1 = x_2 = \dots = x_{k_1-1} = Z.$$

By removing a tubular neighborhood of this intersection with the chain we get the cycle

$\{x_1, \dots, x_{k_1-1}, Z\}$  at the boundary near every point of intersection. Simultaneously this cycle rotates around  $Y$  since  $x_1, \dots, x_{k_1-1}$  have collided with  $Z$ . Hence we have  $[Y, \{x_1, \dots, x_{k_1-1}, Z\}]$  as part of the boundary of our chain. Similarly the intersection with the plane

$$x_1 = x_2 = \dots = x_{k_1-1} = Y$$

produces the cycle  $[\{x_1, \dots, x_{k_1-1}, Y\}, Z]$ . On the other end of the cylinder, the boundary is given by  $\{x_1, \dots, x_{k_1-1}, 1\} \cdot [Y, Z]$  and is 0 in the homology as  $\{x_1, \dots, x_{k_1-1}, 1\} = 0$  in  $H_{>0} \mathcal{B}_d^{(k_1)}(k_1 - 1) = 0$ . There are no other forbidden planes to contribute and thus we obtain (4.2.4). Geometrically we can see the chain  $C$  below.



□

*Case II  $\circ_i$  III:* Let  $\alpha(x_1, \dots, x_{n_1}) \in H_* \mathcal{B}_d^{(k_1)}(n_1)$  be an element of type II and let  $\beta(x_1, \dots, x_{n_2}) \in H_* \mathcal{B}_d^{(k_2)}(n_2)$  be an element of type III. Compositions are as follows:

$$\alpha(x_1, \dots, x_{n_1}) \circ_i \beta(x_1, \dots, x_{n_2}) = \alpha(x_1, \dots, x_{i-1}, \beta(x_i, \dots, x_{i+n_2-1}), x_{i+n_2}, \dots, x_{n_1+n_2-1})$$

If  $i$  is not in the brace of  $\alpha(x_1, \dots, x_{n_1})$  then the brace of  $\alpha(x_1, \dots, x_{n_1})$  bounds a disc, that is,  $\{x_{i_1}, \dots, x_{i_{k_1}}\} = \partial c(x_{i_1}, \dots, x_{i_{k_1}})$ , similar to the example above.

If  $i$  is in the brace of  $\alpha(x_1, \dots, x_{n_1})$  then we can apply Lemma 4.2.2. After iteratively applying (4.2.3) and (4.2.4), although there can be braces inside of braces, each summand in the result must also have  $\{x_{i_1}, \dots, x_{i_{k_2}}\}$  outside of the brace. Indeed, since  $\beta(x_1, \dots, x_{n_2})$  is of type III, and thus has at least two braces. Then  $\{x_{i_1}, \dots, x_{i_{k_2}}\} = \partial c(x_{i_1}, \dots, x_{i_{k_2}})$ . Since the brace  $\{x_{i_1}, \dots, x_{i_{k_2}}\}$ , is the boundary of a disc, composition is trivial. □

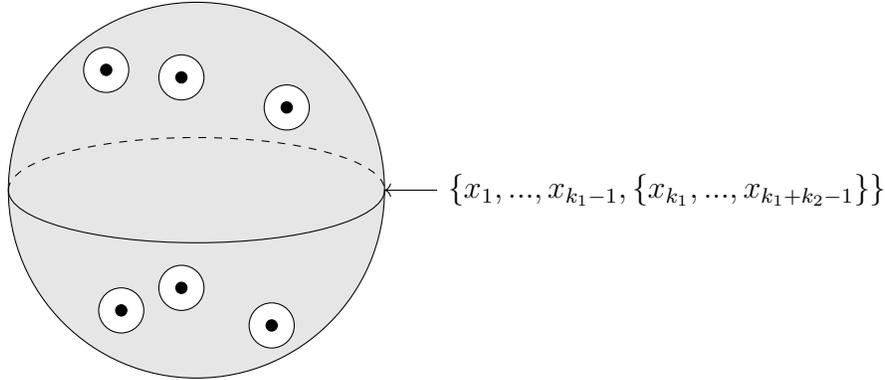
**Theorem 4.2.3.** *The composition of braces inside of other braces has a new relation:*

$$\{x_1, \dots, x_{k_1-1}, \{x_{k_1}, \dots, x_{k_1+k_2-1}\}\} = -(-1)^{(k_1-1)d} \sum_{i=1}^{k_1-1} (-1)^{(i-1)d} [x_i, \{x_1, \dots, \hat{x}_i, \dots, x_{k_1+k_2-1}\}] \quad (4.2.5)$$

$$= (-1)^{(k_1-1)d} \sum_{i=k_1}^{k_1+k_2-1} (-1)^{(i-1)d} [x_i, \{x_1, \dots, \hat{x}_i, \dots, x_{k_1+k_2-1}\}] \quad (4.2.6)$$

Note that the difference of (4.2.5) and (4.2.6) is exactly the generalized Jacobi relation (2.2.4).

*Proof.* We can see the above relation geometrically as follows:



Consider the intersection of the space  $\mathcal{M}_d^{(k_1+k_2-2)}(k_1+k_2-1)$  with the sphere given by  $\sum_{i=1}^{k_1+k_2-1} x_i = 0$  and  $\sum_{i=1}^{k_1+k_2-1} x_i^2 = 1$ . This space is homotopy equivalent to  $\mathcal{M}_d^{(k_1+k_2-2)}(k_1+k_2-1)$ , where translations and rescaling have been killed. The obtained space is  $S^{(k_1+k_2-2)d-1}$  with several subspheres removed. Each removed subsphere is given by the intersection of  $S^{(k_1+k_2-2)d-1}$  with the plane  $x_1 = \dots = \hat{x}_i = \dots = x_{k_1+k_2-1} = 0$  for  $1 \leq i \leq k_1+k_2-1$ . All the removed subspheres are disjoint. We can take tubular neighborhoods around each subsphere, which are also all disjoint. Each tubular neighborhood has boundary which is exactly the cycle  $[x_i, \{x_1, \dots, \hat{x}_i, \dots, x_{k_1+k_2-1}\}]$ . After removing these tubular neighborhoods we are left with a manifold with boundary that we denote  $X_d^{(k_1+k_2-2)}(k_1+k_2-1)$ . The boundary of

$X_d^{(k_1+k_2-2)}(k_1+k_2-1)$  is exactly the generalized Jacobi relation (2.2.4) for  $k = k_1 + k_2 - 2$ . Then  $\{x_1, \dots, x_{k_1-1}, \{x_{k_1}, \dots, x_{k_1+k_2-1}\}\}$  is a submanifold of  $X_d^{(k_1+k_2-2)}(k_1+k_2-1)$  that is of codimension 1. Thus  $\{x_1, \dots, x_{k_1-1}, \{x_{k_1}, \dots, x_{k_1+k_2-1}\}\}$  splits  $X_d^{(k_1+k_2-2)}(k_1+k_2-1)$  into two parts where one part is given by the right hand side of (4.2.5) and the other is given by (4.2.6).

The cycle  $\{x_1, \dots, x_{k_1-1}, \{x_{k_1}, \dots, x_{k_1+k_2-1}\}\}$  can be realized as a product of spheres:  $S^{(k_1-1)d-1} \times S^{(k_2-1)d-1}$ . We can describe the first sphere  $S^{(k_1-1)d-1}$  by the following equations:

$$x_1 + \dots + x_{k_1-1} + Y = 0 \quad x_1^2 + \dots + x_{k_1-1}^2 + Y^2 = c^2 \cdot k_1(k_1 - 1) \quad (4.2.7)$$

where  $Y = \frac{1}{k_1-1}(x_{k_1+1} + \dots + x_{k_1+k_2-1})$ .

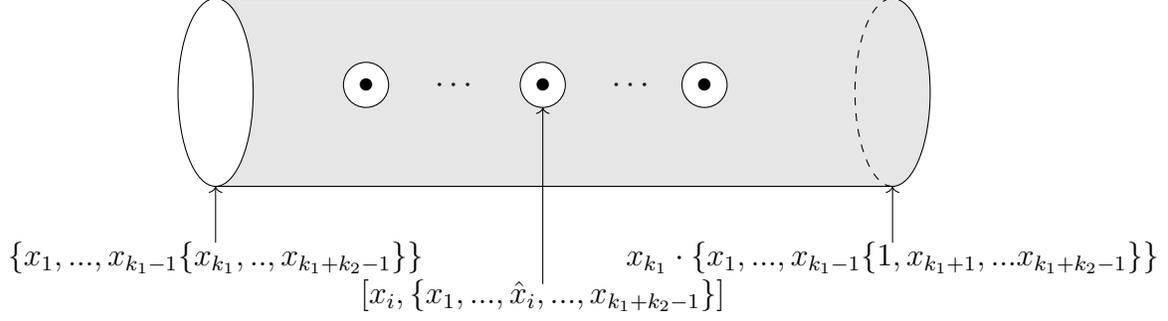
Define  $\bar{x}_{k_1} = x_{k_1} - Y$ ,  $\bar{x}_{k_1+1} = x_{k_1+1} - Y, \dots, \bar{x}_{k_1+k_2-1} = x_{k_1+k_2-1} - Y$ . Then we can describe the second sphere by the following equations:

$$\bar{x}_{k_1} + \dots + \bar{x}_{k_1+k_2-1} = 0 \quad \bar{x}_{k_1}^2 + \dots + \bar{x}_{k_1+k_2-1}^2 = \epsilon^2 \cdot k_2(k_2 - 1), \quad \epsilon^2 \ll c^2. \quad (4.2.8)$$

Next we define a chain one dimension bigger by pulling only one point,  $x_{k_1}$ , in the direction of  $(1, 0, \dots, 0)$ . As we pull  $x_{k_1}$  it can collide with  $x_{k_1+1}, \dots, x_{k_1+k_2-1}$  only when it intersects with  $(k_1 - 1)$  forbidden strata each given by the following set of equations:

$$x_1 = \dots = \hat{x}_i = \dots = x_{k_1+k_2-1}, \quad 1 \leq i \leq k_1 - 1. \quad (4.2.9)$$

Note that  $x_j = x_i$  if and only if  $\bar{x}_j = \bar{x}_i$ . The obtained chain is a cylinder  $S^{(k_1-1)d-1} \times S^{(k_2-1)d-1} \times [0, N]$ , where  $N \gg 0$ . This cylinder intersects the  $(k_1 - 1)$ -forbidden strata (4.2.9) transversely and disjointly.



To get an actual chain in  $\mathcal{M}_d^{(k_1+k_2-2)}(k_1+k_2-1)$ , we remove disjoint tubular neighborhoods of each intersection with the forbidden strata.

As an example, consider the intersection with the stratum  $x_2 = \dots = x_{k_1+k_2-1}$ . It happens when the initial position of  $\bar{x}_{k_1} = (-(k_1-1)\epsilon, 0, \dots, 0)$  and the position of  $\bar{x}_j = (\epsilon, 0, \dots, 0)$  for  $j = k_1+1, \dots, k_1+k_2-1$ . Indeed, when  $x_{k_1}$  is pulled in the direction of  $(1, 0, \dots, 0)$ , the same happens with  $\bar{x}_{k_1}$  and it hits all the other  $\bar{x}_j$ ,  $j = k_1+1, \dots, k_1+k_2-1$  only if  $\bar{x}_{k_1+1} = \dots = \bar{x}_{k_1+k_2-1}$  and are in the position  $(\epsilon, 0, \dots, 0)$ . Now since  $Y = \frac{1}{k_1-1}(x_{k_1+1} + \dots + x_{k_1+k_2-1})$  and all  $x_j$ ,  $j = k_1+1, \dots, k_1+k_2-1$  are equal, we get  $Y = x_j$ ,  $j = k_1+1, \dots, k_1+k_2-1$ . We also need  $Y$  to coincide with  $x_2, \dots, x_{k_1-1}$ , this reduces (4.2.7) to the following equations:

$$x_1 + (k_1 - 1)Y = 0 \tag{4.2.10}$$

$$x_1^2 + (k_1 - 1)Y^2 = c^2 k_1 (k_1 - 1). \tag{4.2.11}$$

The first equation (4.2.10) kills translations and the second equation (4.2.11) kills rescaling. Thus we get a sphere  $S^{d-1}$ , which corresponds to  $[x_1, Y]$ . However since  $Y$  has collided with  $x_2, \dots, x_{k_1+k_2-1}$ , we remove from the attained chain a tubular neighborhood of its intersection with the forbidden stratum  $x_2 = \dots = x_{k_1+k_2-1}$ . This gives us exactly the cycle  $[x_1, \{x_2, \dots, x_{k_1+k_2-1}\}]$ .

Therefore, in general, if  $Y$  has collided with  $x_1, \dots, \hat{x}_i, \dots, x_{k_1+k_2-1}$ , for  $i = i, \dots, k_1-1$ , then

we get similarly

$$\begin{aligned}x_i + (k_1 - 1)Y &= 0 \\x_i^2 + (k_1 - 1)Y^2 &= c^2 k_1 (k_1 - 1).\end{aligned}$$

This gives a sphere  $S^{d-1}$  corresponding to  $[x_i, Y]$ . We remove from the cylinder  $S^{(k_1-1)d-1} \times S^{(k_2-1)d-1} \times [0, N]$  a tubular neighborhood of its intersection with the forbidden strata  $x_1 = \cdots = \hat{x}_i = \cdots = x_{k_1+k_2-1}$ , which yields the boundary cycle  $[x_i, \{x_1, \dots, \hat{x}_i, \dots, x_{k_1+k_2-1}\}]$ .

At the right end of the cylinder,  $S^{(k_1-1)d-1} \times S^{(k_2-1)d-1} \times [0, N]$ , we get a cycle  $x_{k_1} \cdot \{x_1, \dots, x_{k_1-1} \{1, x_{k_1+1}, \dots, x_{k_1+k_2-1}\}\}$  which is homologically trivial since  $\{1, x_1, \dots, x_{k_2-1}\} = 0 \in H_{\geq 0} \mathcal{M}_d^{(k_2)}(k_2-1) = 0$ . All together, this gives us the relation (4.2.5). Note that (4.2.6) minus (4.2.5) is the generalized Jacobi (2.2.4) and therefore (4.2.6) is a consequence of (4.2.5).

We explain how the sign in front of the sum (4.2.6) is found in the next section. It is enough to understand the sign in front of only one of the summands in (4.2.6) (We do it for the very last one). This is due to the argument made at the beginning of the proof.

□

### 4.3 Signs in Theorem 4.2.3

In [12, Section 6], the authors describe the cohomology groups  $H^* \mathcal{M}_d^{(k)}(n)$  as spaces of certain admissible  $k$ -forests, where the  $k$ -forests have two types of vertices square and round. If a  $k$ -forest has only one component, we call it a  $k$ -tree. Every square vertex contains a  $(k-1)$ -elements subset of  $\{1, \dots, n\}$  and every round vertex contains just one element. Each round vertex must be connected by an edge to a single square vertex, or completely disconnected from all other vertices. Square vertices must be connected to at least one round vertex. One orients all the given edges between the vertices. Each  $k$ -forest has an orientation set, which consists of all the edges and square vertices. The order of the orientation set encodes the coorientations of the corresponding chains. The degree of a square vertex is  $(k-2)d$  and

the degree of an edge is  $d - 1$ . The cocycles in  $H^* \mathcal{M}_d^{(k)}(n)$  corresponding to the  $k$ -forests are geometrically realized as an intersection number with cooriented chains in  $\mathcal{M}_d^{(k)}(n)$ , which are defined by a set of (in)equalities as follows. If  $i$  and  $j$  are in the same square vertex, then  $x_i = x_j$ . The authors give a projection  $p_1 : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$  where  $(x^1, \dots, x^d) \mapsto (x^2, \dots, x^d)$ . For two vertices  $A$  and  $B$  in a forest that are connected by an edge oriented from  $A$  to  $B$ , they require that  $B$  is “above”  $A$ . Explicitly, for all  $i \in A$  and all  $j \in B$ ,  $x_i^1 \leq x_j^1$  and  $p_1(x_i) = p_1(x_j)$ .

In [12, Section 8], the authors define a map  $\Psi$ , which describes the intersection between cycles given as products of iterated brackets (and geometrically realized as products of spheres) with the  $k$ -forests cocycles.

As an example,

$$\Psi(\{x_1, \dots, x_k\}) = \sum_{\ell=1}^k (-1)^{(\ell-1)d} \begin{array}{c} 1 \boxed{1, \dots, \hat{\ell}, \dots, k} \\ \downarrow 2 \\ \textcircled{\ell} \end{array} \quad (4.3.1)$$

The formula (4.3.1) means that the intersection of the cycle  $\{x_1, \dots, x_k\}$  and the cocycle  $1 \boxed{1, \dots, \hat{\ell}, \dots, k} \downarrow 2 \textcircled{\ell}$  is  $(-1)^{(\ell-1)d}$ , i.e.  $\{x_1, \dots, x_k\} \cap 1 \boxed{1, \dots, \hat{\ell}, \dots, k} \downarrow 2 \textcircled{\ell} = (-1)^{(\ell-1)d}$ . For this, one needs that the sphere  $\{x_1, \dots, x_k\}$  given by the equations (2.2.7) should be oriented as follows. One projects this sphere to  $(x_1, \dots, x_{k-1})$ . We get an ellipsoid whose orientation is such that the outside normal vector taken as a first one, union the oriented tangent frame gives  $(-1)^{kd}$  times the standard orientation of  $\mathbb{R}^{(k-1)d}$ .<sup>1</sup>

Now to determine the sign in front of (4.2.6) in Theorem 4.2.3, we consider the intersection

of the cocycle  $1 \boxed{1, \dots, \widehat{k_1 - 1}, \dots, k_1 + k_2 - 2} \downarrow 2 \textcircled{k_1 - 1} \downarrow 3 \textcircled{k_1 + k_2 - 1}$  with  $\{x_1, \dots, x_{k_1-1}, \{x_{k_1}, \dots, x_{k_1+k_2-1}\}\}$  and with the right hand side of (4.2.6). The corresponding cochain intersects only the last summand

<sup>1</sup>We do not actually need it, but it is worth mentioning as in the original paper [12], the orientation of  $\{x_1, \dots, x_k\}$  has not been determined. It was just said that the orientation of  $\{x_1, \dots, x_k\}$  is such that the pairing (4.3.1) works, see [12, footnote 3].

of (4.2.6),  $[x_{k_1+k_2-1}, \{x_1, \dots, x_{k_1+k_2-2}\}]$ . The latter intersection is obtained by computing

$$\Psi([x_{k_1+k_2-1}, \{x_1, \dots, x_{k_1+k_2-2}\}]) = (-1)^{(k_1-2)d} \begin{array}{c} \text{---} \textcircled{k_1+k_2-1} \text{---} \xrightarrow{1} \boxed{1, \dots, \widehat{k_1-1}, \dots, k_1+k_2-2} \text{---} \textcircled{k_1-1} \text{---} \\ \downarrow 3 \\ \textcircled{k_1-1} \end{array} + \dots$$

See Section 8 in [12], in particular Example 8.1(a). The other summands are  $(k_1+k_2-2)$ -trees of different shapes (and thus do not contribute). The sign in front comes from (4.3.1). To get the desired intersection, first we reverse the arrow between the square vertex and the round vertex labeled  $k_1+k_2-1$ . This gives the sign  $(-1)^d$ . After reversing the arrow, we have the following  $(k_1+k_2-2)$ -tree:

$$(-1)^{(k_1-1)d} \begin{array}{c} \boxed{1, \dots, \widehat{k_1-1}, \dots, k_1+k_2-2} \\ \downarrow 1 \quad \downarrow 3 \\ \textcircled{k_1+k_2-1} \quad \textcircled{k_1-1} \end{array} .$$

Next, we change the order of the elements 1, 2 and 3 in the orientation set by pulling 2 in front and pushing 1 to the end. The degree of 1 and 3 is  $d-1$ . The degree of 2 is  $(k_1+k_2-4)d$ . So the obtained sign from reordering the orientation set is  $(-1)^{(d-1)(k_1+k_2-4)d} \times (-1)^{(d-1)(d-1)} = (-1)^{d-1}$ . After this change, we now have the  $(k_1+k_2-2)$ -tree that we want:

$$(-1)^{k_1 d-1} \begin{array}{c} \boxed{1, \dots, \widehat{k_1-1}, \dots, k_1+k_2-2} \\ \downarrow 2 \quad \downarrow 3 \\ \textcircled{k_1-1} \quad \textcircled{k_1+k_2-1} \end{array} .$$

Finally, we computed the intersection

$$[x_{k_1+k_2-1}, \{x_1, \dots, x_{k_1+k_2-2}\}] \cap \begin{array}{c} \boxed{1, \dots, \widehat{k_1-1}, \dots, k_1+k_2-2} \\ \downarrow 2 \quad \downarrow 3 \\ \textcircled{k_1-1} \quad \textcircled{k_1+k_2-1} \end{array} = (-1)^{k_1 d-1}. \quad (4.3.2)$$

The sign in front of the last summand of (4.2.6) is  $(-1)^{(k_1-1)d} \times (-1)^{(k_1+k_2-2)d} = (-1)^{(k_2-1)d}$ . In conclusion, we obtained that the intersection with the right hand side of (4.2.6) is

$$(-1)^{k_1 d-1} \times (-1)^{(k_2-1)d} = (-1)^{(k_1+k_2-1)d-1}. \quad (4.3.3)$$

Next we want to check that this is the same sign that we get on the left hand side for  $\{x_1, \dots, x_{k_1-1}, \{x_{k_1}, \dots, x_{k_1+k_2-1}\}\}$ . This cycle is the product of two spheres  $S^{(k_1-1)d} \times S^{(k_2-1)d-1}$ .

The chain  $1 \begin{array}{c} \boxed{1, \dots, \widehat{k_1-1}, \dots, k_1+k_2-2} \\ \downarrow 2 \quad \downarrow 3 \\ \textcircled{k_1-1} \quad \textcircled{k_1+k_2-1} \end{array}$  is the transverse intersection of the following two chains:  $1 \begin{array}{c} \boxed{1, \dots, k_1-2, k_1} \\ \downarrow 2 \\ \textcircled{k_1-1} \end{array}$  and  $1 \begin{array}{c} \boxed{k_1, \dots, k_1+k_2-2} \\ \downarrow 2 \\ \textcircled{k_1+k_2-1} \end{array}$ . The coorientation of

$1 \begin{array}{c} \boxed{1, \dots, \widehat{k_1-1}, \dots, k_1+k_2-2} \\ \downarrow 2 \quad \downarrow 3 \\ \textcircled{k_1-1} \quad \textcircled{k_1+k_2-1} \end{array}$  is equivalent to the coorientation obtained by concatenating the coorientations of the chains corresponding to the  $k_1$ -tree

$1 \begin{array}{c} \boxed{1, \dots, k_1-2, k_1} \\ \downarrow 2 \\ \textcircled{k_1-1} \end{array}$  and the  $k_2$ -tree  $1 \begin{array}{c} \boxed{k_1, \dots, k_1+k_2-2} \\ \downarrow 2 \\ \textcircled{k_1+k_2-1} \end{array}$ . Indeed, the difference in sign is obtained by pulling

the square vertex  $\boxed{k_1, \dots, k_1+k_2-2}$  through the edge of  $1 \begin{array}{c} \boxed{1, \dots, k_1-2, k_1} \\ \downarrow 2 \\ \textcircled{k_1-1} \end{array}$ . This pulling does not affect the sign since the degree of  $\boxed{k_1, \dots, k_1+k_2-2}$  is a multiple of  $d$  and the degree of the edge is  $d-1$ . By (4.3.1), the intersection of  $S^{(k_1-1)d-1}$  (the first factor of  $\{x_1, \dots, x_{k_1-1}, \{x_{k_1}, \dots, x_{k_1+k_2-1}\}\}$ ) with

$1 \begin{array}{c} \boxed{1, \dots, k_1-2, k_1} \\ \downarrow 2 \\ \textcircled{k_1-1} \end{array}$  is  $(-1)^{(k_1-1-1)d-1} = (-1)^{k_1 d-1}$ . Similarly by (4.3.1), the intersection of  $S^{(k_2-1)d-1}$  (the second factor of

$1 \begin{array}{c} \boxed{1, \dots, k_1-2, k_1} \\ \downarrow 2 \\ \textcircled{k_1-1} \end{array}$  is  $(-1)^{(k_1-1-1)d-1} = (-1)^{k_1 d-1}$ . Similarly by (4.3.1), the intersection of  $S^{(k_2-1)d-1}$  (the second factor of

$\{x_1, \dots, x_{k_1-1}, \{x_{k_1}, \dots, x_{k_1+k_2-1}\}\}$  and  $1 \begin{array}{c} \boxed{k_1, \dots, k_1 + k_2 - 2} \\ \downarrow 2 \\ \textcircled{k_1 + k_2 - 1} \end{array}$  is  $(-1)^{k_2 d - d - 1}$ . So the total sign is  $(-1)^{(k_1+k_2-1)d-1}$ , which is exactly the same as (4.3.3). Therefore, the sign  $(-1)^{(k_1-1)d}$  in front of (4.2.6) is correct.

# Chapter 5

## A Cosimplicial Model for

$$\overline{Imm}_\partial^{(k)}(D^1, D^n)$$

### 5.1 Spaces of Non- $k$ -Equal Immersions of Discs

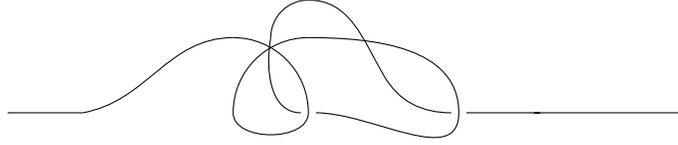
Let  $Imm_\partial^{(k)}(D^m, D^n)$  be the space of smooth immersions  $D^m \hookrightarrow D^n$  of discs that are the standard inclusion in a neighborhood of the boundary and satisfy the condition that the image of any  $k$ -element subset has more than one point. Such spaces are called non- $k$ -equal immersions. The bimodules of non- $k$ -overlapping discs naturally appear in the study of these spaces [12, Section 11]. Note that  $Emb_\partial(D^m, D^n) = Imm_\partial^{(2)}(D^m, D^n)$  is the space of smooth disc embeddings. We also consider the homotopy fiber space over the standard inclusion:  $D^m \subset D^n$

$$\overline{Imm}_\partial^{(k)}(D^m, D^n) = \text{hofib} \left( Imm_\partial^{(k)}(D^m, D^n) \rightarrow Imm_\partial(D^m, D^n) \right). \quad (5.1.1)$$

One has a natural sequence of inclusions:

$$\overline{Emb}_\partial(D^m, D^n) = \overline{Imm}_\partial^{(2)}(D^m, D^n) \subset \overline{Imm}_\partial^{(3)}(D^m, D^n) \subset \cdots \subset \overline{Imm}_\partial^{(\infty)}(D^m, D^n) \quad (5.1.2)$$

Note that  $\overline{Imm}_\partial^{(\infty)}(D^m, D^n) = \overline{Imm}_\partial(D^m, D^n) \simeq *$ .



The spaces  $\overline{Emb}_\partial(D^m, D^n)$ , often called “spaces of embeddings modulo immersions,” have been studied intensively, see [1] and references in within. Each space  $\overline{Imm}_\partial^{(k)}(D^m, D^n)$  is naturally a  $\mathcal{B}_m$ -algebra. Moreover  $\overline{Imm}_\partial^{(2)}(D^m, D^n) = \overline{Emb}_\partial(D^m, D^n)$  is a  $\mathcal{B}_{m+1}$ -algebra [8, 9, 29]. One dimension higher little discs action comes from the fact that embeddings can be pulled through each other. This means, in particular, there is a Browder operation:

$$[\cdot, \cdot] : H_i \overline{Emb}_\partial(D^m, D^n) \times H_j \overline{Emb}_\partial(D^m, D^n) \rightarrow H_{i+j+m} \overline{Emb}_\partial(D^m, D^n). \quad (5.1.3)$$

Exactly the same construction (the idea of pulling immersions through immersions) endows (5.1.2) with a structure of a *filtered  $\mathcal{B}_{m+1}$  – algebra*. Then again, one has a Browder operation:

$$[\cdot, \cdot] : H_i \overline{Imm}_\partial^{(k_1)}(D^m, D^n) \times H_j \overline{Imm}_\partial^{(k_2)}(D^m, D^n) \rightarrow H_{i+j+m} \overline{Imm}_\partial^{(k_1+k_2-2)}(D^m, D^n). \quad (5.1.4)$$

We believe that the computations done in chapter 4 will be helpful in understanding these Browder operations.

## 5.2 Goodwillie-Taylor Tower

Goodwillie and Weiss, in [21, 23], developed a powerful method to study spaces of embeddings, which is called Goodwillie-Weiss manifold calculus of functors. This method provides spaces  $T_\infty \overline{Imm}_\partial^{(k)}(D^m, D^n)$ , which are conjectured to be equivalent to their corresponding spaces  $\overline{Imm}_\partial^{(k)}(D^m, D^n)$  for  $n - m \geq 2$  and  $k \geq 3$ . One says that  $T_\infty \overline{Imm}_\partial^{(k)}(D^m, D^n)$  is the limit of the Goodwillie-Weiss Taylor tower associated to  $\overline{Imm}_\partial^{(k)}(D^m, D^n)$ . For the spaces

of embeddings, this convergence of the Goodwillie-Weiss Taylor tower has been proven to take place for codimension  $n - m \geq 3$ , [22]:

$$T_\infty \overline{Emb}_\partial(D^m, D^n) \simeq \overline{Emb}_\partial(D^m, D^n). \quad (5.2.1)$$

The advantage of  $T_\infty \overline{Imm}_\partial^{(k)}(D^m, D^n)$  is that it has a description in operadic terms [1]. It has been conjectured that for  $n - m \geq 0$  and  $k \geq 2$ :

$$T_\infty \overline{Imm}_\partial^{(k)}(D^m, D^n) \simeq \mathcal{I}bimod_{\mathcal{B}_m}^h(\mathcal{B}_m, \mathcal{B}_n^{(k)}), \quad (5.2.2)$$

where  $\mathcal{I}bimod_{\mathcal{B}_m}^h(\mathcal{B}_m, \mathcal{B}_n^{(k)})$ , is the derived mapping space of infinitesimal bimodules over the operad  $\mathcal{B}_m$ .

The goal of this chapter is to use this operadic approach to obtain a cosimplicial model for the space  $T_\infty \overline{Imm}_\partial^{(k)}(D^1, D^n)$  corresponding to 1-dimensional immersions. Thanks to the natural projection,

$$\pi_0 : \mathcal{B}_1 \rightarrow \mathcal{A}ssoc, \quad (5.2.3)$$

which is an equivalence of operads, we get an adjunction:

$$\pi_0^\dagger : \mathcal{I}bimod_{\mathcal{B}_1} \rightleftarrows \mathcal{I}bimod_{\mathcal{A}ssoc} : \pi_0^*. \quad (5.2.4)$$

Both categories are endowed with the Reedy model structure[15], and moreover (5.2.4) is a Quillen equivalence by [15, Theorem 2.8]. We will describe the Reedy model structure in section 5.3. Then in section 5.4, we describe a cofibrant replacement  $Q_{\mathcal{B}_1}(\mathcal{B}_n^{(k)})$  of  $\mathcal{B}_n^{(k)}$  in  $\mathcal{I}bimod_{\mathcal{B}_1}$ , so then one has an equivalence of spaces:

$$\mathcal{I}bimod_{\mathcal{B}_1}^h(\mathcal{B}_1, \mathcal{B}_n^{(k)}) \simeq \mathcal{I}bimod_{\mathcal{A}ssoc}^h(\mathcal{A}ssoc, \pi_0^\dagger(Q_{\mathcal{B}_1}(\mathcal{B}_n^{(k)}))). \quad (5.2.5)$$

On the other hand, an infinitesimal bimodule over  $\mathcal{A}ssoc$ , when we forget the  $\Sigma$ -action, is nothing but a cosimplicial object and the right hand side is exactly the homotopy totalization

of this object. Therefore, we get

$$T_\infty \overline{Imm}_\partial^{(k)}(D^m, D^n) \simeq Tot^h \pi^!(Q_{\mathcal{B}_1}(\mathcal{B}_n^{(k)}))(\bullet) \quad (5.2.6)$$

In this chapter we give an explicit description of this cosimplicial space  $\pi^!(Q_{\mathcal{B}_1}(\mathcal{B}_n^{(k)}))(\bullet)$ .

### 5.3 Reedy Model Structure

We say an operad  $\mathcal{O}$  is reduced if  $\mathcal{O}(0) = *$ ; it is well-pointed if the unit map  $* \rightarrow \mathcal{O}(1)$  is a cofibration. All operads in the chapter are reduced, well-pointed and  $\Sigma$ -cofibrant. All infinitesimal bimodules  $M$  in this chapter will be  $\Sigma$ -cofibrant. Let  $\mathcal{I}bimod_{\mathcal{O}}$  be the category of infinitesimal bimodules over an operad  $\mathcal{O}$ . We can describe  $\mathcal{I}bimod_{\mathcal{O}}$  as a diagram category  $Top^{\tilde{\Gamma}(\mathcal{O})}$ , where  $\tilde{\Gamma}(\mathcal{O})$  is a topologically enriched category assigned to  $\mathcal{O}$  [1, 15] and  $Top$  is the category of topological spaces. We will be using the Reedy model structure on  $Q_{\mathcal{O}}(M)$ , defined in [1] and further described in section 5.4.

Let  $\Lambda$  be the category with objects finite sets,  $\underline{n} = \{1, \dots, n\}$  for all  $n \geq 0$ , and morphisms injective maps between them. Note that for a map  $\alpha : \underline{k} \rightarrow \underline{n}$  to be a morphism in  $\Lambda$ ,  $k$  must be less than or equal to  $n$ . Then we can form the category of  $\Lambda$ -sequences (denoted  $\Lambda Seq$ ), whose objects are functors  $\Lambda^{op} \rightarrow Top$ . So then, we can say that  $\Lambda Seq = Top^{\Lambda^{op}}$ .

The category  $\Lambda Seq$  has a Reedy model structure [18]. It can be transferred to  $\mathcal{I}bimod_{\mathcal{O}}$  via the following adjunction:

$$\mathcal{F}_{Ib}^\Lambda : \Lambda Seq \rightleftarrows \mathcal{I}bimod_{\mathcal{O}} : \mathcal{U}^\Lambda, \quad (5.3.1)$$

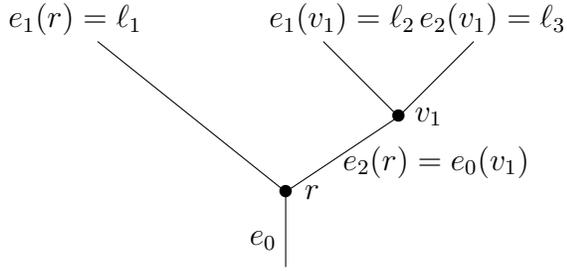
where  $\mathcal{U}^\Lambda$  is the forgetful functor and  $\mathcal{F}_{Ib}^\Lambda$  is its left adjoint. This means a map in  $\mathcal{I}bimod_{\mathcal{O}}$  is a weak equivalence (or fibration) if and only if it is a weak equivalence (or fibration) in  $\Lambda Seq$ . The obtained structure is called the Reedy model structure on  $\mathcal{I}bimod_{\mathcal{O}}$ .

## 5.4 Cofibrant Replacements

Now, we describe a cofibrant replacement  $Q_{\mathcal{O}}(M)$  for an  $\mathcal{O} - \mathcal{I}$ bimodule  $M$  over operad  $\mathcal{O}$ . We will need the language of trees for the construction of the Boardman-Vogt type resolution, which is a functorial way of constructing cofibrant replacements [14]. We mostly follow the notation of the previous reference.

- Let  $T$  be a planar tree that is oriented from top to bottom. There are input edges on top and only 1 output edge on the bottom.
- Denote the vertex connected to the output edge by  $r$ , which is the root of  $T$ . The output edge of the root vertex is called the root edge.
- Using the orientation of  $T$ , the vertex of an edge towards the root edge is the target vertex, denoted  $t(e)$  and the other vertex is the source vertex, denoted  $s(e)$ .
- Input edges have target vertices but not source vertices.
- The output edge has a source vertex but not a target vertex.
- Let  $V(T)$  denote the set of vertices and  $E(T)$  denote the set of edges.
- If an edge has both a source and a target vertex then it is an internal edge. The set of all internal edges is denoted  $E^{int}(T)$ .
- The input edges are also called leaves and are ordered left to right and  $|T|$  is the number of leaves.
- The ordered set of leaves are denoted  $in(T) := \{\ell_1, \dots, \ell_{|T|}\}$
- Let the number of incoming edges of a vertex be denoted  $|v|$  and referred to as arity.
- The set of incoming edges of a vertex, denoted  $in(v) := \{e_1(v), \dots, e_{|v|}(v)\}$ . This is also ordered left to right.
- The unique output edge of a vertex is denoted  $e_0(v)$ .

The following tree gives an example of the notation and vocabulary.



$$E(T) = \{e_0, e_2(r), \ell_1, \ell_2, \ell_3\}$$

$$E^{Int}(T) = \{e_2(r)\}$$

$$V(T) = \{r, v_1\}$$

$$in(T) = \{\ell_1, \ell_2, \ell_3, \}$$

$$in(v_1) = \{\ell_1, \ell_2\}$$

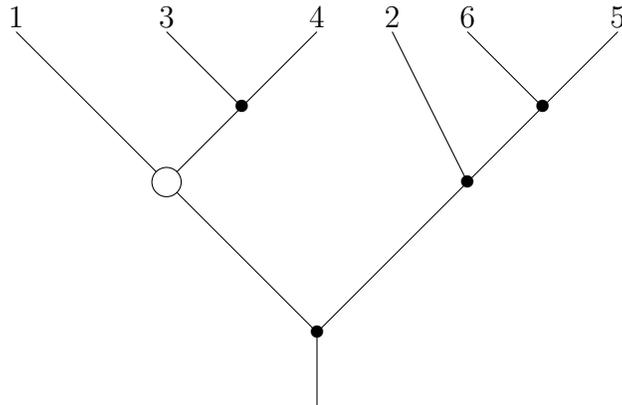
$$in(r) = \{e_1(r), e_2(r)\}$$

Let  $\sigma : \{1, \dots, |T|\} \rightarrow in(T)$  be a bijection labeling the leaves of  $T$  and is an element of  $\Sigma_{|T|}$ . Then a tree is a pair  $(T, \sigma)$ , where  $T$  is a planar tree. The set of trees with  $k$ -leaves is denoted by  $\mathbb{T}_k$ .

The planar trees described above can be used to describe cofibrant replacements for operads. However, we want to describe cofibrant replacements for infinitesimal bimodules over an operad. Instead, we will need *pearled trees*.

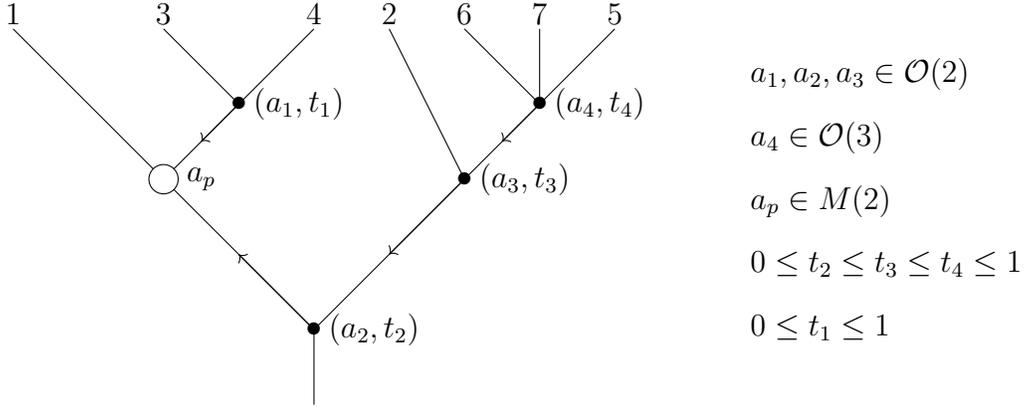
**Definition 5.4.1.** A *pearled tree* is a pair  $(T, p)$  where  $T \in \mathbb{T}_k$  and  $p$  is a unique vertex called the *pearl* and is represented as a white vertex in a tree. We denote the set of pearled trees with  $k$ -leaves and satisfies the following restrictions by  $\mathbb{PT}_k$ :

- The pearl can be of any arity.
- All internal vertices are of arity strictly greater than zero.



We construct a cofibrant replacement. This construction was first introduced in [15] and is given here for completion. Given an operad  $\mathcal{O}$  and an  $\mathcal{O}$ -bimodule  $M$ , the cofibrant replacement is denoted  $Q_{\mathcal{O}}(M)$ , where the points are pearled trees  $(T, p)$  from  $\mathbb{P}\mathbb{T}_k$  with the following data:

- All internal edges are oriented towards the pearl.
- The pearl  $p$  is labeled by an element of  $M$ .
- Every internal vertex is labeled by an element of the operad  $\mathcal{O}$ .
- Every internal vertex is indexed with a real number in the interval  $[0,1]$ .
- For an internal edge  $e$ ,  $s(e) \geq t(e)$ , that is, vertices closer to the pearl are labeled with numbers smaller than those farther away from the pearl.

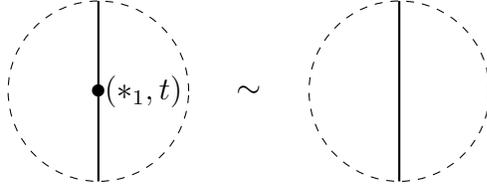


The space  $\mathcal{I}b_{\mathcal{O}}(M)$  is the quotient of the subspace of

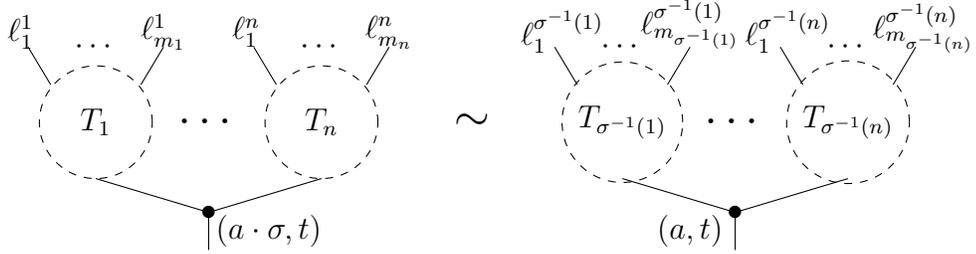
$$\prod_{T \in \mathbb{P}\mathbb{T}_k} \left( M(|p|) \times \prod_{v \in V(T) \setminus \{p\}} [\mathcal{O}(|v|) \times [0, 1]] \right) \quad (5.4.1)$$

where the equivalence relation is given by the following axioms:

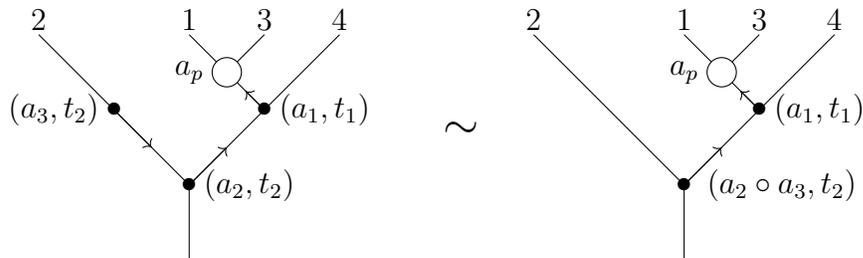
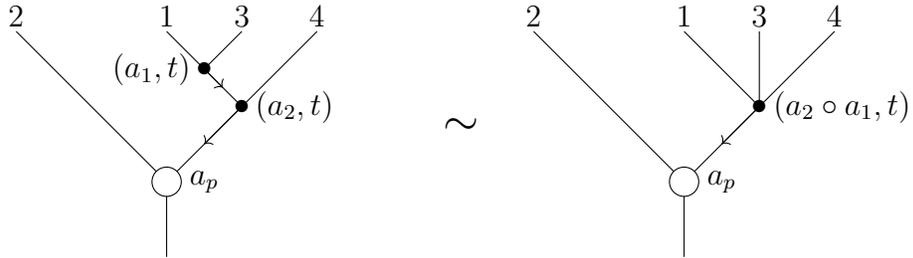
1. If a vertex is labeled by  $*_1 \in \mathcal{O}(1)$ , then one has the identification:



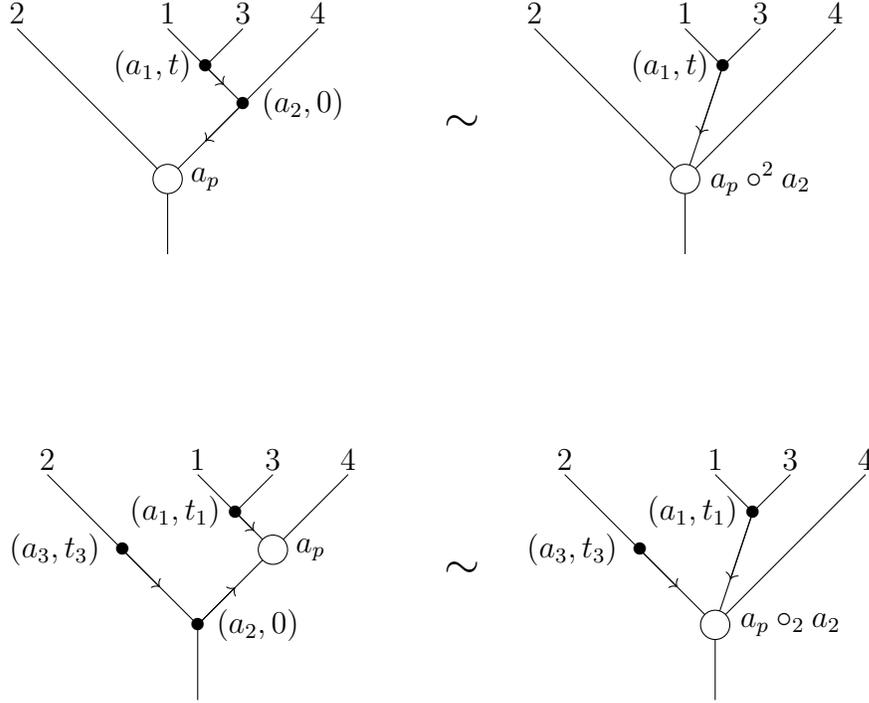
2. If a vertex is indexed by  $a \cdot \sigma$ , with  $\sigma \in \Sigma$ , then



3. For an internal edge  $e$  connecting two vertices (neither of which is the pearl), if the two vertices are indexed by the same real number  $t \in [0, 1]$ , then  $e$  is contracted via the operadic structure (recalled that vertices other than the pearl are labeled by points in the operad  $\mathcal{O}$ ). The obtained vertex after the contraction is then indexed by  $t$ .



4. For a vertex that is indexed by 0, then the output edge (by the orientation towards the pearl) is contracted via the infinitesimal bimodule structure and the obtained vertex becomes the pearl.



Let us now describe the composition structure of  $\mathcal{Ib}_{\mathcal{O}}(M)$ . Let  $a$  be a point of positive arity in  $\mathcal{O}$  and  $M$  a module in  $\mathcal{Ib}_{\mathcal{O}}(M)$ . The composition  $M \circ^i a$  consists of grafting the  $n$ -corolla  $\left( \begin{smallmatrix} i \cdots i+n \\ \vee \end{smallmatrix} \right)$  labeled by  $(a, 1)$  to the  $i$ -th incoming edge of  $M$ . Similarly the composition  $a \circ_i M$  consists of grafting  $M$  to the  $i$ -th leaf of the  $n$ -corolla labeled by  $(a, 1)$ . By [15, Theorem 3.10],  $Q_{\mathcal{O}}(M)$  is a cofibrant replacement of  $M$  in the category  $\text{Ibimod}_{\mathcal{O}}$ . When  $a$  has arity zero, the right action of  $a \circ_i *_0$  is shown by Figure 5.7:

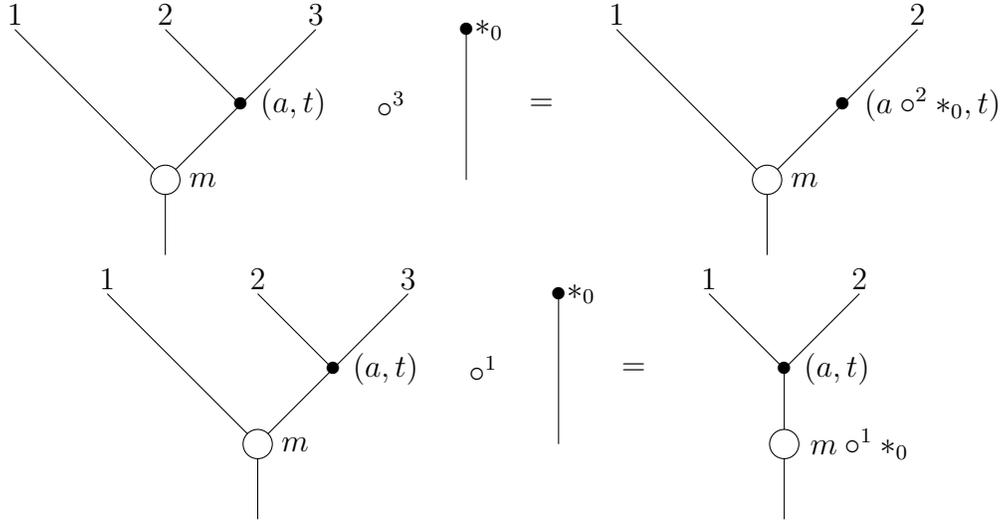


Figure 5.7: Right action

## 5.5 A Cosimplicial Model

### 5.5.1 Restriction-Induction Adjunction

Let  $\phi : P \rightarrow Q$  be a map of operads. Then there is an adjunction:

$$\phi^! : \mathcal{I}bimod_P \rightleftarrows \mathcal{I}bimod_Q : \phi^* \quad (5.5.1)$$

where  $\phi^*$  is the restriction functor and  $\phi^!$  is the left adjoint functor. The functor  $\phi^*$  is easy to be defined. In particular, for any  $Q - \mathcal{I}bimod N$ , one has that  $\phi^*(N)(k) = N(k)$ .

Let  $M$  be an infinitesimal bimodule over  $P$ . Below, we describe  $\phi^!(M)$ , which is a quotient bimodule of the free  $Q$ -Ibimodule generated by  $M$ . Typical elements look like the following trees where each  $q_i \in Q$ , and  $m \in M$ :

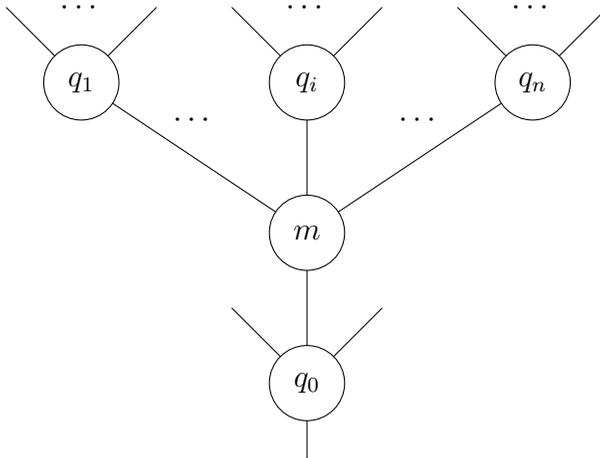


Figure 5.8: Typical Element of  $\phi^!(M)$

Note that some of the  $q_i$ 's may be  $id \in Q(1)$  and thus they can simply be omitted.

If  $q_i = \phi(p_i) \circ_j q_i'$  for  $i \geq 1$ , the relation with respect to the right action is as follows:

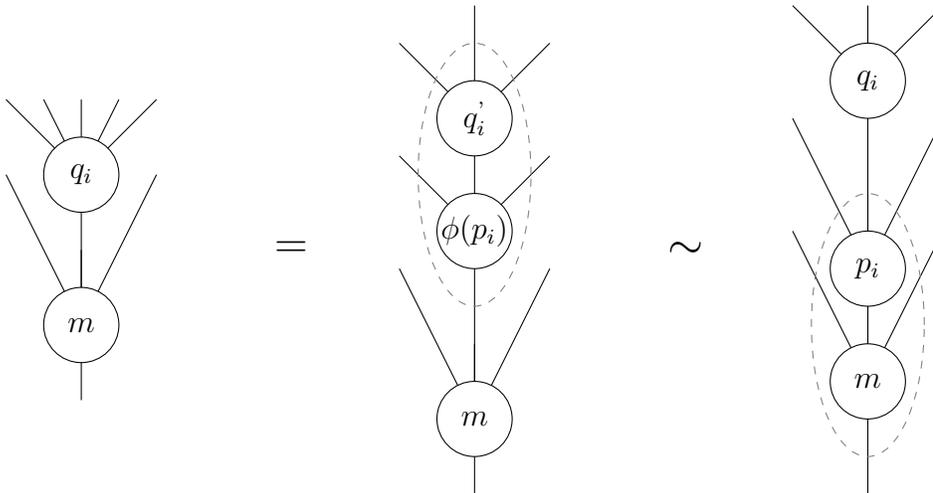


Figure 5.9: Relation with respect to the right action

The dashed circle on the left represents the operadic composition while the right circle is for the infinitesimal right action.

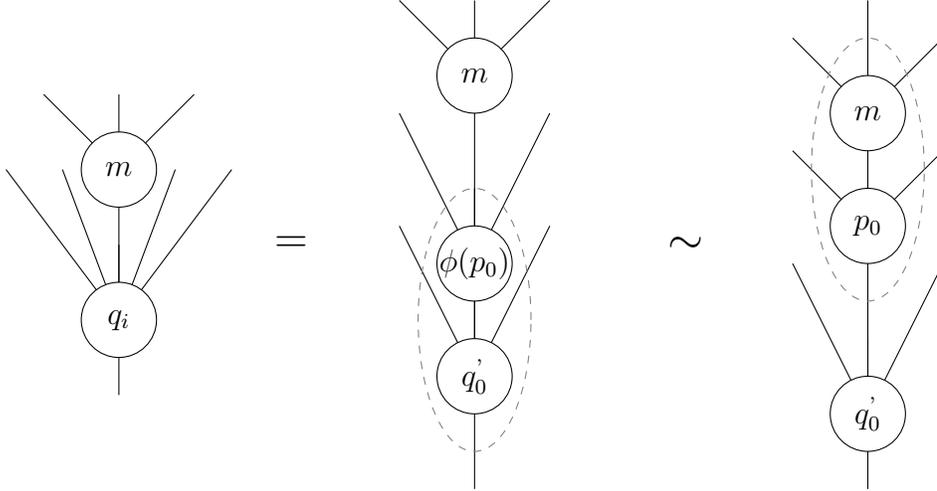


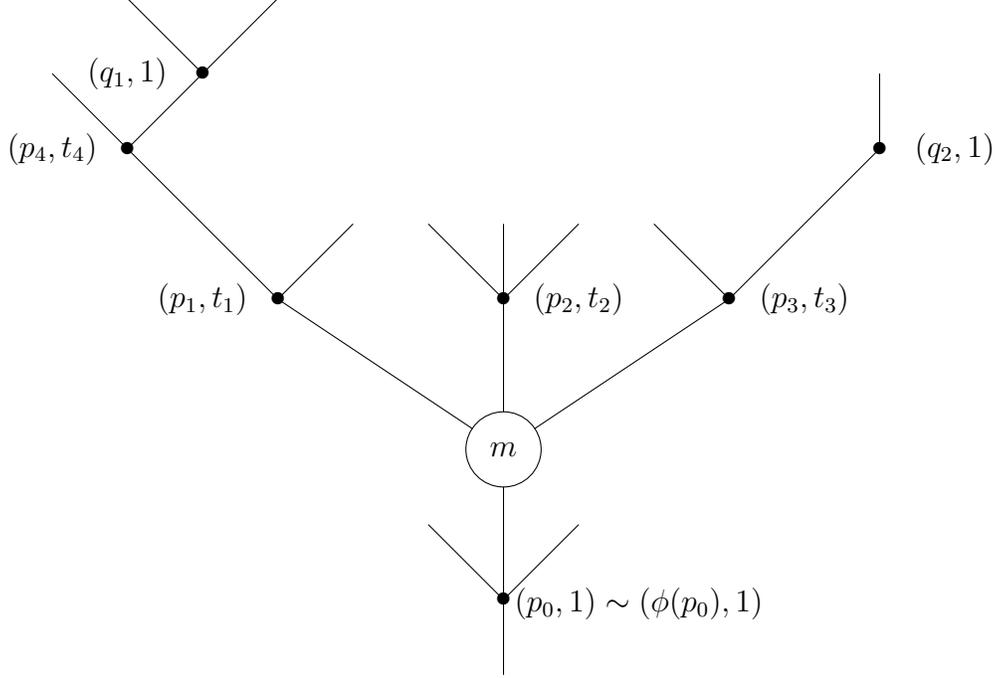
Figure 5.10: Relation with respect to the left action

Similarly, if  $q_0 = q_0' \circ_j \phi(p_0)$ , the dashed circle on the left represents the operadic composition while the right circle is for the infinitesimal left action.

### 5.5.2 Induction Applied to the Boardman-Vogt Resolution

Next we want to examine  $\phi^!(\mathcal{I}b_P(M))$ . Recall that  $\mathcal{I}b_P(M)$  is the Boardman-Vogt resolution for  $M$  in  $\mathcal{I}bimod_P$ , which was described in the section 5.4 using the language of pearled trees. For a pearled tree  $(T, p)$ , the pearl is labeled by an element of  $M$  and the internal vertices are labeled by the operad  $P$  and indexed by a real number in  $[0, 1]$ . Now we apply  $\phi^!$  to  $\mathcal{I}b_P(M)$ . The elements of  $\phi^!(\mathcal{I}b_P(M))$  can also be viewed as pearled trees.

The nodes labeled by the  $q_i$ 's are attached to some of the leaves. If a vertex in  $\mathcal{I}b_P(M)$  is labeled  $(p, 1)$ , then one can use the relations shown in Figures 5.9 and 5.10 to contract the edge. That is, if a vertex of the pearled tree in  $\phi^!(\mathcal{I}b_P(M))$  is indexed by 1, then we label it by  $q$  rather than by  $p$ .



### 5.5.3 A Cosimplicial Model

In this section we give an explicit description of the cosimplicial model for  $T_\infty \overline{Imm}_\partial^{(k)}(D^1, D^n)$ . This model is obtained as  $\pi_0^! \left( Q_{\mathcal{B}_1}(\mathcal{B}_n^{(k)}) \right)$  where  $Q_{\mathcal{B}_1}(\mathcal{B}_n^{(k)})$  is a cofibrant replacement of  $\mathcal{B}_n^{(k)}$  as a  $\mathcal{B}_1$ -Ibimodule. However, there is a slight issue because  $\mathcal{B}_n^{(k)}$  is not  $\Sigma$ -cofibrant. Indeed, the  $\Sigma_\ell$ -action on  $\mathcal{B}_n^{(k)}(\ell)$  is not free for  $k \geq 3$ ,  $\ell \geq 2$ . Therefore we cannot apply the Boardman-Vogt resolution directly to  $\mathcal{B}_n^{(k)}$ . This can be corrected by crossing  $\mathcal{B}_n^{(k)}$  with  $\mathcal{B}_\infty = \bigcup_n \mathcal{B}_n$ . There is a sequence of inclusions:  $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \dots$  and each  $\mathcal{B}_n \subset \mathcal{B}_\infty$ . Note also that  $\mathcal{B}_\infty \simeq *$  for all  $\ell$  and is an infinitesimal bimodule over  $\mathcal{B}_1$  thanks to the inclusion of operads  $\mathcal{B}_1 \hookrightarrow \mathcal{B}_\infty$ . Then  $(\mathcal{B}_n^{(k)} \times \mathcal{B}_\infty)(\ell) = \mathcal{B}_n^{(k)}(\ell) \times \mathcal{B}_\infty(\ell)$  is now  $\Sigma_\ell$ -cofibrant. Now we can apply the Boardman-Vogt resolution to get  $Q_{\mathcal{B}_1}(\mathcal{B}_n^{(k)}) = \mathcal{I}b_{\mathcal{B}_1}(\mathcal{B}_n^{(k)} \times \mathcal{B}_\infty)$ . Finally, the cosimplicial model for  $T_\infty \overline{Imm}_\partial^{(k)}(D^m, D^n)$  is  $\pi_0^!(\mathcal{I}b_{\mathcal{B}_1}(\mathcal{B}_n^{(k)} \times \mathcal{B}_\infty))$ .

# Bibliography

- [1] Gregory Arone and Victor Turchin. On the rational homology of high-dimensional analogues of spaces of long knots. *Geometry & Topology*, 18(3):1261–1322, 2014.
- [2] Gregory Arone and Victor Turchin. Graph-complexes computing the rational homotopy of high dimensional analogues of spaces of long knots. *Ann. Inst. Fourier (Grenoble)*, 65(1):1–62, 2015.
- [3] Yuliy Baryshnikov, Caroline Klivans, and Nicholas Kosar. On the topology of no- $k$ -equal spaces. *Preprint: <http://publish.illinois.edu/yimb/files/2017/08/Treesnokequal.pdf>*, 2017.
- [4] F. Bergeron, G. Labelle, and P. Leroux. *Combinatorial species and tree-like structures*, volume 67 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1998. ISBN 0-521-57323-8. Translated from the 1994 French original by Margaret Readdy, With a foreword by Gian-Carlo Rota.
- [5] François Bergeron. *Algebraic combinatorics and coinvariant spaces*. CMS Treatises in Mathematics. Canadian Mathematical Society, Ottawa, ON; A K Peters, Ltd., Wellesley, MA, 2009. ISBN 978-1-56881-324-0. doi: 10.1201/b10583. URL <http://dx.doi.org/10.1201/b10583>.
- [6] Anders Björner and Volkmar Welker. The homology of “ $k$ -equal” manifolds and related partition lattices. *Adv. Math.*, 110(2):277–313, 1995. ISSN 0001-8708. doi: 10.1006/aima.1995.1012. URL <http://dx.doi.org.er.lib.k-state.edu/10.1006/aima.1995.1012>.
- [7] Angeline Brandt. The free lie ring and lie representations of the full linear group. *Trans. Amer. Math. Soc.*, 56(3):528–536, 1944.

- [8] Ryan Budney. Little cubes and long knots. *Topology*, 46(1):1–27, 2007.
- [9] Ryan Budney. An operad for splicing. *Journal of Topology*, 5(4):945–976, 2012.
- [10] Frédéric Chapoton, Vladimir Dotsenko, and Anton Khoroshkin. Personal communication.
- [11] Frederick Cohen, Thomas Lada, and Peter May. *The homology of iterated loop spaces*, volume 533. Springer, 1976.
- [12] Natalya Dobrinskaya and Victor Turchin. Homology of non- $k$ -overlapping discs. *Homology Homotopy Appl.*, 17(2):261–290, 2015. ISSN 1532-0073. doi: 10.4310/HHA.2015.v17.n2.a13. URL <http://dx.doi.org.er.lib.k-state.edu/10.4310/HHA.2015.v17.n2.a13>.
- [13] Vladimir Dotsenko and Anton Khoroshkin. Character formulas for the operad of two compatible brackets and for the bi-hamiltonian operad. *Functional Analysis and Its Applications*, 41(1):1–17, 2007.
- [14] Julien Ducoulombier. Delooping derived mapping spaces of bimodules over an operad. *Journal of Homotopy and Related Structures*, pages 1–43, 2018.
- [15] Julien Ducoulombier and Victor Turchin. Delooping the functor calculus tower. *arXiv preprint arXiv:1708.02203*, 2017.
- [16] Benoit Fresse. Koszul duality of operads and homology of partition posets, homotopy theory: relations with algebraic geometry, group cohomology, and algebraic  $k$ -theory, 115–215. *Contemp. Math*, 346.
- [17] Benoit Fresse. *Modules over operads and functors*. Springer-Verlag, Berlin, 2009.
- [18] Benoit Fresse. *Homotopy of operads and Grothendieck-Teichmüller groups. Part 1*, volume 217 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017. ISBN 978-1-4704-3481-6. The algebraic theory and its topological background.

- [19] William Fulton and Joe Harris. *Representation theory, A first course, Readings in Mathematics*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. ISBN 0-387-97527-6; 0-387-97495-4. doi: 10.1007/978-1-4612-0979-9. URL <http://dx.doi.org.er.lib.k-state.edu/10.1007/978-1-4612-0979-9>.
- [20] Ezra Getzler and Mikhail Kapranov. Modular operads. *Compositio Mathematica*, 110(1):65–125, 1998.
- [21] Thomas G. Goodwillie. Calculus. III. Taylor series. *Geom. Topol.*, 7:645–711, 2003. ISSN 1465-3060. doi: 10.2140/gt.2003.7.645. URL <https://doi-org.er.lib.k-state.edu/10.2140/gt.2003.7.645>.
- [22] Thomas G. Goodwillie and John R. Klein. Multiple disjunction for spaces of smooth embeddings. *J. Topol.*, 8(3):651–674, 2015. ISSN 1753-8416. doi: 10.1112/jtopol/jtv008. URL <https://doi-org.er.lib.k-state.edu/10.1112/jtopol/jtv008>.
- [23] Thomas G. Goodwillie and Michael Weiss. Embeddings from the point of view of immersion theory. II. *Geom. Topol.*, 3:103–118, 1999. ISSN 1465-3060. doi: 10.2140/gt.1999.3.103. URL <https://doi-org.er.lib.k-state.edu/10.2140/gt.1999.3.103>.
- [24] Keely Grossnickle and Victor Turchin. Cycle index sum for non- $k$ -equal configurations. *Bull. Belg. Math. Soc. Simon Stevin*, 25(1):129–139, 2018. ISSN 1370-1444. URL <https://projecteuclid-org.er.lib.k-state.edu/euclid.bbms/1523412060>.
- [25] Gus Lehrer. Equivariant cohomology of configurations in rd. *Algebras and representation theory*, 3(4), 2000.
- [26] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*, volume 346. Springer Science & Business Media, 2012.
- [27] Ian Macdonald. *Symmetric functions and Hall polynomials*. Oxford university press, 1998.
- [28] MathOverflow. Question:super-plethysm? <https://mathoverflow.net/q/233819>.

- [29] Keiichi Sakai. Deloopings of the spaces of long embeddings. *Fund. Math.*, 227(1):27–34, 2014. ISSN 0016-2736. doi: 10.4064/fm227-1-3. URL <https://doi-org.er.lib.k-state.edu/10.4064/fm227-1-3>.
- [30] Sheila Sundaram and Michelle Wachs. The homology representations of the  $k$ -equal partition lattice. *Trans. Amer. Math. Soc.*, 349(3):935–954, 1997. ISSN 0002-9947. doi: 10.1090/S0002-9947-97-01806-0. URL <http://dx.doi.org.er.lib.k-state.edu/10.1090/S0002-9947-97-01806-0>.