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On Cellular Indecomposable Property of Semi-Fredholm Operators

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Abstract

In this paper we prove that an operator with the cellular indecomposable property has no singular points in the semi-Fredholm domain. Besides its own interests, this fills a gap in [3]. Our proof relies on the 4×4 matrix model of semi-Fredholm operators [2].

Keywords: cellular indecomposable property; semi-Fredholm operators; singular points.

In a series of three papers [3], [4], [5], R. Olin and J. Thomson introduced and studied the **cellular indecomposable property (CIP)** which has become a basic notion in operator theory. An operator $T \in B(H)$ has (CIP) if any two nontrivial invariant subspaces $M_1, M_2 \subset H$ of T have a nontrivial intersection $M_1 \cap M_2 \neq \{0\}$. Note that if T has (CIP), then so does $T - \lambda$ for any $\lambda \in \mathbb{C}$ since T and $T - \lambda$ have the same invariant subspace lattice.

The principle question underlying Olin and Thomson's research is what *the spectral picture* [6] of a CIP operator can look like. For instance, one can show that the Fredholm index of a CIP operator cannot be positive, hence the adjoint

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is quasi-triangular [1], [6]. It is easy to achieve index 0 or -1, and it is still not known whether the index can be -2 or smaller.

Motivated by the spectral picture problem, Olin and Thomson made a thorough analysis of subnormal operators with (CIP). For general operators, they proved a result on semi-Fredholm operators (Lemma 4 in [3], see Theorem 1 in this paper) which is needed in the proof of the main result in [3]. Their proof of Lemma 4, however, contains a gap in handling singular points in the semi-Fredholm domain as explained below.

On the other hand, their result is almost certainly useful for further study of the spectral theory of a general CIP operator. This prompts us to find a complete proof and in this paper we prove a result (Theorem 2) which is enough to fill the gap and is of independent interests–we show that a CIP operator has no singularity at all.

Our main technical tool is the 4×4 matrix model of semi-Fredholm operators developed in [2].

Recall that a **singular point** $\lambda_0 \in \rho_F(T)$ in **the Fredholm domain** $\rho_F(T)$ of an operator $T \in B(H)$ acting on a Hilbert space H is a point λ_0 such that the dimension function of the kernel

$$\lambda \to \dim(ker(T - \lambda))$$

is not continuous at λ_0. When $\lambda_0 \in \rho_{sF}(T)$, **the semi-Fredholm domain**, λ_0 is singular if the projection $P_{ker(T-\lambda)}$ does not converge to $P_{ker(T-\lambda_0)}$ as $\lambda \to \lambda_0$ in the strong operator topology. In this paper, we mainly consider those singular points in the semi-Fredholm domain.

To overcome the complexity caused by a singular point, [3] used a translation argument: For a semi-Fredholm T, possibly singular at 0, they replaced T by $T-\lambda$ for some small λ so they assume that T is regular at 0. However, they implicitly used the following argument: *If T is analytic, then so is $T - \lambda$.* Here an operator T is **analytic** if

$$\cap_{k\geq 0} T^k H = \{0\}.$$

See the first line and the last line of page 402 of [3]. This is not true as illustrated by the following one dimensional extension of a pure isometry $S \in B(H)$,

$$T = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix} \in B(\mathbb{C} \oplus H). \tag{1}$$

The statement of the following Theorem 1 is the same as Lemma 4 in [3].

Theorem 1. *If T is a semi-Fredholm operator such that*

(1) the Fredholm index satisfies $index(T) \notin \{0, -1\}$, and

(2) T is analytic, $\cap_{k\geq 0} T^k H = \{0\}$,

then T is cellular decomposable, that is, it has no (CIP).

A close examination of the proof in [3] shows that the arguments there do not work for the above T in (1). The obstacle is at the end of page 402: After a translation $T - \lambda$, the second analytic condition (2) in Theorem 1 is no longer satisfied. Moreover, [3] actually proved Theorem 1 under an extra condition

(3) T has no singularity at 0.

The main result of this paper is the following.

Theorem 2. *If the Hilbert space H is infinite dimensional, $\dim(H) = \infty$, and $T \in B(H)$ is cellular indecomposable, then T has no singular points in its semi-Fredholm domain.*

So Theorem 1 follows from Theorem 2 and the proof of Olin-Thomson in [3]. Note that Theorem 2 does not hold on a finite dimensional Hilbert space, as illustrated by a single nilpotent Jordan block, which indeed has (CIP) and is singular at the origin.

Corollary 3. *If $T \in B(H)$ is an operator with the cellular indecomposable property, and T is semi-Fredholm, then T has the following matrix decomposition,*

$$T = \begin{pmatrix} T_1 & A \\ 0 & T_2 \end{pmatrix}. \tag{2}$$

Here the decomposition is with respect to $H_1 \oplus H_1^\perp$, with $H_1 = \cap_{k\geq 1} T^k H$, $T_1 \in B(H_1)$ is invertible, and T_2 is a pure shift.

Recall that a **pure shift** is a left-invertible operator which is also analytic [2]. The proof of Corollary 3 is essentially contained in the proof of Theorem 2.

It is an interesting question to see when the T_1 entry in (2) is indeed void. If $index(T) \leq -2$, then Theorem 1 implies that T_1 cannot be void. Again, we do not know whether $index(T) \leq -2$ can happen for a CIP operator.

The rest of this paper is devoted to the proof of Theorem 2.

Proof of Theorem 2. We first recall the 4×4 upper-triangular matrix model of semi-Fredholm operators developed in [2] which we rely on heavily.

For any semi-Fredholm $T \in B(H)$ we can decompose $H = H_1 \oplus H_2 \oplus H_3 \oplus H_4$ into the direct sum of four closed subspaces, with some components possibly void, such that the associated matrix of T has the form

$$T = \begin{pmatrix} T_1 & * & * & * \\ 0 & T_2 & * & * \\ 0 & 0 & T_3 & * \\ 0 & 0 & 0 & T_4 \end{pmatrix}. \tag{3}$$

The properties of T_1, T_2, T_3, T_4 which we will need are listed below.

(i) T_4 is a pure shift semi-Fredholm operator. See the definition after Corollary 3. Or, to be more speficic, recall that a semi-Fredholm operator $S \in B(K)$ is a pure shift if

(a) $ker(S) = \{0\}$, and

(b) S is analytic, $\cap_{k\geq 0} S^k K = \{0\}$.

In particular, if S is a pure shift, then $ker(S^*) \neq \{0\}$ and $\dim(ker(S^* - \lambda))$ is a constant on a small open neighborhood of the origin by general Fredholm theory.

(ii) T_1^* is a pure shift.

(iii) T_2 is invertible,

(iv) T_3 is a finite nilpotent matrix. In particular,

$$\dim(H_3) = N < \infty. \tag{4}$$

It follows that

$$T_3^N = 0. \tag{5}$$

These two conditions will play important roles in the proof.

(v) The origin 0 is a singular point in the semi-Fredholm domain of T if and only if $H_3 \neq \{0\}$. So our goal is to show $H_3 = \{0\}$.

First we show that $H_1 = \{0\}$. Otherwise, $H' = ker(T_1) \neq \{0\}$ is a nontrivial invariant subspace of T. Since T_1^* is a pure shift,

$$\dim(ker(T_1)) = \dim(ker(T_1 - \lambda))$$

when λ is small enough, but nonzero, we have

$$H'' = ker(T_1 - \lambda) \neq \{0\}$$

to be another nontrivial invariant subspace of of T_1, hence of T. Clearly $H' \cap H'' = \{0\}$ since they consist of eigenvectors of different eigenvalues. This is a contradiction since T has (CIP).

Next we show that at most one of H_2 and H_3 can be nonzero. Otherwise, H_2 is a nontrivial invariant subspace. Since H_3 is nonzero, by (v) above, 0 is a singular point of T, hence

$$ker(T) \neq \{0\},$$

which is another nontrivial invariant subspace. Since $T_2 = T|_{H_2}$ is invertible, T is bounded below on H_2. It follows that $H_2 \cap ker(T) = \{0\}$. Again a contradiction with (CIP).

If $H_3 = \{0\}$, then we are done.

Next we assume that $H_2 = \{0\}$, and H_3 is a nontrivial invariant subspace. In this case, $H = H_3 \oplus H_4$.

Since $\dim(H) = \infty$ and $\dim(H_3) = N < \infty$, we know that H_4 is nontrivial. Since T_4 is a pure shift, we can choose a unit vector

$$k \in ker(T_4^*)$$

and let $H_k \subset H$ denote the invariant subspace generated by $\begin{pmatrix} 0 \\ k \end{pmatrix}$ under the action of T.

Claim: $H_k \cap H_3 = \{0\}$.

This will be in contradiction with (CIP), so it follows $H_3 = \{0\}$, and we are done then. The rest of the proof is devoted to prove this claim.

Next we assume that there is a sequence of polynomials $p_t(z) \in \mathbb{C}[z]$ such that

$$\lim_{t\to\infty} p_t(T) \begin{pmatrix} 0 \\ k \end{pmatrix} = \begin{pmatrix} e \\ 0 \end{pmatrix} \in H_k \cap H_3,$$

and we wish to show $e = 0$.

Write

$$T = \begin{pmatrix} T_3 & A \\ 0 & T_4 \end{pmatrix}$$

for some $A \in B(H_4, H_3)$ and for any polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n,$$

we write

$$p(T) \begin{pmatrix} 0 \\ k \end{pmatrix} = \begin{pmatrix} p(T_3) & B_p \\ 0 & p(T_4) \end{pmatrix} \begin{pmatrix} 0 \\ k \end{pmatrix} = \begin{pmatrix} B_p k \\ p(T_4)k \end{pmatrix}.$$

Here B_p is a noncommutative polynomial of T_3, A and T_4. If we can show that for any polynomial p,

$$||B_p k|| \leq C||p(T_4)k|| \tag{6}$$

for some constant C, independent of p, then we can conclude that $e = 0$.

Without loss of generality, we assume

$$n \geq N = \dim(H_3)$$

since otherwise we can choose

$$a_{n+1} = \cdots = a_N = 0,$$

so that p is formally of degree N. This will make the bookkeeping in the proof of (8) easier. Equation (8) is a key step toward the proof of (6).

Next we calculate B_p by direct calculation. For any $i = 1, 2, \cdots, N$, let

$$B_i = a_i T_3^{i-1} A + a_{i+1} T_3^{i-1} A T_4 + \cdots + a_n T_3^{i-1} A T_4^{n-i}.$$

By using

$$T_3^N = 0 \tag{7}$$

and by writing out all terms in B_p, we have

$$B_p = B_1 + \cdots + B_N. \tag{8}$$

The proof of (8) involves some work on bookkeeping, but there is nothing challenging. In writing out all terms of B_p, one just needs to keep (7) in mind.

Note that $N = \dim(H_3)$ is independent of $p = p(z)$. So it suffices to show that for each $i = 1, 2, \cdots, N$,

$$||B_i k|| \leq C||p(T_4)k||$$

for some constant C, independent of p. Let

$$B_i' = a_i + a_{i+1}T_4 + \cdots + a_n T_4^{n-i},$$

then

$$B_i = T_3^{i-1} A B_i',$$

hence it suffices to show

$$||B_i' k|| \leq C||p(T_4)k|| \tag{9}$$

for some constant C, independent of p.

Next we show (9) by induction. First for $i = 1$. Since T_4 is a pure shift, it is bounded below, so we assume

$$||T_4 x|| \geq c||x||$$

for some $c > 0$ and any $x \in H_4$.

Write

$$p(T_4)k = a_0 k + T_4(a_1 + a_2 T_4 + \cdots + a_n T_4^{n-1})k.$$

By our choice of k, $k \perp T_4 H_4$, so we have

$$\begin{aligned} ||p(T_4)k||^2 &= ||a_0 k||^2 + ||T_4(a_1 + a_2 T_4 + \cdots + a_n T_4^{n-1})k||^2 \\ &\geq c^2||(a_1 + a_2 T_4 + \cdots + a_n T_4^{n-1})k||^2, \end{aligned}$$

which is the case of $i = 1$ for (9).

Now replace $p(z)$ by $q(z) = a_1 + a_2 z + \cdots + a_n z^{n-1}$, and apply the $i = 1$ case of (9) to $q(z)$, one obtains the $i = 2$ case of (9) for $p(z)$, with a different constant C. Keep iterating this process and the proof of (9), hence the whole proof, can be completed. $\square$

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