

Discrete modulus in graphs and matroids: theory and algorithms

by

Ngoc Huy Truong

BS, Ho Chi Minh City University of Education, 2021

AN ABSTRACT OF A DISSERTATION

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Abstract

This dissertation develops and extends the theory of discrete modulus for families of combinatorial objects in graphs, matroids, and hypergraphs. Discrete modulus is an optimization framework for studying a finite family of objects through their usage vectors on a common finite ground set. It assigns densities to the elements of the ground set and minimizes a convex energy subject to the requirement that every object in the family has total density at least one. A central theme of this dissertation is that the resulting optimizer often encodes much more than the solution to an optimization problem: it also determines structural invariants of the underlying combinatorial model.

The first part of the dissertation studies modulus for the family of bases of a matroid. We prove that the optimal density of the base modulus problem, called the *universal density*, determines the strength, fractional arboricity, and principal partition of the matroid. We also recover Edmonds's base packing and covering theorems from this perspective, characterize the Fulkerson blocker of the base family, obtain a minimal inequality description of the base dominant, and establish a duality between the base modulus of a matroid and that of its dual.

The second part extends the theory to weighted matroids. In this setting, we introduce matroid reinforcement and matroid sparsification problems, which ask for minimum-cost ways to increase or decrease element weights so that the resulting weighted matroid becomes homogeneous. Here, homogeneous matroids are those that are balanced, in the sense that strength and fractional arboricity coincide. We express the relevant weighted invariants through the geometry of the base dominant and anti-dominant, reduce the optimization problems to polymatroid greedy instances, and obtain explicit algorithms in the graphic case.

The third part studies matroid truncation and dual truncation. We derive explicit formulas expressing the universal density of each truncation in terms of the universal density of the original matroid. These formulas describe how strength, fractional arboricity, and the principal partition evolve across the truncation family. We also characterize the universal density of a weighted matroid as the unique minimizer of a Kullback–Leibler divergence over the convex hull of the base family.

The final part carries the modulus framework to hypergraphs. Hypergraphic matroids generalize spanning trees to hypertrees, but a connected hypergraph need not contain a hypertree. To address this, we introduce the multi-tree family, whose members are multisets of hyperedges that span the vertex set as a hypertree. We show that the modulus of this family reveals the strength, fractional arboricity, and hierarchical decomposition of any connected hypergraph. We also prove a Fulkerson duality between the multi-tree family and a system of partition inequalities.

Across these settings, the same phenomenon recurs: the optimal density arising from discrete modulus is not merely an optimizer of a convex program. It is a structural object that organizes packing, covering, density, duality, and decomposition phenomena in graphs, matroids, and hypergraphs.

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Across these settings, the same phenomenon recurs: the optimal density arising from discrete modulus is not merely an optimizer of a convex program. It is a structural object that organizes packing, covering, density, duality, and decomposition phenomena in graphs, matroids, and hypergraphs.

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Chapter 1

Introduction

1.1 Overview and Motivation

Large-scale networks often fail not because every component is fragile, but because a small number of components are especially important. One example is the 2003 blackout in the United States and Canada, during which a sequence of transmission-line failures in Ohio contributed to a cascading outage affecting approximately fifty million people. This event illustrates a natural question in network analysis: how can we identify the connections whose failure most severely damages the structure or function of a network? Motivated by this and related questions, this dissertation studies structural features, important elements, and hierarchical layers in graphs, matroids, and hypergraphs through the framework of discrete modulus.

Discrete modulus for families of combinatorial objects has received significant attention in recent years [1–5], in part because it unifies several classical quantities within one framework. Modulus provides an optimization-based framework for measuring the richness of a family of objects defined on a finite ground set: each object is represented by a usage vector, and one seeks an optimal density minimizing a convex energy subject to constraints imposed by the family. The modulus of the family of all s - t paths recovers the minimum cut, the shortest path

length, and the effective resistance [2, 4]—three quantities that ordinarily require separate theories. For spanning trees, the modulus of the spanning tree family reveals the same strength and fractional arboricity that appear in the classical packing and covering theorems of Nash-Williams and Tutte [6–8], and drives a hierarchical decomposition of the graph [5, 9].

The purpose of this dissertation is to continue that development across four topics: bases of matroids, weighted matroids with reinforcement and sparsification, matroid truncations, and hypertrees in hypergraphs. In each setting, optimal densities encode structural information, and understanding this encoding leads to new algorithms, new duality results, and new decomposition theorems.

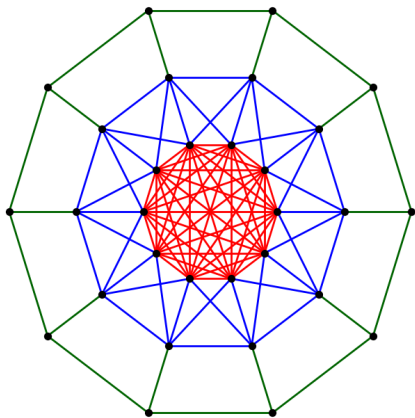


Figure 1.1: An example of a graph whose three colors represent three structural layers revealed by spanning tree modulus

1.2 Historical Background and the Modulus Framework

Spanning trees, packing, and covering

The story begins with spanning trees. A spanning tree of a connected graph is an acyclic subgraph reaching every vertex, and the family of all spanning trees encodes a great deal

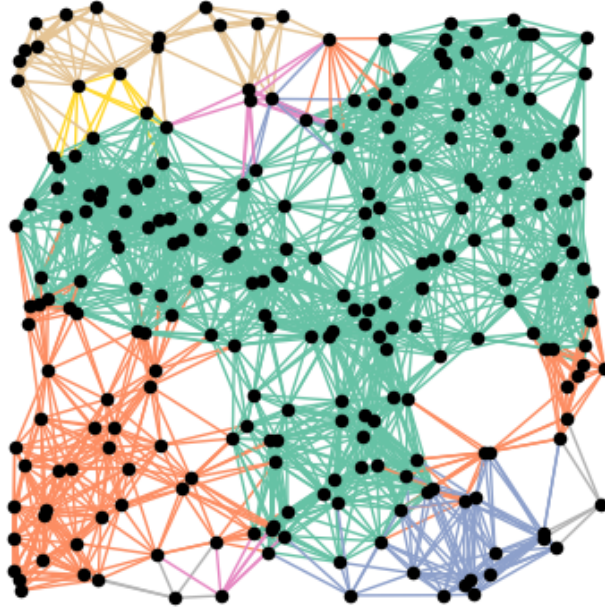


Figure 1.2: Spanning tree modulus on a random geometric graph. Edge colors indicate optimal densities.

about the graph's structure. Two quantities represent some of this information. The *strength* is the minimum, over all edge subsets whose removal increases the number of components, of the ratio of removed edges to new components created: a high-strength network is hard to disconnect easily. The *fractional arboricity* is the maximum, over all non-empty subgraphs, of their density which is the ratio of edges to rank. In other words, it measures how dense is the densest part of the graph. Both quantities are closely related to the theorems of Nash-Williams and Tutte [6–8].

Nash-Williams [6] and Tutte [7] independently proved that a graph contains k edge-disjoint spanning trees if and only if every partition of the vertex set is crossed by at least k times one less than its number of parts. This theorem determines the packing number of the spanning tree family, namely the largest number of spanning trees that can be chosen without sharing edges. Nash-Williams also proved, in [8], the complementary covering result: the edge set decomposes into k forests if and only if the number of edges of every nonempty

subgraph is at most k times the rank of that subgraph. This theorem determines the covering number, namely the smallest number of forests needed to cover all edges. Together these theorems show that the packing and covering numbers of the spanning tree family are both determined by the ratios defining strength and fractional arboricity.

The practical importance of these invariants became clear in network design. Gusfield [10] proposed strength as a measure of network invulnerability. Cunningham [11, 12] gave efficient algorithms for computing strength and for the *strength reinforcement problem*: find the minimum-cost way to increase edge capacities to achieve a target strength. This problem is a direct precursor of the matroid reinforcement problem of Chapter 4, and the polymatroid greedy algorithm Cunningham used reappears as a key tool there.

From graphs to matroids: Edmonds's theorems

Since spanning trees are the bases of the graphic matroid, these results belong to a larger context. Matroids, introduced by Whitney and Nakasawa, provide a common framework for the notion of independence in linear algebra and graph theory. A matroid consists of a finite ground set together with a collection of independent sets satisfying three axioms. Its maximal independent sets are called *bases*, and all bases have the same cardinality, called the *rank* of the matroid. The language of matroids allows results from different settings to be viewed as instances of a single underlying theory. Edmonds [13] proved the matroid analogues of both Nash-Williams–Tutte theorems: base packing and base covering in any matroid are governed by strength and fractional arboricity defined purely through the rank function.

Principal partitions and homogeneous matroids

The theory of *principal partitions* in graphs, matroids, and submodular systems, developed from the late 1960s and surveyed by Fujishige [14], can be considered as a decomposition of a discrete system into its components together with a partially ordered structure of the set

of components. Each level reveals a layer of the matroid’s structure, and the full sequence encodes everything between the global strength and the global fractional arboricity.

A matroid is *homogeneous* (or *uniformly dense*) when strength and fractional arboricity coincide, meaning the principal partition is trivial. Equivalently, one can choose finitely many bases, with repetition allowed, so that each element of the ground set appears equally often among the chosen bases. This class appeared independently in several places. Catlin et al. in [15] gave several characterizations in the context of strength and arboricity. Narayanan and Vartak [16] studied the same class as “uniformly dense matroids.” Independently, homogeneous matroids appeared in a conjecture on cyclic base orderings in [17], proved for the coprime case by Van den Heuvel and Thomassé in [18]. In the graph case, Albin et al. [19] introduced the notion of *strictly homogeneous* graphs, which is stronger than homogeneity. One characterization states that a biconnected graph is strictly homogeneous if it is homogeneous and contains no proper subgraph with the same or greater density.

In the modulus framework, homogeneous matroids correspond exactly to those for which the optimal density is a constant vector, and the reinforcement and sparsification problems of Chapter 4 ask precisely how to transform an arbitrary weighted matroid into a homogeneous one at minimum cost.

Discrete modulus: from conformal analysis to combinatorics

The question of *how* strength, fractional arboricity, and the principal partition are encoded in the spanning tree family is answered particularly well by discrete modulus. These invariants are not only described by combinatorial theorems; they also appear through optimal densities arising from a convex optimization problem over the convex hull of the spanning tree family.

Introduced by Albin and collaborators [2] and inspired by conformal modulus in complex analysis, discrete modulus assigns to any family an *optimal density*: the minimizer of a convex energy functional over the convex hull of the family’s usage vectors. The framework applies to any finite family of objects, once each object is represented by its usage vector on

a common finite ground set.

Its versatility is illustrated by s - t path families: varying the energy parameter recovers the minimum cut, the shortest path, and the effective resistance from the same optimization problem, see [2, 4]. These three quantities had required entirely separate theories; modulus unifies them.

For spanning trees, Albin et al. [5] showed that the optimal density of the 2-modulus problem is tied to strength and fractional arboricity, and that it drives a *deflation process*: contract the densest subgraph, and then repeat. The resulting nested sequence coincides with the classical principal partition of the graphic matroid. This revealed an important feature of the theory: a single optimization problem, formulated purely in terms of the spanning tree family, recovers the entire hierarchical structure.

Fulkerson duality and modulus

One of the central themes of this dissertation is Fulkerson’s theory of blocking and anti-blocking polyhedra, developed in [20, 21]. This theory gives a polyhedral duality between a combinatorial family and the inequalities defining its dominant. The blocker serves as the dual family: it records the essential constraints needed to describe the dominant, and hence captures the polyhedral structure of the original family.

Classical examples already show the importance of this viewpoint. In network theory, paths and cuts form a natural dual pair. For spanning trees, the corresponding dual objects are partitions of the vertex set, and the duality between spanning trees and partitions is closely related to the Nash-Williams–Tutte theorem. Thus, Fulkerson duality gives a polyhedral language for expressing familiar min-max relationships in combinatorics.

This duality also plays an important role in discrete modulus. The modulus problem studies optimal densities associated with a family of usage vectors, while the blocker describes the essential constraints appearing in the dual problem. Thus, understanding the blocker clarifies the dual feasible region of the modulus problem and how the optimal density reflects

the structure of the underlying family.

In this dissertation, we use this perspective in two main settings. First, we study the base family of a matroid and characterize its Fulkerson blocker, thereby obtaining a minimal inequality description of the base dominant. Second, we study hypertrees and multi-trees in hypergraphs, where the corresponding blocker families are described through suitable partition families of the vertex set. These results extend the classical dualities between paths and cuts, and between spanning trees and partitions, to broader matroidal and hypergraph settings.

Weights and the need for a richer theory

The theory described above treats all elements of the ground set equally. In practice, however, elements carry different costs, capacities, or resistances. Weighted graphs and matroids arise naturally throughout combinatorial optimization: in network design, edges have capacities; in matroid intersection, elements have costs; in electrical networks, edges have resistances. The strength and fractional arboricity of a weighted matroid are defined by the same density ratios as in the unweighted case, but with cardinalities replaced by total weights, and they govern the same packing and covering problems.

The modulus framework also accommodates weights in the energy functional. This allows the optimization problem to reflect nonuniform features of the ground set and makes the framework applicable to more practical situations. Extending the modulus framework to the weighted setting is therefore a natural step.

In this dissertation, the preceding results motivate a weighted version of the theory. Modifying weights is a natural and practical way to improve a combinatorial structure without changing the underlying ground set. Given a weighted matroid that is not homogeneous, one may ask how to modify the weights, at minimum cost, so that the resulting weighted matroid becomes homogeneous. Reinforcement addresses this question by increasing weights, while sparsification addresses the complementary question by decreasing weights. Together, these

problems show that weighted modulus can be used not only to measure structure, but also to study cost-sensitive modifications of combinatorial structure.

Matroid truncation and higher-order connectivity

The base modulus theory described above treats a matroid through its full family of bases. A natural further question is how the theory behaves when the matroid is modified by one of the most basic operations in matroid theory: truncation. The *t-truncation* of a matroid replaces the base family by the collection of all independent sets of size exactly t , for a fixed integer t between one and the rank. In the graphic setting, the bases of the t -truncation of a graphic matroid are precisely the spanning forests of the graph with exactly t edges, objects that arise naturally in the study of network reliability and connectivity. More broadly, the notion of *c-order edge toughness* of a graph, introduced as a generalization of strength that measures resistance to disconnection into more than two components, can be expressed directly in terms of truncated matroids. The packing theorem for this generalized toughness, proved in [22, 23], is the truncation analogue of the Nash-Williams–Tutte theorem. This makes it natural to study base modulus under matroid truncation.

Hypergraphs and the limits of the graphic framework

Another natural direction is to ask how far the modulus framework can be pushed beyond graphs and matroids. Hypergraphs, in which each edge may contain more than two vertices, provide a natural setting for this question. They arise when one needs to model higher-order relations among several vertices at once, rather than only pairwise connections. For example, a hyperedge may represent a communication link shared by a group of terminals, a constraint involving several variables, or a cluster of vertices acting as a single unit in a network.

The theory of hypergraphic matroids, initiated by Lorea [24] and developed extensively by Frank, Király, and Kriesell [25], provides the natural matroidal framework for hypergraphs.

In this setting, spanning trees generalize to *hypertrees*. Frank et al. proved the hypergraph analogue of the Nash-Williams–Tutte theorem, characterizing hypergraphs that contain k hyperedge-disjoint hypertrees via a *partition-connectivity* condition: a hypergraph has k disjoint hypertrees if and only if every partition of the vertex set is crossed by at least k times one less than the number of parts. This condition is strictly stronger than ordinary connectivity, which creates an immediate obstacle for the modulus framework: a connected hypergraph may contain no hypertree at all, leaving the natural family of objects to study completely empty.

The strength and fractional arboricity of a hypergraph can still be defined through density ratios analogous to those for graphs, and Fulkerson duality for partition inequalities remains meaningful. This motivates the hypergraph modulus theory developed in this dissertation. The main difficulty is that, unlike connected graphs, connected hypergraphs need not contain hypertrees. Thus, when the hypertree family is empty, one needs a replacement family on which modulus can still be defined. This motivates the introduction of the *multi-tree family*.

This motivates the main work of the dissertation, whose contributions are presented in the next section.

1.3 Main Contributions

The dissertation develops the modulus framework across four main chapters: base modulus for matroids, weighted matroids with reinforcement and sparsification, matroid truncations, and hypergraphs.

Base modulus for matroids

The first main topic studies base modulus for matroids, extending results from the graph setting and establishing foundational properties of the base modulus. In particular, some of these results recover various properties of matroids related to their strength, fractional

arboricity, and principal partition. In the probabilistic interpretation of base modulus, one minimizes the p -energy over the convex hull of the usage vectors of the bases. One of the results is that this optimal density is the same for all $p \in (1, \infty)$. Throughout this dissertation, we call this optimizer the *universal density* of the base modulus. We also provide an alternative derivation of Edmonds’s base packing and covering theorems. We characterize the Fulkerson blocker of the base family of a matroid and give a minimal description of the base dominant. We also establish the duality between the base modulus of a matroid and that of its dual.

Weighted matroids, reinforcement, and sparsification

The second topic extends base modulus to weighted matroids and introduces two optimization problems. The *matroid reinforcement problem* asks for the minimum-cost weight increase making a weighted matroid homogeneous; the *matroid sparsification problem* asks the same for weight decreases. Both are motivated by Cunningham’s strength reinforcement problem in [11] but are more general, requiring new geometric tools: the weighted strength and fractional arboricity are expressed as the largest and smallest dilation factors for which the weight vector lies in appropriate scalings of the base dominant and anti-dominant. Using this reformulation, both problems can be solved using the polymatroid greedy algorithm, with explicit implementations for graphic matroids via minimum-cut computations.

Matroid truncation and universal density

The third topic studies how universal densities transform under matroid truncation and dual truncation. We derive explicit formulas expressing the universal density of every truncation in terms of the original density, and we extend these formulas to dual truncations. In the same chapter, we also provide a new characterization of the universal density of weighted matroids using Kullback–Leibler divergence.

Hypertree and multi-tree modulus in hypergraphs

The fourth topic extends the modulus framework to hypergraphs. For partition-connected hypergraphs, the optimal density of the hypertree family encodes strength and fractional arboricity and drives a hierarchical decomposition analogous to the graph case; homogeneity is again equivalent to the equality of these two invariants.

For connected hypergraphs that are not necessarily partition-connected, the hypertree family may be empty, so hypertree modulus cannot be applied directly. The introduction of the *multi-tree family* resolves this difficulty. Its members are multisets of hyperedges that span the vertex set as a hypertree, and its modulus recovers the corresponding strength, fractional arboricity, and hierarchical decomposition. We establish Fulkerson duality between the multi-tree family and the partition inequality system: the blocker consists of normalized partition indicators over feasible partitions whose contracted hypergraph is vertex-biconnected, giving a minimal description of the system of partition inequalities.

1.4 Organization of the Dissertation

Chapter 1 is the introduction. Chapter 2 provides background and preliminaries. Chapter 3 studies base modulus for matroids. Chapter 4 studies weighted matroids, reinforcement, and sparsification. Chapter 5 studies matroid truncation and dual truncation. Chapter 6 studies hypertree and multi-tree modulus for hypergraphs. Chapter 7 concludes the dissertation. Together, these chapters demonstrate that discrete modulus is a unified framework for combinatorial families across graphs, matroids, and hypergraphs, and that optimal densities encode structural properties of the underlying combinatorial object.

Chapter 2

Preliminaries

2.1 Matroids

Let us begin by revisiting several definitions related to matroids. For a set X we write $|X|$ for its cardinality, and if Y is another set, then $X - Y$ is the relative complement of Y in X .

Definition 2.1.1. Let E be a finite set, the set system $M(E, \mathcal{I})$ is a matroid if the following axioms are satisfied:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $X \in \mathcal{I}$ and $Y \subseteq X$ then $Y \in \mathcal{I}$ (*Hereditary property*).

(I3) If $X, Y \in \mathcal{I}$ and $|X| > |Y|$, then there exists $x \in X - Y$ such that $Y \cup \{x\} \in \mathcal{I}$ (*Exchange property*).

Every set in $\mathcal{I}$ is called an *independent set*.

Let $M(E, \mathcal{I})$ be a matroid on the ground set E with the set of independent sets $\mathcal{I}$. The maximal independent sets are called *bases*, the minimal dependent sets are called *circuits*.

The *rank* function, $r : 2^E \rightarrow \mathbb{Z}_+$, defined on all subsets $X \subset E$ is given by:

$$r(X) := \max \{|Y| : Y \subseteq X, Y \in \mathcal{I}\}.$$

The *closure operator* $\text{cl} : 2^E \rightarrow 2^E$ is a set function, defined as:

$$\text{cl}(X) := \{y \in E : r(X \cup \{y\}) = r(X)\}. \quad (2.1.1)$$

In matroid theory, these concepts play an important role. The following proposition gives some basic properties of matroids that we will use throughout the paper.

Proposition 2.1.2 ([26]). *Given a matroid $M(E, \mathcal{I})$. Let $\mathcal{B}$ be the set of bases of M and let $\mathcal{C}$ be the set of circuits of M . Let r be the rank function. Then, for all subsets $X, Y \subseteq E$, we have:*

1. *If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, then there exists $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$.*
2. *If $B \in \mathcal{B}$ and $x \in E - B$, then there exists a unique circuit $C(x, B)$ contained in $B \cup \{x\}$ and containing x .*
3. *If $B \in \mathcal{B}$, then for any $x \in E - B$, the set $(B - \{y\}) \cup \{x\}$ is a base of M if and only if $y \in C(x, B)$.*
4. $0 \leq r(X) \leq |X|$.
5. *If $Y \subseteq X$ then $r(Y) \leq r(X)$.*
6. $r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y)$.
7. $X \in \mathcal{I} \Leftrightarrow |X| = r(X)$.
8. $X \in \mathcal{B} \Leftrightarrow |X| = r(X) = r(E)$.
9. $\text{cl}(X) \supseteq X$.

10. $\text{cl}(X) = X \cup \{y \in E : y \in C \subseteq (X \cup y) \text{ for some } C \in \mathcal{C}\}$ [27].

Next, we recall dual matroids. Given a matroid $M(E, \mathcal{I})$, the set

$$\mathcal{B}^* := \{X \subseteq E : \text{there exists a base } B \in \mathcal{B} \text{ such that } X = E - B\}$$

is the family of bases of the *dual matroid* M^* on E . The *corank* function r^* of M is defined as the rank function of M^* , and for any subset $X \subseteq E$, we have:

$$r^*(X) = |X| - r(M) + r(E - X). \quad (2.1.2)$$

Let us also recall the operations of *deletion*, *restriction*, and *contraction* in matroids. For a matroid $M(E, \mathcal{I})$ and a subset $X \subseteq E$, the set

$$\mathcal{C}(M \setminus X) = \{C \subseteq E - X : C \in \mathcal{C}(M)\},$$

defines the family of circuits for a matroid on $E - X$. The matroid $M \setminus X$ is called the *deletion* of X from M . The *restriction* to X in M is denoted by $M|X$, and is the matroid on X defined as $M|X := M \setminus (E - X)$. The deletion operation behaves as in the following proposition.

Proposition 2.1.3 ([26]). *Given a matroid $M(E, \mathcal{I})$ and a subset $X \subseteq E$, then:*

1. $\mathcal{I}(M \setminus X) = \{Y \subseteq E - X : Y \in \mathcal{I}(M)\}$.
2. $\mathcal{B}(M \setminus X) = \text{maximal sets in } \{B - X : B \in \mathcal{B}(M)\}$.
3. $r_{M \setminus X}(Y) = r_M(Y)$ for all subsets $Y \subseteq E - X$.

Note that r_M denotes the rank function of a matroid M . When we discuss a matroid M and its deletion and contraction, r is understood as the rank function of M .

For a matroid $M(E, \mathcal{I})$ and a subset $X \subseteq E$, the *contraction* of X in M is the matroid

M/X on $E - X$, which is defined as:

$$M/X = (M^* \setminus X)^*.$$

We have the following properties for the contraction operator:

Proposition 2.1.4 ([26]). *Given a matroid $M(E, \mathcal{I})$ and a subset $X \subseteq E$, then:*

1. $\mathcal{C}(M/X) = \text{minimal sets in } \{C - X : C \in \mathcal{C}(M), C - X \neq \emptyset\}.$
2. $\mathcal{I}(M/X) = \{Y \subseteq E - X : \exists B \in \mathcal{B}(M \setminus (E - X)) \text{ such that } Y \cup B \in \mathcal{I}(M)\}.$
3. $\mathcal{B}(M/X) = \{Y \subseteq E - X : \exists B \in \mathcal{B}(M \setminus (E - X)) \text{ such that } Y \cup B \in \mathcal{B}(M)\}.$
4. $r_{M/X}(Y) = r_M(X \cup Y) - r_M(X)$ for all $Y \subseteq E - X$.

Next, let us recall the definition of polymatroids. From now on, for any vector $x \in \mathbb{R}^E$ and for any subset $A \subset E$, we let $x(A) := \sum_{e \in A} x(e)$. Also, $y \in \mathbb{R}^E$ is another vector, then $y \leq x$ iff $y(e) \leq x(e)$ for all $e \in E$. Polymatroids were introduced by Edmonds [28], who also established many profound insights into the topic. Let E be a finite set. We recall the definition of polymatroids as follows.

Definition 2.1.5. Let $P \subseteq \mathbb{R}^E$. Then P is called a polymatroid if P is a compact non-empty subset of $\mathbb{R}_{\geq 0}^E$ satisfying the following properties:

- (i) If $0 \leq x \leq y$ and $y \in P$, then $x \in P$.
- (ii) For any $x \in \mathbb{R}_{\geq 0}^E$, any maximal vector $y \in P$ with $y \leq x$ (such y is called a P -basis of x) has the same component sum $y(E)$.

A real-valued function f defined on subsets of E is said to be *submodular* if for all subsets A, B of E , we have

$$f(A \cap B) + f(A \cup B) \leq f(A) + f(B). \quad (2.1.3)$$

Conversely, f is said to be *supermodular* if $-f$ is submodular. Let f be a real-valued, normalized, nondecreasing, and submodular function defined on subsets of E , meaning that:

- (i) $f(\emptyset) = 0$;
- (ii) $f(A) \leq f(B)$ if $A \subseteq B \subseteq E$;
- (iii) $f(A \cap B) + f(A \cup B) \leq f(A) + f(B)$ for all subsets $A, B \subseteq E$.

For any such function f , we associate to it a polyhedron P_f :

$$P_f := \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\}. \quad (2.1.4)$$

It is well-known that P_f is a polymatroid [28]. Conversely, every polymatroid P is of the form of P_f for some such function f .

Remark 2.1.6. For a matroid $M(E, \mathcal{I})$ with the rank function r , it is well known that r is a nonnegative, normalized, nondecreasing, and submodular function defined on subsets of E . Hence, the polyhedron P_r as defined in (2.1.4) is a polymatroid associated with M .

We also define the base polytope B_f of f as follows:

$$B_f := \{x \in \mathbb{R}^E : x \in P_f, x(E) = f(E)\}. \quad (2.1.5)$$

Let P be a polymatroid, an interesting problem in this field is finding a P -basis y of a given vector $x \in \mathbb{R}_{\geq 0}^E$. A greedy algorithm for solving this problem starts with $y = 0$ and successively increases each component of y as much as possible, subject to the constraints $y \leq x$ and $y \in P$, see [29]. Given a vector $m \in \mathbb{R}_{\geq 0}^E$, as discussed in [11] and [29], this greedy algorithm can be generalized to solve the following optimization problem:

$$\min \{m^T y : y \text{ is a } P\text{-basis of } x\}. \quad (2.1.6)$$

To solve (2.1.6), we start with $y = 0$ and increase successively each component in the order $j_1, j_2, \dots, j_k$ where $k = |E|$ and $m_{j_1} \leq m_{j_2} \leq \dots \leq m_{j_k}$. The well-known Kruskal's algorithm for finding a minimum spanning tree in a graph [30] is a special case of this greedy

algorithm. It arises by choosing $x = \mathbf{1} \in \mathbb{R}^E$ where $\mathbf{1}$ is the vector containing all ones. Cunningham [11] proved that the generalized algorithm does solve (2.1.6). To implement this algorithm, we need an oracle that computes

$$\max \{ \epsilon : y + \epsilon \mathbb{1}_{\{j\}} \in P \} \quad \text{for any } y \in P \text{ and } j \in E. \quad (2.1.7)$$

For a matroid $M(E, \mathcal{I})$ with the rank function r , let P_r be the polymatroid defined in (2.1.4), and let B_r be the base polytope as defined in (2.1.5). Edmonds [29, Theorem 39] showed that the set of vertices of the polymatroid P_r precisely corresponds to the set of indicator vectors of $\mathcal{I}$. Edmonds [29, Theorem 43] also demonstrated that the set of vertices of the base polytope B_r is the set of indicator vectors of the base family $\mathcal{B}$. In other words, the base polytope B_r is the convex hull $\text{conv}(\mathcal{B})$ of indicator vectors of $\mathcal{B}$.

For simplicity, throughout this dissertation, we only consider loopless matroids with positive rank. Loopless means that $r(X) = 0$ implies $X = \emptyset$ and positive rank means that $r(E) > 0$.

2.1.1 Strength and fractional arboricity of matroids

Let M be a finite matroid with no loops. The ground set of M is denoted by $E(M)$. Let $a_p(M)$ represent the maximum number of disjoint bases of M and let $a_c(M)$ be the minimum number of bases of M whose union equals $E(M)$. The following theorem by Edmonds [13] are well known.

Theorem 2.1.7. [13] *Let M be a matroid with $r(M) > 0$. Each of the following statements holds.*

1. $a_p(M) \geq k$ if and only if $\forall X \subseteq E(M), |E(M) - X| \geq k(r(M) - r(X))$.
2. $a_c(M) \leq k$ if and only if $\forall X \subseteq E(M), |X| \leq k \cdot r(X)$.

For any subset $X \subseteq E(M)$ with $r(X) > 0$, the *density* of X is defined as

$$\theta_M(X) = |X|/r(X).$$

When the matroid M is understood from the context, we often omit the subscript M . We also use $\theta(M)$ for $\theta(E(M))$. The *strength* $S(M)$ and the *fractional arboricity* $D(M)$ of M are defined as follows:

$$S(M) = \min \left\{ \frac{|X|}{r(E) - r(E - X)} : X \subseteq E, r(E - X) < r(E) \right\}, \quad (2.1.8)$$

and

$$D(M) = \max \{ \theta(X) : X \subseteq E(M), r(X) > 0 \}. \quad (2.1.9)$$

Theorem 2.1.7 indicates that $a_p(M) = \lfloor S(M) \rfloor$, and $a_c(M) = \lceil D(M) \rceil$, where $\lfloor \cdot \rfloor$ is the floor function and $\lceil \cdot \rceil$ is the ceiling function. The notions of strength and fractional arboricity are two well-known concepts in matroid theory. Recall that a matroid M is called *uniformly dense* if its strength and fractional arboricity agree:

$$S(M) = D(M). \quad (2.1.10)$$

It was shown in [15] that, equivalently, one can select finitely many bases so that every element of E is used the same number of times.

Next, we recall the base packing problem for the base family $\mathcal{B}$ of a matroid $M = (E, \mathcal{I})$:

$$\begin{aligned} & \underset{\lambda \in \mathbb{R}_{\geq 0}^{\mathcal{B}}}{\text{maximize}} && \sum_{B \in \mathcal{B}} \lambda(B) \\ & \text{subject to} && \sum_{B \in \mathcal{B}: e \in B} \lambda(B) \leq 1, \quad \forall e \in E. \end{aligned} \quad (2.1.11)$$

The covering problem for the base family $\mathcal{B}$ is formulated as follows:

$$\begin{aligned} & \underset{\kappa \in \mathbb{R}_{\geq 0}^{\mathcal{I}}}{\text{minimize}} && \sum_{I \in \mathcal{I}} \kappa(I) \\ & \text{subject to} && \sum_{I \in \mathcal{I}: e \in I} \kappa(I) \geq 1, \quad \forall e \in E. \end{aligned} \tag{2.1.12}$$

Note that the covering problem (2.1.12) can be restricted to only the bases in $\mathcal{B}(M)$ instead of all independent sets in $\mathcal{I}(M)$. This is because for each independent set I , one can replace I with a base that contains it. Consequently, the value of the objective function remains unchanged and the constraints remain satisfactory. In other words, the covering problem (2.1.12) is equivalent to the following problem, which we refer to as the base covering problem:

$$\begin{aligned} & \underset{\kappa \in \mathbb{R}_{\geq 0}^{\mathcal{B}}}{\text{minimize}} && \sum_{B \in \mathcal{B}} \kappa(B) \\ & \text{subject to} && \sum_{B \in \mathcal{B}: e \in B} \kappa(B) \geq 1, \quad \forall e \in E. \end{aligned} \tag{2.1.13}$$

Using two of Edmonds's results from [13], we are able to derive the following theorem.

Theorem 2.1.8. *Let $S(M), D(M), \tau(M), v(M)$ be the strength, the fractional arboricity, the optimal values of the base packing problem, and the base covering problem of a matroid M , respectively. Then, we have:*

$$\tau(M) = S(M), \tag{2.1.14}$$

$$v(M) = D(M). \tag{2.1.15}$$

2.1.2 Principal partition of matroids

In this section, we recall some results from the theory of principal partitions as presented in [14]. Consider a matroid $M(E, \mathcal{I})$ with the rank function r . We start with the following min-max relation, which relates a packing problem with an attack problem. For any positive

integers k and l , we have:

$$\max \left\{ \sum_{i=1}^k |I_i| : \{I_i\}_{i=1}^k \subset \mathcal{I}, \sum_{i=1}^k \mathbb{1}_{\{e \in I_i\}} \leq l, \quad \forall e \in E \right\} = \min \{kr(X) + l|E - X| : X \subseteq E\}.$$

For a nonnegative rational number λ , let $\mathcal{D}_\lambda$ denote the set of minimizers of the submodular function $f_\lambda(X) = r(X) + \lambda|E - X| : 2^E \rightarrow \mathbb{R}$. Then, $\mathcal{D}_\lambda$ is a distributive lattice (closed under unions and intersections). A value λ is called *critical*, if $\mathcal{D}_\lambda$ contains more than one element. It has been shown that the set of all critical values is finite, hence can be denoted by

$$0 \leq \lambda_1 < \cdots < \lambda_q. \quad (2.1.16)$$

For each $i = 1, \dots, q$, define $E_{\lambda_i}^-$ and $E_{\lambda_i}^+$ as the minimum and maximum elements of $\mathcal{D}_{\lambda_i}$, respectively, then $E_{\lambda_i}^- \subsetneq E_{\lambda_i}^+$. Furthermore, we have the following:

$$\emptyset = E_{\lambda_1}^- \subsetneq E_{\lambda_1}^+ = E_{\lambda_2}^- \subsetneq E_{\lambda_2}^+ = \cdots = E_{\lambda_q}^- \subsetneq E_{\lambda_q}^+ = E. \quad (2.1.17)$$

Moreover, each minor $(M|E_{\lambda_i}^+)/E_{\lambda_i}^-$ on $E_{\lambda_i}^+ - E_{\lambda_i}^-$, with critical value $\lambda_i = k/l$, has k bases that uniformly cover every element of $E_{\lambda_i}^+ - E_{\lambda_i}^-$ exactly l times.

Subsequently, the theory of principal partitions of matroids was extended to polymatroids. Let $P(E, r)$ be the polymatroid associated with the matroid $M = (E, \mathcal{I})$. Edmonds demonstrated that for any positive real λ ,

$$\max \{x(E) : x \in P(E, r), x \leq \lambda \mathbf{1}\} = \min \{r(X) + \lambda|E - X| : X \subseteq E\}, \quad (2.1.18)$$

where $\mathbf{1}$ is the column vector of all ones and the inequality holds coordinatewise.

Focusing on the left-hand side of equality (2.1.18), it was shown in [31] that there exists a unique base b^* , called the *universal base* of $P(E, r)$, such that $b^* \wedge \lambda \mathbf{1}$ is a maximizer for each λ , where $(b^* \wedge \lambda \mathbf{1})(e) := \min\{b^*(e), \lambda\}$, for all $e \in E$. Furthermore, all critical values λ_i

as in (2.1.16) are the distinct elements of b^* , and

$$E_{\lambda_i}^+ = \{e \in E : b^*(e) \leq \lambda_i\} \quad \text{for } i = 1, \dots, q.$$

There are two characterizations for the universal base b^* . The first one is that the universal base b^* is an optimal solution of the problem

$$\min \left\{ \sum_{e \in E} x(e)^2 : x \text{ is a base in } P(E, r) \right\}. \quad (2.1.19)$$

On the other hand, for any vector x in $P(E, r)$, enumerate $E = \{e_1, \dots, e_m\}$, and let

$$T(x) := (x(e_{i_1}), \dots, x(e_{i_m}))$$

be a reordering of x such that $x(e_{i_1}) \leq \dots \leq x(e_{i_m})$, where ties are broken by picking the lower index. A base x of $P(E, r)$ is called a *lexicographically optimal base* if it lexicographically maximizes $T(x)$ among all bases in $P(E, r)$. Then, the second characterization is that the universal base b^* is identical to the unique lexicographically optimal base of $P(E, r)$, see [31].

In [15], the authors established that $S(M)$ and $D(M)$ are closely related to the principal partition of M . Specifically, this theory shows that $S(M) = 1/b_{\max}^*$ and $D(M) = 1/b_{\min}^*$, where $b_{\max}^*$ and $b_{\min}^*$ are the maximum and minimum values, respectively, of the universal base b^* .

2.1.3 Modulus

Let E be a finite set with given weights $\sigma \in \mathbb{R}_{\geq 0}^E$ assigned to each element e in E . We say that Γ is a family of objects in E , if each object $\gamma \in \Gamma$ is associated to a function $\mathcal{N}(\gamma, \cdot)^T : E \rightarrow \mathbb{R}_{\geq 0}$, which we think of as a *usage vector* in $\mathbb{R}_{\geq 0}^E$. In other words, Γ is associated with a $|\Gamma| \times |E|$ *usage matrix* $\mathcal{N}$. From now on, we will assume that Γ is non-empty and each object $\gamma \in \Gamma$ uses at least one element in E with a positive and finite

amount.

A *density* $\rho \in \mathbb{R}_{\geq 0}^E$ is a vector where $\rho(e)$ represents the *cost* of using the element $e \in E$. For each object $\gamma \in \Gamma$, we define the *total usage cost* of γ with respect to ρ

$$\ell_\rho(\gamma) := \sum_{e \in E} \mathcal{N}(\gamma, e) \rho(e) = (\mathcal{N}\rho)(\gamma). \quad (2.1.20)$$

A density $\rho \in \mathbb{R}_{\geq 0}^E$ is called *admissible* for Γ , if for all $\gamma \in \Gamma$, $\ell_\rho(\gamma) \geq 1$. In matrix notations, ρ is admissible if $\mathcal{N}\rho \geq \mathbf{1}$. The *admissible set* $\text{Adm}(\Gamma)$ of Γ is defined as the set of all admissible densities for Γ ,

$$\text{Adm}(\Gamma) := \{ \rho \in \mathbb{R}_{\geq 0}^E : \mathcal{N}\rho \geq \mathbf{1} \}. \quad (2.1.21)$$

Fix $1 \leq p < \infty$, the *energy* of the density ρ is defined as follows

$$\mathcal{E}_{p,\sigma}(\rho) := \sum_{e \in E} \sigma(e) \rho(e)^p.$$

When $p = \infty$,

$$\mathcal{E}_{\infty,\sigma}(\rho) := \max_{e \in E} \{ \sigma(e) \rho(e) \}.$$

Definition 2.1.9. The *p-modulus* of Γ is

$$\text{Mod}_{p,\sigma}(\Gamma) := \inf_{\rho \in \text{Adm}(\Gamma)} \mathcal{E}_{p,\sigma}(\rho).$$

Equivalently, the *p-modulus* of Γ is the following optimization problem,

$$\begin{aligned} & \underset{\rho \in \mathbb{R}_{\geq 0}^E}{\text{minimize}} && \mathcal{E}_{p,\sigma}(\rho) \\ & \text{subject to} && \sum_{e \in E} \mathcal{N}(\gamma, e) \rho(e) \geq 1, \quad \forall \gamma \in \Gamma. \end{aligned} \quad (2.1.22)$$

When σ is the vector of all ones, we omit σ and write $\mathcal{E}_p(\rho) := \mathcal{E}_{p,\sigma}(\rho)$ and $\text{Mod}_p(\Gamma) := \text{Mod}_{p,\sigma}(\Gamma)$.

2.1.4 Fulkerson dual families and the MEO problem

Let E be a finite set. Let Γ be a family of objects on E . Let K be a closed convex set in $\mathbb{R}_{\geq 0}^E$ such that $\emptyset \subsetneq K \subsetneq \mathbb{R}_{\geq 0}^E$ and K is *recessive*, meaning that $K + \mathbb{R}_{\geq 0}^E = K$. The *blocker* $\text{BL}(K)$ of K is defined as,

$$\text{BL}(K) = \{\eta \in \mathbb{R}_{\geq 0}^E : \eta^T \rho \geq 1, \forall \rho \in K\}.$$

We will routinely identify Γ with the set of its usage vectors $\{\mathcal{N}(\gamma, \cdot)^T : \gamma \in \Gamma\}$ in $\mathbb{R}_{\geq 0}^E$, hence we can write $\Gamma \subset \mathbb{R}_{\geq 0}^E$. We defined the *dominant* of Γ as

$$\text{Dom}(\Gamma) := \text{conv}(\Gamma) + \mathbb{R}_{\geq 0}^E \quad (2.1.23)$$

where $\text{conv}(\Gamma)$ denotes the convex hull of Γ in $\mathbb{R}^E$.

Note that the admissible set $\text{Adm}(\Gamma)$ defined in (2.1.21) is closed, convex in $\mathbb{R}_{\geq 0}^E$ and recessive. Next, we recall Fulkerson duality for modulus.

Definition 2.1.10. Let Γ be a family of objects on E . The *Fulkerson blocker family* $\widehat{\Gamma}$ of Γ is defined as the set of all the extreme points of $\text{Adm}(\Gamma)$.

$$\widehat{\Gamma} := \text{Ext}(\text{Adm}(\Gamma)) \subset \mathbb{R}_{\geq 0}^E.$$

Fulkerson blocker duality [20] states that

$$\text{Dom}(\widehat{\Gamma}) = \text{Adm}(\Gamma) = \text{BL}(\text{Adm}(\widehat{\Gamma})), \quad (2.1.24)$$

$$\text{Dom}(\Gamma) = \text{Adm}(\widehat{\Gamma}) = \text{BL}(\text{Adm}(\Gamma)). \quad (2.1.25)$$

Moreover, $\widehat{\Gamma}$ has its own Fulkerson blocker family, and

$$\widehat{\widehat{\Gamma}} \subset \Gamma. \quad (2.1.26)$$

Definition 2.1.11. Let Γ and $\tilde{\Gamma}$ be two sets of vectors in $\mathbb{R}_{\geq 0}^E$. We say that Γ and $\tilde{\Gamma}$ are a *Fulkerson dual pair* (or $\tilde{\Gamma}$ is a *Fulkerson dual family* of Γ) if

$$\text{Adm}(\tilde{\Gamma}) = \text{BL}(\text{Adm}(\Gamma)).$$

Remark 2.1.12. If $\tilde{\Gamma}$ is a Fulkerson dual family of Γ , then Γ is also a Fulkerson dual family of $\tilde{\Gamma}$ because

$$\text{BL}(\text{Adm}(\tilde{\Gamma})) = \text{BL}(\text{BL}(\text{Adm}(\Gamma))) = \text{Adm}(\Gamma).$$

Proposition 2.1.13 ([9]). *Let Γ be a set of vectors in $\mathbb{R}_{\geq 0}^E$. Let $\hat{\Gamma}$ be the Fulkerson blocker family of Γ and $\tilde{\Gamma}$ be a Fulkerson dual family of Γ . Then, $\hat{\Gamma}$ is the smallest Fulkerson dual family of Γ , meaning that*

$$\hat{\Gamma} \subset \tilde{\Gamma}.$$

Remark 2.1.14. When all usage vectors of Γ belong to $\{0, 1\}^E$, if the support set $\{e \in E : \mathcal{N}(\gamma, e) \neq 0\}$ of any usage vector of Γ does not contain the support set of any other usage vector of Γ , then Γ is called a *clutter* in the combinatorics literature. An important property of a clutter Γ is $\hat{\hat{\Gamma}} = \Gamma$, see [21]. In this case, Γ is the Fulkerson blocker family of $\hat{\Gamma}$.

When $1 < p < \infty$, let $q := p/(p-1)$ be the Hölder conjugate exponent of p . For any set of weights $\sigma \in \mathbb{R}_{> 0}^E$, define the dual set of weights $\tilde{\sigma}$ as $\tilde{\sigma}(e) := \sigma(e)^{-\frac{q}{p}}$ for all $e \in E$. Let $\tilde{\Gamma}$ be a Fulkerson dual family of Γ . Fulkerson duality for modulus [4, Theorem 3.7] states that

$$\text{Mod}_{p,\sigma}(\Gamma)^{\frac{1}{p}} \text{Mod}_{q,\tilde{\sigma}}(\tilde{\Gamma})^{\frac{1}{q}} = 1. \quad (2.1.27)$$

Moreover, the optimal ρ^* of $\text{Mod}_{p,\sigma}(\Gamma)$ and the optimal η^* of $\text{Mod}_{q,\tilde{\sigma}}(\tilde{\Gamma})$ always exist, are unique, and are related as follows,

$$\eta^*(e) = \frac{\sigma(e)\rho^*(e)^{p-1}}{\text{Mod}_{p,\sigma}(\Gamma)}, \quad \forall e \in E. \quad (2.1.28)$$

When $p = 2$, we have

$$\text{Mod}_{2,\sigma}(\Gamma) \text{Mod}_{2,\sigma^{-1}}(\tilde{\Gamma}) = 1 \quad \text{and} \quad \eta^*(e) = \frac{\sigma(e)}{\text{Mod}_{2,\sigma}(\Gamma)} \rho^*(e) \quad \forall e \in E. \quad (2.1.29)$$

Let $\mathcal{P}(\Gamma)$ be the set of all probability mass functions (pmf) on Γ . According to the probabilistic interpretation of modulus [3], we can express

$$\text{Mod}_2(\Gamma)^{-1} = \min_{\mu \in \mathcal{P}(\Gamma)} \mu^T \mathcal{N} \mathcal{N}^T \mu = \min_{\eta \in \text{conv}(\Gamma)} \sum_{e \in E} \eta^2(e). \quad (2.1.30)$$

Consider the scenario where Γ is a collection of subsets of E with usage vector given by the indicator function. Given a pmf $\mu \in \mathcal{P}(\Gamma)$, let $\underline{\gamma}$ and $\underline{\gamma}'$ be two independent random objects in Γ , identically distributed with law μ . The cardinality of the overlap between $\underline{\gamma}$ and $\underline{\gamma}'$, is $|\underline{\gamma} \cap \underline{\gamma}'|$ and is a random variable whose expectation is denoted by $\mathbb{E}_\mu |\underline{\gamma} \cap \underline{\gamma}'|$, which equals $\mu^T \mathcal{N} \mathcal{N}^T \mu$. Then, the *minimum expected overlap* (MEO) problem for Γ is formulated as $\min_{\mu \in \mathcal{P}(\Gamma)} \mathbb{E}_\mu |\underline{\gamma} \cap \underline{\gamma}'|$, which is equivalent to the middle optimization problem in (2.1.30), see [5]. Moreover, any pmf $\mu \in \mathcal{P}(\Gamma)$ is optimal if and only if

$$(\mathcal{N}\mu)(e) = \rho^*(e) / \text{Mod}_2(\Gamma) \quad \forall e \in E. \quad (2.1.31)$$

And the optimization problem on the right side of (2.1.30) is equivalent to $\text{Mod}_2(\hat{\Gamma})$ since $\text{Adm}(\hat{\Gamma}) = \text{Dom}(\Gamma) = \text{conv}(\Gamma) + \mathbb{R}_{\geq 0}^E$.

Chapter 3

Base Modulus for Matroids

3.1 Overview

This chapter is motivated by the modulus of the spanning tree family of an undirected graph, introduced by Albin et al. [5]. A central role in their analysis is played by the probabilistic form of the spanning tree modulus,

$$\min \left\{ \sum_{e \in E} \eta(e)^2 : \eta \in \text{conv}(\Gamma) \right\}, \quad (3.1.1)$$

where Γ denotes the spanning tree family of a graph $G = (V, E)$ and $\text{conv}(\Gamma)$ is the convex hull of the indicator vectors of its spanning trees. They showed that the optimal density η^* of (3.1.1) encodes the strength and the fractional arboricity of G , and that a deflation process built from η^* exposes a hierarchical decomposition of the graph.

Spanning trees are precisely the bases of the graphic matroid, and, both strength and fractional arboricity are matroidal invariants expressed through the rank function. It is therefore natural to ask how much of this theory survives for matroids. The present chapter addresses this question by studying the modulus of the family $\mathcal{B}$ of all bases of a matroid $M(E, \mathcal{I})$, which we refer to as the *base modulus*.

A second thread concerns the polyhedral side of the theory. Writing

$$\text{Dom}(\mathcal{B}) := \text{conv}(\mathcal{B}) + \mathbb{R}_{\geq 0}^E \quad (3.1.2)$$

for the base dominant, Fulkerson observed [21] that Edmonds's work [13] yields the inequality description

$$\text{Dom}(\mathcal{B}) = \left\{ \eta \in \mathbb{R}_{\geq 0}^E : \sum_{e \in X} \eta(e) \geq r(E) - r(E - X), \text{ for all closed } X \subseteq E \right\}. \quad (3.1.3)$$

In the spanning tree case, Chopra [32] recovered this description by a different argument. Moreover, he also provided a minimal inequality description of the spanning tree dominant. One aim of this chapter is to obtain the analogous minimal description for arbitrary matroids.

It is noteworthy that there is a fundamental relationship between strength and arboricity of a matroid and its dual, see [15, Theorem 1]. This relationship motivates one of our key results, detailed in Theorem 3.6.1 below.

The remainder of the chapter is organized as follows.

- In Sections 3.2, we analyze the base modulus and its probabilistic interpretation. In the process, we introduce the concept of Beurling sets.
- In Section 3.3, we generalize several results from spanning tree modulus to base modulus. In particular, some of these results recover various properties of matroids related to their strength, fractional arboricity, and principal partition.
- In Section 3.4, we present an alternative proof strategy to two of Edmonds's theorems in [13], utilizing results from our study of base modulus.
- In Section 3.5, we provide the Fulkerson blocker family of the base family of a matroid. In other words, we derive a minimal inequality description of the base dominant.
- In Section 3.6, we establish a relationship between the base modulus of matroids and

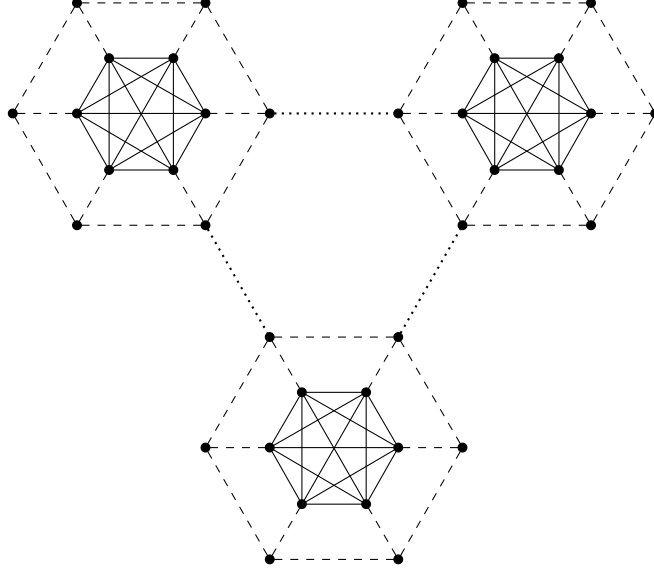


Figure 3.1: A graphic matroid where edges are styled according to the optimal density η^* . Solid edges are present with $\eta^* = \frac{1}{3}$, dashed edges with $\eta^* = \frac{1}{2}$ and dotted edges with $\eta^* = \frac{2}{3}$.

their dual matroids, enriching our understanding of the connection between a matroid and its dual.

- In Section 3.7, we demonstrate that p -modulus for the base family of a matroid can be deduced from the optimal density of the 2-modulus, and this provides a complete understanding of the base modulus across all values of p .

3.2 Modulus for the base family of a matroid

3.2.1 Base modulus and the MEO problem

Let $M = (E, \mathcal{I})$ be a loopless matroid with $r(M) > 0$. Let $\mathcal{B} = \mathcal{B}(M)$ be the family of all bases of M with usage vectors given by the indicator functions. We call $\mathcal{B}$ the *base family* of M . Let $\tilde{\mathcal{B}}$ be a Fulkerson dual family of $\mathcal{B}$ as in Definition 2.1.11. An explicit construction of $\tilde{\mathcal{B}}$ is the Fulkerson blocker family $\hat{\mathcal{B}}$ of $\mathcal{B}$, given in Theorem 3.5.2. Let μ^* be an optimal pmf for $\text{MEO}(\mathcal{B})$. Let ρ^* and η^* be the unique optimal densities for $\text{Mod}_2(\mathcal{B})$ and $\text{Mod}_2(\tilde{\mathcal{B}})$.

Lemma 3.2.1. *Let $\mathcal{B}$ be the family of bases of a matroid M . Let $\mu \in \mathcal{P}(\mathcal{B})$ be a probability mass function (pmf) and let $\eta = \mathcal{N}^T \mu$ be the corresponding element usage probabilities. Then:*

$$\eta(E) = r(E). \quad (3.2.1)$$

Proof. Note that, by part 8 of Proposition 2.1.2, we have that $|B| = r(B) = r(E)$ for any base $B \in \mathcal{B}$. Therefore:

$$\eta(E) := \sum_{e \in E} \eta(e) = \sum_{e \in E} \sum_{B \in \mathcal{B}} \mu(B) \mathcal{N}(B, e) = \sum_{B \in \mathcal{B}} \mu(B) \sum_{e \in E} \mathcal{N}(B, e) = r(E).$$

□

By Lemma 3.2.1, it follows that:

$$\sum_{e \in E} \eta^*(e) = r(E). \quad (3.2.2)$$

Theorem 3.2.2. *Let $M(E, \mathcal{I})$ be a matroid. Let $\mathcal{B}$ be the base family of M , and let $\tilde{\mathcal{B}}$ be a Fulkerson dual family of $\mathcal{B}$. Furthermore, let $\rho \in \mathbb{R}_{\geq 0}^E$, $\eta \in \mathbb{R}_{\geq 0}^E$, and $\mu \in \mathcal{P}(\mathcal{B})$.*

Then ρ , η and μ are optimal, respectively, for $\text{Mod}_2(\mathcal{B})$, $\text{Mod}_2(\tilde{\mathcal{B}})$ and $\text{MEO}(\mathcal{B})$ if and only if the following conditions are satisfied:

- (i) $\rho \in \text{Adm}(\mathcal{B})$, $\eta = \mathcal{N}^T \mu$;
- (ii) ρ and η are parallel, meaning that for some constant $Z > 0$, $\rho(e) = Z\eta(e)$ for all $e \in E$.
- (iii) $\mu(B)(1 - \ell_\rho(B)) = 0 \quad \forall B \in \mathcal{B}$.

In particular, the constant Z in (ii) equals to $\text{Mod}_2(\mathcal{B})$ and

$$1/Z = \text{MEO}(\mathcal{B}) = \text{Mod}_2(\tilde{\mathcal{B}}).$$

Proof. Assume that ρ, η , and μ are optimal for $\text{Mod}_2(\mathcal{B})$, $\text{Mod}_2(\tilde{\mathcal{B}})$ and $\text{MEO}(\mathcal{B})$, respectively. Then, equation (2.1.29) implies that ρ and η are parallel with constant $Z = \text{Mod}_2(\mathcal{B})$, and this establishes (ii). Next, we define $\eta_\mu := \mathcal{N}^T \mu$. Consequently, $\mu^T \mathcal{N} \mathcal{N}^T \mu = \eta_\mu^T \eta_\mu$ is the minimum expected overlap, and by (2.1.31), we have $\rho = \text{Mod}_2(\mathcal{B}) \eta_\mu$. Particularly, by (ii) with Z as above, it follows that $\eta = \eta_\mu$. Furthermore, given that ρ is optimal for $\text{Mod}_2(\mathcal{B})$, we have $\rho \in \text{Adm}(\mathcal{B})$, thus, (i) holds as well. Finally, part (iii) is the complementary slackness condition derived from the probabilistic interpretation for the Lagrange multipliers.

Conversely, assume that ρ, η , and μ satisfy the conditions (i), (ii), and (iii). Define

$$\mathcal{B}^+ := \{B \in \mathcal{B} : \mu(B) > 0\},$$

and let $\mathcal{N}^+$ be the usage matrix for $\mathcal{B}^+$. To demonstrate that ρ is optimal for $\text{Mod}_2(\mathcal{B})$, we apply the Beurling's criterion from [3], for the case $p = 2$, applied to the subfamily $\mathcal{B}^+$. Given condition (iii), we have $\ell_\rho(B) = 1$ for all $B \in \mathcal{B}^+$ and by (i), $\rho \in \text{Adm}(\mathcal{B})$. Therefore, we only need to check the second requirement in Beurling's criterion. Let's consider any $h \in \mathbb{R}^E$ such that $\sum_{e \in E} h(e) \mathcal{N}(B, e) \geq 0$ for every $B \in \mathcal{B}^+$. Our goal is to prove that $h^T \rho \geq 0$. Indeed,

$$\begin{aligned} h^T \rho &= Z h^T \eta && \text{(by (ii))} \\ &= Z h^T \mathcal{N}^T \mu && \text{(by (i))} \\ &= Z h^T \mathcal{N}^+ \mu && \text{(by definition of } \mathcal{B}^+) \\ &= Z \mu^T \mathcal{N}^+ h \geq 0, \end{aligned}$$

since $\mathcal{N}^+ h \geq 0$. Therefore, we have that ρ is the optimal density for $\text{Mod}_2(\mathcal{B})$.

Next, let η^* and μ^* be optimal for $\text{Mod}_2(\tilde{\mathcal{B}})$ and $\text{MEO}(\mathcal{B})$, respectively. Then, by (2.1.29), it follows that $Z \eta = \rho = \text{Mod}_2(\mathcal{B}) \eta^*$. Hence, $Z \eta(E) = \text{Mod}_2(\mathcal{B}) \eta^*(E)$. Note that, by Lemma 3.2.1, $\eta(E) = \eta^*(E) = r(E)$. Consequently, we have $Z = \text{Mod}_2(\mathcal{B})$. Then, we obtain $\eta = \eta^*$.

In addition, by equation (2.1.31), μ is optimal for $\text{MEO}(\mathcal{B})$. □

3.2.2 Beurling sets and serial rule

First, we present some basic properties of the modulus of the base family $\mathcal{B}$. Let $\tilde{\mathcal{B}}$ be a Fulkerson dual family of $\mathcal{B}$ as in Definition 2.1.11. Let μ^* be an optimal pmf for $\text{MEO}(\mathcal{B})$. Let ρ^* and η^* be the unique optimal densities for $\text{Mod}_2(\mathcal{B})$ and $\text{Mod}_2(\tilde{\mathcal{B}})$.

Definition 3.2.3. A base $B \in \mathcal{B}$ is called a *fair base* if there exists an optimal pmf μ^* for $\text{MEO}(\mathcal{B})$ such that $\mu^*(B) > 0$. The set of all fair bases is denoted by $\mathcal{B}^f$.

Remark 3.2.4. If $\mathcal{B}^f \subset \mathcal{B}$ is the family of fair objects in $\mathcal{B}$, then we also have

$$\text{MEO}(\mathcal{B}) = \text{MEO}(\mathcal{B}^f).$$

This follows by restricting optimal pmf's on $\mathcal{B}$ to $\mathcal{B}^f$.

Lemma 3.2.5. *Let B be a fair base, let $x \in E - B$, and let C be the unique circuit contained in $B \cup \{x\}$ and containing x . Then,*

$$\eta^*(x) = \max_{e \in C} \eta^*(e). \tag{3.2.3}$$

Proof. Theorem 3.2.2 implies that (3.2.3) is equivalent to

$$\rho^*(x) = \max_{e \in C} \rho^*(e).$$

Arguing by contradiction, assume that

$$\rho^*(x) < \rho^*(e^*), \quad \text{where } e^* \in \operatorname{argmax}_{e \in C} \rho^*(e).$$

Since M is loopless, C contains at least two elements. Hence, $(C - \{x\}) \subset B$. This implies that $e^* \in B$. By part 3 of Proposition 2.1.2, we have $B' := (B - \{e^*\}) \cup \{x\}$ is a base and,

recalling the total usage from (2.1.20), we have

$$\ell_{\rho^*}(B') = \ell_{\rho^*}(B) - \rho^*(e^*) + \rho^*(x) < \ell_{\rho^*}(B).$$

However, since B is fair, $\ell_{\rho^*}(B) = 1$, by Theorem 3.2.2 (iii) (complementary slackness). But, this contradicts the admissibility of ρ^* . $\square$

Lemma 3.2.6. *We have that $\eta^*(e) > 0$ for all $e \in E$.*

Proof. Assume that there exists an element $x \in E$ such that $\eta^*(x) = 0$. Let B be a fair base, then all elements e in B have positive $\eta^*(e)$. Let C be the unique circuit contained in $B \cup \{x\}$ and containing x , as defined in part 2 of Proposition 2.1.2. By Lemma 3.2.5, this leads to a contradiction. $\square$

Definition 3.2.7. Given a matroid $M(E, \mathcal{I})$, let $\mathcal{B}$ be the family of bases of M , and let $\tilde{\mathcal{B}}$ be a Fulkerson dual family of $\mathcal{B}$. Let ρ^* and η^* be the unique optimal densities for $\text{Mod}_2(\mathcal{B})$ and $\text{Mod}_2(\tilde{\mathcal{B}})$, respectively. Then, M is said to be *homogeneous* if η^* is constant, or equivalently, ρ^* is constant.

Remark 3.2.8. Later, we will show that Definition 3.2.7 is equivalent to the concept of homogeneous matroids mentioned in the introduction. From now on, we will use Definition 3.2.7 when discussing homogeneous matroids.

Theorem 3.2.9. *Let $\mathcal{B}$ be the base family of a matroid M . Let $\tilde{\mathcal{B}}$ be a Fulkerson dual family of $\mathcal{B}$. Let η^* be the optimal density for $\text{Mod}_2(\tilde{\mathcal{B}})$. Define the density*

$$\eta_{\text{hom}}(e) := \frac{r(E)}{|E|} \quad \forall e \in E. \tag{3.2.4}$$

Then,

$$\text{Mod}_2(\tilde{\mathcal{B}}) \geq \mathcal{E}_2(\eta_{\text{hom}}) = \frac{r(E)^2}{|E|}.$$

Moreover, M is homogeneous if and only if $\eta_{\text{hom}} \in \text{Adm}(\tilde{\mathcal{B}})$.

Remark 3.2.10. Let $\mu \in \mathcal{P}(\mathcal{B})$ be a pmf, and let $\eta = \mathcal{N}^T \mu$ be the corresponding element usage probabilities. If η is constant, then η is optimal for $\text{Mod}_2(\tilde{\mathcal{B}})$, because such η is automatically admissible for $\tilde{\mathcal{B}}$. Indeed, $\eta \in \text{Dom}(\mathcal{B})$. Hence by (2.1.25) and Definition 2.1.11, $\eta \in \text{Adm}(\tilde{\mathcal{B}})$.

Proof. First, define the expectation and variance of a vector $\xi \in \mathbb{R}^E$ as:

$$\mathbb{E}(\xi) := \frac{1}{|E|} \sum_{e \in E} \xi(e), \quad \text{and} \quad \text{Var}(\xi) := \mathbb{E}(\xi^2) - (\mathbb{E}(\xi))^2,$$

where the square is taken element-wise. Then, we have that

$$\begin{aligned} \mathcal{E}_2(\eta) &= \sum_{e \in E} \eta(e)^2 = |E|(\text{Var}(\eta) + (\mathbb{E}(\eta))^2) = |E| \left(\text{Var}(\eta) + \frac{r(E)^2}{|E|^2} \right) \\ &\geq 0 + \frac{r(E)^2}{|E|} = \frac{r(E)^2}{|E|}, \end{aligned}$$

where equality holds if and only if η is constant.

If M is homogeneous, then $\eta^* = \frac{r(E)}{|E|} = \eta_{\text{hom}}$. If η_{hom} is admissible for $\tilde{\mathcal{B}}$, then $\text{Mod}_2(\tilde{\mathcal{B}})$ achieves its minimum $\frac{r(E)^2}{|E|}$ at $\eta^* = \eta_{\text{hom}}$. $\square$

To gain a better understanding of η^* , we consider the following lemma:

Lemma 3.2.11. *Let $\mathcal{B}$ be the base family of a matroid M , and let B be a base in $\mathcal{B}$. For any subset $X \subseteq E$, it holds that:*

$$r(X) \geq |B \cap X| \geq r(E) - r(E - X). \quad (3.2.5)$$

Proof. Since B is a base, by the hereditary property in Definition 2.1.1 (I2), both $B \cap X$ and

$B - X$ are independent sets. Then, we have:

$$\begin{aligned}
r(E - X) &\geq r(B - X) && \text{(by Proposition 2.1.2 (5))} \\
&= |B - X| && \text{(by Proposition 2.1.2 (7))} \\
&= |B| - |B \cap X| \\
&= r(E) - |B \cap X| && \text{(by Proposition 2.1.2 (8)).}
\end{aligned}$$

and

$$\begin{aligned}
r(X) &\geq r(B \cap X) && \text{(by Proposition 2.1.2 (5))} \\
&= |B \cap X| && \text{(by Proposition 2.1.2 (7)).}
\end{aligned}$$

□

The following lemma gives a relation between η^* and the set of fair bases $\mathcal{B}^f$.

Lemma 3.2.12. *Given a matroid $M(E, \mathcal{I})$, let $\mathcal{B}$ be the base family of M . Let $\tilde{\mathcal{B}}$ be a Fulkerson dual family of $\mathcal{B}$. Let η^* be the optimal density for $\text{Mod}_2(\tilde{\mathcal{B}})$. Then,*

(i) *For any subset $X \subseteq E$, we have:*

$$\eta^*(X) \geq r(E) - r(E - X). \quad (3.2.6)$$

Moreover, equality holds in (3.2.6), if and only if, every fair base $B \in \mathcal{B}^f$ satisfies $r(E - X) = |B - X|$.

(ii) *For any subset $Y \subseteq E$, we have:*

$$r(Y) \geq \eta^*(Y). \quad (3.2.7)$$

Moreover, equality holds in (3.2.7), if and only if, every fair base $B \in \mathcal{B}^f$ satisfies $r(Y) = |B \cap Y|$.

Let us introduce the notions of *Beurling sets* and *complement-Beurling sets* in the following definition. The term ‘Beurling’ is inspired by Beurling’s Criterion and the notion of Beurling families in the theory of modulus; see [3, Theorem 2.1].

Definition 3.2.13. The set X is said to be a *Beurling set*, if equality holds in (3.2.6). The set Y is said to be a *complement-Beurling set*, if equality holds in (3.2.7).

Remark 3.2.14. Note that, since $r(E) = \eta^*(E)$, a set X is a Beurling set if and only if $E - X$ is a complement-Beurling set.

Proof of Lemma 3.2.12. Write $\eta^* = \mathcal{N}^T \mu^*$ for some pmf μ^* optimal for the MEO problem, as in Theorem 3.2.2. Then,

$$\begin{aligned} \eta^*(X) &= \sum_{e \in X} \eta^*(e) = \sum_{e \in X} \sum_{B \in \mathcal{B}} \mu^*(B) \mathcal{N}(B, e) \\ &= \sum_{B \in \mathcal{B}} \mu^*(B) \sum_{e \in X} \mathcal{N}(B, e) = \sum_{B \in \mathcal{B}} \mu^*(B) |B \cap X| \\ &\geq \sum_{B \in \mathcal{B}} \mu^*(B) (r(E) - r(E - X)), \end{aligned}$$

where the last line follows by applying the second inequality in (3.2.5). Finally, to complete the proof of part (i), we note that $\sum_{B \in \mathcal{B}} \mu^*(B) = 1$.

Moreover, the inequality in the last line holds as equality if and only if the second inequality in (3.2.5) holds as equality for every fair base, see Definition 3.2.3.

To prove part (ii), let X be defined so that $Y = E - X$, then use the same idea as in Remark 3.2.14. □

We recall that a set $X \subseteq E$ is said to be closed if $\text{cl}(X) = X$. Now, we define a *complement-closed set* as follows.

Definition 3.2.15. A set $X \subseteq E$ is said to be complement-closed if $\text{cl}(E - X) = E - X$.

Lemma 3.2.16. *Let $X \subseteq E$. If X is a Beurling set, then X is a complement-closed set.*

Proof. Assume that $\text{cl}(E - X) \supsetneq E - X$. Then, there exists an element $x \in X$ such that $x \in \text{cl}(E - X)$. By definition of the closure operator, in (2.1.1), $r((E - X) \cup \{x\}) = r(E - X)$. Lemma 3.2.12 implies

$$\begin{aligned}
\eta^*(X) &> \eta^*(X) - \eta^*(x) = \eta^*(X - \{x\}) && \text{(by Lemma 3.2.6)} \\
&\geq r(E) - r((E - X) \cup \{x\}) && \text{(by Lemma 3.2.12)} \\
&= r(E) - r(E - X) && \text{(by (2.1.1))} \\
&= \eta^*(X). && \text{(by Definition 3.2.13)}
\end{aligned}$$

This results in a contradiction. □

Next, we aim to apply the serial rule for base modulus. To that end, we introduce the following theorem. Recall that the symbol $\oplus$ for concatenation is defined in (A.1.1).

Theorem 3.2.17. *Given a matroid $M(E, \mathcal{I})$, let $\mathcal{B} = \mathcal{B}(M)$ be the base family of M . Let $X \subseteq E$, let $M \setminus X$ be the deletion of X from M . Let $M/(E - X)$ be the contraction of $E - X$ in M . Let $\mathcal{B}^X$ be the set of bases $B \in \mathcal{B}$ which satisfies $r(E - X) = |B - X|$. Then,*

$$\mathcal{B}(M \setminus X) \oplus \mathcal{B}(M/(E - X)) = \mathcal{B}^X. \quad (3.2.8)$$

Proof. First, we aim to show that

$$\mathcal{B}(M \setminus X) \oplus \mathcal{B}(M/(E - X)) \subset \mathcal{B}^X. \quad (3.2.9)$$

By part 3 of Proposition 2.1.4, we have

$$\mathcal{B}(M \setminus X) \oplus \mathcal{B}(M/(E - X)) \subset \mathcal{B}(M).$$

Let $B_1 \in \mathcal{B}(M \setminus X)$, $B_2 \in \mathcal{B}(M/(E - X))$, and let $B_3 = B_1 \cup B_2$. Then, we have

$$\begin{aligned}
|B_3 - X| &= |B_1| && \text{(by construction)} \\
&= r_{M \setminus X}(B_1) && \text{(by definition of } B_1) \\
&= r_{M \setminus X}(E - X) && \text{(by part 8 of Proposition 2.1.2)} \\
&= r_M(E - X). && \text{(by part 3 of Proposition 2.1.3)}
\end{aligned}$$

Hence, (3.2.9) holds.

Next, we aim to show

$$\mathcal{B}(M \setminus X) \oplus \mathcal{B}(M/(E - X)) \supset \mathcal{B}^X. \quad (3.2.10)$$

Let $B \in \mathcal{B}^X$, then $B = (B - X) \cup (B \cap X)$.

On one hand, by definition of $\mathcal{B}^X$ and part 3 of Proposition 2.1.3, we have

$$|B - X| = r_M(E - X) = r_{M \setminus X}(E - X). \quad (3.2.11)$$

Note that $B - X \subset B$, so $B - X$ is an independent set of M . By part 1 of Proposition 2.1.3, $B - X$ is independent in $M \setminus X$. Finally, by part 8 of Proposition 2.1.2 and (3.2.11), we have

$$(B - X) \in \mathcal{B}(M \setminus X).$$

On the other hand, to show that

$$B \cap X \in \mathcal{B}(M/(E - X)),$$

note that $(B \cap X) \cup (B - X) = B \in \mathcal{B}(M)$, and $B - X \in \mathcal{B}(M \setminus X)$, hence it satisfies the conditions in part 3 of Proposition 2.1.4. Therefore, (3.2.10) holds and the proof is

completed. □

Remark 3.2.18. Note that in Theorem 3.2.17, X is not necessarily closed. This will be useful later.

Now we can state the serial rule for base modulus using the Beurling set property of Lemma 3.2.12, the serial rule in Theorem A.1.1 and the concatenation property in Theorem 3.2.17.

Theorem 3.2.19. *Given a matroid $M(E, \mathcal{I})$, let $\mathcal{B} = \mathcal{B}(M)$ be the base family of M , and let $\tilde{\mathcal{B}}$ be a Fulkerson dual family of $\mathcal{B}$. Let $X \subseteq E$ be a Beurling set, and let $M \setminus X$ be the deletion of X from the matroid M . Let $M/(E - X)$ be the contraction of $E - X$ in M . Let η^* be the optimal density for $\text{Mod}_2(\tilde{\mathcal{B}})$. Then, the following hold:*

(i) *The minimum expected overlap splits as*

$$\text{MEO}(\mathcal{B}(M)) = \text{MEO}(\mathcal{B}(M \setminus X)) + \text{MEO}(\mathcal{B}(M/(E - X))); \quad (3.2.12)$$

(ii) *A pmf $\mu \in \mathcal{P}(\mathcal{B}(M))$ is optimal for $\text{MEO}(\mathcal{B}(M))$ if and only if its marginal pmf $\mu_{\mathcal{B}(M \setminus X)} \in \mathcal{P}(\mathcal{B}(M \setminus X))$ is optimal for $\text{MEO}(\mathcal{B}(M \setminus X))$ and its marginal pmf $\mu_{\mathcal{B}(M/(E - X))} \in \mathcal{P}(\mathcal{B}(M/(E - X)))$ is optimal for $\text{MEO}(\mathcal{B}(M/(E - X)))$;*

(iii) *Conversely, given a pmf $\nu_1 \in \mathcal{P}(\mathcal{B}(M \setminus X))$ that is optimal for $\text{MEO}(\mathcal{B}(M \setminus X))$ and a pmf $\nu_2 \in \mathcal{P}(\mathcal{B}(M/(E - X)))$ that is optimal for $\text{MEO}(\mathcal{B}(M/(E - X)))$, then $\nu_1 \oplus \nu_2$ is an optimal pmf in $\mathcal{P}(\mathcal{B}(M))$ for $\text{MEO}(\mathcal{B}(M))$;*

(iv) *The restriction of η^* onto $E - X$ is optimal for $\text{Mod}_2(\tilde{\mathcal{B}}(M \setminus X))$ and the restriction of η^* onto X is optimal for $\text{Mod}_2(\tilde{\mathcal{B}}(M/(E - X)))$.*

Proof. We apply Theorem A.1.1 for the families $\Gamma_1 = \mathcal{B}(M \setminus X)$ and $\Gamma_2 = \mathcal{B}(M/(E - X))$.

By Theorem 3.2.17, the concatenation of Γ_1 and Γ_2 in this case is given by

$$\mathcal{B}(M \setminus X) \oplus \mathcal{B}(M/(E - X)) = \mathcal{B}^X,$$

where $\mathcal{B}^X$ is the set of bases $B \in \mathcal{B}$ which satisfy $r(E - X) = |B - X|$.

Furthermore, by Lemma 3.2.12, since X is a Beurling set, it follows that $\mathcal{B}^f \subset \mathcal{B}^X$. Consequently, by Remark 3.2.4, $\text{MEO}(\mathcal{B}(M)) = \text{MEO}(\mathcal{B}^X)$. The rest of the proof follows by applying Theorem A.1.1. $\square$

3.3 Strength and Fractional Arboricity

In this section, we demonstrate that base modulus recovers known results in the theory of principal partition of matroids, see [14, 15].

Theorem 3.3.1. *Given a loopless matroid $M = (E, \mathcal{I})$, let $\mathcal{B}$ be the base family of M , and let $\tilde{\mathcal{B}}$ be a Fulkerson dual family of $\mathcal{B}$. Let μ^* be an optimal pmf for $\text{MEO}(\mathcal{B})$, and let $\eta^* = \mathcal{N}^T \mu^*$ be the optimal density for $\text{Mod}_2(\tilde{\mathcal{B}})$. Let $S(M)$ be the strength of M . Define:*

$$X := E_{\max} := \left\{ e \in E : \eta^*(e) = \max_{e' \in E} \eta^*(e') =: \eta_{\max}^* \right\}. \quad (3.3.1)$$

Then, we have the following properties:

(1) X is a Beurling set.

(2) $\text{cl}(E - X) = E - X$.

(3)

$$\eta_{\max}^* = \frac{r(E) - r(E - X)}{|X|}.$$

(4)

$$\eta_{\max}^* = \frac{1}{S(M)}.$$

(5) The matroid $M/(E - E_{\max})$ is homogeneous.

Proof. For part 1, Lemma 3.2.12 (i) indicates that the statement is equivalent to $r(E - X) = |B - X|$ for any fair base $B \in \mathcal{B}^f$. Suppose $r(E - X) > |B - X|$ for some fair base B . Let S be

a maximal independent set in $E - X$, then $|S| = r(E - X) > |B - X|$. Note that $B - X$ and S are independent and $(B - X) \subset (E - X)$. By the exchange property (I3) of Definition 2.1.1, there exists $y \in (S - (B - X)) \subset ((E - X) - (B - X))$ such that $(B - X) \cup \{y\}$ is independent. Since B is a base and $y \notin B$, let C be the unique circuit within $B \cup \{y\}$, containing y . Since $(B - X) \cup \{y\}$ is independent, we have that $C \not\subseteq (B - X) \cup \{y\}$. Consequently, $C \cap X \neq \emptyset$. Let x be an element in $C \cap X$. By Lemma 3.2.5, we have $\eta^*(y) \geq \eta^*(x) = \eta_{max}^*$. This is a contradiction because $y \in S \subset E - X$. Therefore, X is a Beurling set, and part 1 is proved.

Part 2 follows from Lemma 3.2.16.

Furthermore, part 3 holds because

$$\begin{aligned} \eta_{max}^* &= \frac{\eta^*(X)}{|X|} && \text{(by definition of } X) \\ &= \frac{r(E) - r(E - X)}{|X|}. && \text{(since } X \text{ is a Beurling set)} \end{aligned}$$

For part 4, note that for any set $\emptyset \neq Y \subseteq E$, we have

$$\begin{aligned} \eta_{max}^* &\geq \frac{\eta^*(Y)}{|Y|} && \text{(by definition of } \eta_{max}^*) \\ &\geq \frac{r(E) - r(E - Y)}{|Y|}. && \text{(by Lemma 3.2.12 (i))} \end{aligned}$$

This implies that, for any set $Y \subseteq E$ with $r(E) > r(E - Y)$, we can write

$$\frac{1}{\eta_{max}^*} \leq \frac{|Y|}{r(E) - r(E - Y)}.$$

Thus, by definition of the strength, we have

$$\frac{1}{\eta_{max}^*} \leq S(M).$$

Moreover, equality holds by part 3, and the minimum is achieved by the set X .

Finally, by Theorem 3.2.19 (iv), the restriction of η^* to E_{max} is optimal for $\text{Mod}_2(\tilde{\mathcal{B}}(M/(E -$

E_{max})). Also, η^* is constant on E_{max} . Therefore, part 5 holds. □

Theorem 3.3.2. *Given a loopless matroid $M = (E, \mathcal{I})$, let $\mathcal{B}$ be the base family of M , and let $\tilde{\mathcal{B}}$ be a Fulkerson dual family of $\mathcal{B}$. Let μ^* be an optimal pmf for the $\text{MEO}(\mathcal{B})$ problem, and let $\eta^* = \mathcal{N}^T \mu^*$ be the optimal density for $\text{Mod}_2(\tilde{\mathcal{B}})$. We define*

$$Y := E_{min} := \left\{ e \in E : \eta^*(e) = \min_{e' \in E} \eta^*(e') =: \eta_{min}^* \right\}. \quad (3.3.2)$$

Then,

1. Y is a complement-Beurling set.

2. $\text{cl}(Y) = Y$.

3.

$$\eta_{min}^* = \frac{r(Y)}{|Y|} > 0.$$

4.

$$\eta_{min}^* = \frac{1}{D(M)}.$$

5. The matroid $M \setminus (E - E_{min})$ is homogeneous.

Proof. For part 1, by Lemma 3.2.12 (ii), the statement is equivalent to $|B \cap Y| = r(Y)$ for any fair base $B \in \mathcal{B}^f$. By Lemma 3.2.11, we have $|B \cap Y| = r(B \cap Y) \leq r(Y)$ for any base B . Suppose that $|B \cap Y| < r(Y)$ for some fair base B . Let T be a maximal independent set in Y , then $|T| = r(Y) > |B \cap Y|$. Note that $B \cap Y$ and T are independent and $(B \cap Y) \subset Y$. By the exchange property (I3) of Definition 2.1.1, there exists $z \in (T - (B \cap Y)) = T - B$ such that $(B \cap Y) \cup \{z\}$ is independent. Since B is a base and $z \notin B$, let C be the unique circuit within $B \cup \{z\}$, containing z . Since $(B \cap Y) \cup \{z\}$ is independent, it follows that $C \not\subseteq (B \cap Y) \cup \{z\}$. Consequently, $C - Y \neq \emptyset$. Let x be an element in $C - Y$. By Lemma

3.2.5, we have $\eta_{min}^* = \eta^*(z) \geq \eta^*(x)$. This leads to a contradiction as $x \notin Y$. Therefore, Y is a complement-Beurling set.

Part 2 follows from by Lemma 3.2.16.

Since Y is a complement-Beurling set, we have

$$\begin{aligned} \eta_{min}^* &= \frac{\eta^*(Y)}{|Y|} && \text{(by definition of } Y) \\ &= \frac{r(Y)}{|Y|} && \text{(by the definition of complement-Beurling sets)} \\ &> 0, && (Y \text{ is nonempty and } M \text{ is loopless}) \end{aligned}$$

and this proves part 3.

For part 4, consider any nonempty set $Z \subseteq E$, we have

$$\begin{aligned} \eta_{min}^* &\leq \frac{\eta^*(Z)}{|Z|} && \text{(by definition of } \eta_{min}^*) \\ &\leq \frac{r(Z)}{|Z|}. && \text{(by Lemma 3.2.12 (ii))} \end{aligned}$$

This implies that for any nonempty set $Z \subseteq E$ with $r(Z) > 0$, we have

$$\frac{1}{\eta_{min}^*} \geq \frac{|Z|}{r(Z)}.$$

Therefore, the fractional arboricity problem reaches its maximum at the set Y .

Part 5 follows from Theorem 3.2.19 (iv). □

Next, we show that Theorems 3.3.1 and 3.3.2 can be used to recover known characterizations of the set of matroids M for which $S(M) = D(M)$, as given in [15, Theorem 6].

Corollary 3.3.3. *Given a matroid $M(E, \mathcal{I})$ with rank function r . Let $S(M)$ be the strength of M , $D(M)$ be the fractional arboricity of M , and $\theta(M)$ be the density of M . Then, the*

following statements are equivalent:

(i) M is homogeneous.

(ii) $S(M) = \theta(M)$.

(iii) $D(M) = \theta(M)$.

(iv) $S(M) = D(M)$.

Proof. By Definition 3.2.7, M is homogeneous if and only if $\eta_{max}^* = \eta_{min}^*$. By Theorems 3.3.1 and 3.3.2 and definitions of $S(M)$, $D(M)$, $\theta(M)$, we have

$$\frac{1}{\eta_{max}^*} = S(M) \leq \theta(M) \leq D(M) = \frac{1}{\eta_{min}^*}. \quad (3.3.3)$$

Therefore, it is enough to show that (ii) implies (i) and (iii) implies (i).

Assume that we have (ii), then

$$\begin{aligned} \eta_{max}^* &\geq \frac{\eta^*(E)}{|E|} && \text{(by definition of } \eta_{max}^*) \\ &\geq \frac{r(E) - r(E \setminus E)}{|E|} && \text{(by Lemma 3.2.12 (i))} \\ &= \eta_{max}^*. && \text{(by the assumption)} \end{aligned}$$

This implies that equality holds throughout, hence η^* is constant, because its average is equal to its max.

Finally, assume that we have (iii), then we obtain

$$\begin{aligned} \eta_{min}^* &\leq \frac{\eta^*(E)}{|E|} && \text{(by definition of } \eta_{min}^*) \\ &\leq \frac{r(E)}{|E|} && \text{(by Lemma 3.2.12 (ii))} \\ &= \eta_{min}^*. && \text{(by the assumption)} \end{aligned}$$

This implies that equality holds throughout and η^* is a constant.

□

3.4 Base packing and base covering problems

In this section, we present an alternative proof of Theorem 2.1.8, using the theory of base modulus. To lay the groundwork for this proof, we provide several lemmas.

Given a matroid $M(E, \mathcal{I})$, let $\mathcal{B} = \mathcal{B}(M)$ be the base family of M , and let $\tilde{\mathcal{B}}$ be a Fulkerson dual family of $\mathcal{B}$. Let $\eta^* = \mathcal{N}^T \mu^*$ be the optimal density for $\text{Mod}_2(\tilde{\mathcal{B}})$. Let $S(M)$, $D(M)$, $\theta(M)$, $\tau(M)$, and $v(M)$ be the strength, the fractional arboricity, the density, the optimal values of the base packing problem (2.1.11), and the base covering problem (2.1.13) of M , respectively.

Lemma 3.4.1. *We have the following chain of inequalities:*

$$\frac{1}{\eta_{max}^*} = S(M) \leq \tau(M) \leq \theta(M) \leq v(M) \leq D(M) = \frac{1}{\eta_{min}^*}. \quad (3.4.1)$$

Proof. Note that in (3.3.3) we have already established the two equalities. Therefore, it remains to show the four inequalities.

1. We aim to show that

$$\frac{1}{\eta_{max}^*} \leq \tau(M).$$

Let μ^* be an optimal probability mass function (pmf) for $\text{MEO}(\mathcal{B})$. We will show that the density $\lambda = \mu^*/\eta_{max}^* \in \mathbb{R}_{\geq 0}^{\mathcal{B}}$ satisfies all the constraints of the base packing problem (2.1.11). Indeed, for a fixed $e \in E$, using Theorem 3.2.2 (i), we see that

$$\sum_{B \in \mathcal{B}: e \in B} \frac{\mu^*(B)}{\eta_{max}^*} = \frac{\eta^*(e)}{\eta_{max}^*} \leq 1.$$

Therefore,

$$\tau(M) \geq \sum_{B \in \mathcal{B}} \frac{\mu^*(B)}{\eta_{max}^*} = \frac{1}{\eta_{max}^*}.$$

2. Likewise, we want to show that

$$v(M) \leq \frac{1}{\eta_{min}^*}.$$

Let μ^* be an optimal probability mass function (pmf) for $\text{MEO}(\mathcal{B})$. Following the approach in part 1, we show that the density $\kappa = \mu^*/\eta_{min}^* \in \mathbb{R}_{\geq 0}^{\mathcal{B}}$ satisfies all the constraints of the base covering problem (2.1.13). Again, for a fixed $e \in E$, by Theorem 3.2.2 (i), we have

$$\sum_{B \in \mathcal{B}: e \in B} \frac{\mu^*(B)}{\eta_{min}^*} = \frac{\eta^*(e)}{\eta_{min}^*} \geq 1.$$

Therefore,

$$v(M) \leq \sum_{B \in \mathcal{B}} \frac{\mu^*(B)}{\eta_{min}^*} = \frac{1}{\eta_{min}^*}.$$

3. Next, we aim to show that

$$\tau(M) \leq \theta(M).$$

Let λ^* be an optimal solution for the base packing problem (2.1.11). We have

$$\begin{aligned} \tau(M)r(E) &= \sum_{B \in \mathcal{B}} r(E)\lambda^*(B) && \text{(by definition of } \lambda^*) \\ &= \sum_{B \in \mathcal{B}} \sum_{e \in E} \lambda^*(B)\mathcal{N}(B, e) && \text{(since } r(E) = |E|) \\ &= \sum_{e \in E} \sum_{B \in \mathcal{B}} \lambda^*(B)\mathcal{N}(B, e) \leq \sum_{e \in E} 1 = |E|, && \text{(by constraint in (2.1.11))} \end{aligned}$$

(3.4.2)

where equality holds if and only if

$$\sum_{B \in \mathcal{B}} \lambda^*(B)\mathcal{N}(B, e) = 1, \quad \forall e \in E.$$

Consequently, we conclude that $\tau(M) \leq |E|/r(E) = \theta(M)$.

4. Lastly, we establish that

$$\theta(M) \leq v(M).$$

Let $\kappa^* \in \mathbb{R}_{\geq 0}^{\mathcal{B}}$ be an optimal solution for the base covering problem (2.1.13). We derive

$$\begin{aligned} v(M)r(E) &= \sum_{B \in \mathcal{B}} r(E)\kappa^*(B) && \text{(by definition of } \kappa^*) \\ &= \sum_{B \in \mathcal{B}} \sum_{e \in E} \kappa^*(B)\mathcal{N}(B, e) && \text{(since } r(E) = |B|) \\ &= \sum_{e \in E} \sum_{B \in \mathcal{B}} \kappa^*(B)\mathcal{N}(B, e) \geq \sum_{e \in E} 1 = |E|, && \text{(by constraint in (2.1.13))} \end{aligned} \tag{3.4.3}$$

where the equality holds if and only if

$$\sum_{B \in \mathcal{B}} \kappa^*(B)\mathcal{N}(B, e) = 1, \quad \forall e \in E.$$

Consequently, this implies that $v(M) \geq |E|/r(E) = \theta(M)$.

□

Remark 3.4.2. Based on the chain of inequalities (3.4.1), if η^* is constant, then it follows that $\tau(M) = \theta(M) = v(M)$.

The next corollary gives a converse for Remark 3.4.2.

Corollary 3.4.3. *The following statements hold:*

1. *If $\tau(M) = \theta(M)$, then η^* is constant.*
2. *If $v(M) = \theta(M)$, then η^* is constant.*

Proof. Assume that $\tau(M) = \theta(M)$. Let λ^* be an optimal solution for the base packing problem. Consider the pmf defined as

$$\mu^* := \frac{\lambda^*}{\sum_{B \in \mathcal{B}} \lambda^*(B)}.$$

Then, the inequality in (3.4.2) holds as equality. Consequently, the pmf μ^* induces constant element usage probabilities $\eta := \mathcal{N}^T \mu^*$. By Remark 3.2.10, it follows that $\eta^* = \eta$.

The proof for the part 2. follows a similar approach. □

Next, we prove some helpful properties of deletions and contractions.

Lemma 3.4.4. *Given a matroid $M(E, \mathcal{I})$ and subsets $X, Y \subseteq E$, the following properties hold:*

1. *If $\text{cl}(E - X) \neq E$, then $S(M) \leq S(M/(E - X))$.*
2. *If $Y \neq \emptyset$, then $D(M) \geq D(M \setminus (E - Y))$.*
3. *If $\text{cl}(E - X) \neq E$, then $\tau(M) \leq \tau(M/(E - X))$.*
4. *If $Y \neq \emptyset$, then $v(M) \geq v(M \setminus (E - Y))$.*

Proof. We prove each part in the order given.

1. We have

$$\begin{aligned} S(M) &= \min \left\{ \frac{|E - T|}{r(E) - r(T)} : T \subseteq E, \text{cl}(T) \subsetneq E \right\} \\ &\leq \min \left\{ \frac{|E - T|}{r(E) - r(T)} : (E - X) \subseteq T \subseteq E, \text{cl}(T) \subsetneq E \right\}. \end{aligned}$$

Since $E - T \subset X$, we can rewrite the right-hand side as

$$\begin{aligned} & \min \left\{ \frac{|X| - |T \cap X|}{[r(E) - r(E - X)] - [r(T) - r(E - X)]} : (E - X) \subseteq T \subseteq E, \text{cl}(T) \subsetneq E \right\} \\ &= \min \left\{ \frac{|X| - |T - (E - X)|}{[r(E) - r(E - X)] - [r(T) - r(E - X)]} : (E - X) \subseteq T \subseteq E, \text{cl}(T) \subsetneq E \right\}. \end{aligned}$$

By part 4 of Proposition 2.1.4, we have

$$r_{M/(E-X)}(X) = r_M(E) - r_M(E - X),$$

and

$$r_{M/(E-X)}(T - (E - X)) = r_M(T) - r_M(E - X).$$

Therefore,

$$\begin{aligned} S(M) &\leq \min \left\{ \frac{|X| - |T - (E - X)|}{r_{M/(E-X)}(X) - r_{M/(E-X)}(T - (E - X))} : (E - X) \subseteq T \subseteq E, \text{cl}(T) \subsetneq E \right\} \\ &= S(M/(E - X)). \end{aligned}$$

2. We have

$$\begin{aligned} D(M) &= \max \left\{ \frac{|T|}{r(T)} : T \subseteq E, r(T) > 0 \right\} \\ &\geq \max \left\{ \frac{|T|}{r(T)} : T \subseteq Y, r(T) > 0 \right\}. \end{aligned}$$

Thus, by part 3 of Proposition 2.1.3, the right hand-side equals

$$\begin{aligned} & \max \left\{ \frac{|T|}{r_{M \setminus (E-Y)}(T)} : T \subseteq Y, r(T) > 0 \right\} \\ &= D(M \setminus (E - Y)). \end{aligned}$$

3. First, we construct a map f from $\mathcal{B}(M)$ to $\mathcal{B}(M/(E - X))$ such that $f(B) \subset B$ for

any base $B \in \mathcal{B}(M)$. Let $B \in \mathcal{B}(M)$, by part 4 of Proposition 2.1.4, we have

$$\begin{aligned} r_{M/(E-X)}(B - (E - X)) &= r_M(B) - r_M(E - X) \\ &= r_M(E) - r_M(E - X) \\ &= r_{M/(E-X)}(X). \end{aligned}$$

Therefore, there exists a base $B_0 \in \mathcal{B}(M/(E - X))$ such that $B_0 \subset B - (E - X) = B \cap X \subset B$. Then, we define $f(B) := B_0$. If B_0 is not unique, we pick one randomly.

Next, let λ^* be an optimal solution for the base packing problem (2.1.11) of M . We define a vector $\omega \in \mathbb{R}_{\geq 0}^{\mathcal{B}(M/(E-X))}$ as follows:

$$\omega(\varsigma) := \sum_{B \in \mathcal{B}(M)} \lambda^*(B) \mathbb{1}_{f(B)=\varsigma}, \quad \forall \varsigma \in \mathcal{B}(M/(E - X)).$$

Then, we obtain

$$\begin{aligned} \tau(M) &= \sum_{B \in \mathcal{B}} \lambda^*(B) && \text{(by definition of } \lambda^*) \\ &= \sum_{\varsigma \in \mathcal{B}(M/(E-X))} \sum_{B \in \mathcal{B}} \lambda^*(B) \mathbb{1}_{f(B)=\varsigma} \\ &= \sum_{\varsigma \in \mathcal{B}(M/(E-X))} \omega(\varsigma). \end{aligned} \tag{3.4.4}$$

However, for every fixed $e \in X$:

$$\begin{aligned}
\sum_{\varsigma \in \mathcal{B}(M/(E-X))} \omega(\varsigma) \mathbb{1}_{e \in \varsigma} &= \sum_{\varsigma \in \mathcal{B}(M/(E-X))} \mathbb{1}_{e \in \varsigma} \sum_{B \in \mathcal{B}(M)} \lambda^*(B) \mathbb{1}_{f(B)=\varsigma} \\
&= \sum_{\varsigma \in \mathcal{B}(M/(E-X))} \sum_{B \in \mathcal{B}(M)} \lambda^*(B) \mathbb{1}_{e \in \varsigma} \mathbb{1}_{f(B)=\varsigma} \\
&= \sum_{B \in \mathcal{B}(M)} \sum_{\varsigma \in \mathcal{B}(M/(E-X))} \lambda^*(B) \mathbb{1}_{e \in \varsigma} \mathbb{1}_{f(B)=\varsigma} \\
&= \sum_{B \in \mathcal{B}(M)} \lambda^*(B) \sum_{\varsigma \in \mathcal{B}(M/(E-X))} \mathbb{1}_{e \in \varsigma} \mathbb{1}_{f(B)=\varsigma} \\
&\leq \sum_{B \in \mathcal{B}(M)} \lambda^*(B) \mathbb{1}_{e \in B} && \text{(since } f(B) \subset B \text{)} \\
&\leq 1. && \text{(by definition of } \lambda^* \text{)}
\end{aligned}$$

Consequently, the vector ω satisfies all constraints of the base packing problem of $M/(E-X)$. By (3.4.4), we obtain that $\tau(M) \leq \tau(M/(E-X))$.

4. Let κ^* be an optimal solution for the base covering problem (2.1.13) of M . For any $B \in \mathcal{B}(M)$, by part 1 of Proposition 2.1.3, we have that $B \cap Y$ is an independent set in the matroid $M \setminus (E-Y)$. Then, for each $I \in \mathcal{I}(M \setminus (E-Y))$, we define

$$\nu(I) := \sum_{B \in \mathcal{B}(M)} \kappa^*(B) \mathbb{1}_{B \cap Y = I}.$$

Then,

$$\begin{aligned}
v(M) &= \sum_{B \in \mathcal{B}(M)} \kappa^*(B) \\
&= \sum_{I \in \mathcal{I}(M \setminus (E-Y))} \sum_{B \in \mathcal{B}(M)} \kappa^*(B) \mathbb{1}_{B \cap Y = I} \\
&= \sum_{I \in \mathcal{I}(M \setminus (E-Y))} \nu(I). \tag{3.4.5}
\end{aligned}$$

Moreover, we have that, for each $e \in Y$

$$\begin{aligned}
\sum_{I \in \mathcal{I}(M \setminus (E - Y))} \nu(I) \mathbb{1}_{e \in I} &= \sum_{I \in \mathcal{I}(M \setminus (E - Y))} \mathbb{1}_{e \in I} \sum_{B \in \mathcal{B}(M)} \kappa^*(B) \mathbb{1}_{B \cap Y = I} \\
&= \sum_{I \in \mathcal{I}(M \setminus (E - Y))} \sum_{B \in \mathcal{B}(M)} \kappa^*(B) \mathbb{1}_{B \cap Y = I} \mathbb{1}_{e \in I} \\
&= \sum_{B \in \mathcal{B}(M)} \kappa^*(B) \sum_{I \in \mathcal{I}(M \setminus (E - Y))} \mathbb{1}_{B \cap Y = I} \mathbb{1}_{e \in I} \\
&= \sum_{B \in \mathcal{B}} \kappa^*(B) \mathbb{1}_{e \in B \cap Y} \\
&= \sum_{B \in \mathcal{B}} \kappa^*(B) \mathbb{1}_{e \in B} \\
&\geq 1. \tag{by definition of κ^* }
\end{aligned}$$

Therefore, vector ν satisfies all constraints of the covering problem (2.1.13) of the matroid $M \setminus (E - Y)$. By (3.4.5), we conclude that $v(M) \geq v(M \setminus (E - Y))$.

□

Proof for Theorem 2.1.8. Let $E_{\min}$ and $E_{\max}$ be defined as in (3.3.2) and (3.3.1). We have

$$\begin{aligned}
D(M) &= D(M \setminus (E - E_{\min})) \quad (\text{by part 4, part 5 of Theorem 3.3.2 and Corollary 3.3.3}) \\
&= \theta(M \setminus (E - E_{\min})) \quad (\text{by part 5 of Theorem 3.3.2 and Corollary 3.3.3}) \\
&= v(M \setminus (E - E_{\min})) \quad (\text{by Remark 3.4.2}) \\
&\leq v(M) \quad (\text{by part 4 of Lemma 3.4.4}) \\
&\leq D(M), \quad (\text{by (3.4.1)})
\end{aligned}$$

and

$$\begin{aligned}
S(M) &= S(M/(E - E_{\max})) && \text{(by part 4, part 5 of Theorem 3.3.1 and Corollary 3.3.3)} \\
&= \theta(M/(E - E_{\max})) && \text{(by part 5 of Theorem 3.3.1 and Corollary 3.3.3)} \\
&= \tau(M/(E - E_{\max})) && \text{(by Remark 3.4.2)} \\
&\geq \tau(M) && \text{(by part 3 Lemma 3.4.4)} \\
&\geq S(M). && \text{(by (3.4.1))}
\end{aligned}$$

Therefore, $v(M) = D(M)$ and $\tau(M) = S(M)$. □

3.5 Fulkerson duality for the base family

Given a matroid $M(E, \mathcal{I})$, let $\mathcal{B}$ be the base family of M . In this section, we introduce the Fulkerson blocker family and a Fulkerson dual family for the family $\mathcal{B}$.

For a matroid $M(E, \mathcal{I})$, a set $X \subseteq E$ is called a *separator* of M if any circuit $C \in \mathcal{C}(M)$ is contained in either X or $E - X$. A matroid $M(E, \mathcal{I})$ is called *connected* if it has no separators other than E and $\emptyset$.

Definition 3.5.1. Let M be a matroid on E with rank function r . Let Φ be the family of all nonempty complement-closed sets $X \subseteq E$ with usage vectors:

$$\tilde{\mathcal{N}}(X, \cdot)^T = \frac{1}{r(E) - r(E - X)} \mathbb{1}_X. \quad (3.5.1)$$

Also, let $\Theta \subset \Phi$ be the family of all nonempty complement-closed sets $X \subseteq E$ with usage vectors as in (3.5.1) for which the matroid $M/(E - X)$ is connected.

Theorem 3.5.2. Let $M = (E, \mathcal{I})$ be a loopless matroid with $r(M) > 0$. Let $\mathcal{B}$ be the base family of M and let Θ be as in Definition 3.5.1. Then, the Fulkerson blocker family of $\mathcal{B}$ is $\hat{\mathcal{B}} = \Theta$.

Remark 3.5.3. By Proposition 2.1.13, the Fulkerson blocker family $\widehat{\mathcal{B}}$ gives a minimal inequality description $\text{Adm}(\widehat{\mathcal{B}})$ of the base dominant $\text{Dom}(\mathcal{B})$. This is a common theme in the study of various convex polytopes.

Proof of Theorem 3.5.2. The proof follows from three lemmas shown below. By Lemma 3.5.4, $\widehat{\mathcal{B}} \subset \Phi$. By Lemma 3.5.9, $\Theta \subset \widehat{\mathcal{B}}$. And by Lemma 3.5.12, $\Phi \cap \widehat{\mathcal{B}} \subseteq \Theta$. $\square$

Lemma 3.5.4. *We have $\widehat{\mathcal{B}} \subset \Phi$.*

To prove Lemma 3.5.4, we first recall the connection between the strength $S(M)$, the base packing value $\tau(M)$, and $\text{Mod}_1(\mathcal{B})$. Note that, we have $S(M) = \tau(M) = \text{Mod}_1(\mathcal{B})$ since the base packing problem is the dual problem of $\text{Mod}_1(\mathcal{B})$. For arbitrary weights $\sigma \in \mathbb{R}_{>0}^E$, we define the weighted strength of M with weights σ :

$$S_\sigma(M) := \min \left\{ \frac{\sigma(X)}{r(E) - r(E - X)} : X \subseteq E, r(E) > r(E - X) \right\}. \quad (3.5.2)$$

We also define the weighted base packing problem:

$$\begin{aligned} & \underset{\lambda \in \mathbb{R}_{\geq 0}^{\mathcal{B}}}{\text{maximize}} && \sum_{B \in \mathcal{B}} \lambda(B) \\ & \text{subject to} && \sum_{B \in \mathcal{B}: e \in B} \lambda(B) \leq \sigma(e), \quad \forall e \in E. \end{aligned} \quad (3.5.3)$$

This is the dual problem of $\text{Mod}_{1,\sigma}(\mathcal{B})$ and we denote its optimal value by $\tau_\sigma(M)$.

Lemma 3.5.5. *We have that $S_\sigma(M) = \tau_\sigma(M) = \text{Mod}_{1,\sigma}(\mathcal{B})$ for any $\sigma \in \mathbb{R}_{>0}^E$.*

Proof. Let $\sigma \in \mathbb{Z}_{>0}^E$. We assign each element e with weight $\sigma(e)$. For each $e \in E(M)$, let $X_e = \{e_1, e_2, \dots, e_{\sigma(e)}\}$ be a set such that $X_e \cap X_{e'} = \emptyset$, for all $e, e' \in E(M)$ with $e \neq e'$. We can think of X_e as the set of $\sigma(e)$ copies of e . The σ -parallel extension M_σ of M is obtained by replacing each element $e \in E(M)$ by X_e . Specifically, the ground set $E(M_\sigma)$ of M_σ is $\bigcup_{e \in E(M)} X_e$. A subset $Y \in E(M_\sigma)$ is independent in M_σ if and only if

$\forall e \in E(M), |X_e \cap Y| \leq 1$ and the set $\{e \in E(M) : X_e \cap Y \neq \emptyset\}$ is independent in M . If every element e is assigned a constant weight r , we write M_r for $M_{\sigma \equiv r}$ and call M_r the r -parallel extension of M . Let $E' = \{e_1 : e \in E(M)\} \subseteq E(M_\sigma)$. Consequently, there is a matroid isomorphism between M and $M_\sigma|E'$ with the bijection $e \leftrightarrow e_1$ between $E(M)$ and E' . So, we shall see M as the restriction $M_\sigma|E'$ of M_σ . Then, we have that $S_\sigma(M) = S(M_\sigma)$ and $\tau_\sigma(M) = \tau(M_\sigma)$. Therefore, we establish that $S_\sigma(M) = \tau_\sigma(M) = \text{Mod}_{1,\sigma}(\mathcal{B}(M))$ for $\sigma \in \mathbb{Z}_{>0}^E$. Note that $S_\sigma(M)$ and $\text{Mod}_{1,\sigma}(\mathcal{B}(M))$ are homogeneous with respect to σ , meaning that $S_{r\sigma}(M) = rS_\sigma(M)$ and $\text{Mod}_{1,r\sigma}(\mathcal{B}(M)) = r \text{Mod}_{1,\sigma}(\mathcal{B}(M))$ for any positive number r . Therefore, $S_\sigma(M) = \tau_\sigma(M) = \text{Mod}_{1,\sigma}(\mathcal{B}(M))$ for $\sigma \in \mathbb{Q}_{>0}^E$. And by continuity, $S_\sigma(M) = \tau_\sigma(M) = \text{Mod}_{1,\sigma}(\mathcal{B}(M))$ for $\sigma \in \mathbb{R}_{>0}^E$. $\square$

Using Lemma 3.5.5 and the Proof of Claim 4.1 in [9], we obtain Lemma 3.5.4. Next, we consider the following well-known proposition for connected matroids.

Proposition 3.5.6. *A matroid M is connected if and only if every two elements of M lie in a common circuit of M .*

Using Proposition 3.5.6, we derive the following lemma.

Lemma 3.5.7. *Let $M(E, \mathcal{I})$ be a connected matroid. Then for any two elements e_1 and e_2 of M there exist two bases B_1 and B_2 such that*

$$B_1 - B_2 = \{e_1\} \quad \text{and} \quad B_2 - B_1 = \{e_2\}.$$

Proof. Let C be a circuit that contains e_1 and e_2 , then $C - \{e_2\}$ is a independent set. There exists a base B_1 such that $(C - \{e_2\}) \in B_1$. By part 3 of Proposition 2.1.2, the set $B_2 := B_1 - \{e_1\} \cup \{e_2\}$ is a base of M . $\square$

Remark 3.5.8. Note that, by Lemma 3.2.11, for any $X \subset E$ such that $r(E) - r(E - X) > 0$, the vector in (3.5.1) is necessarily admissible for $\mathcal{B}$. Indeed, if $B \in \mathcal{B}$,

$$\tilde{\mathcal{N}}(X, \cdot) \mathbb{1}_B = \frac{1}{r(E) - r(E - X)} \mathbb{1}_X^T \mathbb{1}_B = \frac{|B \cap X|}{r(E) - r(E - X)} \geq 1.$$

Lemma 3.5.9. *We have $\Theta \subset \widehat{\mathcal{B}}$.*

Proof. Every $X \in \Theta$ is non-empty and complement-closed, so $r(E) - r(E - X) > 0$. Hence, by Remark 3.5.8, we have that $\Theta \subset \text{Adm}(\mathcal{B})$.

Consider $X \in \Theta$, we aim to show that

$$w := \frac{1}{r(E) - r(E - X)} \mathbb{1}_X \in \text{Ext}(\text{Adm}(\mathcal{B})) = \widehat{\mathcal{B}}.$$

Assume that there are two densities $\rho_1, \rho_2 \in \text{Adm}(\mathcal{B})$ such that

$$w = \frac{1}{2}(\rho_1 + \rho_2).$$

For every element $e \in E - X$, we have $\tilde{\mathcal{N}}(X, e) = 0$, so

$$\frac{1}{2}(\rho_1(e) + \rho_2(e)) = w(e) = 0 \Rightarrow \rho_1(e) = \rho_2(e) = 0.$$

Now, assume that $|X| \geq 2$. Let e_1 and e_2 be two arbitrary distinct elements in X . Since $M/(E - X)$ is connected, by Lemma 3.5.7, there exists two bases B_1, B_2 of $M/(E - X)$ such that $B_1 - B_2 = \{e_1\}$ and $B_2 - B_1 = \{e_2\}$. By part 3 of Proposition 2.1.4, there exists $B'_1 \in \mathcal{B}(M)$ and $B'_2 \in \mathcal{B}(M)$ such that $B_1 = B'_1 - (E - X)$ and $B_2 = B'_2 - (E - X)$.

For any given density $\rho \in \mathbb{R}_{\geq 0}^E$ and $A \subset E$, denote $\rho(A) := \sum_{e \in A} \rho(e)$. Since $\rho_1, \rho_2 \in \text{Adm}(\mathcal{B})$, we have

$$\rho_1(B'_1), \rho_1(B'_2), \rho_2(B'_1), \rho_2(B'_2) \geq 1.$$

Since $\rho_1 + \rho_2 = 2w$, we have $\rho_1(B'_1) + \rho_2(B'_1) = 2w(B'_1) = 2w(B_1) = 2$ and $\rho_1(B'_2) + \rho_2(B'_2) = 2w(B'_2) = 2w(B_2) = 2$. This implies

$$\rho_1(B'_1) = \rho_2(B'_1) = \rho_1(B'_2) = \rho_2(B'_2) = 1.$$

Since $\rho_1(e) = \rho_2(e) = 0$ for all $e \in E - X$, we obtain

$$\rho_1(B_1) = \rho_2(B_1) = \rho_1(B_2) = \rho_2(B_2) = 1.$$

Consequently,

$$\rho_1(e_1) + \rho_1(B_1 \cap B_2) = \rho_1(e_2) + \rho_1(B_1 \cap B_2) = 1,$$

$$\rho_2(e_1) + \rho_2(B_1 \cap B_2) = \rho_2(e_2) + \rho_2(B_1 \cap B_2) = 1.$$

Hence, $\rho_1(e_1) = \rho_1(e_2)$ and $\rho_2(e_1) = \rho_2(e_2)$. Thus, since e_1 and e_2 were chosen arbitrarily, we conclude that ρ_1, ρ_2 are constant in X . Given that $\rho_1(B_1) = \rho_2(B_1) = 1$ and B_1 has $r(E) - r(E - X)$ elements, we derive that

$$\rho_1(e) = \rho_2(e) = \frac{1}{r(E) - r(E - X)} = w(e), \quad \forall e \in X.$$

Therefore, $\rho_1 = \rho_2 = w$. So we have shown that w is an extreme point of $\text{Adm}(\mathcal{B})$.

If $|X| = 1$, suppose $X = \{e\}$. Let B be a base M that contains e . Using the same argument as above, we have $\rho_1(e) = \rho_2(e) = \frac{1}{r(E) - r(E - X)} = 1$. Therefore, $\rho_1 = \rho_2 = w$, and once again, w is an extreme point of $\text{Adm}(\mathcal{B})$. In conclusion, $w \in \widehat{\mathcal{B}}$. $\square$

Remark 3.5.10. Note that the connectedness of $M/(E - X)$ implies that X is complement-closed. This is because if $E - X$ is not closed, then $M/(E - X)$ will have some loops, which act as nontrivial separators.

Proposition 3.5.11 ([26]). *For a matroid $M(E, \mathcal{I})$, a set $X \subseteq E$ is a separator of M if and only if $r(X) + r(E - X) = r(E)$.*

Using Proposition 3.5.11, we derive the following lemma:

Lemma 3.5.12. *We have $\Phi \cap \widehat{\mathcal{B}} \subseteq \Theta$.*

Proof. Let $X \in \Phi \cap \widehat{\mathcal{B}}$. Assume that the matroid $M/(E - X)$ is not connected. Then, let A

be a nontrivial separator of $M/(E - X)$ and define $D := X - A$. Using part 4 of Proposition 2.1.4, we derive that

$$r_{M/(E-X)}(A) = r_M(E - D) - r_M(E - X), \quad (3.5.4)$$

$$r_{M/(E-X)}(D) = r_M(E - A) - r_M(E - X),$$

and

$$r_{M/(E-X)}(X) = r_M(E) - r_M(E - X).$$

Since X is nonempty and complement-closed, $E - X$ is closed in M . Therefore, since $E - D \supsetneq E - X$, we deduce $r(E - D) > r(E - X)$, which, by (3.5.4), implies that $r_{M/(E-X)}(A) > 0$. Similarly, we also have $r_{M/(E-X)}(D) > 0$. Consequently, by Proposition 3.5.11, we have

$$r_M(E) - r_M(E - D) = r_{M/(E-X)}(X) - r_{M/(E-X)}(A) = r_{M/(E-X)}(D) > 0, \quad (3.5.5)$$

$$r_M(E) - r_M(E - A) = r_{M/(E-X)}(X) - r_{M/(E-X)}(D) = r_{M/(E-X)}(A) > 0, \quad (3.5.6)$$

and adding these two identities and using Proposition 3.5.11 again, we obtain

$$2r_M(E) - r_M(E - A) - r_M(E - D) = r_{M/(E-X)}(X) = r_M(E) - r_M(E - X). \quad (3.5.7)$$

Therefore, we have that $\frac{1}{r_M(E) - r_M(E - X)} \mathbb{1}_X$ is equal to

$$\frac{r_M(E) - r_M(E - A)}{r_M(E) - r_M(E - X)} \cdot \frac{1}{r_M(E) - r_M(E - A)} \mathbb{1}_A + \frac{r_M(E) - r_M(E - D)}{r_M(E) - r_M(E - X)} \cdot \frac{1}{r_M(E) - r_M(E - D)} \mathbb{1}_D.$$

Hence, by (3.5.7), we see that $\frac{1}{r_M(E) - r_M(E - X)} \mathbb{1}_X$ is a convex combination of $\frac{1}{r(E) - r(E - A)} \mathbb{1}_A$ and $\frac{1}{r(E) - r(E - D)} \mathbb{1}_D$ and these two vectors are in $\text{Adm}(\mathcal{B})$, by (3.5.5) and (3.5.6) and Remark 3.5.8. This is a contradiction, since $X \in \widehat{\mathcal{B}}$. $\square$

3.6 Modulus for dual matroids

Given a loopless matroid $M(E, \mathcal{I})$ with $r(E) > 0$, let $\mathcal{B}$ be the base family of M . We recall that the set

$$\mathcal{B}^* = \{X \subseteq E : \text{there exists a base } B \in \mathcal{B} \text{ such that } X = E - B\}$$

is the family of bases of the dual matroid M^* of M with the rank function r^* . We also have that $r^*(E) = |E| - r(M)$. The dual of the dual of a matroid M is the matroid M itself, in other words, $(M^*)^* = M$.

For a loopless matroid M , the dual matroid M^* is not necessarily loopless. If we consider the base modulus for a matroid with loops, the optimal element usage probabilities η^* of those loops are zero because they are not contained in any bases. Furthermore, the dual matroid M^* does not necessarily have positive rank. Specifically, $r^*(E) > 0$ if and only if $r(E) < |E|$.

Note that for a loopless matroid $M(E, \mathcal{I})$, by Lemma 3.2.6, we have $0 < \eta^*(e) \leq 1$ for all $e \in E$, where η^* are the optimal element usage probabilities. In this section, we investigate the relationships between the base modulus of a matroid and the base modulus of its dual. Consider a loopless matroid $M(E, \mathcal{I})$ with $r(E) > 0$. If $r(E) = |E|$, then the dual matroid M^* has rank zero. Now, assuming $r(E) < |E|$, the concept of dual matroids motivates the following theorem.

Theorem 3.6.1. *Let E be a finite set. Let Γ be a family of non-empty proper subsets of E of equal size with usage vectors given by indicator functions. Define $\Gamma^* := \{E - \gamma : \gamma \in \Gamma\}$ with usage vectors given by indicator functions. Let $\widehat{\Gamma}$ and $\widehat{\Gamma}^*$ be the Fulkerson blocker families of Γ and Γ^* , respectively. Let η^* and η_o^* be the optimal densities for $\text{Mod}_2(\widehat{\Gamma})$ and $\text{Mod}_2(\widehat{\Gamma}^*)$, respectively. Then we have*

$$\eta^* + \eta_o^* = \mathbf{1},$$

where $\mathbf{1}$ is the vector of all ones.

Proof. Let $\mathcal{N}$ be the usage matrix of Γ . Then, all row sums of $\mathcal{N}$ are equal to k . Let $\mathbf{1}_{|\Gamma| \times |E|}$ be the $|\Gamma| \times |E|$ matrix of all ones. Consequently, $\mathbf{1}_{|\Gamma| \times |E|} - \mathcal{N}$ is the usage matrix of Γ^* .

Note that by the proof of Lemma 3.2.1, we have

$$(\eta^*)^T \mathbf{1} = k, \quad (3.6.1)$$

and

$$(\eta_o^*)^T \mathbf{1} = |E| - k. \quad (3.6.2)$$

Additionally, we have

$$\mathcal{N}\mathbf{1} = k\mathbf{1}, \quad (\mathbf{1}_{|\Gamma| \times |E|})\eta^* = k\mathbf{1}, \quad (\mathbf{1}_{|\Gamma| \times |E|})\eta_o^* = (|E| - k)\mathbf{1}. \quad (3.6.3)$$

By Theorem 3.2.2, we have that $\rho^* = \eta^*/\mathcal{E}(\eta^*)$ is the optimal solution for $\text{Mod}_2(\Gamma)$ and $\rho_o^* = \eta_o^*/\mathcal{E}(\eta_o^*)$ is the optimal solution for $\text{Mod}_2(\Gamma^*)$. Hence, by admissibility, we have

$$\mathcal{N} \frac{\eta^*}{\mathcal{E}(\eta^*)} \geq \mathbf{1}, \quad (3.6.4)$$

and

$$(\mathbf{1}_{|\Gamma| \times |E|} - \mathcal{N}) \frac{\eta_o^*}{\mathcal{E}(\eta_o^*)} \geq \mathbf{1}, \quad (3.6.5)$$

where these two inequalities hold element-wise. Let $a = \mathbf{1} - \eta_o^* \leq \mathbf{1}$, then

$$\begin{aligned} (\mathbf{1}_{|\Gamma| \times |E|} - \mathcal{N}) \frac{\eta_o^*}{\mathcal{E}(\eta_o^*)} &= (\mathbf{1}_{|\Gamma| \times |E|} - \mathcal{N}) \frac{\mathbf{1} - a}{\mathcal{E}(\mathbf{1} - a)} \\ &= \frac{|E|\mathbf{1} - k\mathbf{1} - k\mathbf{1} + \mathcal{N}a}{\mathcal{E}(\mathbf{1} - a)} && \text{(by (3.6.3))} \\ &= \frac{(|E| - 2k)\mathbf{1} + \mathcal{N}a}{|E| - 2k + \mathcal{E}(a)}, && \text{(by (3.6.2))} \end{aligned}$$

where in the denominator we expanded the squares and used (3.6.2). Therefore, (3.6.5) is

equivalent to

$$(|E| - 2k)\mathbf{1} + \mathcal{N}a \geq (|E| - 2k + \mathcal{E}(a))\mathbf{1}$$

which we rewrite as,

$$\mathcal{N} \frac{(\mathbf{1} - \eta_{\circ}^*)}{\mathcal{E}(\mathbf{1} - \eta_{\circ}^*)} = \frac{\mathcal{N}a}{\mathcal{E}(a)} \geq \mathbf{1},$$

Similarly, the inequality (3.6.4) is equivalent to

$$(\mathbb{1}_{|\Gamma| \times |E|} - \mathcal{N}) \frac{\mathbf{1} - \eta^*}{\mathcal{E}(\mathbf{1} - \eta^*)} \geq \mathbf{1}.$$

In other words, the vector $\frac{\mathbf{1} - \eta_{\circ}^*}{\mathcal{E}(\mathbf{1} - \eta_{\circ}^*)}$ is admissible for Γ and $\frac{\mathbf{1} - \eta^*}{\mathcal{E}(\mathbf{1} - \eta^*)}$ is admissible for Γ^* . Hence, since ρ^* is optimal for $\text{Mod}_2(\Gamma)$, we have

$$\frac{1}{\mathcal{E}(\mathbf{1} - \eta_{\circ}^*)} = \mathcal{E}\left(\frac{\mathbf{1} - \eta_{\circ}^*}{\mathcal{E}(\mathbf{1} - \eta_{\circ}^*)}\right) \geq \mathcal{E}(\rho^*) = \frac{1}{\mathcal{E}(\eta^*)}, \quad (3.6.6)$$

and, since $\rho_{\circ}^*$ is optimal for $\text{Mod}_2(\Gamma^*)$,

$$\frac{1}{\mathcal{E}(\mathbf{1} - \eta^*)} = \mathcal{E}\left(\frac{\mathbf{1} - \eta^*}{\mathcal{E}(\mathbf{1} - \eta^*)}\right) \geq \mathcal{E}(\rho_{\circ}^*) = \frac{1}{\mathcal{E}(\eta_{\circ}^*)}. \quad (3.6.7)$$

These are equivalent to

$$\mathcal{E}(\eta^*) \geq \mathcal{E}(\mathbf{1} - \eta_{\circ}^*) = |E| - 2(|E| - k) + \mathcal{E}(\eta_{\circ}^*), \quad (\text{by (3.6.2)})$$

and

$$\mathcal{E}(\eta_{\circ}^*) \geq \mathcal{E}(\mathbf{1} - \eta^*) = |E| - 2k + \mathcal{E}(\eta^*). \quad (\text{by (3.6.1)})$$

Consequently, we have shown that

$$\mathcal{E}(\eta^*) - k \geq \mathcal{E}(\eta_{\circ}^*) - (|E| - k) \geq \mathcal{E}(\eta^*) - k.$$

Therefore, two inequalities in (3.6.6) and (3.6.7) holds as equalities. Hence, by uniqueness of η^* and η_o^* , we obtain that $\mathbf{1} - \eta_o^* = \eta^*$. $\square$

Applying this to the case when Γ is the base family of a matroid, we recover a known result [15] about strength and fractional arboricity of dual matroids.

Corollary 3.6.2. *Let $M = (E, \mathcal{I})$ be a loopless matroid and $0 < r(M) < |E|$. Let M^* be the dual matroid of M . Let η^* be the optimal density for $\text{Mod}_2(\widehat{\mathcal{B}(M)})$ and let η_o^* be the optimal density for $\text{Mod}_2(\widehat{\mathcal{B}(M^*)})$. Then, we have*

$$\eta^* + \eta_o^* = \mathbf{1},$$

where $\mathbf{1}$ is the vector of all ones. Moreover, if the dual matroid M^* is loopless, we have

$$\frac{1}{D(M)} + \frac{1}{S(M^*)} = 1,$$

and

$$\frac{1}{S(M)} + \frac{1}{D(M^*)} = 1.$$

Proof. Given that M is loopless, we have

$$\eta_{\min}^* = \frac{1}{D(M)} \quad \text{and} \quad \eta_{\max}^* = \frac{1}{S(M)}.$$

If M^* is also loopless, we get

$$(\eta_o^*)_{\min} = \frac{1}{D(M^*)} \quad \text{and} \quad (\eta_o^*)_{\max} = \frac{1}{S(M^*)}.$$

The proof follows by applying Theorem 3.6.1. $\square$

3.7 Deflation process and other values of p

Let $M(E, \mathcal{I})$ be a matroid with base family $\mathcal{B} = \mathcal{B}(M)$. We want to show that $\text{Mod}_p(\mathcal{B})$ can be deduced from $\text{Mod}_2(\mathcal{B})$.

Theorem 3.7.1. *Let $M(E, \mathcal{I})$ be a matroid with base family $\mathcal{B} = \mathcal{B}(M)$. Let $\tilde{\mathcal{B}}$ be a Fulkerson dual family of $\mathcal{B}$. Let $p \in (1, \infty) \setminus \{2\}$. Let q be the Hölder conjugate exponent of p , so that $(p-1)(q-1) = 1$. Then $\text{Mod}_2(\tilde{\mathcal{B}})$ and $\text{Mod}_q(\tilde{\mathcal{B}})$ have the same optimal density η^* . Moreover,*

$$\text{Mod}_p(\mathcal{B}) = \left(\sum_{e \in E} (\eta^*(e))^q \right)^{1-p}.$$

Proof. Let η^* be the unique optimal density for $\text{Mod}_2(\tilde{\mathcal{B}})$. We recall $E_{\min}$ defined as in (3.3.2). According to part 1 of Theorem 3.3.2, $E_{\min}$ is a complement-Beurling set. Our objective is to iteratively apply Theorem 3.2.19 to the Beurling set $E - E_{\min}$. Initially, we decompose the matroid M into $M_1 := M|_{E_{\min}}$ and $M/E_{\min}$ by utilizing Theorem 3.2.19. Then, M_1 is homogeneous, we proceed to further decompose the matroid $M/E_{\min}$ using the minimum value of η^* , continuing this deflation process. Then, this deflation process yields the principal partitions of the matroid M . Let us denote $M_1, M_2, \dots, M_k$ as the sequence of homogeneous matroids resulting from this process, alongside the corresponding sequence of ground sets $E_1, E_2, \dots, E_k$.

Note that $E_1, E_2, \dots, E_k$ are subsets of E , and η^* is equal to the constant $\eta_j^* := \frac{r(M_j)}{|E_j|}$ on each E_j , where $r(M_j)$ is the rank of matroid M_j . Moreover, we have $\eta_1^* < \eta_2^* < \dots < \eta_k^*$. Then, we establish that

$$(\text{Mod}_2(\mathcal{B}(M)))^{-1} = \sum_{j=1}^k \frac{r(M_j)^2}{|E_j|}. \quad (3.7.1)$$

The q -energy of η^* is

$$\mathcal{E}_q(\eta^*) = \sum_{j=1}^k |E_j| \left(\frac{r(M_j)}{|E_j|} \right)^q. \quad (3.7.2)$$

We define a density ρ by the formula

$$\rho := \frac{(\eta^*)^{q-1}}{\mathcal{E}_q(\eta^*)}.$$

We want to check that $\rho \in \text{Adm}(\mathcal{B})$. Note that for every fair base $B \in \mathcal{B}$, we have $|B \cap E_j| = r(M_j)$ for every j . This is because since E_1 is a complement-Beurling set, we have $|B \cap E_1| = r_M(E_1) = r(M_1)$ (by Lemma 3.2.11 and part 3 of Proposition 2.1.3). Then $B \cap (E - E_1)$ is a base in M/E_1 (by part 3 of Proposition 2.1.4), and it is also a fair base in M/E_1 (by Theorem 3.2.19 (ii)). By induction, we achieve that $|B \cap E_j| = r(M_j)$ for every j . Therefore, for any fair base B , we have

$$\begin{aligned} \ell_\rho(B) &= \frac{1}{\mathcal{E}_q(\eta^*)} \sum_{j=1}^k r(M_j) \left(\frac{r(M_j)}{|E_j|} \right)^{q-1} \\ &= \frac{1}{\mathcal{E}_q(\eta^*)} \sum_{j=1}^k |E_j| \left(\frac{r(M_j)}{|E_j|} \right)^q \\ &= 1, \end{aligned} \tag{3.7.3}$$

where the last equality follows from (3.7.2).

Let B be a base in $\mathcal{B}(M)$ that has minimum total usage $\ell_\rho(B)$. We have that $|B \cap E_1| \leq r_M(E_1) = r(M_1)$ by Lemma 3.2.11. Assume that $|B \cap E_1| < r_M(E_1)$. Let T be a maximal independent set in E_1 , then $|T| = r_M(E_1) > |B \cap E_1|$. Note that $B \cap E_1$ and T are independent and $(B \cap E_1) \subset E_1$. By the exchange property (I3) of Definition 2.1.1, there exists $z \in (T - (B \cap E_1)) = T - B$ such that $(B \cap E_1) \cup \{z\}$ is independent in M . Since B is a base and $z \notin B$, let C be the unique circuit within $B \cup \{z\}$, containing z . Since $(B \cap E_1) \cup \{z\}$ is independent in M , it follows that $C \not\subseteq (B \cap E_1) \cup \{z\}$. Consequently, $C - E_1 \neq \emptyset$. Let x be an element in $C - E_1$. Then $\rho(x) > \rho(z)$ by definitions of ρ and E_1 . Note that $x \in C$ and $x \neq z$, then $x \in B$. By part 3 of Proposition 2.1.2, we have $B' := (B - \{x\}) \cup \{z\}$ is a

base and, recalling the total usage from (2.1.20), we have

$$\ell_\rho(B') = \ell_\rho(B) - \rho(x) + \rho(z) < \ell_\rho(B).$$

This contradicts the definition of B . Therefore, we have that $|B \cap E_1| = r(M_1)$. Then $B \cap (E - E_1)$ is a base in M/E_1 (by part 3 of Proposition 2.1.4). Then, we use the same argument, by induction, we achieve that $|B \cap E_j| = r(M_j)$ for every j . Therefore, by (3.7.3), we have $\ell_p(B) = 1$. Thus, $\rho \in \text{Adm}(\mathcal{B})$.

Next, since $\rho \in \text{Adm}(\mathcal{B})$ and $\eta^* \in \text{Adm}(\tilde{\mathcal{B}})$, we have

$$\begin{aligned} \text{Mod}_p(\mathcal{B}) &\leq \mathcal{E}_p(\rho) \\ &= \sum_{e \in E} \rho(e)^p = \frac{1}{\mathcal{E}_q(\eta^*)^p} \sum_{e \in E} (\eta^*(e))^{qp-p} \\ &= \frac{1}{\mathcal{E}_q(\eta^*)^p} \sum_{e \in E} (\eta^*(e))^q = \mathcal{E}_q(\eta^*)^{1-p} \\ &\leq \left(\text{Mod}_q(\tilde{\mathcal{B}}) \right)^{1-p}. \end{aligned} \tag{3.7.4}$$

But Fulkerson duality for p -modulus says that

$$\left(\text{Mod}_q(\tilde{\mathcal{B}}) \right)^{1-p} = (\text{Mod}_p(\mathcal{B}))^{-(q-pq)/p} = (\text{Mod}_p(\mathcal{B})).$$

Therefore, equality holds in (3.7.4) and we have shown that ρ is extremal for $\text{Mod}_p(\mathcal{B})$. By (2.1.28), we have η^* is the optimal density for $\text{Mod}_q(\tilde{\mathcal{B}})$. In particular, we have the following formula

$$\text{Mod}_p(\mathcal{B}) = \left(\sum_{j=1}^k \frac{r(M_j)^q}{|E_j|^{q-1}} \right)^{1-p}.$$

□

Chapter 4

Matroid Reinforcement, and Sparsification

4.1 Overview

The base modulus theory of Chapter 3 treats every element of the ground set on equal footing. In many applications, however, the elements of a matroid carry weights, and it is the interplay between these weights and the combinatorial structure that one wishes to control. This chapter develops base modulus for weighted matroids and uses it to study two cost-sensitive modifications of the weights: reinforcement and sparsification.

Our starting point is a network-theoretic one. For a connected graph $G = (V, E)$, Cunningham [11] studied the *strength reinforcement problem*: given a per-unit cost $m(e)$ for raising the weight of each edge e and a target value s_0 , increase the edge weights as cheaply as possible so that the resulting network has strength at least s_0 . His solution rests on the polymatroid greedy algorithm, together with an algorithm for the associated *attack problem*, and the same circle of ideas underlies the algorithms developed in this chapter. We take the reinforcement viewpoint a step further: rather than reaching a prescribed strength, we ask how to modify the weights so that the matroid becomes homogeneous, and we treat both

increases and decreases of weight.

To make this precise we work with the weighted analogues of strength and fractional arboricity. For a weighted matroid $M = (E, \mathcal{I})$ with weight vector $\sigma \in \mathbb{R}_{>0}^E$, these invariants $S_\sigma(M)$ and $D_\sigma(M)$ are obtained from the density ratios of Section 2.1.1 by replacing cardinalities with total weights. The matroid is called σ -homogeneous when

$$S_\sigma(M) = D_\sigma(M),$$

which extends the notion of homogeneity from the unweighted setting. A guiding fact, established in Section 4.2, is the characterization

$$\sigma \in \text{conic}(\mathcal{B}) \iff S_\sigma(M) = D_\sigma(M), \quad (4.1.1)$$

where $\text{conic}(\mathcal{B})$ denotes the conical hull in $\mathbb{R}^E$ of the indicator vectors of the bases in $\mathcal{B}$. Thus σ -homogeneity is equivalent to membership of σ in the cone generated by the base family.

With (4.1.1) in hand, the two problems of the chapter can be stated as follows. Given a weighted matroid M that is not σ -homogeneous and a per-unit modification cost $m(e) \geq 0$, the *matroid reinforcement problem* seeks the cheapest weight increase that makes M homogeneous,

$$\begin{aligned} & \underset{z \in \mathbb{R}_{\geq 0}^E}{\text{minimize}} && m^T z \\ & \text{subject to} && \sigma + z \in \text{conic}(\mathcal{B}), \end{aligned} \quad (4.1.2)$$

while the *matroid sparsification problem* seeks the cheapest weight decrease achieving the same,

$$\begin{aligned} & \underset{z \in \mathbb{R}_{\geq 0}^E}{\text{minimize}} && m^T z \\ & \text{subject to} && \sigma - z \in \text{conic}(\mathcal{B}). \end{aligned} \quad (4.1.3)$$

Problem (4.1.2) generalizes Cunningham's reinforcement problem, and problem (4.1.3) is its complementary counterpart.

The approach to (4.1.2) and (4.1.3) is geometric, and rests on two dilation identities for the base dominant $\text{Dom}(\mathcal{B})$. Writing $hK := \{hx : x \in K\}$ for the dilation of a set $K \subseteq \mathbb{R}_{\geq 0}^E$ by $h > 0$, we show in Section 4.3 that the weighted strength is the largest dilation of the dominant still containing σ ,

$$S_\sigma(M) = \max \{h > 0 : \sigma \in h \text{ Dom}(\mathcal{B})\}. \quad (4.1.4)$$

Dually, letting

$$\text{Antidom}(\mathcal{B}) := (\text{conv}(\mathcal{B}) - \mathbb{R}_{\geq 0}^E) \cap \mathbb{R}_{\geq 0}^E \quad (4.1.5)$$

be the *anti-dominant* of $\mathcal{B}$, consisting of the points of the nonnegative orthant lying below the convex hull of the bases, we show that the weighted fractional arboricity is the smallest such dilation,

$$D_\sigma(M) = \min \{k > 0 : \sigma \in k \text{ Antidom}(\mathcal{B})\}. \quad (4.1.6)$$

Together, (4.1.4) and (4.1.6) reduce the feasible regions of the two problems to a single scalar each: if $\sigma + z \in h \text{ conv}(\mathcal{B})$ then $\sigma \in h \text{ Antidom}(\mathcal{B})$, so $h \geq D_\sigma(M)$, and $D_\sigma(M)$ turns out to be the most cost-effective choice; symmetrically, $S_\sigma(M)$ is the most cost-effective scale for the sparsification problem. Once the scale is fixed, the polymatroid greedy algorithm delivers an optimal redistribution of weight. Figure 4.1 illustrates these three facts.

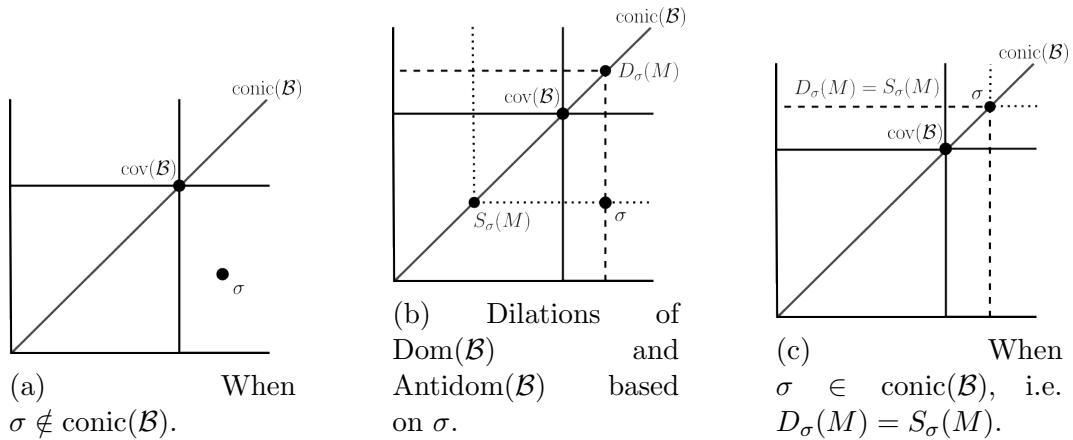


Figure 4.1: Subfigures (a) and (b) illustrate (4.1.4) and (4.1.6); subfigure (c) illustrates (4.1.1).

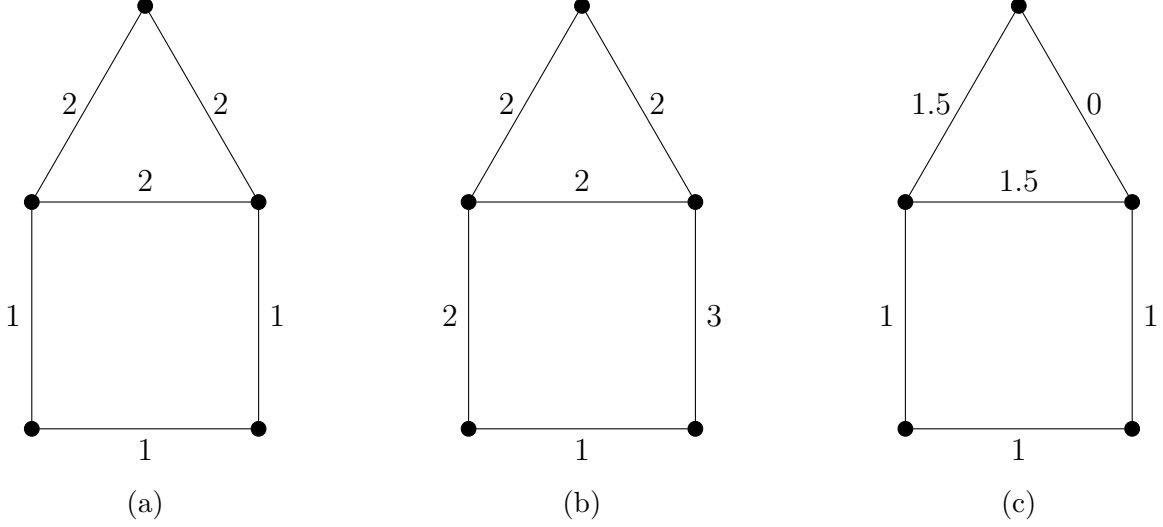


Figure 4.2: An example with the House Graph: (a) original edge weights, (b) reinforced edge weights, and (c) sparsified edge weights.

Because the arguments draw on properties of polymatroids rather than matroids alone, we carry out the analysis in the polymatroid setting: throughout the chapter, the reinforcement and sparsification problems (4.1.2) and (4.1.3) are studied in their generalized, polymatroidal form.

The contributions of the chapter are organized as follows.

- In Section 4.2, we extend base modulus from unweighted to weighted matroids by means of parallel elements, and establish the conical-hull characterization of σ -homogeneity in (4.1.1).
- In Section 4.3, we treat the generalized versions of (4.1.2) and (4.1.3) in the polymatroid setting. We first reduce their feasible sets via the dilation identities (4.1.4) and (4.1.6) (Theorem 4.3.8 and Theorem 4.3.18), and then show that the polymatroid greedy algorithm yields optimal solutions. In Section 4.3.4, we relate our problems to Cunningham's strength reinforcement problem [11].
- In Section 4.4, we specialize the two algorithms to graphic matroids, giving explicit implementations of Algorithm 1 and Algorithm 2 through Cunningham's minimum-cut formulations. We further give Algorithm 3 for computing fractional arboricity and, as

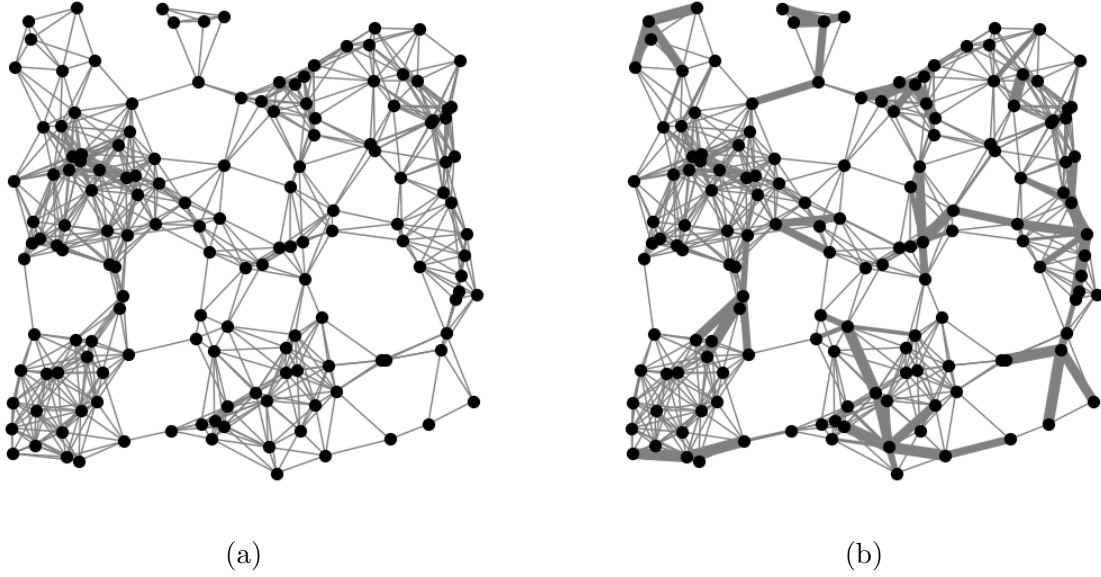


Figure 4.3: Homogeneous reinforcement on a random geometric graph. Edge thickness represents weights: (a) original uniform edge weights (all 1), (b) reinforced edge weights.

a consequence, Algorithm 4 for computing the spanning tree modulus from it.

Remark 4.1.1. Detailed algorithms for reinforcement and sparsification are presented here for graphic matroids; Figures 4.2 and 4.3 work through two examples of Algorithms 1 and 2 on graphs. Extending these methods to hypergraphic matroids is a natural direction for future work.

4.2 Weighted Base Modulus via Parallel Extensions

In this section, we extend the base modulus theory from unweighted matroids to weighted matroids. Let $M = (E, \mathcal{I})$ be a matroid with rank function r , and let $\sigma \in \mathbb{R}_{>0}^E$ be a positive weight vector. For $X \subseteq E$, write $\sigma(X) := \sum_{e \in X} \sigma(e)$.

The weighted strength and weighted fractional arboricity of M are obtained from the unweighted definitions by replacing cardinality with total weight:

$$S_\sigma(M) = \min \left\{ \frac{\sigma(X)}{r(E) - r(E - X)} : X \subseteq E, r(E) > r(E - X) \right\}, \quad (4.2.1)$$

and

$$D_\sigma(M) = \max \left\{ \frac{\sigma(X)}{r(X)} : X \subseteq E, r(X) > 0 \right\}. \quad (4.2.2)$$

We also write

$$\theta_\sigma(X) := \frac{\sigma(X)}{r(X)}$$

whenever $r(X) > 0$, and set $\theta_\sigma(M) := \theta_\sigma(E)$.

The weighted theory can be understood through parallel extensions. We first assume that $\sigma \in \mathbb{Z}_{>0}^E$. For each $e \in E$, let

$$X_e := \{e_1, e_2, \dots, e_{\sigma(e)}\}$$

be a set of $\sigma(e)$ parallel copies of e , with the sets X_e pairwise disjoint. The σ -parallel extension M_σ is the matroid with ground set

$$E(M_\sigma) = \bigcup_{e \in E} X_e,$$

where a set $Y \subseteq E(M_\sigma)$ is independent if and only if $|Y \cap X_e| \leq 1$ for every $e \in E$ and

$$\{e \in E : Y \cap X_e \neq \emptyset\}$$

is independent in M . Thus, M_σ is obtained by replacing each element e by $\sigma(e)$ parallel copies. If $\sigma(e) = q$ for all $e \in E$, we write M_q and call it the q -parallel extension of M .

This construction converts weights into multiplicities. The weighted quantities for M agree with the corresponding unweighted quantities for M_σ :

$$S_\sigma(M) = S(M_\sigma), \quad D_\sigma(M) = D(M_\sigma), \quad \text{Mod}_{2,\sigma}(\mathcal{B}(M)) = \text{Mod}_2(\mathcal{B}(M_\sigma)). \quad (4.2.3)$$

For strength, an optimal set in M_σ may be taken to be a union of entire parallel classes. For fractional arboricity, adding the remaining copies in a parallel class does not change rank

and can only increase the density. Hence the extremal ratios in M_σ are exactly the weighted ratios in M .

The same parallel-extension idea transfers the unweighted base modulus results to the weighted setting. For integer weights, this follows directly from M_σ . For rational weights, one first clears denominators. For real positive weights, the result follows by approximation and continuity of the weighted modulus problem. Thus, the weighted theory is the natural extension of the unweighted theory obtained by replacing elements with parallel copies. Next, we present the weighted generalization of base modulus for matroids.

Let $\mathcal{B} = \mathcal{B}(M)$ be the base family of M , and let $\eta^* = \eta_M^*$ denote the weighted universal density of the base modulus problem. We define $\sigma^{-1}\eta^*$ componentwise by

$$(\sigma^{-1}\eta^*)(e) := \frac{\eta^*(e)}{\sigma(e)}.$$

The following definition extends homogeneity from the unweighted setting to the weighted setting.

Definition 4.2.1. A weighted matroid (M, σ) is called σ -homogeneous if the vector $\sigma^{-1}\eta^*$ is constant on E .

Equivalently, (M, σ) is σ -homogeneous if the universal density is proportional to the weight vector. Since every vector in $\text{conv}(\mathcal{B})$ has total mass $r(E)$, the proportional density is

$$n_\sigma(e) := \frac{\sigma(e)}{\sigma(E)} r(E), \quad e \in E. \quad (4.2.4)$$

The parallel-extension construction gives the weighted analogue of the homogeneous matroid characterization as follows.

Theorem 4.2.2. Let $M(E, \mathcal{I})$ be a matroid with weights $\sigma \in \mathbb{R}_{>0}^E$, and let $\mathcal{B}$ be its base family. Let $S_\sigma(M)$ be the strength of M and let $D_\sigma(M)$ be the fractional arboricity of M . Define the density n_σ by

$$\eta_\sigma(e) := \frac{\sigma(e)}{\sigma(E)} r(E), \quad \forall e \in E. \quad (4.2.5)$$

Then, the following statements are equivalent:

1. The matroid M is σ -homogeneous.
2. $\eta_\sigma \in \text{conv}(\mathcal{B})$, where $\text{conv}(\mathcal{B})$ denotes the convex hull of $\mathcal{B}$.
3. $\sigma \in \text{conic}(\mathcal{B})$, where $\text{conic}(\mathcal{B})$ is the conical hull of $\mathcal{B}$.
4. $S_\sigma(M) = D_\sigma(M)$.

Next, the weighted universal density also determines the weighted principal partition. Let

$$s_1 < s_2 < \cdots < s_k$$

be the distinct values of the vector $\sigma^{-1}\eta^*$. For $i = 1, \dots, k$, define

$$A_i := \left\{ e \in E : \frac{\eta^*(e)}{\sigma(e)} = s_i \right\}, \quad (4.2.6)$$

and

$$E_i := \left\{ e \in E : \frac{\eta^*(e)}{\sigma(e)} \geq s_i \right\}. \quad (4.2.7)$$

Then $E_1 = E$ and

$$E_i = E - \bigcup_{j=1}^{i-1} A_j, \quad i = 2, \dots, k.$$

For each i , the universal density of the contraction $M/(E - E_i)$ is the restriction of η^* to E_i .

Moreover,

$$S_\sigma(M/(E - E_i)) = \frac{1}{s_k}, \quad D_\sigma(M/(E - E_i)) = \frac{1}{s_i}. \quad (4.2.8)$$

Here the weights on the contraction are the restrictions of σ to E_i . In particular, E_k is the unique maximal optimal set for $S_\sigma(M)$, and A_1 is the unique maximal optimal set for $D_\sigma(M)$. We call A_1 the *core* of M . The sets $A_1, \dots, A_k$ are the blocks of the weighted principal partition.

Finally, the largest value of $\sigma^{-1}\eta^*$ has a useful dual interpretation. Since $S_\sigma(M) =$

$\text{Mod}_{1,\sigma}(\mathcal{B})$ and $S_\sigma(M) = 1/s_k$, Fulkerson duality gives

$$s_k = \frac{1}{S_\sigma(M)} = \frac{1}{\text{Mod}_{1,\sigma}(\mathcal{B})} = \text{Mod}_{\infty,\sigma^{-1}}(\widehat{\mathcal{B}}) = \min_{\eta \in \text{conv}(\mathcal{B})} \max_{e \in E} \frac{\eta(e)}{\sigma(e)}. \quad (4.2.9)$$

Thus, the weighted universal density η^* is also optimal for minimization problem in (4.2.9).

4.3 Polymatroid Reinforcement and Polymatroid Sparsification

In this section, we introduce and study the generalized versions of the matroid reinforcement problem (4.1.2) and the matroid sparsification (4.1.3) in the context of polymatroids.

Let f be a real-valued, normalized, nondecreasing, and submodular function defined on subsets of a finite ground set E . From now on, we assume additionally that $f(\{e\}) \neq 0$ for all $e \in E$. Let $s \in \mathbb{R}_{>0}^E$ be a set of given weights and let $m(e) \geq 0$ be a per-unit increase cost for each $e \in E$. We introduce a generalized version of problem (4.1.2), the so-called *polymatroid reinforcement* problem:

$$\begin{aligned} & \underset{z \in \mathbb{R}_{\geq 0}^E}{\text{minimize}} && m^T z \\ & \text{subject to} && s + z \in hB_f \text{ for some } h > 0. \end{aligned} \quad (4.3.1)$$

Similarly, we introduce a generalized version of problem (4.1.3), the so-called *polymatroid sparsification* problem:

$$\begin{aligned} & \underset{z \in \mathbb{R}_{\geq 0}^E}{\text{minimize}} && m^T z \\ & \text{subject to} && s - z \in hB_f \text{ for some } h > 0. \end{aligned} \quad (4.3.2)$$

In the next two sections, we will study these two generalized problems.

4.3.1 Polymatroid Reinforcement

Let f be a real-valued, normalized, nondecreasing, and submodular function defined on subsets of a finite ground set E . Let $P = P_f$ be the associated polymatroid of f . Let B_f be the base polytope defined as in (2.1.5). Let $s \in \mathbb{R}_{\geq 0}^E$ be a set of given weights and a per-unit increase cost $m(e) \geq 0$ for each $e \in E$. In this section, we shall show that we can reduce the feasible set of the polymatroid reinforcement problem (4.3.1) to the set

$$\{z \in \mathbb{R}_{\geq 0}^E : s + z \in \alpha B_f\}$$

where

$$\alpha := \min \{h > 0 : s \in hP_f\}. \quad (4.3.3)$$

The value of α can be characterized in the following remark.

Remark 4.3.1. By the definition of P_f , we have that $s \in hP_f$ if and only if $s(A) \leq hf(A)$, for all $A \subseteq E$. This is equivalent to

$$h \geq \frac{s(A)}{f(A)} \quad \text{for all nonempty subsets } A \subseteq E.$$

Hence, the value of α in (4.3.3) is equal to

$$\max \left\{ \frac{s(A)}{f(A)} : \emptyset \neq A \subseteq E, \right\}$$

We call this value the *fractional arboricity* of f and denote it by $D_s(f)$.

We start by characterizing maximal vectors in P_f .

Lemma 4.3.2. *Let f be a real-valued, normalized, nondecreasing, and submodular function defined on subsets of a finite ground set E . Let $P = P_f$ be the associated polymatroid of f . Let B_f be the base polytope of f . Given $x \in P$, the vector x is a maximal vector in P if and only if $x \in B_f$.*

This fact should be standard, but we provide a proof for the reader's convenience.

Proof. Assume that $x \in B_f$. Then, we have $x(E) = f(E)$. Hence, x is maximal because if we increase any component of x , then $x(E) > f(E)$, contradiction with the definition of P in (2.1.4).

Assume that $x \in P$ is a maximal vector in P . Let c be a number such that $c \geq f(E)$. Let $\mathbf{c} \in \mathbb{R}^E$ be the column vector of all c . We have

$$x(e) \leq f(\{e\}) \leq c, \quad \forall e \in E \quad (\text{by the definition (2.1.4)}).$$

Hence, we obtain $x \leq \mathbf{c}$. Combining with the fact that x is maximal in P , we achieve that x is a P -basis of $\mathbf{c}$. Let $y \in B_f$, then y is maximal and is a P -basis of $\mathbf{c}$. By Definition 2.1.5 of polymatroids, x and y have the same component sum, in other words, $x(E) = y(E) = f(E)$. Thus, we obtain $x \in B_f$. $\square$

The following lemma defines a polymatroid generated by any element $s \in P_f$.

Lemma 4.3.3. *Let f be a real-valued, normalized, nondecreasing, and submodular function defined on subsets of a finite ground set E . Let $P = P_f$ be the associated polymatroid of f . Let B_f be the base polytope of f . Given $s \in P$, we define the following polyhedron*

$$P^s := \{x \in \mathbb{R}_{\geq 0}^E : s + x \in P\}. \quad (4.3.4)$$

Then, P^s is a polymatroid.

Proof. We aim to show that P^s satisfies Definition 2.1.5. Since $s \in P$, then P^s contains the vector 0. Moreover, P^s is compact by its definition. For any vectors x, y such that $x \leq y$ and $y \in P^s$, we have $s + y \in P$ and $s + x \leq s + y$. Then, $s + x \in P$ by Definition 2.1.5 (i), which implies $x \in P^s$. Hence, P^s satisfies Definition 2.1.5 (i).

For any $y \in \mathbb{R}_{\geq 0}^E$, let x be a maximal vector among all vectors satisfying $x \leq y$ and $x \in P^s$. By the definition of P^s , $s + x \in P$. We want to show that: $s + x$ is a P -basis of

$s + y$. Indeed, suppose that $s + x$ is not a P -basis of $s + y$. Then, there exists $\epsilon > 0$ and $e \in E$ satisfying

$$s + x + \epsilon \mathbb{1}_{\{e\}} < s + y \quad \text{and} \quad s + x + \epsilon \mathbb{1}_{\{e\}} \in P.$$

This is equivalent to

$$x + \epsilon \mathbb{1}_{\{e\}} < y \quad \text{and} \quad x + \epsilon \mathbb{1}_{\{e\}} \in P^s,$$

contradicting the definition of x . Thus, $s + x$ is a P -basis of $s + y$. By Definition 2.1.5 (ii), we have $(s + x)(E)$ does not depend on x , it only depends on y and s . This implies that $x(E) = (s + x)(E) - s(E)$ does not depend on x . Therefore, P^s satisfies Definition 2.1.5 (ii). In conclusion, P^s is a polymatroid. $\square$

The following lemma characterizes maximal vectors in the polymatroid P^s .

Lemma 4.3.4. *Let f be a real-valued, normalized, nondecreasing, and submodular function defined on subsets of a finite ground set E . Let $P = P_f$ be the associated polymatroid of f . Let B_f be the base polytope of f . Given $s \in P$, let P^s be defined as in (4.3.4). Given $x \in P^s$, then, x is maximal in P^s if and only if $(s + x)(E) = f(E)$.*

Proof. Assume that $(s + x)(E) = f(E)$. By Lemma 4.3.2, $s + x$ is maximal in P , this implies that x is maximal in P^s by the definition of P^s .

Conversely, assume that x is maximal in P^s . To prove that $(s + x)(E) = f(E)$, we first claim that there exists $x' \in P^s$ such that $(s + x')(E) = f(E)$.

If s is maximal in P , by Lemma 4.3.2, we obtain $s(E) = f(E)$. Then, we choose $x' = 0 \in \mathbb{R}^E$. If s is not maximal in P , then we can increase successively each component of s as much as possible subject to $s \in P$, the resulting vector has the form $s + x'$ for some $x' \geq 0$. The vector $s + x'$ is maximal in P because if there exists some $e \in E$ and $\epsilon > 0$ such that $(s + x' + \epsilon \mathbb{1}_{\{e\}}) \in P$, then during the process, we must have been increased the e -component of s to at least $x'(e) + \epsilon$, a contradiction. Hence, by Lemma 4.3.2, $(s + x')(E) = f(E)$. Therefore, there exists $x' \in P^s$ such that $(s + x')(E) = f(E)$.

By the above argument, x' is also maximal in P^s . By the Definition 2.1.5 (ii), x and x' have the same component sum, this implies that $(s + x)(E) = (s + x')(E) = f(E)$. $\square$

Next, we recall that the feasible set of the polymatroid reinforcement problem (4.3.1) is equal to

$$\bigcup_{h>0} A_h$$

where

$$A_h := \{z \in \mathbb{R}_{\geq 0}^E : s + z \in hB_f\}. \quad (4.3.5)$$

The next remark characterizes A_h .

Remark 4.3.5. For $h > 0$, we denote the dilation of P_f by $hP_f := \{hx : x \in P_f\}$. Then, we have

$$A_h = \{z \in \mathbb{R}_{\geq 0}^E : s + z \in hP_f, (s + z)(E) = hf(E)\}.$$

Note that $(hP_f)^s := \{x \in \mathbb{R}_{\geq 0}^E : s + x \in hP_f\}$. By Lemma 4.3.4, we obtain

$$A_h = \{z \in (hP_f)^s : z \text{ is maximal in } (hP_f)^s\}. \quad (4.3.6)$$

We want to show that we can reduce this feasible set to A_α , where α is defined in (4.3.3). To do this, we first give the following lemma.

Lemma 4.3.6. *Let f be a real-valued, normalized, nondecreasing, and submodular function defined on subsets of a finite ground set E . Let $P = P_f$ be the associated polymatroid of f . Let B_f be the base polytope of f . Let A_h be defined as in (4.3.5). Let α be defined as in (4.3.3). Then, $A_h \neq \emptyset$ if and only if $h \geq \alpha$.*

Proof. Let $h > 0$. If there exists $z \in A_h$, then $s + z \in hB_f \subset hP_f$. Because $s \leq s + z$, we have $s \in hP_f$, hence $h \geq \alpha$ by the definition of α . If $h \geq \alpha$, then $s \in hP_f$. If s is maximal in hP_f , by Lemma 4.3.2, we obtain $s \in hB_f$ and $0 \in A_h$. If s is not maximal in hP_f , then we can increase successively each component of s as much as possible subject to $s \in hP_f$,

the resulting vector has the form $s + z$ for some $z \geq 0$. The vector $s + z$ is maximal in hP_f because if there exists some $e \in E$ and $\epsilon > 0$ such that $(s + z + \epsilon \mathbb{1}_{\{e\}}) \in hP_f$, then during the process, we must have been increased the e -component of s to at least $z(e) + \epsilon$, a contradiction. Hence, by Lemma 4.3.2, $(s + z) \in hB_f$ and $z \in A_h$. $\square$

The following remark shows that α is the most cost-effective value for h if the costs are uniform.

Remark 4.3.7. Let $h_1 \geq h_2 \geq \alpha$, and let $z_1 \in A_{h_1}$, $z_2 \in A_{h_2}$. Then

$$z_1(E) = h_1 f(E) - s(E) \geq h_2 f(E) - s(E) = z_2(E).$$

Therefore, if the cost $m(e) = 1$ for all $e \in E$, then

$$\min_{z \in \bigcup_{h > 0} A_h} \mathbf{1}^T z = \min_{z \in A_\alpha} \mathbf{1}^T z = \alpha f(E) - s(E).$$

The following theorem shows that α is the most cost-effective value for any costs $m \in \mathbb{R}_{\geq 0}^E$.

Theorem 4.3.8. *Let f be a real-valued, normalized, nondecreasing, and submodular function defined on subsets of a finite ground set E . Let $P = P_f$ be the associated polymatroid of f . Let B_f be the base polytope of f . Given $s \in \mathbb{R}_{> 0}^E$ and $m \in \mathbb{R}_{\geq 0}^E$. Let A_h and α be defined as in (4.3.5) and (4.3.3), respectively. For every $h \geq \alpha$, let z_h^* be an optimal solution of the problem $\min_{z \in A_h} m^T z$. Then,*

$$m^T z_h^* \geq m^T z_\alpha^* \quad \forall h \geq \alpha. \quad (4.3.7)$$

In other words, the polymatroid reinforcement problem (4.3.1) is equivalent to $\min_{z \in A_\alpha} m^T z$.

Proof. Let $h \geq \alpha$. If we assume that there exists $y \in A_\alpha$ such that $y \leq z_h^*$, then

$$m^T y \leq m^T z_h^*, \quad (4.3.8)$$

and by the definition of z_α^* , we obtain

$$m^T z_\alpha^* \leq m^T y. \quad (4.3.9)$$

Hence, we have (4.3.7).

The rest of the proof is to show that such y exists. Let $(\alpha P_f)^s$ be defined as in (4.3.4), and let y be an $(\alpha P_f)^s$ -basis of z_h^* . Then, we obtain that $y \leq z_h^*$ and $s + y$ is an (αP_f) -basis of $s + z_h^*$. The proof is completed if we can show that $y \in A_\alpha$, i.e., $(s + y)(E) = \alpha f(E)$. To that end, we denote

$$x := \frac{\alpha}{h}(s + z_h^*) \leq (s + z_h^*).$$

By the definition of z_h^* , we have $s + z_h^* \in hB_f$. Thus, $x \in \alpha B_f$, this implies that $x(E) = \alpha f(E)$. By Lemma 4.3.2, x is maximal in αP_f . Moreover, since $x \leq (s + z_h^*)$, we have that x is an (αP_f) -basis of $s + z_h^*$. Note that $s + y$ is also an (αP_f) -basis of $s + z_h^*$. By Definition 2.1.5 (ii), $(s + y)(E) = x(E) = \alpha f(E)$. The proof is completed. □

By Remark 4.3.5, we have reduced the feasible set of the polymatroid reinforcement problem (4.3.1) to the set

$$\{z \in (\alpha P_f)^s : z \text{ is maximal in } (\alpha P_f)^s\}.$$

Assuming α is known, since $(\alpha P_f)^s$ is a polymatroid, we can use the greedy algorithm to solve this problem. To implement the greedy algorithm, we have that the oracle

$$\max \left\{ \epsilon : z + \epsilon \mathbb{1}_{\{j\}} \in (\alpha P_f)^s \right\}, \quad (4.3.10)$$

is equivalent to

$$\max \left\{ \epsilon : s + z + \epsilon \mathbb{1}_{\{j\}} \in \alpha P_f \right\}. \quad (4.3.11)$$

Note that the oracle (4.3.11) is equivalent to

$$\min \{ \alpha f(A) - (s + z)(A) : j \in A \subset E \}. \quad (4.3.12)$$

Assume that we have a method to implement the oracle (4.3.12). Suppose $m_{j_1} \leq m_{j_2} \leq \dots \leq m_{j_k}$ where $k = |E|$. We introduce Algorithm 1 for the polymatroid reinforcement problem (4.3.1).

Algorithm 1 Algorithm for the polymatroid reinforcement problem

Input: f, s, α

Output: Optimal z

```

1:  $z \leftarrow 0$ 
2: for  $i \in \{1, 2, \dots, k\}$  do
3:    $z(j_i) \leftarrow z(j_i) + \min \{ \alpha f(A) - (s + z)(A) : j_i \in A \subset E \}$ 
4: end for
5: return  $z$ 

```

Next, we want to show that when we increase weights, the fractional arboricity is non-decreasing.

Proposition 4.3.9. *Let f be a real-valued, normalized, nondecreasing, and submodular function defined on subsets of a finite ground set E with weights $\sigma \in \mathbb{R}_{>0}^E$. Let $D_\sigma(f)$ be the fractional arboricity of f . Then, the function $\sigma \mapsto D_\sigma(f)$ is Lipschitz continuous and monotonically nondecreasing.*

Proof. Let $\sigma_1, \sigma_2 \in \mathbb{R}_{>0}^E$, and let X_1 and X_2 be optimal for $D_{\sigma_1}(f)$ and $D_{\sigma_2}(f)$ respectively. Then, without loss of generality, we assume that $D_{\sigma_1}(f) \leq D_{\sigma_2}(f)$. So,

$$\begin{aligned}
D_{\sigma_2}(f) - D_{\sigma_1}(f) &\leq \frac{\sigma_2(X_2)}{f(X_2)} - \frac{\sigma_1(X_2)}{f(X_2)} \\
&= \frac{1}{f(X_2)} \sum_{e \in X_2} (\sigma_2(e) - \sigma_1(e)) \\
&\leq \frac{1}{f(X_2)} \sum_{e \in X_2} |\sigma_2(e) - \sigma_1(e)| \\
&\leq \frac{1}{\min_{\emptyset \neq A \subset E} f(A)} \sum_{e \in X_2} |\sigma_2(e) - \sigma_1(e)| \leq \frac{1}{\min_{\emptyset \neq A \subset E} f(A)} \sum_{e \in E} |\sigma_2(e) - \sigma_1(e)| \\
&\leq \frac{1}{\min_{\emptyset \neq A \subset E} f(A)} |E| \|\sigma_2 - \sigma_1\|_\infty.
\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the maximum norm on $\mathbb{R}^E$. So, the function $\sigma \mapsto D_\sigma(f)$ is Lipschitz continuous.

For the monotonicity, assume that $\sigma_1 \leq \sigma_2$, then $\sigma_1(A) \leq \sigma_2(A)$ for all $A \subset E$. Therefore $D_{\sigma_1}(f) \leq D_{\sigma_2}(f)$. $\square$

The following theorem describes the output of Algorithm 1.

Theorem 4.3.10. *Let f be a real-valued, normalized, nondecreasing, and submodular function defined on subsets of a finite ground set E with given weights $s \in \mathbb{R}_{\geq 0}^E$. Let $m \in \mathbb{R}_{\geq 0}^E$ be the cost of increasing each element weight by one unit. Let $\alpha = D_s(f)$ be the fractional arboricity of f defined in Remark 4.3.1. Then, after each iteration of Step 3 in Algorithm 1, the polymatroid P_f with the new weights $s + z$ has the same fractional arboricity α . When the algorithm terminates and outputs the optimal z , the vector $s + z$ belongs to the base polytope αB_f .*

Proof. After each iteration of Step (3) of Algorithm 1, the weights are nondecreasing. Then, by Proposition 4.3.9, the fractional arboricity is nondecreasing as well. In contrast, since the new weights are always in αP_f , the fractional arboricity never exceeds α during the process by Remark 4.3.1. Therefore, the fractional arboricity remains unchanged during the entire

algorithm. When the algorithm terminates, $s + z$ is maximal in αP_f , which means that $s + z \in \alpha B_f$. $\square$

4.3.2 Polymatroid Sparsification

In this section, we will show that we can reduce the feasible set of the polymatroid sparsification (4.3.2) problem in the same manner as the polymatroid reinforcement problem.

We start by recalling the concept of contrapolymatroid. Let f be a real-valued, normalized, nondecreasing, and submodular function defined on subsets of a finite ground set E . Let $s \in \mathbb{R}_{>0}^E$ be a set of given weights and a per-unit decrease cost $m(e) \geq 0$ for each $e \in E$. We define the set function g as follows:

$$g(U) := f(E) - f(E \setminus U) \quad \text{for all subsets } U \subseteq E. \quad (4.3.13)$$

It follows that $g(\emptyset) = f(E) - f(E) = 0$. For any set $A \subseteq B \subseteq E$, we have

$$g(A) = f(E) - f(E \setminus A) \leq f(E) - f(E \setminus B) = g(B).$$

Furthermore, for all subsets $A, B \subseteq E$, by definition of g and the submodularity of f , we have

$$g(A \cap B) + g(A \cup B) \geq g(A) + g(B).$$

Hence, the function g is normalized, nondecreasing and supermodular. Then, we associate the following polyhedron with g :

$$Q_g := \{x \in \mathbb{R}^E : x \geq 0, x(A) \geq g(A), \forall A \subseteq E\}. \quad (4.3.14)$$

In the literature, see for instance [33], the polyhedron Q_g is known as a contrapolymatroid. Notice that for any $e \in E$, we have $g(\{e\}) \geq g(\emptyset) = 0$. Thus, the polyhedron Q_g can be

described as

$$Q_g = \{x \in \mathbb{R}^E : x(A) \geq g(A), \forall A \subseteq E\}.$$

We also define

$$C_g := \{x \in Q_g : x(E) = g(E)\}. \quad (4.3.15)$$

It is well-known that $B_f = C_g$, where B_f is the base polytope associated with f , see [33, Section 44.5].

Next, we want to show that we can reduce the feasible set of the polymatroid sparsification problem (4.3.2) to

$$\{z \in \mathbb{R}_{\geq 0}^E : s - z \in \beta C_g\}$$

where

$$\beta := \max \{h > 0 : s \in hQ_g\}. \quad (4.3.16)$$

The following remark characterizes the value of β .

Remark 4.3.11. By definition of Q_g , we have $s \in hQ_g$ if and only if $s(A) \geq hg(A) = h(f(E) - f(E \setminus A))$, for all $A \subseteq E$. This is equivalent to

$$h \leq \frac{s(A)}{f(E) - f(E \setminus A)} \quad \text{for all } A \subseteq E, f(E) > f(E - A).$$

Hence, the value of β in (4.3.16) is equal to

$$\min \left\{ \frac{s(A)}{f(E) - f(E - A)} : A \subseteq E, f(E) > f(E - A) \right\}. \quad (4.3.17)$$

We call this value the *strength* of f and denote it by $S_s(f)$.

To deal with Q_g , we bound the set Q_g from above as follows. Let c be a constant such that $c \geq f(E)$, and let $\mathbf{c} := c\mathbf{1}$. Consider the following polyhedron:

$$Q_{g,c} := \{x \in \mathbb{R}^E : \mathbf{c} \geq x, x(A) \geq g(A), \forall A \subseteq E\}. \quad (4.3.18)$$

This polyhedron is bounded and hence is a polytope. Additionally, we define

$$C_{g,c} := \{x \in Q_{g,c} : x(E) = g(E)\}. \quad (4.3.19)$$

We aim to demonstrate that, under some translation, the polytope $Q_{g,c}$ maps to a polymatroid.

Theorem 4.3.12. *Let $c \geq f(E)$, and let $\mathbf{c}$ be the vector of all c . Let $t: \mathbb{R}^E \rightarrow \mathbb{R}^E$ be a map such that $t(x) = -x + \mathbf{c}$ for $x \in \mathbb{R}^E$. Then, $t(Q_{g,c})$ is a polymatroid.*

Proof. Let $x \in Q_{g,c}$, and $y = -x + \mathbf{c}$. Then, we have $y \geq 0$. Next, note that

$$\begin{aligned} x(A) \geq g(A) &\Leftrightarrow -y(A) + c|A| \geq f(E) - f(E \setminus A) \\ &\Leftrightarrow y(A) \leq -f(E) + f(E \setminus A) + c|A|. \end{aligned}$$

We define a set function $h(A) := -f(E) + f(E \setminus A) + c|A|$. We want to show that h is normalized, nondecreasing, and submodular. Indeed, by definition of h and the submodularity of f , we have that h is normalized and submodular. For $A \subsetneq B \subseteq E$, we have

$$\begin{aligned} -f(E) + f(E \setminus A) + c|A| &\leq -f(E) + f(E \setminus B) + c|B| \\ &\Leftrightarrow f(E \setminus A) - f(E \setminus B) \leq c|B \setminus A|, \end{aligned}$$

which is true because $c \geq f(E) \geq f(E \setminus A)$. So, the function h is nondecreasing. Therefore, $t(Q_{g,c})$ is a polymatroid. $\square$

We apply Theorem 4.3.12 to provide corresponding results from Section 4.3.1.

Lemma 4.3.13. *Let $Q := Q_{g,c}$ be defined as in (4.3.18). A vector $x \in Q$ is minimal in Q if and only if $x(E) = g(E) = f(E)$.*

Lemma 4.3.14. *Let $Q := Q_{g,c}$ be defined as in (4.3.18). Given $s \in Q$, then $Q^s := \{z \in \mathbb{R}_{\geq 0}^E : s - z \in Q\}$ is a polymatroid under some translation.*

Lemma 4.3.15. *Let $Q := Q_{g,c}$ be defined as in (4.3.18). Let $s \in Q$ and $z \in Q^s$ where $Q^s := \{z \in \mathbb{R}_{\geq 0}^E : s - z \in Q\}$. Then, z is maximal in Q^s if and only if $(s - z)(E) = f(E)$.*

Given a per-unit increase cost $m(e) \geq 0$ for each $e \in E$ and $s \in \mathbb{R}_{\geq 0}^E$. Let c be a constant such that $c \geq \max\{f(E), \|s\|_\infty\}$, see Remark 4.3.16 for the reason we choose such a c . We consider another version of the polymatroid sparsification problem (4.3.2) where we replace $B_f (= C_g)$ by $C_{g,c}$. In fact, we will prove that $C_g = C_{g,c}$ (see Proposition 4.3.22), so this version is equivalent to (4.3.2). The feasible set of the polymatroid sparsification problem (4.3.2) is

$$\bigcup_{h>0} F_h$$

where

$$F_h := \{z \in \mathbb{R}_{\geq 0}^E : s - z \in hQ_{g,c}\}.$$

Remark 4.3.16. Let $\beta := \max\{h > 0 : s \in hQ_g\}$ be defined as in (4.3.16). Since we assume that $c \geq \max\{f(E), \|s\|_\infty\}$, it follows that $\beta = \max\{h > 0 : s \in hQ_{g,c}\}$.

Lemma 4.3.17. *Let $\beta = \max\{h > 0 : s \in hQ_{g,c}\}$. Then, $F_h \neq \emptyset$ if and only if $h \leq \beta$.*

Theorem 4.3.18. *Given $s \in \mathbb{R}_{\geq 0}^E$ and $m \in \mathbb{R}_{\geq 0}^E$. For every $0 < h \leq \beta$, let z_h^* be an optimal solution of the problem $\min_{z \in F_h} m^T z$. Then,*

$$m^T z_h^* \geq m^T z_\beta^* \quad \forall h \leq \beta.$$

In other words, the polymatroid sparsification problem is equivalent to $\min_{z \in F_\beta} m^T z$.

Proof. This can be proved in the same manner as in the proof of Theorem 4.3.8. □

By Lemma 4.3.15, we have reduced the feasible set of the polymatroid sparsification problem (4.3.2) to the set

$$\{z \in (\beta Q_{g,c})^s : z \text{ is maximal in } (\beta Q_{g,c})^s\},$$

and we can use the greedy algorithm to solve this problem. Suppose $m_{j_1} \leq m_{j_2} \leq \dots \leq m_{j_k}$ where $k = |E|$. Following the same method as in Section 4.3.1, we obtain Algorithm 2 for the polymatroid sparsification problem (4.1.3).

Algorithm 2 Algorithm for the polymatroid sparsification problem

Input: g, s, β

Output: Optimal z

```

1:  $z \leftarrow 0$ 
2: for  $i \in \{1, 2, \dots, k\}$  do
3:    $z(j_i) \leftarrow z(j_i) + \min \{(s - z)(A) - \beta g(A) : j_i \in A \subset E\}$ 
4: end for
5: return  $z$ 

```

Remark 4.3.19. Similar to the case of $D_\sigma(f)$, it can be shown in the same manner of Proposition 4.3.9 that the map $\sigma \mapsto S_\sigma(f)$ is Lipschitz continuous and monotonically nondecreasing.

Theorem 4.3.20. *Let f be a real-valued, normalized, nondecreasing, and submodular function defined on subsets of a finite ground set E . Let $m \in \mathbb{R}_{\geq 0}^E$ be the cost of decreasing each element weight by one unit. Let $\beta = S_s(f)$ be the strength of f defined in Remark 4.3.11. Then, after each iteration of Step 3 in Algorithm 2, the function f with the new weights $s - z$ has the same strength. When the algorithm terminates and outputs the optimal z , the vector $s - z$ belongs to the base polytope βB_f .*

Proof. After each iteration of Step 3 of Algorithm 2, the weights are nonincreasing. By Remark 4.3.19, the strength is nonincreasing as well. In contrast, since the new weights are always in βQ_g , the strength never becomes smaller than β during the process by Remark 4.3.11. Therefore, the strength remains unchanged during the entire algorithm. When the algorithm terminates, $s - z$ is minimal in βQ_g , which means that $s - z \in \beta C_g = \beta B_f$. $\square$

4.3.3 Relationship between P_f and Q_g

Let f be a real-valued, normalized, nondecreasing, and submodular function defined on subsets of a finite ground set E . Let g be the set function defined in equation (4.3.13). Let

Q_g be defined as in (4.3.14). In this section, we explore several properties of P_f and Q_g . We start by providing some propositions describing P_f and Q_g .

Proposition 4.3.21. *Let f be a real-valued, normalized, nondecreasing, and submodular function defined on a finite ground set E . Let P_f be the associated polymatroid of f and B_f be the base polytope associated with f . Then,*

$$P_f = (B_f - \mathbb{R}_{\geq 0}^E) \cap \mathbb{R}_{\geq 0}^E. \quad (4.3.20)$$

Proof. Let $x \in (B_f - \mathbb{R}_{\geq 0}^E) \cap \mathbb{R}_{\geq 0}^E$, we have $0 \leq x = y - r$ for some $y \in B_f$ and $r \in \mathbb{R}_{\geq 0}^E$. Hence, $x \leq y$. Since $y \in P_f$, by Property (i) of Definition 2.1.5, we have $x \in P_f$. Therefore,

$$(B_f - \mathbb{R}_{\geq 0}^E) \cap \mathbb{R}_{\geq 0}^E \subset P_f.$$

Conversely, let $x \in P_f$. Then, by the greedy algorithm for polymatroids, there exists $x' \geq 0$ such that $x + x'$ is maximal in P_f . By Lemma 4.3.2, we have that

$$x + x' \in B_f.$$

Hence,

$$0 \leq x = (x + x') - x' \in B_f - \mathbb{R}_{\geq 0}^E.$$

Therefore, $P_f \subset (B_f - \mathbb{R}_{\geq 0}^E) \cap \mathbb{R}_{\geq 0}^E$. The proof is completed. $\square$

Proposition 4.3.22. *Let f be a real-valued, normalized, nondecreasing, and submodular function defined on E . Let g be the set function defined in equation (4.3.13). Let c be a real number such that $c \geq f(E) = g(E)$, and let $\mathbf{c} := c\mathbf{1}$ where $\mathbf{1}$ is the constant vector of all ones. Let $Q_g, C_g, Q_{g,c}$, and $C_{g,c}$ be defined as in (4.3.14), (4.3.15), (4.3.18), and (4.3.19), respectively. Then, the following statements hold:*

(i)

$$C_{g,c} = C_g, \quad (4.3.21)$$

(ii)

$$Q_{g,c} = (C_{g,c} + \mathbb{R}_{\geq 0}^E) \cap \{x \in \mathbb{R}_{\geq 0}^E : \mathbf{c} \geq x\}, \quad (4.3.22)$$

(iii)

$$Q_g = C_g + \mathbb{R}_{\geq 0}^E. \quad (4.3.23)$$

Proof. For (i), denote $J := \{x \in \mathbb{R}_{\geq 0}^E : x(E) = g(E)\}$ and $K_c := \{x \in \mathbb{R}_{\geq 0}^E : x \leq \mathbf{c}\}$. It follows that $J \subset K_c$. Hence

$$C_{g,c} = Q_{g,c} \cap J = Q_g \cap K_c \cap J = Q_g \cap J = C_g.$$

By the proof of Theorem 4.3.12 and Proposition 4.3.21, we obtain (ii). Finally, for (iii),

$$\begin{aligned} Q_g &= \bigcup_{c \geq f(E)} Q_{g,c} \\ &= \bigcup_{c \geq f(E)} ((C_{g,c} + \mathbb{R}_{\geq 0}^E) \cap K_c) \\ &= \bigcup_{c \geq f(E)} (C_g + \mathbb{R}_{\geq 0}^E) \cap K_c \\ &= (C_g + \mathbb{R}_{\geq 0}^E) \cap \left(\bigcup_{c \geq f(E)} K_c \right) \\ &= C_g + \mathbb{R}_{\geq 0}^E. \end{aligned}$$

□

Next, we present the relationship between the notions of strength and fractional arboricity in the context of polymatroids.

Theorem 4.3.23. *Let f be a real-valued, normalized, nondecreasing, and submodular function defined on E . Let g be the set function defined in equation (4.3.13). Given $s \in \mathbb{R}_{> 0}^E$, let $D_s(f)$ be the fractional arboricity of f defined as in Remark 4.3.1 and let $S_s(f)$ be the*

strength of f defined as in Remark 4.3.11. Let P_f, B_f, Q_g , and C_g be defined as in (2.1.4), (2.1.5), (4.3.14), and (4.3.15). Then, for $h > 0$,

(i) $s \in hP_f$ if and only if $h \geq D_s(f)$.

(ii) $s \in hQ_g$ if and only if $h \leq S_s(f)$.

(iii) $D_s(f) \geq S_s(f)$, and $s \in hB_f$ if and only if $h = D_s(f) = S_s(f)$.

Proof. By the definitions of $D_s(f)$ and $S_s(f)$, we have (i) and (ii). For (iii), we have that $s \in D_s(f)P_f$ and $s \in S_s(f)Q_g$, then $D_s(f)f(E) \geq s(E) \geq S_s(f)g(E)$. Since $f(E) = g(E)$, we have that $D_s(f) \geq S_s(f)$. It is well-known that $B_f = C_g$, we will prove this fact for the reader's convenience. This is true because given that $x(E) = f(E)$, for any subset $A \subseteq E$,

$$x(A) \leq f(A) \Leftrightarrow x(E) - x(E - A) \leq f(A) \quad (4.3.24)$$

$$\Leftrightarrow f(E) - x(E - A) \leq f(A) \quad (4.3.25)$$

$$\Leftrightarrow x(E - A) \geq f(E) - f(A). \quad (4.3.26)$$

Next, we assume that $h = D_s(f) = S_s(f)$. Then, $D_s(f)f(E) = s(E) = S_s(f)g(E)$ and $s \in hB_f = hC_g$.

Finally, we assume that $s \in hB_f = hC_g$. By definition of $D_s(f)$ and $S_s(f)$, we have $S_s(f) \geq h \geq D_s(f)$. Note that $hf(E) = s(E) = hg(E)$, we obtain $D_s(f) \geq h \geq S_s(f)$. Therefore, $h = D_s(f) = S_s(f)$.

□

The following remark interprets the polymatroid reinforcement problem and the polymatroid sparsification problem in the case when f is the rank function of a matroid.

Remark 4.3.24. When f is the rank function of a weighted matroid $M(E, \mathcal{I})$ with given weights $s \in \mathbb{R}_{>0}^E$, by definition, the strength $S_s(M)$ is equal to the strength $S_s(f)$ of f , and

the fractional arboricity $D_s(M)$ is equal to the fractional arboricity $D_s(f)$ of f .

Let $\mathcal{B}$ be the family of bases of M . Let g be the set function defined in equation (4.3.13). Then, we have $B_f = C_g = \text{conv}(\mathcal{B})$. By Proposition 4.3.22, we have $\text{Dom}(\mathcal{B}) = Q_g$ where $\text{Dom}(\mathcal{B})$ is defined in (2.1.23). And by Proposition 4.3.21, it follows that $\text{Antidom}(\mathcal{B}) = P_f$, where $\text{Antidom}(\mathcal{B})$ is defined as in (4.1.5).

4.3.4 Connection with the Strength Reinforcement problem

In this section, we study connections between problems (4.3.1), (4.3.2), and the strength reinforcement problem by Cunningham [11].

Let f be a real-valued, normalized, nondecreasing, and submodular function defined on subsets of a finite ground set E . Let $P = P_f$ be the associated polymatroid of f . Let B_f be the base polytope defined as in (2.1.5). Let $s \in \mathbb{R}_{\geq 0}^E$ be a set of given weights and consider a per-unit increase cost $m(e) \geq 0$ for each $e \in E$. Let the set function g be defined as in equation (4.3.13). Let $S_s(f)$ be the strength of f , and let $D_s(f)$ be the fractional arboricity of f .

Next, we generalize the strength reinforcement problem by Cunningham [11] in the context of polymatroids. Let S_0 be a desired strength, the strength reinforcement problem given a function f is defined as follows:

$$\min\{m^T z : z \in \mathbb{R}_{\geq 0}^E, S_{s+z}(f) \geq S_0\}. \quad (4.3.27)$$

We recall that, by Theorem 4.3.23, for a real positive number h , we have that

- (i) $s \in hP_f$ if and only if $h \geq D_s(f)$.
- (ii) $s \in hQ_g$ if and only if $h \leq S_s(f)$.

Hence, a vector $z \in \mathbb{R}_{\geq 0}^E$ is feasible for problem (4.3.27) if and only if

$$s + z \in S_0 Q_g.$$

Cunningham [11] provided an algorithm to solve this problem. The first step is to find an $(S_0 P_f)$ -basis w of s . Then, we apply the greedy algorithm to increase each component of w until we obtain a vector $w + z$ which belongs to $S_0 B_f$. Thus, we have $s + z \in S_0 Q_g$ because $s + z \geq w + z$, and this choice of z is our optimal solution.

Let us analyze this algorithm. Note that w is an $(S_0 P_f)$ -basis of s . We have $w \in S_0 P_f$ and $w \in D_w(f) P_f$. Hence, it follows that $S_0 \geq D_w(f)$. We have two cases:

- If $s \notin S_0 P_f$, hence $w \neq s$, it must follow that $S_0 = D_w(f)$, because otherwise, if $S_0 > D_w(f)$, we can increase a small enough amount at some j -component of w where $w(j) < s(j)$ such that the output still belongs to $S_0 P_f$.
- If $s \in S_0 P_f$, then $w = s$. We can only conclude that $S_0 \geq D_s(f)$.

The following remark presents the connection between the problem (4.3.27) and our polymatroid reinforcement problem (4.3.1).

Remark 4.3.25. In the case where $S_0 = D_s(f)$, we have $s \in D_s(f) P_f = S_0 P_f$. It follows that $w = s$. The strength reinforcement algorithm for this case is precisely identical to the one for the polymatroid reinforcement problem, where we increase s element-wise to get a vector $s + z \in S_0 B_f$.

Next, motivated by the strength reinforcement problem (4.3.27), we introduce the *fractional arboricity sparsification* problem, where we seek the most cost-effective way to decrease element weights so that the resulting weights induce a fractional arboricity of at most a given number D_0 . This problem is defined as follows:

$$\min\{m^T z : z \in \mathbb{R}_{\geq 0}^E, D_{s-z}(f) \leq D_0\}. \quad (4.3.28)$$

Following the strategy for the strength reinforcement problem by Cunningham [11], we provide the following algorithm.

First, note that a vector $z \in \mathbb{R}_{\geq 0}^E$ is feasible for problem (4.3.28) if and only if it satisfies

$$s - z \in D_0 P_f.$$

This is equivalent to the existence of a vector y such that $y \geq s - z$ and $y \in D_0 B_f$. Hence, we first find a vector $w \in \mathbb{R}_{\geq 0}^E$ such that w is minimal with respect to $w \geq s$ and $w \in D_0 Q_g$. Then, we decrease w element-wise to obtain a vector $y = w - z$ which belongs to $D_0 B_f$. In each step, we need to solve this oracle problem:

$$\min\{s(A) - D_0 g(A) : j \in A \subset E\}.$$

It follows that this choice of z is an optimal solution for problem (4.3.28).

In Section 4.4, we present detailed algorithms for the case of graphic matroids.

4.3.5 A Remark when Weights Attain Values of 0

In this section, we study the case when weights are allowed to attain values of 0. Let f be a real-valued, normalized, nondecreasing, and submodular function defined on subsets of a finite ground set E . We also assume that $f(\{e\}) \neq 0$ for all $e \in E$. Let $P = P_f$ be the associated polymatroid of f , and let B_f be the base polytope of f . Let $s \in \mathbb{R}_{\geq 0}^E$ be a set of nonnegative weights, and we assume that s is not identically zero. Let $D_s(f)$ be the fractional arboricity of f defined as in Remark 4.3.1, and let $S_s(f)$ be the strength of f defined as in Remark 4.3.11.

Let $E_0 \subset E$ be the set of elements e that attain weights 0. First, we have that Theorem 4.3.23 still holds for any set of nonnegative weights $s \in \mathbb{R}_{\geq 0}^E$ (the proof remains identical),

with the following conventions:

$$0P_f := \{0\} \subset \mathbb{R}_{\geq 0}^E \quad \text{and} \quad 0Q_g := \{0\} \subset \mathbb{R}_{\geq 0}^E.$$

We consider two cases as follows:

- Assume that $f(E - E_0) < f(E)$, then

$$S_s(f) \leq \frac{s(E_0)}{f(E) - f(E - E_0)} = 0, \quad \text{and} \quad D_s(f) \geq \frac{s(E)}{f(E)} > 0.$$

Hence, $S_s(f) = 0 < D_s(f)$. Thus, we have that $s \notin hB_f$ for any $h > 0$. In other words, if $s \in hB_f$ for some $h > 0$, then $f(E - E_0) = f(E)$.

- Assume that $f(E - E_0) = f(E)$. For any $X \subset E$ such that $f(E - X) < f(E)$, we have that $X \cap (E - E_0) \neq \emptyset$. Hence, $s(X) > 0$. By the definition of $S_s(f)$, it follows that

$$S_s(f) > 0.$$

We have that Theorem 4.3.23 still holds for any set of nonnegative weights. We know that Theorem 4.3.23 is a generalization of characterizations for weighted homogeneous matroids (in Section 4.2) in the context of polymatroids. In the rest of this section, we will analyze the base modulus with nonnegative weights.

Consider a matroid $M(E, \mathcal{I})$ with the rank function f . Let $0 \neq s \in \mathbb{R}_{\geq 0}^E$ be a set of nonnegative weights. Let $\mathcal{B}$ be the base family of M . Let $E_0 \subset E$ be the set of elements e that attain weights 0. For the 2-modulus $\text{Mod}_{2,s}(\mathcal{B})$, let $\mathcal{B}'$ be the set of bases B of M which satisfy $B \cap E_0 = \emptyset$. Since s attains values of 0 in E_0 , for a density $\rho \in \mathbb{R}_{\geq 0}^E$, we can assign $\rho(e) = 1$ for $e \in E_0$ and this ρ is admissible for $\mathcal{B} \setminus \mathcal{B}'$. Note that $\rho(e)$ for any $e \in E_0$ does not contribute to the objective function, it follows that

$$\text{Mod}_{2,s}(\mathcal{B}) = \text{Mod}_{2,s}(\mathcal{B}').$$

We have two cases as follows:

- Assume that $f(E - E_0) < f(E)$. Let Y be a maximal independent set contained in $E - E_0$, and let B be an arbitrary base of M . Then, $|Y| = f(E - E_0) < f(E) = |B|$. By (I3) in Definition 2.1.1, there exists $x \in B - Y$ such that $Y \cup \{x\} \in \mathcal{I}$. It follows that x must not be in $E - E_0$. Hence, $x \in B \cap E_0$. Therefore, $\mathcal{B}' = \emptyset$ and $\text{Mod}_{2,s}(\mathcal{B}) = \text{Mod}_{2,s}(\mathcal{B}') = 0$. In this scenario, we say that M is *not s-homogeneous*.
- Assume that $f(E - E_0) = f(E)$. Then $\mathcal{B}' \neq \emptyset$. The matroid M is said to be *s-homogeneous* if any optimal density ρ^* of $\text{Mod}_{2,s}(\mathcal{B})$ is constant in $E - E_0$.

For the second case when $f(E - E_0) = f(E)$, we remove E_0 from the matroid M as follows. Recall that the matroid $M \setminus E_0$ is called the *deletion* of E_0 from M . The base family $\mathcal{B}(M \setminus E_0)$ of $M \setminus E_0$ is the collection of maximal sets in $\{B - E_0 : B \in \mathcal{B}(M)\}$, which is equal to $\mathcal{B}'$. Let s' be the projection of s to $\mathbb{R}^{E-E_0}$. It follows that

$$\text{Mod}_{2,s}(\mathcal{B}(M)) = \text{Mod}_{2,s'}(\mathcal{B}(M \setminus E_0)).$$

Hence, M is s -homogeneous if and only if $M \setminus E_0$ is s' -homogeneous. Finally, we want to show that Theorem 4.2.2 still holds for nonnegative weights.

Theorem 4.3.26. *Let $M(E, \mathcal{I})$ be a matroid with weights $s \in \mathbb{R}_{\geq 0}^E$. Let $\mathcal{B}$ be its base family, and let f be its rank function. Let $E_0 \subset E$ be the set of elements e that attain weights 0. Assume that $f(E - E_0) = f(E)$. Let $S_s(M)$ be the strength of M , and let $D_s(M)$ be the fractional arboricity of M .*

Then, the following statements are equivalent:

1. M is s -homogeneous.
2. $s \in \text{conic}(\mathcal{B})$, where $\text{conic}(\mathcal{B})$ is the conical hull of $\mathcal{B}$.
3. $S_s(M) = D_s(M)$.

Proof. First, (ii) and (iii) are equivalent because of Theorem 4.3.23 with the set of nonnegative weights s .

Next, let s' be the projection of s to $\mathbb{R}^{E-E_0}$. Then, the matroid M is s -homogeneous if and only if the matroid $M \setminus E_0$ is s' -homogeneous. This is equivalent to $s' \in \text{conic}(\mathcal{B}(M \setminus E_0))$, which is equivalent to $s \in \text{conic}(\mathcal{B}(M))$. Indeed, $s \in \text{conic}(\mathcal{B}(M))$ is equivalent to

$$s = \sum_{B \in \mathcal{B}(M)} \mu(B) \mathbb{1}_B,$$

where $\mu \in \mathbb{R}_{\geq 0}^{\mathcal{B}(M)}$. Because s attains 0 on E_0 , we have that $\mu(B) = 0$ for any base $B \subset E$ of M which is not a base of the matroid $M \setminus E_0$. Thus, this is equivalent to $s' \in \text{conic}(\mathcal{B}(M \setminus E_0))$. $\square$

4.4 Application for Graphs

In this section, we study the matroid reinforcement and matroid sparsification problems for graphic matroids.

4.4.1 An Algorithm for Computing Fractional Arboricity

In this section, we present an algorithm for computing the fractional arboricity of graphs. We begin by recalling some notations.

We consider an undirected, connected, and weighted graph $G = (V, E, \sigma)$ with edge weights $\sigma \in \mathbb{R}_{>0}^E$. The graphic matroid associated with G is the matroid with the ground set E , where a subset $A \subseteq E$ is independent if and only if A does not contain any cycles in G . For every subset $A \subset E$, its rank $f(A)$ is given by $|V(A)| - r_A$, where $V(A)$ is the set of vertices of the edge-induced subgraph H_A induced by A , and r_A is the number of connected components in H_A [29]. The family of all spanning trees of the graph G is the base family of M .

Let f be the rank function of the graphic matroid associated with G . The fractional arboricity $D_\sigma(G)$ of G is defined as:

$$D_\sigma(G) := \max \left\{ \frac{\sigma(X)}{f(X)} : X \subseteq E, r(X) > 0 \right\}. \quad (4.4.1)$$

The following theorem characterizes $D_\sigma(G)$.

Theorem 4.4.1. *Let $G = (V, E, \sigma)$ be a weighted connected graph. Let $D_\sigma(G)$ be the fractional arboricity of G . Let $\mathcal{H}$ be the set of all vertex-induced connected subgraphs of G that contain at least one edge. Let E_B denote the set of edges that have both endpoints in B , for each $B \subseteq V$. Then*

$$D_\sigma(G) = \max \left\{ \frac{\sigma(E_B)}{|B| - 1} : B \subseteq V, |B| \geq 2 \right\} = \max_{H \in \mathcal{H}} \frac{\sigma(E_H)}{|V_H| - 1}. \quad (4.4.2)$$

Proof. We want to show that the following three definitions for fractional arboricity are equivalent:

$$D_1 := D_\sigma(G), \quad D_2 := \max \left\{ \frac{\sigma(E_B)}{|B| - 1} : B \subseteq V, |B| \geq 2 \right\}, \quad D_3 := \max_{H \in \mathcal{H}} \frac{\sigma(E_H)}{|V_H| - 1}.$$

Let $H' = (V_{H'}, E_{H'})$ be a subgraph that optimizes D_3 . Choose $X = E_{H'}$, then $f(X) = |V_{H'}| - 1$. Thus, we obtain $D_3 \leq D_1$. Choose $B = V_{H'}$, then $D_3 \leq D_2$.

Let X be an optimizer for D_1 . Then $H = (V_X, X)$ is the edge-induced subgraph generated by X . Suppose H has m connected components $H_1, H_2, \dots, H_m$. We want to show that

$$\frac{\sigma(X)}{f(X)} \leq \max_{i=1, \dots, m} \frac{\sigma(E_{H_i})}{|V_{H_i}| - 1}.$$

Notice that for $a_1, \dots, a_n > 0$ and $b_1, \dots, b_n > 0$, we have

$$\sum_{i=1}^n a_i \leq \left(\max_{i=1, \dots, n} \frac{a_i}{b_i} \right) \sum_{i=1}^n b_i.$$

Therefore,

$$\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \leq \max_{i=1, \dots, n} \frac{a_i}{b_i}.$$

In particular,

$$\begin{aligned} \frac{\sigma(X)}{f(X)} &\leq \frac{\sigma(E_{H_1}) + \cdots + \sigma(E_{H_m})}{(|V_{H_1}| - 1) + \cdots + (|V_{H_m}| - 1)} \\ &\leq \max_{i=1, \dots, m} \frac{\sigma(E_{H_i})}{|V_{H_i}| - 1}. \end{aligned}$$

Thus, $D_1 \leq D_3$.

Let B be an optimizer for D_2 . Then $K = (B, E_B)$ is the vertex-induced subgraph generated by B . Suppose K has m connected components $K_1, K_2, \dots, K_m$, where each component has at least one edge. We want to show that

$$\frac{\sigma(E_B)}{|B| - 1} \leq \max_{i=1, \dots, m} \frac{\sigma(E_{K_i})}{|V_{K_i}| - 1}.$$

We have,

$$\begin{aligned} \frac{\sigma(E_B)}{|B| - 1} &\leq \frac{\sigma(E_{K_1}) + \cdots + \sigma(E_{K_m})}{|V_{K_1}| + \cdots + |V_{K_m}| - 1} \\ &\leq \frac{\sigma(E_{K_1}) + \cdots + \sigma(E_{K_m})}{|V_{K_1}| + \cdots + |V_{K_m}| - m} \\ &\leq \max_{i=1, \dots, m} \frac{\sigma(E_{K_i})}{|V_{K_i}| - 1}. \end{aligned}$$

Thus, $D_2 \leq D_3$.

□

It is well known that the polytope P_f , defined as in (2.1.4), can be described as follows:

$$\begin{aligned} x(E_B) &\leq |B| - 1 \quad \forall B \text{ such that } \emptyset \neq B \subseteq V; \\ x &\geq 0. \end{aligned} \tag{4.4.3}$$

where E_B denotes the set of edges that have both endpoints in B . This fact gives rise to a second proof for the first equality in (4.4.2).

Next, we give an algorithm for computing the fractional arboricity $D_\sigma(G)$. There is a well-known method for dealing with quotients like $D_\sigma(G)$, which is described as follows.

Denote $\alpha := D_\sigma(G)$ and let $b > 0$. By (4.4.2), we have

$$\begin{aligned}
\alpha < b &\Leftrightarrow \max \left\{ \frac{\sigma(E_B)}{|B| - 1} : B \subseteq V, |B| \geq 2 \right\} < b \\
&\Leftrightarrow \frac{\sigma(E_B)}{|B| - 1} < b \quad \forall B \text{ such that } B \subseteq V, |B| \geq 2 \\
&\Leftrightarrow b(|B| - 1) - \sigma(E_B) > 0 \quad \forall B \text{ such that } B \subseteq V, |B| \geq 2 \\
&\Leftrightarrow Y(b) := \min \{ b(|B| - 1) - \sigma(E_B) : B \subseteq V, |B| \geq 2 \} > 0.
\end{aligned} \tag{4.4.4}$$

Note that we also have similar statements:

$$\alpha = b \Leftrightarrow Y(b) = 0, \tag{4.4.5}$$

$$\alpha > b \Leftrightarrow Y(b) < 0. \tag{4.4.6}$$

Next, we assume that we have a lower bound b of α , in other words, $\alpha \geq b$. Then, we have $Y(b) \leq 0$. Let us consider two cases:

- If $Y(b) = 0$, then $\alpha = b$.
- If $Y(b) < 0$, let B^* be a minimizer of $Y(b)$. Then,

$$\begin{aligned}
&b(|B^*| - 1) - \sigma(E_{B^*}) < 0 \\
&\Leftrightarrow b' := \frac{\sigma(E_{B^*})}{|B^*| - 1} > b.
\end{aligned}$$

Thus, b' is a better lower bound of α . Based on this argument, we obtain an algorithm for computing α :

Algorithm 3 Algorithm for computing $D_\sigma(G)$

Input: $G = (V, E, \sigma)$

Output: $D_\sigma(G)$

```
1:  $B \leftarrow V$ 
2:  $b \leftarrow \sigma(E)/(|V| - 1)$ 
3: while  $Y(b) < 0$  do
4:    $B \leftarrow$  a minimizer of  $Y(b)$ 
5:    $b \leftarrow \sigma(E_B)/(|B| - 1)$ 
6: end while
7: return  $b, B$ 
```

The following lemma shows that Algorithm 3 takes at most $n = |V|$ steps.

Lemma 4.4.2. *Assume that $Y(b_0) < 0$ and B_0 is a minimizer of $Y(b_0)$. Denote*

$$b_1 := \frac{\sigma(E_{B_0})}{|B_0| - 1}.$$

Let B_1 be a minimizer of $Y(b_1)$. If $Y(b_1) < 0$, then $|B_1| < |B_0|$.

Proof. Because $Y(b_0) < 0$ and $Y(b_1) < 0$, we have

$$\frac{\sigma(E_{B_1})}{|B_1| - 1} > b_1 = \frac{\sigma(E_{B_0})}{|B_0| - 1} > b_0.$$

Then,

$$\begin{aligned} 0 &< \sigma(E_{B_1}) - b_1(|B_1| - 1) \\ &= \sigma(E_{B_1}) - b_0(|B_1| - 1) + b_0(|B_1| - 1) - b_1(|B_1| - 1) \\ &\leq \sigma(E_{B_0}) - b_0(|B_0| - 1) + b_0(|B_1| - 1) - b_1(|B_1| - 1) && \text{(because } B_0 \text{ is a minimizer of } Y(b_0)) \\ &= b_1(|B_0| - 1) - b_0(|B_0| - 1) + b_0(|B_1| - 1) - b_1(|B_1| - 1) \\ &= (b_0 - b_1)(|B_1| - |B_0|). \end{aligned}$$

Since $b_0 - b_1 < 0$, we obtain $|B_1| < |B_0|$. □

By Lemma 4.4.2, we conclude that Algorithm 3 will compute $D_\sigma(G)$ within at most $|V|$

iterations. The remaining work is to find a way to compute $Y(b)$ given that $Y(b) \leq 0$. Denote $x := \sigma/b$, then finding $Y(b)$ is equivalent to finding

$$\min \{(|B| - 1) - x(E_B) : B \subseteq V, |B| \geq 2\}. \quad (4.4.7)$$

Let us recall the problem of finding a most-violated inequality for the description (4.4.3) of P_f :

$$\min \{(|B| - 1) - x(E_B) : \emptyset \neq B \subseteq V\}. \quad (4.4.8)$$

In general, the problem (4.4.7) and the problem (4.4.8) are not equivalent. However, note that we have $Y(b) \leq 0$, and when $|B| = 1$, we have $|B| - 1 - x(E_B) = 0$. Therefore, in our case, these two problems are equivalent. A network flow algorithm for this problem is known [34].

A Cunningham's minimum cut formulation.

In [34], Cunningham gives an algorithm for finding a most-violated inequality for the description (4.4.3). We recall the algorithm for completeness.

Given $x \in \mathbb{R}_{>0}^E$. As described in [34], we construct a capacitated undirected graph G' from $G = (V, E)$ as follows.

- The vertex set of G' is $V \cup \{r, s\}$.
- Every edge e of G is an edge of G' with the same endpoints and it has capacity $\frac{1}{2}x(e)$.
- For every $v \in V$, there is an edge connecting v and s and it has capacity 1.
- For every $v \in V$, let $\delta(v) := \{e \in E : e \text{ is incident with } v\}$, there is an edge connecting v and r and it has capacity $\frac{1}{2}x(\delta(v))$.

Denote the capacity of edges in G' by w and $B = V \setminus A$. Consider a cut separating r and s in G' determined by

$A \cup \{r\}$ for any $A \subset V$.

Then, the value of this cut is:

$$\begin{aligned}
& \sum_{e=\{a,b\}:a \in A, b \in B} w(e) + \sum_{e=\{r,b\}:b \in B} w(e) + \sum_{e=\{a,s\}:a \in A} w(e) \\
&= \sum_{e=\{a,b\}:a \in A, b \in B} \frac{1}{2}x(e) + \sum_{b \in B} \frac{1}{2}x(\delta(b)) + |A| \\
&= x(E) - x(E_A) + |A|.
\end{aligned}$$

where E_A is the set of edges that have both endpoints in A . Therefore, the problem (4.4.8) is equivalent to find a minimum cut $A \cup \{r\}$ satisfying $A \neq \emptyset$. To overcome the condition $A \neq \emptyset$, we just need to solve $|V|$ minimum cut problems, each one determined by changing the capacity of an edge connecting v and r to ∞ . This is because any cut $A \cup \{r\}$ with $A \neq \emptyset$ will not use at least one of these edges.

Remark 4.4.3. Run-time complexity for Algorithm 3: $|V|^2$ minimum-cut calculations. This is because the problem of finding a most-violated inequality for the description (4.4.3) of P_f takes $|V|$ minimum-cut calculations and the number of iterations is at most $|V|$.

4.4.2 Matroid Reinforcement for Graphs

In this section, we give a detailed method to implement Algorithm 1.

Cunningham's minimum cut formulation modification

In [11], Cunningham gives a network-flow algorithm that solves (4.3.12). Given $x \in P_f$ and $j \in E$, we want to solve

$$\min \{f(A) - x(A) : j \in A \subset E\}. \quad (4.4.9)$$

Because we have that $x \in P_f$, then a set A that minimizes $f(A) - x(A)$ over $j \in A \subseteq E$ can be required to be of the form E_B . Hence, Cunningham modified his minimum cut formulation as follows.

We construct an undirected capacitated graph G' from $G = (V, E)$ as follows.

- The vertex set for G' is $V \cup \{r, s\}$, where r and s are new vertices that will be the source and sink respectively.
- Every edge e of G is an edge of G' with the same endpoints and it has capacity $\frac{1}{2}x(e)$.
- For every $v \in V$, there is an edge connecting v and s and it has capacity 1.
- For every $v \in V$, let $\delta(v) := \{e \in E : e \text{ is incident with } v\}$, there is an edge connecting v and r . It has capacity $\frac{1}{2}x(\delta(v))$ if v is not an endpoint of j and has capacity ∞ if v is an endpoint of j .

As shown in [11], (4.4.9) can be recovered from the value of a minimum cut separating r and s in G' and the edges of a minimum cut (after removing any edges incident with r or s) form a minimizer for (4.4.9).

Remark 4.4.4. Run-time complexity for Algorithm 1: $|V|^2 + |E|$ minimum-cut calculations. This is due to finding $D_\sigma(G)$ and the number of iterations is $|E|$ where each iteration takes 1 minimum-cut calculation.

4.4.3 Matroid Sparsification for Graphs

In this section, we provide a detailed method to implement Algorithm 2. Given $x \in Q_g$ and $j \in E$, we want to solve

$$\min \{x(A) - g(A) : j \in A \subseteq E\}.$$

This is equivalent to

$$\min \{x(A) + f(E - A) : j \in A \subseteq E\}. \quad (4.4.10)$$

By a change of variable $B := E - A$, the problem (4.4.10) is equivalent to

$$\min \{f(B) - x(B) : j \notin B \subset E\}. \quad (4.4.11)$$

We can deal with this problem by defining a vector $x' \in \mathbb{R}_{\geq 0}^E$ such that $x'(e) = x(e)$ for all $e \neq j$, and $x'(j) = 0$. Then (4.4.11) is equivalent to

$$\min \{f(B) - x'(B) : B \subset E\}. \quad (4.4.12)$$

This equivalence holds because if an optimal B for problem (4.4.12) contains j , then $B - \{j\}$ is also optimal.

Alternatively, we can handle problem (4.4.11) by deleting the edge j from G . Let $G' = (V_{G'}, E_{G'}) := G - \{j\}$, then (4.4.11) is equivalent to

$$\min \{f_{G'}(B) - x(B) : B \subset E_{G'}\}. \quad (4.4.13)$$

These equivalent problems are known as the attack problem, see [11]. Cunningham gives an algorithm for it in [11]. The algorithm requires $|E|$ minimum-cut calculations.

Remark 4.4.5. Run-time complexity for Algorithm 2: $|V||E| + |E|^2$ minimum-cut calculations. This is due to finding $S_\sigma(G)$ (we use the algorithm in [11], it requires $|V||E|$ minimum-cut calculations) and the number of iterations is $|E|$ where each iteration takes $|E|$ minimum-cut calculations.

Remark 4.4.6. A natural direction for further research is to speed up Algorithms 1 and 2. Currently, Algorithms 1 and 2 depend on computations involving the fractional arboricity, the strength, the attack problem (note that the strength can be solved by $|V|$ attack problems), the oracle (4.4.9) (note that the attack problem can be solved by $|E|$ of these problems), and the oracle (4.4.11) (this is equivalent to one attack problem). In fact, significant progress has been made in accelerating these problems; see, for example, [12, 35–37]. Integrating such advancements might substantially improve the efficiency of Algorithms 1

and 2.

4.4.4 An Algorithm for Computing Spanning Tree Modulus

Let $G = (V, E, \sigma)$ be a weighted graph with edge weights $\sigma \in \mathbb{R}_{>0}^E$. Let $\mathcal{B}$ be the family of spanning trees of G . Let η^* be the optimal density for $\text{Mod}_{2, \sigma^{-1}}(\widehat{\mathcal{B}})$. In [38], they propose an algorithm for computing spanning tree modulus based on the maximum value of η^* using Cunningham's algorithm for computing the strength of an unweighted graph. Applying results in [9], we suggest an algorithm for computing spanning tree modulus utilizing Algorithm 3 based on the minimum value of η^* . This algorithm is described as follows.

Any subgraph that is optimal for the fractional arboricity problem (4.4.2) is said to be a D-optimal subgraph. First, we find a D-optimal subgraph H of G using Algorithm 3. Then, [9, Theorem 7.7] shows that $\sigma^{-1}\eta^*$ takes the value $(|V_H| - 1)/\sigma(E_H)$ on all edges in H .

Now, we shrink H to a vertex; this results in a shrunk graph $G_1 = G/H$. Next, find a D-optimal subgraph H_1 of G_1 , and $\sigma^{-1}\eta^*$ (for the spanning tree modulus of G) takes the value $(|V_{H_1}| - 1)/\sigma(E_{H_1})$ on the edges of H_1 . Repeating this procedure iteratively computes $\sigma^{-1}\eta^*$ for at least one edge. Thus, after a finite number of iterations, $\sigma^{-1}\eta^*$ will be computed for all edges.

A more detailed description of this algorithm is provided in Algorithm 4.

Algorithm 4 Using D-optimal subgraphs to compute spanning tree modulus

Input: $G = (V, E, \sigma)$

Output: η^*

```

1: while  $G$  is nontrivial do
2:   compute  $D_\sigma(G)$  and a D-optimal subgraph  $H = (V_H, E_H)$ 
3:   for all  $e \in E_H$  do
4:      $\eta^*(e) \leftarrow \sigma(e)(D_\sigma(G))^{-1}$ 
5:   end for
6:    $G \leftarrow G/H$ 
7: end while
8: return  $\eta^*$ 

```

Remark 4.4.7. Run-time complexity for Algorithm 4: $|V|^3$ minimum-cut calculations. This

is because solving $D_\sigma(G)$ takes $|V|^2$ minimum-cut calculations and the number of iterations is at most $|V|$.

Chapter 5

Matroid Truncation

5.1 Overview

The base modulus of Chapters 3 and 4 treats a matroid through its full family of bases. This chapter asks how the theory behaves under one of the most basic operations in matroid theory, truncation, and how the universal density is transformed by it.

The motivation comes from graphs. The strength of a graph admits a natural generalization, the *c-order edge toughness*

$$\tau_c(G) = \min_{X \subseteq E: \omega(G-X) > c} \left\{ \frac{|X|}{\omega(G-X) - c} \right\}, \quad (5.1.1)$$

where c is a positive integer and $\omega(G-X)$ is the number of components of $G-X$; the case $c = 1$ recovers the ordinary strength. It was shown in [22] that $\tau_c(G) \geq k$ if and only if G has k edge-disjoint spanning forests with exactly c components, a common generalization of the theorems of Nash-Williams [6] and Tutte [7]. The same result was obtained in [23] through matroid truncation, which is the operation we now recall. For a matroid M and a fixed integer $1 \leq t \leq r(M)$, a *t-independent set* is an independent set of size at most t , and the *t-truncation* M_t is the matroid on E whose independent sets are exactly the *t-independent*

Table 5.1: Values for the universal densities η_t^* of matroid truncation M_t for $t = 1, \dots, 84$, of the graphic matroid presented in Figure 3.1.

Styles (#)	$t = 1 \rightarrow 28$	$t = 29 \rightarrow 34$	$t = 35$	$t = 36 \rightarrow 42$	$t = 43 \rightarrow 55$	$t = 56 \rightarrow 84$
Solid (45)	$t/84$	$1/3$	$1/3$	$(t - 20)/45$	$(t - 2)/81$	$t/84$
Dashed (36)	$t/84$	$(t - 15)/39$	$1/2$	$1/2$	$(t - 2)/81$	$t/84$
Dotted (3)	$t/84$	$(t - 15)/39$	$2/3$	$2/3$	$2/3$	$t/84$

sets of M . In the graphic case, the bases of $M(G)_t$ are precisely the spanning forests of G with t edges, for $1 \leq t \leq |V| - 1$; truncation therefore lets one study such forests as the base family of a matroid, and (5.1.1) places them squarely within the modulus framework.

The first contribution of the chapter is a description of how the universal density behaves under this operation: for every $t \leq r(M)$, the universal density $\eta_{M_t}^*$ of the truncation can be obtained explicitly from the universal density η_M^* of M . We then extend the family in the opposite direction. For $t = r(M), r(M) + 1, \dots, |E| - 1$, define the *dual truncation*

$$M_t := ((M^*)_{|E|-t})^*, \quad (5.1.2)$$

whose base family is

$$\mathcal{B}(M_t) = \{X \subseteq E : |X| = t, X \supseteq B \text{ for some } B \in \mathcal{B}(M)\}. \quad (5.1.3)$$

By a slight abuse of notation, M_t denotes the t -truncation when $t \leq r(M)$ and the dual truncation when $t > r(M)$, the intended meaning being clear from the range of t . We give the universal densities of the dual truncations as well, so that $\eta_{M_t}^*$ is determined for the entire range $t = 1, \dots, |E| - 1$; Figure 3.1 and Table 5.1 illustrate this for a graphic matroid.

A second theme is a new variational characterization of the universal density in the weighted setting. Recall that for a matroid M weighted by $\sigma \in \mathbb{R}_{>0}^E$, the universal density

η_M^* is the unique minimizer of the weighted energy over the base convex hull,

$$\min \left\{ \sum_{e \in E} \sigma^{-1}(e) \eta(e)^2 : \eta \in \text{conv}(\mathcal{B}(M)) \right\}. \quad (5.1.4)$$

We show that the same density arises from a logarithmic energy motivated by the Kullback–Leibler divergence,

$$\min_{\eta \in \text{conv}(\mathcal{B}(M))} - \sum_{e \in E} \sigma(e) \log \eta(e), \quad (5.1.5)$$

with the convention $\log \eta(e) = -\infty$ when $\eta(e) = 0$; that is, the minimizers of (5.1.4) and (5.1.5) coincide (Theorem 5.3.4).

The contributions of the chapter are summarized as follows.

- In Section 5.2, we give the universal densities of all matroid truncations, and of the dual truncations in (5.1.2).
- In Section 5.3, we establish the Kullback–Leibler characterization of the universal density of weighted matroids (Theorem 5.3.4).

5.2 Truncation of matroids and modulus

Throughout this section we work in the *unweighted* setting, i.e. we take $\sigma \equiv \mathbf{1}$, and $s_1 < \dots < s_k$ denote the distinct values of the universal density η^* itself. See Remark 5.2.9 for a discussion of the weighted case.

5.2.1 Homogeneous matroids and balancity

Let $M = (E, \mathcal{I})$ be a matroid with the base family $\mathcal{B}$. Let $\mathcal{N}$ be the usage matrix whose rows are the indicator vectors of the bases. First, we recall the notion of *matroid truncation*. Let t denote a fixed integer such that $1 \leq t \leq r(M)$. The t -truncation of M , denoted by M_t , is a matroid on E whose collection of independent sets is the set of all independent sets of

M with size at most t . For $X \subset E$, the rank of M_t is

$$r_{M_t}(X) = \min\{t, r_M(X)\},$$

and the density of M_t is

$$\theta(M_t) = \frac{|E|}{t} =: \theta_t(M). \quad (5.2.1)$$

Moreover, the strength and fractional arboricity of M_t satisfy:

$$S(M_t) = \min \left\{ \frac{|X|}{t - r_M(E - X)} : X \subset E, r_M(E - X) < t \right\} =: S_t(M) \quad (5.2.2)$$

and

$$D(M_t) = \max \left\{ \frac{|X|}{\min\{t, r_M(X)\}} : \emptyset \neq X \subset E \right\} =: D_t(M). \quad (5.2.3)$$

The main goal of this section is to show that the universal density $\eta_t^* := \eta_{M_t}^*$ of M_t can be computed using the universal density $\eta^* := \eta_M^*$ of M . Recall that the matroid M is homogeneous if and only if the universal density η^* is constant. Since $\eta^*(E) = r(M)$, then $\eta^*(e) = r(M)/|E|$ for all $e \in E$ in this case. We start with the following lemma.

Lemma 5.2.1. *If M is homogeneous, then M_t is homogeneous for any $t = 1, \dots, r(M) - 1$.*

Proof. Assume that M is homogeneous. Let $\mu^* \in \mathcal{P}(\Gamma)$ be an η^* -induced pmf, i.e., $\mathcal{N}^T \mu^* = \eta^*$. Denote $r := r(M)$, then

$$\eta^*(e) = \sum_{B \in \mathcal{B}: e \in B} \mu^*(B) = r/|E|, \quad \forall e \in E.$$

Let γ be an independent set of size t . We define μ as follows:

$$\mu(\gamma) := \sum_{B \in \mathcal{B}: \gamma \subset B} \mu^*(B) \frac{1}{\binom{r}{t}}.$$

We want to show that μ is a pmf in $\mathcal{P}(M_t)$ with constant element-usages. Indeed, μ is a pmf

because any base B contains exactly $\binom{r}{t}$ subsets of size t which are independent. Moreover, if we fix $e \in E$:

$$\begin{aligned}
\sum_{\gamma \in \mathcal{B}(M_t): e \in \gamma} \mu(\gamma) &= \sum_{\gamma \in \mathcal{B}(M_t): e \in \gamma} \sum_{B \in \mathcal{B}(M): \gamma \subset B} \mu^*(B) \frac{1}{\binom{r}{t}} \\
&= \frac{1}{\binom{r}{t}} \sum_{\gamma \in \mathcal{B}(M_t)} \sum_{B \in \mathcal{B}(M)} \mathbb{1}_{e \in \gamma} \mathbb{1}_{\gamma \subset B} \mu^*(B) \\
&= \frac{1}{\binom{r}{t}} \sum_{B \in \mathcal{B}(M)} \binom{r-1}{t-1} \mathbb{1}_{e \in B} \mu^*(B) \\
&= \frac{1}{\binom{r}{t}} \binom{r-1}{t-1} \eta^*(e) \\
&= \frac{t}{r} \frac{r}{|E|} = \frac{t}{|E|}.
\end{aligned}$$

Hence, M_t is homogeneous. □

The following lemma provides several properties of the sequence of truncations M_t .

Lemma 5.2.2. *Let $M = (E, \mathcal{I})$ be a matroid. Let t be an integer such that $1 \leq t < r(M)$ and let M_t be the t -truncation of M . Then, $D(M_t) \geq D(M_{t+1})$, $\theta(M_t) > \theta(M_{t+1})$, and $S(M_t) > S(M_{t+1})$.*

Proof. First, we have

$$D(M_t) = \max_{\emptyset \neq X \subset E} \frac{|X|}{\min\{t, r_M(X)\}} \geq \max_{\emptyset \neq X \subset E} \frac{|X|}{\min\{t+1, r_M(X)\}} = D(M_{t+1}).$$

Next, we have

$$\theta(M_t) = \frac{|E|}{t} > \frac{|E|}{t+1} = \theta(M_{t+1}).$$

Finally, let A be an optimal set for $S(M_t)$. Then

$$S(M_t) = \frac{|E| - |A|}{t - r_M(A)} > \frac{|E| - |A|}{t+1 - r_M(A)} \geq S(M_{t+1}).$$

□

We note that M_1 is always homogeneous. By Lemma 5.2.1, it follows that if M_{t_0} is homogeneous for some $1 < t_0 \leq r(M)$, then M_t is also homogeneous for $t < t_0$. Hence, we introduce the following notion.

Definition 5.2.3. Let $M = (E, \mathcal{I})$ be a matroid. The largest integer t such that M_t is homogeneous is called the *balancity* of M and is denoted by $b(M)$.

This definition is a generalization of the balancity of graphs in [23]. The following proposition provides a characterization of the balancity $b(M)$ of M . Recall that, for a matroid M , the unique maximal optimal set for $D(M)$ is called the core of M .

Proposition 5.2.4. Let $M = (E, \mathcal{I})$ be a matroid. Let t be an integer such that $1 \leq t \leq r(M)$. Then, M_t is homogeneous if and only if $D(M) \leq \theta_t(M)$. So,

$$b(M) = \max \{t \in \mathbb{Z} \cap [1, r(M)] : D(M) \leq \theta_t(M)\}.$$

If $t > b(M)$, then M_t is not homogeneous, and $D(M) = D(M_t)$, furthermore, M and M_t share the same core X^* with $r(X^*) < t$.

Proof. For the case $t = r(M)$, the statement holds because Corollary 3.3.3. Next, for an integer t such that $t \leq r(M) - 1$, we have

$$D(M) = \max \left\{ \max_{0 < r(X) < t} \frac{|X|}{r(X)}, \max_{r(X) \geq t} \frac{|X|}{r(X)} \right\}. \quad (5.2.4)$$

By (5.2.3), we also have

$$D_t(M) = \max \left\{ \max_{0 < r(X) < t} \frac{|X|}{r(X)}, \max_{r(X) \geq t} \frac{|X|}{t} \right\} = \max \left\{ \max_{0 < r(X) < t} \frac{|X|}{r(X)}, \frac{|E|}{t} \right\}. \quad (5.2.5)$$

Note that we always have

$$\max_{r(X) \geq t} \frac{|X|}{r(X)} \leq \frac{|E|}{t}. \quad (5.2.6)$$

Therefore, $D(M) \leq \frac{|E|}{t}$, if and only if, $\max_{0 < r(X) < t} \frac{|X|}{r(X)} \leq \frac{|E|}{t}$, and if and only if, $D_t(M) = \frac{|E|}{t} = \theta_t(M)$. Hence, $D(M) \leq \frac{|E|}{t} = \theta_t(M)$, if and only if, M_t is homogeneous.

Assume that M_t is not homogeneous, then $D_t(M) > \frac{|E|}{t}$. By (5.2.5) and (5.2.6), we obtain

$$D_t(M) = \max_{0 < r(X) < t} \frac{|X|}{r(X)} > \frac{|E|}{t} \geq \max_{r(X) \geq t} \frac{|X|}{r(X)}.$$

Combining this with (5.2.4), we have

$$D(M) = \max_{0 < r(X) < t} \frac{|X|}{r(X)} = D_t(M) > \frac{|E|}{t} \geq \max_{r(X) \geq t} \frac{|X|}{r(X)}.$$

Therefore, the sets of solutions of $D(M)$ and $D(M_t)$ are identical to the set of solutions of $\max_{0 < r(X) < t} \frac{|X|}{r(X)}$. Note that, as mentioned after (4.2.8), the fractional arboricity problems $D(M)$ and $D(M_t)$ admit unique maximal cores X^* and X_t^* , respectively. It follows that $X^* = X_t^*$ and $r(X^*) < t$. $\square$

5.2.2 Universal densities for all truncations

The main goal of this section is to find the universal density η_t^* of M_t for all $t \leq r(M)$. For $t \leq b(M)$, the truncation M_t is homogeneous by definition of $b(M)$, and in this case, the universal density of M_t is the constant $t/|E|$. Therefore, we shall focus on the range when $t = b + 1, \dots, r(M)$. In Section 5.2.3, we will address the case $t > r(M)$.

Theorem 5.2.5. *Let $M = (E, \mathcal{I})$ be a loopless matroid with $r(M) \geq 1$, unweighted (i.e. $\sigma \equiv \mathbf{1}$), with rank function r and universal density η^* . Let $s_1 < s_2 < \dots < s_k$ be the distinct values of η^* . For $i = 1, \dots, k$, denote $E_i := \{e \in E : \eta^*(e) \geq s_i\}$, $b_i := \lfloor |E_i| s_i \rfloor$, and $c_i := \lfloor |E_i| s_i \rfloor + r(E - E_i)$, where $\lfloor \cdot \rfloor$ is the floor function. Denote $c_0 := 0$.*

For $t = 1, \dots, r(M)$, let η_t^* be the universal density of the t -truncation M_t of M . Then, we have the following:

- (i) b_i is the balancity of $M/(E - E_i)$ for $i = 1, \dots, k$;
- (ii) $0 = c_0 < c_1 \leq c_2 \leq \dots \leq c_{k-1} < c_k = r(M)$;
- (iii) $c_i - b_i = r(E - E_i) \leq c_{i-1}$ for $i = 1, \dots, k$;
- (iv) For $i = 1, \dots, k$, and t such that $c_{i-1} < t \leq c_i$, then

$$\eta_t^*(e) = \begin{cases} \frac{t - r(E - E_i)}{|E_i|} & \text{if } e \in E_i; \\ \eta^*(e) & \text{if } e \in E - E_i. \end{cases} \quad (5.2.7)$$

In order to prove Theorem 5.2.5, we first want to study the relation between contractions using the core and truncations of a matroid. When $b(M) < r(M)$, the matroids $M_{b+1}, \dots, M_{r(M)}$ are not homogeneous. By Proposition 5.2.4, these matroids share the same core X^* with $r(X^*) \leq b$. The following lemma shows that the matroid contraction $(M_t)/X^*$ is identical to the $(t - r(X^*))$ -truncation $(M/X^*)_{t-r(X^*)}$ of M/X^* . With this lemma, we will be able to study the contraction $(M_t)/X^*$ by applying Proposition 5.2.4 to matroid truncations of M/X^* .

Lemma 5.2.6. *Let $b = b(M)$ be the balancity of M . Assume that $b < r(M)$. Let X^* be the core of M . For $t = b + 1, \dots, r(M)$, we have that $(M_t)/X^*$ is identical to the $(t - r(X^*))$ -truncation of M/X^* .*

Proof. First, by Proposition 5.2.4, all matroid truncations $M_{b+1}, \dots, M_{r(M)}$ share the same core X^* where $r(X^*) \leq b$, and X^* is the unique maximal optimal set for

$$\max_{0 < r(X) \leq b} \frac{|X|}{r(X)}.$$

We first verify that $M|X^* = M_t|X^*$ for $t = b + 1, \dots, r(M)$. Since $r(X^*) \leq b < t$, for every $Y \subset X^*$ we have $r_M(Y) \leq r_M(X^*) = r(X^*) < t$, and therefore

$$r_{M_t|X^*}(Y) = r_{M_t}(Y) = \min\{t, r_M(Y)\} = r_M(Y) = r_{M|X^*}(Y).$$

Two matroids on the same ground set with the same rank function are equal, so $M|X^* = M_t|X^*$.

For any $t \geq b+1$, by the definition of contraction, a subset $I \subset E - X^*$ is independent in $(M_t)/X^*$ if and only if $\exists B \in \mathcal{B}(M_t|X^*)$ such that $B \cup I$ is independent in M and $r(X^* \cup I) \leq t$. Conversely, a subset $I \in E$ is independent in $(M/X^*)_{t-r(X^*)}$ if and only if $\exists B \in \mathcal{B}(M|X^*)$ such that $B \cup I$ is independent in M and $r(X^* \cup I) - r(X^*) \leq t - r(X^*)$, which is equivalent to $r(X^* \cup I) \leq t$. Therefore, $(M_t)/X^*$ and $(M/X^*)_{t-r(X^*)}$ share the same collection of independent sets. $\square$

Next, we want to study the balanciness of M/X^* . If $r(X^*) = b(M)$, then $(M/X^*)_{b(M)-r(X^*)}$ has rank 0. If $r(X^*) < b(M)$, the following lemma shows that the balanciness of M/X^* is at least $b(M) - r(X^*)$.

Lemma 5.2.7. *Let $b = b(M)$ be the balanciness of a matroid M . Let X^* be the core of M . If $r(X^*) < b(M)$, then $(M/X^*)_{b-r(X^*)}$ is homogeneous.*

Proof. Denote $t = b - r(X^*)$ and $\mathcal{M} := M/X^*$. We want to show that $D(\mathcal{M}_t) = \theta(\mathcal{M}_t)$. Recall that $\mathcal{M}$ has ground set $E - X^*$, and for any nonempty subset $A \subset E - X^*$, we have $r_{\mathcal{M}}(A) = r(X^* \cup A) - r(X^*)$. By (5.2.5), we have

$$D(\mathcal{M}_t) = \max \left\{ \max_{0 < r_{\mathcal{M}}(A) < t} \frac{|A|}{r_{\mathcal{M}}(A)}, \frac{|E - X^*|}{t} \right\}.$$

We want to show that

$$\max_{0 < r_{\mathcal{M}}(A) < t} \frac{|A|}{r_{\mathcal{M}}(A)} < \frac{|E - X^*|}{t} = \theta(\mathcal{M}_t). \quad (5.2.8)$$

Since X^* is closed in M , it is enough to consider sets $A \neq \emptyset$ such that $r(X^* \cup A) - r(X^*) < t$. By the definition of X^* , we have

$$\frac{|X^* \cup A|}{r(X^* \cup A)} < \frac{|X^*|}{r(X^*)}. \quad (5.2.9)$$

Since $r(X^*) < b(M)$, we have $r(X^*) = r_{M_b}(X^*)$. And since M_b is homogeneous, we have

$$\frac{|X^*|}{r(X^*)} = \frac{|X^*|}{r_{M_b}(X^*)} \leq \frac{|E|}{b}. \quad (5.2.10)$$

Hence,

$$\begin{aligned} \frac{|A|}{r_{\mathcal{M}}(A)} &= \frac{|A|}{r(X^* \cup A) - r(X^*)} = \frac{|X^* \cup A| - |X^*|}{r(X^* \cup A) - r(X^*)} < \frac{|X^*|}{r(X^*)} && \text{(by (5.2.9))} \\ &\leq \frac{|E|}{b} \leq \frac{|E| - |X^*|}{b - r(X^*)} && \text{(by (5.2.10)).} \end{aligned}$$

Hence, $D(\mathcal{M}_t) = \theta(\mathcal{M}_t)$. It follows that $\mathcal{M}_t$ is homogeneous. $\square$

Finally, we are ready to prove Theorem 5.2.5. By Lemma 5.2.6, every contraction M_t/X^* is identical to a truncation of M/X^* . So, applying Proposition 5.2.4 and Lemma 5.2.6 to M/X^* will reveal additional values of η_t^* . Continuing by contracting the core of M/X^* and applying these two lemmas to further reveal values of η_t^* . This strategy determines all the values of η_t^* .

Proof of Theorem 5.2.5. Denote $A_i := \{e \in E : \eta^*(e) = s_i\}$. By definition, we have $E_1 = E$, and for $j = 2, \dots, k$,

$$E_i = E - \bigcup_{j=1}^{i-1} A_j.$$

For $i = 1, \dots, k$, write $N^{(i)} := M/(E - E_i)$, a matroid on the ground set E_i , so that $N^{(1)} = M$, and put $t_i := t - r(E - E_i)$. By (4.2.8), the universal density of $N^{(i)}$ is the restriction $\eta^*|_{E_i}$, we have $D(N^{(i)}) = 1/s_i$, and A_i is the core of $N^{(i)}$; consequently the minimum value of the universal density of $N^{(i)}$ is s_i and is attained exactly on A_i . We prove by induction on i that (i)–(iv) hold for i , together with the following statement

(*) for every integer t with $c_{i-1} < t \leq r(M)$, the contraction $M_t/(E - E_i)$ equals $(N^{(i)})_{t_i}$, and $\eta_t^*|_{E_i}$ is the universal density of $M_t/(E - E_i)$.

The statement (*) is what allows the argument to be repeated at the next level, since it lets

us apply (4.2.8) to $M_t/(E - E_i)$.

The case $i = 1$. By (4.2.8), we have $s_1 = 1/D(M)$ and M has the core A_1 . By Proposition 5.2.4, the balancity of M is equal to $\lfloor |E|/D(M) \rfloor = \lfloor |E_1|s_1 \rfloor = b_1$. Thus, for $c_0 = 0 < t \leq c_1 = b_1$, we have that M_t is homogeneous, equivalently, η_t^* is a constant vector; since $\eta_t^*(E) = r(M_t) = t$ and $E_1 = E$, this constant is

$$\eta_t^*(e) = \frac{t}{|E_1|} = \frac{t - r(E - E_1)}{|E_1|}.$$

So, (i),(ii),(iii), and (iv) hold for $i = 1$; here (ii) uses $c_1 = b_1 = \lfloor |E|/D(M) \rfloor \geq 1$, which holds because $r(M) \geq 1$. Finally, $(*)_1$ holds: $E - E_1 = \emptyset$, so $M_t/(E - E_1) = M_t = (N^{(1)})_{t_1}$ and $\eta_t^*|_{E_1} = \eta_t^*$ is by definition the universal density of M_t .

The inductive step. Let $1 \leq i \leq k - 1$ and assume (i)–(iv) and $(*)_i$ hold for i . Let t be an integer with $c_i < t \leq r(M)$; then $t > c_{i-1}$, so $(*)_i$ applies and $M_t/(E - E_i) = (N^{(i)})_{t_i}$ with

$$t_i = t - r(E - E_i) > c_i - r(E - E_i) = b_i.$$

By Proposition 5.2.4 applied to $N^{(i)}$, since A_i is the core of $N^{(i)}$, it is also the core of $(N^{(i)})_{t_i} = M_t/(E - E_i)$ for $t_i > b_i$, and $D(M_t/(E - E_i)) = D(N^{(i)}) = 1/s_i$. By $(*)_i$ the universal density of $M_t/(E - E_i)$ is $\eta_t^*|_{E_i}$, so by (4.2.8) its minimum value s_i is attained exactly on A_i . Hence

$$\eta_t^*(e) = \eta^*(e) = s_i \text{ for } e \in A_i = E_i - E_{i+1}, \quad \eta_t^*(e) > s_i \text{ for } e \in E_{i+1},$$

which, with the inductive hypothesis (iv), gives $\eta_t^* = \eta^*$ on $E - E_{i+1}$. Next, we consider the matroid $N^{(i)}/A_i = M/(E - E_{i+1}) = N^{(i+1)}$. By Proposition 5.2.4, the balancity of $M/(E - E_{i+1})$ is equal to $\lfloor |E_{i+1}|/D(M/(E - E_{i+1})) \rfloor = \lfloor |E_{i+1}|s_{i+1} \rfloor = b_{i+1}$, and $M/(E - E_{i+1})$ has the core A_{i+1} ; this is (i) for $i + 1$. By Proposition 5.2.4 applied to $N^{(i)}$ with $t = b_i + 1$,

its core satisfies $r_{N^{(i)}}(A_i) \leq b_i$; since $r(E - E_{i+1}) = r(E - E_i) + r_{N^{(i)}}(A_i)$, this gives

$$r(E - E_{i+1}) \leq r(E - E_i) + b_i = c_i,$$

which is (iii) for $i + 1$. If $r_{N^{(i)}}(A_i) = b_i$, then $c_{i+1} \geq c_i$ because $b_{i+1} \geq 0$. Otherwise $r_{N^{(i)}}(A_i) < b_i$ and, by Lemma 5.2.7, the balancity b_{i+1} of $M/(E - E_{i+1})$ is at least $b_i - r_{N^{(i)}}(A_i)$. Therefore,

$$c_{i+1} = b_{i+1} + r(E - E_{i+1}) \geq b_i + r(E - E_i) = c_i,$$

which is (ii) for $i + 1$. To obtain $(*)_{i+1}$, note that $b_i < r(N^{(i)})$ (as $\eta^*|_{E_i}$ is non-constant for $i < k$, so $N^{(i)}$ is not homogeneous), and $t_i > b_i$; so Lemma 5.2.6 applies to $N^{(i)}$ and gives

$$M_t/(E - E_{i+1}) = (M_t/(E - E_i))/A_i = (N^{(i)})_{t_i}/A_i = (N^{(i+1)})_{t_i - r_{N^{(i)}}(A_i)} = (N^{(i+1)})_{t_{i+1}}.$$

Moreover, applying (4.2.8) to the matroid $M_t/(E - E_i)$, whose core is A_i , the restriction of its universal density to $E_i - A_i = E_{i+1}$ is the universal density of $(M_t/(E - E_i))/A_i = M_t/(E - E_{i+1})$; by $(*)_i$ that restriction is $\eta_t^*|_{E_{i+1}}$. This is $(*)_{i+1}$.

Finally, if t satisfies $c_i < t \leq c_{i+1}$, then by $(*)_{i+1}$, $M_t/(E - E_{i+1}) = (M/(E - E_{i+1}))_{t - r(E - E_{i+1})}$ and this truncation is homogeneous since $t - r(E - E_{i+1}) \leq b_{i+1}$; equivalently, its universal density is constant, and as it sums to its rank $t - r(E - E_{i+1})$,

$$\eta_t^*(e) = \frac{t - r(E - E_{i+1})}{|E_{i+1}|}, \quad \text{for } e \in E_{i+1}.$$

So, (i),(ii),(iii), and (iv) hold for $i + 1$.

Finally, by (4.2.8) the universal density of $N^{(i)}$ is $\eta^*|_{E_i}$, which attains the values $s_i < \dots < s_k$ and is constant if and only if $i = k$. Since $(N^{(i)})_{r(N^{(i)})} = N^{(i)}$, the definition of balancity gives $b_i = r(N^{(i)})$ if $i = k$, and $b_i < r(N^{(i)})$ if $i < k$. As $r(N^{(i)}) = r(M) - r(E - E_i)$, adding $r(E - E_i)$ yields $c_k = r(M)$ and $c_i < r(M)$ for $i < k$; in particular $c_{k-1} < c_k$.

□

Remark 5.2.8. With the assumptions in Theorem 5.2.5, if $c_{i-1} < t \leq c_i$ for $i > 1$, then η_t^* attains i distinct values, and η_t^* agrees with η^* in its first $i - 1$ distinct values in increasing order. Indeed, by (5.2.7) the vector η_t^* agrees with η^* on $E - E_i = A_1 \cup \dots \cup A_{i-1}$, where it attains the values $s_1 < \dots < s_{i-1}$, and is constant on E_i with value $(t - r(E - E_i))/|E_i|$; moreover this constant is strictly larger than s_{i-1} , since the inductive step of the proof, applied at $i - 1$, shows that $\eta_t^*(e) > s_{i-1}$ for every $e \in E_i$. Hence η_t^* attains exactly the i distinct values $s_1 < \dots < s_{i-1} < (t - r(E - E_i))/|E_i|$, and the partition of E induced by η_t^* is

$$A_1, \dots, A_{i-1}, E_i = A_i \cup A_{i+1} \cup \dots \cup A_k.$$

In other words, it is obtained from the partition induced by η^* by merging the blocks $A_i, \dots, A_k$ into the single block E_i . When we decrease t from $c_k = r(M)$ to $c_0 = 0$, the blocks A_i , which form a partition of E induced by η^* , start to merge together, in the sense that, the partition induced by η_t^* is obtained by successively merging blocks of the principal partition as t passes through the critical values c_i . Note that c_{i-1} may equal c_i for several i , so two or more blocks can merge at a time.

Remark 5.2.9. Theorem 5.2.5 is stated for $\sigma \equiv \mathbf{1}$. A weighted version of the theorem is not treated here. We expect it to be formulated along the same lines, replacing $|X|$ by $\sigma(X)$ throughout: the truncation M_t would be studied through its weighted density $\theta_\sigma(M_t) = \sigma(E)/t$, the σ -balancity $b_\sigma(M)$ would be the largest t such that M_t is σ -homogeneous, and the quantities b_i, c_i would be replaced by $\lfloor \sigma(E_i)s_i \rfloor$ and $\lfloor \sigma(E_i)s_i \rfloor + r(E - E_i)$, with s_i the distinct values of $\sigma^{-1}\eta^*$. The expected form of (5.2.7) on E_i is then $\eta_t^*(e) = \sigma(e)(t - r(E - E_i))/\sigma(E_i)$, which reduces to (5.2.7) when $\sigma \equiv \mathbf{1}$. We expect the weighted analogues of Lemmas 5.2.1, 5.2.6 and 5.2.7 to hold, and leave the details to future work.

5.2.3 Dual truncations

In Theorem 5.2.5, we provided the universal density of all matroid truncations of a given matroid M . Next, we consider a different collection of matroids. Let M^* be the dual matroid

of M . It was shown in Chapter 3 that the universal density of M^* is equal to $\mathbf{1} - \eta^*$, where $\mathbf{1}$ is the vector of all ones. For $t = r(M), r(M) + 1, \dots, |E| - 1$, we define the *dual truncation* M_t as follows:

$$M_t := ((M^*)_{|E|-t})^*, \quad (5.2.11)$$

where M^* denotes the dual matroid of M . For consistency, when $t = |E|$, we denote by M_t the matroid whose ground set E is a base. It follows that the base family of M_t is

$$\mathcal{B}(M_t) = \{X \subset E : |X| = t, X \supset B \text{ for some base } B \in \mathcal{B}(M)\}. \quad (5.2.12)$$

Moreover, the following proposition provides the fractional arboricity of M_t .

Proposition 5.2.10. *Let $M = (E, \mathcal{I})$ be a matroid with the rank $r(M) < |E|$. For $t = r(M), r(M) + 1, \dots, |E| - 1$, let M_t be the matroid defined as in (5.2.11). Then,*

$$D(M_t) = \max_{\emptyset \neq X \subset E} \frac{|X|}{t - r(M) + r(X)}.$$

Proof. For $X \subset E$,

$$r_{M_t}(X) = |X| - r((M^*)_{|E|-t}) + r_{(M^*)_{|E|-t}}(E - X) \quad (5.2.13)$$

$$= |X| - (|E| - t) + \min \{|E| - t, |E| - |X| - r(M) + r(X)\} \quad (5.2.14)$$

$$= \min \{|X|, t - r(M) + r(X)\}. \quad (5.2.15)$$

If $|X| \leq t - r(M) + r(X)$, then

$$\frac{|X|}{r_{M_t}(X)} = 1 < \frac{|E|}{t} = \frac{|E|}{r_{M_t}(E)}.$$

Hence

$$D(M_t) = \max_{\emptyset \neq X \subset E} \frac{|X|}{r_{M_t}(X)} = \max_{\emptyset \neq X \subset E} \frac{|X|}{t - r(M) + r(X)}.$$

□

Remark 5.2.11. Since the universal density η^* of M encodes the universal densities of all truncations (Theorem 5.2.5) and of the dual matroid Chapter 3, it also encodes the universal density η_t^* of the dual truncation M_t for $t = r(M), r(M) + 1, \dots, |E|$. In particular, η^* also determines $D(M_t)$ and $S(M_t)$. We do not state a separate formula for η_t^* in this range: it is obtained by applying Theorem 5.2.5 to the dual matroid M^* and then taking complements.

See Figure 3.1 and Table 5.1 for an example with all universal densities of M_t , for $t = 1, \dots, |E|$.

5.3 Kullback–Leibler divergence

In this section, we study an optimization problem on the base polytope $\text{conv}(\mathcal{B}(M))$ of a matroid M , namely the convex hull of the indicator functions of all bases in $\mathcal{B}(M)$. First, we recall the relevant definitions for probability vectors (i.e., discrete distributions on finite sets). Let $p, q \in \mathbb{R}_{\geq 0}^n$ be two vectors. The *Kullback–Leibler divergence* between p and q is defined as

$$D_{\text{KL}}(p||q) := - \sum_{i=1}^n p_i \log \frac{q_i}{p_i} = - \sum_{i=1}^n p_i \log q_i + \sum_{i=1}^n p_i \log p_i.$$

Here $\log$ denotes the logarithm of base e and $0 \log 0 := 0$. The first sum on the right-hand side is the cross entropy between p and q . This motivated Problem [19, Equation (6.1)] in the context of graphic matroids. Here we generalize it to the following problem. Let $M(E, \mathcal{I})$ be a matroid with weights $\sigma \in \mathbb{R}_{\geq 0}^E$ and base family $\mathcal{B}$. Let $\mathcal{N}$ be the matrix whose rows are the indicator vectors of the bases. Let $\mathcal{P}(\mathcal{B})$ be the set of all probability mass functions (or pmfs) on $\mathcal{B}$. Now, we consider the *Minimum Kullback–Leibler divergence*

problem $(\text{MKL}_\sigma(\mathcal{B}))$:

$$\begin{aligned}
& \underset{\eta, \mu}{\text{minimize}} && - \sum_{e \in E} \sigma(e) \log \eta(e) \\
& \text{subject to} && \eta = \mathcal{N}^T \mu, \\
& && \eta(e) > 0 \quad \forall e \in E, \\
& && \mu \in \mathcal{P}(\mathcal{B}).
\end{aligned} \tag{5.3.1}$$

Lemma 5.3.1. *Let $M = (E, \mathcal{I})$ be a loopless matroid with weights $\sigma \in \mathbb{R}_{>0}^E$. Then (5.3.1) has a minimizer, and the optimal vector η is unique and strictly positive. We denote it by $\bar{\eta}$; a pmf μ with $\mathcal{N}^T \mu = \bar{\eta}$ need not be unique.*

Proof. In the variable η alone, the feasible set of (5.3.1) is $F := \text{conv}(\mathcal{B}) \cap \mathbb{R}_{>0}^E$. We first check that $F \neq \emptyset$. Let $e \in E$. Since M is loopless, $\{e\}$ is independent, so it extends to a base $B_e \in \mathcal{B}$; thus every element of E lies in at least one base. Let μ_0 be the uniform pmf on $\mathcal{B}$ and $\eta_0 := \mathcal{N}^T \mu_0$, so that for every $e \in E$

$$\eta_0(e) = \sum_{B \in \mathcal{B}: e \in B} \mu_0(B) = \frac{|\{B \in \mathcal{B} : e \in B\}|}{|\mathcal{B}|} \geq \frac{1}{|\mathcal{B}|} > 0.$$

Hence $\eta_0 \in F$.

Now consider the problem over $\text{conv}(\mathcal{B})$ with objective

$$f(\eta) := - \sum_{e \in E} \sigma(e) \log \eta(e) \in (-\infty, +\infty], \quad -\log 0 := +\infty,$$

so that $f = +\infty$ exactly on $\text{conv}(\mathcal{B}) \setminus F$. Every $\eta \in \text{conv}(\mathcal{B})$ is a convex combination of 0–1 vectors, so $\eta(e) \leq 1$ and hence $f \geq 0$; together with $f(\eta_0) < \infty$, this shows the infimum of f over $\text{conv}(\mathcal{B})$ is finite. Consequently

$$\inf_{\eta \in F} \left(- \sum_{e \in E} \sigma(e) \log \eta(e) \right) = \inf_{\eta \in \text{conv}(\mathcal{B})} f(\eta), \tag{5.3.2}$$

and the two problems have the same minimizers.

Each map $x \mapsto -\sigma(e) \log x$ is continuous on $(0, 1]$ and tends to $+\infty$ as $x \rightarrow 0^+$, hence is lower semicontinuous on $[0, 1]$; therefore so is f . Hence, there is $\bar{\eta} \in \text{conv}(\mathcal{B})$ with $f(\bar{\eta}) = \min_{\text{conv}(\mathcal{B})} f \leq f(\eta_0) < \infty$. Finiteness forces $\bar{\eta}(e) > 0$ for all e , i.e. $\bar{\eta} \in F$; by (5.3.2), $\bar{\eta}$ minimizes (5.3.1).

Uniqueness. The function $-\log$ is strictly convex, so f is strictly convex on $\mathbb{R}_{>0}^E$. Hence, the optimal η , denoted by $\bar{\eta}$, is unique. $\square$

Remark 5.3.2. Since each row sum of $\mathcal{N}$ equals to $r(E)$, we have $\eta(E) = r(E)$.

Remark 5.3.3. We recall the Gibbs's inequality for Kullback–Leibler divergence:

$$\begin{aligned} -\sum_{e \in E} \sigma(e) \log \eta(e) &= -\sum_{e \in E} \sigma(e) \log \left(\frac{\eta(e)}{\eta(E)} \right) - \sum_{e \in E} \sigma(e) \log \eta(E) \\ &= \sigma(E) \left(-\sum_{e \in E} \frac{\sigma(e)}{\sigma(E)} \log \left(\frac{\eta(e)}{\eta(E)} \right) \right) - \sigma(E) \log r(M) \\ &\geq \sigma(E) \left(-\sum_{e \in E} \frac{\sigma(e)}{\sigma(E)} \log \left(\frac{\sigma(e)}{\sigma(E)} \right) \right) - \sigma(E) \log r(M). \end{aligned}$$

Hence, if there exists a vector $\eta \in \text{conv}(\mathcal{B})$ such that it is parallel to σ , meaning that $\eta = t\sigma$ for some $t > 0$, it follows that this vector must be the unique optimal solution of MKL_σ .

The main theorem in this section is presented as follows.

Theorem 5.3.4. *Let $M(E, \mathcal{I})$ be a matroid with weights $\sigma \in \mathbb{R}_{>0}^E$ and the base family $\mathcal{B}$. Let $\bar{\eta}$ be the unique optimal solution for the $\text{MKL}_\sigma(\mathcal{B})$ problem. Then, $\bar{\eta}$ equals to the universal density of M .*

Below, we provide three different proofs for this theorem. The first proof uses the serial rule for the MKL problem, analogous to the serial rule for modulus. The other two proofs use two characterizations of the universal density.

5.3.1 Proof 1 of Theorem 5.3.4

Let $v \in \mathbb{R}_{\geq 0}^E$ and let B be a base in $\mathcal{B}$, the v -length of B is defined as $\ell_v(B) := \sum_{e \in B} v(e)$. First, we give the following lemma.

Lemma 5.3.5. *Suppose that $(\bar{\eta}, \mu^*)$ is a minimizer for the MKL_σ problem (5.3.1). Define a vector $v \in \mathbb{R}_{\geq 0}^E$ where $v(e) = \sigma(e)/\bar{\eta}(e)$ for $e \in E$. Then, for all $B \in \mathcal{B}$,*

$$\ell_v(B) \leq \sigma(E).$$

If $\mu^*(B) > 0$, then equality holds.

Proof. Let $B \in \mathcal{B}$ and let $\delta_B \in \mathcal{P}(\mathcal{B})$ be the Dirac mass on B . For the real parameter t , define

$$\mu_t = (1 - t)\mu^* + t\delta_B.$$

Defining $\eta_t := \mathcal{N}^T \mu_t$ yields

$$\eta_t(e) = (1 - t)\bar{\eta}(e) + t\mathcal{N}(B, e).$$

The entries of μ_t sum to 1 for every t , so $\mu_t \in \mathcal{P}(\mathcal{B})$ if and only if $\mu_t \geq 0$. For $0 \leq t \leq 1$ all entries of μ_t are nonnegative, so $\mu_t \in \mathcal{P}(\mathcal{B})$. While for $t < 0$ with sufficiently small $|t|$, $\mu_t \geq 0$ when $\mu^*(B) > 0$.

Moreover $\bar{\eta} > 0$ by Lemma 5.3.1 and $\eta_t \rightarrow \bar{\eta}$ as $t \rightarrow 0$, so $\eta_t > 0$ for all t in a neighborhood of 0. Hence (η_t, μ_t) is feasible for (5.3.1) for all small $t > 0$, and also for all small $t < 0$ precisely when $\mu^*(B) > 0$.

Note that

$$\log \eta_t(e) = \log \bar{\eta}(e) + \log \left(1 - t + t \frac{\mathcal{N}(B, e)}{\bar{\eta}(e)} \right) = \log \bar{\eta}(e) - t \left(1 - \frac{\mathcal{N}(B, e)}{\bar{\eta}(e)} \right) + O(t^2),$$

so

$$-\sum_{e \in E} \sigma(e) \log \eta_t(e) = -\sum_{e \in E} \sigma(e) \log \bar{\eta}(e) + t(\sigma(E) - \ell_v(B)) + O(t^2).$$

If $\mu^*(B) > 0$, then (η_t, μ_t) is feasible for t in a two-sided neighborhood of 0, so the derivative at $t = 0$ must vanish, implying $\ell_v(B) = \sigma(E)$. If $\mu^*(B) = 0$, then (η_t, μ_t) is feasible only for small positive t , so the one-sided derivative at $t = 0$ must be nonnegative, implying $\ell_v(B) \leq \sigma(E)$. $\square$

The following proposition provides the relationship between the core of M and the unique optimal solution $\bar{\eta}$ of $\text{MKL}_\sigma(\mathcal{B})$.

Proposition 5.3.6. *Let $\bar{\eta}$, μ^* and v be defined as in Lemma 5.3.5. Denote $v_{\max} = \max_e v(e)$ and let $V_{\max}$ be the set of elements where $v(e)$ attains this maximum. Let $D_\sigma(M)$ be the fractional arboricity of M . We have the following:*

(i)

$$\bar{\eta}(V_{\max}) = r(V_{\max});$$

(ii)

$$v_{\max} = \frac{\sigma(V_{\max})}{r(V_{\max})};$$

(iii) $V_{\max}$ is the unique maximal optimal set for $D_\sigma(M)$ and

$$v_{\max} = D_\sigma(M).$$

Proof. To prove (i), by Lemma 3.2.12, we have

$$r(V_{\max}) \geq \bar{\eta}(V_{\max}), \tag{5.3.3}$$

and the equality holds, if and only if, every base $B \in \text{supp } \mu^*$ satisfies $r(V_{\max}) = |B \cap V_{\max}|$. Suppose that a base B satisfies $|B \cap V_{\max}| < r(V_{\max})$. Then, by the proof of Part 1 in

Theorem 3.3.2, there exist $z \in V_{max} \setminus B$ and $x \in B \setminus V_{max}$ such that $B' = (B \cup \{z\}) \setminus \{x\}$ is a base. Then,

$$\sigma(E) \geq \ell_v(B') = \ell_v(B) + v(z) - v(x) > \ell_v(B).$$

By Lemma 5.3.5, we have $\mu^*(B) = 0$. Therefore, (i) holds.

For (ii), by the definition of V_{max} , we have

$$v_{max}\bar{\eta}(V_{max}) = v_{max} \sum_{e \in V_{max}} \bar{\eta}(e) = \sum_{e \in V_{max}} \sigma(e) = \sigma(V_{max}).$$

Hence, by (i)

$$v_{max} = \frac{\sigma(V_{max})}{\bar{\eta}(V_{max})} = \frac{\sigma(V_{max})}{r(V_{max})}.$$

For (iii), consider any nonempty set $Z \subset E$ with $r(Z) > 0$, we have

$$v_{max}\bar{\eta}(Z) = \sum_{e \in Z} v_{max}\bar{\eta}(e) \geq \sum_{e \in Z} \sigma(e) = \sigma(Z).$$

Then, by (5.3.3),

$$v_{max} \geq \frac{\sigma(Z)}{\bar{\eta}(Z)} \geq \frac{\sigma(Z)}{r(Z)}.$$

Therefore, the fractional arboricity problem $D_\sigma(M)$ reaches its maximum at the set V_{max} . Note that $D_\sigma(M)$ attains its maximum at a unique maximal optimal set. If $Z - V_{max} \neq \emptyset$, then

$$v_{max}\bar{\eta}(Z) = \sum_{e \in Z} v_{max}\bar{\eta}(e) > \sum_{e \in Z} \sigma(e) = \sigma(Z),$$

and, by (5.3.3),

$$v_{max} > \frac{\sigma(Z)}{\bar{\eta}(Z)} \geq \frac{\sigma(Z)}{r(Z)}.$$

Hence, Z is not optimal for $D_\sigma(M)$. Since $D_\sigma(M) = v_{max}$, the set Z is not optimal for $D_\sigma(M)$. Hence every optimal set for $D_\sigma(M)$ is contained in V_{max} . As V_{max} is itself optimal, it contains every optimal set and is therefore the largest optimal set; in particular it is the unique maximal one. (Proper subsets of V_{max} may also be optimal, but each is contained in the optimal set V_{max}) Therefore, V_{max} is the unique maximal optimal set for $D_\sigma(M)$. $\square$

Proposition 5.3.6 shows that V_{max} , defined as in Proposition 5.3.6, is the core of M with weights σ . Now, we provide the first proof of Theorem 5.3.4. In the proof, we use the serial rule for the MKL problem, see Theorem A.2.1 in the appendix, to show that the unique optimal solution $\bar{\eta}$ of $\text{MKL}_\sigma(\mathcal{B})$ preserves after contracting the core of M .

Proof 1 for Theorem 5.3.4. Here $\eta^* = \eta_M^*$ is the universal density of M with weights σ , i.e. the unique optimal solution of (5.1.4). We argue by induction on $|E|$, the inductive statement being: for every loopless matroid with positive weights, the optimal solution of its MKL problem is its universal density. Let $\bar{\eta}$, μ^* , v and V_{max} be defined as in Proposition 5.3.6, and recall from that proposition that $v_{max} = D_\sigma(M)$ and that V_{max} is the core of M .

The value of $\bar{\eta}$ on V_{max} . By the definition of V_{max} , $v(e) = v_{max} = D_\sigma(M)$ for $e \in V_{max}$; since $v(e) = \sigma(e)/\bar{\eta}(e)$,

$$\bar{\eta}(e) = \frac{\sigma(e)}{D_\sigma(M)} \quad \text{for } e \in V_{max}. \quad (5.3.4)$$

On the other hand, by (4.2.8) the universal density η^* of M satisfies $\sigma^{-1}(e)\eta^*(e) = s_1 = 1/D_\sigma(M)$ for e in the core $A_1 = V_{max}$, i.e. $\eta^*(e) = \sigma(e)/D_\sigma(M)$ there. Hence $\bar{\eta} = \eta^*$ on V_{max} .

If $V_{max} = E$, then (5.3.4) holds for all $e \in E$, so $\bar{\eta} = \eta^*$ on E and we are done.

If $V_{max} \subsetneq E$, by Theorem 3.2.17,

$$\mathcal{B}(M|V_{max}) \oplus \mathcal{B}(M/V_{max}) = \mathcal{B}',$$

where $\mathcal{B}'$ is the set of bases $B \in \mathcal{B}$ satisfying $|B \cap V_{max}| = r(V_{max})$. By the proof of Proposition 5.3.6, every base $B \in \text{supp } \mu^*$ satisfies $|B \cap V_{max}| = r(V_{max})$, i.e. $\text{supp } \mu^* \subseteq \mathcal{B}'$. Consequently $\mu^* \in \mathcal{P}(\mathcal{B}')$ and $\bar{\eta} = \mathcal{N}^T \mu^* \in \text{conv}(\mathcal{B}')$; since $\text{conv}(\mathcal{B}') \subseteq \text{conv}(\mathcal{B})$, the minimum of the MKL_σ objective over $\text{conv}(\mathcal{B}')$ is attained at $\bar{\eta}$ as well. Hence $\text{MKL}_\sigma(\mathcal{B}) = \text{MKL}_\sigma(\mathcal{B}')$, with the same unique optimal vector $\bar{\eta}$.

Applying the serial rule, Theorem A.2.1, to $\Gamma_1 = \mathcal{B}(M|V_{max})$ on V_{max} and $\Gamma_2 = \mathcal{B}(M/V_{max})$ on $E - V_{max}$, we obtain that the restriction $\bar{\eta}|_{E-V_{max}}$ is the unique optimal solution of $\text{MKL}_{\sigma'}(\mathcal{B}(M/V_{max}))$, where $\sigma' := \sigma|_{E-V_{max}}$.

Since V_{max} is the core of M , it is closed. Hence the matroid M/V_{max} is loopless. Since $V_{max} \neq \emptyset$, the ground set $E - V_{max}$ is strictly smaller than E , and the induction hypothesis applies to M/V_{max} with weights σ' : the optimal solution of $\text{MKL}_{\sigma'}(\mathcal{B}(M/V_{max}))$ is the universal density of M/V_{max} . Therefore

$$\bar{\eta}|_{E-V_{max}} = \eta_{M/V_{max}}^* = \eta^*|_{E-V_{max}},$$

the last equality by (4.2.8), since $V_{max} = A_1 = E - E_2$ and the universal density of $M/(E - E_2)$ is the restriction of η^* to E_2 . Combined with $\bar{\eta} = \eta^*$ on V_{max} , this gives $\bar{\eta} = \eta^*$ on all of E . $\square$

Remark 5.3.7. Let $\bar{\eta}$ be the universal density of M with weights σ . By Theorem 5.3.4 and Lemma 5.3.5, we have

$$\sum_{e \in B} \frac{\sigma(e)}{\bar{\eta}(e)} \leq \sigma(E) \quad \text{for all } B \in \mathcal{B},$$

and equality holds if B is supported by an optimal pmf inducing $\bar{\eta}$.

5.3.2 Proof 2 and 3 of Theorem 5.3.4

We first recall some notations from [31]. Define the *dependence function* dep from $\text{conv}(\mathcal{B}) \times E$ into 2^E as follows: For a vector $x \in \text{conv}(\mathcal{B})$, $\text{dep}(x, u)$ is the set of all elements $v \in E$ such

that, for some $\epsilon > 0$, the vector $y \in \mathbb{R}^E$, defined by

$$y = x + \epsilon \mathbb{1}_u - \epsilon \mathbb{1}_v,$$

is a vector in $\text{conv}(\mathcal{B})$. Basically, the set $\text{dep}(x, u)$ consists of all coordinate elements of x from which mass can be transported to u while keeping the vector in the convex hull.

Note that $u \in \text{dep}(x, u)$ for any $u \in E$ where $x(u) > 0$. The author in [31] gives the following theorem.

Theorem 5.3.8. [31] *Let x be a vector in $\text{conv}(\mathcal{B})$, and let σ be given weights in $\mathbb{R}_{>0}^E$. Define*

$$c(e) = \frac{x(e)}{\sigma(e)} \quad (e \in E),$$

and let the distinct values of $c(e)$ for $e \in E$ be given by

$$c_1 < c_2 < \cdots < c_k. \tag{5.3.5}$$

Define subsets $S_i \subset E$ ($i = 1, 2, \dots, k$) by

$$S_i = \{e \in E \mid c(e) \leq c_i\} \quad (i = 1, 2, \dots, k). \tag{5.3.6}$$

Then, the following three conditions are equivalent:

- (i) x is the universal density of M with respect to the weight vector σ .*
- (ii) $x(S_i) = r(S_i)$ for all $i = 1, 2, \dots, k$.*
- (iii) $\emptyset \subset \text{dep}(x, e) \subset S_i$ for all $e \in S_i$ and $i = 1, 2, \dots, k$.*

The characterizations of the universal density in (ii) and (iii) of Theorem 5.3.8 provide the second and third proof of Theorem 5.3.4.

Proof 2 for Theorem 5.3.4. Let $\bar{\eta}$, μ^* and v be as in Lemma 5.3.5 and define $c(e) = 1/v(e)$ for every $e \in E$. We aim to apply Part (ii) of Theorem 5.3.8 for $\bar{\eta}$. Define S_i as in (5.3.6). By Lemma 3.2.12, we have $r(S_i) \geq \bar{\eta}(S_i)$ for every S_i and the equality holds if and only if, every base $B \in \text{supp } \mu^*$ satisfies $r(S_i) = |B \cap S_i|$. Suppose that a base B satisfies $|B \cap S_i| < r(S_i)$. Then by the proof of Part 1 in Theorem 3.3.2, there exist $z \in S_i \setminus B$ and $x \in B \setminus S_i$ such that $B' = (B \cup \{z\}) \setminus \{x\}$ is a base. Then,

$$\sigma(E) \geq \ell_v(B') = \ell_v(B) + v(z) - v(x) > \ell_v(B).$$

By Lemma 5.3.5, $\mu^*(B) = 0$. Therefore, we have $\bar{\eta}(S_i) = r(S_i)$ for all $i = 1, 2, \dots, k$. By Part (ii) of Theorem 5.3.8, the proof is completed. $\square$

Proof 3 for Theorem 5.3.4. Let $\bar{\eta}$, μ^* and v be as in Lemma 5.3.5. We aim to apply Part (iii) of Theorem 5.3.8 for $\bar{\eta}$. Define $c(e) = 1/v(e)$ for every $e \in E$ and let the distinct values of $c(e)$ for $e \in E$ be given by

$$c_1 < c_2 < \dots < c_k.$$

Denote $f(\eta)$ the objective function of the $\text{MKL}_\sigma(\mathcal{B})$ problem. Then f is differentiable on the relative interior of $\text{conv}(\mathcal{B})$, and

$$\left. \frac{\partial}{\partial \eta(e)} f(\eta) \right|_{\eta=\bar{\eta}} = -\frac{\sigma(e)}{\bar{\eta}(e)} = -\frac{1}{c(e)}. \quad (5.3.7)$$

Let $e \in S_i$ for some i and let $e' \in \text{dep}(\bar{\eta}, e)$. By definition of $\text{dep}(\bar{\eta}, e)$, there exists $\epsilon > 0$ such that

$$\bar{\eta} + t(\mathbb{1}_e - \mathbb{1}_{e'}) \in \text{conv}(\mathcal{B}) \quad \text{for all } t \in [0, \epsilon].$$

Let $d := \mathbb{1}_e - \mathbb{1}_{e'}$ and define $g : [0, \epsilon] \rightarrow \mathbb{R}$ by

$$g(t) := f(\bar{\eta} + td).$$

Then g is differentiable and

$$g'(0) = \left. \frac{\partial f}{\partial \eta(e)} \right|_{\eta=\bar{\eta}} - \left. \frac{\partial f}{\partial \eta(e')} \right|_{\eta=\bar{\eta}} = -\frac{1}{c(e)} + \frac{1}{c(e')} = \frac{1}{c(e')} - \frac{1}{c(e)}.$$

If $c_i < c(e')$, then $c(e) \leq c_i < c(e')$, hence $g'(0) < 0$. Therefore, for some $0 < \delta < \epsilon$ small enough, we have

$$f(\bar{\eta} + \delta \mathbb{1}_e - \delta \mathbb{1}_{e'}) < f(\bar{\eta}),$$

contradicting optimality of $\bar{\eta}$. Hence we must have $c(e') \leq c_i$. So, for each $i = 1, 2, \dots, k$, we have

$$\emptyset \subset \text{dep}(\bar{\eta}, e) \subset S_i \tag{5.3.8}$$

for all $e \in S_i$. By Theorem [5.3.8](#), the proof is completed. $\square$

Chapter 6

Hypertree and Multi-Tree Modulus

6.1 Overview

The hypertrees of a hypergraph are the bases of its hypergraphic matroid, so the base modulus theory developed in Chapters 3–5 can be applied directly to them. This raises the guiding question of the chapter: does the modulus of the hypertree family encode the strength and fractional arboricity of the hypergraph, as base modulus does for a matroid? The answer is subtler than in the graph case, and this subtlety is what the chapter is organized around.

The difficulty is that connectivity of a hypergraph does not guarantee the existence of a hypertree. For a connected hypergraph H , the bases of the associated hypergraphic matroid are the maximal hyperforests, so base modulus sees the strength and fractional arboricity of the *matroid*, which need not coincide with those of the *hypergraph*. To recover the hypergraph invariants one must restrict to partition-connected hypergraphs, where hypertrees exist; and to handle the general connected case, where the hypertree family may be empty, we introduce a replacement, the multi-tree family, on which modulus is always defined.

Hypergraphs, hypertrees, and their invariants

We recall only what is needed to state the results; the hypergraphic matroid and its place in the modulus framework were discussed in Chapter 1. Let $H = (V, E)$ be a hypergraph. For $X \subseteq V$ and $F \subseteq E$, write $F[X]$ for the hyperedges of F contained in X . A set $F \subseteq E$ is a *hyperforest* if $|F[X]| \leq |X| - 1$ for every non-empty $X \subseteq V$, and a *hypertree* if in addition $|F| = |V| - 1$; for graphs these are exactly the spanning trees. Lorea [24] showed that the hyperforests of a hypergraph are the independent sets of a matroid, called the *hypergraphic matroid*. Frank et al. [25] gave a rank formula for this matroid and the maximum number of disjoint hypertrees contained in a hypergraph. Let $H = (V, E)$ be a hypergraph. If $\mathcal{P} = \{V_1, \dots, V_m\}$ is a partition of V and $F \subseteq E$, define $\delta_F(\mathcal{P})$ to be the set of hyperedges in F that meet at least two parts of $\mathcal{P}$. We write $|\mathcal{P}|$ for the number of parts of $\mathcal{P}$, and we write $\delta(\mathcal{P})$ instead of $\delta_E(\mathcal{P})$ when $F = E$. For $F \subseteq E$, the rank of F in the hypergraphic matroid is

$$r(F) = \min \{ |V| - |\mathcal{P}| + |\delta_F(\mathcal{P})| : \mathcal{P} \text{ is a partition of } V \}. \quad (6.1.1)$$

For a positive integer k , the hypergraph H is called *k-partition-connected* if

$$|\delta(\mathcal{P})| \geq k(|\mathcal{P}| - 1) \quad (6.1.2)$$

for every partition $\mathcal{P}$ of V . When $k = 1$, we simply say that H is *partition-connected*. A hypergraph contains a hypertree if and only if it is partition-connected. More generally, H contains k edge-disjoint hypertrees if and only if (6.1.2) holds for every partition $\mathcal{P}$ of V .

By analogy with the matroidal invariants of Section 2.1.2, we attach to a hypergraph a strength and a fractional arboricity through partition and subhypergraph density ratios,

$$S(H) := \min_{\mathcal{P}: |\mathcal{P}| \geq 2} \frac{|\delta(\mathcal{P})|}{|\mathcal{P}| - 1}, \quad D(H) := \max_{X \subseteq V: |X| \geq 2} \frac{|E[X]|}{|X| - 1}, \quad (6.1.3)$$

and, for hyperedge weights $\sigma \in \mathbb{R}_{>0}^E$, their weighted counterparts

$$S_\sigma(H) := \min_{\mathcal{P}: |\mathcal{P}| \geq 2} \frac{\sigma(\delta(\mathcal{P}))}{|\mathcal{P}| - 1}, \quad D_\sigma(H) := \max_{X \subseteq V: |X| \geq 2} \frac{\sigma(E[X])}{|X| - 1}. \quad (6.1.4)$$

We write $\Gamma = \Gamma(H)$ for the *hypertree family* of H , identified with the indicator vectors $\mathbb{1}_\gamma$ of its hypertrees.

Hypertree modulus

For a partition-connected hypergraph, we relate the 2-modulus of $\Gamma(H)$ to the invariants (6.1.3). Following the notion of homogeneity for graphs and matroids used throughout this dissertation, we call H *homogeneous* when the optimal density η^* of $\text{Mod}_2(\widehat{\Gamma})$ is a constant vector, where $\widehat{\Gamma}$ is the Fulkerson blocker family of Γ . This is again governed by the equality of the two invariants.

Theorem 6.1.1. *A partition-connected hypergraph H is homogeneous if and only if $S(H) = D(H)$.*

More than the homogeneous case, the optimal density of the hypertree modulus drives a hierarchical decomposition of H tied to its strength and fractional arboricity, the *Hypergraph Decomposition Process* of Section 6.3, mirroring the deflation process for graphs and the principal partition for matroids.

The multi-tree family and the general connected case

When H is connected but not partition-connected, $\Gamma(H)$ may be empty and the hypertree modulus is unavailable. To address this case, we introduce the family of multitrees below. For every multiset A from E , we denote $V[A] \subset V$ as the set of vertices in $\bigcup_{e \in A} e$. The hypergraph $(V[A], A)$ is called the *subhypergraph induced by the hyperedge-multiset A* and denoted by $H[A]$. The indicator vector $x^A \in \mathbb{R}_{\geq 0}^E$ of A is defined such that $x^A(e)$ represents

the multiplicity of e in A , see Example 6.1.5 for instance. We introduce the following definition.

Definition 6.1.2. The family of all indicator vectors of multisets A from E such that $H[A]$ is a hypertree with vertex set V is called the *multi-tree* family of H and is denoted by $\Omega = \Omega(H)$.

The multi-tree modulus recovers the same structure and, when hypertrees exist, agrees with the hypertree modulus at the level of optimal densities.

Theorem 6.1.3. *If H is partition-connected, then $\text{Mod}_2(\widehat{\Omega})$ and $\text{Mod}_2(\widehat{\Gamma})$ have the same optimal density.*

This lets us extend homogeneity to every connected hypergraph, by declaring H *homogeneous* when the optimal density of $\text{Mod}_2(\widehat{\Omega}(H))$ is constant; the definition agrees with the partition-connected one by Theorem 6.1.3. The characterization of Theorem 6.1.1 then persists in full generality.

Theorem 6.1.4. *A connected hypergraph H is homogeneous if and only if $S(H) = D(H)$.*

As with hypertrees, the multi-tree modulus induces a hierarchical decomposition of H ; Section 6.4 develops this process and works through an explicit example.

Example 6.1.5 includes an example of the hypertree family and the multitree family.

Example 6.1.5. Consider the partition-connected hypergraph H in Figure 6.1. Table 6.1 presents the usage matrices, whose rows correspond to vectors from the hypertree family $\Gamma(H)$ and the multi-tree family $\Omega(H)$, respectively. The table also illustrates the following relationships. First, the unweighted modulus $\text{Mod}_2(\Gamma)$ and $\text{Mod}_2(\Omega)$ share both the same value and optimal density, as guaranteed by Theorem 6.1.3; the table also illustrates Theorem 6.4.2. Second, the unweighted modulus $\text{Mod}_1(\Gamma)$ and $\text{Mod}_1(\Omega)$ coincide with the strength $S(H)$, consistent with Remarks 6.5.11 and 6.5.13. Third, the weighted modulus $\text{Mod}_{1,\sigma}(\Omega)$ equals the weighted strength $S_\sigma(H)$, as follows from Remark 6.5.11. Finally, we

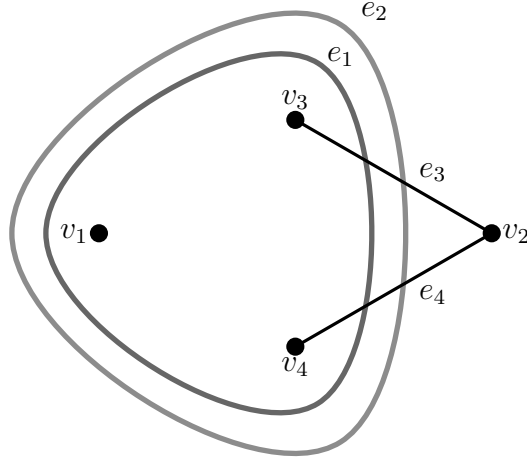


Figure 6.1: A hypergraph $H = (V, E)$ with $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4\}$, where $e_1 = \{v_1, v_3, v_4\}$ and $e_2 = \{v_1, v_3, v_4\}$ are represented by two nested rounded triangular curves, $e_3 = \{v_2, v_3\}$, and $e_4 = \{v_2, v_4\}$.

conjecture that Theorem 6.4.2 extends to the weighted case, as suggested by the table:

$$S_\sigma(H) = \frac{1}{(\sigma^{-1}\tilde{\eta}^*)_{\max}}, \quad D_\sigma(H) = \frac{1}{(\sigma^{-1}\tilde{\eta}^*)_{\min}},$$

where $\sigma^{-1}\tilde{\eta}^*$ denotes the element-wise product.

Next, we present an example of $\text{Mod}_2(\hat{\Omega})$ revealing a hierarchical structure of a hypergraph H .

Example 6.1.6. Figure 6.2 describes an example of the Hypergraph Decomposition Process discussed in Section 6.4. Consider a connected hypergraph H in the top of Figure 6.2. Here, the optimal density $\tilde{\eta}^*$ of $\text{Mod}_2(\hat{\Omega}(H))$ takes three distinct values, indicated by the edge styles as described in the figure caption. First, we remove the dotted hyperedges from H . As the result, we obtain two identical connected hypergraphs, denoted by H_1 and H_2 , displayed in the middle right of the figure, and two isolated vertices. On the other hand, we shrink H_1 and H_2 in the original hypergraph H , the resulting shrunk graph is the hypergraph, denoted by K , displayed in the middle left. Notice that the hyperedge set of H is the disjoint union of the hyperedge sets of H_1 , H_2 and K . So, we say that H decomposes into H_1 , H_2 , two isolated

	Hypertree family Γ	Multi-tree family Ω
Usage matrix	$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix}$
Unweighted Mod_2	$\text{Mod}_2(\Gamma) = 4/9$ $\eta^* = (3/4, 3/4, 3/4, 3/4)$	$\text{Mod}_2(\Omega) = 4/9$ $\tilde{\eta}^* = (3/4, 3/4, 3/4, 3/4)$
Unweighted Mod_1	$\text{Mod}_1(\Gamma) = 4/3$	$\text{Mod}_1(\Omega) = 4/3$
Unweighted $S(H), D(H)$	$S(H) = 4/3$	$D(H) = 4/3$
Weighted $\sigma = (4, 1, 1, 1)$ $\text{Mod}_{2,\sigma}$	$\text{Mod}_{2,\sigma}(\Gamma) = 12/19$ $\sigma^{-1}\eta^* = (1/4, 2/3, 2/3, 2/3)$	$\text{Mod}_{2,\sigma}(\Omega) = 10/13$ $\sigma^{-1}\tilde{\eta}^* = (2/5, 2/5, 1/2, 1/2)$
Weighted $\sigma = (4, 1, 1, 1)$ $\text{Mod}_{1,\sigma}$	$\text{Mod}_{1,\sigma}(\Gamma) = 3/2$	$\text{Mod}_{1,\sigma}(\Omega) = 2$
Weighted $\sigma = (4, 1, 1, 1)$ $S_\sigma(H), D_\sigma(H)$	$S_\sigma(H) = 2$	$D_\sigma(H) = 5/2$

Table 6.1: Families $\Gamma(H)$ and $\Omega(H)$, their usage matrices, optimal densities η^* defined in (2.1.28), modulus values, and strength, fractional arboricity for the hypergraph H in Figure 6.1.

vertices and K . Next, for H_1 (similarly for H_2), we continue to decompose it by removing the dashed hyperedges from H_1 . Then, we obtain a connected hypergraph displayed in the bottom right, denoted by N , and two isolated vertices. In H_1 , we shrink N to a single vertex to get the line hypergraph L in the bottom left. The decomposition process stops here since all the created hypergraphs (isolated vertices, K , N and L) are homogeneous.

In the end of this section, we give a remark about $\text{Mod}_p(\widehat{\Gamma})$ and $\text{Mod}_p(\widehat{\Omega})$ for $p \in (1, \infty) \setminus \{2\}$.

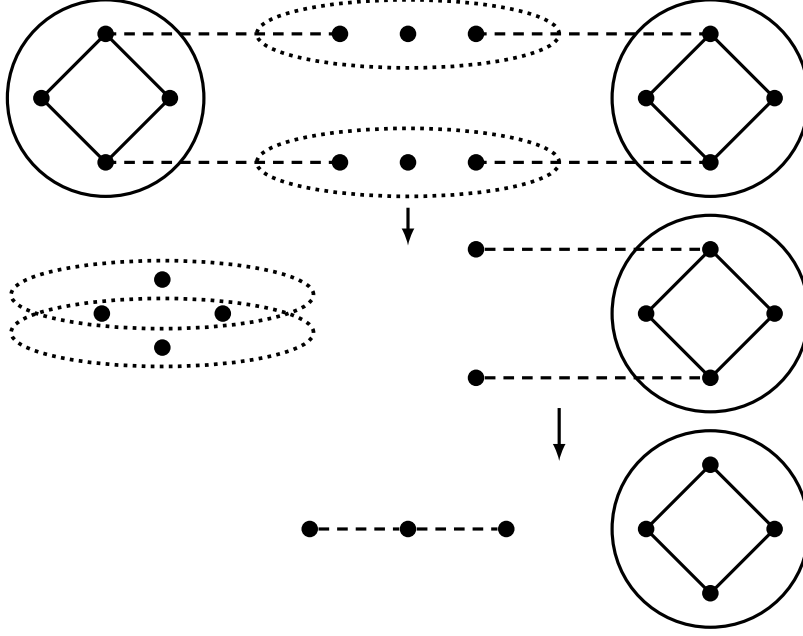


Figure 6.2: A hypergraph where hyperedges are styled according to the optimal density $\tilde{\eta}^*$. Solid edges are present with $\tilde{\eta}^* = \frac{3}{5}$, dashed edges with $\tilde{\eta}^* = 1$ and dotted edges with $\tilde{\eta}^* = \frac{3}{2}$.

Remark 6.1.7. Let $p \in (1, \infty) \setminus \{2\}$, and let M be a matroid with family of bases $\mathcal{B}$. Theorem 3.7.1 states that $\text{Mod}_2(\widehat{\mathcal{B}})$ and $\text{Mod}_p(\widehat{\mathcal{B}})$ have the same optimal density. By applying this theorem to the case of hypergraphic matroids, it follows that $\text{Mod}_2(\widehat{\Gamma})$ and $\text{Mod}_p(\widehat{\Gamma})$ have the same optimal density. Furthermore, it follows from Theorem 6.4.2 below that $\text{Mod}_2(\widehat{\Omega})$ and $\text{Mod}_p(\widehat{\Omega})$ have the same optimal density.

Fulkerson duality for partition inequalities

The final contribution extends the classical Fulkerson duality between spanning trees and vertex partitions, recorded in Chapter 1, to hypergraphs. Building on the partition characterization of Frank et al., the authors of [39] showed that the *partition polyhedron*

$$P(H) := \{x \in \mathbb{R}_{\geq 0}^E : x(\delta(\mathcal{P})) \geq |\mathcal{P}| - 1 \text{ for every partition } \mathcal{P} \text{ of } V\} \quad (6.1.5)$$

has integral extreme points, all lying in $\Omega(H)$; for graphs these are the spanning trees. We identify the Fulkerson blocker of the multi-tree family exactly, and thereby a minimal

description of the partition inequality system. Call a partition *feasible* if each part induces a connected subhypergraph, and, for a feasible $\mathcal{P}$, let the shrunk hypergraph $H_{\mathcal{P}} := H/(E \setminus \delta(\mathcal{P}))$ be the contraction onto the crossing hyperedges.

Theorem 6.1.8. *The Fulkerson blocker family $\widehat{\Omega}$ of Ω consists of the vectors $\frac{1}{|\mathcal{P}|-1} \mathbb{1}_{\delta(\mathcal{P})}$ over all feasible partitions $\mathcal{P}$ whose shrunk hypergraph $H_{\mathcal{P}}$ is vertex-biconnected.*

A consequence of this duality is the identification of 1-modulus with strength: for every $\sigma \in \mathbb{R}_{>0}^E$,

$$\text{Mod}_{1,\sigma}(\Omega(H)) = S_{\sigma}(H), \quad (6.1.6)$$

and in particular $\text{Mod}_1(\Omega(H)) = S(H)$. The hypertree family behaves less uniformly. Since $\Gamma(H) \subseteq \Omega(H)$, one might expect the same identity for hypertrees, but in the weighted setting it can fail,

$$\text{Mod}_{1,\sigma}(\Gamma(H)) \neq S_{\sigma}(H) \quad \text{for some } \sigma,$$

as an example in Section 6.5 shows. For partition-connected hypergraphs we nonetheless obtain the unweighted equality $\text{Mod}_1(\Gamma(H)) = S(H)$, a statement that does not appear to follow from the partition characterization of Frank et al. and requires a separate argument.

The chapter is organized as follows.

- Section 6.2 fixes the definitions and notation specific to hypergraphs used in this chapter.
- Section 6.3 studies the hypertree modulus, proving Theorem 6.1.1 and developing the decomposition of partition-connected hypergraphs.
- Section 6.4 studies the multi-tree modulus, establishing Theorem 6.1.3 and Theorem 6.1.4 and the decomposition of general connected hypergraphs.
- Section 6.5 develops the Fulkerson duality between the partition inequality system and $\Omega(H)$, proving Theorem 6.1.8, deriving the strength identity (6.1.6), and comparing the 1-modulus of the hypertree and multi-tree families.

6.2 Some background

6.2.1 Hypergraphs and hypertrees

A hypergraph H is defined as a pair (V, E) , where V is the set of vertices and E is a collection of nonempty subsets of V , called hyperedges. The hyperedges in E are not necessarily distinct. A hyperedge consisting of a single vertex is called a loop, and two hyperedges containing the same vertices are called parallel hyperedges. In this paper, we consider hypergraphs with no loops.

Let $H = (V, E)$ be a hypergraph. For a subset $F \subseteq E$, we denote $V[F] \subset V$ as the set of vertices in $\bigcup_{e \in F} e$. The hypergraph $(V[F], F)$ is referred to as the *subhypergraph induced by the hyperedge subset F* and is denoted by $H[F]$. Next, let t be a positive integer. The hypergraph $H^t = (V, E^t)$ is defined to have the same vertex set V , where E^t is constructed by replacing each hyperedge $e \in E$ with a set of t parallel hyperedges $\{e^1, \dots, e^t\}$. The set of t parallel hyperedges $\{e^1, \dots, e^t\}$ in E^t is called the *parallel group* corresponding to the hyperedge e and is denoted by g_e . Let F be a subset of E^t . We define a multiset F' from E generated by F , where the multiplicity of each hyperedge $e \in E$ in F' is the cardinality of $F \cap g_e$.

A hypergraph H is said to be *forest-representable* or *wooded*, if it is possible to select two distinct vertices from each hyperedge of H such that the chosen pairs, when considered as graph edges on V , form a forest. If the representing forest can be made connected, then H is called *tree-representable*. Lovász showed in [40] that a hypergraph H is a hyperforest if and only if it is wooded.

For any nonempty subset $F \subseteq E(H)$, the density of F is defined to be

$$\theta_H(F) = \frac{|F|}{|V(H[F])| - 1}.$$

We often use $\theta(H)$ to denote $\theta_H(E(H))$. Let $S(H)$ and $D(H)$ be the strength and the fractional arboricity of H defined as in (6.1.3), respectively. It follows immediately that for

any loopless nontrivial hypergraph H , we have

$$S(H) \leq \theta(H) \leq D(H).$$

Moreover, it is also straightforward that

$$S(H) = \min\{\theta(H/F) : F \subset E(H), |V(H/F)| \geq 2\},$$

and

$$D(H) = \max\{\theta_H(F) : F \subset E(H), |V(H[F])| \geq 2\}.$$

6.3 Modulus of the hypertree family

In this section, we study the modulus of the hypertree family of a partition-connected hypergraph. Specifically, we analyze 2-modulus of this family, and the main tool we utilize is the theory of the base modulus in matroids.

Let $H = (V, E)$ be a partition-connected hypergraph. This implies that H has at least one hypertree. Let $\Gamma = \Gamma(H)$ be the hypertree family of H , where usage vectors are indicator functions. Let $\Omega = \Omega(H)$ be the multi-tree family of H . Note that $\Gamma(H) \subset \Omega(H)$.

Let $M = M(H)$ be the hypergraphic matroid associated with H . An interesting question is how to interpret known results in the theory of modulus for bases of matroids on hypergraph. Let r be the rank function of M . Then, the hypertree family $\Gamma = \Gamma(H)$ is the family of bases of M . The following theorem is one of our main results.

Theorem 6.3.1. *Let H be a partition-connected hypergraph, and let $\Gamma = \Gamma(H)$ be the hypertree family of H . Let η^* be the optimal density of $\text{Mod}_2(\widehat{\Gamma})$, and let $E_{\max}$ be the set of elements where $\eta^*(e)$ attains the maximum value of η^* . Then, $E_{\max} = \delta_H(\mathcal{P})$ for some feasible partition $\mathcal{P} = \{V_1, \dots, V_k\}$ of V such that each subhypergraph $H[V_i]$ is partition-connected, for $i = 1, \dots, k$.*

Proof. Denote η_{max}^* as the maximum value of η^* . Let $M = M(H)$ be the hypergraphic matroid associated with H . Recall the strength $s(M)$ of the matroid M defined as:

$$s(M) = \min \left\{ \frac{|X|}{r(E) - r(E \setminus X)} : X \subseteq E, r(E) > r(E \setminus X) \right\}.$$

Here, $s(M)$ is known as the strength of the matroid M , and it coincides with graph strength in the graphs setting.

By Theorem 3.3.1, we have that the set $E - E_{max}$ is closed in M , and

$$\frac{1}{\eta_{max}^*} = s(M), \quad (6.3.1)$$

where $s(M)$ is the strength of the unweighted matroid M . Moreover, Theorem 3.3.1 shows that E_{max} is optimal for the strength problem $s(M)$. Furthermore, E_{max} is the maximal set that is optimal for $s(M)$ and this can be deduced from the proof of Theorem 3.3.1 as follows. For any set $\emptyset \neq Y \not\subseteq E_{max}$, the set Y is not optimal for $s(M)$ because

$$\begin{aligned} \eta_{max}^* &> \frac{\eta^*(Y)}{|Y|} && \text{(by definition of } \eta_{max}^*) \\ &\geq \frac{r(E) - r(E - Y)}{|Y|} && \text{(by Lemma 3.2.12).} \end{aligned}$$

Now, we remove E_{max} from the hypergraph H to obtain the subhypergraph $H_1 := H[E - E_{max}]$. Assume that H_1 has k connected components $L_1, \dots, L_k$ with $L_i = (V_i, C_i)$, $k \geq 1$. Let $\mathcal{P}$ be the partition of V where each class of $\mathcal{P}$ is the vertex set V_i of L_i , for $i = 1, \dots, k$. Then $\bigcup_{i=1, \dots, k} C_i = E - E_{max}$. Note that the set $E - E_{max}$ is closed in M , and E_{max} is the maximal set that is optimal for the strength problem $s(M)$, and

$$1 \leq \frac{1}{\eta_{max}^*} = s(M) = \frac{|E_{max}|}{(|V| - 1) - r(E - E_{max})}. \quad (6.3.2)$$

Assume that L_1 has at least one hyperedge and is not partition-connected. Hence, $r(C_1) < |V_1| - 1$. Using Theorem 6.1.1 for the rank function of the matroid $M(H)$ restricted to C_1 , there exists a feasible partition $J = \{J_1, J_2, \dots, J_l\}$ of V_1 such that

$$r(C_1) = |V_1| - |J| + |\delta_{L_1}(J)|.$$

Thus,

$$|\delta_{L_1}(J)| = |J| - |V_1| + r(C_1). \quad (6.3.3)$$

Since $r(C_1) < |V_1| - 1$, it follows that $|J| \geq 2$, $\delta_{L_1}(J)$ is not empty, and $|\delta_{L_1}(J)| < |J| - 1$. For the matroid strength problem $s(M)$, we choose a candidate $X := E_{max} \cup \delta_{L_1}(J)$. For convenience, we also denote

$$a = |E_{max}|, \quad b = |V| - 1 - r(E \setminus E_{max}), \quad f = |\delta_{L_1}(J)| = |J| - |V_1| + r(C_1).$$

Because $\delta_{L_1}(J)$ is not empty, we obtain $E_{max} \subsetneq X$. Since the set E_{max} is the maximal set that is optimal for the strength problem $s(M)$, we have

$$1 \leq s(M) = \frac{a}{b} < \frac{|X|}{|V| - 1 - r(E \setminus X)} = \frac{a + f}{b + r(E \setminus E_{max}) - r(E \setminus X)}. \quad (6.3.4)$$

Now, we aim to prove

$$r(E \setminus E_{max}) - r(E \setminus X) \geq |J| - |V_1| + r(C_1) = f. \quad (6.3.5)$$

Indeed, we have

$$\begin{aligned}
r(E \setminus E_{max}) - r(E \setminus X) - f &= r(C_1) - r(C_1 \setminus \delta_{L_1}(J)) - f \\
&= r(C_1) - r(C_1 \setminus \delta_{L_1}(J)) - (|J| - |V_1| + r(C_1)) \\
&= |V_1| - r(C_1 \setminus \delta_{L_1}(J)) - |J| \\
&= (|J_1| - r(E(L_1[J_1]))) + \cdots + (|J_l| - r(E(L_l[J_l]))) - |J| \\
&\geq 1 + \cdots + 1 - |J| = |J| - |J| = 0.
\end{aligned}$$

Therefore, (6.3.4) implies that

$$\frac{a}{b} < \frac{a+f}{b+f}.$$

This is equivalent to $a < b$, which contradicts $a/b \geq 1$. Therefore, $\delta_{L_1}(J)$ must be empty, $|J| = 1$, and L_1 is partition-connected. Similarly, every other connected component L_i with at least one hyperedge of the subhypergraph $H[E - E_{max}]$ is partition-connected.

To conclude, we want to show that $E_{max} = \delta_H(\mathcal{P})$. If there exists a hyperedge $e \in E_{max}$ such that it lies in $H[V_i]$, then adding e to $E(L_i)$ does not change the rank of $E(L_i)$ since L_i is partition-connected. This contradicts the fact that $E - E_{max}$ is closed in M . Therefore, we must have $E_{max} \subset \delta_H(\mathcal{P})$. Let $x \in \delta_H(\mathcal{P})$. Then $x \in E_{max}$; because otherwise, x would connect two connected components of H_1 , which is a contradiction. Thus, we conclude that $E_{max} = \delta_H(\mathcal{P})$. $\square$

The next remark describes the difference between matroid strength and hypergraph strength.

Remark 6.3.2. If H is not partition-connected, then H has no hypertree and every base of the hypergraphic matroid $M(H)$ has rank less than $|V| - 1$. In general, we have $s(M(H)) \neq S(H)$. Here is an example, consider the hypergraph $H = (V, E)$ with $V = \{v_1, v_2, v_3\}$, $E = \{e\}$ and $e = \{v_1, v_2, v_3\}$. Then, $S(H) = 1/2$ while $s(M(H)) = 1$.

In the case where H is partition-connected, the following proposition shows that matroid

strength and hypergraph strength coincide.

Proposition 6.3.3. *Let H be a partition-connected hypergraph. Let $M = M(H)$ be the hypergraphic matroid associated with H . Then,*

$$s(M(H)) = S(H). \quad (6.3.6)$$

Proof of Proposition 6.3.3. Let $M = M(H)$ be the hypergraphic matroid associated with H . By Theorem 6.3.1, $E_{max} = \delta_H(\mathcal{P})$ for some feasible partition $\mathcal{P} = \{V_1, \dots, V_k\}$ of V such that each subhypergraph $H[V_i]$ is partition-connected, for $i = 1, \dots, k$. Note that E_{max} is the maximal set that is optimal for the strength problem $s(M)$, then

$$s(M(H)) = \frac{|E_{max}|}{r(E) - r(E - E_{max})} = \frac{|\delta_H(\mathcal{P})|}{|\mathcal{P}| - 1} \geq S(H).$$

For any $X \subset E$ such that $r(E) - r(X) > 0$, assume that hypergraph $H[X]$ has k connected components $H_i = (V_i, X_i)$, for $i = 1, \dots, k$. Then

$$|V(H/X)| = |V(H)| - \sum_{i=1}^k (|V_i| - 1) \leq |V(H)| - \sum_{i=1}^k r(X_i) = |V(H)| - r(X) = r(E) + 1 - r(X).$$

Hence,

$$s(M) \leq \frac{|E - X|}{r(E) - r(X)} \leq \frac{|E - X|}{|V(H/X)| - 1}.$$

Thus, $s(M) \leq S(H)$. In conclusion, we have $s(M) = S(H)$. $\square$

The next theorem describes the serial rule for the modulus of the hypertree family $\Gamma(H)$, this theorem is motivated by the serial rule for the modulus for bases of matroids.

Theorem 6.3.4. *Let H be a partition-connected hypergraph. Let $\Gamma = \Gamma(H)$ be the family of hypertrees of H . Assume that the optimal solution η^* of $\text{Mod}_2(\widehat{\Gamma})$ is not a constant vector and let E_{max} be the set of hyperedges e where $\eta^*(e)$ attains the maximum value of η^* . Let*

$H/(E - E_{max})$ be the hypergraph obtained from H by contracting $E - E_{max}$. Let the hypergraph $H[E - E_{max}]$ have k connected components $L_1, \dots, L_k$ with $L_i = (V_i, C_i)$, for $i = 1, \dots, k$. Then,

(i) The hypergraph $H/(E - E_{max})$ and every hypergraph L_i are partition-connected, for $i = 1, \dots, k$.

(ii) The modulus problem $\text{Mod}_2(\widehat{\Gamma}(H))$ splits as follows:

$$\text{Mod}_2(\widehat{\Gamma}(H)) = \text{Mod}_2(\widehat{\Gamma}(L_1)) + \text{Mod}_2(\widehat{\Gamma}(L_2)) + \dots + \text{Mod}_2(\widehat{\Gamma}(L_k)) + \text{Mod}_2(\widehat{\Gamma}(H/(E - E_{max}))); \quad (6.3.7)$$

(iii) The restriction of η^* onto C_i is optimal for $\text{Mod}_2(\widehat{\Gamma}(L_i))$ for $i = 1, 2, \dots, k$ and the restriction of η^* onto E_{max} is optimal for $\text{Mod}_2(\widehat{\Gamma}(H/(E - E_{max})))$.

Proof. By Remark 6.5.6 and Theorem 6.3.1, we obtain (i). Denote the base family of a matroid M by $\mathcal{B}(M)$. We recall the serial rule for the base modulus $\mathcal{B}(M) = \Gamma(H)$ of $M(H)$. Let η^* be the optimal density for $\text{Mod}_2(\widehat{\Gamma}(H))$. Let $M|(E - E_{max})$ be the restriction of $M(H)$ to $E - E_{max}$. Let $M/(E - E_{max})$ be the contraction of $E - E_{max}$ in M . By Theorem 3.2.19 and Theorem 3.3.1, we have

$$\text{Mod}_2(\widehat{\mathcal{B}}(M)) = \text{Mod}_2(\widehat{\mathcal{B}}(M|(E - E_{max})) + \text{Mod}_2(\widehat{\mathcal{B}}(M/(E - E_{max}))). \quad (6.3.8)$$

Moreover, the restriction of η^* onto $E - E_{max}$ is optimal for $\text{Mod}_2(\widehat{\mathcal{B}}(M|(E - E_{max})))$, and the restriction of η^* onto E_{max} is optimal for $\text{Mod}_2(\widehat{\mathcal{B}}(M/(E - E_{max})))$. Note that the matroid $M/(E - E_{max})$ is the hypergraphic matroid associated with $H/(E - E_{max})$ because $E_{max} = \delta_H(\mathcal{P})$ and (i), where $\mathcal{P}$ is the partition of V where each class of $\mathcal{P}$ is the vertex set V_i of L_i for $i = 1, \dots, k$. We also have that the restriction $M|(E - E_{max})$ is the hypergraphic matroid associated with $H[E - E_{max}]$. Therefore, we achieve (ii) and (iii). $\square$

Theorem 6.3.4 uncovers a hierarchical structure within hypergraphs, this is described as follows.

Hypergraph Decomposition Process (HDP): Let H be a partition-connected hypergraph that is not homogeneous. First, using Theorem 6.3.4, we decompose H into $H/(E - E_{max})$ and k partition-connected hypergraphs L_i as defined in Theorem 6.3.4. It follows that $H/(E - E_{max})$ is homogeneous. By Proposition 6.3.3 and Equation (6.3.1), we have

$$\frac{1}{S(H)} = \eta_{max}^* = \frac{1}{S(H/(E - E_{max}))}.$$

We proceed to further decompose each hypergraph L_i using the maximum value of η^* , continuing this decomposition process until we only have a collection of homogeneous hypergraphs. Let E_{min} be the set of elements where $\eta^*(e)$ attains the minimum value of η^* . At the end of this process, we obtain a collection of connected components of $H[E_{min}]$ that are vertex-induced, partition-connected, and homogeneous. We call each of these a *homogeneous core*. Furthermore, each homogeneous core K of $H[E_{min}]$ satisfies

$$\eta_{min}^* = \frac{1}{S(K)}.$$

This process yields a collection of subhypergraphs and shrunk hypergraphs which correspond to a hierarchical structure of H , see Example 6.1.6 for an example.

Next, similar to Proposition 6.3.3, we have a corresponding theorem for fractional arboricity. Recall the fractional arboricity of the hypergraphic matroid $M = M(H)$ which is defined as:

$$D(M) = \max \left\{ \frac{|X|}{r(X)} : X \subseteq E, r(X) > 0 \right\}.$$

Theorem 6.3.5. *Let H be a partition-connected hypergraph. Let $M = M(H)$ be the hypergraphic matroid associated with H . Then, $D(M(H)) = D(H)$.*

Proof. We have that the hypertree family $\Gamma = \Gamma(H)$ of H is the family of bases of the matroid $M(H)$. Let η^* be the optimal density of $\text{Mod}_2(\widehat{\Gamma})$. Let E_{min} be the set of elements

where $\eta^*(e)$ attains the minimum value η_{min}^* of η^* . By Theorem 3.3.2, we have

$$D(M(H)) = \frac{1}{\eta_{min}^*}.$$

By the Hypergraph Decomposition Process of H , we have that every connected component of $H[E_{min}]$ is vertex-induced, partition-connected, and homogeneous. Let K be a connected component among these components. Then,

$$D(M(H)) = \frac{|E_{min}|}{r(E_{min})} = \frac{1}{\eta_{min}^*} = \frac{|E(K)|}{r(M(K))} = \frac{|E(K)|}{|V(K)| - 1} \leq D(H).$$

On the other hand, for all $X \subseteq E$ such that $r(X) > 0$, we have

$$D(M(H)) \geq \frac{|X|}{r(X)} \geq \frac{|X|}{|V(H[X])| - 1}.$$

Thus, this yields $D(M(H)) \geq D(H)$. In conclusion,

$$\frac{1}{\eta_{min}^*} = D(M(H)) = D(H).$$

□

Next, we prove Theorem 6.1.1.

Proof for Theorem 6.1.1. Let $\Gamma = \Gamma(H)$ be the hypertree family of H . Let η^* be the optimal density of $\text{Mod}_2(\widehat{\Gamma})$. Let $M = M(H)$ be the hypergraphic matroid associated with H . By the theory of modulus for bases of matroids, η^* is a constant vector if and only the strength of M coincides with the fractional arboricity of M . By Proposition 6.3.3 and Theorem 6.3.5, the proof is completed. □

Remark 6.3.6. We also have a reverse version of the serial rule for the modulus of hypertrees in Theorem 6.3.4, where we can shrink any homogeneous core K and split the modulus

problem as follows:

$$\text{Mod}_2(\widehat{\Gamma}(H)) = \text{Mod}_2(\widehat{\Gamma}(K)) + \text{Mod}_2(\widehat{\Gamma}(H/K)). \quad (6.3.9)$$

Finally, based on Remark 6.3.6, we present a reverse version for the Hypergraph Decomposition Process.

Remark 6.3.7. Hypergraph Shrinking Process (HSP): The hierarchical structure established by the Hypergraph Decomposition Process can also be uncovered by reversing the process as follows. Note that every homogeneous core K of H in $H[E_{min}]$ is vertex-induced, partition-connected, homogeneous, and satisfies

$$\frac{1}{D(H)} = \eta_{min}^* = \frac{1}{D(K)} = \frac{1}{S(K)}.$$

In the first step, we shrink every homogeneous core K of H to get a shrunk hypergraph H_1 which is partition-connected. We proceed shrinking with the same strategy for H_1 using the minimum value of η^* , and continue this process until we only have a shrunk hypergraph, denoted by L , which is homogeneous. It follows that L satisfies

$$\frac{1}{S(H)} = \eta_{max}^* = \frac{1}{S(L)} = \frac{1}{D(L)}.$$

This process reveals the same collection of subhypergraph and shrunk hypergraphs as in the Hypergraph Decomposition Process.

6.4 Modulus for the multi-tree family

In this section, we answer the following questions: What happens when H is not partition-connected? Does it still have any hierarchical structure? Let $H = (V, E)$ be a connected hypergraph that is not necessarily partition-connected. The idea is to build a hypergraph $H^t = (V, E^t)$ from H , where E^t is obtained by replacing each hyperedge e in E with a set

of t parallel hyperedges. Let $\Gamma(H^t)$ be the hypertree family of the hypergraph H^t where usage vectors are indicator vectors. By Remark 6.5.9, there is a one to one correspondence between the families $\Omega(H)$ and $\Gamma(H^t)$. Then, we will use results from hypertree modulus in the hypergraphs H^t to study the multi-tree modulus of $\Omega(H)$. The main tool for this strategy is presented in the following section.

6.4.1 Symmetry property for modulus

In this section, we provide a symmetry property for modulus. Given a ground set E with weights $\sigma \in \mathbb{R}_{>0}^E$. A *symmetry* is an involution $Q : E \rightarrow E$ such that

$$\sigma(Q(e)) = \sigma(e) \quad \forall e \in E. \quad (6.4.1)$$

Given a symmetry Q and a family of objects Γ with usage matrix $\mathcal{N}$, we say that Γ is Q -invariant if for every object $\gamma_1 \in \Gamma$, there is an object $\gamma_2 \in \Gamma$ such that

$$\mathcal{N}(\gamma_2, Q(e)) = \mathcal{N}(\gamma_1, e), \quad \forall e \in E. \quad (6.4.2)$$

We set

$$\text{Adm}_Q(\Gamma) := \{\rho \in \text{Adm}(\Gamma) : \rho(e) = \rho(Q(e)), \forall e \in E\}$$

as the set of Q -invariant admissible densities.

Proposition 6.4.1. *Assume that Γ is a family of objects with weights $\sigma \in \mathbb{R}_{>0}^E$ that is also Q -invariant. Then, for $1 \leq p < \infty$,*

$$\text{Mod}_{p,\sigma}(\Gamma) = \inf_{\rho \in \text{Adm}_Q(\Gamma)} \mathcal{E}_{p,\sigma}(\rho).$$

Proof. Suppose ρ is extremal for $\text{Mod}_{p,\sigma}(\Gamma)$, so that in particular $\rho \in \text{Adm}(\Gamma)$. Define

$\rho_1 := \rho \circ Q$. Then, by (6.4.2), for every $\gamma_1 \in \Gamma$, there exists $\gamma_2 \in \Gamma$ such that

$$\ell_{\rho_1}(\gamma_1) = \sum_{e \in E} \mathcal{N}(\gamma_1, e) \rho_1(e) = \sum_{e \in E} \mathcal{N}(\gamma_2, Q(e)) \rho(Q(e)) = \ell_{\rho}(\gamma_2) \geq 1.$$

Therefore, $\rho_1 \in \text{Adm}(\Gamma)$. Moreover, by 6.4.1,

$$\mathcal{E}_{p,\sigma}(\rho_1) = \sum_{e \in E} \sigma(e) |\rho_1(e)|^p = \sum_{e \in E} \sigma(Q(e)) |\rho(Q(e))|^p = \mathcal{E}_{p,\sigma}(\rho).$$

Define $\rho_2 := \frac{\rho + \rho_1}{2}$. Then, $\rho_2 \in \text{Adm}(\Gamma)$. Also, since Q is an involution, we have

$$\rho_2 \circ Q = \frac{\rho \circ Q + \rho_1 \circ Q}{2} = \frac{\rho_1 + \rho}{2} = \rho_2.$$

So, $\rho_2 \in \text{Adm}_Q(\Gamma)$.

By convexity,

$$\mathcal{E}_{p,\sigma}(\rho_2) \leq \frac{\mathcal{E}_{p,\sigma}(\rho) + \mathcal{E}_{p,\sigma}(\rho_1)}{2} = \text{Mod}_{p,\sigma}(\Gamma).$$

Therefore,

$$\inf_{\rho \in \text{Adm}_Q(\Gamma)} \mathcal{E}_{p,\sigma}(\rho) \leq \text{Mod}_{p,\sigma}(\Gamma).$$

The other direction follows from the fact that $\text{Adm}_Q(\Gamma) \subset \text{Adm}(\Gamma)$.

□

Let e and f be two distinct elements in E . We say that e and f are *symmetric* if Γ is Q -invariant for some symmetry Q and $Q(e) = f$. Assume that $P = \{E_1, E_2, \dots, E_k\}$ is a partition of the ground set E such that all elements in each class of P are symmetric to each other. Then, we define a ground set E_P as the set of classes of P , and we enumerate E_P as $\{q_1, \dots, q_k\}$. Then, $|E_P| = |P|$. Let Γ_P be the family of all vectors $\gamma_P \in \mathbb{R}_{\geq 0}^{E_P}$ associated with $\gamma \in \Gamma$ and defined as

$$\gamma_P(q_i) = \sum_{e \in E_i} \mathcal{N}(\gamma, e)$$

for $i = 1, \dots, k$. We also define weights $\sigma_P \in \mathbb{R}_{\geq 0}^{E_P}$ with

$$\sigma_P(q_i) = \sigma(e)|E_i|$$

for some $e \in E_i$. Using the symmetry property in Proposition 6.4.1 for modulus, we obtain that

$$\text{Mod}_{p,\sigma}(\Gamma) = \text{Mod}_{p,\sigma_P}(\Gamma_P).$$

Moreover, let $\rho^* \in \mathbb{R}_{\geq 0}^E$ be an optimal density for $\text{Mod}_{p,\sigma}(\Gamma)$. Then, a density $\tilde{\rho}^*$, defined as $\tilde{\rho}^*(q_i) = \rho^*(e)$ for some $e \in E_i$, is optimal for $\text{Mod}_{p,\sigma_P}(\Gamma_P)$. Recall that the optimal density η^* for the Fulkerson dual problem of $\text{Mod}_{p,\sigma}(\Gamma)$ satisfies

$$\eta^*(e) = \frac{\sigma(e)\rho^*(e)^{p-1}}{\text{Mod}_{p,\sigma}(\Gamma)}.$$

It follows that the optimal density $\tilde{\eta}^*$ for the Fulkerson dual problem of $\text{Mod}_{p,\sigma_P}(\Gamma_P)$ satisfies

$$\tilde{\eta}^*(q_i) = \frac{\sigma_P(q_i)\tilde{\rho}^*(q_i)^{p-1}}{\text{Mod}_{p,\sigma_P}(\Gamma_P)} = |E_i|\eta^*(e) \text{ for some } e \in E_i.$$

In the following section, we will use these properties to study the modulus of the multi-tree family of hypergraphs.

6.4.2 Modulus for the multi-tree family

Let $H = (V, E)$ be a connected hypergraph that is not necessarily partition-connected. Let $\Omega(H)$ be the multi-tree family of H . Let $t = |V|$, we recall the hypergraph $H^t = (V, E^t)$ where E^t is obtained by replacing each hyperedge e in E by a set of t parallel hyperedges. By Remark 6.5.9, there is a one to one correspondence between two families $\Omega(H)$ and $\Gamma(H^t)$.

For the hypergraph H^t , consider a partition P of E where each class of P is the set of t parallel hyperedges in $E(H^t)$ corresponding to each hyperedge e in $E(H)$. The hyperedges in each class are symmetric with respect to the family $\Gamma(H^t)$. Hence, the set $E(H)$ can be

identified with the set of all classes of P and $\Omega(H)$ is the family of usage vectors in $\mathbb{R}_{\geq 0}^{E(H)}$ associated with the family $\Gamma(H^t)$ generated by the partition P .

First, let us analyze 1-modulus of the multi-tree family. By the symmetry property for modulus in Section 6.4.1, it follows that

$$t \operatorname{Mod}_1(\Omega) = \operatorname{Mod}_1(\Gamma(H^t)).$$

By the relationship between modulus of hypertrees and strength of hypergraphs in Proposition 6.3.3, we have

$$\operatorname{Mod}_1(\Gamma(H^t)) = S(H^t) = tS(H).$$

Therefore, we obtain that $\operatorname{Mod}_1(\Omega) = S(H)$, which also was shown in (6.1.6).

Next, we apply the symmetry property of modulus in Section 6.4.1 to deduce results of the 2-modulus of $\Omega(H)$ from the 2-modulus of $\Gamma(H^t)$.

Theorem 6.4.2. *Let H be a connected hypergraph. Let $\Omega = \Omega(H)$ be the multi-tree family of H . Let $\tilde{\eta}^*$ be the optimal density of $\operatorname{Mod}_2(\widehat{\Omega})$. Let $E_{\max}$ be the set of hyperedges e where $\tilde{\eta}^*(e)$ attains the maximum value $\tilde{\eta}_{\max}^*$ of $\tilde{\eta}^*$, and let $E_{\min}$ be the set of hyperedges e where $\tilde{\eta}^*(e)$ attains the minimum value $\tilde{\eta}_{\min}^*$ of $\tilde{\eta}^*$. Let $S(H)$ and $D(H)$ be the strength and fractional arboricity of H .*

Then,

$$S(H) = \frac{1}{\tilde{\eta}_{\max}^*}, \quad D(H) = \frac{1}{\tilde{\eta}_{\min}^*}.$$

Moreover, $E_{\max}$ and $E_{\min}$ are optimal for the strength problem and fractional arboricity problem, respectively.

Proof. Let $t = |V|$, we recall the hypergraph $H^t = (V, E^t)$ where E^t is obtained by replacing each hyperedge e in E by a set of t parallel hyperedges. For the hypergraph H^t , consider a partition P of $E(H^t)$ where each class of P is the set of t parallel hyperedges in $E(H^t)$ corresponding to each hyperedge e in $E(H)$. Note that every class of P has size t , by the

symmetry property of modulus, we obtain

$$t \operatorname{Mod}_2(\Omega(H)) = \operatorname{Mod}_2(\Gamma(H^t)).$$

Moreover, let η^* be the optimal density for $\operatorname{Mod}_2(\widehat{\Gamma}(H^t))$. It follows that the optimal density $\tilde{\eta}^*$ of $\operatorname{Mod}_2(\widehat{\Omega})$ satisfies

$$\tilde{\eta}^*(e) = t\eta^*(e_1),$$

for every $e \in E$ and for some copy e_1 in $E(H^t)$ of e .

Now, we apply the modulus for the hypertree family $\Gamma(H^t)$ as follows. Apply Theorem 6.3.1 to $\Gamma(H^t)$ and noting that E_{max} corresponds to hyperedges in H^t that attain the maximum value of η^* , we obtain

$$E_{max} = \delta_H(\mathcal{P}),$$

for some feasible partition $\mathcal{P} = \{V_1, \dots, V_k\}$ of $V(H)$ in the hypergraph H . Moreover,

$$tS(H) = S(H^t) = \frac{1}{\eta_{max}^*} = \frac{t}{\tilde{\eta}_{max}^*}.$$

Hence,

$$S(H) = 1/\tilde{\eta}_{max}^*,$$

and E_{max} is optimal for the strength problem $S(H)$. Similarly, by Theorem 6.3.5, we also have

$$D(H) = 1/\tilde{\eta}_{min}^*,$$

and E_{min} is optimal for the fractional arboricity problem $D(H)$. □

Proof for Theorem 6.1.4. As a corollary of Theorem 6.4.2, the optimal density $\tilde{\eta}^*$ of $\operatorname{Mod}_2(\widehat{\Omega})$ is a constant vector if and only if the strength of H coincides with the fractional arboricity of H . □

We can also deduce a serial rule for 2-modulus of $\Omega(H)$ from the serial rule for 2-modulus

of $\Gamma(H^t)$ in Theorem 6.3.4, this is deduced in the same manner as the proof of Theorem 6.4.2.

Theorem 6.4.3. *Let H be a connected hypergraph. Let $\Omega = \Omega(H)$ be the multi-tree family of H . Assume that the optimal $\tilde{\eta}^*$ of $\text{Mod}_2(\hat{\Omega})$ is not a constant vector, and let $E_{\max}$ be the set of hyperedges e where $\tilde{\eta}^*(e)$ attains the maximum value $\tilde{\eta}_{\max}^*$ of $\tilde{\eta}^*$. Let $H/(E - E_{\max})$ be the hypergraph obtained from H by contracting $(E - E_{\max})$. Let the hypergraph $H[E - E_{\max}]$ have k connected components $L_1, \dots, L_k$ with $L_i = (V_i, C_i)$, for $i = 1, \dots, k$. Then,*

(i) *The modulus problem $\text{Mod}_2(\hat{\Omega}(H))$ splits as follows:*

$$\text{Mod}_2(\hat{\Omega}(H)) = \text{Mod}_2(\hat{\Omega}(L_1)) + \text{Mod}_2(\hat{\Omega}(L_2)) + \dots + \text{Mod}_2(\hat{\Omega}(L_k)) + \text{Mod}_2(\hat{\Omega}(H/(E - E_{\max}))); \quad (6.4.3)$$

(ii) *The restriction of $\tilde{\eta}^*$ onto C_i is optimal for $\text{Mod}_2(\hat{\Omega}(L_i))$ for $i = 1, 2, \dots, k$, and the restriction of $\tilde{\eta}^*$ onto $E_{\max}$ is optimal for $\text{Mod}_2(\hat{\Omega}(H/(E - E_{\max})))$.*

Remark 6.4.4. Hypergraph decomposition process: Let H be a connected hypergraph that is not homogeneous. The hypergraph decomposition process for the partition-connected hypergraph H^t , see Remark 6.3.7, corresponds to a hypergraph decomposition process for the connected hypergraph H by scaling. This yields a hierarchical structure of H .

Proof for Theorem 6.1.3. Theorem 6.4.2 and the hypergraph decomposition process of H imply that if H is partition-connected, $\text{Mod}_2(\hat{\Omega})$ and $\text{Mod}_2(\hat{\Gamma})$ have the same optimal density and they reveal the same hierarchical structure of H . $\square$

Remark 6.4.5. If H is not partition-connected, the hierarchical structure of H is not necessarily related to the hierarchical structure of $M(H)$. Here is an example: $H = (V = \{v_1, v_2, v_3, v_4\}, E = \{e_1, e_2\})$ where $e_1 = \{v_1, v_2, v_3\}$, $e_2 = \{v_3, v_4\}$. This remark is similar to Remark 6.3.2.

6.5 Fulkerson duality for hypertrees

In this section, we prove Theorem 6.1.8. To start, we define the usage vectors of the family of feasible partitions.

Definition 6.5.1. Let $H = (V, E)$ be a connected hypergraph. Let Φ denote the family of all feasible partitions $\mathcal{P}$ of H , with the usage vectors defined as:

$$\tilde{\mathcal{N}}(\mathcal{P}, \cdot)^T := \frac{1}{|\mathcal{P}| - 1} \mathbb{1}_{\delta(\mathcal{P})}. \quad (6.5.1)$$

Then, the admissible set of Φ is given by:

$$\text{Adm}(\Phi) = \left\{ \eta \in \mathbb{R}_{\geq 0}^E : \tilde{\mathcal{N}}\eta \geq \mathbf{1} \right\}.$$

Let $H = (V, E)$ be a connected hypergraph. Let $\Omega = \Omega(H)$ be the multi-tree family defined in Definition 6.1.2. We want to study Fulkerson duality between two families Φ and Ω . We start with the following lemma.

Lemma 6.5.2. *The admissible set $\text{Adm}(\Phi)$ of Φ is equal to the partition polyhedron P defined as in (6.1.5)*

Proof. By definition, we have that $P \subset \text{Adm}(\Phi)$. Let $\mathcal{P} = \{V_1, \dots, V_k\}$ be a partition of V . For each $i = 1, \dots, k$, if $H[V_i]$ is not connected and has at least 2 connected components, we divide the class V_i into smaller classes corresponding to the vertex sets of those connected components. This new partition is feasible, the number of classes is increased and the cut set $\delta(\mathcal{P})$ remains unchanged. Therefore $\text{Adm}(\Phi) \subset P$. Thus, we conclude that $\text{Adm}(\Phi) = P$. $\square$

Since $\text{Adm}(\Phi) = P$ where P defined as in (6.1.5), as mentioned in the introduction, we have that

$$\widehat{\Phi} \subset \Omega.$$

For instance, consider a hypergraph $H = (V, E)$ with $V = \{v_1, v_2, v_3\}$, $E = \{e_1, e_2\}$, and

$e_1 = e_2 = \{v_1, v_2, v_3\}$. This hypergraph H has the partition polyhedron $P = \{(x_1, x_2) : x_1 + x_2 \geq 2, x \geq 0\}$. The extreme points of P are $(2, 0)$ and $(0, 2)$. We use this fact to achieve the following result.

Proposition 6.5.3. *We have that*

$$\text{Dom}(\Omega) = \text{Adm}(\Phi).$$

In other words, Ω and Φ form a Fulkerson dual pair.

Proof. In [39], it was shown that every extreme point of the partition polyhedron P defined as in (6.1.5) belongs to $\Omega(H)$. Hence, by Lemma 6.5.2, we have $\widehat{\Phi} \subset \Omega$. This implies that $\text{Adm}(\Phi) = \text{Dom}(\widehat{\Phi}) \subset \text{Dom}(\Omega)$.

Let x^A be an indicator vector in Ω , where A is a multiset from E . By the definition of Ω , the hypergraph $H[A] = (V(H[A]), E(H[A]))$ is a hypertree with vertex set V , and hence it is partition-connected. Thus, for any feasible partition $\mathcal{P}$ of V in $\Phi(H)$,

$$|\delta_{E(H[A])}(\mathcal{P})| \geq |\mathcal{P}| - 1.$$

This is equivalent to

$$x^A \cdot \mathbb{1}_{\delta_{E(H)}(\mathcal{P})} \geq |\mathcal{P}| - 1,$$

for any feasible partition $\mathcal{P}$ of V in $\Phi(H)$.

Therefore, $x^A \in \text{Adm}(\Phi)$. Thus, $\text{Dom}(\Omega) \subset \text{Adm}(\Phi)$. In conclusion, we have $\text{Dom}(\Omega) = \text{Adm}(\Phi)$. $\square$

By the definition of Fulkerson dual pairs, Proposition 6.5.3 implies that $\widehat{\Omega} \subset \Phi$. Hence, the remaining task is to describe $\widehat{\Omega}$ precisely.

Next, we provide the proof of Theorem 6.1.8. This proof follows a similar strategy from Fulkerson duality for spanning trees in [9].

Proof of Theorem 6.1.8. Let Θ represent the family of all feasible partitions $\mathcal{P}$ of H such that the shrunk hypergraphs $H_{\mathcal{P}}$ are vertex-biconnected.

The proof follows from the fact that $\widehat{\Omega} \subset \Phi$ and the two lemmas stated below. From Lemma 6.5.5, we have $\Phi \cap \widehat{\Omega} \subseteq \Theta$. Furthermore, Lemma 6.5.7 establishes that $\Theta \subset \widehat{\Omega}$. $\square$

Remark 6.5.4. The Fulkerson blocker family $\widehat{\Omega}$ gives a minimal inequality description $\text{Adm}(\widehat{\Omega})$ of the partition polyhedron P defined as in (6.1.5). This is a common theme in the study of various convex polytopes.

Lemma 6.5.5. *We have $\Phi \cap \widehat{\Omega} \subseteq \Theta$.*

Proof. Let $\mathcal{P} \in \Phi \cap \widehat{\Omega}$. Suppose $\mathcal{P} = \{V_1, V_2, \dots, V_k\}$, each $H[V_i]$ shrinks to a vertex denoted by v_i in $H_{\mathcal{P}}$. If $H_{\mathcal{P}}$ is vertex-biconnected, then $\mathcal{P} \in \Theta$. Assume that $H_{\mathcal{P}}$ is not vertex-biconnected. Then, there is a vertex v_j such that the deletion of v_j disconnects $H_{\mathcal{P}}$ into two subhypergraphs H_1 and H_2 of $H_{\mathcal{P}}$, each of them can have one or several connected components. Denote $A := \{i : v_i \in \text{the vertex set of } H_1\}$. Denote $B := \{i : v_i \in \text{the vertex set of } H_2\}$. Then, A, B and $\{j\}$ are disjoint and $A \cup B \cup \{j\} = \{1, 2, \dots, k\}$. Let

$$V_B := \left(\bigcup_{i \in B \cup \{j\}} V_i \right) \subset V(H),$$

we define a partition $\mathcal{P}_1$ of the vertex set V as follows:

$$\mathcal{P}_1 := \{V_i : i \in A\} \cup \{V_B\}.$$

Let

$$V_A := \left(\bigcup_{i \in A \cup \{j\}} V_i \right) \subset V(H),$$

we define a partition $\mathcal{P}_2$ of the vertex set V as follows:

$$\mathcal{P}_2 := \{V_i : i \in B\} \cup \{V_A\}.$$

Since v_j is connected with all the connected components of H_1 within $H_{\mathcal{P}} \setminus H_2$, then the subhypergraph of $H_{\mathcal{P}}$ induced by $A \cup \{j\}$ is connected. Therefore, the subhypergraph of H induced by V_A is connected. Hence, the partition $\mathcal{P}_2$ is feasible. Similarly, the partition $\mathcal{P}_1$ is feasible. Note that we have $|\mathcal{P}_1| = |A| + 1$ and $|\mathcal{P}_2| = |B| + 1$. Thus $|\mathcal{P}_1| + |\mathcal{P}_2| = |A| + 1 + |B| + 1 = |\mathcal{P}| + 1$. We also have that $\delta(\mathcal{P}_1) \cup \delta(\mathcal{P}_2) = \delta(\mathcal{P})$ and $\delta(\mathcal{P}_1) \cap \delta(\mathcal{P}_2) = \emptyset$. Therefore,

$$\frac{1}{|\mathcal{P}| - 1} \mathbb{1}_{\delta(\mathcal{P})} = \frac{|\mathcal{P}_1| - 1}{|\mathcal{P}| - 1} \cdot \frac{1}{|\mathcal{P}_1| - 1} \mathbb{1}_{\delta(\mathcal{P}_1)} + \frac{|\mathcal{P}_2| - 1}{|\mathcal{P}| - 1} \cdot \frac{1}{|\mathcal{P}_2| - 1} \mathbb{1}_{\delta(\mathcal{P}_2)}. \quad (6.5.2)$$

Notice that $\frac{|\mathcal{P}_1| - 1}{|\mathcal{P}| - 1} + \frac{|\mathcal{P}_2| - 1}{|\mathcal{P}| - 1} = 1$. So $\frac{1}{|\mathcal{P}| - 1} \mathbb{1}_{\delta(\mathcal{P})}$ is a convex combination of $\frac{1}{|\mathcal{P}_1| - 1} \mathbb{1}_{\delta(\mathcal{P}_1)}$ and $\frac{1}{|\mathcal{P}_2| - 1} \mathbb{1}_{\delta(\mathcal{P}_2)}$. Since $\frac{1}{|\mathcal{P}_1| - 1} \mathbb{1}_{\delta(\mathcal{P}_1)}, \frac{1}{|\mathcal{P}_2| - 1} \mathbb{1}_{\delta(\mathcal{P}_2)} \in \text{Adm}(\Omega)$ and $\frac{1}{|\mathcal{P}_1| - 1} \mathbb{1}_{\delta(\mathcal{P}_1)} \neq \frac{1}{|\mathcal{P}_2| - 1} \mathbb{1}_{\delta(\mathcal{P}_2)}$, we have $\frac{1}{|\mathcal{P}| - 1} \mathbb{1}_{\delta(\mathcal{P})} \notin \widehat{\Omega}$, this is a contradiction. Therefore, $H_{\mathcal{P}}$ is vertex-biconnected and $\mathcal{P} \in \Theta$. $\square$

Before providing the next lemma, we give a remark as follows.

Remark 6.5.6. Let $\mathcal{P}$ be a feasible partition of a hypergraph H . If H is partition-connected, then the shrunk hypergraph $H_{\mathcal{P}}$ is also partition-connected.

Lemma 6.5.7. *We have $\Theta \subset \widehat{\Omega}$.*

To prove this lemma, we first introduce additional notations and a useful theorem. Let u, v be two vertices in V . A *walk* W of length k connecting u and v in H is a sequence $v_0 e_1 v_1 e_2 v_2 \dots v_{k-1} e_k v_k$ of vertices and hyperedges (possibly repeated) such that $v_0, v_1, \dots, v_k \in V$, $e_1, \dots, e_k \in E$, $v_0 = u$, $v_k = v$, and for all $i = 1, 2, \dots, k$, the vertices v_{i-1} and v_i are adjacent in H via the edge e_i . Then, the hypergraph H is connected if and only if there exists a walk connecting u and v for every pair of distinct vertices u and v . For a walk $W = v_0 e_1 v_1 e_2 v_2 \dots v_{k-1} e_k v_k$ in H , if $k \geq 2$, $v_0 = v_k$, the vertices $v_0, v_1, \dots, v_{k-1}$ are pairwise distinct and the edges $e_1, \dots, e_k$ are also pairwise distinct, then W is called a *cycle*. A *cut hyperedge* in H is a hyperedge $e \in E$ such that its removal increases the number of connected components of H . A *cut vertex* in a hypergraph $H = (V, E)$ with $|V| \geq 2$ is a vertex $v \in V$ such that its deletion increases the number of connected components of H .

We have the following theorem.

Theorem 6.5.8. [41, Theorem 3.45] *Let $H = (V, E)$ be a connected hypergraph with $|V| \geq 2$, without hyperedges of cardinality less than 2, and without vertices of degree less than 2. Then the following are equivalent:*

1. *H has no cut vertices and no cut hyperedges.*
2. *Every pair of elements from $V \cup E$ lie on a common cycle.*
3. *Every pair of vertices lie on a common cycle.*
4. *Every pair of hyperedges lie on a common cycle.*

Let $\Gamma = \Gamma(H)$ be the family of all hypertrees where usage vectors are indicator vectors. Let $t = |V|$, and recall the hypergraph $H^t = (V, E^t)$, where E^t is constructed by replacing each hyperedge e in E with a set of t parallel hyperedges $\{e^1, \dots, e^t\}$. Let $\Gamma(H^t)$ be the family of all hypertrees of H^t with indicator vectors as usage vectors. The following remark gives a connection between $\Gamma(H^t)$ and $\Omega(H)$

Remark 6.5.9. Each hypertree $\gamma \in \Gamma(H^t)$ corresponds to a multiset γ' of E with the indicator vector $x^{\gamma'}$, where $x^{\gamma'} \in \Omega(H)$. Conversely, every vector in $\Omega(H)$ corresponds to a hypertree in $\Gamma(H^t)$. Hence, there is a one to one correspondence between the families $\Omega(H)$ and $\Gamma(H^t)$.

Using this theorem, we provide a useful lemma for the proof of Lemma 6.5.7.

Lemma 6.5.10. *Let H be a vertex-biconnected hypergraph with $|V| \geq 2$ and $|E| \geq 2$. Let e and f be two arbitrary distinct and possibly parallel hyperedges. Let $\Omega = \Omega(H)$ be the multi-tree family of H . Then there exist two vectors x_1, x_2 in Ω such that $x_1 - \mathbb{1}_e + \mathbb{1}_f = x_2$.*

Proof. We consider the hypergraph H^t with $t = |V|$. Since H is vertex-biconnected, it follows that H^t is also vertex-biconnected. By definition of H^t , H^t has no cut hyperedges and has no hyperedges with cardinality less than 2, and no vertices with degree less than 2.

Let e and f be two arbitrary distinct and possibly parallel hyperedges in H . The hyperedge e corresponds the group g_e of t parallel hyperedges $\{e^1, \dots, e^t\}$ in H^t and f corresponds

to the group g_f of t parallel hyperedges $\{f^1, \dots, f^t\}$ in H^t . Applying Theorem 6.5.8, we have that e^1 and f^1 lie on a common cycle $\pi = v_0 a_1 v_1 a_2 v_2 \dots a_k v_k$ where $v_k = v_0$ and $k \geq 2$. Let $A = \bigcup_{i=1, \dots, k} a_i$. Enumerate the set $A \setminus \{v_0, \dots, v_{k-1}\}$ as $\{p_1, \dots, p_m\}$ and we extend the cycle π as follows. To extend the cycle π , note that $p_1 \in A$, we pick an hyperedge a_i such that $p_1 \in a_i$, and extend the cycle

$$\pi = v_0 a_1 v_1 \dots v_{i-1} a_i v_i \dots a_k v_k$$

to

$$v_0 a_1 v_1 \dots v_{i-1} a_i p_1 a'_i v_i \dots a_k v_k,$$

where a'_i is some hyperedge in the group of t parallel hyperedges corresponding to a_i but not included in the previous cycle. Note that the number of hyperedges in the cycle is at most $|V| = t$, and the previous cycle includes e^1 and f^1 , which belong to different groups g_e and g_f . Thus, there always exists such an a'_i , as each group has t elements. We proceed to extend the cycle with $p_2, \dots, p_m$. In the end, this yields a cycle of size $|A|$. Thus, the set C of $|A|$ hyperedges in the cycle forms a circuit in the hypergraphic matroid $M(H^t)$, as we now explain. Indeed, note that the size of C is equal to $|\bigcup_{c \in C} c|$, implying that C is dependent. Furthermore, if any hyperedge c is removed from C , the set $C \setminus \{c\}$ is tree-representable, with its representing tree being a path connecting all vertices in $\bigcup_{c \in C} c$ derived from the cycle. Therefore $C \setminus \{c\}$ is independent. Hence, C is a minimal dependent set, forming a circuit of the matroid $M(H^t)$.

Given that C is a circuit of the matroid $M(H^t)$ containing e^1 and f^1 , it follows that $C - \{f^1\}$ is an independent set. There exists a base B_1 of $M(H^t)$ such that $(C - \{f^1\}) \subset B_1$. Now, define the set $B_2 := (B_1 - \{e^1\}) \cup \{f^1\}$ and it is also a base of $M(H^t)$. The corresponding indicator vectors $x^{B'_1}$ and $x^{B'_2}$ in Ω of two multisets B'_1 and B'_2 corresponding to B_1 and B_2 satisfy the following relation:

$$x^{B'_1} - \mathbb{1}_e + \mathbb{1}_f = x^{B'_2}.$$

This completes the proof. □

Proof for Lemma 6.5.7. Let $\mathcal{P} \in \Theta$, we want to show that

$$w := \frac{1}{|\mathcal{P}| - 1} \mathbb{1}_{\delta(\mathcal{P})} \in \text{Ext}(\text{Adm}(\Omega)) = \widehat{\Omega}.$$

Assume that there are two densities $\rho_1, \rho_2 \in \text{Adm}(\Omega)$ such that

$$w = \frac{1}{2}(\rho_1 + \rho_2).$$

For every hyperedge $e \in E \setminus \delta(\mathcal{P})$, we have

$$\frac{1}{2}(\rho_1(e) + \rho_2(e)) = w(e) = 0 \Rightarrow \rho_1(e) = \rho_2(e) = 0.$$

Since $\mathcal{P} \in \Theta$, we have that $H_{\mathcal{P}}$ is vertex-biconnected. First, assume that $|\delta(\mathcal{P})| \geq 2$. Let e and f be two arbitrary distinct hyperedges in $\delta(\mathcal{P})$. By Lemma 6.5.10, there exists two vectors x_1, x_2 in $\Omega(H_{\mathcal{P}})$ such that $x_1 - \mathbb{1}_e + \mathbb{1}_f = x_2$.

For each $i = 1, \dots, |\mathcal{P}|$, the hypergraph $H[V_i]$ is connected, there exists a vector y_i in $\Omega(H[V_i])$. We concatenate all vectors y_i with vector x_1 to form $z_1 \in \mathbb{R}_{\geq 0}^E$ and we have that $z_1 \in \Omega(H)$. Similarly, concatenating all vectors y_i with x_2 yields $z_2 \in \mathbb{R}_{\geq 0}^E$ and $z_2 \in \Omega(H)$.

Because $\rho_1, \rho_2 \in \text{Adm}(\Omega(H))$, we have

$$\rho_1 \cdot z_1, \rho_1 \cdot z_2, \rho_2 \cdot z_1, \rho_2 \cdot z_2 \geq 1.$$

Since $\rho_1 + \rho_2 = 2w$, we have $(\rho_1 + \rho_2) \cdot z_1 = 2w \cdot z_1 = 2w \cdot x_1 = 2$ and $(\rho_1 + \rho_2) \cdot z_2 = 2w \cdot z_2 = 2w \cdot x_2 = 2$. This implies

$$\rho_1 \cdot z_1 = \rho_2 \cdot z_1 = \rho_1 \cdot z_2 = \rho_2 \cdot z_2 = 1.$$

Because $\rho_1(e) = \rho_2(e) = 0$ for all $e \in E \setminus \delta(\mathcal{P})$, we obtain

$$\rho_1 \cdot x_1 = \rho_2 \cdot x_1 = \rho_1 \cdot x_2 = \rho_2 \cdot x_2 = 1.$$

Thus,

$$0 = \rho_1 \cdot (x_1 - x_2) = \rho_1 \cdot (\mathbb{1}_e - \mathbb{1}_f)$$

Hence, since e and f were chosen arbitrarily, ρ_1 must be constant in $\delta(\mathcal{P})$. Similarly, ρ_2 is also constant in $\delta(\mathcal{P})$. Because $\rho_1 \cdot x_1 = \rho_2 \cdot x_1 = 1$ and $x_1(E) = |\mathcal{P}| - 1$, we obtain that

$$\rho_1(e) = \rho_2(e) = \frac{1}{|\mathcal{P}| - 1} = w(e), \quad \forall e \in \delta(\mathcal{P}).$$

Therefore, $\rho_1 = \rho_2 = w$ and w is an extreme point of $\text{Adm}(\Omega(H))$.

If $|\delta(\mathcal{P})| = 1$. By the same argument above, w is an extreme point of $\text{Adm}(\Omega(H))$. In conclusion, we have shown that $w \in \widehat{\Omega}$. $\square$

If H is partition-connected, equivalently, the family $\Gamma(H)$ of all hypertrees is not empty. Then, the Fulkerson blocker family $\widehat{\Gamma}(H)$ can be found by considering it as the base family of the associated hypergraphic matroids. For instance, consider a hypergraph $H = (V, E)$ with $V = \{v_1, v_2, v_3\}$, $E = \{e_1, e_2\}$, and $e_1 = e_2 = \{v_1, v_2, v_3\}$. This hypergraph H has only one hypertree $\{e_1, e_2\}$ with the usage vector $(1, 1)$, and $\widehat{\Gamma}(H)$ includes $(1, 0)$ and $(0, 1)$, which are not related to feasible partitions.

Finally, we provide a corollary of the Fulkerson duality between multi-trees and partitions. Let $\sigma \in \mathbb{R}_{>0}^E$ be a set of weights assigned to the hyperedges in E . The next remarks provide the 1-modulus of both the hypertree family and the multi-tree family.

Remark 6.5.11. By Proposition 6.5.3, two families Φ and Ω form a Fulkerson dual pair. Since the 1-modulus is a linear program, at least one optimal density of $\text{Mod}_{1,\sigma}(\Omega(H))$ must occur

at an extreme point of $\text{Adm}(\Omega(H))$. Hence, we have

$$\text{Mod}_{1,\sigma}(\Omega(H)) = S_\sigma(H),$$

for any set of weights $\sigma \in \mathbb{R}_{>0}^E$.

Remark 6.5.12. Let H be a partition-connected hypergraph. The hypertree family $\Gamma = \Gamma(H)$ is the family of bases of the matroid $M(H)$. Recall the weighted strength $s_\sigma(M)$ of the matroid M with weights $\sigma \in \mathbb{R}_{>0}^E$ defined as:

$$s_\sigma(M) = \min \left\{ \frac{\sigma(X)}{r(E) - r(E \setminus X)} : X \subseteq E, r(E) > r(E \setminus X) \right\}.$$

Here, $s_\sigma(M)$ is known as the strength of the matroid M , and it coincides with graph strength in the graphs setting. Lemma 3.5.5 shows that

$$s_\sigma(M) = \text{Mod}_{1,\sigma}(\Gamma(H)).$$

Remark 6.5.13. Let H be a partition-connected hypergraph. From Remark 6.5.12, we have $s(M) = \text{Mod}_1(\Gamma(H))$. Combining with the Proposition 6.3.3, we obtain that

$$\text{Mod}_1(\Gamma(H)) = S(H). \tag{6.5.3}$$

Chapter 7

Conclusion and Future Work

7.1 Modulus as a Unifying Framework

This dissertation develops the theory of discrete modulus for several families of combinatorial objects arising from graphs, matroids, and hypergraphs. The central theme is that modulus provides a flexible optimization framework for studying families of discrete objects through their optimal densities. These optimal densities often encode structural information about the underlying object, including strength, fractional arboricity, homogeneity, principal partitions, duality, reinforcement, sparsification, truncation, and hypergraph decompositions. This point of view unifies several themes from graph theory, combinatorics, convex optimization, and combinatorial duality.

7.2 Future Directions

There are several natural directions for future research.

One direction is to further develop the weighted version of matroid truncation, hypertree, and multi-tree modulus.

A second direction is the algorithmic study of modulus for matroids and hypergraphs.

While several algorithms are developed in special settings, especially for graphic matroids, it would be useful to understand which parts of the theory admit efficient computation for broader classes of matroids and hypergraphs. In particular, algorithmic versions of the hypergraph decomposition processes would be valuable.

A third direction is to study other matroid operations through discrete modulus. This dissertation focuses on truncation and dual truncation, but many other operations may interact meaningfully with universal density. Examples include restriction, contraction, deletion, direct sum, matroid union, and matroid duality.

A fourth direction is to investigate probabilistic interpretations of the universal density. In the base modulus setting, probability measures on the base family whose marginals equal the universal density play an important role. A deeper understanding of these measures may connect modulus to random bases, entropy, negative dependence, and related probabilistic questions in matroid theory.

A fifth direction is to extend the game-theoretic interpretation of spanning tree modulus to matroids and hypergraphs. Such interpretations may provide new applications of modulus to network security, resource allocation, and robustness problems.

Finally, a long-term goal is to determine which structural features of a combinatorial family can be detected by its modulus, and to extend this study to broader classes of combinatorial families.

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Appendix A

Appendix: Serial rule for the MKL problem

A.1 The MEO problem

In this section, we want to recall the serial rule for the MEO problem. Let us recall some notations in a general setting. Let E be a finite ground set, and let Γ be a collection of subsets of E . Let $\mathcal{N}$ be the matrix whose rows are the indicator vectors of subsets in Γ . Given a set $A \subset E$, let ψ_A be the restriction operator defined as

$$\begin{aligned}\psi_A : 2^E &\rightarrow 2^A \\ \gamma \subset E &\mapsto \gamma \cap A.\end{aligned}$$

Then, for each set $A \subset E$, ψ_A induces a family of objects $\psi_A(\Gamma) = \{\gamma \cap A : \gamma \in \Gamma\}$. Let $\{E_1, E_2\}$ be a partition of the edge set E . For each $i = 1, 2$, we define an induced family of objects $\Gamma_i := \psi_{E_i}(\Gamma)$. Next, we say that a partition $\{E_1, E_2\}$ of the edge set E *divides* Γ , if

Γ coincides with the *concatenation*

$$\Gamma_1 \oplus \Gamma_2 := \{\gamma_1 \cup \gamma_2 : \gamma_i \in \Gamma_i, i = 1, 2\}. \quad (\text{A.1.1})$$

Given a partition $\{E_1, E_2\}$ that divides Γ and a pmf $\mu \in \mathcal{P}(\Gamma)$. For each $i = 1, 2$, we define the *marginal* $\mu_i \in \mathcal{P}(\Gamma_i)$ as follows,

$$\mu_i(\gamma_i) := \sum \{\mu(\gamma) : \gamma \in \Gamma, \psi_i(\gamma) = \gamma_i\} \quad \forall \gamma_i \in \Gamma_i.$$

On the other hand, given measures $\nu_i \in \mathcal{P}(\Gamma_i)$ for $i = 1, 2$, we define their *product measure* in $\mathcal{P}(\Gamma)$ as follows,

$$(\nu_1 \oplus \nu_2)(\gamma) := \nu_1(\gamma_1)\nu_2(\gamma_2), \quad (\text{A.1.2})$$

for all $\gamma = \gamma_1 \cup \gamma_2$ where $\gamma_i \in \Gamma_i, i = 1, 2$.

In this case, the modulus and MEO problems split into two smaller subproblems.

Theorem A.1.1 ([5]). *Let Γ be a family of subsets of the ground set E . Let $E = E_1 \cup E_2$ be a partition that divides Γ . Let Γ_1 and Γ_2 be the family induced by the restriction operators ψ_{E_1} and ψ_{E_2} . Then:*

(i) *We have*

$$\text{MEO}(\Gamma) = \text{MEO}(\Gamma_1) + \text{MEO}(\Gamma_2);$$

(ii) *A pmf $\mu \in \mathcal{P}(\Gamma)$ is optimal for $\text{MEO}(\Gamma)$ if and only if its marginal pmfs $\mu_i \in \mathcal{P}(\Gamma_i), i = 1, 2$ are optimal for $\text{MEO}(\Gamma_i)$ respectively;*

(iii) *Conversely, given pmfs $\nu_i \in \mathcal{P}(\Gamma_i)$ that are optimal for $\text{MEO}(\Gamma_i)$ for $i = 1, 2$ then $\nu_1 \oplus \nu_2$ is an optimal pmf in $\mathcal{P}(\Gamma)$ for $\text{MEO}(\Gamma)$;*

(iv) *For any pmf μ with marginals μ_i , if $e \in E_i, i = 1, 2$, then*

$$\mathbb{P}_\mu(e \in \underline{\gamma}) = \mathbb{P}_{\mu_i}(e \in \underline{\gamma}_i).$$

A.2 Serial rule for the MKL problem

In this section, we prove the serial rule for the MKL problem. Here, for $i = 1, 2$, the notation $\text{MKL}_\sigma(\Gamma_i)$ means the MKL problem on the family Γ_i of subsets of E_i with the weight vector σ restricted to E_i , that is, with objective $-\sum_{e \in E_i} \sigma(e) \log \eta_i(e)$. With these notations, the MKL_σ problem defined on the family Γ with weights $\sigma \in \mathbb{R}_{>0}^E$ splits into two smaller subproblems as follows.

Theorem A.2.1. *Let Γ be a family of subsets of the ground set E with weights $\sigma \in \mathbb{R}_{>0}^E$. Let $E = E_1 \cup E_2$ be a partition that divides Γ . Let Γ_1 and Γ_2 be the family induced by the restriction operators ψ_{E_1} and ψ_{E_2} . Then:*

(i) *We have*

$$\text{MKL}_\sigma(\Gamma) = \text{MKL}_\sigma(\Gamma_1) + \text{MKL}_\sigma(\Gamma_2);$$

(ii) *A pmf $\mu \in \mathcal{P}(\Gamma)$ is optimal for $\text{MKL}_\sigma(\Gamma)$ if and only if its marginal pmfs $\mu_i \in \mathcal{P}(\Gamma_i)$, $i = 1, 2$, are optimal for $\text{MKL}_\sigma(\Gamma_i)$ respectively;*

(iii) *Conversely, given pmfs $\nu_i \in \mathcal{P}(\Gamma_i)$ that are optimal for $\text{MKL}_\sigma(\Gamma_i)$ for $i = 1, 2$, then $\nu_1 \oplus \nu_2$ is an optimal pmf in $\mathcal{P}(\Gamma)$ for $\text{MKL}_\sigma(\Gamma)$;*

(iv) *For any pmf μ with marginals μ_i , if $e \in E_i$, $i = 1, 2$, then*

$$\sum_{\gamma \in \Gamma: e \in \gamma} \mu(\gamma) = \sum_{\gamma_i \in \Gamma_i: e \in \gamma_i} \mu_i(\gamma_i).$$

Proof. Let μ be a pmf in $\mathcal{P}(\Gamma)$ and let $\eta(e) = \sum_{\gamma \in \Gamma: e \in \gamma} \mu(\gamma)$. We denote $\underline{\gamma}$ a random subset in Γ distributed by μ . Since the partition $\{E_1, E_2\}$ divides Γ , we see that for any $\gamma, \gamma' \in \Gamma$, there exist $\gamma_i, \gamma'_i \in \Gamma_i$ for $i = 1, 2$, such that

$$\gamma = \gamma_1 \cup \gamma_2 \quad \text{and} \quad \gamma' = \gamma'_1 \cup \gamma'_2.$$

For $e \in E_1$,

$$\mathbb{P}_\mu(e \in \underline{\gamma}) = \sum_{\gamma \in \Gamma} \mathcal{N}(\gamma, e) \mu(\gamma) = \sum_{\gamma_1, \gamma_2} \mathcal{N}(\gamma_1, e) \mu(\gamma_1 \cup \gamma_2) = \sum_{\gamma_1} \mathcal{N}(\gamma_1, e) \mu_1(\gamma_1) = \mathbb{P}_{\mu_1}(e \in \underline{\gamma_1}). \quad (\text{A.2.1})$$

The case $e \in E_2$ yields a similar expression with μ_1 replaced by the second marginal μ_2 . This implies that, if $\mu \in P(\Gamma)$, and μ_i , $i = 1, 2$, are its marginals. Then

$$-\sum_{e \in E} \sigma(e) \log \mathbb{P}_\mu(e \in \underline{\gamma}) = -\sum_{e \in E_1} \sigma(e) \log \mathbb{P}_{\mu_1}(e \in \underline{\gamma_1}) - \sum_{e \in E_2} \sigma(e) \log \mathbb{P}_{\mu_2}(e \in \underline{\gamma_2}).$$

If we choose μ to be an optimal pmf for $\text{MKL}_\sigma(\Gamma)$, we get that

$$\text{MKL}_\sigma(\Gamma_1) + \text{MKL}_\sigma(\Gamma_2) \leq -\sum_{e \in E_1} \sigma(e) \log \mathbb{P}_{\mu_1}(e \in \underline{\gamma_1}) + -\sum_{e \in E_2} \sigma(e) \log \mathbb{P}_{\mu_2}(e \in \underline{\gamma_2}) \quad (\text{A.2.2})$$

$$= -\sum_{e \in E} \sigma(e) \log \mathbb{P}_\mu(e \in \underline{\gamma}) = \text{MKL}_\sigma(\Gamma). \quad (\text{A.2.3})$$

Conversely, suppose $\nu_i \in P(\Gamma_i)$, $i = 1, 2$, are optimal for their respective MKL problems. Let $\nu := \nu_1 \oplus \nu_2$ be defined as in (A.1.2).

Since the partition $\{E_1, E_2\}$ divides Γ , this implies that the support set of ν is contained in Γ . Moreover,

$$\sum_{\gamma_1 \in \Gamma_1} \sum_{\gamma_2 \in \Gamma_2} \nu(\gamma_1 \cup \gamma_2) = \sum_{\gamma_1 \in \Gamma_1} \sum_{\gamma_2 \in \Gamma_2} \nu_1(\gamma_1) \nu_2(\gamma_2) = \left(\sum_{\gamma_1 \in \Gamma_1} \nu_1(\gamma_1) \right) \left(\sum_{\gamma_2 \in \Gamma_2} \nu_2(\gamma_2) \right) = 1.$$

Thus, ν is a pmf in $P(\Gamma)$. Furthermore,

$$\sum_{\gamma_2 \in \Gamma_2} \nu(\gamma_1 \cup \gamma_2) = \nu_1(\gamma_1) \sum_{\gamma_2 \in \Gamma_2} \nu_2(\gamma_2) = \nu_1(\gamma_1),$$

which shows that ν_1 is a marginal of ν . A similar argument applies to ν_2 . Then, we have

$$\text{MKL}_\sigma(\Gamma) \leq - \sum_{e \in E} \sigma(e) \log \mathbb{P}_\nu(e \in \underline{\gamma}) \quad (\text{A.2.4})$$

$$= - \sum_{e \in E_1} \sigma(e) \log \mathbb{P}_{\nu_1}(e \in \underline{\gamma}_1) + - \sum_{e \in E_2} \sigma(e) \log \mathbb{P}_{\nu_2}(e \in \underline{\gamma}_2) \quad (\text{A.2.5})$$

$$= \text{MKL}_\sigma(\Gamma_1) + \text{MKL}_\sigma(\Gamma_2). \quad (\text{A.2.6})$$

This proves parts 1 and 3.

For part 2, assume $\mu \in P(\Gamma)$ is optimal. By part 1, we obtain

$$0 = \left(- \sum_{e \in E_1} \sigma(e) \log \mathbb{P}_{\mu_1}(e \in \underline{\gamma}_1) - \text{MKL}_\sigma(\Gamma_1) \right) + \left(- \sum_{e \in E_2} \sigma(e) \log \mathbb{P}_{\mu_2}(e \in \underline{\gamma}_2) - \text{MKL}_\sigma(\Gamma_2) \right).$$

Since both terms in parentheses are non-negative, they must each equal zero. This implies that the marginals of μ are also optimal for their respective MKL problems.

Conversely, suppose that μ is a pmf on Γ such that both of its marginals μ_i are optimal for their respective MKL problems. Then, it follows that μ is optimal for $\text{MKL}_\sigma(\Gamma)$. This completes the proof of part 2.

Part 4 is obtained from equation (A.2.1). □

Next, even when a partition does not divide the family, the serial rule still provides a bound.

Corollary A.2.2. *Let $\Gamma \subset 2^E$ be a family of objects on a ground set E , let $E = E_1 \cup E_2$ be a partition of E . Let Γ_1 and Γ_2 be the family induced by the restriction operators ψ_{E_1} and ψ_{E_2} . Then*

$$\text{MKL}_\sigma(\Gamma) \geq \text{MKL}_\sigma(\Gamma_1) + \text{MKL}_\sigma(\Gamma_2),$$

with equality if the partition divides Γ .

Proof. This follows from Theorem A.2.1 and the following monotonicity property of the

MKL problem. Given an arbitrary partition of E , it always holds that $\Gamma \subset \Gamma_1 \oplus \Gamma_2$. This implies that $\text{conv}(\Gamma) \subset \text{conv}(\Gamma_1 \oplus \Gamma_2)$, and therefore,

$$\text{MKL}_\sigma(\Gamma) \geq \text{MKL}_\sigma(\Gamma_1 \oplus \Gamma_2).$$

□

Remark A.2.3. The serial rule in Theorem A.2.1 can be generalized to partitions with a larger number of parts.

Remark A.2.4. Theorem A.2.1 can be applied to the direct sum of two matroids where the direct sum of two matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ with disjoint ground sets is the matroid

$$M_1 \oplus M_2 := (E_1 \cup E_2, \{I_1 \cup I_2 : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}). \quad (\text{A.2.7})$$

Hence, the theorem can also be applied to a matroid with at least two connectivity components since it is equal to the direct sum of its connectivity components.