PREDICTION OF THE DOMINANT CHARACTERISTIC ROOTS OF A MATRIX

by

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INTRODUCTION

Determination of the characteristic roots of a matrix is often of fundamental importance in the theory of matrices. The power method is commonly used when the matrix has a single dominant root. This method can also be generalized to determine any number of equally dominant roots. The main deterrent to its use is that the multiplicity of the dominant root must be known in advance. Since this information is rarely available, the development of some method to determine the number and form of the dominant characteristic roots in advance would permit selection of the appropriate numerical procedure.

The power method was applied to matrices with predetermined dominant roots and the behavior of the results observed through several iterations. It was the objective of this research to identify those factors which give definite indication of the size, multiplicity and form of the dominant roots. Then the dominant roots of a general matrix might be predicted by observing the effect of these factors when the power method is applied to the general matrix.

THE POWER METHOD

This discussion is drawn from class notes used in Leonard Fuller's course, "Theory of Matrices", at Kansas State University.

If $r_1, r_2, \ldots, r_j$ are the distinct characteristic roots of the square matrix A, let them have the ordering:

$$|r_1| \geq |r_2| \geq \ldots \geq |r_j| \quad .$$

The minimum polynomial for A (the monic polynomial of least degree satisfied by A) may be written:

$$m(x) = (x-r_1)^{t_1}(x-r_2)^{t_2} \ldots (x-r_j)^{t_j} \quad .$$

If $|r_1| > |r_2|$, then $(x-r_1)^{t_1}$ is defined to be the dominant elementary divisor of the matrix A. If $|r_1| = |r_2| = \ldots = |r_k| > |r_{k+1}|$, then $(x - r_1)^{t_1}$, $(x-r_2)^{t_2}$, ..., $(x-r_k)^{t_k}$ are the k dominant elementary divisors.

Theorem I. If the matrix A has the k dominant elementary divisors $(x-r_1)^{t_1}$, $(x-r_2)^{t_2}$, ..., $(x-r_k)^{t_k}$, then for a specified accuracy, there exists an N such that, for all $m > N$,

$$(1) \quad A^m(A-r_1I)^{t_1} \ldots (A-r_kI)^{t_k} \approx Z \ .$$

The matrix Z is the zero matrix, with all components equal to zero. The power method is developed from a simplification of this basic theorem.

Theorem II. If the matrix A has a single dominant root and the dominant elementary divisor is linear (i.e. $|r_1| > |r_2|$, $k = 1$, $t_1 = 1$), then for a specified accuracy there exists an N such that, for all $m > N$,

$$(2) \quad A^m(A - r_1 I) \approx Z \qquad \text{or} \qquad A^{m+1} \approx r_1 A^m \ .$$

Corollary. For any matrix satisfying the conditions of the theorem and for any vector Y not in the null space of any power of A, A^mY is a nonzero characteristic vector of A corresponding to r_1.

Define: $Y_0 = Y$

$s_i Y_i = AY_{i-1}$, $i = 1,2,\ldots$.

The product AY_{i-1} is "normalized" by dividing each of the components of AY_{i-1} by a given component. This component is called the normalizing factor. The product AY_{i-1} is now expressed as the product of the normalizing factor s_i and a new vector Y_i.

It can be shown that $s_1 s_2 \ldots s_m Y_m = A^m Y$ and

$$\left(\prod_{k=1}^{m+1} s_k\right)Y_{m+1} = A^{m+1}Y = AA^mY \approx r_1 A^m Y = r_1\left(\prod_{k=1}^{m} s_k\right)Y_m \ .$$

This reduces to:

$$(3) \quad s_{m+1}Y_{m+1} \approx r_1 Y_m$$

and is called the vector equation. If m is large enough so that Y_{m+1} is approximately equal to Y_m, then for any component of Y_m,

(4) $s_{m+1} \approx r_1$.

To apply the power method, an arbitrary nonzero vector Y is chosen. The vector is multiplied by A and the product normalized to determine a new vector Y_1. This vector Y_1 is in turn multiplied by A and the result normalized to get Y_2. This procedure is repeated until the same normalizing factor and the same vector (to a specified degree of accuracy) are obtained for at least two consecutive iterations. The normalizing factor at this point will equal r_1 and the vector is a characteristic vector corresponding to this root.

Normalization can be done in several acceptable ways. For this specific investigation, the component of largest absolute value in AY_o was determined. Normalization was then fixed on this component throughout the iterative procedure. This method of normalization must be used in order to derive a system of equations from the vector equation for the solution with multiple dominant roots.

PREPARATION OF DATA

Discussion

The objective of this research was to determine a way to predict the dominant characteristic roots of a matrix when the power method was applied to the matrix. In an attempt to restrict the variable factors in the problem, the data matrices were specialized in several ways.

The results are limited to companion matrices, which were constructed in the following manner. The roots for the matrix were selected and the characteristic polynomial formed by taking the combined product of corresponding linear factors:

$$f(x) = (x-r_1)(x-r_2)\ldots(x-r_n) = x^n - a_{n-1}x^{n-1} - \ldots -a_1x - a_0 \quad .$$

The companion matrix for this polynomial then has the form:

$$\begin{bmatrix} 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & & 0 & 1 \\ a_0 & a_1 & a_2 & & a_{n-2} & a_{n-1} \end{bmatrix}$$

The companion matrices tested, listed by included roots and letter code used for reference, are displayed in the following section.

To facilitate comparison of behavior patterns in the normalizing factors, dominant roots had the same absolute value in all matrices. The values ± 10, $\pm 8 \pm 6i$, $\pm 6 \pm 8i$, and $\pm 10i$ were used as dominant roots; the pair of roots ± 2 was also used in each matrix.

The letter code of a matrix has no inherent significance. In the explanation of results that follows, a matrix will be labeled by its dominant roots enclosed in parentheses. Unless otherwise indicated, it may be assumed that the matrix also has the roots ± 2.

The power method was applied to each of the 54 matrices and for each matrix results were obtained through 99 iterations. The initial Y vector used in the normalizing process had all components equal to one. In addition, some of these matrices were tested with different initial vectors. The degree of accuracy used to determine convergence was $\pm .00005$.

In the ensuing discussion, normalizing factors will frequently be referred to as "S". "Even S" or "odd S" are normalizing factors generated at even- or odd-numbered iterations, respectively.

To facilitate discussion of the results from matrices with complex roots, some convention must be understood in referring to the "sign" of a complex number. Because of the similarity in the patterns produced by the normalizing factors, it has been decided to give a complex number the sign of its real part. Thus, in the ensuing discussion, the pair of complex conjugates, $8\pm 6i$, can be thought of as a pair of "positive" numbers.

Reference will be made to the "balance" or "division" in the signs of the roots. If there is an equal number of positive and negative roots, the division in signs is equal. If there are more roots of one sign than the other, there is an unequal division or imbalance in the signs of the roots. As the ratio of roots of one sign to roots of the opposite sign is increased, the imbalance in the signs is increased.

List of Matrices Tested

Included Roots	Code Letters
10, 10, 2, -2	X
9, 10, 2, -2	V
10, 10, 5, -2	1C
10, -10, 2, -2	I
9, -10, 2, -2	W
-10, -10, 2, -2	Y
8±6i, 2, -2	Z
-8±6i, 2, -2	U
6±8i, 2, -2	1A
-6±8i, 2, -2	1B
±10i, 2, -2	1D
10, 10, 10, 2, -2	AA
10, 10, 10, 5, -2	NN
5, 5, 5, 2, -2	PP
20, 20, 20, 2, -2	QQ
10, 10, -10, 2, -2	BB
10, 10, -10, 5, -2	OO
10, -10, -10, 2, -2	CC
-10, -10, -10, 2, -2	DD
10, 8±6i, 2, -2	GG
10, 6±8i, 2, -2	KK
10, -8±6i, 2; -2	II

List of Matrices Tested--Continued

Included Roots	Code Letters
10, ±10i, 2, -2	RR
-10, 8±6i, 2, -2	JJ
-10, 6±8i, 2, -2	LL
-10, -8±6i, 2, -2	HH
-10, -6±8i, 2, -2	MM
-10, ±10i, 2, -2	SS
10, 10, 10, 10, 2, -2	AAA
10, 10, 10, -10, 2, -2	BBB
10, 10, -10, -10, 2, -2	CCC
10, -10, -10, -10, 2, -2	DDD
-10, -10, -10, -10, 2, -2	EEE
10, 10, 8±6i, 2, -2	FFF
10, 10, 6±8i, 2, -2	QQQ
10, 10, ±10i, 2, -2	OOO
10, 10, -8±6i, 2, -2	GGG
10, 10, -6±8i, 2, -2	RRR
10, -10, 8±6i, 2, -2	HHH
10, -10, 6±8i, 2, -2	VVV
10, -10, ±10i, 2, -2	PPP
10, -10, -8±6i, 2, -2	III
-10, -10, 8±6i, 2, -2	JJJ
-10, -10, -8±6i, 2, -2	KKK

List of Matrices Tested--Continued

Included Roots	Code Letters
$-10, -10, \pm 10i, 2, -2$	WWW
$8\pm 6i, 8\pm 6i, 2, -2$	LLL
$8\pm 6i, 6\pm 8i, 2, -2$	TTT
$8\pm 6i, -8\pm 6i, 2, -2$	MMM
$8\pm 6i, -6\pm 8i, 2, -2$	UUU
$6\pm 8i, -6\pm 8i, 2, -2$	SSS
$-8\pm 6i, -8\pm 6i, 2, -2$	NNN
$\pm 10i, \pm 10i, 2, -2$	XXX
$\pm 10i, 8\pm 6i, 2, -2$	YYY
$\pm 10i, -8\pm 6i, 2, -2$	ZZZ

ANALYSIS OF EFFECTS OF DOMINANT ROOTS

Discussion

The output results have been sorted into four groups, according to the dominant roots of the matrix: (1) all real, (2) all complex, (3) both real and complex, and (4) imaginary roots. Within each of these four major groups are two subgroups determined by the signs of the roots. Roots with all signs the same form one subgroup. The subgroup formed by roots with both positive and negative signs can be further classified according to the balance in the number of positive and negative roots.

Three principal patterns emerged when normalizing factors were compared. For matrices with all dominant roots real, imaginary, or a mixture of real and imaginary, the normalizing factors increase or decrease monotonically. When the normalizing factors are graphed with the corresponding iteration numbers, the resulting curves approach the real root as a limit asymptotically. For matrices whose dominant roots are all complex conjugate pairs, a zig-zag or "sawtooth" curve results when the normalizing factors are graphed. For matrices with both real and complex dominant roots, the curve formed by the normalizing factors has either the "sawtooth" pattern or the appearance of a damped wave. If there are more complex than real roots, the curve is a "sawtooth". It is also a "sawtooth" when there is an equal number of real and complex roots but an unequal division in their signs. The damped wave pattern occurs when there is an equal number of real and complex roots and their signs are all the same or evenly divided.

The balance in the number of positive and negative roots has a definite effect on the appearance of the curves. When all the roots of the matrix have the same sign, the normalizing factors form a single curve. If signs of the roots are mixed there are two separate but very similar patterns formed by the odd- and even-numbered normalizing factors. The separation of the two curves is determined by the balance in the signs of the roots. It is greater when the division of signs is close or equal and is lessened considerably when the imbalance in signs is more pronounced.

With just two exceptions, matrices whose dominant roots are additive inverses of each other produced curves which are reflections with respect to the X-axis. All later references to such reflections may be understood to mean reflections with respect to the X-axis.

The angle between the complex conjugates $a\pm bi$ is defined to be the angle enclosed by the two vectors emanating from the origin and terminating at the points $a\pm bi$. This angle is a definite factor in the results. Its effect is most clearly noted in the appearance of stronger "outlining" curves which overlay the basic curves. The number of the "outline" curves seems to be determined by the multiple of the angle which gives 360°.

Three other factors were investigated but not exhaustively: the actual numerical value of the root, the effect of the smaller, non-dominant roots, and the effect of the initial vector. These factors were checked only in matrices with all roots real.

A more detailed description of the results from each matrix tested follows.

Results

The figures referred to below are in the following section entitled "Graphs".

All roots real

The normalizing factors of matrices with all roots real follow a pattern of monotonic convergence. If all signs are the same, there is a single curve generated. Matrix (10, 10) produced a monotonically decreasing sequence of normalizing factors. When graphed, the curve approaches the value 10 asymptotically from above (fig. 1). At the 99th iteration, the value of S is 10.102264. It may be assumed that if the procedure were continued long enough, the process would eventually converge, within the desired range of accuracy, to the value 10. For comparison purposes, the matrix (9,10) was tested--the sequence of normalizing factors converged to 10 after 75 iterations.

Almost identical curves are produced by matrices (10,10,10) and (10,10,10,10), except that they are delayed or lag behind that for (10,10) (fig. 1). The first differences for (10,10,10) are approximately twice the first differences for (10,10). For (10,10,10,10) the first differences are approximately three times those for (10,10). It appears from figure 1 that the curvature is decreased with increasing multiplicity of the root.

When the smaller roots are changed from ±2 to 5, -2 in matrices (10,10) and (10,10,10) the curves for matrices with roots (10,10,5,-2) and (10,10,10,5,-2) are almost identical with the curves for (10,10) and (10,10,10) but lag behind by approximately one iteration.

When the real dominant roots are numerically the same but opposite in sign, the normalizing factors generated are numerically the same but opposite in sign. The curves of the normalizing factors for negative roots are the same as would be produced by rotating the curves for the positive roots through 180^{o} with respect to the X-axis. The normalizing factors approach -10 asymptotically through monotonically increasing negative values. Because of this correspondence, graphs for (-10, -10), (-10,-10,-10) and (-10,-10,-10,-10) have not been included.

Matrices (5,5,5) and (20,20,20) were tested to observe the effect of the size of the dominant root on the rate of convergence. Values of S from (10,10,10) are just twice those for (5,5,5). Normalizing factors from (20,20,20) are four times the corresponding S-values in (5,5,5). Thus the curves are similar in shape to that for (10,10,10) but are spaced on a graph in relation to the limiting values. The decimal portion of S for (5,5,5) is almost identically equal to the decimal portion of S from (10,10). As a consequence, these two curves are nearly identical but separated by five units on the vertical axis. Graphs have not been included because there is too much loss in detail in adjusting the scale on the vertical axis to include all the curves.

If the roots of the matrix are all real and equal in absolute value, but of different sign, a variation in the basic monotonic asymptotic pattern develops. The normalizing factors for even- and odd-numbered iterations form two separate convergences, each having the monotonic and asymptotic properties. When there is an equal number of positive and negative dominant roots, the two sequences converge to separate values. If the division of signs is uneven, the two sequences converge to the same value, but at differing rates.

In matrix (10,-10) even S (normalizing factors of even index) converge to 1--in fact, after the eighth iteration the value of the even normalizing factors remains constant at 0.999999. Odd S (normalizing factors of odd index) converge to 100--after the 13th iteration they have the constant value 100.000030. For matrix (10,10,-10,-10) the limiting values of the two sequences are the same, but at iteration 52 even S begin to go beyond 1. However, the rate of increase is so slow (5-6 iterations to produce a .000001 increase in the normalizing factor) as to suggest the possibility of round-off error in the calculations. Graphs for (10,-10) and (10,10,-10,-10) have not been included. The curves for both are essentially two horizontal lines, one at 1 and one at 100.

When the division of signs in the real dominant roots is uneven, odd and even S each form convergent sequences which have the same limiting value, but the rate of convergence is different. The limit is always the root with the greater multiplicity. In general, the convergence of the even S seems more rapid than that for odd S.

For matrix (10,10,-10) both sequences decrease monotonically toward 10 (fig. 2). The first differences of odd S are approximately 10 times as large as the first differences of even S. For matrix (10,10,10,-10) the pattern of convergence is the same but the first differences of odd and even S are approximately equal. The curves for these two convergences are very close to each other and lag a little behind the curve formed by even S of (10,10,-10) (fig. 2).

It was stated above that matrices whose roots are additive inverses produce curves which are reflections of each other. Matrix (10,-10,-10) is an exception to this general rule. Both sequences approach -100, but the even S decrease monotonically from above and odd S increase monotonically from below (fig. 3). Thus the curves are not the reflection of those for (10,10,-10) (fig. 2). The curves formed by the even S of both (10,10,-10) and (10,-10,-10) are essentially the same but separated vertically by 20 units on the graph. The rate of convergence of odd S of (10,-10,-10) is somewhat slower than that for the odd S of (10,10,-10). Thus, these curves are reflections of each other in shape but are displaced horizontally. The normalizing factors of matrix (10,-10,-10,-10) have a pattern which is the same as that for (10,10,10,-10) with all signs reversed. Both sequences for (10,-10,-10,-10) increase monotonically to -10 and odd and even first differences are approximately equal. The two curves for (10,-10,-10,-10) are quite similar to and somewhat precede the curve for odd S of (10,-10,-10) (fig. 3).

From figures 2 and 3, it is indicated that as the imbalance between positive and negative roots is increased, the separation is lessened between the two curves generated.

A matrix with dominant roots (9,-10) was tested but convergence was not obtained in 99 iterations. It was assumed that the pattern of convergence would be similar to that for (10,-10); however, the results were more similar to those obtained from (10,-10,-10) (fig. 3). Even S decrease monotonically to -10; odd S increase monotonically to -10. Unlike matrix (10,-10,-10) though, the two curves of (9,-10) are almost reflections of each other with respect to the value -10.

The effect of the smaller roots was again tested by changing the roots ±2 in matrix (10,10,-10) to 5, -2. The resulting effect was a smaller separation in the two convergences--values were consistently higher for even and lower for odd S. However, this variation between the normalizing factors of the two matrices decreased with increasing iteration number.

A brief test was made of the effect of the initial vector. The initial vector used with all the matrices had all components equal to one. Matrix (10,10,-10) was also tested with an initial vector with components (1,15,1,1,1). In this case, the rate of convergence of both even and odd S was slower and the odd S increased monotonically to 10, rather than decreasing (fig. 4). The resulting curves are thus more widely separated than the original pair. When a vector with components (1,1,1,15,1) was used with matrix (10,10,-10), the <u>even</u> S increased monotonically and the separation in the two curves is increased. However, the separation is not as great as when initial vector (1,15,1,1,1) is used.

The same vectors were also tested with matrix (10,-10,-10). The effect with vector (1,15,1,1,1) was to slow both convergences and thus produce a greater separation in the curves. With vector (1,1,1,15,1) the rate of convergence for even S was slower but the rate for odd S was increased. However, even S were also reversed in direction of approach. The net result was to bring the two curves closer together.

Matrix (10,-10,-10,-10) was tested with vectors (1,15,1,1,1,1) and (1,1,15,1,1,1). The two curves for this matrix with the original vector have very little separation (fig. 3). Using either of the two new vectors produced only small variations in this general pattern.

All Roots Complex

The angle between a pair of complex conjugates appears to have a definite influence on the behavior of the normalizing factors. In particular, the actual factor is the multiple of the angle which gives 360°. Results obtained from matrices with complex roots have in common two features which indicate such influence. The general pattern produced by the normalizing factors is interrupted by an irregularity; then the pattern begins to repeat itself. It is tentatively theorized that these irregularities occur because the angle between the complex roots is not a perfect divisor of 360°--the interruption is in a sense a "correction" factor. In most matrices studied, the irregularity occurred each time after a sequence of 44 iterations; however, this phenomenon was sometimes noted after sequences of 39 iterations.

The second feature is the appearance of "outlining" curves which dominate or supersede the curves anticipated from the data. The number of these outline curves is directly related to the multiple of the angle between the complex conjugates which gives 360°. The complex roots $8 \pm 6i$ enclose an angle of approximately $73^\circ\ 44'$; when 360° is divided by 72°, the quotient is 5. The complex roots $6 \pm 8i$ enclose an angle of approximately $106^\circ\ 16'$; when 360° is divided by 120°, the quotient is 3. In results described below, the factors of 5 and 3 will repeatedly appear in the results for $\pm 8 \pm 6i$ and $\pm 6 \pm 8i$, respectively. However, when the matrix contains both, the resultant factor is 4.

It was stipulated above that the sign of a complex number is that of its real part. The normalizing factors generated by matrices whose dominant roots are all pairs of complex conjugates with the same sign produced a pattern designated as "sawtooth". Because of the rather large spread in numerical values of the normalizing factors, it was difficult to construct graphs with any real degree of accuracy. But the general pattern referred to as "sawtooth" can be illustrated by figure 5, which is the graph of the normalizing factors of matrix ($8 \pm 6i$). A "sawtooth" curve has several values, decreasing in absolute value, followed by a single value of opposite sign. The number of values in a group appears to be a function of the angle between the complex conjugates. The grouping of values is repeated with noticeable "drift". It is broken at definite intervals by an irregularity (discussed above), usually a group of fewer values. Following such an interruption, the initial value of the next group produces a new extreme peak and the cycle is begun again.

For matrix ($8 \pm 6i$) (fig. 5), there are five values in each group: four positive values, decreasing in absolute value, and a single negative value. The sequence, containing nine peaks, appears to have a cycle of 44 iterations. The cycle is illustrated by the portion of the curve between iterations 50 and 94. The pattern for matrix ($8 \pm 6i$, $8 \pm 6i$) is very similar to that of matrix ($8 \pm 6i$) but the peak values are more extreme. For matrix ($6 \pm 8i$) (fig. 6), the same sawtooth pattern occurs, but there are only two or three values of the same sign, followed by one value of opposing sign. This produces a sawtooth graph with "teeth" which are more frequent and irregular in shape. It is thus more difficult to note the irregularity which interrupts the sequence. The drift of the peaks is also reversed. For matrix $(8 \pm 6i)$ the peaks within a given sequence are lower with increasing iteration; for $(6 \pm 8i)$, the peaks are higher as the iteration number increases. This drift appears to depend upon the angle between the complex conjugate roots. For $8 \pm 6i$, the angle is slightly larger than the nearest integral divisor of 360°. The drift is downward. For $6 \pm 8i$ the angle is somewhat smaller than the nearest integral divisor of 360°. The drift is upward.

However, another pattern seems to dominate the sawtooth for $(6 \pm 8i)$. It is formed by joining those values of S whose iteration numbers have the formula $3m + j$, $m = 0,1,2, \ldots$ and $j = 0,1$, and 2, forming three separate sawtooth curves (fig. 7). This phenomenon occurs many times with complex roots; the curve anticipated from the data is dominated or superseded by a stronger set of curves.

Changing the "signs" of the complex roots produces curves which are just reflections of the original curves. However, correspondence of actual numerical values is not as close as for real roots, particularly at the peaks. Thus the curves for matrices $(-8 \pm 6i)$, $(-8 \pm 6i, -8 \pm 6i)$ and $(-6 \pm 8i)$ are simply the reflections of those for their positive counter parts, discussed above. The graph for matrix $(-8 \pm 6i)$ (fig. 8) has been included for later reference.

The normalizing factors of matrix ($8\pm6i$, $6\pm8i$) produced a nearly level sawtooth curve with almost no drift. Apparently the opposing drifts for $8\pm6i$ and $6\pm8i$ "cancel" or negate each other when the roots appear in the same matrix. There are four values in each group; the second and third values are usually approximately 8.0 and 2.0 and the initial value is approximately 16.0, except when extreme peaks are generated.

As in the case of real roots, when the roots of the matrix have different signs, even and odd S produced separate patterns. For matrices ($8\pm6i$, $-8\pm6i$) and ($6\pm8i$, $-6\pm8i$) even S were constant at .999999 or 1.000000, and odd S produced a variation of the sawtooth pattern. For ($8\pm6i$, $-8\pm6i$) there are five values in a group: two positive, one negative, one positive, one negative. This grouping of five values is repeated, with downward drift in the values, then interrupted by an extra pair: one positive, one negative. Then the pattern is begun again. Rather than relating to its own roots, the results from matrix ($8\pm6i$, $-8\pm6i$) more closely resemble the reflection of the results for matrix ($6\pm8i$, $-6\pm8i$). The pattern formed by the odd S of ($8\pm6i$, $-8\pm6i$) is very similar to the curve for matrix ($6\pm8i$) (fig. 6). The curve for ($8\pm6i$, $-8\pm6i$) has two outline curves, formed by joining every other odd S. These two curves resemble the three outline curves for matrix ($6\pm8i$) (fig. 7). For matrix ($6\pm8i$, $-6\pm8i$) the pattern is the same as the curve ($8\pm6i$, $-8\pm6i$) with all signs reversed, but numerical values do not particularly correspond. The odd S of matrix ($6\pm8i$, $-6\pm8i$) have a pattern very similar to that for matrix ($-6\pm8i$). The two outline curves are very similar to the three outline curves for ($-6\pm8i$).

Matrix ($8\pm6i$, $-6\pm8i$) produced a sawtooth curve which is much like that for matrix ($6\pm8i$) (fig. 6). There are four outline curves, one of which is nearly constant at 3.0. The other three curves are similar to the outline curves for ($6\pm8i$) (fig. 7). These curves each show a cycle of 44 iterations.

Both Real and Complex Roots

One real root, one complex conjugate pair

For all matrices with three dominant roots, one real root and one pair of complex conjugate roots, the pattern exhibited by the normalizing factors is some variation of the sawtooth curve which was produced in the case of the complex conjugate pair alone. When all signs of the roots are the same, there are more values in each group. This gives the appearance of having "stretched" the curve (figs. 8 and 9). For matrix (10, $8\pm6i$) the single negative value in a group does not occur; all normalizing factors are positive. For (10, $6\pm8i$) there is an occasional negative value, but not with every group. The graphs for both are similar to each other, but peaks occur more frequently for (10, $6\pm8i$). The graphs for (-10, $-8\pm6i$) (fig. 9) and (-10, $-6\pm8i$) are also similar, except for frequency of peaks. They are almost exact reflections of (10, $8\pm6i$) and (10, $6\pm8i$) except that the peaks are more extreme.

When the signs of the roots are not the same, there must be an uneven division of signs, since there are only three dominant roots. As with a matrix with all roots real, odd and even S form two separate sequences. The shape is determined by the complex roots. The patterns produced by matrices $(10, -8\pm6i)$ and $(-10, 8\pm6i)$ are essentially negatives of each other. For matrix $(10, -8\pm6i)$ (fig. 10), odd and even S each form a sawtooth curve similar to that for $(-8\pm6i)$ (fig. 8). The curves produced by matrix $(-10, 6\pm8i)$ (fig. 11) are similar to the curves generated by matrix $(6\pm8i)$ (fig. 6). Again, a pattern of three curves, formed by joining every third normalizing factor (fig. 12), seems to overlay this pattern, as was the case with matrix $(6\pm8i)$ (fig. 7).

Two real roots, one complex conjugate pair

When the matrix has four dominant roots, two real roots and one complex conjugate pair, there is one of three patterns generated. If all signs are the same, a single damped wave-type curve results (fig. 13). If signs are equally divided, there are two wave-type curves (fig. 14). If there is an imbalance in the signs of the roots, there are two sawtooth curves (fig. 18).

For matrix $(10,10,8\pm6i)$ (fig. 13), all normalizing factors are greater than 10. The curve oscillates above a baseline of 10 and is damped asymptotically from above. When the curves for matrices (10,10) and (10,10,10) (fig. 1) are superimposed on this graph, the curve for (10,10) appears to bisect the wave and the curve for (10,10,10) seems to be tangent to the peaks of the wave. Matrix ($-10,-10,-8\pm6i$) produces a curve which is a reflection of that for ($10,10,8\pm6i$). The curve generated by matrix ($10,10,6\pm8i$) has the same general structure as that of ($10,10,8\pm6i$) but peaks occur more frequently. Eight values form an individual wave for ($10,10,8\pm6i$) and there are six values in an individual wave in ($10,10,6\pm8i$).

If the signs of the roots are mixed but equally divided between positive and negative (i.e. both real roots have the same sign), even and odd S each produce a damped wave-type curve, which oscillates around the curve generated in the case of the real roots alone. The peaks of one curve coincide with the valleys of the other. This is illustrated by matrices ($10,10,-8\pm6i$) (fig. 14) and ($10,10,-6\pm8i$) (fig. 16). For matrix ($-10,-10,8\pm6i$) the curves are reflections of those for ($10,10,-8\pm6i$). This is the expected result if conclusions are extrapolated from previous cases. However, in sketching these curves, others predominate. For matrix ($10,10,-8\pm6i$) there are five outline curves, formed by joining every fifth normalizing factor (fig. 15). Three decrease monotonically and asymptotically, one increases monotonically and asymptotically, and the fifth may be either monotonic increasing or have a slight wave form. The appearance of outline curves is even more dramatically illustrated in matrix ($10,10,-6\pm8i$) (fig. 17). Dominating the wave-type curves formed by odd and even S are three other wave-type curves, formed by joining every third normalizing factor. The results from this matrix first called attention to the outline curves in other results. Again, the number of outline curves apparently bears a direct relationship to the size of the angle between the complex roots.

If the division of signs in the roots is uneven (i.e. the two real roots have different signs), odd and even S form separate curves and each has the sawtooth shape. The two curves are very similar; the curve for odd S is displaced just one iteration behind that for the even S. It was noted in the discussion of real roots that matrices with greater imbalance in the signs of the roots generated curves with a smaller separation. This same effect can be noted with mixed roots by comparing the results for matrices (10,10,-10) and (10,10,10,-10) (fig. 3) with those for matrices (10,-8±6i) (fig. 10) and (10,-10,-8±6i) (fig. 18). For matrix (10,-10,-8±6i) (fig. 18), the curves have a shape very similar to that for (-8±6i) (fig. 8), indicating again that the shape is determined by the complex roots. Matrix (10,-10,6±8i) produced a pair of curves similar in shape to that for (6±8i) (fig. 6). The curves of matrix (10,-10,8±6i) are reflections of those for (10,-10,-8+6i).

Imaginary Roots

Matrices that have imaginary dominant roots give responses which may be either like those for real roots or those for complex roots. For matrix (±10i) even S are constant at 1.0 and odd S constant at -100.0. The two sequences of the normalizing factors of ($\pm$10i,$\pm$10i) have the same limits, but the even S of this matrix begin to go below 1.0. At the 98th iteration S has the value .999992. This is probably due to round-off error. These two patterns are very similar to those for matrices (10,-10) and (10,10,-10,-10).

For matrix (10,$\pm$10i) there are four separate sequences formed by the normalizing factors, converging monotonically to four different limits. After iteration 45, the four limits have the constant values: 3.665746, 182.797110, 10.945287, and 1.363456. Two of the four limits are negative for matrix (-10,$\pm$10i): 8.992830, -221.200370, -9.045211, 0.555775. These values are constant after the 17th iteration. For matrix (10,-10,$\pm$10i) there are four separate curves decreasing monotonically to the values 1.0, 2000.0, 1.0, and 5.0.

Matrices (10,10,$\pm$10i) and (-10,-10,$\pm$10i) also show four separate convergences but to the same limiting value: the value of the real root. For matrix (10,10,$\pm$10i) (fig. 19) there are four curves decreasing monotonically and asymptotically toward 10. When this graph is superimposed upon the graph for matrix (10,10,-10) (fig. 2), the four curves for (10,10,$\pm$10i) "bracket" the two curves from (10,10,-10). As with matrix (10,-10,-10) (fig. 3), the curves for matrix (-10,-10,$\pm$10i) are not reflections of those for (10,10,$\pm$10i). Three of the curves for (-10,-10,$\pm$10i) increase monotonically and asymptotically to -10, but one curve approaches -10 through monotonically decreasing values.

In all these cases, the product of the four limiting values is 10,000. The number of curves generated by these matrices has particular significance. Imaginary numbers have an argument of 90°, which divides 360° into 4, and there are four curves generated.

If the matrix has both complex and imaginary roots, a very irregular sawtooth curves results, but there is a definite cycle in the pattern. For matrix ($\pm 10i, 8 \pm 6i$) the sawtooth curve has a cycle of 34 iterations. Matrix ($\pm 10i$, $-8 \pm 6i$) produced a curve which is essentially the reflection of that for ($\pm 10i, 8 \pm 6i$).

Graphs

All of the graphs in this section are of curves resulting when the normalizing factors of a matrix are plotted corresponding to the iteration number. Vertical axes are labeled "S" and horizontal axes labeled "N". Portions of the curves are occasionally deleted in order to maintain a scale on the axes which would permit the characteristic shape of the curve to be displayed. The matrices whose normalizing factors are plotted in each graph are listed below with the figure number. Not all matrices tested have graphs included in this section.

Figure 1	(10,10), (10,10,10), (10,10,10,10)
2	(10,10,-10), (10,10,10,-10)
3	(10,-10,-10), (10,-10,-10,-10)
4	(10,10,-10)--two different initial vectors
5	($8\pm6i$)
6	($6\pm8i$)
7	($6\pm8i$)--outline curves
8	($-8\pm6i$)
9	($-10,-8\pm6i$)
10	($10,-8\pm6i$)
11	($-10,6\pm8i$)
12	($-10,6\pm8i$)--outline curves
13	($10,10,8\pm6i$)
14	($10,10,-8\pm6i$), (10,10)
15	($10,10,-8\pm6i$)--outline curves

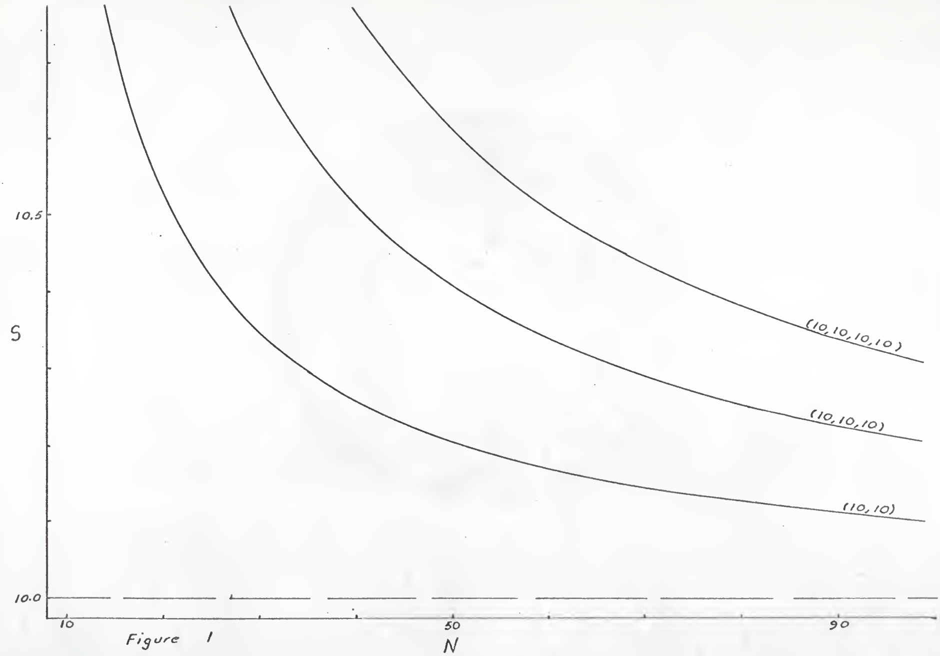

Figure 1

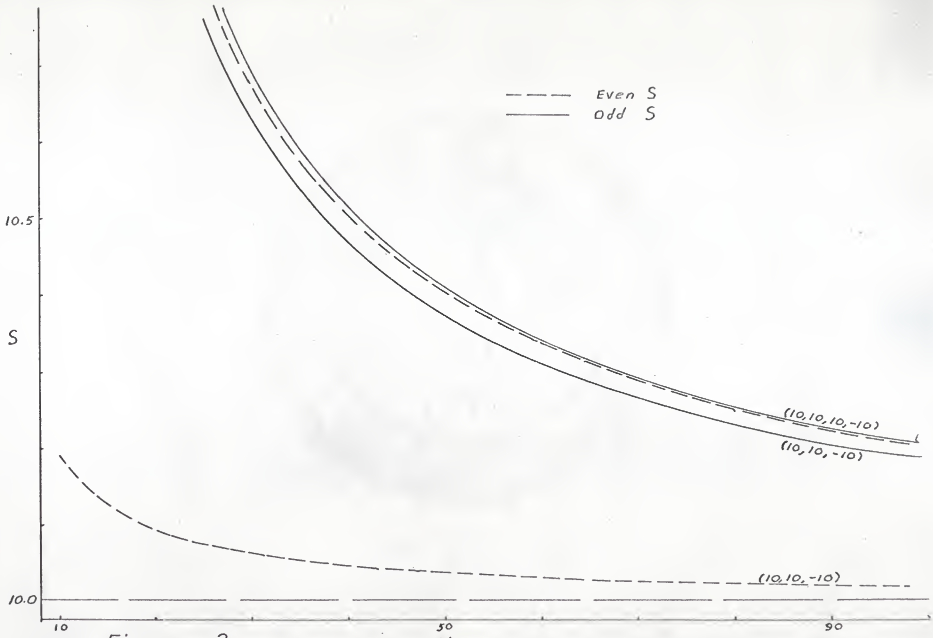

Figure 2

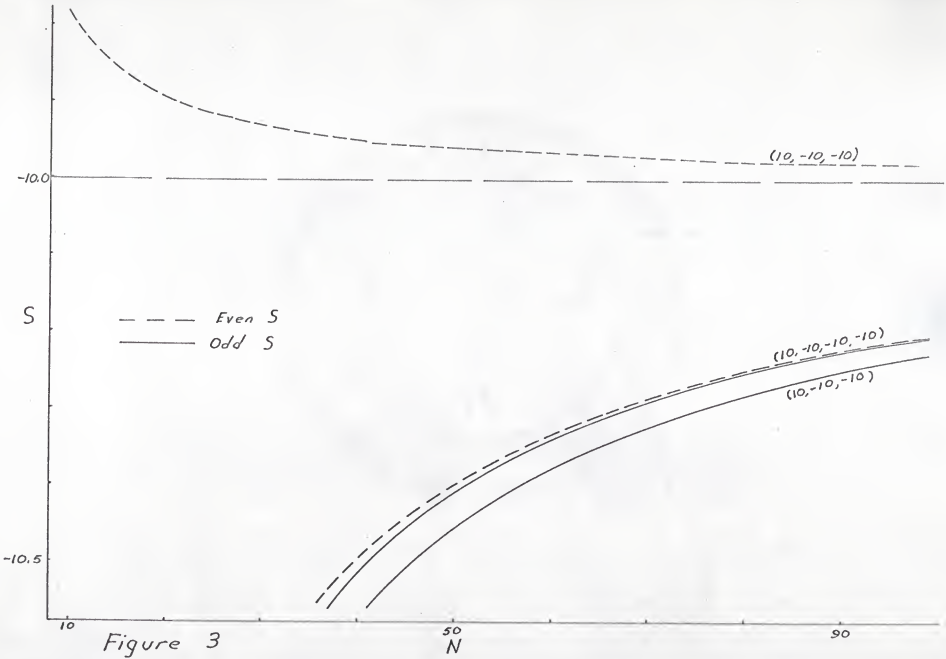

Figure 3

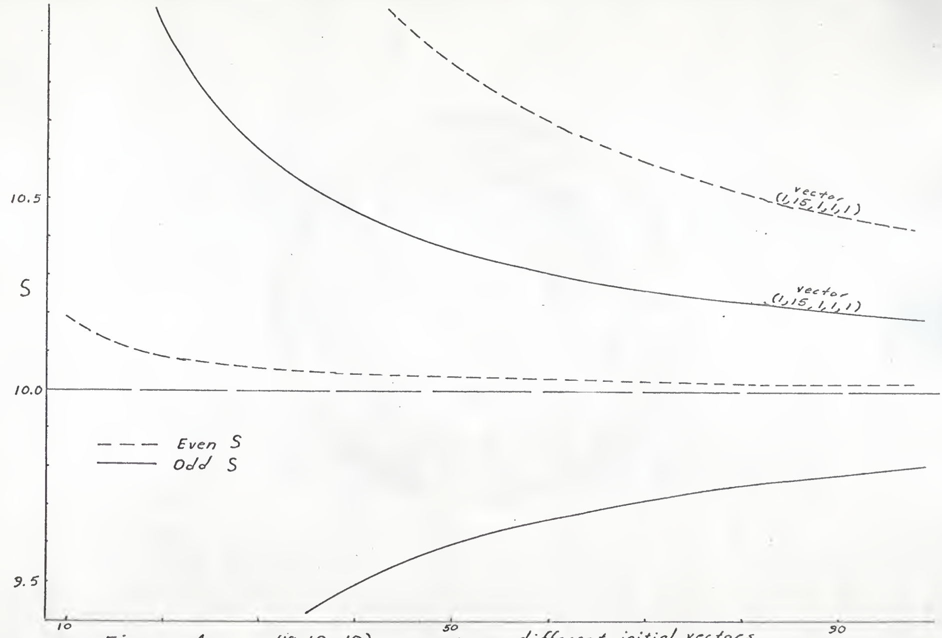

Figure 4 (10,10,-10) different initial vectors

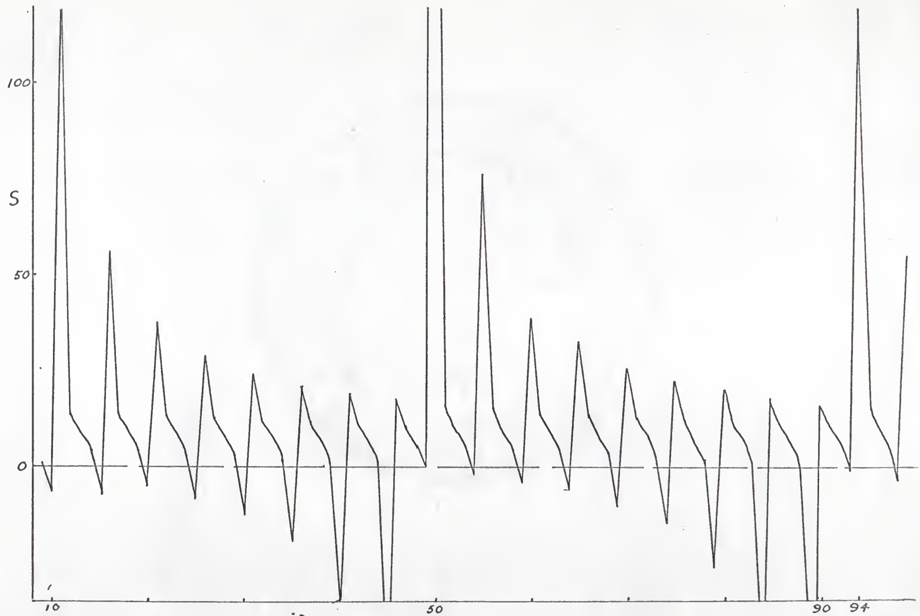

Figure 5 $(8 \pm 6i)$

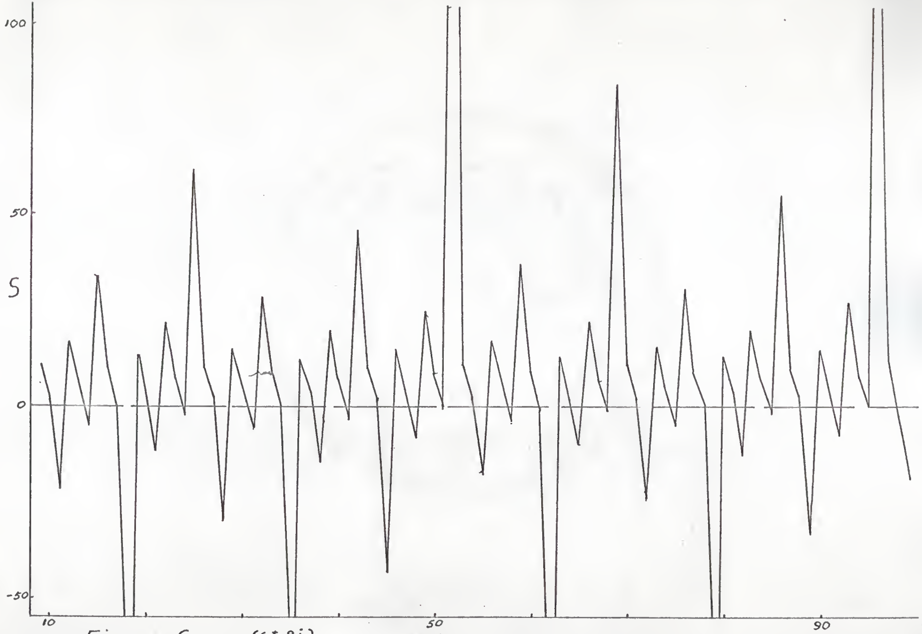

Figure 6 (6 ± 8i)

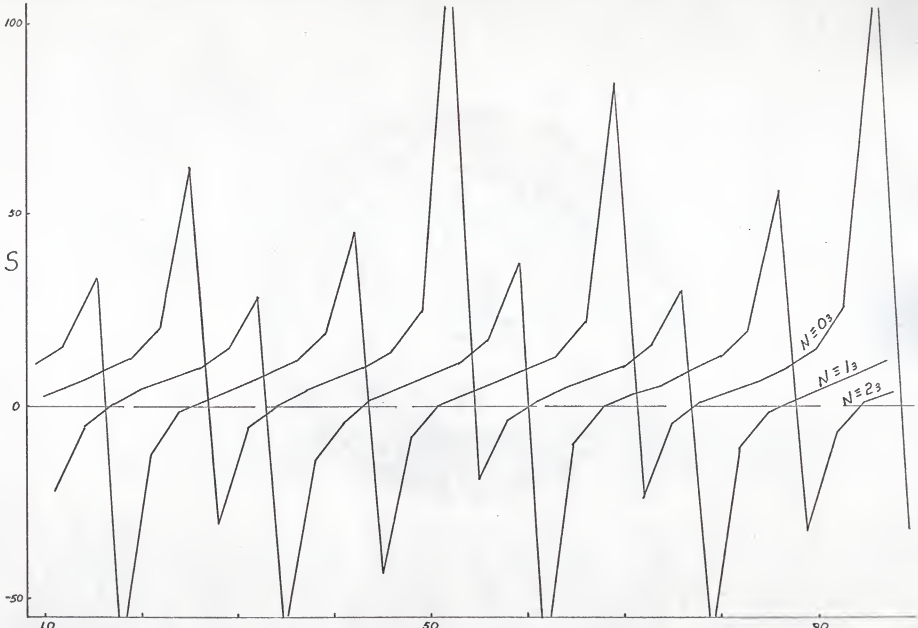

Figure 7 $(6 \pm 8i)$ N Outline Curves

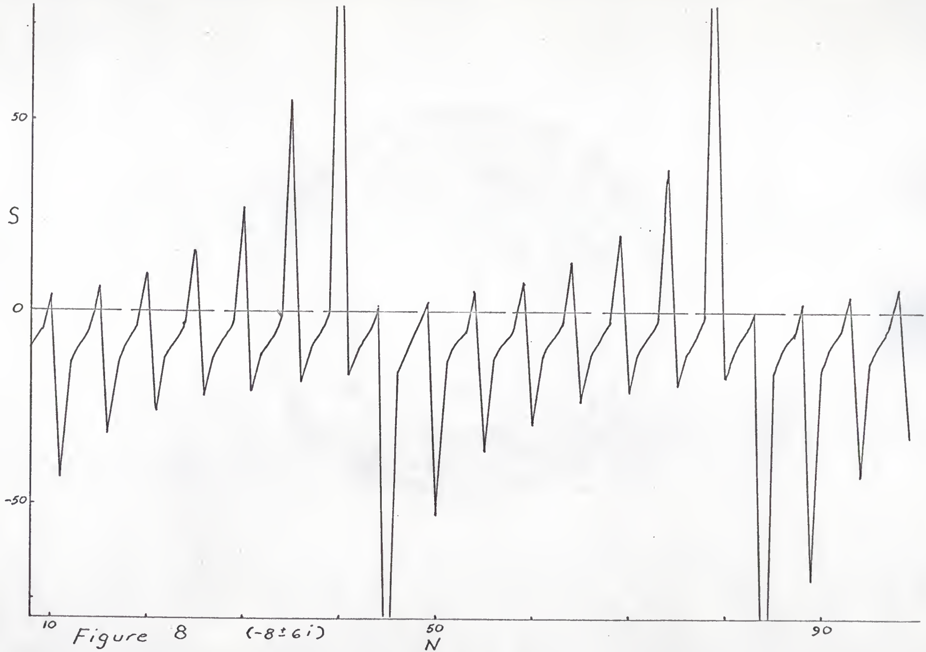

Figure 8 (-8±6i)

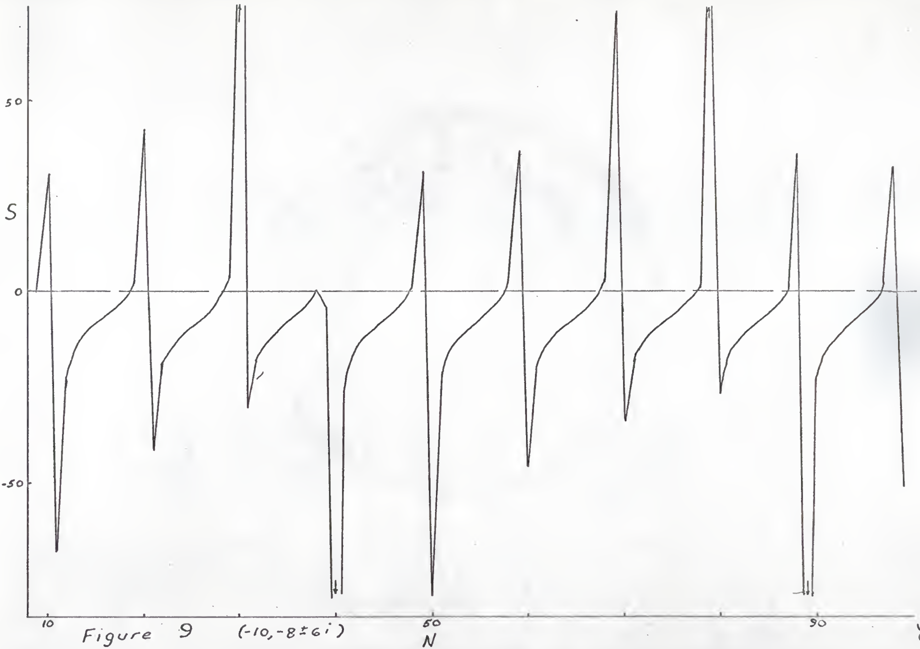

Figure 9 (-10, -8 ± 6i)

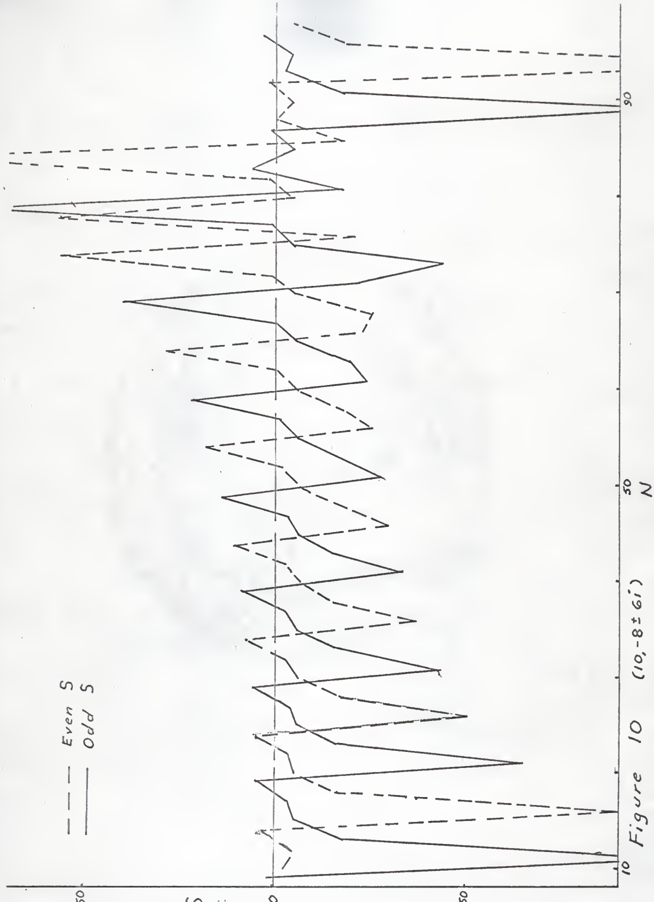

Figure 10 (10, -8 ± 6i)

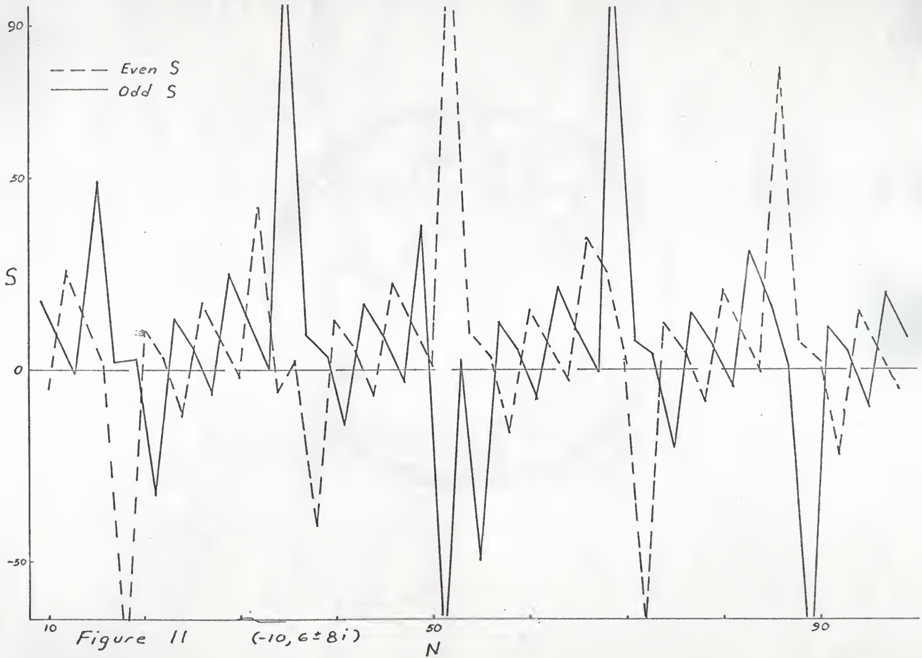

Figure 11 $(-10, 6 \pm 8i)$

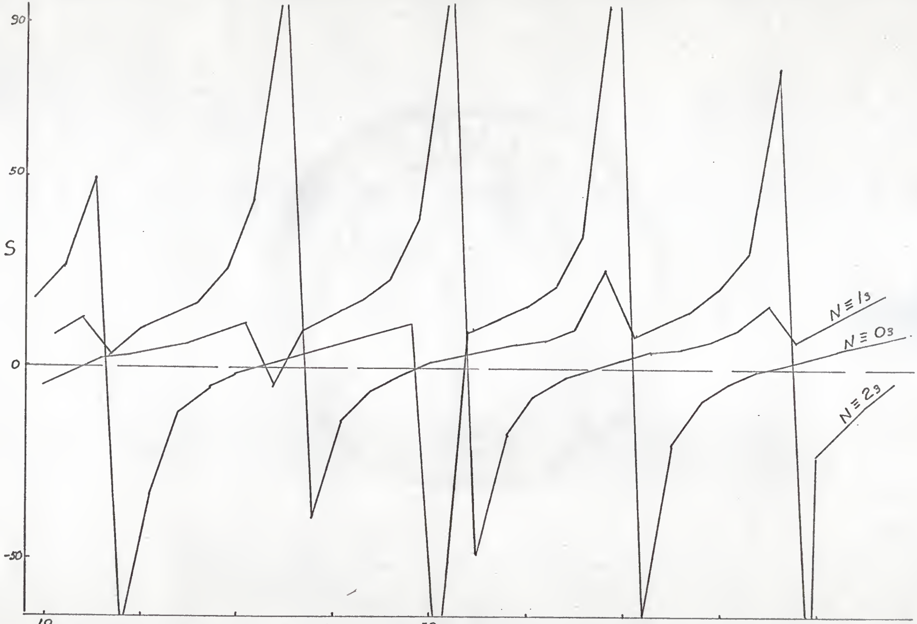

Figure 12 (-10, 6 ± 8i) Outline Curves

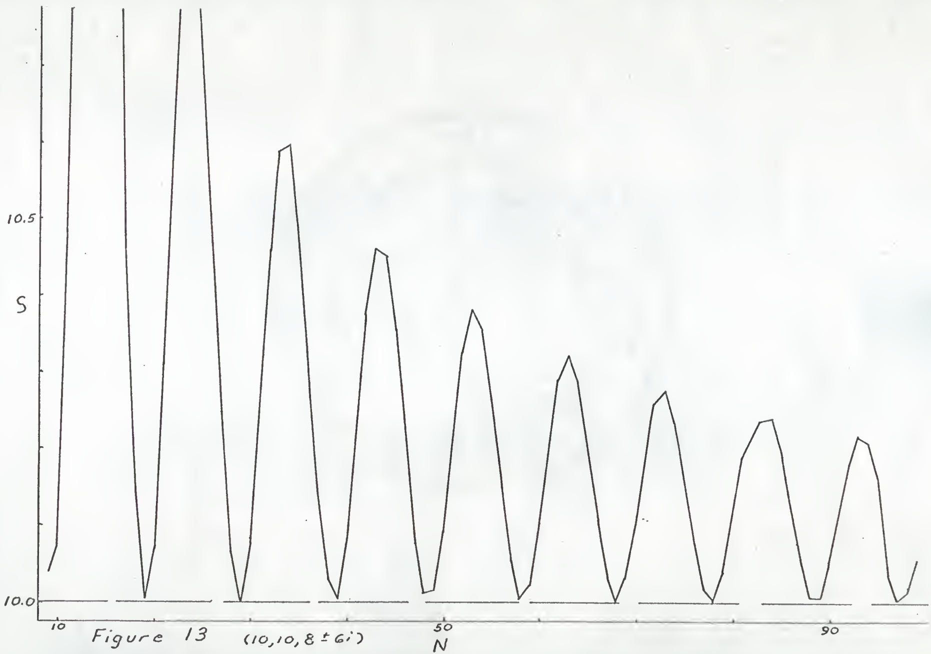

Figure 13 $(10,10,8 \pm 6i)$

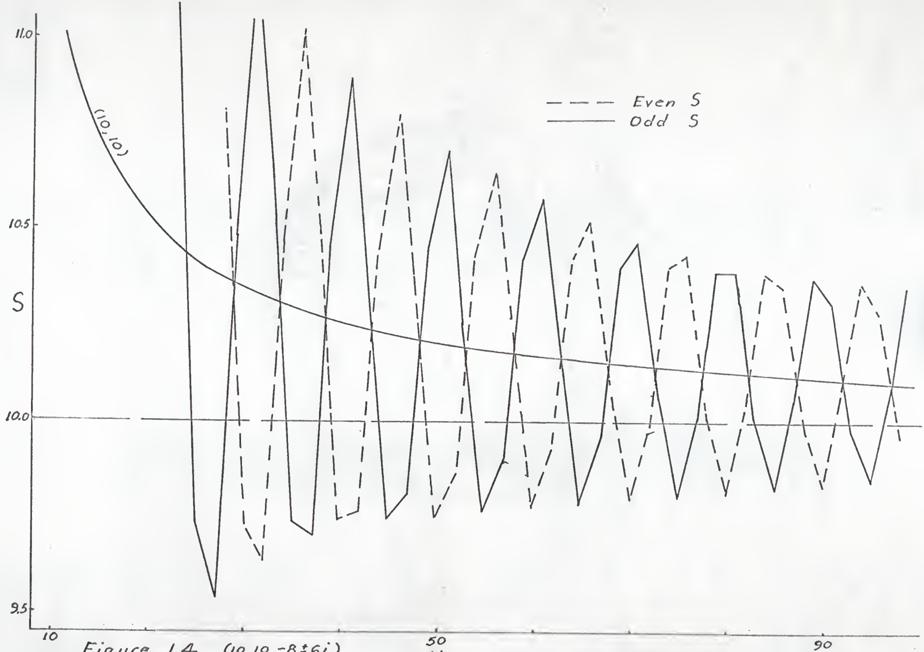

Figure 14 (10,10,-8±6i)

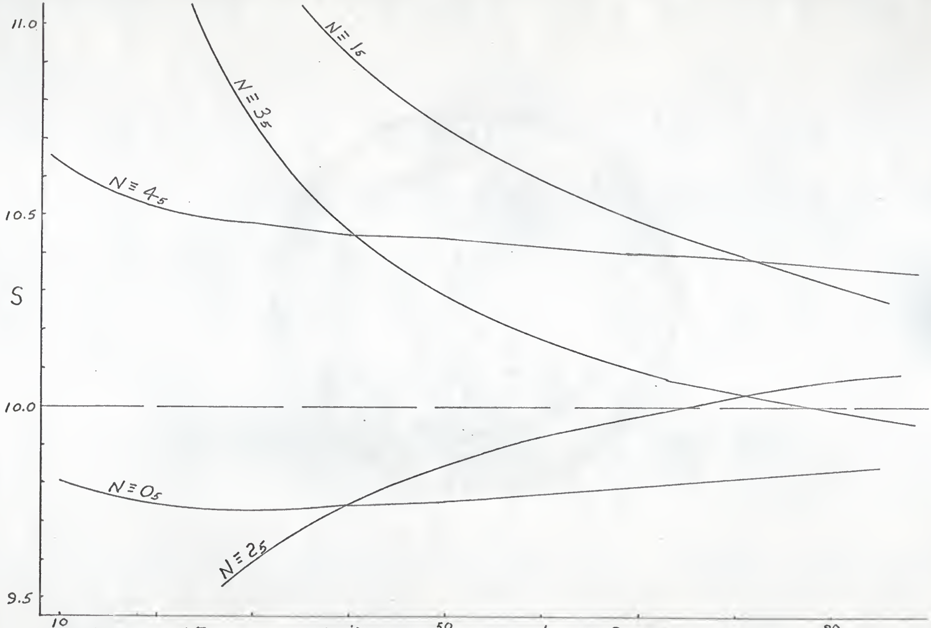

Figure 15 (10,10,-8 ± 6i) Outline Curves

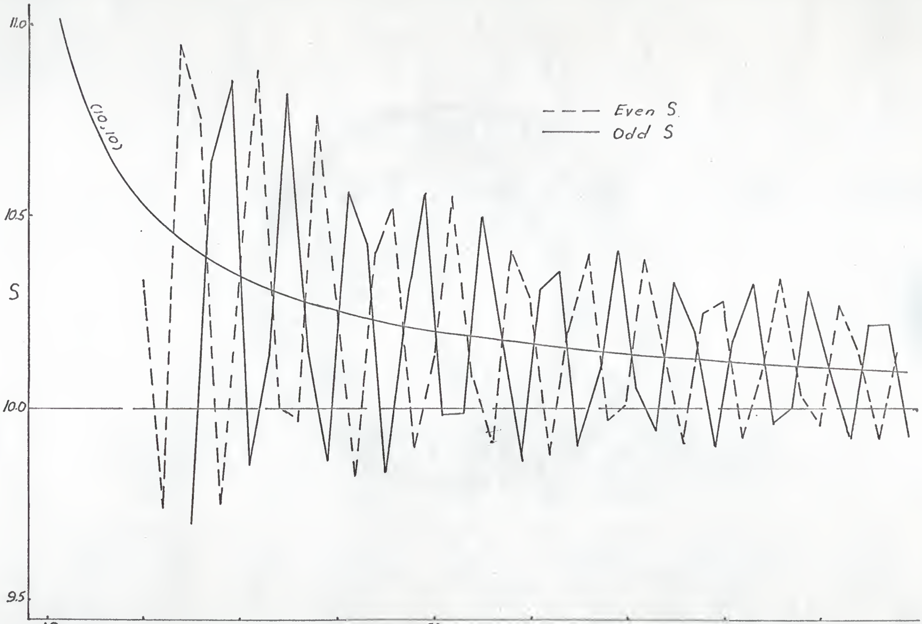

Figure 16 (10,10,-6±8i)

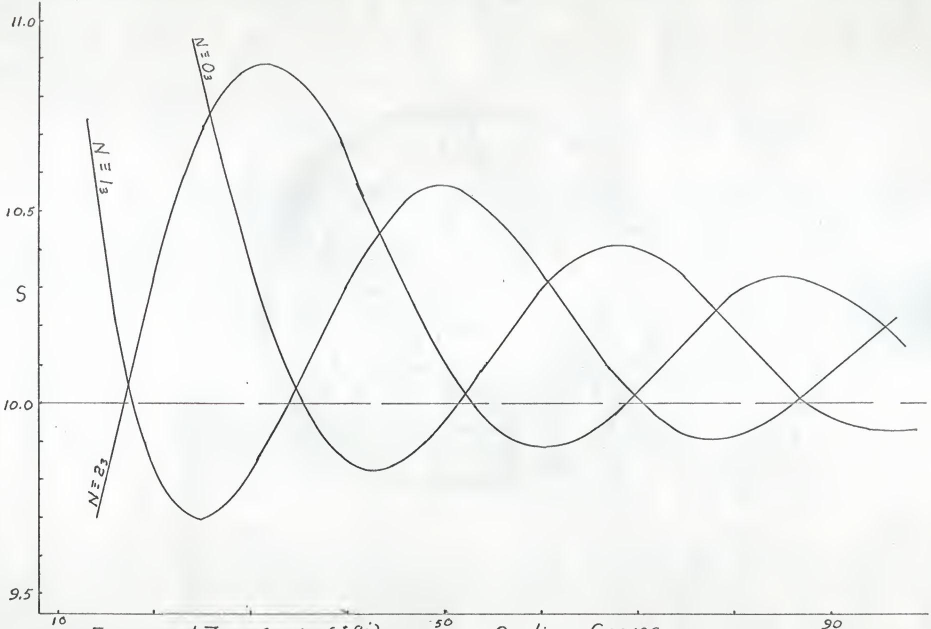

Figure 17 (10,10,−6±8i) Outline Curves

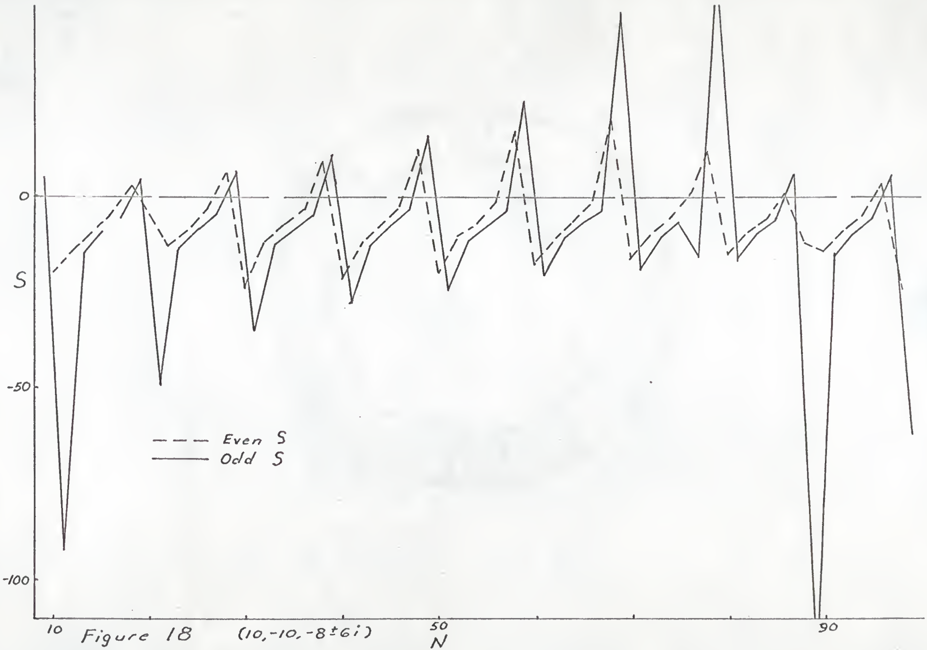

Figure 18 $(10, -10, -8 \pm 6i)$

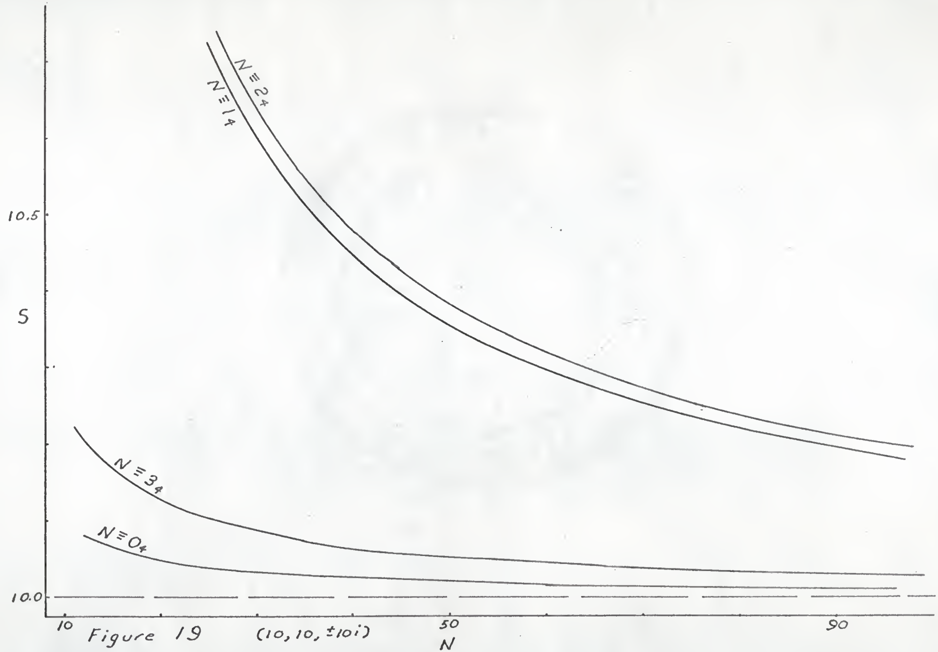

Figure 19 (10, 10, ±10i)

CONCLUSIONS

The largest matrices tested were of dimension six and had four equally dominant roots. Extrapolation of results and conclusions to non-companion matrices or to matrices with greater than four equally dominant roots can only be suggested.

The power method is applied to a matrix and the pattern produced by the normalizing factors over several iterations is noted. Because of the cyclical behavior noted, a minimum number of 50 iterations should probably be used. Based upon the data matrices discussed above, the following predictions might be made about the dominant characteristic roots of the matrix.

If the method converges, the matrix has a single real dominant root.

A curve which approaches a limit monotonically and asymptotically indicates real or imaginary roots. If there is just a single curve, there are two or more real roots with the same sign. The limit is the value of one of these roots.

Two such curves can be formed by the odd and even normalizing factors approaching different limits. If the limits have the same sign, there are an equal number of positive and negative real roots. If the two limits have opposite sign, the roots are pairs of imaginary numbers. If there are two curves approaching the same limiting value, the roots are real and there is an imbalance in their signs. The limit is the value of the root with the greater multiplicity. The separation between the two curves gives some indication of the division in signs. The separation is decreased as the imbalance is more pronounced.

If there are four monotonic curves approaching four limits, there are both real and imaginary roots. If there is more than one real root, the signs will be mixed. If there are four such curves approaching the same limit, the roots are a mixture of real and imaginary and the real roots have the same sign. The limiting value is that of a real root.

A sawtooth curve indicates the presence of complex conjugate roots. There will be a cycle in the pattern which repeats itself. There may also be outlining or dominating curves. The length of the cycle, the "drift" of the curve, and the number of outlining curves are thought to be determined in some manner by the angle between the complex roots.

If there is a single sawtooth curve, the roots can be all complex or complex and imaginary or complex and real. If real roots are present, they will have the same sign as the complex conjugate pair.

If there are two sawtooth curves, formed by the odd- and even-numbered normalizing factors, there are both real and complex roots and the signs are mixed. The separation in the two curves gives some indication of the imbalance in signs. If the two curves are evenly spaced, there is one real root, opposite in sign to that of the complex pair. If the two curves are separated by just one iteration, there are two real roots which are additive inverses.

A wave-like curve indicates both real and complex roots, with an equal number of each. If there is a single curve, the roots all have the same sign. The curve oscillates above a baseline equal to the real root.

If there are two curves, formed by odd and even normalizing factors, the signs of the real roots are the same and opposite to those of the complex roots. The two curves oscillate around the curve generated by the real roots alone.

The 54 matrices tested are only a small portion of the possibilities inherent in research of this nature. Within the limitations of this study are several suggested extensions.

For real roots, fig. 1 would probably be enhanced by the addition of the curve generated by the matrix with roots $(10, \pm 2)$. It would also be of interest to determine if the curve approaches a straight line as the multiplicity of the dominant root is increased beyond 4. If matrices with more than 4 equally dominant roots were tested, the effect of the balance in the signs of the roots could be further analyzed. The only ratios possible in this study were 2:1 and 3:1. The influence of the angle between a pair of complex conjugate roots could be further investigated by choosing complex conjugates whose included angle is some factor of 360° other than 3 or 5.

Such factors as the actual numerical value of the root, the effect of the smaller, non-dominant roots, and the effect of the initial vector were mentioned but certainly not investigated exhaustively. Each factor in itself might provide material for extensive study.

The behavior of non-companion matrices has not even been attempted in this research.

The ultimate goal, of course, would be the derivation of one or more formulas incorporating all essential factors. These formulas would then be used to predict the multiplicity and character of the dominant roots of any matrix.

EXTENSION OF THE POWER METHOD

Two Dominant Roots

The power method can be expanded to determine any number of equally dominant roots. If the matrix A has two linear dominant elementary divisors, the procedure is developed from the approximate equation:

$$A^m(A-r_1I)(A-r_2I) \approx Z .$$

This equation can be viewed either as a simplification of equation (1) or an expansion of equation (2). Applying the iterative procedure to this equation yields the vector equation:

$$(5) \quad s_{m+2}s_{m+1}Y_{m+2} - (r_1 + r_2)s_{m+1}Y_{m+1} + r_1r_2Y_m \approx 0 \quad ,$$

corresponding to equation (3).

When m is large enough, then for any component of the Y_k:

$$(6) \quad s_{m+2}s_{m+1} - (r_1 - r_2)\, s_{m+1} + r_1r_2 \approx 0 \; ,$$

corresponding to equation (4).

When a matrix has two equally dominant roots in its minimum polynomial, $|r_1| = |r_2|$ and three possibilities exist: $r_1 = r_2$; $r_1 = -r_2$; or $r_1 = \overline{r}_2$. Substitution into equation (6) of each of these possibilities provides the three specialized equations:

$$(7) \quad s_{m+2}s_{m+1} - 2r_1s_{m+1} + r_1^2 \approx 0$$

$$(8) \quad s_{m+2}s_{m+1} - r_1^2 \approx 0$$

$$(9) \quad s_{m+2}s_{m+1} - 2as_{m+1} + a^2 + b^2 \approx 0 ,$$

where $a \pm bi$ has been used to denote a pair of complex conjugates.

If $r_1 = r_2$, the matrix A actually has a single dominant elementary divisor which is quadratic. Thus equation (7) is also derived from Theorem I (p. 2) with $k = 1$ and $t_1 = 2$. Equations (8) and (9) can be derived from Theorem I with $k = 2$, $t_1, t_2 = 1$, and the appropriate relationship of the two roots.

For a matrix with a single dominant root, solving for the root is fairly straightforward, because the normalizing factor generated at each iteration _is_ an approximation to the root. For multiple dominant roots, finding the roots necessitates first generating the normalizing factors and then solving a system of linear equations. Two consecutive equations like equation (7) can be generated by increasing the index on the normalizing factors by one. The solution of this pair of equations, linear in r_1 and r_1^2, can be obtained and compared for accuracy with the solution obtained from the preceding pair of equations. The same method is used to find a pair of complex conjugate roots (equation (9)). When the real roots are additive inverses (equation (8)), the procedure is somewhat simpler. At each iteration an approximation to the root can be obtained by taking the square root of the product of two successive normalizing factors.

When the root has been approximated to the desired degree of accuracy, a double check should be made with the vector equation to assure convergence.

It must be known in advance that the matrix has just two dominant roots if this procedure is to be applied. Knowing the form of the roots simplifies the work somewhat by indicating which equation (7,8,9) is applicable. However, a solution can be obtained from the general equation (6).

Three Dominant Roots

A matrix with three equally dominant roots in its minimum polynomial requires procedures much like those for two dominant roots. The equations become more complicated and three consecutive equations must be generated to obtain a solution. Corresponding to equations (5) and (6) are the equations:

$$(10)\quad s_{m+3}s_{m+2}s_{m+1}Y_{m+3} - (r_1 + r_2 + r_3)\, s_{m+2}s_{m+1}Y_{m+2} + (r_1r_2 + r_2r_3 + r_1r_3)s_{m+1}Y_{m+1} - r_1r_2r_3Y_m \approx 0$$

$$(11)\quad s_{m+3}s_{m+2}s_{m+1} - (r_1 + r_2 + r_3)s_{m+2}s_{m+1} + (r_1r_2 + r_2r_3 + r_1r_3)s_{m+1} - r_1r_2r_3 \approx 0 .$$

When there are three equally dominant roots in the minimum polynomial, $|r_1| = |r_2| = |r_3|$ and there are three possible relationships: $r_1 = r_2 = r_3$; $r_1 = r_2 = -r_3$; r_1 real, $r_2 = \overline{r}_3$. For these possibilities equation (11) has the three separate forms:

$$(12)\quad s_{m+3}s_{m+2}s_{m+1} - 3r_1 s_{m+2}s_{m+1} + 3r_1^2 s_{m+1} - r_1^3 \approx 0$$

$$(13)\quad s_{m+3}s_{m+2}s_{m+1} - r_1 s_{m+2}s_{m+1} - r_1^2 s_{m+1} + r_1^3 \approx 0$$

$$s_{m+3}s_{m+2}s_{m+1} - (2a + r_1)s_{m+2}s_{m+1}$$
$$+ (a^2 + b^2 + 2ar_1)s_{m+1} - r_1(a^2 + b^2) \approx 0 .$$

The last equation can be simplified by noting that $a^2 + b^2 = r_1^2$:

$$(14)\quad s_{m+3}s_{m+2}s_{m+1} - (2a + r_1)s_{m+2}s_{m+1} + r_1(2a + r_1)s_{m+1} - r_1^3 \approx 0.$$

Theorem I can be used to derive these same three equations. If $r_1 = r_2 = r_3$, equation (12) is obtained by setting $k = 1$, $t_1 = 3$. In this case, the matrix A has a single cubic dominant elementary divisor. When $r_1 = r_2 = -r_3$, $k = 2$, $t_1 = 2$, $t_2 - 1$, equation (13) is the result. The matrix has two dominant elementary divisors, one quadratic and one linear. Equation (14) evolves when r_1 is real, $r_2 = \overline{r}_3$, $k = 3$, $t_1 = t_2 = t_3 = 1$. The matrix in this case has three linear dominant elementary divisors.

Equations (12) and (13) are linear in r_1, r_1^2, and r_1^3. In equation (14) the ratio of the factors $r_1(2a + r_1)$ and $(2a + r_1)$ gives an approximation to r_1.

After obtaining a solution to the accuracy desired, the results should be checked with the vector equation.

By extrapolation from the double and triple root cases just described, it is obvious that the power method can be extended to determine dominant roots of any multiplicity. Thus the dominant roots of a matrix could be determined by applying the power method successively for 1, 2, ..., n roots until convergence is obtained. However, such a method would be both tedious and highly impractical for matrices of large dimension.

Computer Programs

A program for the IBM 1410 computer executing the power method for a single dominant root was already available when this research was undertaken. Six other computer programs were written to permit determination of each of the three separate cases for two or three equally dominant roots. This required only minor changes in the basic program and the addition of the algebra necessary for the solution of a system of equations in each particular case. The method of generating the normalizing factors remains the same for all seven programs. The only difference in the programs is in the determination of the dominant roots.

Incorporated in the program for two dominant roots, $r_1 = -r_2$, are formulas to give corresponding characteristic vectors for both roots. These formulas are:

$$P_1 = r_1Y_k + s_{k+1}Y_{k+1} = r_1Y_k + AY_k, \text{ corresponding to } r_1, \text{ and}$$

$$P_2 = r_2Y_k - s_{k+1}Y_{k+1} = r_1Y_k - AY_k, \text{ corresponding to } r_2.$$

When the power method has converged for the roots $r_1 = -r_2$, vectors of alternate index will be equal: $Y_k = Y_{k+2}$. The product of two normalizing factors equals the square of the root or the negative of the product of the two roots: $s_{k+1}s_{k+2} = r_1^2 = -r_1r_2$. Using this information and the definition, $s_iY_i = AY_{i-1}$, it can be verified that these are characteristic vectors:

$$\begin{aligned} AP_1 &= A(r_1Y_k + s_{k+1}Y_{k+1}) = r_1AY_k + s_{k+1}AY_{k+1} \\ &= r_1s_{k+1}Y_{k+1} + s_{k+1}s_{k+2}Y_{k+2} = r_1s_{k+1}Y_{k+1} + r_1^2Y_k \\ &= r_1(r_1Y_k + s_{k+1}Y_{k+1}) = r_1P_1; \end{aligned}$$

$$\begin{aligned} AP_2 &= A(r_1Y_k - s_{k+1}Y_{k+1}) = r_1AY_k - s_{k+1}AY_{k+1} \\ &= r_1s_{k+1}Y_{k+1} - s_{k+1}s_{k+2}Y_{k+2} = -r_2s_{k+1}Y_{k+1} + r_1r_2Y_k \\ &= r_2(r_1Y_k - s_{k+1}Y_{k+1}) = r_2P_2. \end{aligned}$$

Similarly, since normalizing factors of alternate index will be equal, it can be shown that the formulas:

$$P_1' = r_1Y_{k+1} + s_kY_k = r_1Y_{k+1} + AY_{k-1}$$

$$P_2' = r_1Y_{k+1} - s_kY_k = r_1Y_{k+1} - AY_{k-1}$$

will also give characteristic vectors.

Matrices (10,10,10) and (10,10,10,10) were tested in the program written to determine two equal dominant roots. The value computer for the root in (10,10,10) is very close to the value of the normalizing factor for (10,10) at the same iteration. The values computed for the root in (10,10,10,10) are close to the values of the normalizing factors for (10,10,10). However, when matrix (10,10,10,10) was tested in the program for three equal dominant roots, no similar correlation was noted.

When matrices (10,10,10), (-10,-10,-10) and (10,-10,-10) were first tested in the programs for finding three equally dominant roots, the process did not converge. When the required degree of accuracy was reduced from ±.00005 to ±.005, convergence was obtained in all three cases after 12 or 13 iterations. With the accuracy level at ±.00005, matrix (10,10,-10) converged after 41 iterations; with the lower accuracy requirement, it converged in 12 iterations. The degree of accuracy obtainable in the programs can be increased by increasing the number of significant digits carried by the computer. For these programs, provision was not made to increase the eight significant digits normally handled by the 1410.

```
MON$$     JOB  FIND DOMINANT CHAR. ROOT AND CORRESP. VECTOR

      DIMENSIONA(10,10),Y(10),Z(10),COMP(10)
   14 FORMAT(1H1,39X,53HDOMINANT CHARACTERISTIC ROOT AND CORRESPONDING V
     1ECTOR)
    1 FORMAT(2I5)
    2 FORMAT(8F10.3)
   17 FORMAT(10F8.6)
    3 FORMAT(1HL,62X,8HMATRIX A)
    4 FORMAT(1HK,10F13.3)
   12 FORMAT(1HL,62X,9HINITIAL Y)
   13 FORMAT(1HK,60X,F10.6)
    5 FORMAT(1HL,37X,1HN,5X,7HAY(N-1),7X,4HY(N),10X,1HS,8X,11HS(N)-S(N-1
     1))
    6 FORMAT(//34X,I5,4F13.6)
    7 FORMAT(39X,2F13.6)
    8 FORMAT(//1H ,12HRESULT AFTER,I3,11H ITERATIONS)
    9 FORMAT(1HK,10X,5HR(1)=,F13.6,5X,5HP(1)=,F13.6)
   10 FORMAT(39X,F13.6)
   11 FORMAT(1HK,22HNONCONVERGENCE ON S(1))
C     READ IN N, THE DIMENSION OF THE SQUARE MATRIX A, AND M, THE
C     MAXIMUM NUMBER OF ITERATIONS.
  175 READ(1,1)N,M
C     THE PROGRAM TERMINATES WHEN THE LAST CARD, WHICH MUST BE BLANK, IS
C     READ.
      IF(N.EQ.0)GOTO150
      WRITE(3,14)
C     READ IN THE ELEMENTS OF THE MATRIX A, ROW-WISE.
      DO15I=1,N
   15 READ(1,2)(A(I,J),J=1,N)
C     READ IN THE INITIAL Y VECTOR
      READ(1,17)(Y(I),I=1,N)
      STORE=0.
      WRITE(3,3)
C     PRINT OUT THE MATRIX A.
      DO30I=1,N
   30 WRITE(3,4)(A(I,J),J=1,N)
      WRITE(3,12)
C     PRINT OUT INITIAL Y.
      DO31I=1,N
   31 WRITE(3,13)Y(I)
      WRITE(3,5)
      DO100KK=1,M
      DO40I=1,N
      Z(I)=0.
C     COMPUTE AY AND PLACE THE RESULT IN Z.
      DO40K=1,N
   40 Z(I)=Z(I)+A(I,K)*Y(K)
C     IF FIRST ITERATION, FIX THE ELEMENT OF LARGEST ABS. VALUE IN Z, S.
      IF(KK.NE.1)GOTO27
      S=ABS(Z(1))
      IT=1
      DO50I=2,N
      IF(S.LT.ABS(Z(I)))GOTO160
   50 CONTINUE
   27 S=Z(IT)
C     COMPUTE Y=Z/S FOR ALL ELEMENTS OF Y AND Z.
      DO60I=1,N
   60 Y(I)=Z(I)/S
      IF(KK.EQ.1)GOTO70
      DIFF=S-STORE
      GOTO80
  160 S=ABS(Z(I))
      IT=I
      GOTO50
   70 DIFF=0.
C     PRINT OUT THE ITERATION NUMBER, N, AY(N-1)=Z, Y(N), S, AND THE
C     DIFFERENCE BETWEEN THE VALUES OF THE CHAR. ROOT,S, FROM THIS
```

```
C       ITERATION AND THE PREVIOUS ITERATION,DIFF.
   80 WRITE(3,6)KK,Z(1),Y(1),S,DIFF
      WRITE(3,7)(Z(I),Y(I),I=2,N)
      IF(KK.EQ.1)GOTO310
C     IF THE ABSOLUTE VALUE OF DIFF IS LESS THAN .00005 CHECK THE CHAR.
C     VECTOR.
      IF(ABS(DIFF).GE..00005)GOTO310
C     IF THE DIFFERENCE BETWEEN THE ABSOLUTE VALUES OF ALL THE
C     CORRESPONDING ELEMENTS OF THE CHAR. VECTOR,Y, FROM THIS ITERATION
C     AND THE PREVIOUS ITERATION,COMP, IS NOT LESS THAN .00005 CONTINUE
C     THE ITERATIVE PROCEDURE.
      DO320I=1,N
  320 IF(ABS(Y(I)-COMP(I)).GE..00005)GOTO310
      GOTO200
C     STORE THE CHAR. VECTOR IN COMP.
  310 DO330I=1,N
  330 COMP(I)=Y(I)
C     STORE THE CHAR. ROOT IN STORE.
  100 STORE=S
C     M HAS BEEN EXCEEDED WITH NONCONVERGENCE ON S(1).
      WRITE(3,11)
      GOTO175
C     PRINT OUT RESULTS, ITERATION NO., CHAR. ROOT AND VECTOR.
  200 WRITE(3,8)KK
  125 WRITE(3,9)S,Y(1)
      WRITE(3,10)(Y(I),I=2,N)
      GOTO175
  150 STOP
      END
```

```
MON$$     JOB   TWO EQUAL CHARACTERISTIC ROOTS

      DIMENSIONA(10,10),Y(10),COMP1(10),COMP2(10),Z(10)
    1 FORMAT(2I5)
    2 FORMAT(8F10.3)
    3 FORMAT(10F8.6)
   11 FORMAT(1H1,48X,24HTWO EQUAL DOMINANT ROOTS)
   12 FORMAT(1HL,62X,8HMATRIX A)
   13 FORMAT(1HK,10F13.3)
   14 FORMAT(1HL,62X,9HINITIAL Y)
   15 FORMAT(1HK,60X,F10.6)
   16 FORMAT(1HL,30X,1HN,5X,7HAY(N-1),7X,4HY(N),10X,1HS,12X,1HR,8X,11HR(
     1N)-R(N-1))
   17 FORMAT(//27X,I5,5F13.6)
   18 FORMAT(32X,2F13.6)
   19 FORMAT(1HK,14HNONCONVERGENCE)
   20 FORMAT(//1H ,12HRESULT AFTER,I5,11H ITERATIONS)
   21 FORMAT(1HK,10X,5HR(1)=,F13.6,5X,5HP(1)=,F13.6)
   22 FORMAT(39X,F13.6)
C     READ IN N, THE DIMENSION OF THE SQUARE MATRIX A, AND M, THE
C     MAXIMUM NUMBER OF ITERATIONS
  175 READ(1,1)N,M
C     THE PROGRAM TERMINATES WHEN THE LAST CARD, WHICH MUST BE BLANK,
C     IS READ.
      IF(N.EQ.0)GOTO150
      WRITE(3,11)
C     READ IN THE ELEMENTS OF THE MATRIX A, ROW-WISE
      DO25I=1,N
   25 READ(1,2)(A(I,J),J=1,N)
C     READ IN THE INITIAL Y VECTOR
      READ(1,3)(Y(I),I=1,N)
      STORE=0
      DIFF=0
      PDIFF=0
      PRIOR=0
      VAR=0
C     PRINT OUT THE MATRIX A
      WRITE(3,12)
      DO30I=1,N
   30 WRITE(3,13)(A(I,J),J=1,N)
C     PRINT OUT THE INITIAL Y VECTOR
      WRITE(3,14)
      DO31I=1,N
   31 WRITE(3,15)Y(I)
      WRITE(3,16)
      DO100KK=1,M
      DO40I=1,N
      Z(I)=0
C     COMPUTE AY AND PLACE THE RESULT IN Z.
      DO40K=1,N
   40 Z(I)=Z(I)+A(I,K)*Y(K)
C     IF FIRST ITERATION, FIX THE ELEMENT OF LARGEST ABS. VALUE IN Z, S.
      IF(KK.NE.1)GOTO27
      S=ABS(Z(1))
      IT=1
      DO50I=2,N
      IF(S.LT.ABS(Z(I)))GOTO160
   50 CONTINUE
   27 S=Z(IT)
C     COMPUTE Y=Z/S FOR ALL ELEMENTS OF Y AND Z.
      DO60I=1,N
   60 Y(I)=Z(I)/S
C     COMPUTE THE DOMINANT ROOT BY SOLVING A PAIR OF LINEAR EQUATIONS.
      IF(KK.LT.3)GOTO80
      DIFF=S-STORE
      ROOT=.5*STORE*(DIFF+PDIFF)/PDIFF
      IF(KK.EQ.3)GOTO80
      VAR=ROOT-PRIOR
```

```
      GOTO80
  160 S=ABS(Z(I))
      IT=I
      GOTO50
C     PRINT OUT THE ITERATION NUMBER N, AY(N-1)=Z, Y(N), S, ROOT, AND
C     THE DIFFERENCE BETWEEN THIS AND THE PRIOR APPROX. TO THE ROOT, VAR
   80 WRITE(3,17)KK,Z(1),Y(1),S,ROOT,VAR
      WRITE(3,18)(Z(I),Y(I),I=2,N)
      IF(KK.LT.4)GOTO310
      IF(ABS(VAR).GE..00005)GOTO310
C     IF THE ROOT HAS BEEN APPROXIMATED TO THE DESIRED DEGREE OF
C     ACCURACY (.00005), CHECK THE VECTOR EQUATION.
      DO475I=1,N
  475 IF(ABS(S*STORE*Y(I)-2.*ROOT*STORE*COMP2(I)+(ROOT**2)*COMP1(I)).GE.
     1.00005)GOTO310
      GOTO200
C     SHIFT THE STORED VECTORS, DISCARDING THE VECTOR OF LOWEST INDEX,
C     AND STORE THE CURRENT VECTOR IN COMP2
  310 DO510I=1,N
      COMP1(I)=COMP2(I)
  510 COMP2(I)=Y(I)
C     STORE THE DIFFERENCE OF THE NORMALIZING FACTORS IN PDIFF, THE
C     NORMALIZING FACTOR, S, IN STORE, AND THE ROOT IN PRIOR.
      PDIFF=DIFF
      STORE=S
  100 PRIOR=ROOT
C     M HAS BEEN EXCEEDED WITH NONCONVERGENCE.
      WRITE(3,19)
      GOTO175
C     PRINT OUT THE RESULTS, ITERATION NUMBER, CHAR. ROOT, AND VECTOR.
  200 WRITE(3,20)KK
      WRITE(3,21)ROOT,Y(1)
      WRITE(3,22)(Y(I),I=2,N)
      GOTO175
  150 STOP
      END
```

```
      MON$$     JOB  TWO EQUAL DOM. ROOTS, NEGATIVES

      DIMENSIONA(10,10),Y(10),Z(10),COMP(10),P1(10),P2(10),CHECK(10),SHI
     1FT(10)
    1 FORMAT(2I5)
    2 FORMAT(8F10.3)
    3 FORMAT(10F8.6)
   11 FORMAT(1H1,38X,55HTWO DOMINANT ROOTS, EQUAL ABSOLUTE VALUE, OPPOSI
     1TE SIGN)
   12 FORMAT(1HL,62X,8HMATRIX A)
   13 FORMAT(1HK,10F13.3)
   14 FORMAT(1HL,62X,9HINITIAL Y)
   15 FORMAT(1HK,60X,F10.6)
   16 FORMAT(1HL,30X,1HN,5X,7HAY(N-1),7X,4HY(N),10X,1HS,12X,1HR,8X,11HR(
     1N)-R(N-1))
   17 FORMAT(//27X,I5,5F13.6)
   18 FORMAT(32X,2F13.6)
   19 FORMAT(1HK,39HM HAS BEEN EXCEEDED WITH NONCONVERGENCE)
   20 FORMAT(//1H ,12HRESULT AFTER,I5,11H ITERATIONS)
   21 FORMAT(1HK,10X,5HR(1)=,F13.6,5X,5HP(1)=,F13.6,5X,6H-R(1)=,F13.6,5X
     1,5HP(2)=,F13.6)
   22 FORMAT(39X,F13.6,34X,F13.6)
C     READ IN N, THE DIMENSION OF THE SQUARE MATRIX A, AND M, THE
C     MAXIMUM NUMBER OF ITERATIONS.
  175 READ(1,1)N,M
C     THE PROGRAM TERMINATES WHEN THE LAST CARD WHICH MUST BE BLANK,
C     IS READ.
      IF(N.EQ.0)GOTO150
      WRITE(3,11)
C     READ IN THE ELEMENTS OF THE MATRIX A, ROW-WISE
      DO25I=1,N
   25 READ(1,2)(A(I,J),J=1,N)
C     READ IN THE INITIAL Y VECTOR
      READ(1,3)(Y(I),I=1,N)
      ROOT=0
      STORE=0
      SAVE=0
C     PRINT OUT THE MATRIX A
      WRITE(3,12)
      DO30I=1,N
   30 WRITE(3,13)(A(I,J),J=1,N)
C     PRINT OUT THE INITIAL Y VECTOR
      WRITE(3,14)
      DO31I=1,N
   31 WRITE(3,15)Y(I)
      WRITE(3,16)
      DO100KK=1,M
      DO40I=1,N
      Z(I)=0
C     COMPUTE AY AND PLACE THE RESULT IN Z.
      DO40K=1,N
   40 Z(I)=Z(I)+A(I,K)*Y(K)
C     IF FIRST ITERATION, FIX THE ELEMENT OF LARGEST ABS. VALUE IN Z, S.
      IF(KK.NE.1)GOTO27
      S=ABS(Z(1))
      IT=1
      DO50I=2,N
      IF(S.LT.ABS(Z(I)))GOTO160
   50 CONTINUE
   27 S=Z(IT)
C     COMPUTE Y=Z/S FOR ALL ELEMENTS OF Y AND Z.
      DO60I=1,N
   60 Y(I)=Z(I)/S
C     COMPUTE THE SQUARE ROOT OF THE PRODUCT OF THE CHAR. ROOT FROM THIS
C     ITERATION AND THE PREVIONS ITERATION
      IF(KK.EQ.1)GOTO70
      ROOT=SQRT(STORE*S)
      DIFF=ROOT-SAVE
```

```
      DO350I=1,N
      P1(I)=ROOT*COMP(I)+Z(I)
  350 P2(I)=ROOT*COMP(I)-Z(I)
      GOTO80
  160 S=ABS(Z(I))
      IT=1
      GOTO50
   70 DIFF=0
C     PRINT OUT THE ITERATION NUMBER N, AY(N-1)=Z, Y(N), S,
C     SQRT(S(N)*S(N-1))=ROOT,AND THE DIFFERENCE BETWEEN THIS AND THE
C     PRIOR APPROXIMATION TO THE ROOT, DIFF.
   80 WRITE(3,17)KK,Z(1),Y(1),S,ROOT,DIFF
      WRITE(3,18)(Z(I),Y(I),I=2,N)
      IF(KK.LE.3)GOTO310
C     IF TWO SUCCESSIVE APPROXIMATIONS FOR ROOT ARE WITHIN THE DESIRED
C     RANGE OF ACCURACY, (.00005), CHECK THE CHAR. VECTOR.  IF IT IS NOT
C     WITHIN THE DESIRED ACCURACY RANGE, (.00005), CONTINUE THE
C     ITERATIVE PROCEDURE
      IF(ABS(DIFF).GE..00005)GOTO310
      IF(ABS(VAR).GE..00005)GOTO310
      DO320I=1,N
  320 IF(ABS(P1(I)-CHECK(I)).GE..00005)GOTO310
      GOTO200
C     STORE Y(N) AND THE APPROX. TO THE CHAR. VECTOR.
  310 DO510I=1,N
      COMP(I)=Y(I)
      CHECK(I)=SHIFT(I)
  510 SHIFT(I)=P1(I)
C     STORE THE CHAR. ROOT, S, IN STORE, THE GEOM. MEAN, R, IN SAVE, AND
C     DIFF IN VAR.
      VAR=DIFF
      STORE=S
  100 SAVE=ROOT
C     M HAS BEEN EXCEEDED WITH NONCONVERGENCE
      WRITE(3,19)
      GOTO175
C     PRINT OUT RESULTS, ITERATION NUMBER, CHAR. ROOTS, AND CHAR. VECTOR
  200 SECOND=-ROOT
      WRITE(3,20)KK
      WRITE(3,21)ROOT,P1(1),SECOND,P2(1)
      WRITE(3,22)(P1(I),P2(I),I=2,N)
      GOTO175
  150 STOP
      END
```

```
MON$$      JOB   TWO DOMINANT ROOTS  COMPLEX CONJUGATES

      DIMENSIONA(10,10),Y(10),Z(10),COMP1(10),COMP2(10)
    1 FORMAT(2I5)
    2 FORMAT(8F10.3)
    3 FORMAT(10F8.6)
   11 FORMAT(1H1,48X,38HTWO DOMINANT ROOTS  COMPLEX CONJUGATES)
   12 FORMAT(1HL,62X,8HMATRIX A)
   13 FORMAT(1HK,10F13.3)
   14 FORMAT(1HL,62X,9HINITIAL Y)
   15 FORMAT(1HK,60X,F10.6)
   16 FORMAT(1HL,24X,1HN,5X,7HAY(N-1),7X,4HY(N),10X,1HS,10X,4HREAL,8X,4H
     1IMAG,6X,11HC(N)-C(N-1))
   17 FORMAT(//21X,I5,6F13.6)
   18 FORMAT(26X,2F13.6)
   19 FORMAT(1HK,14HNONCONVERGENCE)
   20 FORMAT(//1H ,12HRESULT AFTER,I5,11H ITERATIONS)
   21 FORMAT(1HK,10X,10HREAL COOR=,F13.6,5X,10HIMAG COOR=,F13.6,5X,5HP(1
     1)=,F13.6)
   22 FORMAT(72X,F13.6)
C     READ IN N, THE DIMENSION OF THE SQUARE MATRIX A, AND M, THE
C     MAXIMUM NUMBER OF ITERATIONS.
  175 READ(1,1)N,M
C     THE PROGRAM TERMINATES WHEN THE LAST CARD, WHICH MUST BE BLANK,
C     IS READ.
      IF(N.EQ.0)GOTO150
      WRITE(3,11)
C     READ IN THE ELEMENTS OF THE MATRIX A, ROW-WISE.
      DO25I=1,N
   25 READ(1,2)(A(I,J),J=1,N)
C     READ IN THE INITIAL VECTOR, Y.
      READ(1,3)(Y(I),I=1,N)
      STORE=0
      DIFF=0
      PDIFF=0
      PRIOR=0
      VAR=0
      PREV=0
      REAL=0
      XIMAG=0
C     PRINT OUT THE MATRIX A.
      WRITE(3,12)
      DO30I=1,N
   30 WRITE(3,13)(A(I,J),J=1,N)
C     PRINT OUT THE INITIAL Y VECTOR.
      WRITE(3,14)
      DO31I=1,N
   31 WRITE(3,15)(Y(I))
      WRITE(3,16)
      DO100KK=1,M
      DO40I=1,N
      Z(I)=0
C     COMPUTE AY AND PLACE THE RESULT IN Z.
      DO40K=1,N
   40 Z(I)=Z(I)+A(I,K)*Y(K)
C     IF FIRST ITERATION, FIX THE ELEMENT OF LARGEST ABS. VALUE IN Z, S.
      IF(KK.NE.1)GOTO27
      S=ABS(Z(1))
      IT=1
      DO50I=2,N
      IF(S.LT.ABS(Z(I)))GOTO160
   50 CONTINUE
   27 S=Z(IT)
C     COMPUTE Y=Z/S FOR ALL ELEMENTS OF Y AND Z.
      DO60I=1,N
   60 Y(I)=Z(I)/S
C     COMPUTE THE DOMINANT ROOT BY SOLVING A PAIR OF LINEAR EQUATIONS.
      IF(KK.LT.3)GOTO80
```

```
      DIFF=S-STORE
      REAL=.5*STORE*(DIFF+PDIFF)/PDIFF
      XIMAG=SQRT(2.*REAL*STORE-S*STORE-REAL**2)
      IF(KK.EQ.3)GOTO80
      VAR=REAL-PRIOR
      GOTO80
  160 S=ABS(Z(I))
      IT=I
      GOTO50
C     PRINT OUT THE ITERATION NUMBER N, AY(N-1)=Z, Y(N), S, REAL, XIMAG,
C     AND THE DIFFERENCE BETWEEN THIS AND THE PRIOR APPROXIMATION TO THE
C     REAL PART OF THE COMPLEX ROOT, VAR.
   80 WRITE(3,17)KK,Z(1),Y(1),S,REAL,XIMAG,VAR
      WRITE(3,18)(Z(I),Y(I),I=2,N)
      IF(KK.LT.4)GOTO310
C     WHEN THE REAL AND IMAG COEFF OF THE COMPLEX ROOT HAVE BEEN
C     APPROXIMATED TO THE DESIRED DEGREE OF ACCURACY (.00005), CHECK
C     THE VECTOR EQUATION.
      IF(ABS(VAR).GE..00005)GOTO310
      IF(ABS(XIMAG-PREV).GE..00005)GOTO310
      DO475I=1,N
  475 IF(ABS(S*STORE*Y(I)-2.*REAL*STORE*COMP2(I)+COMP1(I)*(REAL**2+XIMAG
     1**2)).GE..00005)GOTO80
      GOTO200
C     SHIFT THE STORED VECTORS, DISCARDING THE VECTOR OF LOWEST INDEX,
C     AND STORE THE CURRENT VECTOR IN COMP2.
  310 DO510I=1,N
      COMP1(I)=COMP2(I)
  510 COMP2(I)=Y(I)
C     STORE THE DIFFERENCE OF THE NORMALIZING FACTORS IN PDIFF, THE
C     NORMALIZING FACTOR, S, IN STORE, THE REAL PART OF THE COMPLEX ROOT
C     IN PRIOR, AND THE IMAGINARY PART IN PREV.
      PDIFF=DIFF
      STORE=S
      PRIOR=REAL
  100 PREV=XIMAG
C     M HAS BEEN EXCEEDED WITH NONCONVERGENCE.
      WRITE(3,19)
      GOTO175
C     PRINT OUT THE RESULTS, ITERATION NO., CHAR. ROOTS, AND VECTOR.
  200 WRITE(3,20)KK
      WRITE(3,21)REAL,XIMAG,Y(1)
      WRITE(3,22)(Y(I),I=2,N)
      GOTO175
  150 STOP
      END
```

```
MON$$     JOB   THREE EQUAL DOMINANT ROOTS

      DIMENSIONA(10,10),Y(10),VECT1(10),VECT2(10),VECT3(10),Z(10)
    1 FORMAT(2I5)
    2 FORMAT(8F10.3)
    3 FORMAT(10F8.6)
   11 FORMAT(1H1,47X,26HTHREE EQUAL DOMINANT ROOTS)
   12 FORMAT(1HL,62X,8HMATRIX A)
   13 FORMAT(1HK,10F13.3)
   14 FORMAT(1HL,62X,9HINITIAL Y)
   15 FORMAT(1HK,60X,F10.6)
   16 FORMAT(1HL,24X,1HN,5X,7HAY(N-1),7X,4HY(N),10X,1HS,8X,11HS(N)-S(N-1
     1),8X,1HR,8X,11HR(N)-R(N-1))
   17 FORMAT(//21X,I5,6F13.6)
   18 FORMAT(26X,2F13.6)
   19 FORMAT(1HK,14HNONCONVERGENCE)
   20 FORMAT(//1H ,12HRESULT AFTER,I5,11H ITERATIONS)
   21 FORMAT(1HK,10X,5HR(1)=,F13.6)
C     READ IN N, THE DIMENSION OF THE SQUARE MATRIX A, AND M, THE
C     MAXIMUM NUMBER OF ITERATIONS.
  175 READ(1,1)N,M
C     THE PROGRAM TERMINATES WHEN THE LAST CARD, WHICH MUST BE BLANK, IS
C     READ.
      IF(N.EQ.0)GOTO150
      WRITE(3,11)
C     READ IN THE ELEMENTS OF THE MATRIX A, ROW-WISE.
      DO25I=1,N
   25 READ(1,2)(A(I,J),J=1,N)
C     READ IN THE INITIAL Y VECTOR.
      READ(1,3)(Y(I),I=1,N)
      ES2=0
      ES3=0
      ES4=0
      DIF2=0
      DIF3=0
      DIF4=0
      DIFF=0
      ROOT=0
      PRIOR=0
      VAR=0
C     PRINT OUT THE MATRIX A.
      WRITE(3,12)
      DO30I=1,N
   30 WRITE(3,13)(A(I,J),J=1,N)
C     PRINT OUT THE INITIAL Y VECTOR.
      WRITE(3,14)
      DO31I=1,N
   31 WRITE(3,15)Y(I)
      WRITE(3,16)
      DO100KK=1,M
      DO40I=1,N
      Z(I)=0
C     COMPUTE AY AND PLACE THE RESULT IN Z.
      DO40K=1,N
   40 Z(I)=Z(I)+A(I,K)*Y(K)
C     IF FIRST ITERATION, FIX THE ELEMENT OF LARGEST ABS. VALUE IN Z, S.
      IF(KK.NE.1)GOTO27
      S=ABS(Z(I))
      IT=1
      DO50I=2,N
      IF(S.LT.ABS(Z(I)))GOTO160
   50 CONTINUE
   27 S=Z(IT)
C     COMPUTE Y=Z/S FOR ALL ELEMENTS OF Y AND Z.
      DO60I=1,N
   60 Y(I)=Z(I)/S
C     COMPUTE THE DOMINANT ROOT BY SOLVING A SYSTEM OF THREE LINEAR
C     EQUATIONS.
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      IF(KK.LT.5)GOTO80
      DIFF=S-ES4
      ROOT=(ES3*(ES2*(DIF4+DIF3+DIF2)*(DIF3+DIF2)-DIF2*(ES4*(DIFF+DIF4)+
     1ES3*(DIF4+DIF3)+ES2*(DIF3+DIF2))))/(3.0*(ES2*(DIF3+DIF2)**2-DIF2*(
     1ES3*(DIF4+DIF3)+ES2*(DIF3+DIF2))))
      IF(KK.EQ.5)GOTO80
      VAR=ROOT-PRIOR
      GOTO80
  160 S=ABS(Z(I))
      IT=I
      GOTO50
C     PRINT OUT THE ITERATION NO. N, AY(N-1)=Z, Y(N), S, S(N)-S(N-1),
C     ROOT, AND THE DIFFERENCE BETWEEN THIS AND THE PRIOR APPROXIMATION
C     TO THE ROOT, VAR.
   80 WRITE(3,17)KK,Z(1),Y(1),S,DIFF,ROOT,VAR
      WRITE(3,18)(Z(I),Y(I),I=2,N)
      IF(KK.LT.6)GOTO310
      IF(ABS(VAR).GE..005)GOTO310
C     IF THE ROOT HAS BEEN APPROXIMATED TO THE DESIRED DEGREE OF
C     ACCURACY, CHECK THE VECTOR EQUATION.
      DO475I=1,N
  475 IF(ABS(S*ES4*ES3*Y(I)-3.0*ROOT*ES4*ES3*VECT3(I)+3.0*(ROOT**2)*ES3*
     1VECT2(I)-(ROOT**3)*VECT1(I)).GE..005)GOTO310
      GOTO200
C     SHIFT THE STORED VECTORS, DISCARDING THE VECTOR OF LOWEST INDEX,
C     AND STORE THE CURRENT VECTOR IN VECT3.
  310 DO510I=1,N
      VECT1(I)=VECT2(I)
      VECT2(I)=VECT3(I)
  510 VECT3(I)=Y(I)
C     SHIFT THE STORED NORMALIZING FACTORS AND DIFFERENCES, DISCARDING
C     THOSE OF LOWEST INDEX, AND STORE THE CURRENT FACTOR AND DIFFERENCE
C     IN ES4 AND DIF4.  SOTRE THE ROOT IN PRIOR.
      ES2=ES3
      ES3=ES4
      ES4=S
      DIF2=DIF3
      DIF3=DIF4
      DIF4=DIFF
  100 PRIOR=ROOT
C     M HAS BEEN EXCEEDED WITH NONCONVERGENCE.
      WRITE(3,19)
      GOTO175
C     PRINT OUT THE RESULTS, ITERATION NUMBER AND ROOTS.
  200 WRITE(3,20)KK
      WRITE(3,21)ROOT
      GOTO175
  150 STOP
      END
```

```
MON$$     JOB  THREE REAL DOM ROOTS, MIXED SIGNS

      DIMENSIONA(10,10),Y(10),VECT1(10),VECT2(10),VECT3(10),Z(10)
    1 FORMAT(2I5)
    2 FORMAT(8F10.3)
    3 FORMAT(10F8.6)
   11 FORMAT(1H1,30X,62HTHREE DOMINANT ROOTS, EQUAL IN ABSOLUTE VALUE, S
     1IGN DIFFERENCE)
   12 FORMAT(1HL,62X,8HMATRIX A)
   13 FORMAT(1HK,10F13.3)
   14 FORMAT(1HL,62X,9HINITIAL Y)
   15 FORMAT(1HK,60X,F10.6)
   16 FORMAT(1HL,24X,1HN,5X,7HAY(N-1),7X,4HY(N),10X,1HS,8X,11HS(N)-S(N-1
     1),8X,1HR,8X,11HR(N)-R(N-1))
   17 FORMAT(//21X,I5,6F13.6)
   18 FORMAT(26X,2F13.6)
   19 FORMAT(1HK,14HNONCONVERGENCE)
   20 FORMAT(//1H ,12HRESULT AFTER,I5,11H ITERATIONS)
   21 FORMAT(1HK,10X,10HR(1),R(2)=,F13.6,5X,5HR(3)=,F13.6)
C     READ IN N, THE DIMENSION OF THE SQUARE MATRIX A, AND M, THE
C     MAXIMUM NUMBER OF ITERATIONS.
  175 READ(1,1)N,M
C     THE PROGRAM TERMINATES WHEN THE LAST CARD, WHICH MUST BE BLANK, IS
C     READ.
      IF(N.EQ.0)GOTO150
      WRITE(3,11)
C     READ IN THE ELEMENTS OF THE MATRIX A, ROW-WISE.
      DO25I=1,N
   25 READ(1,2)(A(I,J),J=1,N)
C     READ IN THE INITIAL Y VECTOR.
      READ(1,3)(Y(I),I=1,N)
      ES2=0
      ES3=0
      ES4=0
      DIF2=0
      DIF3=0
      DIF4=0
      DIFF=0
      ROOT=0
      PRIOR=0
      VAR=0
C     PRINT OUT THE MATRIX A.
      WRITE(3,12)
      DO30I=1,N
   30 WRITE(3,13)(A(I,J),J=1,N)
C     PRINT OUT THE INITIAL Y VECTOR.
      WRITE(3,14)
      DO31I=1,N
   31 WRITE(3,15)Y(I)
      WRITE(3,16)
      DO100KK=1,M
      DO40I=1,N
      Z(I)=0
C     COMPUTE AY AND PLACE THE RESULT IN Z.
      DO40K=1,N
   40 Z(I)=Z(I)+A(I,K)*Y(K)
C     IF FIRST ITERATION, FIX THE ELEMENT OF LARGEST ABS. VALUE IN Z, S.
      IF(KK.NE.1)GOTO27
      S=ABS(Z(I))
      IT=1
      DO50I=2,N
      IF(S.LT.ABS(Z(I)))GOTO160
   50 CONTINUE
   27 S=Z(IT)
C     COMPUTE Y=Z/S FOR ALL ELEMENTS OF Y AND Z.
      DO60I=1,N
   60 Y(I)=Z(I)/S
C     COMPUTE THE DOMINANT ROOT BY SOLVING A SYSTEM OF THREE LINEAR
```

```
C     EQUATIONS.
      IF(KK.LT.5)GOTO80
      DIFF=S-ES4
      ROOT=(ES3*(ES2*(DIF4+DIF3+DIF2)*(DIF3+DIF2)-DIF2*(ES4*(DIFF+DIF4)+
     1ES3*(DIF4+DIF3)+ES2*(DIF3+DIF2))))/(ES2*(DIF3+DIF2)**2-DIF2*(ES3*(
     1DIF4+DIF3)+ES2*(DIF3+DIF2)))
      IF(KK.EQ.5)GOTO80
      VAR=ROOT-PRIOR
      GOTO80
  160 S=ABS(Z(I))
      IT=I
      GOTO50
C     PRINT OUT THE ITERATION NO. N, AY(N-1)=Z, Y(N), S, S(N)-S(N-1),
C     ROOT, AND THE DIFFERENCE BETWEEN THIS AND THE PRIOR APPROXIMATION
C     TO THE ROOT, VAR.
   80 WRITE(3,17)KK,Z(1),Y(1),S,DIFF,ROOT,VAR
      WRITE(3,18)(Z(I),Y(I),I=2,N)
      IF(KK.LT.6)GOTO310
      IF(ABS(VAR).GE..005)GOTO310
C     IF THE ROOT HAS BEEN APPROXIMATED TO THE DESIRED DEGREE OF
C     ACCURACY, CHECK THE VECTOR EQUATION.
      DO475I=1,N
  475 IF(ABS(S*ES4*ES3*Y(I)-ROOT*ES4*ES3*VECT3(I)-(ROOT**2)*ES3*VECT2(I)
     1+(ROOT**3)*VECT1(I)).GE..005)GOTO310
      GOTO200
C     SHIFT THE STORED VECTORS, DISCARDING THE VECTOR OF LOWEST INDEX,
C     AND STORE THE CURRENT VECTOR IN VECT3.
  310 DO510I=1,N
      VECT1(I)=VECT2(I)
      VECT2(I)=VECT3(I)
  510 VECT3(I)=Y(I)
C     SHIFT THE STORED NORMALIZING FACTORS AND DIFFERENCES, DISCARDING
C     THOSE OF LOWEST INDEX, AND STORE THE CURRENT FACTOR AND DIFFERENCE
C     IN ES4 AND DIF4.  SOTRE THE ROOT IN PRIOR.
      ES2=ES3
      ES3=ES4
      ES4=S
      DIF2=DIF3
      DIF3=DIF4
      DIF4=DIFF
  100 PRIOR=ROOT
C     M HAS BEEN EXCEEDED WITH NONCONVERGENCE.
      WRITE(3,19)
      GOTO175
C     PRINT OUT THE RESULTS, ITERATION NUMBER AND ROOTS.
  200 OPP=-ROOT
      WRITE(3,20)KK
      WRITE(3,21)ROOT,OPP
      GOTO175
  150 STOP
      END
```

```
      MON$$      JOB  THREE DOM ROOTS, ONE REAL, COMPLEX CONJ PAIR

      DIMENSIONA(10,10),Y(10),VECT1(10),VECT2(10),VECT3(10),Z(10)
    1 FORMAT(2I5)
    2 FORMAT(8F10.3)
    3 FORMAT(10F8.6)
   11 FORMAT(1H1,35X,53HTHREE DOMINANT ROOTS, COMPLEX CONJUGATES AND ONE
     1 REAL)
   12 FORMAT(1HL,62X,8HMATRIX A)
   13 FORMAT(1HK,10F13.3)
   14 FORMAT(1HL,62X,9HINITIAL Y)
   15 FORMAT(1HK,60X,F10.6)
   16 FORMAT(1HL,7X,1HN,5X,7HAY(N-1),7X,4HY(N),10X,1HS,8X,11HS(N)-S(N-1)
     1,7X,1HR,7X,11HR(N)-R(N-1),5X,4HREAL,5X,13HRE(N)-RE(N-1),4X,4HIMAG)
   17 FORMAT(//4X,I5,9F13.6)
   18 FORMAT(9X,2F13.6)
   19 FORMAT(1HK,14HNONCONVERGENCE)
   20 FORMAT(//1H ,12HRESULT AFTER,I5,11H ITERATIONS)
     1=,F13.6)
   21 FORMAT(1HK,5X,5HR(1)=,F13.6,5X,10HREAL COOR=,F13.6,5X,10HIMAG COOR
C     READ IN N, THE DIMENSION OF THE SQUARE MATRIX A, AND M, THE
C     MAXIMUM NUMBER OF ITERATIONS.
  175 READ(1,1)N,M
C     THE PROGRAM TERMINATES WHEN THE LAST CARD, WHICH MUST BE BLANK, IS
C     READ.
      IF(N.EQ.0)GOTO150
      WRITE(3,11)
C     READ IN THE ELEMENTS OF THE MATRIX A, ROW-WISE.
      DO25I=1,N
   25 READ(1,2)(A(I,J),J=1,N)
C     READ IN THE INITIAL Y VECTOR.
      READ(1,3)(Y(I),I=1,N)
      ES2=0
      ES3=0
      ES4=0
      DIF2=0
      DIF3=0
      DIF4=0
      DIFF=0
      ROOT=0
      PRIOR=0
      VAR=0
      PREV=0
      REAL=0
      CHNG=0
      OLD=0
      XIMAG=0
C     PRINT OUT THE MATRIX A.
      WRITE(3,12)
      DO30I=1,N
   30 WRITE(3,13)(A(I,J),J=1,N)
C     PRINT OUT THE INITIAL Y VECTOR.
      WRITE(3,14)
      DO31I=1,N
   31 WRITE(3,15)Y(I)
      WRITE(3,16)
      DO100KK=1,M
      DO40I=1,N
      Z(I)=0
C     COMPUTE AY AND PLACE THE RESULT IN Z.
      DO40K=1,N
   40 Z(I)=Z(I)+A(I,K)*Y(K)
C     IF FIRST ITERATION, FIX THE ELEMENT OF LARGEST ABS. VALUE IN Z, S.
      IF(KK.NE.1)GOTO27
      S=ABS(Z(I))
      IT=1
      DO50I=2,N
      IF(S.LT.ABS(Z(I)))GOTO160
```

```
   50 CONTINUE
   27 S=Z(IT)
C     COMPUTE Y=Z/S FOR ALL ELEMENTS OF Y AND Z.
      DO60I=1,N
   60 Y(I)=Z(I)/S
C     COMPUTE THE DOMINANT ROOT BY SOLVING A SYSTEM OF THREE LINEAR
C     EQUATIONS.
      IF(KK.LT.5)GOTO80
      DIFF=S-ES4
      DUM1=(ES3*(ES2*(DIF4+DIF3+DIF2)*(DIF3+DIF2)-DIF2*(ES4*(DIFF+DIF4)+
     1ES3*(DIF4+DIF3)+ES2*(DIF3+DIF2))))/(ES2*(DIF3+DIF2)**2-DIF2*(ES3*(
     1DIF4+DIF3)+ES2*(DIF3+DIF2)))
      DUM2=(ES3*ES2*((DIF4+DIF3+DIF2)*(ES3*(DIF4+DIF3)+ES2*(DIF3+DIF2))-
     1(DIF3+DIF2)*(ES4*(DIFF+DIF4)+ES3*(DIF4+DIF3)+ES2*(DIF3+DIF2))))/(E
     1S2*(DIF3+DIF2)**2-DIF2*(ES3*(DIF4+DIF3)+ES2*(DIF3+DIF2)))
      ROOT=DUM2/DUM1
      REAL=.5*(DUM1-ROOT)
      XIMAG=SQRT(ROOT**2-REAL**2)
      IF(KK.EQ.5)GOTO80
      VAR=ROOT-PRIOR
      CHNG=REAL-PREV
      GOTO80
  160 S=ABS(Z(I))
      IT=I
      GOTO50
C     PRINT OUT THE ITERATION NO.N, AY(N-1)=Z, Y(N), NORMALIZING FACTOR
C     S, ROOT R, REAL AND IMAG. COEFF. FOR THE COMPLEX ROOTS, AND
C     FIRST DIFFERENCES FOR S, R, REAL.
   80 WRITE(3,17)KK,Z(1),Y(1),S,DIFF,ROOT,VAR,REAL,CHNG,XIMAG
      WRITE(3,18)(Z(I),Y(I),I=2,N)
      IF(KK.LT.6)GOTO310
C     WHEN THE ROOT AND REAL AND IMAG. COEFF. HAVE BEEN APPROXIMATED TO
C     THE DESIRED DEGREE OF ACCURACY (.00005), CHECK THE VECTOR EQUATION
      IF(ABS(VAR).GE..00005)GOTO310
      IF(ABS(CHNG).GE..00005)GOTO310
      IF(ABS(XIMAG-OLD).GE..00005)GOTO310
      DO475I=1,N
  475 IF(ABS(S*ES4*ES3*Y(I)-(2.0*REAL+ROOT)*ES4*ES3*VECT3(I)+(ROOT**2+2.
     10*REAL*ROOT)*ES3*VECT2(I)-(ROOT**3)*VECT1(I)).GE..00005)GOTO310
      GOTO200
C     SHIFT THE STORED VECTORS, DISCARDING THE VECTOR OF LOWEST INDEX,
C     AND STORE THE CURRENT VECTOR IN VECT3.
  310 DO510I=1,N
      VECT1(I)=VECT2(I)
      VECT2(I)=VECT3(I)
  510 VECT3(I)=Y(I)
C     SHIFT THE STORED NORMALIZING FACTORS AND DIFFERENCES, DISCARDING
C     THOSE OF LOWEST INDEX, AND STORE THE CURRENT FACTOR AND DIFFERENCE
C     IN ES4 AND DIF4.  STORE ROOT IN PRIOR, REAL IN PREV, AND XIMAG IN
C     OLD.
      ES2=ES3
      ES3=ES4
      ES4=S
      DIF2=DIF3
      DIF3=DIF4
      DIF4=DIFF
      OLD=XIMAG
      PREV=REAL
  100 PRIOR=ROOT
C     M HAS BEEN EXCEEDED WITH NONCONVERGENCE.
      WRITE(3,19)
      GOTO175
C     PRINT OUT THE RESULTS, ITERATION NUMBER AND ROOTS.
  200 WRITE(3,20)KK
      WRITE(3,21)ROOT,REAL,XIMAG
      GOTO175
  150 STOP
```

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PREDICTION OF THE DOMINANT CHARACTERISTIC ROOTS OF A MATRIX

by

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Determination of the characteristic roots of a matrix is often of fundamental importance in the theory of matrices. The power method is commonly used when the matrix has a single dominant root. This method can also be generalized to determine any number of equally dominant roots. The main deterrent to its use is that the multiplicity of the dominant root must be known in advance. This information is rarely available.

The power method was applied to matrices with predetermined dominant roots and the behavior of the results observed through several iterations. It was the objective of this research to identify those factors which give definite indication of the size, multiplicity and form of the dominant roots.

The power method for a single dominant root is described and restrictions placed upon the matrices tested. The results are limited to companion matrices of maximum dimension six and no more than four equally dominant roots. To facilitate comparisons, dominant roots had the same absolute value in all matrices.

Three principal patterns emerged when normalizing factors were compared. For matrices with all dominant roots real, imaginary, or a mixture of real and imaginary, the normalizing factors increase or decrease monotonically. The resulting curves approach the real root as a limit asymptotically. For matrices whose dominant roots are all complex conjugate pairs, a zig-zag or "sawtooth" pattern results when the normaling factors are graphed. For matrices with both real and complex dominant roots, the curve formed by the normalizing factors has either the "sawtooth" pattern or the appearance of a damped wave.

When all the roots of the matrix have the same sign, the normalizing factors form a single curve. If signs of the roots are mixed, there are two separate but very similar patterns formed by the odd- and even-numbered normalizing factors. The separation of the two curves is determined by the balance in the signs of the roots.

With just two exceptions, matrices whose dominant roots are negatives of each other produced curves which are reflections with respect to the X-axis.

The angle between the complex roots also is a definite factor. Its effect is most clearly noted in the appearance of stronger "outlining" curves which overlay the basic curves. It also produces a cyclical pattern in the curve.

A detailed description is given of the results obtained from each matrix tested, and graphs are included for several. Based upon these results, predictions might be made about the dominant characteristic roots of any matrix when the power method has been applied.

The power method is extended to solve for two or three equally dominant roots. Computer programs for execution of the power method for one, two or three dominant roots are included.