APPLICATIONS OF THE DISCRETE MAXIMUM PRINCIPLE TO ONE-DIMENSIONAL PROCESSES

by

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1. INTRODUCTION

During the past decade, there has been a remarkable growth of interest in problems of dynamic optimization. This has given rise to a number of methods useful for rendering systems optimum. One such method is Pontryagin's maximum principle. Originally it was developed, in 1956, for continuous processes and has been chiefly applied in the field of optimum system control [8].

The first attempt to extend the maximum principle to the optimization of stagewise processes was made by Rozoner in 1959. The various versions of the discrete maximum principle were proposed by Chang [2], Katz [7], and Fan and Wang [4]. Most of the published literature on the application of the maximum principle is in the field of control and process design. Examples of its application to management and operations research problems, however, are still limited.

The aim of this report is to demonstrate the applicability of the discrete maximum principle to one-dimensional multi-stage non-linear as well as linear processes in management and industry. Only deterministic processes are considered here.

A multi-stage decision process may be considered as an abstract notion by which a large number of human activities can be represented. A stage may represent any real or abstract entity (a space unit, a time period or an economic activity) in which a transformation takes place. Those variables which are transformed in each stage are called state variables. The desired transformation for the state variables is achieved through manipulation of decision variables which remain, or may be considered to remain, constant within each stage of the process. The equations which completely describe the transformation at each stage are called performance equations. A homogeneous process is one in which the state variables and the decision variables are inter-related by the same set of performance equations. A process is called heterogeneous if it is not a homogeneous process. A process with a single state variable is called a one-dimensional process. Any process whose performance equations are linear in state variables is called a linear process. A process which is not linear is called a non-linear process.

The basic algorithm of the discrete maximum principle is first stated. The general form of performance equations, both for non-linear and linear one-dimensional processes, is then given. The recurrence relation of the optimal state and decision for non-linear as well as linear processes are next presented. Examples (1) through (5) are the case studies of non-linear processes. Examples (5a) through (7) are the case studies of linear processes.

In each of the examples considered, the discrete maximum principle has lead to the optimality condition represented by a recurrence relation of the control variable. Such a recurrence relation is generally valid for an n-stage system. For each of the special cases treated, a general solution is reduced immediately to a specific solution which agrees with the available results obtained by means of the Lagrange multiplier technique and by dynamic programming.

At the end optimum recurrence equations for multi-dimensional processes are presented.

2. THE DISCRETE MAXIMUM PRINCIPLE

STATEMENT OF THE ALGORITHM

The following is an outline of the general algorithm of the discrete maximum principle [4].

A multistage decision process consisting of N stages in sequence is schematically shown in Fig. 1. The state of the process stream denoted by an s-dimensional vector, $x = (x_1, x_2, \ldots, x_s)$, is transformed at each stage according to an r-dimensional decision vector, $\theta = (\theta_1, \theta_2, \ldots, \theta_r)$, which represents the decision made at that stage. The transformation of the process stream at the n^{th} stage is described by a set of performance equations

$$x_i^n = T_i^n (x_1^{n-1}, x_2^{n-1}, \ldots, x_s^{n-1}; \theta_1^n, \theta_2^n, \ldots, \theta_r^n),$$

$$x_i^0 = \alpha_i, \qquad i = 1, 2, \ldots, s; \quad n = 1, 2, \ldots, N.$$

or in vector form

$$x^n = T^n (x^{n-1}; \theta^n), \qquad n = 1, 2, \ldots, N, \tag{1}$$

$$x^0 = \alpha.$$

A typical optimization problem associated with such a process is to find a sequence of θ^n, $n = 1, 2, \ldots, N$, subject to constraints

$$\phi_i^n [\theta_1^n, \theta_2^n, \ldots, \theta_r^n] \leq 0, \tag{2}$$

$$n = 1, 2, \ldots, N,$$

$$i = 1, 2, \ldots, r,$$

which makes a function of the state variable of the final stage N

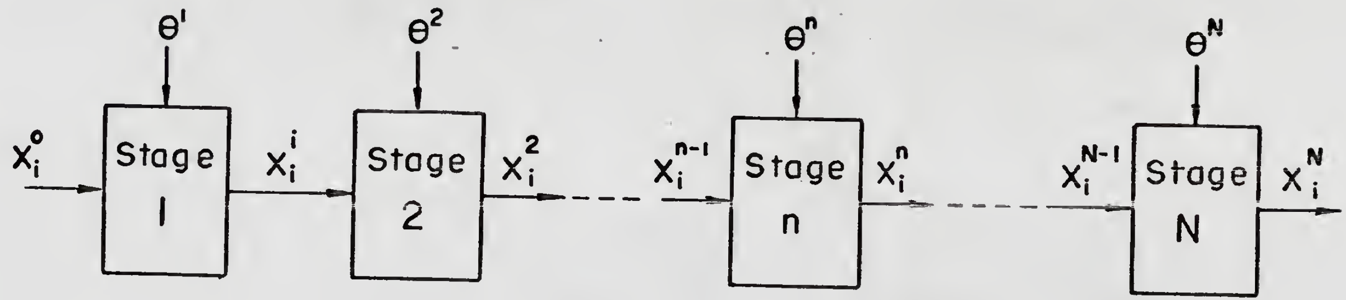

Fig. 1. Multistage decision process

$$S = \sum_{i=1}^{s} c_i x_i^N, \qquad c_i = \text{constant,} \tag{3}$$

an extremum when the initial condition $x^0 = \alpha$ is given. The function S which is to be maximized (or minimized) is the objective function of the process.

The procedure for solving such an optimization problem by the discrete maximum principle is to introduce an s-dimensional adjoint vector z^n and a Hamiltonian function H^n satisfying the following relations

$$H^n = (z^n)^T x^n = \sum_{i=1}^{s} z_i^n \, T_i^n \, (x^{n-1}; \theta^n), \qquad n = 1, 2, \ldots, N, \tag{4}$$

$$z_i^{n-1} = \frac{\partial H^n}{\partial x_i^{n-1}}, \qquad i = 1, 2, \ldots, s; \quad n = 1, 2, \ldots, N, \tag{5}$$

and

$$z_i^N = c_i, \qquad i = 1, 2, \ldots, s. \tag{6}$$

If the optimal decision vector function θ^n, which makes the objective function S an extremum (maximum or minimum), is interior to the set of admissible decisions θ^n, the set given by equation (2), a necessary condition for S to be a (local) extremum with respect to θ^n is

$$\frac{\partial H^n}{\partial \theta^n} = 0, \qquad n = 1, 2, \ldots, N. \tag{7}$$

If $\bar{\theta}^n$ is at a boundary of the set, it can be determined from the condition that H^n is (locally) extremum. The following special cases can be considered [6]:

(i) A necessary condition for S to be a (local) extremum with respect to θ^n, is

$$\frac{\partial H^n}{\partial \theta^n} = 0 , \qquad n = 1, 2, \ldots, N .$$

(ii) When the performance equation is linear in state variables x_i^{n-1}, namely

$$T_i^n(x^{n-1}; \theta^n) = \sum_{i=1}^{s} A_{ji}^n(\theta^n) x_j^{n-1} + f_i^n(\theta^n) , \tag{8}$$

a local maximum (or minimum) of the objective function corresponds to local maximum (or minimum) of the Hamiltonian function. In other words,

$$H^n = \text{maximum (or minimum)}$$

is the necessary condition for the objective function to be locally maximum (or minimum).

(iii) When A_{ji}^n in equation (8) is constant, or when the optimal decision is always known to be on the boundary of its admissible decision, the objective function is absolutely maximum (or minimum) if and only if H^n is absolutely maximum (or minimum).

(iv) When the performance equation is linear in their arguments, that is,

$$T_i^n(x^{n-1}; \theta^n) = \sum_{j=1}^{s} A_{ji}^n x_j^{n-1} + \sum_{j=1}^{r} B_{ji}^n \theta_j^n , \tag{8a}$$

then,

$$H^n = \text{maximum (or minimum)},$$

is necessary as well as sufficient for the objective function, S, to be absolutely (or globaly) maximum (or minimum).

For the optimization problems in which some of the final values of state variables, x_i^N, are pre-assigned, such as $x_a^N = c_1$, $x_b^N = c_2$, and the objective function is specified as

$$S = \sum_{\substack{i=1 \\ i \neq a \\ i \neq b}}^{s} c_i x_i^N ,$$

the basic algorithm represented by equations (4) through (7) is still applicable, except that equation (6) is replaced by

$$z_i^N = c_i, \qquad i = 1, 2, \ldots, s. \quad i \neq a, b. \tag{9}$$

THE PERFORMANCE EQUATIONS OF ONE-DIMENSIONAL PROCESSES [4]

If a multistage decision process can be completely characterized for the purpose of optimization by a single state variable, the process is called a one-dimensional multistage decision process.

For a one-dimensional process, there is only one state variable x_1 satisfying the performance equation

$$x_1^n = T\ (x_1^{n-1};\ \theta^n), \qquad n = 1, 2, \ldots, N \tag{10}$$

where T is the transformation operator and θ is the decision variable. In general, the objective function to be maximized is the sum of a certain function of x_1 and θ over all stages of the system such as

$$\sum_{n=1}^{N} G\ (x_1^{n-1};\ \theta^n).$$

The optimization problem associated with such a process is to find a sequence of decision variables θ^n, $n = 1, 2, \ldots, N$ so as to maximize

$$\sum_{n=1}^{N} G\,(x_1^{n-1};\ \theta^n)$$

with x_1^0 given. Introducing a new state variable x_2^n satisfying

$$x_2^n = x_2^{n-1} + G\,(x_1^{n-1};\ \theta^n)\ , \qquad x_2^0 = 0\ , \qquad n = 1, 2, \ldots, N, \tag{11}$$

such that

$$\sum_{n=1}^{N} G\,(x_1^{n-1};\ \theta^n) = x_2^N\ .$$

Thus the problem is transformed into the standard form in which a sequence of θ^n, $n = 1, 2, \ldots, N$, is to be chosen so as to maximize x_2^N for a process described by equations (10) and (11). x_1^n is called the primary state variable and x_2^n the secondary state variable.

A number of one-dimensional processes with single state variables has the following form of performance equations.

$$x_1^n = x_1^{n-1} + \alpha\, F_1\,(\theta^n)\ , \qquad x_1^0 = 0\ , \tag{12}$$

$$x_2^n = x_2^{n-1} + \beta\,(x_1^n - x_1^{n-1}) + F_2\,(\theta^n)\ , \qquad x_2^0 = 0\ . \tag{13}$$

Here $F_1(\theta^n)$ and $F_2(\theta^n)$ are two arbitrary functions of the decision variable θ^n; α and β are arbitrary constants.

A comparison of equations (12) and (13) with equations (10) and (11) immediately shows that

$$T\ (x_1^{n-1};\ \theta^n) = x_1^{n-1} + \alpha\,[F_1\ (\theta^n)]\ ,$$

$$G\ (x_1^{n-1};\ \theta^n) = \alpha\,\beta\,[F_1\ (\theta^n)] + F_2\ (\theta^n)\ .$$

Performance equations (12) and (13) are linear in state variables. Hence the processes represented by these equations are referred to as one-dimensional multistage linear processes. Any process whose performance equations are not linear in state variables is called a non-linear process.

THE RECURRENCE RELATION OF THE OPTIMAL STATE AND DECISION FOR THE NONLINEAR PROCESSES [4]

Let the nonlinear process be represented by performance equations (10) and (11) with the objective function

$$S = x_2^N\ . \tag{14}$$

Then the Hamiltonian function H^n given by equation (4) can be written as

$$H^n = z_1^n\ T\ (x_1^{n-1};\ \theta^n) + z_2^n\ [x_2^{n-1} + G\ (x_1^{n-1};\ \theta^n)]\ . \tag{15}$$

According to equation (5), the recurrence relations for the adjoint vector elements z_1 and z_2 are found to be

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = \frac{\partial T\ (x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}}\ z_1^n + \frac{\partial G\ (x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}}\ z_2^n\ , \tag{16}$$

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n\ , \qquad n = 1, 2, \ldots, N\ . \tag{17}$$

Since the objective function is $S = \sum_{i=1}^{2} c_i\ x_i^N = x_2^N$, that is, $c_1 = 0$, $c_2 = 1$,

then, we obtain

$$z_1^N = 0 \, , \tag{18a}$$

$$z_2^N = 1 \, . \tag{18b}$$

Combining equation (18b) and (17) and substituting in equation (16) gives

$$z_2^n = 1 \, , \qquad n = 1, 2, \ldots, N \, , \tag{19}$$

and

$$z_1^{n-1} = \frac{\partial T \, (x_1^{n-1}; \, \theta^n)}{\partial x_1^{n-1}} \, z_1^n + \frac{\partial G \, (x_1^{n-1}; \, \theta^n)}{\partial x_1^{n-1}} \tag{20}$$

$$n = 1, 2, \ldots, N \, .$$

Combining equations (19) and (15), we obtain

$$H^n = z_1^n \, T \, (x_1^{n-1}; \, \theta^n) + G \, (x_1^{n-1}; \, \theta^n) + x_2^{n-1} \, ,$$

$$n = 1, 2, \ldots, N \, .$$

According to equation (7), θ^n may be found where

$$\frac{\partial H^n}{\partial \theta^n} = z_1^n \, \frac{\partial T \, (x_1^{n-1}; \, \theta^n)}{\partial \theta^n} + \frac{\partial G \, (x_1^{n-1}; \, \theta^n)}{\partial \theta^n} = 0 \, .$$

Solving this equation for z_1^n, we obtain

$$z_1^n = - \frac{\dfrac{\partial G \, (x_1^{n-1}; \, \theta^n)}{\partial \theta^n}}{\dfrac{\partial T \, (x_1^{n-1}; \, \theta^n)}{\partial \theta^n}} \, . \tag{21}$$

Substitution of equation (21) into equation (20) gives the recurrence relation

$$\frac{\dfrac{\partial G(x_1^{n-1};\,\theta^n)}{\partial\theta^n}}{\dfrac{\partial T(x_1^{n-1};\,\theta^n)}{\partial\theta^n}} = \frac{\dfrac{\partial G(x_1^{n};\,\theta^{n+1})}{\partial\theta^{n+1}}}{\dfrac{\partial T(x_1^{n};\,\theta^{n+1})}{\partial\theta^{n+1}}} \cdot \frac{\partial T(x_1^n;\,\theta^{n+1})}{\partial x_1^n} - \frac{\partial G(x_1^n;\,\theta^{n+1})}{\partial x_1^n}, \quad (22)$$

$$n = 1, 2, \ldots, N-1.$$

Combining equations (18a) and (21) gives

$$\frac{\partial G(x_1^{N-1};\,\theta^N)}{\partial\theta^N} = 0. \quad (23)$$

Making use of recurrence relation (22) along with performance equation (10) and relation (23), a number of optimization problems associated with one dimensional processes can be solved.

For processes with fixed end point x_1^N, the condition $z_1^N = 0$ (equation 18a) or equivalent relation (23) is deleted.

GENERAL SOLUTION OF LINEAR PROCESSES

Comparing performance equations (12) and (13) of a linear process with equations (10) and (11) respectively, it follows

$$T(x_1^{n-1};\,\theta^n) = x_1^{n-1} + \alpha F_1(\theta^n) \quad (24)$$

$$G(x_1^{n-1};\,\theta^n) = \beta(x_1^n - x_1^{n-1}) + F_2(\theta^n)$$

$$= \alpha\beta F_1(\theta^n) + F_2(\theta^n), \quad n = 1, 2, \ldots, N. \quad (25)$$

The partial derivatives of equations (24) and (25) with respect to x_1^{n-1} and θ^n yields

$$\frac{\partial T(x_1^{n-1};\,\theta^n)}{\partial\theta^n} = \frac{\partial F_1(\theta^n)}{\partial\theta^n}, \quad (26)$$

$$\frac{\partial T(x_1^{n-1};\,\theta^n)}{\partial x_1^{n-1}} = 1, \quad (27)$$

$$\frac{\partial G\,(x_1^{n-1};\,\theta^n)}{\partial\,\theta^n} = \alpha\beta\,\frac{\partial\,F_1(\theta^n)}{\partial\theta^n} + \frac{\partial\,F_2(\theta^n)}{\partial\theta^n}\,, \tag{28}$$

$$\frac{\partial G\,(x_1^{n-1};\,\theta^n)}{\partial\,x_1^{n-1}} = 0. \tag{29}$$

Substituting equations (26) through (29) into equation (22), we obtain

$$\left[\alpha\beta\frac{\partial\,F_1(\theta^n)}{\partial\,\theta^n} + \frac{\partial F_2(\theta^n)}{\partial\,\theta^n}\right] \Big/ \alpha\frac{\partial F_1(\theta^n)}{\partial\,\theta^n} = \left[\alpha\beta\frac{\partial F_1(\theta^{n+1})}{\partial\theta^{n+1}} + \frac{\partial F_2(\theta^{n+1})}{\partial\theta^{n+1}}\right] \Big/ \alpha\frac{\partial F_1(\theta^{n+1})}{\partial\theta^{n+1}} \tag{30}$$

or

$$\frac{\partial\,F_2(\theta^n)}{\partial\theta^n} \Big/ \frac{\partial F_1(\theta^n)}{\partial\theta^n} = \frac{\partial F_2(\theta^{n+1})}{\partial\,\theta^n} \Big/ \frac{\partial F_1(\theta^{n+1})}{\partial\,\theta^{n+1}}$$

from which we conclude that

$$\theta^n = \theta^{n+1}, \qquad n = 1, 2, \ldots, N\text{-}1. \tag{31}$$

Hence the optimal policy for the class of linear processes described by equations (12) and (13) requires using the same value of the decision variable for all stages.

A number of one-dimensional processes may also be represented by the following form of the performance equations [4].

$$x_1^n = x_1^{n-1}\,F_1(\theta^n) + \alpha\,[F_1(\theta^n) - 1], \qquad x_1^0 = 0, \tag{32}$$

$$x_2^n = x_2^{n-1} + \beta\,(x_1^n - x_1^{n-1}) + F_2(\theta^n), \qquad x_2^0 = 0. \tag{33}$$

It is seen [4] that the optimal policies for such a class of processes is to apply an equal value of the decision variable at each stage as given by equation (31).

3. CASE STUDIES OF NON-LINEAR PROCESSES

The use of the recurrence equations of the optimal state and decision is demonstrated by solving some management problems with non-linear cost functions.

The problems are first stated as two stage processes; however, the general solutions obtained by the maximum principle are for multi-stages ($N > 2$). Then, the solution for the two stage processes is considered as a special case of general solution.

EXAMPLE (1). A PRODUCTION SCHEDULING PLAN

PROBLEM [5]:

A company must produce a total quantity s in N-month periods. Let n be the index for months and p^n be the quantity that will be produced in the n^{th} month of N-month long production period. There are many production plans that meet the requirements

$$\sum_{n=1}^{N} p^n = s \; ; \qquad p^n \geqslant 0. \tag{1}$$

Therefore to make the problem definite, a non-negative weighting factor w^n is assigned to the production in the n^{th} month, and it is required then to determine a production plan which minimizes the sum of weighted squares of production over the N-month period. The effect of the weighting factor would be to allow management to penalize production in some months more than others.

Thus the problem is to minimize

$$\underset{p^n}{\mathrm{Min}} \sum_{n=1}^{N} w^n (p^n)^2 \tag{2}$$

subject to constraints in equation (1).

SOLUTION BY THE MAXIMUM PRINCIPLE

Let each month represent one stage, and let us define

$\theta^n = p^n$ = production during n^{th} month of N-month period,

x_1^n = quantity that remains to be produced in (N-n) months,

x_2^n = sum of weighted squares of production up to and including n^{th} month.

The N-stage process can then be represented by the following performance equations:

$$x_1^n = x_1^{n-1} - \theta^n, \qquad x_1^0 = s, \qquad x_1^N = 0, \qquad n = 1, 2, \ldots, N, \tag{3}$$

$$x_2^n = x_2^{n-1} + w^n(\theta^n)^2, \qquad x_2^0 = 0. \tag{4}$$

Constraints: $\theta^n \geqslant 0, \qquad n = 1, 2, \ldots, N.$

Objective function:

$$\text{Minimize} \quad \sum_{n=1}^{N} w^n(\theta^n)^2 = x_2^N.$$

The solution is obtained by comparing equations (3) and (4) with equations (10) and (11) of Sec. 2 of one-dimensional process, and we obtain

$$T\,(x_1^{n-1};\, \theta^n) = x_1^{n-1} - \theta^n,$$

$$G\,(x_1^{n-1};\, \theta^n) = w^n(\theta^n)^2.$$

Differentiating these functions with respect to x_1^{n-1} and θ^n gives

$$\frac{\partial T\,(x_1^{n-1};\ \theta^n)}{\partial \theta^n} = -1, \tag{5}$$

$$\frac{\partial T\,(x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}} = 1, \tag{6}$$

$$\frac{\partial G\,(x_1^{n-1};\ \theta^n)}{\partial \theta^n} = 2\, w^n \theta^n, \tag{7}$$

$$\frac{\partial G\,(x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}} = 0. \tag{8}$$

Substituting equations (5) through (8) into recurrence equation (22) Sec. 2 yields

$$2\,\frac{w^n\theta^n}{-1} = 2\,\frac{w^{n+1}\theta^{n+1}}{-1} -0,$$

or

$$\theta^n = \theta^{n+1}\,\frac{w^{n+1}}{w^n}\,. \tag{9}$$

Equation (9) is the recurrence relation of the optimal decision.

With x_1^N given, θ^N can be computed from equation (3) by assuming a value of x_1^{N-1}. The corresponding value of x_1^0 is obtained from equations (3) and (9) and is then directly compared with the given value of x_1^0. The trial calculations are repeated until the computed value is sufficiently close to the given one. The sequence of computed θ_1^n, n = 1, 2,..., N for each assumed value of x_1^{N-1} is the optimal sequence corresponding to the initial condition x_1^0 obtained in each run of trial calculations.

To find the optimal sequence of θ^n, n = 1, 2,..., N for $x_1^0 = s$ we proceed as follows:

Let c = assumed value of x_1^{N-1}. For n = N equation (3) becomes

$$x_1^N = x_1^{N-1} - \theta^N$$

Since $x_1^N = 0$ and $x_1^{N-1} = c$, we obtain

$$\theta^N = c. \tag{10}$$

Substituting equation (10) into equation (9) for n = N-1 yields

$$\theta^{N-1} = \theta^N \frac{w^N}{w^{N-1}}$$

$$= c \frac{w^N}{w^{N-1}} . \tag{11}$$

For n = N-1 equation (3) becomes

$$x_1^{N-1} = x_1^{N-2} - \theta^{N-1} . \tag{12}$$

Substituting equation (11) into equation (12) yields

$$x_1^{N-1} = c = x_1^{N-2} - c \frac{w^N}{w^{N-1}} ,$$

or

$$x_1^{N-2} = c + c \frac{w^N}{w^{N-1}} . \tag{13}$$

Again from equation (9) and equation (3) for n = N-2, we obtain

$$\theta^{N-2} = \theta^{N-1} \frac{w^{N-1}}{w^{N-2}} ,$$

or

$$\theta^{N-2} = c\,\frac{w^N}{w^{N-1}} \cdot \frac{w^{N-1}}{w^{N-2}}$$

$$= c\,\frac{w^N}{w^{N-2}}\,, \tag{14}$$

and

$$x_1^{N-2} = x_1^{N-3} - \theta^{N-2}\,,$$

or

$$x_1^{N-3} = x_1^{N-2} + \theta^{N-2}$$

$$= c + c\,\frac{w^N}{w^{N-1}} + c\,\frac{w^N}{w^{N-2}}\,.$$

Similarly

$$x_1^{N-4} = c + c\,\frac{w^N}{w^{N-1}} + c\,\frac{w^N}{w^{N-2}} + c\,\frac{w^N}{w^{N-3}}\,,$$

$$\cdot\ \cdot\ \cdot\ \cdot$$

$$x_1^1 = c + c\,\frac{w^N}{w^{N-1}} + c\,\frac{w^N}{w^{N-2}} + \ldots + c\,\frac{w^N}{w^2}\,,$$

$$x_1^0 = c + c\,\frac{w^N}{w^{N-1}} + c\,\frac{w^N}{w^{N-2}} + \ldots + c\,\frac{w^N}{w^1}\,,$$

or

$$x_1^0 = c\left[1 + \frac{w^N}{w^{N-1}} + \ldots + \frac{w^N}{w^1}\right]$$

$$= c\,w^N\left[\frac{1}{w^N} + \frac{1}{w^{N-1}} + \ldots + \frac{1}{w^1}\right]$$

$$= c\,w^N\left[\sum_{n=1}^{N} \frac{1}{w^n}\right].$$

Since $x_1^0 = s$ [from equation (3)],

$$s = c\, w^N \left[\sum_{n=1}^{N} \frac{1}{w^n} \right] ,$$

or

$$c = \left[\frac{s}{\sum_{n=1}^{N} \frac{1}{w^n}} \right] \frac{1}{w^N} = \theta^N ,$$

or generalizing

$$\theta^n = \left[\frac{s}{\sum_{n=1}^{N} \frac{1}{w^n}} \right] \frac{1}{w^n} . \qquad (15)$$

A Numerical Example:

Let

$$s = 9$$

$$N = 3$$

$$w^1 = 6, \quad w^2 = 3, \quad w^3 = 2,$$

then

$$\sum_{n=1}^{3} \frac{1}{w^n} = \frac{1}{6} + \frac{1}{3} + \frac{1}{2} = 1,$$

$$\theta^1 = \left[\frac{s}{\sum_{n=1}^{3} \frac{1}{w^n}} \right] \frac{1}{w^n} = \frac{s}{w^1} = \frac{9}{6} = 1.5,$$

$$\theta^2 = \frac{s}{w^2} = \frac{9}{3} = 3, \ [\Sigma \frac{1}{w^n} = 1] ,$$

$$\theta^3 = \frac{s}{w^3} = \frac{9}{2} = 4.5 .$$

SOLUTION BY THE LAGRANGE MULTIPLIER METHOD

The following is the solution to the same problem using the Lagrange multiplier method.

To minimize

$$\sum_{n=1}^{N} w^n (p^n)^2, \qquad (p^n = \theta^n), \tag{16}$$

subject to constraint

$$\sum_{n=1}^{N} p^n = s, \tag{17}$$

the following quantity is first formed from equation (16) and (17).

$$\sum_{n=1}^{N} w^n (p^n)^2 + \lambda \left(\sum_{n=1}^{N} p^n - s \right) \tag{18}$$

where λ is an undetermined coefficient (multiplier). Differentiating it with respect to p^n and equating to zero yields

$$2\, w^n\, p^n + \lambda = 0, \tag{19}$$

or

$$p^n = \frac{-\lambda}{2\, w^n} . \tag{20}$$

Substituting equation (20) into equation (17), we obtain

$$\sum_{n=1}^{N} \frac{-\lambda}{2\, w^n} = s,$$

or

$$\frac{-\lambda}{2} \sum_{n=1}^{N} \frac{1}{w^n} = s,$$

or

$$\lambda = \frac{-2\ s}{\sum_{n=1}^{N} \frac{1}{w^n}} . \tag{21}$$

Combination of equations (20) and (21) yields

$$p^n = \left[\frac{s}{\sum_{n=1}^{N} \frac{1}{w^n}} \right] \frac{1}{w^n}, \qquad n = 1, 2, \ldots, N \tag{22}$$

which is equivalent to equation (15), the same result as obtained by maximum principle. However the constraint $p^n \geqslant 0$ may not always be satisfied by this approach and this method does not guarantee that all p^n's, $n = 1, 2, \ldots, N$, shall be positive. The requirement that all production quantities must be positive is difficult to incorporate in the Lagrange multiplier method. The maximum principle approach and dynamic programming approach can handle this type of difficulty.

SOLUTION BY DYNAMIC PROGRAMMING [5]

Let

$p(m|n)$ = amount to be produced in the n^{th} month from the end if a quantity m remains to be made in n monthly periods.

$v(m|n)$ = the minimum weighted sum of squares obtainable if the total amount to be made is m and n periods remain.

Then according to Bellman's principle of optimality, the optimal policy to minimize cost function (weighted squares of production) should minimize weighted square of production $w^n(p^n)^2$ for the current month and total weighed squares of production $v(m-p^n|n-1)$ for the remainder of (n-1) months (subsequent stages resulting from this decision).

Also production should be positive. Then

$$v(m|n) = \min_{0 \leqslant p^n \leqslant m} 2^n(p^n)^2 + v(m-p^n|n-1) \qquad (23)$$

$$m \geqslant 0, \qquad n = 2, 3, \ldots$$

Consider the situation when there is only one month remaining in which to make total quantity m^1. There is not then much choice but to produce m^1 in this month and the total sum of the squares will be $w^1(m^1)^2$, giving

$$p^1 = p(m^1|1) = m^1 = \frac{\frac{1}{w^1} m^1}{\frac{1}{w^1}}, \qquad (24)$$

$$v(m^1|1) = w^1(m^1)^2$$

$$= \frac{(m^1)^2}{\frac{1}{w^1}}. \qquad (25)$$

With two months remaining and the total quantity which remains to be produced being m^2, we have from equation (23)

$$v(m^2|2) = \min_{0 \leqslant p^2 \leqslant m^2} [w^2(p^2)^2 + v(m^2-p^2|1)],$$

but

$$v(m^2-p^2|1) = v(m^1|1) = w^1(m^1)^2 = w^1(m^2-p^2)^2.$$

Since

$$m^1 = m^2-p^2$$

hence

$$v(m^2|2) = \min_{0 \leqslant p^2 \leqslant m^2} [w^2(p^2)^2 + w^1(m^2-p^2)^2]. \qquad (26)$$

Differentiating equation (26) partially with respect to p^2 and equating to zero, we obtain

$$2\, w^2 p^2 - 2\, w^1 (m^2 - p^2) = 0,$$

or

$$p^2 = p(m^2 | 2) = \frac{w^1 m^2}{(w^1 + w^2)} = \frac{\frac{1}{w^2}\, m^2}{\frac{1}{w^2} + \frac{1}{w^1}} . \tag{27}$$

Substituting equation (27) into equation (26) gives

$$\begin{aligned} v(m^2 | 2) &= w^2 \left(\frac{w^1 m^2}{w^1 + w^2} \right)^2 + w^1 \left(m^2 - \frac{w^1 m^2}{w^1 + w^2} \right)^2 \\ &= \frac{w^2 (w^1)^2 (m^2)^2}{(w^1 + w^2)^2} + \frac{w^1 (w^2)^2 (m^2)^2}{(w^1 + w^2)^2} \\ &= \frac{w^1 w^2 (m^2)^2}{(w^1 + w^2)} = \frac{(m^2)^2}{\frac{1}{w^2} + \frac{1}{w^1}} . \end{aligned} \tag{28}$$

For n=3, that is, three months remain and a quantity m^3 remains to be produced

$$m^3 = s \text{ and } m^2 = m^3 - p^3 = s - p^3,$$

hence

$$v(m^3 | 3) = v(s | 3) = \min_{0 \leq p^3 \leq s} \left[w^3 (p^3)^2 + v(s - p^3 \;\; 2) \right] .$$

Substituting equation (28) into above equation yields

$$v(s | 3) = w^3 (p^3)^2 + \frac{w^1 w^2}{(w^1 + w^2)} (s - p^3)^2 . \tag{29}$$

Differentiating equation (29) with respect to p^3 and equating to zero yields

$$2\, w^3 p^3 - 2 \frac{w^1 w^2}{(w^1 + w^2)} (s - p^3) = 0,$$

or

$$p^3 (w^1 w^2 + w^1 w^3 + w^2 w^3) = w^1 w^2 s,$$

or

$$p^3 = \frac{w^1 w^2\, s}{w^1 w^2 + w^1 w^3 + w^2 w^3} \,. \tag{30}$$

Dividing equation (30) by $\frac{1}{w^1 w^2 w^3}$ yields

$$p^3 = p(s|\,3) = \frac{\frac{1}{w^3}\, s}{\frac{1}{w^3} + \frac{1}{w^2} + \frac{1}{w^1}} \,. \tag{31}$$

Substituting equation (30) into equation (29), we obtain

$$v(s|3) = [w^3 (p^3)^2 + \frac{w^1 w^2}{w^1 + w^2} (s - p^3)^2]$$

$$= \frac{w^1 w^2 w^3 (s)^2}{(w^1 w^2 + w^1 w^3 + w^2 w^3)}$$

$$= \frac{(s)^2}{\frac{1}{w^3} + \frac{1}{w^2} + \frac{1}{w^1}} \,. \tag{32}$$

It can be shown from equations (24), (27) and (31), by induction, that

$$p(m^B \mid B) = \left[\frac{m^B}{\sum_{n=1}^{B} \frac{1}{w^n}} \right] \frac{1}{w^B} , \qquad B = 1,\ 2,\ 3 \ldots \tag{33}$$

where B are the months remaining and m^B is the quantity remaining to be produced in B periods. Also from equations (25), (28) and (32), we obtain

$$v(m^B \mid B) = \frac{(m^B)^2}{\sum_{n=1}^{N} \frac{1}{w^n}} , \qquad B = 1,\ 2, \ldots \tag{34}$$

For the given numerical example, the solution is as follows. From equation (32), we obtain

$$v(s \mid 3) = v(9 \mid 3) = \frac{1}{\frac{1}{2} + \frac{1}{3} + \frac{1}{6}} (81) = 81 .$$

From equation (31)

$$p^3 = p(9 \mid 3) = \frac{1/6}{\frac{1}{2} + \frac{1}{3} + \frac{1}{6}} (9) = 1.5 ,$$

$$m^2 = s - p^3 = 9 - 1.5 = 7.5 .$$

From equations (27) and (28) respectively we obtain

$$p^2 = p(7.5 \mid 2) = \frac{1/w^2}{\frac{1}{w^1} + \frac{1}{w^2}} \cdot (7.5) = \frac{1/3}{\frac{1}{2} + \frac{1}{3}} (7.5) = 3 ,$$

and

$$v(7.5 \mid 2) = \frac{(7.5)^2}{\frac{1}{w^1} + \frac{1}{w^2}} = \frac{56.25}{\frac{1}{2} + \frac{1}{3}} = 67.5 ,$$

$$m^1 = m^2 - p^2 = 7.5 - 3 = 4.5 .$$

From equation (25) then

$$p(4.5|1) = 4.5,$$

$$v(4.5|1) = \frac{(4.5)^2}{\frac{1}{w^1}} = 2(20.25) = 40.5.$$

These results agree with those obtained by the maximum principle. Equations (15) and (33) respectively, i.e.,

$$\theta^n = \left[\frac{s}{\sum_{n=1}^{N} \frac{1}{w^n}}\right] \frac{1}{w^n}, \qquad n = 1, 2, \ldots, N,$$

$$p[m^B|B] = p^B = \left[\frac{m^B}{\sum_{n=1}^{B} \frac{1}{w^n}}\right] \frac{1}{w^B}, \qquad B = 1, 2, \ldots$$

however, should not create confusion. In the first equation obtained by the maximum principle, s is the total quantity to be produced in total N-month periods, θ^n being the production in n^{th} month from beginning; whereas in the second equation, m^B is the quantity remains to be produced in remaining B months, and p^B is the production in the B^{th} month from the end.

EXAMPLE (2). AN ALLOCATION PROBLEM

PROBLEM [Ref. 1, p. 252]:

A firm has two production facilities producing one product. The total production costs of each facility are related to the output levels as follows

$$Tc_1(y_1) = a_1 + b_1 y_1 + c_1 y_1^2, \qquad a_1, b_1, c_1 > 0,$$

$$Tc_2(y_2) = a_2 + b_2y_2 + c_2y_2^2, \qquad a_2,\ b_2,\ c_2 > 0,$$

where y_1 and y_2 represent the volumes of output. The firm must produce exactly s units per time period and desires to split this production load between the two facilities so as to minimize total product cost.

SOLUTION BY THE MAXIMUM PRINCIPLE:

Consider a general case of N-production facilities. Let each production facility represent a stage, and let

$\theta^n = y^n$ = volume of output at n^{th} stage,

x_1^n = volume of product which remains unproduced up to n^{th} stage,

x_2^n = total cost of production up to and including n^{th} stage (production facility) where cost for n^{th} stage is

$$Tc^n(\theta^n) = a^n + b^n\theta^n + c^n(\theta^n)^2.$$

Then the process may be described by the following two performance equations:

$$x_1^n = x_1^{n-1} - \theta^n; \qquad x_1^0 = s, \qquad x_1^N = 0,$$

$$n = 1, 2, \ldots, N, \tag{1}$$

$$x_2^n = x_2^{n-1} + [a^n + b^n\,(\theta^n) + c^n\,(\theta^n)^2], \qquad x_2^0 = 0. \tag{2}$$

Comparing equations (1) and (2) with performance equations, (10) and (11) Sec. 2, of one-dimensional processes, we find

$$T(x_1^{n-1};\ \theta^n) = x_1^{n-1} - \theta^n, \tag{3}$$

$$G(x_1^{n-1};\ \theta^n) = a^n + b^n(\theta^n) + c^n(\theta^n)^2. \tag{4}$$

Taking partial derivatives of equations (3) and (4) with respect to x_1^{n-1}, θ^n, we obtain

$$\frac{\partial T(x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}} = 1, \tag{5}$$

$$\frac{\partial T(x_1^{n-1};\ \theta^n)}{\partial \theta^n} = -1, \tag{6}$$

$$\frac{\partial G(x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}} = 0, \tag{7}$$

and

$$\frac{\partial G(x_1^{n-1};\ \theta^n)}{\partial \theta^n} = b^n + 2\ c^n(\theta^n). \tag{8}$$

Substituting equations (5) through (8) into the recurrence equation (22) Sec. 2, we obtain

$$\frac{b^n + 2\ c^n\theta^n}{-1} = \frac{b^{n+1} + 2\ c^{n+1}\theta^{n+1}}{-1}\ (1),$$

or

$$\theta^n = \frac{b^{n+1} - b^n}{2\ c^n} + \frac{c^{n+1}}{c^n}\ \theta^{n+1}. \tag{9}$$

Iterative use of recurrence equation (9) and equation (1) until conditions in equation (1) are satisfied gives the solution to this problem. However, a solution without iterative procedure can be obtained as below.

Let us assume a value of $x_1^{N-1} = K$ such that the corresponding value of x_1^0 obtained by iterative applications of equations (9) and (1) is equal to given value of $x_1^0 = s$. From equation (1)

$$x_1^{N-1} = \theta^N = K, \qquad (x_1^N = 0, \text{ given}). \tag{10}$$

From equations (9) and (10)

$$\theta^{N-1} = \frac{b^N - b^{N-1}}{2\, c^{N-1}} + \frac{c^N}{c^{N-1}} K, \qquad (\theta^N = K). \tag{11}$$

Again

$$\theta^{N-2} = \frac{b^{N-1} - b^{N-2}}{2\, c^{N-2}} + \frac{c^{N-1}}{c^{N-2}} \theta^{N-1}. \tag{12}$$

Substituting equation (11) into equation (12), we obtain

$$\theta^{N-2} = \frac{b^{N-1} - b^{N-2}}{2\, c^{N-2}} + \frac{b^N - b^{N-1}}{2\, c^{N-2}} + \frac{c^N}{c^{N-2}} K$$

or simplifying,

$$\theta^{N-2} = \frac{b^N - b^{N-2}}{2\, c^{N-2}} + \frac{c^N}{c^{N-2}} K. \tag{13}$$

Again from equations (9) and (13)

$$\begin{aligned}
\theta^{N-3} &= \frac{b^{N-2} - b^{N-3}}{2\, c^{N-3}} + \frac{c^{N-2}}{c^{N-3}} \theta^{N-2} \\
&= \frac{b^{N-2} - b^{N-3}}{2\, c^{N-3}} + \frac{b^N - b^{N-2}}{2\, c^{N-3}} + \frac{c^N}{c^{N-3}} K \\
&= \frac{b^N - b^{N-3}}{2\, c^{N-3}} + \frac{c^N}{c^{N-3}} K. \qquad (14)
\end{aligned}$$

From equations (11), (13) and (14), we can generalize for decision at n^{th} stage in terms of decision at N^{th} stage K,

$$\theta^n = \frac{b^N - b^n}{2\,c^n} + \frac{c^N}{c^n} K, \quad \text{(where } \theta^N = K\text{)}, \tag{15}$$

and

$$\theta^1 = \frac{b^N - b^1}{2\,c^1} + \frac{c^N}{c^1} K. \tag{16}$$

Once the decision variables for all stages have been obtained, we calculate x_1^0 iteratively as follows. From equations (1) and (10), we obtain

$$\begin{aligned} x_1^{N-2} &= x_1^{N-1} + \theta^{N-1} \\ &= K + \frac{b^N - b^{N-1}}{2\,c^{N-1}} + \frac{c^N}{c^{N-1}} K \end{aligned} \tag{17}$$

From equations (1), (13) and (17), we get

$$\begin{aligned} x_1^{N-3} &= x_1^{N-2} + \theta^{N-2} \\ &= K + \frac{b^N - b^{N-1}}{2\,c^{N-1}} + \frac{c^N}{c^{N-1}} K + \frac{b^N - b^{N-2}}{2\,c^{N-2}} + \frac{c^N}{c^{N-2}} K \\ &= [K + \frac{c^N}{c^{N-1}} K + \frac{c^N}{c^{N-2}} K] + [\frac{b^N - b^{N-1}}{2c^{N-1}} + \frac{b^N - b^{N-2}}{2c^{N-2}}]. \end{aligned} \tag{18}$$

Combining equations (1), (14) and (18), we obtain

$$\begin{aligned} x_1^{N-4} &= x_1^{N-3} + \theta^{N-3} \\ &= [K + \frac{c^N}{c^{N-1}} K + \frac{c^N}{c^{N-2}} K] + [\frac{b^N - b^{N-1}}{2c^{N-1}} + \frac{b^N - b^{N-2}}{2c^{N-2}}] \\ &\quad + \frac{b^N - b^{N-3}}{2\,c^{N-3}} + \frac{c^N}{c^{N-3}} K \end{aligned}$$

$$= \left[K + \frac{c^N}{c^{N-1}} K + \frac{c^N}{c^{N-2}} K + \frac{c^N}{c^{N-3}} K\right]$$

$$+ \left[\frac{b^N - b^{N-1}}{2\ c^{N-1}} + \frac{b^N - b^{N-2}}{2\ c^{N-2}} + \frac{b^N - b^{N-3}}{2\ c^{N-3}}\right],$$

or

$$x_1^{N-4} = K\ c^N \left[\sum_{n=N-3}^{N} \frac{1}{c^n}\right] + \sum_{n=N-3}^{N-1} \left(\frac{b^N - b^n}{2\ c^n}\right). \tag{19}$$

In general, we obtain

$$x_1^n = K\ c^N \left[\sum_{i=n+1}^{N} \frac{1}{c^i}\right] + \sum_{i=n+1}^{N-1} \left(\frac{b^N - b^i}{2\ c^i}\right) \tag{20}$$

and in particular for n=0, we obtain

$$x_1^0 = K\ c^N \left[\sum_{n=1}^{N} \frac{1}{c^n}\right] + \sum_{n=1}^{N-1} \left(\frac{b^N - b^n}{2\ c^n}\right) . \tag{21}$$

As $x_1^0 = s$, we obtain

$$K\ c^N \left[\left(\sum_{n=1}^{N} \frac{1}{c^n}\right)\right] + \sum_{n=1}^{N-1} \left(\frac{b^N - b^n}{2\ c^n}\right) = s,$$

or

$$K = \theta^N = \left[\frac{s - \sum_{n=1}^{N-1} \left(\frac{b^N - b^n}{2\ c^n}\right)}{\sum_{n=1}^{N} \frac{1}{c^n}}\right] \frac{1}{c^N} . \tag{22}$$

Equations (9) and (22) give a complete sequence of decision variables θ^n; $n = 1, 2, \ldots, N$.

Let

$$\frac{s - \sum_{n=1}^{N-1} \frac{b^N - b^n}{2\, c^n}}{\sum_{n=1}^{N} \frac{1}{c^n}} = A, \tag{23}$$

equation (22) then becomes

$$\theta^N = \frac{A}{c^N} . \tag{24}$$

Substituting equation (24) into equation (11) yields

$$\theta^{N-1} = \frac{b^N - b^{N-1}}{2\, c^{N-1}} + \frac{A}{c^{N-1}} . \tag{25}$$

Substituting equation (24) into equation (13) yields

$$\theta^{N-2} = \frac{b^N - b^{N-2}}{2\, c^{N-2}} + \frac{A}{c^{N-2}} . \tag{26}$$

Similarly substituting equation (24) into equation (14) gives

$$\theta^{N-3} = \frac{b^N - b^{N-3}}{2\, c^{N-3}} + \frac{A}{c^{N-3}} . \tag{27}$$

From equations (24) through (27), by comparison, it may be concluded that

$$\theta^n = \frac{b^N - b^n}{2\, c^n} + \frac{A}{c^n} , \qquad n = 1, 2, \ldots, N, \tag{28}$$

where A is given by equation (23). For the given problem N = 2, from equation (23), then, we obtain

$$A = [s - (\frac{b^2 - b^1}{2\,c^1})]\,\frac{c^1c^2}{c^1 + c^2}\,. \tag{29}$$

From equations (28) and (29), we obtain for n = 1:

$$\theta^1 = \frac{b^2 - b^1}{2\,c^1} + [s - \frac{(b^2 - b^1)}{2\,c^1}]\,\frac{c^1c^2}{(c^1 + c^2)} \cdot \frac{1}{c^1}\,,$$

or

$$\theta^1 = \frac{c^2}{c^1 + c^2}\,s + \frac{(b^2 - b^1)}{2\,c^1}\,[1 - \frac{c^2}{c^1 + c^2}]$$

$$= \frac{c^2}{c^1 + c^2}\,s + \frac{(b^2 - b^1)}{2\,(c^1 + c^2)}\,, \tag{30}$$

for n = 2

$$\theta^2 = \frac{A}{c^2} = [s - \frac{b^2 - b^1}{2\,c^1}]\;\frac{c^1c^2}{(c^1 + c^2)} \cdot \frac{1}{c^2}$$

$$= \frac{c^1}{c^1 + c^2}\,s - \frac{b^2 - b^1}{2\,(c^1 + c^2)}\,. \tag{31}$$

SOLUTION BY THE LAGRANGE MULTIPLIER METHOD:

The function to be minimized is

$$f(y^1, y^2) = a^1 + b^1y^1 + c^1(y^1)^2 + a^2 + b^2y^2 + c^2(y^2)^2,$$

and the constraint is

$$y^1 + y^2 = s.$$

The Lagrangian function for this problem is

$$L(y^1, y^2, \lambda) = a^1 + b^1 y^1 + c^1 (y^1)^2 + a^2 + b^2 y^2 + c^2 (y^2)^2$$
$$+ \lambda (s - y^1 - y^2). \tag{32}$$

Taking partial derivatives with respect to y^1, y^2, and λ yields

$$\frac{\partial L}{\partial y^1} = b^1 + 2c^1 y^1 - \lambda = 0, \tag{33}$$

$$\frac{\partial L}{\partial y^2} = b^2 + 2c^2 y^2 - \lambda = 0, \tag{34}$$

and

$$\frac{\partial L}{\partial \lambda} = s - y^1 - y^2 = 0. \tag{35}$$

Assuming that these derivatives will equal zero for some positive y^1 and y^2, from equations (33) and (34), by subtraction, we obtain

$$y^1 = \frac{b^2 - b^1}{2c^1} + \frac{c^2}{c^1} y^2,$$

since

$$y^2 = s - y^1,$$

$$y^1 = \frac{c^2}{c^1 + c^2} s + \frac{b^2 - b^1}{2(c^1 + c^2)}, \tag{36}$$

and

$$y^2 = \frac{c^1}{c^1 + c^2} s - \frac{b^2 - b^1}{2(c^1 + c^2)}. \tag{37}$$

Because of the great complexity involved it is very difficult to find a general relation for N-production facilities by Lagrange multiplier method.

The results obtained above may be applied to problems such as the one dealt below.

EXAMPLE (2a). A CONSTRUCTION COMPANY'S PROBLEM

PROBLEM [Ref. 1, p. 104]:

A construction company is building a large dam in a desert. There are two roads which provide access to the dam site, both roads originating at a town where the construction company owns a cement plant which is to provide all the cement for the dam. The construction company ships ready mixed concrete in its own trucks from the plant to the dam site. In fact, it is the only user of the two roads. It knows that the more concrete it ships along any road, the higher the shipping cost will be, because of increasing congestion which causes bunching of traffic, etc. A careful study of costs uncovers the following: (1) the total cost of shipping x cubic yards per day along road 1 is $c_1(x) = ax + bx^2$ where a, b > 0; (2) the total cost of shipping y cubic yards per day along road 2 is $c_2(y) = cy + dy^2$ where c, d > 0. Each day exactly K cubic yards must be shipped from plant to dam. Determine x and y for most economical shipping of concrete.

SOLUTION BY THE MAXIMUM PRINCIPLE:

To make it a general case let there be N roads providing access to the dam. In order to apply the maximum principle, let each road represent a stage. Considering the n^{th} road, let

θ^n = cubic yards of concrete shipped per day along the n^{th} road,

x_1^n = cubic yards of concrete which remain to be shipped after the first n stages,

x_2^n = total cost of shipping up to and including the n^{th} stage where the cost of shipping at n^{th} stage is

$$T^n(\theta^n) = b^n\theta^n + c^n(\theta^n)^2, \qquad b^n, c^n > 0$$

Then the system may be described by the following performance equations,

$$x_1^n = x_1^{n-1} - \theta^n, \qquad x_1^0 = K, \qquad x_1^N = 0, \tag{38}$$

$$n = 1, 2, \ldots, N.$$

$$x_2^n = x_2^{n-1} + b^n\theta^n + c^n(\theta^n)^2, \qquad x_2^0 = 0, \tag{39}$$

$$n = 1, 2, \ldots, N.$$

From equations (38) and (39), we obtain

$$T\,(x_1^{n-1};\, \theta^n) = x_1^{n-1} - \theta^n, \tag{40}$$

$$G\,(x_1^{n-1};\, \theta^n) = b^n\theta^n + c^n(\theta^n)^2. \tag{41}$$

Partial differentiation of equations (40) and (41) with respect to x_1^{n-1} and θ^n gives

$$\frac{\partial T\,(x_1^{n-1};\, \theta^n)}{\partial x_1^{n-1}} = 1, \tag{42}$$

$$\frac{\partial T\,(x_1^{n-1};\, \theta^n)}{\partial \theta^n} = -1, \tag{43}$$

$$\frac{\partial G\,(x_1^{n-1};\, \theta^n)}{\partial x_1^{n-1}} = 0, \tag{44}$$

and

$$\frac{\partial G(x_1^{n-1}; \theta^n)}{\partial \theta^n} = b^n + 2c^n\theta^n. \tag{45}$$

Substituting equations (42) through (45) into the recursive equation (22) Sec. 2, we obtain

$$\frac{b^n + 2c^n\theta^n}{-1} = \frac{b^{n+1} + 2c^{n+1}\theta^{n+1}}{-1} \quad (1),$$

or

$$\theta^n = \frac{b^{n+1} - b^n}{2c^n} + \frac{c^{n+1}}{c^n}\theta^{n+1}. \tag{46}$$

As equations (42) through (46) are the same as equations (5) through (9), we can use the results obtained before. From equation (23), we obtain

$$A = \frac{K - \sum_{n=1}^{N-1} \left(\frac{b^N - b^n}{2c^n}\right)}{\sum_{n-1}^{N} \frac{1}{c^n}}. \tag{47}$$

For the given problem N=2, i.e., two roads, from equation (30), we obtain

$$x = \theta^1 = \frac{c^2}{c^1 + c^2} s + \frac{b^2 - b^1}{2(c^1 + c^2)},$$

or in terms of constants given for this problem,

$$x = \frac{d}{b + d} K + \frac{c - a}{2(b + d)}. \tag{48}$$

And equation (31) yields

$$y = \theta^2 = \frac{b}{b + d} K - \frac{c - a}{2(b + d)} \tag{49}$$

EXAMPLE (3). CONSULTING ENGINEER'S PROBLEM

PROBLEM [Ref. 1, p. 20]:

A consulting engineer whose services are in great demand charges w_o dollars per day and has decided to limit his work schedule to B days per year. He works in both Canada (C_1) and the United States (C_2) and pays taxes in each country on the income earned in that country only. The tax schedule for each country can be adequately approximated by functions $T_1(y_1)$ and $T_2(y_2)$, which are defined as follows (letting y_i represent total income earned in country C_i):

$$T_1(y_1) = p_1\, y_1 + q_1\, y_1^2 \,,$$

$$T_2\,(y_2) = p_2\, y_2 + q_2\, y_2^2 \,.$$

The problem, of course, is: How should the engineer split his time between the two countries so as to make the greatest after-tax income.

SOLUTION BY THE MAXIMUM PRINCIPLE:

To make the above problem a general case, an N-country decision situation is considered below.

In order to apply the maximum principle let each country represent a stage.

Let

θ^n = number of days the engineer works in the n^{th} country,

x_1^n = number of days remaining after having worked in the first n countries,

x_2^n = after-tax income up to and including the n^{th} country.

Further, to make the problem more general, let w^n be the engineer's charge in dollars per day in n^{th} country. Then the total earnings in the n^{th} country are

$$y^n = \theta^n w^n. \tag{1}$$

Tax expenses in the n^{th} country are

$$T^n(y^n) = p^n y^n + q^n (y^n)^2 \tag{2}$$

Substituting equation (1) into (2) yields

$$T^n(y^n) = p^n w^n \theta^n + q^n (w^n)^2 (\theta^n)^2. \tag{3}$$

After-tax income in the n^{th} country is then

$$\begin{aligned} G(\theta^n; x_1^{n-1}) &= y^n - T^n(y^n) \\ &= \theta^n w^n - p^n w^n \theta^n - q^n (w^n)^2 (\theta^n)^2 \\ &= (1-p^n) w^n \theta^n - q^n (w^n)^2 (\theta^n)^2 \\ &= a^n w^n \theta^n - q^n (w^n)^2 (\theta^n)^2, \end{aligned} \tag{4}$$

where

$$a^n = 1 - p^n$$

The performance equations are

$$x_1^n = x_1^{n-1} - \theta^n; \qquad x_1^0 = B, \qquad x_1^N = 0,$$
$$n = 1, 2, \ldots, N, \tag{5}$$

$$x_2^n = x_2^{n-1} + [a^n w^n \theta^n - q^n (w^n)^2 (\theta^n)^2], \qquad x_2^0 = 0,$$
$$n = 1, 2, \ldots, N. \tag{6}$$

Objective function:

$$\max.\ x_2^N .$$

Constraints:

$$0 \leqslant \theta^n \leqslant B. \tag{7}$$

In equation (6), a^n, w^n and q^n, $n = 1, 2, \ldots, N$ are given constants.

Let

$$a^n w^n = b^n,$$

$$q^n (w^n)^2 = c^n. \tag{8}$$

Then equation (6) becomes

$$x_2^n = x_2^{n-1} + [b^n \theta^n - c^n (\theta^n)^2], \qquad x_2^0 = 0, \tag{9}$$

$$n = 1, 2, \ldots, N.$$

From equations (5) and (9)

$$T (x_1^{n-1}; \theta^n) = x_1^{n-1} - \theta^n, \tag{10}$$

$$G (x_1^{n-1}; \theta^n) = b^n \theta^n - c^n (\theta^n)^2. \tag{11}$$

Differentiating equations (10) and (11) partially with respect to x_1^{n-1} and θ^n yields

$$\frac{\partial T (x_1^{n-1}; \theta^n)}{\partial x_1^{n-1}} = 1, \tag{12}$$

$$\frac{\partial G\,(x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}} = 0, \tag{13}$$

$$\frac{\partial T\,(x_1^{n-1};\ \theta^n)}{\partial \theta^n} = -1, \tag{14}$$

and

$$\frac{\partial G\,(x_1^{n-1};\ \theta^n)}{\partial \theta^n} = b^n - 2\,c^n\,\theta^n. \tag{15}$$

Substituting equations (12) through (15) into the recursive equation (22) Sec. 2 yields

$$b^n - 2\,c^n\,\theta^n = b^{n+1} - 2\,c^{n+1}\,\theta^{n+1},$$

or

$$\theta^n = \frac{c^{n+1}}{c^n}\,\theta^{n+1} - \frac{b^{n+1} - b^n}{2\,c^n}\,. \tag{16}$$

As equations (12) through (15) are the same as equations (5) through (9) of example (2), except that c^n in example (2) is now replaced by $-c^n$, we obtain the solution by making use of results obtained in example (2).

From equation (23) of example (2), replacing c^n by $-c^n$, we obtain

$$A = \frac{B + \sum\limits_{n=1}^{N-1} \frac{b^N - b^n}{2\,c^n}}{\sum\limits_{n=1}^{N} \frac{1}{c^n}}\,. \tag{17}$$

The general solution given by equation (28) of example (2) then reduces to

$$\theta = \frac{b^N - b^n}{-2\,c^n} - \frac{A}{c^n}\,, \qquad n = 1,\ 2, \ldots,\ N, \tag{18}$$

where

$$b^n = a^n\,w^n, \tag{19a}$$

$$c^n = q^n\,(w^n)^2, \tag{19b}$$

and

$$a^n = 1 - p^n, \qquad n = 1,\ 2, \ldots,\ N. \tag{19c}$$

For the given problem N = 2, and

$$w^1 = w^2 = w^0, \tag{20}$$

substituting equations (19) and (20) into equations (17) and (18) yields

$$A = \frac{B + \dfrac{1}{w^0}\,\dfrac{p^1 - p^2}{2\,q^1}}{-\left(\dfrac{1}{q^1} + \dfrac{1}{q^2}\right)}\,, \tag{21}$$

$$\theta^n = \frac{p^n - p^2}{-2w^0\,q^n} - \frac{A}{w^0 q^n} \qquad n = 1,\ 2, \tag{22}$$

Substituting equation (21) into equation (22), we obtain

$$\theta^1 = \frac{q^2}{q^1 + q^2}\,B - \frac{p^1 - p^2}{2\,w^0\,(q^1 + q^2)}\,, \tag{23}$$

and

$$\theta^2 = \frac{q^1}{q^1 + q^2}\,B + \frac{p^1 - p^2}{2\,w^0\,(q^1 + q^2)}\,. \tag{24}$$

θ^1 and θ^2 given by equations (23) and (24) are the optimum number of days the consulting engineer must work in Canada and the United States respectively in order to make the greatest after-tax income. For instance if $p^1 = q^1 = 0$ (i.e. no tax in Canada), then equation (24) gives $\theta^2 = 0$, ($\theta^1 \geqslant 0$), i.e., the engineer should not work in the United States, which is quite obvious.

EXAMPLE (4). AN ADVERTISING INVESTMENT SCHEME

PROBLEM [Ref. 1, p. 372]:

You are the sales manager for a firm which sells two products. You have been alloted B dollars with which to promote two products. Some market research has shown that the most likely total sales (in units of products) over the next year as a function of the quantity of funds spent in promotion will be

$$s_1\,(x_1) = a_1 + b_1\,(x_1)^2,$$

$$s_2\,(x_2) = a_2 + b_2\,(x_2)^2,$$

where s_i is the number of units of product and x_i is the amount spent in promotion and a_i and b_i are positive constants. Product 1 yields a constant gross profit per unit of w_1 dollars and product 2, a gross profit per unit of w_2 dollars. How should you invest to maximize the gross profit?

SOLUTION BY THE DISCRETE MAXIMUM PRINCIPLE:

To generalize the problem, we consider that B dollars are to be alloted to N-products. To apply the maximum principle let each product represent a stage and considering the n^{th} stage let

$\theta^n = x_n$ = amount spent in promoting the n^{th} product yielding corresponding sales

$$s^n(\theta^n) = a^n + b^n(\theta^n)^2,$$

x_1^n = amount that has not been invested in the first n products,

x_2^n = total profit up to and including the first n stages where gross profit for the n^{th} stage is $s^n(\theta^n)w^n = [a^n + b^n(\theta^n)^2]\, w^n$.

The performance equations are then given by,

$$x_1^n = x_1^{n-1} - \theta^n, \qquad x_1^0 = B, \qquad x_1^N = 0, \qquad n = 1, 2, \ldots, N, \tag{1}$$

$$x_2^n = x_2^{n-1} + a^n w^n + b^n w^n (\theta^n)^2, \qquad x_2^0 = 0, \qquad n = 1, 2, \ldots, N. \tag{2}$$

Constraints:

$$0 \leqslant \theta^n \leqslant B, \qquad n = 1, 2, \ldots, N.$$

Objective function:

$$\text{Max.} \sum_{n=1}^{N} [a^n w^n + b^n w^n (\theta^n)^2] = x_2^N, \qquad c_1 \neq 0 \; (x_1^N = 0), \qquad c_2 = 1. \tag{3}$$

The adjoint vector z^n and the Hamiltonian are given by

$$\begin{aligned} H^n &= z_1^n\, x_1^n + z_2^n\, x_2^n \\ &= z_1^n\,(x_1^{n-1} - \theta^n) + z_2^n\,[x_2^{n-1} + a^n w^n + b^n w^n (\theta^n)^2], \end{aligned} \tag{4}$$

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = z_1^n , \tag{5}$$

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n, \qquad z_2^N = c_2 = 1. \tag{6}$$

From equation (6), we obtain

$$z_2^n = 1, \qquad n = 1, 2, \ldots, N. \tag{7}$$

Substituting equation (7) into (4), H^n becomes

$$H^n = z_1^n (x_1^{n-1} - \theta^n) + a^n w^n + b^n w^n (\theta^n)^2. \tag{8}$$

As z_1^n, x_1^{n-1}, a^n and w^n for a given problem are constant, the variable part of Hamiltonian becomes

$$H_v^n = b^n w^n (\theta^n)^2 - z_1^n \theta^n. \tag{9}$$

It should be noted that as the performance equations are linear in state variables x_i^{n-1}, namely of the form [see equation (8) of Sec. 2],

$$T_i^n (x^{n-1}; \theta^n) = \sum_{i=1}^{s} A_{ji}^n (\theta^n) x_j^{n-1} + f_i^n(\theta^n),$$

a local maximum (or minimum) of the objective function corresponds to a local maximum (or minimum) of the Hamiltonian function. In other words

$$H^n = \text{maximum (or minimum)}$$

is the necessary condition for the objective function to be locally maximum (or minimum). Further, when A_{ji}^n is constant (as is in this case), or when the optimal decision is always known to be on the boundary of its admissible region, the objective function is absolutely maximum (or minimum) if and only if H^n is absolutely maximum (or minimum). Furthermore there is a direct correspondence between H^n and S^n and their respective derivatives, i.e.,

$$\max. H^n \rightleftharpoons \max. S^n,$$

$$\frac{\partial H^n}{\partial \theta^n} \rightleftharpoons \frac{\partial S^n}{\partial \theta^n},$$

and

$$\frac{\partial^2 H^n}{\partial (\theta^n)^2} \rightleftharpoons \frac{\partial^2 S^n}{\partial (\theta^n)^2}.$$

From equation (9), we obtain

$$H^n = b^n w^n (\theta^n)^2 - z_1^n \theta^n. \tag{10}$$

Differentiating with respect to θ^n for maximum H^n gives

$$\frac{\partial H^n}{\partial \theta^n} = 2\, b^n w^n \theta^n - z_1^n = 0, \tag{11}$$

$$\frac{\partial^2 H^n}{\partial (\theta^n)^2} = 2\, b^n w^n > 0, \tag{12}$$

since b^n and w^n are both positive. Hence equation (11) yields a value of θ^n which gives H^n minimum, and the maximum value of H^n lies at the boundaries of θ^n.

<u>Stage 1</u>:

From equation (10), we obtain

$$H^1 = b^1 w^1 (\theta^1)^2 - z_1^1 \theta^1, \tag{13}$$

and the stationary points can be found from

$$\frac{\partial H^1}{\partial \theta^1} = 0 = 2\ b^1 w^1 \theta^1 - z_1^1. \tag{14}$$

The stationary points are located at

$$\theta^1 = 0 \quad \text{when} \quad z_1^1 = 0, \tag{15}$$

$$\theta^1 = B \quad \text{when} \quad z_1^1 = 2\ b^1 w^1 B, \tag{16}$$

and

$$\theta^1 = \frac{B}{2} \quad \text{when} \quad z_1^1 = b^1 w^1 B. \tag{17}$$

The three conditions are illustrated in the plot of H_1^1 versus θ^1 in Fig. 4. Therefore, H_v^1 = maximum at

$$\left.\begin{array}{lll} \theta^1 = 0 & \text{when} & z_1^1 \geqslant b^1 w^1 B, \\ \theta^1 = B & \text{when} & z_1^1 \leqslant b^1 w^1 B. \end{array}\right. \tag{18}$$

<u>For Stage 2</u>:

Similarly for stage 2, we obtain H_v^2 = maximum at

$$\theta^2 = 0 \quad \text{when} \quad z_1^2 \geqslant b^2 w^2 B, \tag{19}$$

$$\theta^2 = B \quad \text{when} \quad z_1^2 \leqslant b^2 w^2 B, \tag{20}$$

and in general then H^n = maximum at

$$\theta^n = 0 \quad \text{when} \quad z_1^n \geqslant b^n w^n B, \tag{21}$$

$$\theta^n = B \quad \text{when} \quad z_1^n \leqslant b^n w^n B, \tag{22}$$

$$n = 1, 2, \ldots, N.$$

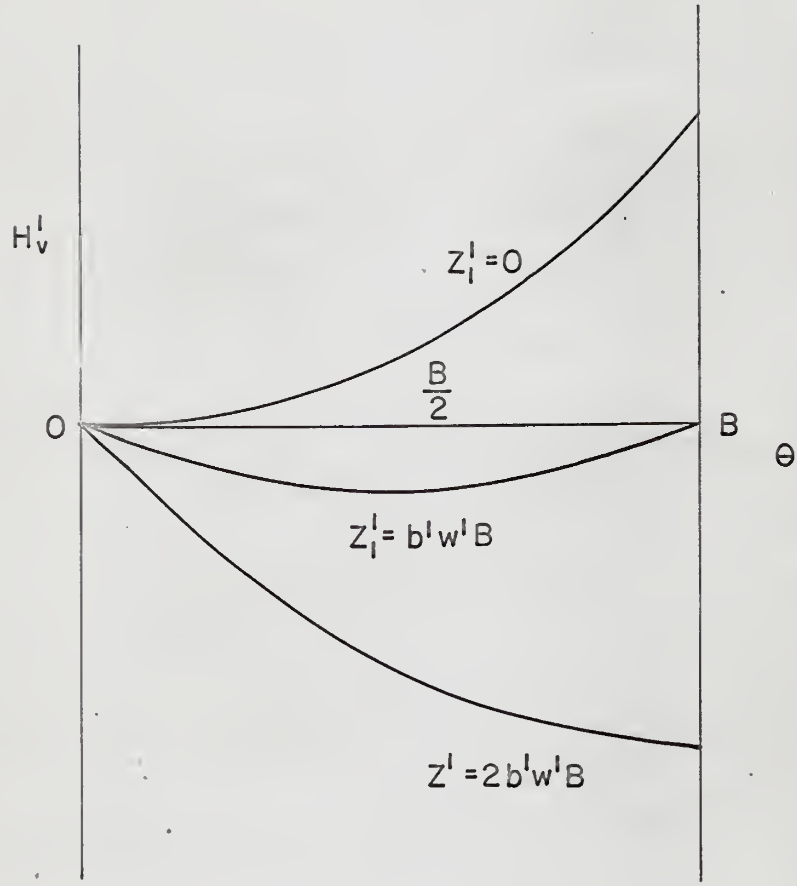

Fig. 4. H'_v versus θ' for various values of Z'_1.

Furthermore from equation (5), we obtain

$$z_1^{n-1} = z_1^n, \qquad n = 2, 3, \ldots, N. \tag{23}$$

Assuming that

$$b^1w^1 > b^2w^2 > b^3w^3 > \ldots > b^Nw^N, \tag{24}$$

then from equations (21) and (22) we see that

if $\quad z_1^n < b^n w^n B,$

$$\theta^n = \theta^{n-1} = \ldots = \theta^1 = B,$$

$$\theta^{n+1} = \theta^{n+2} = \ldots = \theta^N = 0.$$

However,

$$\theta^1 + \theta^2 + \theta^3 + \ldots + \theta^N = B.$$

Hence, if $w^2b^2B < z_1^n \leqslant w^1b^1B$, we obtain

$$\theta^1 = B,$$

and

$$\theta^2 = \theta^3 = \ldots = \theta^N = 0.$$

In general, if

$$w^kb^k > w^nb^n,$$

$$n = 1, 2, \ldots, N,$$

$$n \neq k.$$

then the optimal solution is given by

$$\theta^k = B,$$

$$\theta^n = 0, \qquad n = 1, 2, \ldots, N,$$
$$n \neq k.$$

For the given two product problem then

$$\theta^1 = B \quad \text{if} \quad w^1 b^1 > w^2 b^2,$$

$$\theta^1 = 0 \quad \text{if} \quad w^1 b^1 < w^2 b^2,$$

and

$$\theta^2 = B - \theta^1.$$

EXAMPLE (5). A PROBLEM IN PRODUCTION SMOOTHING; A PERISHABLE COMMODITY

PROBLEM [Ref. 9, p. 282; Ref. 1, p. 371]

The manufacturing process for a perishable commodity is such that the cost of changing the level of production from one month to the next is twice the square of difference in production levels. Any production not sold by the end of the month is wasted at a cost of $20 per unit. Given the sales forecast below, which must be met, determine the optimal production schedule. Assume the December production was α.

Month:	January	February	March	April
Sales:	210	220	195	180

SOLUTION BY THE MAXIMUM PRINCIPLE:

Let each month represent a stage and considering the n^{th} stage let us define

x_1^n = production during the n^{th} month,

θ^n = the difference in production levels of the $(n-1)^{th}$ and n^{th} month,

s^n = sales forecast for the n^{th} month,

x_2^n = total cost of production up to and including the n^{th} month where the cost of the n^{th} month is

$$G^n = 2(\theta^n)^2 + 20(x_1^n - s^n), \quad \text{if} \quad (x_1^n)_{Calc.} > s^n,$$

$$= 2(\theta^n)^2, \quad \text{if} \quad (x_1^n)_{Calc.} \leqslant s^n.$$

Hence three cases arising are

(i) $(x_1^n)_{Calc.} > s^n$,

(ii) $(x_1^n)_{Calc.} \leqslant s^n$,

(iii) $(x_1^n)_{Calc.} > s^n$, for some n's,

$(x_1^n)_{Calc.} \leqslant s^n$, for other n's.

Each case is dealt with as follows.

Case (i): $(x_1^n)_{Calc.} > s^n$.

The performance equations for this case are

$$x_1^n = x_1^{n-1} + \theta^n, \qquad x_1^0 = \alpha = 300, \qquad (1)$$

$$n = 1, 2, \ldots, N,$$

$$x_2^n = x_2^{n-1} + 2(\theta^n)^2 + 20(x_1^n - s^n)$$

$$= x_2^{n-1} + 2(\theta^n)^2 + 20(x_1^{n-1} + \theta^n - s^n). \qquad (2)$$

Constraint:

$$x_1^n \geqslant s^n, \qquad \text{(Demand must be met for each month).}$$

Objective faction:

$$\text{Max.} \sum_{i=1}^{2} C_i x_i^N = x_2^N, \qquad C_1 = 0, \qquad C_2 = 1. \tag{3}$$

From equations (1) and (2), we obtain

$$T\ (x_1^{n-1};\ \theta^n) = x_1^{n-1} + \theta^n, \tag{4}$$

$$G\ (x_1^{n-1};\ \theta^n) = 2\ (\theta^n)^2 + 20\ (x_1^{n-1} + \theta^n - s^n). \tag{5}$$

Partial differentiation of equations (4) and (5) with respect to x_1^{n-1} and θ^n yields

$$\frac{\partial T\ (x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}} = 1, \tag{6}$$

$$\frac{\partial G\ (x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}} = 20, \tag{7}$$

$$\frac{\partial T\ (x_1^{n-1};\ \theta^n)}{\partial \theta^n} = 1, \tag{8}$$

and

$$\frac{\partial G\ (x_1^{n-1};\ \theta^n)}{\partial \theta^n} = 4\ \theta^n + 20. \tag{9}$$

Substituting equations (6) through (9) into the recurrence equation, equation

(22) in Sec. 2, yields

$$\frac{4\ \theta^{n} + 20}{1} = \frac{4\ \theta^{n+1} + 20}{1} - 20$$

or

$$\theta^{n} = \theta^{n+1} - 5. \tag{10}$$

Since the end point is not fixed, from equation (23) in Sec. 2, we obtain

$$\frac{\partial G\ (x_1^{N-1};\ \theta^{N})}{\partial\ \theta^{N}} = 0 = 4\ \theta^{N} + 20$$

or

$$\theta^{N} = -5. \tag{11}$$

Combining equations (10) and (11) yields

$$\theta^{N} = \theta^{4} = -5, \tag{12}$$

$$\theta^{3} = -10, \tag{13}$$

$$\theta^{2} = -15, \tag{14}$$

and

$$\theta^{1} = -20. \tag{15}$$

$(x_1^n)_{Calc.}$ is found from the relation

$$(x_1^n)_{Calc.} = x_1^{n-1} + \theta^{n}, \tag{16}$$

and if

$$(x_1^n)_{Calc.} > s^{n}, \qquad x_1^n = (x_1^n)_{Calc.}$$

Table 1 illustrates this case.

Table 1. Optimum Production Plan for $(x_1^n)_{calc.} > s^n$.

n	s^n	θ^n	$(x_1^n)_{calc.} = x_1^{n-1} + \theta^n$	$(x_1^n)_{calc.} > s^n$	(x_1^n)	Production cost	Commu. cost
0					300		
1	210	-20	280	yes	280	2200	2200
2	220	-15	265	yes	265	1350	3550
3	195	-10	255	yes	255	1400	4950
4	180	- 5	250	yes	250	1450	6400

Case (ii): $(x_1^n)_{Calc} \leqslant s^n$.

Equation (2) in this case becomes

$$x_2^{n-1} = x_2^{n-1} + 2\,(\theta^n)^2, \qquad x_2^0 = 0. \tag{17}$$

Equation (1) remains unchanged, that is

$$x_1^n = x_1^{n-1} + \theta^n, \qquad x_1^0 = \propto = 200. \tag{18}$$

Hence

$$T\,(x_1^{n-1};\,\theta^n) = x_1^{n-1} + \theta^n, \tag{19}$$

$$G\,(x_1^{n-1};\,\theta^n) = 2\,(\theta^n)^2. \tag{20}$$

Differentiating the above two equations partially with respect to x_1^{n-1} and θ^n yields

$$\frac{\partial T\,(x_1^{n-1};\,\theta^n)}{\partial x_1^{n-1}} = 1, \tag{21}$$

$$\frac{\partial G\,(x_1^{n-1};\,\theta^n)}{\partial x_1^{n-1}} = 0, \tag{22}$$

$$\frac{\partial T\,(x_1^{n-1};\,\theta^n)}{\partial \theta^n} = 1, \tag{23}$$

and

$$\frac{\partial G\,(x_1^{n-1};\,\theta^n)}{\partial \theta^n} = 4\,\theta^n. \tag{24}$$

Substituting equations (21) through (24) into the recurrence equation yields

$$4\ \theta^n = 4\ \theta^{n+1}$$

or

$$\theta^n = \theta^{n+1}, \qquad n = 1, 2, 3. \tag{25}$$

And from equation (23) in Sec. 2, we obtain

$$\frac{\partial G\ (x_1^{N-1};\ \theta^N)}{\partial \theta^N} = 0 = 4\ \theta^N$$

or

$$\theta^N = 0. \tag{26}$$

Combining equations (25) and (26), we obtain

$$\theta^n = 0, \qquad n = 1,\ldots, 4. \tag{27}$$

Hence $(x_1^n)_{Calc.} = x_1^{n-1}$, and since the sales must be met, if $(x_1^n)_{Calc.} \leqslant s^n$, $x_1^n = s^n$.

An example is shown in Table 2.

Case (iii): Mixed cases.

$$(x_1^n)_{Calc.} > s^n \quad \text{for some n's.}$$

$$(x_1^n)_{Calc.} \leqslant s^n \quad \text{for other n's.}$$

This case is best solved by combining the results obtained in cases (i) and (ii). First it is assumed that $(x_1^n)_{Calc.} > s^n$ where $(x_1^n)_{Calc.} = x_1^{n-1} + \theta^n$, and θ^n is given by equations (12) through (15); then

$$(x_1^n) = (x_1^n)_{Calc.}$$

Table 2. Optimum Production Plan for $(x_1^n)_{calc.} \leq s^n$.

n	s^n	θ^n	$(x_1^n)_{calc.} = x_1^{n-1} + \theta^n$	$(x_1^n)_{calc.} \leqslant s^n$	x_1^n	Cost	Comm. cost
0					200		
1	210	0	200	yes	210	200	200
2	220	0	210	yes	220	200	400
3	220	0	220	yes	220	0	400
4	225	0	220	yes	225	50	450

Seco[illegible], if $(x_1^n)_{Calc.} \leqslant s^n$, x_1^n will be

$$(x_1^n) = \text{Max} \begin{cases} x_1^{n-1} \\ s^n \end{cases} .$$

The two examples shown in Tables 3 and 4 illustrate this case.

It is noted that if the sequence of sales forecast is further changed from that above, the maximum principle and dynamic programming will not necessarily yield an optimum production plan without further analysis.

SOLUTION BY DYNAMIC PROGRAMMING

Let $f_n(p_n)$ be the minimum achievable cost when last (previous) month's production was p_n and there are n months to go, and let

s = sales forecast for the current month,

x = production for the current month,

$x_n = P_{n-1}$.

According to Bellman's principle of optimality, the optimal policy to minimize cost function should minimize cost for the current month and total cost for the rest of the (n-1) stages (subsequent stages resulting from this decision).

Also demand must be met at every stage, then we obtain the functional relation,

$$f_n(P_n) = \min_{x_n \geqslant s_n} \left[2(x_n-P_n)^2 + 20(x_n-s_n) + f_{n-1}(P_{n-1})\right]. \qquad (1)$$

Considering April as the first month and January as the fourth,

$$f_1(P_1) = \min_{x \geqslant 180} \left[2(x_1-P_1)^2 + 20(x_1-180)\right] \qquad (2)$$

Table 3.

n	s^n	θ^n	$(x_1^n)_{calc.} = x_1^{n-1} + \theta^n$	$(x_1^n)_{calc.} > s^n$	$(x_1^n)_{calc.} \leqslant s^n$	θ^n	$x_1^{n-1} \leqslant s^n$	x_1^n optimal	Cost	Comm. cost
0								200		
1	210	-20	190		yes	0	no	210	200	200
2	220	-15	195		yes	0	no	220	200	400
3	195	-10	210	yes		-10	no	210	500	900
4	180	- 5	205	yes		- 5	no	205	550	1450

Table 4.

n	s^n	θ^n	(x_1^n)calc. $= x_1^{n-1} + \theta^n$	(x_1^n)calc. $> s^n$	(x_1^n)calc. $\leqslant s^n$	θ^n	$x_1^{n-1} \leqslant s^n$	x_1^n	Cost	Comm. cost
0								200		
1	190	-20	180		yes	0	no	200	200	200
2	210	-15	175		yes	0	yes	210	200	400
3	195	-10	200	yes		-10	no	200	300	700
4	180	- 5	195	yes		- 5	no	195	350	1050

where

$$P_1 = \text{production during March,}$$

$$x_1 = \text{production during April.}$$

The necessary conditions of minimizing $f_1(P_1)$ is

$$\frac{\partial f_1(P_1)}{\partial x_1} = 0 = 4\,(x_1 - P_1) + 20$$

or

$$x_1^* = P_1 - 5 \qquad (x_1 \geqslant 180). \tag{3}$$

Substituting equation (3) into equation (2) gives

$$f_1^*(P_1) = 2\,(P_1 - 5 - P_1)^2 + 20\,(P_1 - 5 - 180)$$

$$= 20\,P_1 - 3650. \tag{4}$$

For March (n = 2) we obtain

$$f_2(P_2) = 2\,(x_2 - P_2)^2 + 20\,(x_2 - 195) + f_1^*(P_1)$$

$$= 2\,(x_2 - P_2)^2 + 20\,(x_2 - 195) + 20\,x_2 - 3650$$

since $P_1 = x_2$.

The necessary condition of minimizing $f_2(P_2)$ is

$$\frac{\partial f_2(P_2)}{\partial x_2} = 0 = 4\,(x_2 - P_2) + 20 + 20$$

or

$$x_2^* = P_2 - 10 \qquad (x_2 \geqslant 195). \tag{5}$$

Then we obtain

$$f_2^*(P_2) = 2(P_2 - 10 - P_2)^2 + 20(P_2 - 10 - 195) + 20(P_2 - 10) - 3650$$

$$= 40P_2 - 7750. \qquad (6)$$

For February (n = 3), $f_3(P_3)$ becomes

$$f_3(P_3) = 2(x_3 - P_3)^2 + 20(x_3 - 220) + 40P_2 - 7750 \qquad (7)$$

$$= 2(x_3 - P_3)^2 + 20(x_3 - 220) + 40x_3 - 7750,$$

and

$$\frac{\partial f_3(P_3)}{\partial x_3} = 4(x_3 - P_3) + 20 + 40 = 0$$

or

$$x_3^* = P_3 - 15 = P_2 \qquad (x_3 \geqslant 220). \qquad (8)$$

Substituting equation (8) into equation (7) yields

$$f_3^*(P_3) = 2(P_3 - 15 - P_3)^2 + 20(P_3 - 15 - 220) + 40(P_3 - 15) - 7750$$

$$= 60P_3 - 12{,}600. \qquad (9)$$

For January (n=4), we obtain

$$f_4(P_4) = 2(x_4 - P_4)^2 + 20(x_4 - 210) + 60x_4 - 12600\ ,$$

and

$$\frac{\partial f_4(P_4)}{\partial x_4} = 0 = 4(x_4 - P_4) + 20 + 60 = 0,$$

or

$$x_4^* = P_4 - 20 \qquad (x_4 \geqslant 210). \tag{10}$$

However, P_4 is the production level in December and equal to 200($=\alpha$), therefore,

$$(x_4)_{\text{calculated}} = 200 - 20 = 180.$$

Since the demand is 210, the production level should be

$$x_4 = 210.$$

Equations (3), (5), (8) and (10) give optimal values of x's.

$$\begin{aligned} x_3 &= P_3 - 15 \\ &= 210 - 15 = 195 < 220 = s_3, \end{aligned}$$

therefore,

$$x_3 = 220$$

$$\begin{aligned} x_2 &= P_2 - 10 \\ &= 220 - 10 = 210 > 195 = s_2, \end{aligned}$$

or

$$x_2 = 210,$$

$$\begin{aligned} x_1 &= P_1 - 5 \\ &= 210 - 5 = 205 > 180 = s_1 \end{aligned}$$

or

$$x_1 = 205.$$

Hence the optimal production plan is

January	February	March	April
210	220	210	205

which is the same as that obtained by the maximum principle.

4. CASE STUDIES OF LINEAR ONE-DIMENSIONAL PROCESSES

In general the one-dimensional multistage linear processes have the following form of performance equations

$$x_1^n = x_1^{n-1} + \alpha\,[F_1(\theta^n)] \tag{1}$$

$$x_2^n = x_2^{n-1} + \beta\,(x_1^n - x_1^{n-1}) + F_2(\theta^n), \qquad x_2^0 = 0 \tag{2}$$

where $F_1(\theta^n)$ and $F_2(\theta^n)$ are two arbitrary functions of the decision variable θ^n; α and β are arbitrary constants. It was shown in Sec. 2 that the optimal policies for such a class of processes are to apply an equal value of the decision variable at each stage.

EXAMPLE (5a).

Let us consider the production smoothing problem of the perishable commodity; which we studied in example (5) and consider the case of $(x_1^n)_{calc} \leqslant s^n$. The process described by the performance equations are

$$x_1^n = x_1^{n-1} + \theta^n; \qquad x_1^0 = 200 \tag{3}$$

$$x_2^n = x_2^{n-1} + 2\,(\theta^n)^2 \quad \text{for} \quad x_1^n \leqslant s^n, \qquad x_2^0 = 0 \tag{4}$$

A comparison of equations (3) and (4) with equations (1) and (2) shows that

$$\alpha = 1,$$

$$\beta = 0,$$

$$F_1(\theta^n) = \theta^n,$$

and

$$F_2(\theta^n) = 2\,(\theta^n)^2.$$

Hence the problem belongs to a class of one-dimensional linear processes and the optimal policy is to apply an equal value of decision variable at each stage, i.e.

$$\theta^n = \theta^{n+1}, \qquad n = 1, \ldots, N-1. \tag{5}$$

This may be verified as follows. From equations (3) and (4), we obtain

$$T\,(x_1^{n-1};\, \theta^n) = x_1^{n-1} + \theta^n, \tag{6}$$

$$G\,(x_1^{n-1};\, \theta^n) = 2\,(\theta^n)^2, \tag{7}$$

and differentiating equations (6) and (7) with respect to x_1^{n-1} and θ^n gives

$$\frac{\partial T\,(x_1^{n-1};\, \theta^n)}{\partial x_1^{n-1}} = 1, \tag{8}$$

$$\frac{\partial G\,(x_1^{n-1};\, \theta^n)}{\partial x_1^{n-1}} = 0, \tag{9}$$

$$\frac{\partial T\,(x_1^{n-1};\, \theta^n)}{\partial \theta^n} = 1, \tag{10}$$

and

$$\frac{\partial G\,(x_1^{n-1};\, \theta^n)}{\partial \theta^n} = 4\,\theta^n. \tag{11}$$

Substituting equations (8) through (11) into the recursive relation of one-dimensional processes, equation (22) of Sec. 2, we obtain

$$4\,\theta^{n} = 4\,\theta^{n+1}$$

or

$$\theta^{n} = \theta^{n+1} \tag{12}$$

that is the solution presented by equation (5). To find θ^N, equation (23) of Sec. 2 gives

$$\frac{\partial G\,(x_1^{N-1};\ \theta^N)}{\partial \theta^N} = 4\,\theta^N = 0$$

or

$$\theta^N = 0. \tag{13}$$

Combining equations (11) and (13) yields,

$$\theta^n = 0, \qquad n = 1, 2, \ldots, N. \tag{14}$$

As a further illustration of one-dimensional linear multistage processes, two examples are discussed next, the first is a purely mathematical problem while the second one is an economic problem.

EXAMPLE (6). AN OPTIMAL SUBDIVISION PROBLEM

PROBLEM:

A positive quantity B is to be divided into N parts in such a way that the product of the N parts is to be a maximum. Obtain the optimal subdivision.

SOLUTION BY THE DISCRETE MAXIMUM PRINCIPLE

In order to use the maximum principle, let each part represent a stage, and let us define

θ^n = part of quantity B alloted to n^{th} stage,

then the objective function is

$$\text{Maximize } p = \prod_{n=1}^{N} \theta^n, \tag{1}$$

with the constraints

$$0 \leqslant \theta^N \leqslant B. \tag{2}$$

Taking the logarithm on both sides of equation (1), we obtain

$$\ln p = \sum_{n=1}^{N} \ln \theta^n. \tag{3}$$

Let us define the state variables as

x_1^n = part of quantity B left after allocation to first n stages,

x_2^n = sum of the logarithmic θ^n which is equivalent to product of first n parts.

Then the performance equations for the process are

$$x_1^n = x_1^{n-1} - \theta^n, \qquad x_1^0 = B, \qquad x_1^N = 0, \qquad n = 1, 2, \ldots, N, \tag{4}$$

and

$$x_2^n = x_2^{n-1} + \ln \theta^n, \qquad x_2^0 = 0. \tag{5}$$

A simple comparison shows that equations (4) and (5) are in the forms of linear one-dimensional processes. Here

$$\alpha = -1,$$

$$\beta = 0,$$

$$F_1(\theta^n) = \theta^n,$$

$$F_2(\theta^n) = \ln \theta^n.$$

Hence the optimal policy is

$$\theta^n = \theta^{n+1}, \qquad n = 1, 2, \ldots, N-1. \tag{6}$$

This is verified as follows. From equations (4) and (5), we obtain

$$T\ (x_1^{n-1};\ \theta^n) = x_1^{n-1} - \theta^n, \tag{7}$$

$$G\ (x_1^{n-1};\ \theta^n) = \ln \theta^n. \tag{8}$$

Differentiating equations (7) and (8) with respect to x_1^{n-1} and θ^n yields

$$\frac{\partial T\ (x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}} = 1, \tag{9}$$

$$\frac{\partial T\ (x_1^{n-1};\ \theta^n)}{\partial \theta^n} = -1, \tag{10}$$

$$\frac{\partial G\ (x_1^{n-1};\ \theta^n)}{\partial x_1^{n-1}} = 0, \tag{11}$$

and

$$\frac{\partial G\ (x_1^{n-1};\ \theta^n)}{\partial \theta^n} = \frac{1}{\theta^n}\,. \tag{12}$$

Substituting equations (9) through (12) into recursive equation (22) Sec. 1 yields

$$\frac{-1}{\theta^n} = \frac{-1}{\theta^{n+1}} \cdot 1$$

or

$$\theta^n = \theta^{n+1}, \tag{13}$$

which is the same as in equation (6) above. As the end point is fixed ($x_1^N = 0$), we cannot use the relation

$$\frac{\partial G\,(x_1^{N-1};\, \theta^N)}{\partial \theta^N} = 0;$$

however, from equations (4) and (13)

$$\sum_{n=1}^{N} \theta^n = N\,\theta^n = B,$$

or

$$\theta^n = \frac{B}{N}, \qquad n = 1, 2, \ldots, N, \tag{14}$$

and the corresponding objective function

$$x_2^N = (\frac{B}{N})^N.$$

SOLUTION BY THE LAGRANGE MULTIPLIER METHOD:

Another approach would be to use the method of dynamic programming or the method of Lagrangian multiplier. The second method is discussed as follows.

Let B be divided into N parts, $x_1, x_2, \ldots, x_n$ so that we have to maximize the function

$$y = x_1 . x_2 \ldots x_N,$$

subject to the constraints

$$\sum_{i=1}^{N} x_i = B; \qquad x_i \geqslant 0, \qquad i = 1, 2, \ldots, N.$$

Let

$$Y = x_1 . x_2 \ldots x_N + \lambda \left[B - \sum_i x_i \right], \tag{15}$$

the maximum value of y, can be obtained from the conditions

$$\frac{\partial Y}{\partial x_i} = 0, \qquad i = 1, 2, \ldots, N, \tag{16}$$

$$\frac{\partial Y}{\partial \lambda} = 0. \tag{17}$$

From equation (15),

$$\frac{\partial Y}{\partial x_i} = \prod_{\substack{j=1 \\ j \neq i}}^{N} x_j - \lambda = 0,$$

or

$$\lambda = \prod_{\substack{j=1 \\ j \neq i}}^{N} x_j = \frac{y}{x_i}, \tag{18}$$

or

$$x_i = \frac{y}{\lambda}, \tag{19}$$

and equation (17) gives

$$\frac{\partial Y}{\partial \lambda} = B - \Sigma x_i = 0$$

or

$$B = \Sigma x_i. \tag{20}$$

Substituting equation (19) into equation (20), we obtain

$$B = \Sigma \frac{y}{\lambda} = \frac{N y}{\lambda}$$

or

$$\lambda = \frac{N y}{B} \tag{21}$$

Substituting the value of λ from equation (21) into equation (19), we obtain

$$x_i = \frac{B}{N}, \qquad i = 1, 2, \ldots, N, \tag{22}$$

and

$$y = (\frac{B}{N})^N. \tag{23}$$

The results obtained above agree with those obtained by the maximum principle.

EXAMPLE (7). STATIC CONSUMER-CHOICE PROBLEM

PROBLEM [Ref. 1, p. 371]:

Consider the decision problem of an individual who has a fixed sum of money Y available for the purchase and consumption of N different commodities (in the amounts $x_1, x_2, \ldots, x_N$) over some definite single planning period. Assume that none of the money is to be saved and that the consumer seeks to maximize his utility. Assume that his utility for a purchase plan $x \in x = \{(x_1, x_2, \ldots, x_N): x_i \in R \text{ and } x_i \geqslant 0\}$ may be represented in terms of mapping

$U: x \rightarrow R$ which is defined by

$$U = x_1 . x_2 \ldots x_N = \prod_{i=1}^{N} x_i .$$

We assume that the various commodities are available in any amounts at the known and constant prices $p_1, p_2, p_3, \ldots, p_N$. Solve for the optimal purchase plan.

SOLUTION BY THE MAXIMUM PRINCIPLE:

In order to formulate this problem into the discrete maximum principle form, let each commodity represent a stage, and let us define

θ^n = money spent in purchasing n^{th} commodity,

with the constraints,

$$0 \leqslant \theta^n \leqslant Y, \tag{1}$$

then the objective function becomes

$$\text{Max.} \quad S = \prod_{n=1}^{N} \frac{\theta^n}{p^n} \tag{2}$$

Taking the logarithm on both sides of equation (2), we obtain

$$\ln S = \sum_{n=1}^{N} \ln \frac{\theta^n}{p^n} \tag{3}$$

Let us define the state variables as

x_1^n = money left after purchasing first n commodities,

x_2^n = sum of logarithmic $\dfrac{\theta^n}{p^n}$ which is equivalent to product of amounts of up to and including n^{th} commodity where amount of n^{th}

$$\text{commodity} = \frac{\theta^n}{p^n} .$$

The performance equations are

$$x_1^n = x_1^{n-1} - \theta^n; \qquad x_1^0 = Y, \qquad x_1^N = 0, \tag{4}$$

$$x_2^n = x_2^{n-1} + \ln \frac{\theta^n}{p^n}, \qquad x_2^0 = 0 \tag{5}$$

and the objective function is

$$\text{Max.} \quad x_2^N \tag{6}$$

Comparing equations (4) and (5) with equations (12) and (13) of linear processes, we have

$$\alpha = -1,$$

$$\beta = 0,$$

$$F_1(\theta^n) = \theta^n,$$

$$F_2(\theta^n) = \ln \frac{\theta^n}{p^n} .$$

Hence the optimal policy is to apply an equal value of the decision variable at each stage, i.e.

$$\theta^n = \theta^{n+1}, \qquad n = 1, 2, \ldots, N-1. \tag{7}$$

This is verified as follows. From equations (4) and (5), we obtain

$$T\,(x_1^{n-1}; \theta^n) = x_1^{n-1} - \theta^n, \tag{8}$$

$$G\,(x_1^{n-1}; \theta^n) = \ln \frac{\theta^n}{p^n} . \tag{9}$$

Partial differentiation of these two equations with respect to x_1^{n-1} and θ^n

gives

$$\frac{\partial T\,(x_1^{n-1};\,\theta^n)}{\partial x_1^{n-1}} = 1, \tag{10}$$

$$\frac{\partial T\,(x_1^{n-1};\,\theta^n)}{\partial \theta^n} = -1, \tag{11}$$

$$\frac{\partial G\,(x_1^{n-1};\,\theta^n)}{\partial x_1^{n-1}} = 0, \tag{12}$$

and

$$\frac{\partial G\,(x_1^{n-1};\,\theta^n)}{\partial \theta^n} = \frac{1}{\theta^n}\,, \qquad (p^n \text{ is constant}). \tag{13}$$

Substitution of equations (10) through (13) into the recurrence relation of one-dimensional processes equation (22) Sec. 3 yields

$$\frac{-1}{\theta^n} = \frac{-1}{\theta^{n+1}} \quad (1)$$

or

$$\theta^n = \theta^{n+1}, \tag{14}$$

Again, as $x_1^N = 0$, we cannot make use of the relation

$$\frac{\partial G\,(x_1^{N-1};\,\theta^N)}{\partial \theta^N} = 0.$$

However, from equations (4) and (13)

$$\sum_{n=1}^{N} \theta^n = N\theta^n = Y$$

or

$$\theta^n = \frac{Y}{N} . \tag{15}$$

5. OPTIMUM RECURRENCE EQUATION FOR MULTI-DIMENSIONAL PROCESSES

Let an N-stage system be described by a set of performance equations

$$x_i^n = T_i^n (x_1^{n-1}, x_2^{n-1}, \ldots, x_s^{n-1}; \theta_1^n, \theta_2^n, \ldots, \theta^n),$$

$$i = 1, 2, \ldots, s,$$

or in vector form

$$x^n = T^n (x^{n-1}; \theta^n), \tag{1}$$

where x^n is an s-dimensional vector and θ^n is an r-dimensional vector which is to be chosen at each stage to maximize the objective function

$$S = \sum_{i=1}^{s} c_i \, x_i^N . \tag{2}$$

The procedure for solving such an optimization problem by the discrete maximum principle is to introduce an s-dimensional adjoint vector z^n and a Hamiltonian function H^n satisfying

$$H^n = \sum_{i=1}^{s} z_i^n \, T_i^n (x^{n-1}; \theta^n), \qquad n = 1, 2, \ldots, N, \tag{3}$$

or in vector form

$$H^n = (z^n)^T x^n,$$

and

$$z_i^{n-1} = \frac{\partial H^n}{\partial x_i^{n-1}}, \qquad i = 1, 2,\ldots, s; \quad n = 1, 2,\ldots, N. \tag{4}$$

Substituting equation (3) into equation (4), yields

$$z_i^{n-1} = \sum_{j=1}^{s} z_j^n \frac{\partial T_j^n}{\partial x_i^{n-1}}, \qquad i = 1, 2,\ldots, s. \tag{5}$$

If θ^n is unconstrained, then for the objective function to be a maximum, the weak maximum principle for staged systems requires that

$$\frac{\partial H^n}{\partial \theta_k^n} = 0, \qquad k = 1, 2,\ldots, r, \quad n = 1, 2,\ldots, N. \tag{6}$$

Substituting equation (3) into equation (6) yields

$$\sum_{j=1}^{s} z_j^n \frac{\partial T_j^n}{\partial \theta_k^n} = 0, \qquad k = 1, 2,\ldots, r, \quad n = 1, 2,\ldots, N, \tag{7}$$

or in vector form

$$(z^n)^T \frac{\partial T^n}{\partial \theta^n} = 0,$$

which is a short hand form of the matrix

$$(z_1^n, z_2^n, \ldots, z_k^n, \ldots, z_s^n) \begin{bmatrix} \frac{\partial T_1^n}{\partial \theta_1^n} & \frac{\partial T_1^n}{\partial \theta_2^n} & \cdots & \frac{\partial T_1^n}{\partial \theta_j^n} & \cdots & \frac{\partial T_1^n}{\partial \theta_r^n} \\ \frac{\partial T_2^n}{\partial \theta_1^n} & \frac{\partial T_2^n}{\partial \theta_2^n} & \cdots & \frac{\partial T_2^n}{\partial \theta_j^n} & \cdots & \frac{\partial T_2^n}{\partial \theta_r^n} \\ \vdots & & & & & \\ \frac{\partial T_k^n}{\partial \theta_1^n} & \frac{\partial T_k^n}{\partial \theta_2^n} & \cdots & \frac{\partial T_k^n}{\partial \theta_j^n} & \cdots & \frac{\partial T_k^n}{\partial \theta_r^n} \\ \vdots & & & & & \\ \frac{\partial T_s^n}{\partial \theta_1^n} & \frac{\partial T_s^n}{\partial \theta_2^n} & \cdots & \frac{\partial T_s^n}{\partial \theta_j^n} & \cdots & \frac{\partial T_s^n}{\partial \theta_r^s} \end{bmatrix} = 0. \quad (8)$$

From equation (8) r-simultaneous equations are obtained as follows

$$\begin{aligned} z_1^n \frac{\partial T_1^n}{\partial \theta_1^n} + z_2^n \frac{\partial T_2^n}{\partial \theta_1^n} + \ldots + z_k^n \frac{\partial T_k^n}{\partial \theta_1^n} + \ldots + z_s^n \frac{\partial T_s^n}{\partial \theta_1^n} &= 0 \\ z_1^n \frac{\partial T_1^n}{\partial \theta_2^n} + z_2^n \frac{\partial T_2^n}{\partial \theta_2^n} + \ldots + z_k^n \frac{\partial T_k^n}{\partial \theta_2^n} + \ldots + z_k^n \frac{\partial T_k^n}{\partial \theta_2^n} &= 0 \qquad (9) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \quad & \;\; \vdots \\ z_1^n \frac{\partial T_1^n}{\partial \theta_r^n} + z_2^n \frac{\partial T_2^n}{\partial \theta_r^n} + \ldots + z_k^n \frac{\partial T_k^n}{\partial \theta_r^n} + \ldots + z_s^n \frac{\partial T_s^n}{\partial \theta_r^n} &= 0 \end{aligned}$$

If $r \geqslant s - 1$, then considering $z_1^n \frac{\partial T_1^n}{\partial \theta_j^n}$, $j = 1, 2, \ldots, r$ as constant, equation (9) may be written

$$
\begin{array}{l}
z_2^n \dfrac{\partial T_2^n}{\partial \theta_1^n} + z_3^n \dfrac{\partial T_3^n}{\partial \theta_1^n} + \cdots + z_k^n \dfrac{\partial T_k^n}{\partial \theta_1^n} + \cdots + z_s^n \dfrac{\partial T_s^n}{\partial \theta_1^n} = -z_1^n \dfrac{\partial T_1^n}{\partial \theta_1^n} \\[2ex]
z_2^n \dfrac{\partial T_2^n}{\partial \theta_2^n} + z_3^n \dfrac{\partial T_3^n}{\partial \theta_2^n} + \cdots + z_k^n \dfrac{\partial T_k^n}{\partial \theta_2^n} + \cdots + z_s^n \dfrac{\partial T_s^n}{\partial \theta_2^n} = -z_1^n \dfrac{\partial T_1^n}{\partial \theta_2^n} \\
\quad \vdots \\
z_2^n \dfrac{\partial T_2^n}{\partial \theta_r^n} + z_3^n \dfrac{\partial T_3^n}{\partial \theta_r^n} + \cdots + z_k^n \dfrac{\partial T_k^n}{\partial \theta_r^n} + \cdots + z_s^n \dfrac{\partial T_s^n}{\partial \theta_r^n} = -z_1^n \dfrac{\partial T_1^n}{\partial \theta_r^n}
\end{array}
\tag{10}
$$

From equation (10), we obtain

$$
z_k^n = \frac{\begin{vmatrix}
\dfrac{\partial T_2^n}{\partial \theta_1^n} & \dfrac{\partial T_3^n}{\partial \theta_1^n} & \cdots & -z_1^n \dfrac{\partial T_1^n}{\partial \theta_1^n} & \cdots & \dfrac{\partial T_s^n}{\partial \theta_1^n} \\[2ex]
\dfrac{\partial T_2^n}{\partial \theta_2^n} & \dfrac{\partial T_3^n}{\partial \theta_2^n} & \cdots & -z_1^n \dfrac{\partial T_1^n}{\partial \theta_2^n} & \cdots & \dfrac{\partial T_s^n}{\partial \theta_2^n} \\
\vdots & \vdots & & \vdots & & \vdots \\
\dfrac{\partial T_2^n}{\partial \theta_r^n} & \dfrac{\partial T_s^n}{\partial \theta_r^n} & \cdots & -z_1^n \dfrac{\partial T_1^n}{\partial \theta_r^n} & \cdots & \dfrac{\partial T_s^n}{\partial \theta_r^n}
\end{vmatrix}}{\begin{vmatrix}
\dfrac{\partial T_2^n}{\partial \theta_1^n} & \dfrac{\partial T_3^n}{\partial \theta_1^n} & \cdots & \dfrac{\partial T_k^n}{\partial \theta_1^n} & \cdots & \dfrac{\partial T_s^n}{\partial \theta_1^n} \\[2ex]
\dfrac{\partial T_2^n}{\partial \theta_2^n} & \dfrac{\partial T_3^n}{\partial \theta_2^n} & \cdots & \dfrac{\partial T_k^n}{\partial \theta_2^n} & \cdots & \dfrac{\partial T_s^n}{\partial \theta_2^n} \\
\vdots & \vdots & & \vdots & & \\
\dfrac{\partial T_2^n}{\partial \theta_r^n} & \dfrac{\partial T_3^n}{\partial \theta_r^n} & & \dfrac{\partial T_3^n}{\partial \theta_r^n} & \cdots & \dfrac{\partial T_s^n}{\partial \theta_r^n}
\end{vmatrix}}
\tag{11}
$$

or equation (11) reduces to

$$z_k^n = -z_1^n \frac{\begin{vmatrix} \frac{\partial T_2^n}{\partial \theta_1^n} & \frac{\partial T_3^n}{\partial \theta_1^n} & \cdots & \frac{\partial T_1^n}{\partial \theta_1^n} & \cdots & \frac{\partial T_s^n}{\partial \theta_1^n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \frac{\partial T_2^n}{\partial \theta_r^n} & \frac{\partial T_3^n}{\partial \theta_r^n} & \cdots & \frac{\partial T_1^n}{\partial \theta_r^n} & & \frac{\partial T_s^n}{\partial \theta_r^n} \end{vmatrix}}{B} \tag{12}$$

where B is the determinant of denominator in equation (11),

or

$$z_k^n = - z_1^n R_k^n \tag{13}$$

where R_k^n is the ratio of determinants in equation (12), and $R_1^n = -1$.

From equation (5), we obtain

$$z_k^{n-1} = \sum_{i=1}^{s} z_i^n \frac{\partial T_i^n}{\partial x_k^{n-1}}, \qquad k = 1, 2, \ldots, s, \tag{14}$$

then equation (13) yields

$$z_k^{n-1} = - z_1^{n-1} R_k^{n-1}, \tag{15}$$

and

$$z_i^n = - z_1^n R_i^n. \tag{16}$$

Substituting equations (15) and (16) into equation (14), we obtain

$$z_1^{n-1} R_k^{n-1} = z_1^n \sum_{i=1}^{s} \frac{\partial T_i^n}{\partial x_k^{n-1}} R_i^n. \tag{17}$$

Equation (17) is homogeneous in z_1^{n-1} and z_1^n, so they may be eliminated to obtain generalized multidimensional recurrence equations as follows.

From equations (13) and (5) respectively we obtain

$$z_j^{n-1} = - z_1^{n-1} R_j^{n-1}, \tag{18}$$

and

$$z_j^{n-1} = \sum_{i=1}^{s} z_i^n \frac{\partial T_i^n}{\partial x_j^{n-1}} . \tag{19}$$

Equating equations (18) and (19) yields

$$z_1^{n-1} = - \sum_{i=1}^{s} z_i^n \frac{\partial T_i^n}{\partial x_j^{n-1}} \cdot \frac{1}{R_j^{n-1}} . \tag{20}$$

Substituting the value of z_i^n from equation (13) into equation (20), we obtain

$$z_1^{n-1} = z_1^n \sum_{i=1}^{s} R_i^n \frac{\partial T_i^n}{\partial x_j^{n-1}} \cdot \frac{1}{R_j^{n-1}} . \tag{21}$$

Substituting equation (21) into equation (17) gives

$$z_1^n \sum_{i=1}^{s} R_i^n \frac{\partial T_i^n}{\partial x_j^{n-1}} \frac{1}{R_j^{n-1}} \cdot R_k^{n-1} = z_1^n \sum_{i=1}^{s} \frac{\partial T_i^n}{\partial x_k^{n-1}} R_i^n$$

or

$$R_k^{n-1} \sum_{i=1}^{s} \frac{\partial T_i^n}{\partial x_j^{n-1}} R_i^n - R_j^{n-1} \sum_{i=1}^{s} \frac{\partial T_i^n}{\partial x_k^{n-1}} R_i^n = 0 \tag{22}$$

Equation (22) is the optimum recurrence equation for multi-dimensional processes. It is called the generalized Euler Equations [3].

For the special case s = 2, equation (22) reduces to

$$R_k^{n-1}\left[\frac{\partial T_1^n}{\partial x_j^{n-1}} R_1^n + \frac{\partial T_2^n}{\partial x_j^{n-1}} R_2^n\right] - R_j^{n-1}\left[\frac{\partial T_1^n}{\partial x_k^{n-1}} R_1^n + \frac{\partial T_2^n}{\partial x_k^{n-1}} R_2^n\right] = 0. \quad (23)$$

Substituting $R_1^n = -1$ and

$$R_2^n = -\frac{z_2^n}{z_1^n} = \frac{\dfrac{\partial T_1^n}{\partial \theta^n}}{\dfrac{\partial T_2^n}{\partial \theta^n}} = \frac{A}{B} \text{ Say,}$$

into equation (23), we obtain

$$R_k^{n-1}\left[\frac{\partial T_2^n}{\partial x_j^{n-1}} \frac{A}{B} - \frac{\partial T_1^n}{\partial x_j^{n-1}}\right] + \left[\frac{\partial T_1^n}{\partial x_k^{n-1}} - \frac{\partial T_2^n}{\partial x_k^{n-1}} \frac{A}{B}\right] R_j^{n-1} = 0 \quad (24)$$

For k = 1, and j = 2, equation (24) reduces to

$$\frac{\partial T_1^n}{\partial x_2^{n-1}} - \frac{\partial T_2^n}{\partial x_2^{n-1}} \frac{A}{B} + \left[\frac{\partial T_1^n}{\partial x_1^{n-1}} - \frac{\partial T_2^n}{\partial x_1^{n-1}} \frac{A}{B}\right] \frac{C}{D} = 0 \quad (25)$$

where

$$R_1^{n-1} = -1,$$

and

$$R_2^{n-2} = \frac{\dfrac{\partial T_1^{n-1}}{\partial \theta^{n-1}}}{\dfrac{\partial T_2^{n-1}}{\partial \theta^{n-1}}} = \frac{C}{D}\,. \quad (26)$$

Assuming that performance equations are of the form

$$x_1^n = T_1^n = T\ (x_1^{n-1};\ \theta^n) \tag{27}$$

$$x_2^n = T_2^n = x_2^{n-1} + G\ (x_1^{n-1};\ \theta^n) \tag{28}$$

partial differentiation of equations (27) and (28) yields,

$$\frac{\partial T_1^n}{\partial x_2^{n-1}} = 0, \tag{29}$$

$$\frac{\partial T_2^n}{\partial x_2^{n-1}} = 1. \tag{30}$$

Substituting equations (29) and (30) into equation (25), we obtain

$$-\ (1)\ \frac{A}{B} = -\ \frac{C}{D}\left[\frac{\partial T_1^n}{\partial x_1^{n-1}} - \frac{\partial G}{\partial x_1^{n-1}}\ \frac{A}{B}\right], \tag{31}$$

or

$$\frac{D}{C} = \frac{B}{A}\left[\frac{\partial T_1^n}{\partial x_1^{n-1}} - \frac{\partial G}{\partial x_1^{n-1}}\ \frac{A}{B}\right]. \tag{32}$$

Substituting values of A, B, C and D from equations (23a) and (26) into equation (32), we obtain

$$\frac{\dfrac{\partial G}{\partial \theta^{n-1}}}{\dfrac{\partial T}{\partial \theta^{n-1}}} = \frac{\dfrac{\partial G}{\partial \theta^n}}{\dfrac{\partial T}{\partial \theta^n}}\left[\frac{\partial T^n}{\partial x_1^{n-1}} - \frac{\partial G}{\partial x_1^{n-1}} \cdot \frac{\dfrac{\partial T_1^n}{\partial \theta^n}}{\dfrac{\partial G^n}{\partial \theta^n}}\right]. \tag{33}$$

Substituting n = n+1 into equation (33), yields

$$\frac{\dfrac{\partial G}{\partial \theta^n}}{\dfrac{\partial T}{\partial \theta^n}} = \frac{\dfrac{\partial G}{\partial \theta^{n+1}}}{\dfrac{\partial G}{\partial \theta^{n+1}}} \cdot \frac{\partial T}{\partial x_1^n} - \frac{\partial G}{\partial x_1^n}, \tag{34}$$

$$n = 1, 2, \ldots, N-1.$$

Equation (34) is the same as the recurrence equation obtained for one-dimensional processes shown in Sec. 2.

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APPLICATIONS OF THE DISCRETE MAXIMUM PRINCIPLE
TO ONE-DIMENSIONAL PROCESSES

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The objective of this report is to demonstrate the applicability of the discrete maximum principle to one-dimensional multistage non-linear as well as linear processes frequently encountered in management and industry. Only deterministic cases are considered.

The basic algorithm of the discrete maximum principle is first stated. The general form of performance equations both for linear and non-linear one-dimensional processes is then given. The recurrence relation of the optimal state and decision for linear as well as non-linear processes are presented. Case studies, examples (1) through (5), are the problems of non-linear processes. These include a production scheduling plan, an allocation problem, a construction company's problem, a consulting engineer's problem, an advertising investment scheme and a production smoothing problem. Case studies, examples (5a) through (7), are the problems of linear processes. These include production smoothing, optimal sub-division and static consumer-choice problems.

In each of the examples considered, the discrete maximum principle leads to the optimality condition represented by a recurrence relation of the decision variable. Such a recurrence relation is generally valid for an n-stage system. For each of the special cases considered, such a general solution is reduced immediately to a specific solution which agrees with available results obtained by means of the Lagrange multiplier technique and by dynamic programming.

Although a particular method may not always be superior to all others it may be superior to others in solving a certain problem. In order to find the best technique for solving a particular problem, it is necessary to study comparatively all available techniques. Therefore, some examples are solved by more than one method.