SOME TRANSFORMATIONS OF THE COMPLEX PLANE CORRESPONDING TO A ROTATION OF THE SPHERE IN STEREOGRAPHIC PROJECTION

by

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INTRODUCTION

Mapping configurations from one plane to another, and the stereographic projection of circles are thoroughly discussed in the ordinary textbooks on functions of a complex variable.

In this paper we have established the well known relationships between the points of the complex plane and the tangent sphere. Using these relationships we have projected several simple configurations upon the sphere, rotated the sphere about any diameter, and projected them back upon the plane. We were especially interested in the transformation of a parabola by these methods.

TRANSFORMATION EQUATIONS FOR STEREOGRAPHIC PROJECTION

Complex numbers are commonly given geometric significance by interpreting them as points on a plane, but it is frequently convenient to interpret them as points on a sphere. The desired result may be accomplished by assuming the complex plane as before and supposing that we have a sphere tangent to this plane at the origin. We shall refer to the point of tangency as the south pole of the sphere, while the opposite pole will be spoken of as the north pole. If we now take the north pole O' (Fig. 1)

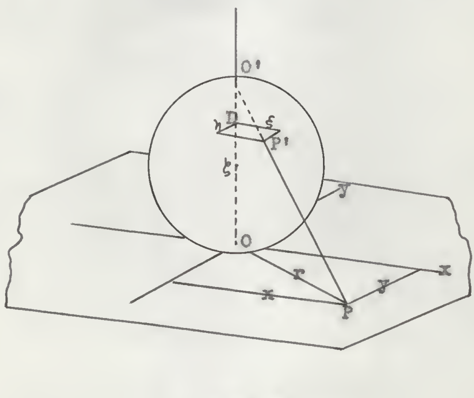

Fig. 1.

as the center of projection we can project in a definite manner every point of the plane upon the sphere. Thus in Fig. 1 the point P in the complex plane corresponds to the point P' of the sphere. In this way there corresponds to each point of the plane a definite point of the sphere, and conversely. This method of mapping the complex plane upon the sphere is called stereographic projection.

In order to determine the character of a configuration on the sphere and its relation to the corresponding configuration in the plane, we deduced the equations of transformation by means of which the cartesian space coordinates of any point upon the sphere can be expressed in terms of the cartesian coordinates of the corresponding point in the plane. Let ξ, η, ζ denote the coordinates of a point on the sphere. Let the ξ-axis and the η-axis coincide respectively with the axis of reals and the axis of imaginaries of the plane. Let the ζ-axis be perpendicular to the complex plane. Suppose the radius of the given sphere to be $\frac{1}{2}$. The equation of the sphere is

$$\xi^2 + \eta^2 = \zeta(1-\zeta).$$

If we now denote by x, y the coordinates of any point P in the plane, the coordinates ξ, η, ζ of the projection P' upon the sphere of the point P are readily found in terms of

x, y, r, where $r = \sqrt{x^2 + y^2}$.

In Fig. 1 from the similar triangles O'DP, O'OP, we get

$$\frac{1 - \zeta}{DP'} = \frac{1}{r} \quad \text{and} \quad r = \frac{DP'}{1-\zeta}.$$

Squaring each side of the equation we get

$$r^2 = \frac{DP'^2}{(1-\zeta)^2} \quad \text{but} \quad DP'^2 = \xi^2 + \eta^2 = \zeta(1-\zeta) \quad \text{and} \quad r^2 = \frac{\zeta}{1-\zeta}.$$

Again from Fig. 1 in the $\xi\zeta$-plane we have

$$\frac{\xi}{x} = \frac{1-\zeta}{1}, \quad x = \frac{\xi}{1-\zeta}$$

and from the $\eta\zeta$-plane

$$y = \frac{\eta}{1-\zeta}.$$

Also, we can solve for ξ, η, ζ and get

$$\xi = \frac{x}{1+r^2}, \quad \eta = \frac{y}{1+r^2}, \quad \zeta = \frac{r^2}{1+r^2}$$

SOME CONFIGURATIONS IN STEREOGRAPHIC PROJECTION

The equation of a circle in the plane may be written

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

When projected on the sphere it becomes

$$\frac{\xi^2}{(1-\zeta)^2} + \frac{\eta^2}{(1-\zeta)^2} + 2g\frac{\xi}{1-\zeta} + 2f\frac{\eta}{1-\zeta} + c = 0$$

but from the equation of the sphere $\xi^2 + \eta^2 = \zeta(1-\zeta)$. Replacing $\xi^2 + \eta^2$ by this value, and multiplying each term by $(1-\zeta)$ we get

$$\zeta + 2g\xi + 2f\eta + c(1-\zeta) = 0.$$

This is the equation of a plane, and the curve of intersection of this plane and the given sphere is a circle. So, any circle on the plane projects into a circle of the sphere.

Let us now consider the following proposition:

Great circles through the ends of a fixed diameter of a sphere project into a family of coaxal circles, and the small circles whose planes are perpendicular to this diameter project into the orthogonal system of circles.

To prove this statement, let us use the sphere $\xi^2 + \eta^2 = \zeta(1-\zeta)$ whose ξ-axis and η-axis coincide respectively with the x-axis and y-axis of the plane; and let (a,b,c) be the space coordinates of one end of the fixed diameter. The equation of the planes through the center of the sphere, $(0,0,\frac{1}{2})$, is

$$\xi + A\eta + B(\zeta-\tfrac{1}{2}) = 0$$

where A and B are any constants. If these planes are also to pass through (a,b,c) we can write

$$a + Ab + B(c-\tfrac{1}{2}) = 0.$$

When B is eliminated between these two eq uations we have

$$\xi(1-2c) + A\eta(1-2c) + 2\zeta(a+Ab) - (Ab+a) = 0 \qquad (1)$$

which is the equation of a sheaf of planes through $(0,0,\frac{1}{2})$ and (a,b,c). Projected on the plane we have

$$(x^2+y^2)(Ab+a) + x(1-2c) + Ay(1-2c)-(Ab+a) = 0. \qquad (2)$$

Equation (2) defines a family of circles with centers at the points $\frac{2c-1}{Ab+a}\ \frac{A(2c-1)}{Ab+a}$. As A takes different values these points define a line of slope $\frac{a(2c-1)}{b(1-2c)}$. Hence, the circles (2) are coaxal.

Let the equation of the family of parallel planes which are perpendicular to the sheaf (1) be

$$\xi + H\eta + K\zeta = 0 \qquad (3)$$

where H and K are constants to be determined. One of these planes must pass through the point (a,b,c); so (3) may be written

$$a + Hb + Kc = 0 \qquad (4)$$

for a particular plane. Now if equations (3) and (4) are to represent perpendicular planes, their coefficients must have this relationship*

$$(1-2c) + HA(1-2c) + 2K(a+Ab) = 0. \qquad (5)$$

Solving equation (5) for H in terms of a,b,c,A,K and substituting this value into (4) we can solve for K in terms a,b,c,A. Substituting this value in (3) we get

* See Roberts and Colpitts, Analytic Geometry, Art. 116.

$$\xi\left[A(Ac-4Ac^2+4Ac^3-2ab+4abc-2Ab^2+4Ab^2c)\right]$$
$$-\eta\left[A(4Aabc+4a^2c-2Aab-2a^2+4c^3-4c^2+c)\right]$$
$$+\zeta\left[A(4Aac+4bc^2-4Aac^2-4bc-Ac+b)\right] = 0 \qquad (6)$$

which is the equation of the parallel planes perpendicular to the sheaf (1). When the circles of intersection of these planes and the sphere are projected upon the plane we have

$$(x^2+y^2)\left[A(4Aac+4bc^2-4Aac^2-4bc-Ac+b)\right]$$
$$+x\left[A(Ac-4Ac^2+4Ac^3-2ab+4abc-2Ab^2+4Ab^2c)\right]$$
$$-y\left[A(4Aabc+4a^2c-2Aab-2a^2+4c^3-4c^2+c)\right]$$
$$-A(4Aac+4bc^2-4Aac^2-4bc-Ac+b) = 0 \qquad (7)$$

which is a family of circles in the plane.

If the equations of two circles are represented by

$$x^2+y^2+2f_1x+2g_1y+c_1 = 0$$
$$x^2+y^2+2f_2x+2g_2y+c_2 = 0$$

then the circles are orthogonal if $2f_1f_2+2g_1g_2-c_1-c_2 = 0$.* Applying this test to equation (2) and (7) we prove the two families of circles are orthogonal. This result is obvious in light of the fact that mapping by stereographic projection is always conformal.**

Let us now prove that if z_1, z_2 are the stereographic projections of the ends of a diameter of the sphere, then $z_1\bar{z}_2 = -1$. ($\bar{z}_2$ is the conjugate of z_2.)

* See W. B. Smith, Co-ordinate Geometry, Art. 81.
** See Townsend, Functions of a Complex Variable, Art. 40.

Let (a,b,c) and (-a,-b,-c) be the coordinates of the ends of the given diameter. Substituting for these values their values after projection on the plane we get

$$x_1 = \frac{a}{1-c}, \qquad y_1 = \frac{b}{1-c}, \qquad z_1 = \frac{a+ib}{1-c}$$

$$x_2 = \frac{-a}{1+c}, \qquad y_2 = \frac{-b}{1+c}, \qquad z_2 = \frac{-a-ib}{1+c}, \qquad \bar{z}_2 = \frac{-a+ib}{1+c}$$

$$z_1\bar{z}_2 = \frac{a+ib}{1-c}\;\frac{-a+ib}{1+c} = \frac{-a^2-b^2}{1-c^2} = -1$$

since a,b,c satisfy the equation of the sphere.

STEREOGRAPHIC PROJECTION AND ROTATION OF THE SPHERE

Let us consider now the following proposition:

If the sphere be rotated through an angle θ about any diameter the corresponding transformation of the complex plane is given by

$$\frac{z'-z_1}{z'-z_2} = (\cos\theta + i\sin\theta)\frac{z-z_1}{z-z_2} \tag{8}$$

where z_1 and z_2 are the projections of the ends of the diameter.

In proving this it is easily seen that when the sphere is rotated any point moves along a circle of the family (6), hence, its projection on the plane must necessarily move along a circle of the family (7), say circle C, Fig. 2.

In Fig. 2 (we are here using the notation of ordinary geometry for points, lines, and angles.) $\angle z_1Oz = \angle z_1Oz$, and $\angle Ozz_1 = \angle Oz_2z$ since each is measured by one-half of the same arc. Therefore, $\Delta Oz_1z \sim \Delta Oz_2z$ and $Oz_1:Oz = Oz:Oz_2$. But $Oz = OR$, and the circle C divides line z_1z_2 so that $Oz_1:OR = OR:Oz_2$. Then from elementary geometry the ratio $z_1P:z_2P$ is a constant where P is any point on the circle C. It follows that $z_1z':z'z_2 = z_1z:zz_2$, or (using the notation of complex variable) $\left|\frac{z'-z_1}{z'-z_2}\right| = \left|\frac{z-z_1}{z-z_2}\right|$.

Now consider the amplitudes of equation (8). Equation (1) is a sheaf of planes intersecting the sphere in the great circles through the ends of the diameter of rotation. If the sphere is rotated through an angle θ, a point on the sphere moves from one of these circles to another separated from the first by angle θ. The projection of this point in the plane must move between the projections of these two circles (S and S', Fig. 2) along one of the circles (7), (C, Fig. 2). Since stereographic projection is conformal, θ is the angle between the tangents to S and S' at their points of intersection, and is therefore the angle between their radii at those points.

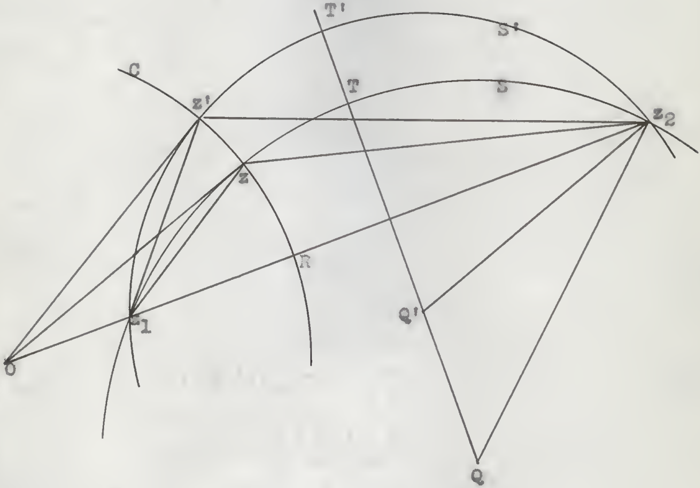

Fig. 2.

From Fig. 2,

$$\angle z_1z_2z' + \angle z_2z_1z' = \angle z_2Q'T' \tag{9}$$

$$\angle z_1z_2z + \angle z_2z_1z = \angle z_2QT \tag{10}$$

subtracting (10) from (9)

$$\angle z_1z_2z' + \angle z_2z_1z' - \angle z_1z_2z - \angle z_2z_1z = \angle z_2Q'T' - \angle z_2QT \tag{11}$$

substituting in (11) from the figure we get

$$\angle zz_1z' + \angle zz_2z' = \angle Qz_2Q'. \tag{12}$$

Now we rewrite (8) to read

$$\frac{z'-z_1}{z-z_1}\cdot\frac{z-z_2}{z'-z_2} = \cos\theta + i\sin\theta.$$

From the figure we write

$$\operatorname{amp}\frac{z'-z_1}{z-z_1} = \angle zz_1z', \quad \operatorname{amp}\frac{z-z_2}{z'-z_2} = \angle zz_2z', \quad \operatorname{amp}(\cos\theta + i\sin\theta) = \theta$$

and compare with (12). The proposition is therefore proved.

It is interesting to note that if (8) is solved for z' we get

$$z' = \frac{z\left[z_1 - z_2(\cos\theta + i\sin\theta)\right] - \left[z_1z_2 - z_1z_2(\cos\theta + i\sin\theta)\right]}{z\left[1 - (\cos\theta + i\sin\theta)\right] - \left[z_2 - z_1(\cos\theta + i\sin\theta)\right]}$$

and on dividing each term of the right member by $\cos(\frac{1}{2}\theta + \operatorname{amp} z_1) + i\sin(\frac{1}{2}\theta + \operatorname{amp} z_1)$ we get a transformation of the form $z' = \frac{az+b}{cz+d}$ where $d = \bar{a}$ and $b = -\bar{c}$; hence, we can write $z' = \frac{az-\bar{c}}{cz+\bar{a}}$.

To project configurations from the plane to the sphere, rotate the sphere, and project back on the plane, it would seem desirable to solve (8) for x' and y' in terms of z_1, z_2, and θ, ($z=x+iy$, $z_1=x_1+iy_1$, $z_2=x_2+iy_2$) but we get

$$\begin{aligned}
x' = {} & x^2\left[x_2+x_1-\cos\theta(x_1+x_2)+\sin\theta(y_2-y_1)\right] \\
& +y^2\left[x_2+x_1+\sin\theta(y_2-y_1)-\cos\theta(x_2+x_1)\right] \\
& +x\left[-4x_1x_2+\cos\theta(2x_1x_2-2y_1y_2+x_1^2+y_1^2-x_2^2+y_2^2)\right] \\
& +y\left[-2x_2y_1-2x_1y_2+\sin\theta(x_1^2+y_1^2-x_2^2-y_2^2)+2\cos\theta(x_2y_1+x_1y_2)\right] \\
& +x_1^2x_2+x_1x_2^2+x_1y_2^2+x_2y_1^2 \\
& +\sin\theta(x_2^2y_1-x_1^2y_2+y_1y_2^2-y_1^2y_2) \\
& -\cos\theta(x_1x_2^2+x_1^2x_2+x_1y_2^2+x_2y_1^2)
\end{aligned}$$

over this denominator

$$\begin{aligned}(x^2+y^2)&\left[2-2\cos\theta\right]\\ &+x\left[2(\cos\theta-1)(x_2+x_1)+2\sin\theta(y_2-y_1)\right]\\ &+y\left[2(\cos\theta-1)(y_1+y_2)+2\sin\theta(x_1-x_2)\right]\\ &+x_1^2+y_1^2+x_2^2+y_2^2-2\cos\theta(x_1x_2+y_1y_2)+2\sin\theta(x_2y_1-x_1y_2)\end{aligned} \tag{13}$$

and y' is a similar expression:

$$y' = f(x, y) \tag{14}$$

In these equations we can replace θ by $-\theta$, interchange x' and x, and y' and y, and have transformation equations solved for x and y. (For if a rotation of the sphere through an angle θ transforms x into x', then another rotation through an angle $-\theta$ will put x' back to its original position, x.) Now we can write the equation of a configuration in the plane, replace x and y by their values in equations (13) and (14), and have the equation of the curve resulting from a rotation of the sphere. But equations (13) and (14) are so unwieldy this would be a laborious method of procedure. We can get the same effects by a less tedious method.

ELLIPTIC TRANSFORMATIONS

When the sphere is rotated about a diameter, the projections of the ends of the diameter, z_1 and z_2, remain invariant in the plane. The projection of any other point on the sphere will move along one of the circles of equation (7) around either z_1 or z_2. This circular motion of points in the plane about two, finite, distinct, invariant points is an elliptic transformation of the plane by definition. If z_1 and z_2 are not finite and distinct, the diameter of rotation must be vertical. We disregard this special case. We will now refer to the transformation represented by a rotation of the sphere about a diameter as an elliptic transformation of the plane.

It is easily shown* that the equation

$$z' = \frac{z}{uz+v} \tag{15}$$

represents an elliptic transformation of the plane when u is any complex number, and $v = re^{i\theta}$ where $r = 1$, $\theta \neq 0$.

If we write $u = a+ib$, $v = c+id$, and solve (15) for x' and y', we get

$$x' = \frac{a(x^2+y^2)+cx+dy}{(a^2+b^2)(x^2+y^2)+2x(ac+bd)+2y(ad-cb)+c^2+d^2}$$

$$y' = -\frac{b(x^2+y^2)+dx-cy}{(a^2+b^2)(x^2+y^2)+2x(ac+bd)+2y(ad-cb)+c^2+d^2}. \tag{16}$$

If we solve (15) for x and y, we get

$$x = -\frac{(ac+bd)(x'^2+y'^2)-cx'+dy'}{(a^2+b^2)(x'^2+y'^2)-2ax'+2by'+1}$$

$$y = \frac{(cb-ad)(x'^2+y'^2)+dx+cy}{(a^2+b^2)(x'^2+y'^2)-2ax'+2by'+1}. \tag{17}$$

* See E. J. Townsend, Functions of a Complex Variable, Art. 41.

Now we can write the equation of a curve in the plane as

$$f(x, y) = 0 \tag{18}$$

and substituting for x and y their values in (17) we have the equation of the curve resulting from the transformation (15).

Observe that in (17) x and y are of first degree, and the right members are of second degree in x' and y'. In general then, a curve of degree n in x and y will become of degree 2n in x' and y' after an elliptic transformation. Circles, however, are an exception to this general rule for all circles transform into circles. It is here understood that a straight line is to be considered a circle of infinite radius.

If any point is transformed into itself by the transformation (15), we must have $z = \frac{z}{uz+v}$, that is $z^2u+z(v-1) = 0$. This equation has two roots, 0 and $\frac{1-v}{u}$. These are the points which remain invariant by the transformation. Hence a curve through either of these points passes through the same point after transformation.

If we let $z' = \infty$ in (15), we get $z = -\frac{v}{u}$. This is the point which goes to infinity by the transformation. Hence a curve through this point becomes infinite after transformation.

If we let $z = \infty$ in (15), we get $z' = \frac{1}{u}$. This is the point into which infinite values of z are transformed.

In the case of a parabola, either branch of the curve is infinite for infinite values of z. Therefore the transformation puts each branch of the parabola through the point $\frac{1}{u}$. Now the two tangents to a parabola are parallel for infinite values of z. Hence, by conformal mapping the transformed curve must have two parallel tangents through the point $\frac{1}{u}$, i.e., a cusp at that point

We now assign specific values to u and v and examine the transformation of a few simple curves.

We find it convenient to plot these curves in parametric form, i.e., assigning to x and y values satisfying (18), and substituting these values in (16) we get the corresponding points on the transformed curve.

For $u = 1$ and $v = i$

$y = x$	transforms	into	a	circle	through	$(0,0)$, $(1,0)$, $(0,1)$;
$y = -x$	"	"	"	"	"	$(0,0)$, $(1,0)$, $(0,-1)$;
$x = 1$	"	"	"	"	"	$(1,0)$, $(\frac{1}{2},-\frac{1}{2})$, $(1,-1)$;
$x = -1$	"	"	"	"	"	$(1,0)$, $(\frac{1}{2},\frac{1}{2})$, $(1,1)$;
$y = 1$	"	"	"	"	"	$(1,0)$, $(3/4,\frac{1}{2})$, $(\frac{1}{2},0)$;
$y = -1$	"	"	the line			$x' = \frac{1}{2}$;
$x^2+y^2 = 1$	"	"	"	"		$x' = \frac{1}{2}$;

$x^2+y^2 = 4$ transforms into a circle through $(2/3,0)$, $(2,0)$, $(4/5, 2/5)$;

$y^2 = x$	" " the quartic	(Plate I, Fig. 1.);
$y^2 = -x$	" " " "	(" I, Fig. 2.);
$x^2 = y$	" " " "	(" I, Fig. 3.);
$x^2 = -y$	" " " "	(" I, Fig. 4.);

$x^2-x\sqrt{2}+y\sqrt{2}+y^2+2xy = 0$ transforms into the quartic (" II, Fig. 1.);

$x^2+x\sqrt{2}+y\sqrt{2}+y^2-2xy = 0$ transforms into the quartic (" II, Fig. 2.);

for $u = 1$ and $v = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$y^2 = x$ transforms into the quartic (" II, Fig. 3.);

for $u = 2$ and $v = i$

$y^2 = x$ transforms into the quartic (" II, Fig. 4.).

Plate I

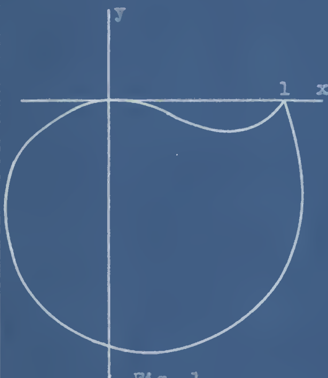

Fig. 1.

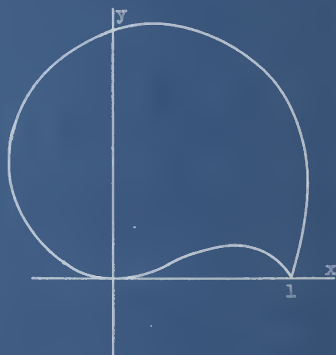

Fig. 2

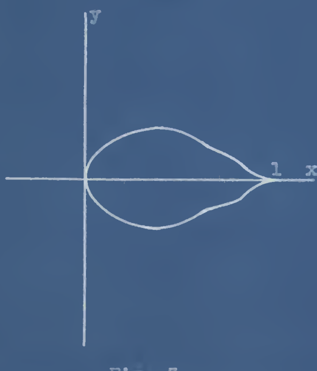

Fig. 3.

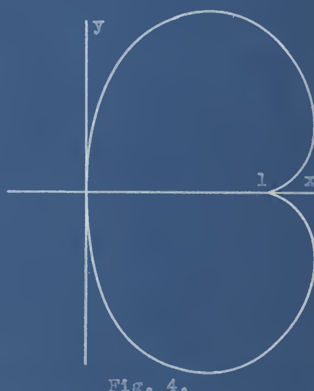

Fig. 4.

Plate II

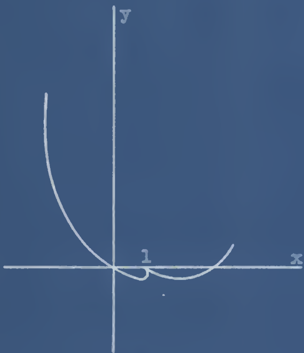

Fig. 1.

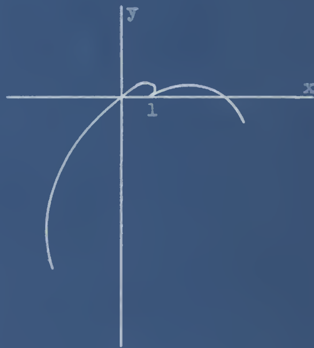

Fig. 2.

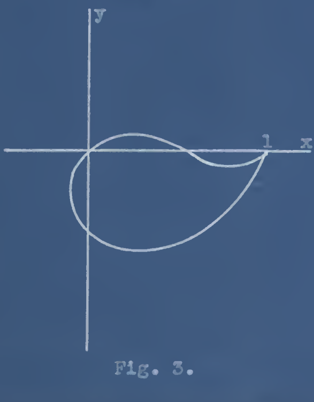

Fig. 3.

Fig. 4.

CONCLUSION

If the sphere be rotated through an angle θ about a fixed diameter, the corresponding transformation of the plane is given by

$$\frac{z'-z_1}{z'-z_2} = (\cos\theta + i\sin\theta)\frac{z-z_1}{z-z_2}$$

where z_1 and z_2 are the stereographic projections of the ends of the fixed diameter.

The transformation of the plane corresponding to a rotation of the sphere about a diameter is always elliptic, and may be written

$$z' = \frac{z}{uz+v}$$

when u is any complex number, and $v = re^{i\theta}$ with $r = 1$ and $\theta \neq 0$.

By an elliptic transformation a parabola becomes a quartic with a cusp at the point into which infinite values of z are transformed.

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