BEAMS ON ONE-WAY ELASTIC FOUNDATIONS

by 1264

KOU-KWANG LIN

B.S., National Taiwan University, 1964

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Civil Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1970

Approved by:

Wayne W. Williams

Major Professor

CONTENTS

INTRODUCTION

Mat foundations under certain structures, such as silos, water-storage tanks, coal-storage towers, etc., and footing foundations supporting a group of columns, are frequently designed and constructed in the form of beams resting on soil. This analysis requires some assumptions on properties and behavior of the soil-foundation system. The theory of the bending of beams on an elastic foundation, developed by E. Winkler in 1867, is based on the assumption that the intensity of the continuously distributed reaction of the foundation at every point is proportional to the deflection at that point (1). Its application to the design of foundations has received considerable attention. Since then, some refinements and various assumptions have been made by others, notably Hetenyi, Biot, Vesić, Levinton, Malter, and Bowles (2). One common feature of these works is that the foundation can take tensile stress. Recently, Nien-chien Tsai and Russell A. Westmann have indicated an approach based on tensionless foundation assumption to account for the effects of beam uplift (3). No doubt, it satisfies the actual conditions of real soil under elastic theory. A study of the beam-foundation problem, following this latter approach will be presented and developed by a matrix formulation for the numerical evaluation of the problem. Consequently, this process can be applied to the design of foundations.

PURPOSE OF THE STUDY

In this report, an analysis is made of the response of a loaded beam resting on a Winkler type foundation wherein the foundation properties of tension and compression at the interface will be relaxed by assuming the subgrade can take compression only. A review of Winkler's assumption modified by Tsai (3) (i.e., the tensionless foundation) is examined in terms of a matrix formulation. An iterative solution of a typical beam resting on a micaceous silt subgrade is presented. The subgrade is represented by equally spaced springs with a stiffness per unit length the same as the actual subgrade. The cases of finite and infinite beams under the action of concentrated center loads are examined. The results are compared with test results obtained by Vesic(1), Bowles' classical solution(2), and by Tsai and Westmann(3) to verify the effects of the assumption that the foundation interface can support compression only and to investigate the equally spaced spring analogy.

REVIEW OF LITERATURE

The analysis of beams on elastic foundation has been a classical problem in engineering mechanics. A theory of bending of beams on an elastic foundation appears to have been developed first by Winkler who used an elastic solution of the problem by assuming that the continuous reaction of the foundation is proportional to the deflection. Following this simple assumption, Timoshenko successively established a general solution to the differential equation which expressed the beam deflection in terms of foundation reaction (4). In 1946, Hetenyi made a comprehensive study of beam-foundation problems following the theory of elasticity and the basic mathematical relationship between the subgrade reaction and settlement (5). He summarized the analysis of the response of beam-foundation systems acted upon by different load conditions. Some notable mathematic techniques in solving the problem (such as redundant reaction method, and finite difference method) have been developed by Levinton and Malters in 1947 and 1960, respectively (6, 7). Both approaches, initially assumed the type of pressure distribution under the foundation (for instance, parabolic, stepped, or linear etc.) and then computed the equivalent reactions. About 1959, Leonards and Harr simplified the problem formulation and solution by assuming that the foundation could take tension(8).

A further refinement was made by Kerr in 1964, assuming that the subgrade properties are identical in tension and compression (9). The common feature to all of these works is the assumed mode of stress transfer across the beam-foundation interface. Usually the resulting analysis based on this classical solution is not acceptable, particularly in dealing with the infinite beam, because of beam uplift. Recently, Tsai and Russell indicated an approach which considered both the Winkler assumption and the uplift effects of the beam and simplified the problem formulation and solution by assuming that the foundation can take compression only. Apparently, this tensionless foundation solution is more compatible with the condition of real soil.

STATEMENT OF THE PROBLEM

In the classical solution for beams on foundations, it is usual to assume that foundation properties are identical in tension and compression. Often the resulting analysis then indicates an alternating reaction thus implying the foundation can support a tensile stress. Usually this is not an acceptable result for real soil. Therefore, the Winkler model should be modified to take into account the effect of beam uplift. This will then lead to a non-linear solution(3). As the beam is supported along its entire length by a continuous elastic medium, the problem formulation and solution can be made by assuming that the beam rests on "one-way", equally spaced, elastic springs. The more springs chosen along the length of the beam, the closer the analogy is to the continuous medium. The subgrade tensile stress in the uplift portion of beam can be relaxed simply by setting the spring constants of those portions equal to zero. In short, two basic assumptions are made: (1) the subgrade can take compression only; and, (2) the compressive stress in the foundation is proportional to the deflection.

OUTLINE OF THE STUDY

1. Present the matrix formulation of beams on elastic spring supports which are regarded as analogous to beams on elastic foundations(10).

2. Choose a typical beam on a micaceous silt subgrade as an illustration of the application of matrix formulation and numerical evaluation.

3. Perform the iteration process using a computer program written in Fortran IV to obtain the deflections of long and short beams under the action of concentrated center loads. Hence, visualize the behavior of the beams resting upon elastic foundations based on the modified Winkler assumption-i.e., the tensionless foundation proposed by Tsai and Westmann(3).

4. Compare results obtained with Vesic's results(1).

5. Compare results obtained with Bowles' elastic solutions(2).

6. Compare results obtained with classical solutions and non-linear solutions of Tsai and Westmann(3).

PROBLEM FORMULATION

Elastic solutions of beam-foundation problems are based on the assumption that soil behaves as an elastic, homogeneous, infinite, and isortropic solid, defined by a modulus of deformation, Es, and a Poisson's ratio, v, and obeying Hooke's law. It is further assumed that there are no shearing stresses at the contact between beam and soil. In addition, possible influences of soil overburden on pressure distribution are neglected. If the problem is so posed, Winkler's model can be replaced by a continuous beam resting on a set of springs with stiffness constant K. (2) Its value is defined by

$$K = Ks' \times a$$

where

$Ks' = Ks \times B$ = modulus of subgrade reaction x width of beam.

a = cell length of beam (distance between springs equally spaced).

Once the problem is set up, it can be visualized as a continuous beam of a finite number of spans supported by a row of springs. The solution of this problem then can be expressed by a matrix formulation as follows. Let us consider a beam supported by five equally spaced springs, shown in Fig. 1, where a = cell length of beam, γ = uniform dead load, and Q = concentrated center load.

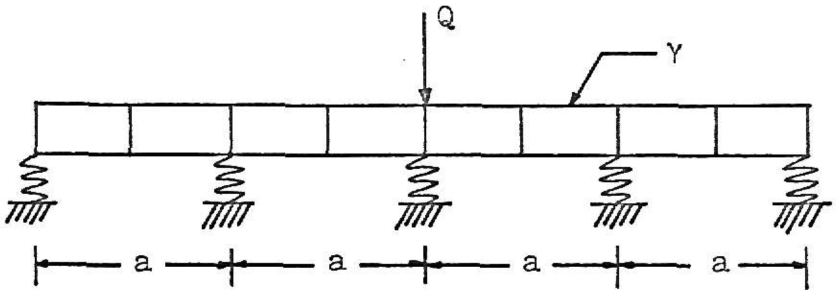


Fig. 1 - The Given Beam

1. Load Matrix [P] and Displacement Matrix [X]

The load matrix [P] is defined as a column vector whose elements are the externally applied loads. Each load P_i, accordingly, is a component of the load matrix [P]. The displacement matrix [X] consists of the displacements at the points of application of the load vector components measured in the same directions as the loads. Referring to Fig. 2, the load matrix [P] is expressed by

$$[P] = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \\ P_{10} \end{bmatrix} \tag{1}$$

and the displacement matrix $[X]$ is expressed by

$$[X] = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \\ X_9 \\ X_{10} \end{bmatrix} \qquad (2)$$

Consider the beam shown in Fig. 1. The load matrix $[P]$ can be obtained easily by solving the joint equilibrium equation.

Refer to Fig. 3.

$$P_1 = \frac{ra^2}{12}$$

$$P_2 = P_3 = P_4 = \frac{ra^2}{12} - \frac{ra^2}{12} = 0$$

$$P_5 = -\frac{ra^2}{12}$$

$$P_6 = \frac{ra}{2}$$

$$P_7 = ra$$

$$P_8 = ra + Q$$

$$P_9 = ra$$

$$P_{10} = \frac{ra}{2}$$

Substituting into Eq. 1,

$$[P] = \begin{bmatrix} \frac{ra^2}{12} \\ 0 \\ 0 \\ 0 \\ \frac{-ra^2}{12} \\ \frac{ra}{2} \\ ra \\ ra + Q \\ ra \\ \frac{ra}{2} \end{bmatrix}$$

2. Deformation Matrix [e] and Force Matrix [F]

A deformation matrix [e] consisting of member deformations e_i at any joint can be defined for any structure. There will be a subset for each member. All relative movements of the end joints of the member are included in the subset of the deformation matrix for the member. Referring to Fig. 4 and Fig. 5, matrix [e] can be expressed by

$$[e] = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_9 \\ e_{10} \\ e_{11} \\ e_{12} \\ e_{13} \end{bmatrix} \qquad (3)$$

A force matrix [F] corresponding to the deformation matrix [e] is defined to consist of components that are the end forces of the members of a structure. (springs are also considered as members, taking axial force on the end.) Therefore, it can be expressed by

$$[F] = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ F_{11} \\ F_{12} \\ F_{13} \end{bmatrix} \qquad (4)$$

3. The Force-Load Matrix [A]

The transformation matrix necessary to transform member forces to loads can be found by observing joint equilibrium. Refer to Fig. 6.

Then

$$P_1 = F_1$$

$$P_2 = F_2 + F_3$$

$$P_3 = F_4 + F_5$$

$$P_4 = F_6 + F_7$$

$$P_5 = F_8$$

$$P_6 = -F_a + \frac{F_1 + F_2}{a}$$

$$P_7 = -F_{10} - \frac{F_1 + F_2}{a} + \frac{F_3 + F_4}{a}$$

$$P_8 = -F_{11} - \frac{F_3 + F_4}{a} + \frac{F_5 + F_6}{a}$$

$$P_9 = -F_{12} - \frac{F_5 + F_6}{a} + \frac{F_7 + F_8}{a}$$

$$P_{10} = -F_{13} - \frac{F_7 + F_8}{a}$$

In matrix notation, $[P] = [A][F]$ (5)

in which

[A] =

P \ F	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1												
2		1	1										
3				1	1								
4						1	1						
5								1					
6	1/a	1/a							-1				
7	-1/a	-1/a	1/a	1/a						-1			
8			-1/a	-1/a	1/a	1/a					-1		
9					-1/a	-1/a	1/a	1/a				-1	
10							-1/a	-1/a					-1

4. Stiffness Matrix [S]

A stiffness matrix is a transformation matrix [S] which transforms deformation into forces according to Eq. 6.

$$[F] = [S]\ [e] \tag{6}$$

Where the matrix [S] for a single member can be established as follows

$$F_i = ({}^{4EI}/_a)\ e_i + ({}^{2EI}/_a)\ e_j$$

$$F_j = ({}^{2EI}/_a)\ e_i + ({}^{4EI}/_a)\ e_j$$

in which EI = flexural rigidity of beam

a = cell length of beam (distance between springs equally spaced)

These are basically just the slope-deflection equations for the beam segment. For the given beam, the stiffness matrix [S] is expressed by

[S]=

F \ e	1	2	3	4	5	6	7	8	9	10	11	12	13
1	$\frac{4EI}{a}$	$\frac{2EI}{a}$											
2	$\frac{2EI}{a}$	$\frac{4EI}{a}$											
3			$\frac{4EI}{a}$	$\frac{2EI}{a}$									
4			$\frac{2EI}{a}$	$\frac{4EI}{a}$									
5					$\frac{4EI}{a}$	$\frac{2EI}{a}$							
6					$\frac{2EI}{a}$	$\frac{4EI}{a}$							
7							$\frac{4EI}{a}$	$\frac{2EI}{a}$					
8							$\frac{2EI}{a}$	$\frac{4EI}{a}$					
9									K				
10										K			
11											K		
12												K	
13													K

in which K is a spring constant related to the axial force and deformation of a spring by $K = {}^{F}/_{e}$

Let NP be the degrees of freedom in rotation and translation of the elastic supports, NF be the number of internal end forces, and NLC be the number of loading conditions (10).

Then

$$[P]_{NP \times NLC} = [A]_{NP \times NF} [F]_{NF \times NLC} \tag{5}$$

$$[F]_{NF \times NLC} = [S]_{NF \times NF} [e]_{NF \times NLC} \tag{6}$$

and $$[e]_{NF \times NLC} = [B]_{NF \times NP} [X]_{NP \times NLC} \tag{7}$$

where [B] is a transformation matrix which transforms joint displacements [X] to member deformation [e]. A proof that [B] is always the transpose of the force-load matrix [A] follows: A structure in equilibrium undergoes an arbitrary virtual displacement δX compatible with the boundary conditions. The corresponding virtual deformations of the ends of the members are δe.

By the principle of virtual work:

$$[P^T] \cdot [\delta X] = [F^T] \cdot [\delta e]$$

From Eq. 5, $[P] = [A] \cdot [F]$ Then

$$[F^T][A^T] \cdot [\delta X] = [F^T] \cdot [\delta e]$$
$$= [F^T][B][\delta X]$$

Therefore, $[B] = [A^T]$

$$[e]_{NF \times NLC} = [A^T]_{NF \times NP} [X]_{NP \times NLC}$$

Substituting (7) to (6)

$$[F]_{NF \times NLC} = [SA^T]_{NF \times NP} [X]_{NP \times NLC} \tag{8}$$

Substituting (8) to (5)

$$[P]_{NP \times NLC} = [ASA^T]_{NP \times NP} [X]_{NP \times NLC} \tag{9}$$

from which

$$[X]_{NP \times NLC} = [ASA^T]^{-1}_{NP \times NP} [P]_{NP \times NLC} \tag{10}$$

As an illustration of the matrix solution to the present problem, numerical examples were analyzed with the IBM 360-50 digital computer using a program written in Fortran IV. The iteration for the tensionless foundation solution is accomplished by setting the spring constants in the stiffness matrix [S] equal to zero for those points wherein the deflections are upward and then recalculating the deflections.

NUMERICAL EXAMPLES

Four numerical examples are presented. The computer program, as shown in Appendix C is used to obtain the displacement matrix [X].

Example 1. - A short beam which has unit weight included in the analysis (Fig. 7), for the purpose of a comparison with results (1) shown in Fig. 11. Beam length L = 72 in., center load Q = 8250#, unit weight γ = 31 #/ft. spring constant $\bar{K}$ = 0.00215 (K/ft.).* Cross section properties of the beam and subgrade are shown in Table 1 and Table II.

Example 2. -

Case A - A short beam with unit weight not included in the analysis (Fig. 8). The results are compared with Bowles finite and infinite beam solutions (2) and plotted in Fig. 12. L = 2.4 ft., Q = 32.688 kips, K = 43.584 K/ft. Cross section properties of beam and subgrade are shown in Table I and Table II.

Case B - A long beam with the unit weight not included in the analysis (Fig. 9). The results are compared with Bowles' infinite solution (2) and plotted in Fig. 13. L = 84 ft., Q = 32.688 kips, a = 3 ft., K = 196 K/ft. Cross section properties of beam and subgrade are shown in Table I and Table II.

---

* $$\bar{K} = \frac{K}{30.5 \times 10^6} \times \frac{215}{144}$$

Example 3. - A long beam with unit weight included in the analysis (Fig. 10). As the problem of the tensionless foundation is of prime interest, attention has been concentrated on the solutions for the 8WF31 steel beam resting on a micaceous silt subgrade and loads of 8.6 kips, 12.9 kips, 17.2 kips, 34.4 kips. This corresponds to the cases of $n = 1.0$, 1.5, 2.0, 4.0, $\alpha = 1$, discussed by Nien-chien Tsai (3), and the results are compared to those tensionless foundation solutions and shown on Fig. 14 to Fig. 21.

$L = 84$ ft., $a = 3$ ft., $\gamma = 31$ lbs/ft. $K = 196$ kips/ft.

Cross section properties of beam and subgrade are shown in Table I and Table II.

Table I - Data on Beam Section

Beam	Width B inch	Depth inch	Area inch^2	Moment Inertia I; inch^4	Modulus of Elasticity E; psi
Wide-Flange 8WF31	8.0	8.0	9.12	109.7	30×10^6

Table II - Properties of Micaceous Silt Subgrade

Modulus of Elasticity of Soil Es, psi	Poisson's Ratio "v"	Modulus of Subgrade Reaction Ks, psi	Length Character-istic λL
1192	0.25	454	0.98

SUMMARY OF NUMERICAL RESULTS AND COMPARISON WITH REFERENCES

Table Ex. 1 - Output of computer for Example 1, with comparison to (1). L = 72 in.

Input Data		W lbs	Q lbs	CL ft	XK K/ft	NC
		31	8250	0.75	0.00215	8
Distance from Center, in.		0	9	18	27	36
Deflection (in.)	Computer	0.251	0.255	0.259	0.262	0.264
	Soil Test	0.253	0.253	0.253	0.253	0.253

Numerical results and comparison are plotted in Fig. 11.

Table Ex. 2a - Output of computer for Example 2A, with comparison to (2). L = 24 ft.

Input Data		W kips	Q kips	CL ft	XK K/ft	EIL K-ft^2	NC
		0	32.688	2.4	43.584	22896	10
Distance from Center, ft.		0	2.4	4.8	7.2	9.6	12
Deflection (in.)	Computer	1.40	1.30	1.08	0.79	0.476	0.158
	Bowles Finite	1.41	1.32	1.10	0.83	0.52	0.206
	Bowles Infinite	1.28	1.13	0.82	0.502	0.252	0.09

Numerical results and comparison are plotted in Fig. 12.

Table Ex. 2b - Output of computer for Example 2B, with comparison to (2).

L = 84 ft.

Input Data	W kips		Q kips			CL ft.			XK K/ft.		EIL K-ft²		NC		
	0		32.688			3.0			196		22896		28		
Distance from Center, ft.	0	3	6	9	12	15	18	21	24	27	30	33	36	39	42
Deflection (in.) Computer Tension Case	+ .491	+ .407	+ .255	+ .124	+ .038	- .006	- .02	- .02	- .014	- .007	- .003	- 0	+ 0	+ 0	+ 0
Deflection (in.) Bowles Infinite Solution	+ .50	+ .41	+ .25	+ .12	+ .033	- .008	- .02	- .02	- .013	- .007	- .003	- 0	+ 0	+ 0	+ 0

Numerical results and comparison are plotted in Fig. 13.

Table Ex. 3. - Input data of computer program, corresponding to Nien-chien Tsai's tensionless foundation model (3).

Case A - $n = 1.0$, $\alpha = 0$

$W = 3.1 \times 10^{-2}$ kips $Q = 8.6$kips $CL = 3.0$ft

$XK = 196$K/ft $EIL = 22896$k - ft^2 $NC = 28$

Case B - $n = 1.5$, $\alpha = 0$, $\alpha = 1$

$W = 3.1 \times 10^{-2}$kips $Q = 12.9$kips $CL = 3$ft

$XK = 196$k/ft $EIL = 22896$k - ft^2 $NC = 28$

Case C - $n = 2.0$, $\alpha = 0$, $\alpha = 1$

$W = 3.1 \times 10^{-2}$kips $Q = 17.2$kips $CL = 3$ft

$XK = 196$k/ft $EIL = 22896$k - ft $NC = 28$

Case D - $n = 4.0$, $\alpha = 0$, $\alpha = 1$

$W = 3.1 \times 10^{-2}$kips $Q = 34.4$kips $CL = 3$ft

$XK = 196$k/ft $EIL = 22896$k -ft $NC = 28$

*Note

Case 3a. results are shown in Fig. 14, and Fig. 18. Data is collected in the following tables.

Case 3b. results are shown in Fig. 15, Fig. 18, and Fig. 19. Data is collected in the following tables.

Case 3c. results are shown in Fig. 16, Fig. 18, and Fig. 20. Data is collected in the following tables.

Case 3d. results are shown in Fig. 17, Fig. 18, and Fig. 21. Data is collected in the following tables.

Table Ex. 3a. - Output data for Case 3a. $n = 1$, $\alpha = 0$, $\alpha = 1$

THE MATRIX X

ROW 1	2.8396084E-05
ROW 2	2.1422689E-05
ROW 3	8.8527822E-06
ROW 4	-6.5819995E-06
ROW 5	-2.4084453E-05
ROW 6	-4.1256964E-05
ROW 7	-5.0146948E-05
ROW 8	-3.3583972E-05
ROW 9	3.6211146E-05
ROW 10	1.9272984E-04
ROW 11	4.5774761E-04
ROW 12	8.0432533E-04
ROW 13	1.0965588E-03
ROW 14	1.0147179E-03
ROW 15	1.6126791E-08
ROW 16	-1.0146936E-03
ROW 17	-1.0965550E-03
ROW 18	-8.0434140E-04
ROW 19	-4.5776647E-04
ROW 20	-1.9273958E-04
ROW 21	-3.6215250E-05
ROW 22	3.3582881E-05
ROW 23	5.0147559E-05
ROW 24	4.1258580E-05
ROW 25	2.4086243E-05
ROW 26	6.5835429E-06
ROW 27	-8.8517309E-06
ROW 28	-2.1422456E-05
ROW 29	-2.8396040E-05
ROW 30	3.3917930E-04
ROW 31	4.1587185E-04
ROW 32	4.6212017E-04
ROW 33	4.6612369E-04
ROW 34	4.2056013E-04
ROW 35	3.2194681E-04
ROW 36	1.8130179E-04
ROW 37	4.6518137E-05
ROW 38	3.3027754E-05
ROW 39	3.5051000E-04
ROW 40	1.2979165E-03
ROW 41	3.1785737E-03
ROW 42	6.0695112E-03
ROW 43	9.3838386E-03
ROW 44	1.1224985E-02
ROW 45	9.3839392E-03
ROW 46	6.0696565E-03
ROW 47	3.1786885E-03
ROW 48	1.2979733E-03
ROW 49	3.5053003E-04
ROW 50	3.3030432E-05
ROW 51	4.6512432E-05
ROW 52	1.8129489E-04
ROW 53	3.2194331E-04
ROW 54	4.2056269E-04
ROW 55	4.6613044E-04
ROW 56	4.6213064E-04
ROW 57	4.1588419E-04
ROW 58	3.3919327E-04

Table Ex. 3b. - Output data for Case 3b. $n = 1.5$, $\alpha = 1$

THE MATRIX X

ROW	1	2.7415095E-05	
ROW	2	1.9424173E-05	
ROW	3	2.9366347E-06	
ROW	4	-2.0624910E-05	
ROW	5	-5.0242699E-05	
ROW	6	-7.9898571E-05	
ROW	7	-9.1958718E-05	
ROW	8	-5.5543453E-05	
ROW	9	6.5905479E-05	
ROW	10	3.0894484E-04	
ROW	11	7.0392410E-04	
ROW	12	1.2180246E-03	
ROW	13	1.6511150E-03	
ROW	14	1.5246486E-03	
ROW	15	-8.2972242E-09	
ROW	16	-1.5246458E-03	
ROW	17	-1.6510894E-03	
ROW	18	-1.2180018E-03	
ROW	19	-7.0392177E-04	
ROW	20	-3.0895323E-04	
ROW	21	-6.5916160E-05	
ROW	22	5.5532868E-05	
ROW	23	9.1950802E-05	
ROW	24	7.9894584E-05	
ROW	25	5.0242059E-05	
ROW	26	2.0625361E-05	
ROW	27	-2.9365774E-06	
ROW	28	-1.9424668E-05	
ROW	29	-2.7415721E-05	
ROW	30	3.6559440E-04	
ROW	31	4.3832557E-04	
ROW	32	4.7364319E-04	
ROW	33	4.4887187E-04	
ROW	34	3.4383917E-04	
ROW	35	1.4737912E-04	
ROW	36	-1.1795576E-04	
ROW	37	-3.5589701E-04	
ROW	38	-3.6618137E-04	
ROW	39	1.6112671E-04	
ROW	40	1.6394344E-03	
ROW	41	4.5037940E-03	
ROW	42	8.8665821E-03	
ROW	43	1.3850953E-02	
ROW	44	1.6616315E-02	
ROW	45	1.3850942E-02	
ROW	46	8.8665970E-03	
ROW	47	4.5038797E-03	
ROW	48	1.6359762E-03	
ROW	49	1.6125472E-04	
ROW	50	-3.6608614E-04	
ROW	51	-3.5583880E-04	
ROW	52	-1.1792489E-04	
ROW	53	1.4739415E-04	
ROW	54	3.4384825E-04	
ROW	55	4.4888095E-04	
ROW	56	4.7365320E-04	
ROW	57	4.3833512E-04	
ROW	58	3.6560278E-04	

Table Ex. 3c. - Output data for Case 3c. $n = 2.0$, $\alpha = 1$

THE MATRIX X

ROW 1	1.9357234E-05
ROW 2	8.9689975E-06
ROW 3	-1.6085090E-05
ROW 4	-5.5498705E-05
ROW 5	-1.0469297E-04
ROW 6	-1.4588716E-04
ROW 7	-1.4730885E-04
ROW 8	-7.2401177E-05
ROW 9	1.1539398E-04
ROW 10	4.5263651E-04
ROW 11	9.7587542E-04
ROW 12	1.6496715E-03
ROW 13	2.2158062E-03
ROW 14	2.0388991E-03
ROW 15	1.0039344E-08
ROW 16	-2.0388907E-03
ROW 17	-2.2158052E-03
ROW 18	-1.6496836E-03
ROW 19	-9.7587379E-04
ROW 20	-4.5263069E-04
ROW 21	-1.1539506E-04
ROW 22	7.2396680E-05
ROW 23	1.4730377E-04
ROW 24	1.4588219E-04
ROW 25	1.0469009E-04
ROW 26	5.5497236E-05
ROW 27	1.6084494E-05
ROW 28	-8.9692330E-06
ROW 29	-1.9357554E-05
ROW 30	4.2783539E-04
ROW 31	4.7399499E-04
ROW 32	4.6698167E-04
ROW 33	3.6312477E-04
ROW 34	1.2420962E-04
ROW 35	-2.5703688E-04
ROW 36	-7.1134302E-04
ROW 37	-1.0645604E-03
ROW 38	-1.0328614E-03
ROW 39	-2.2273931E-04
ROW 40	1.8689679E-03
ROW 41	5.7830624E-03
ROW 42	1.1659332E-02
ROW 43	1.8334892E-02
ROW 44	2.2030797E-02
ROW 45	1.8334944E-02
ROW 46	1.1659425E-02
ROW 47	5.7831183E-03
ROW 48	1.8690091E-03
ROW 49	-2.2269343E-04
ROW 50	-1.0328088E-03
ROW 51	-1.0645131E-03
ROW 52	-7.1131135E-04
ROW 53	-2.5701616E-04
ROW 54	1.2421986E-04
ROW 55	3.6312873E-04
ROW 56	4.6698260E-04
ROW 57	4.7399383E-04
ROW 58	4.2783283E-04

Table Ex. 3d. - Output data for Case 3d. n = 4.0, α = 1

THE MATRIX X

ROW	1	-3.5181828E-04
ROW	2	-3.7999591E-04
ROW	3	-4.4015725E-04
ROW	4	-4.9573951E-04
ROW	5	-5.1018083E-04
ROW	6	-4.4692634E-04
ROW	7	-2.6940275E-04
ROW	8	5.8948412E-05
ROW	9	5.7467935E-04
ROW	10	1.3143700E-03
ROW	11	2.3145939E-03
ROW	12	3.5439029E-03
ROW	13	4.5660585E-03
ROW	14	4.1336827E-03
ROW	15	2.1412234E-08
ROW	16	-4.1336529E-03
ROW	17	-4.5660213E-03
ROW	18	-3.5438733E-03
ROW	19	-2.3145450E-03
ROW	20	-1.3143173E-03
ROW	21	-5.7463930E-04
ROW	22	-5.8924532E-05
ROW	23	2.6941649E-04
ROW	24	4.4692284E-04
ROW	25	5.1015639E-04
ROW	26	4.9568876E-04
ROW	27	4.4008926E-04
ROW	28	3.7991884E-04
ROW	29	3.5173749E-04
ROW	30	8.8965031E-04
ROW	31	-1.9552752E-04
ROW	32	-1.4223435E-03
ROW	33	-2.8318989E-03
ROW	34	-4.3556318E-03
ROW	35	-5.8152787E-03
ROW	36	-6.9229305E-03
ROW	37	-7.2808638E-03
ROW	38	-6.3818507E-03
ROW	39	-3.6088447E-03
ROW	40	1.7649357E-03
ROW	41	1.0507792E-02
ROW	42	2.2821240E-02
ROW	43	3.6449831E-02
ROW	44	4.3922286E-02
ROW	45	3.6449946E-02
ROW	46	2.2821479E-02
ROW	47	1.0508124E-02
ROW	48	1.7653427E-03
ROW	49	-3.6082603E-03
ROW	50	-6.3811317E-03
ROW	51	-7.2800890E-03
ROW	52	-6.9220848E-03
ROW	53	-5.8144145E-03
ROW	54	-4.3548048E-03
ROW	55	-2.8311659E-03
ROW	56	-1.4217789E-03
ROW	57	-1.9519652E-04
ROW	58	8.8971830E-04

Table 3e. - Output data for Ex. 3 of $\alpha = 0$, $n = 1.5, 2.0, 4.0$

THE MATRIX X (n = 1.5)

0.16600E - 01	0.13839E - 01	0.88675E - 02	0.45310E - 02
0.17099E - 02	0.38835E - 03	-0.18873E - 03	-0.16988E - 03
0.30505E - 04	0.24002E - 03	0.38851E - 03	0.46235E - 03
0.47068E - 03	0.42796E - 03	0.35149E - 03	

THE MATRIX X (n = 2.0)

0.21976E - 01	0.18294E - 01	0.11665E - 01	0.58834E - 02
0.21218E - 02	0.22616E - 03	-0.41048E - 03	-0.38628E - 03
-0.12028E -03	0.15810E - 03	0.35645E - 03	0.45857E - 03
0.47924E - 03	0.44004E - 03	0.36379E - 03	

THE MATRIX X (n = 4.0)

0.43477E - 01	0.36113E - 01	0.22857E - 01	0.11293E - 01
0.37694E - 02	-0.22579E - 04	-0.12975E - 02	-0.12519E - 02
-0.72344E-03	-0.16959E - 03	0.22824E - 03	0.44346E - 03
0.51345E - 03	0.48836E - 03	0.41298E - 03	

Dimensionless Deflection Calculation;

In order to compare the computer results with tensionless foundation solutions, the above table deflections (ft) should be modified to become dimensionless coordinates by the following equations (3).

$$X = x/\ell \qquad (11)$$

$$\delta = w/n \cdot x \;(\gamma \ell^4 e^{\pi}/4EI) \tag{12}$$

where X and δ are dimensionless distance and deflection, respectively.

W = deflection(ft)

ℓ = characteristic length(ft)

γ = unit weight of beam

EI = flexural rigidity of beam

and $\ell^4 = 4EI/Ks'$ (13)

where Ks' = modulus of subgrade x width of beam (Ks x B)
Substituting EI = 22896K-ft, Ks' = 454psi (Table II) into Eq. 13.

Then

$$\ell = \sqrt[4]{\frac{4EI}{Ks'}} = \sqrt[4]{\frac{4 \times 22896}{0.454 \times 144}} \cong 6(\mathrm{ft})$$

$$\gamma \ell^4 e^{\pi}/4EI = \frac{0.031 \times 6^4 \times 23.14}{4 \times 22896} \cong 1 \times 10^{-2}(\mathrm{ft})$$

These results are graphically presented in Fig. 14 to Fig. 21.

CONCLUSIONS

1. The study described herein shows that the matrix solution for beams on elastic foundations gives good agreement with the classical linear and non-linear (tensionless foundation) solutins, for the number of springs chosen.

2. The matrix solution of the problem shows its simplicity not only in matrix formulation but also in the numerical evaluation by computer.

3. The beam and subgrade properties were chosen arbitrarily. The method of investigation is completely general.

4. Due to the nonlinearity of the problem, the principle of superposition is not valid. Instead the problem arising with each different loading must be considered separately. This is clearly illustrated in Fig. 19, Fig. 20, and Fig. 21.

5. Physical properties of real soils are more complicated than that represented by Winkler's assumptions.

6. The degree of continuity in subgrade can be assumed, in a given case, only by physical testing of real soil.

ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation and gratitude to Dr. Jack B. Blackburn for his advice and suggestions during the preparation of this report.

He would also like to thank his major advisor Professor Wayne W. Williams and Dr. Stuart E. Swartz for their encouragement and suggestions.

REFERENCES

1. A. B. Vesić, "Model Studies of Beam Resting on a Silt Subgrade," Journal of the Soil Mechanics and Foundations Division, American Society of Civil Engineering, Feb., 1963, Part I.

2. Joseph E. Bowles, Foundation Analysis and Design, MacGraw-Hill Co., New York, 1968.

3. Nien-chien Tsai and Russell A. Westmann, "Beam on Elastic Foundation," Journal of Engineering Mechanics Division, American Society of Civil Engineers, Oct., 1967.

4. S. Timoshenko, Strength of Materials, Part II, D. Van Nostran Co., Princeton, N.J., 1965.

5. M. Hetenyi, Beams on Elastic Foundation, The Univ. of Michigan Press, Ann Arbor, 1946.

6. Z. Levinton, "Elastic Foundations Analyzed by the Method of Redundant Reactions," Trans. American Society of Civil Engineers, 1949.

7. H. Malter, "Numerical Solutions for Beams on Elastic Foundations," Trans. American Society of Civil Engineers, 1960.

8. G. A. Leonards and M. E. Harr, "Analysis of Concrete Slabs on Ground," Journal of the Soil Mechanics and Foundation Division, American Society of Civil Engineers, June, 1959.

9. A. D. Kerr, "Elastic and Viscoelastic Foundation Models," Journal of Applied Mechanics, American Society of Mechanical Engineers, Sept., 1964.

10. C. K. Wang, Matrix Methods of Structural Analysis International Textbook Co., 1966, Chapter 4 and Chapter 10.

APPENDIX A - NOTATION

The following symbols are used in this report:

EI = flexural rigidity of beam

Ks = subgrade modulus

B = width of beam

$Ks' = Ks \times B$ = subgrade modulus include the effect of beam width

ℓ = characteristic length of beam-foundation

n = loading parameter

Q = magnitude of concentrated center load

W = dimensionless deflection

w = transverse deflection

X = dimensionless length coordinate

x = length coordinate

γ = unit weight of beam

K = spring constant

a = cell length of beam-foundation

Es = modulus of elasticity of soil

v = Poisson's ratio

L = total length of beam

δ = dimensionless deflection $= w/(\gamma \ell^4 e^{\pi}/4EI)$

λL = length characteristic

APPENDIX B - FIGURES

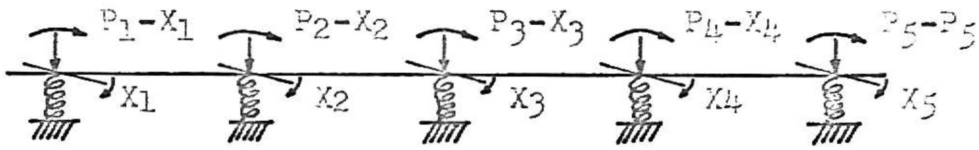


Fig. 2 - Force-Deflection Diagram

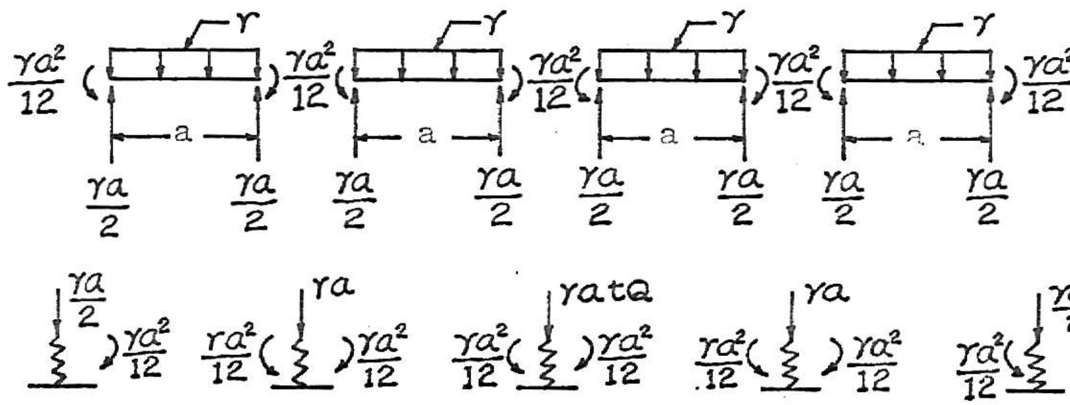


Fig. 3 - Load Diagram for the Given Beam

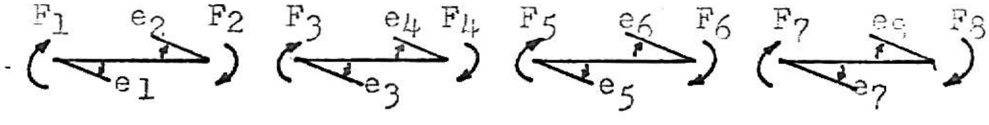


Fig. 4 - Internal Moments & Rotations

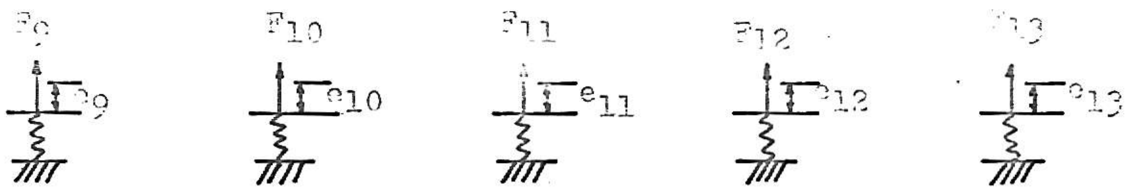


Fig. 5 - Spring Forces & Deflections

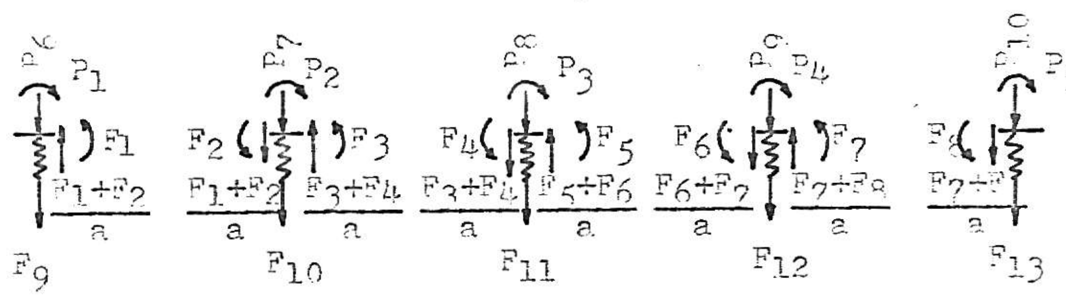


Fig. 6 - Joint Equilibrium Diagram

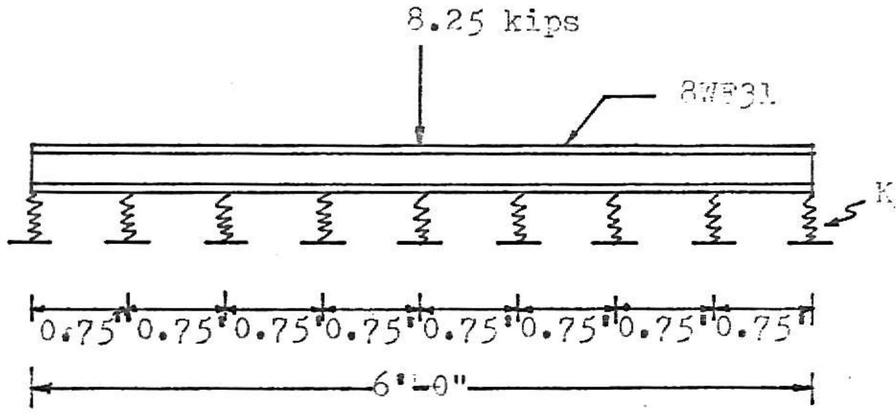


Fig. 7. - Example 1. Beam tested by Vesic'(1)

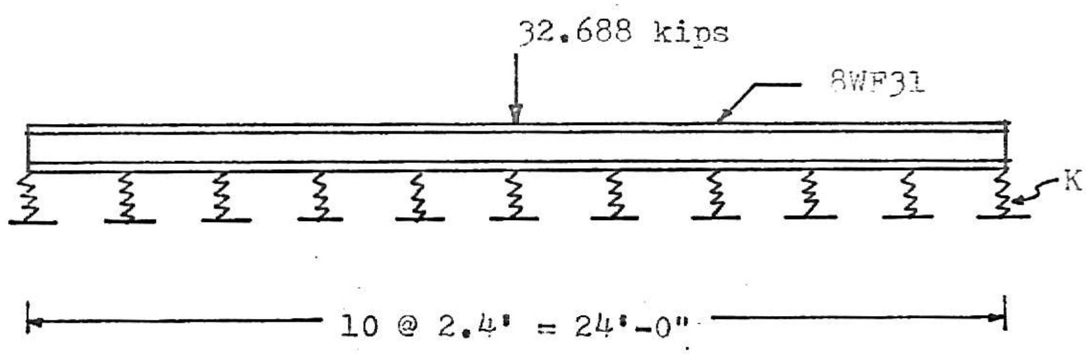


Fig. 8. - Example 2A Short Beam to Compare with
Bowles Finite and Infinite Beam Analysis.(2)

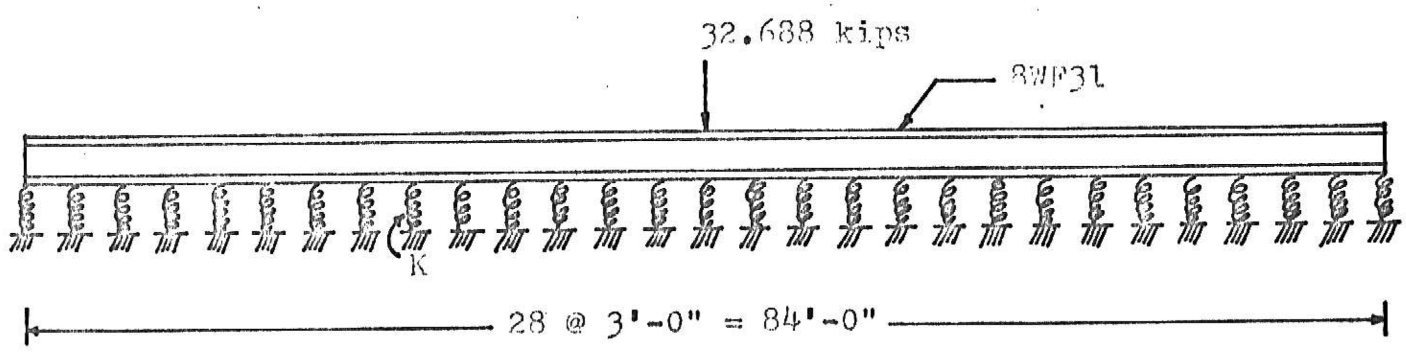


Fig. 9. Example 2B Long Beam to Compare with Bowles Infinite Beam Analysis(2)

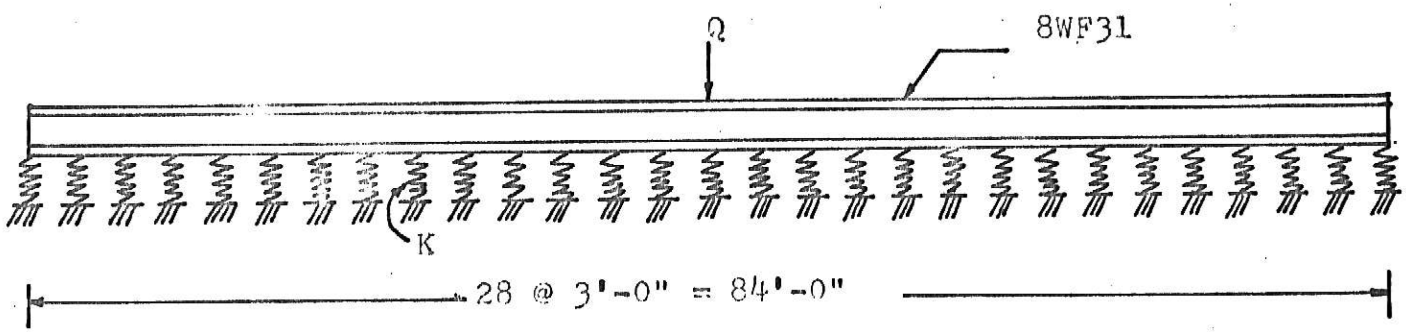


Fig. 10. Example 3 Long Beam to Compare with Infinite Beam Analysis of Tasi & Russell (3) for Q = 8.6^k, 12.9^k, 17.2^k, 34.4^k

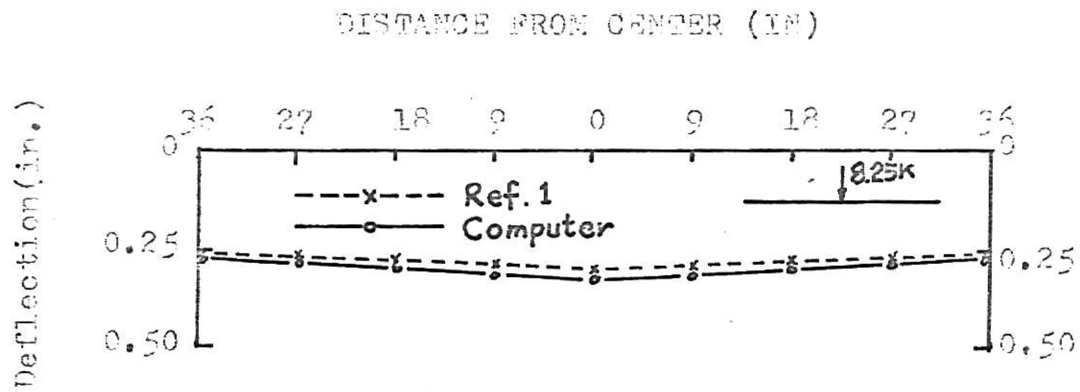


Fig. 11. - Comparison of Deflection with Soil Test Result for Example 1

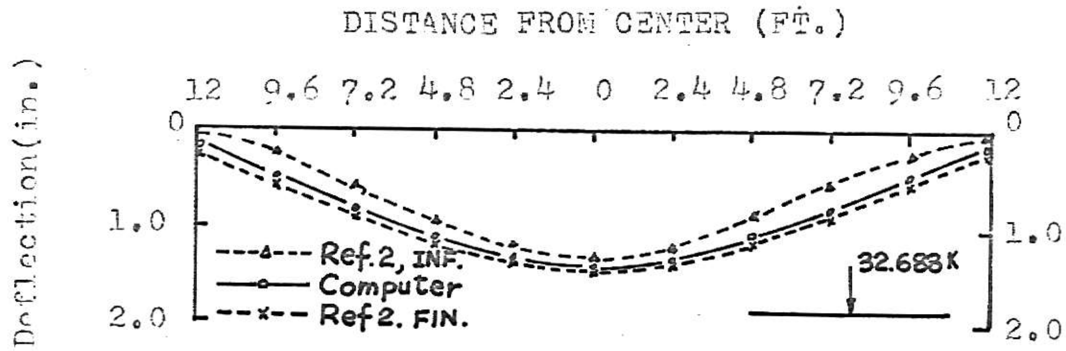


Fig. 12. - Comparison of Deflection with Bowles Finite and Infinite Beam Solution for Example 2A

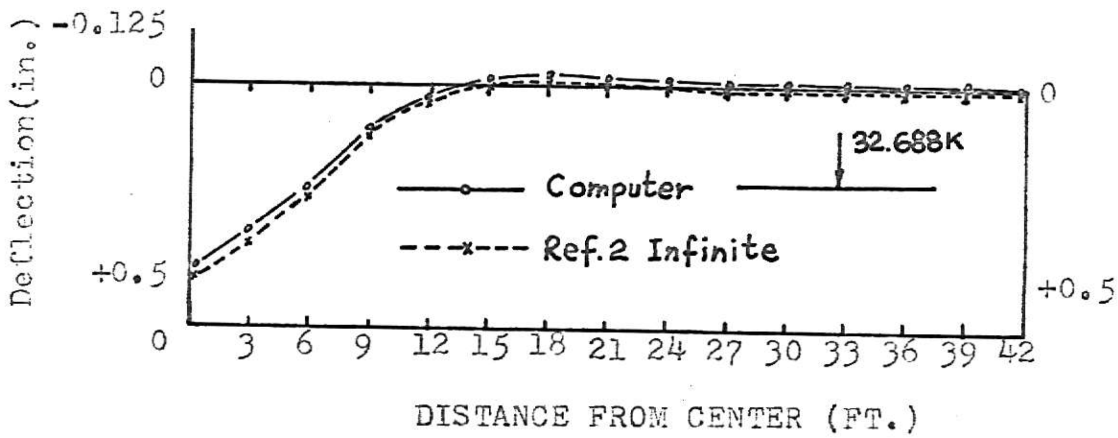


Fig. 13. - Comparison of Deflection with Bowles Infinite Beam Solution for Example 2B

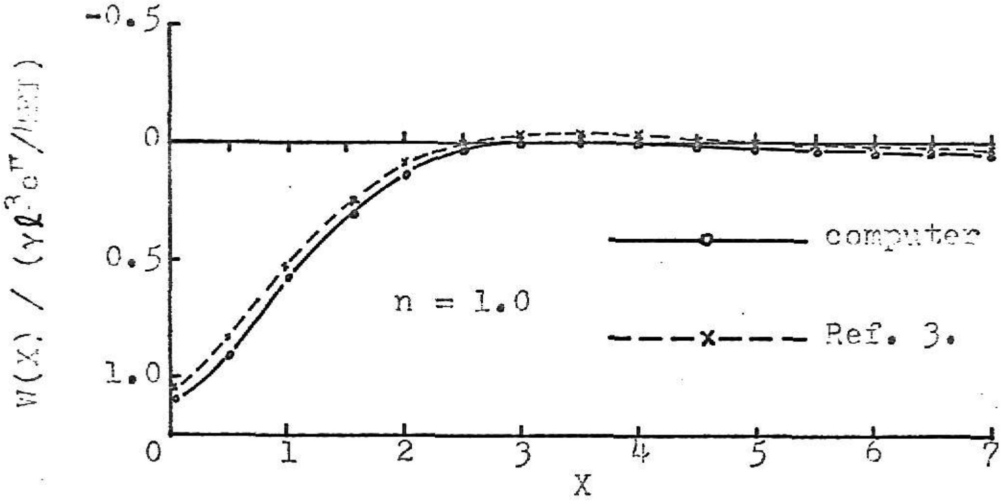


Fig. 14. - Dimensionless Deflection for Example 3 with Uniform Dead Load $\gamma = 31\ ^{lbs}/_{ft}$ & $Q = 8.6^{k}$

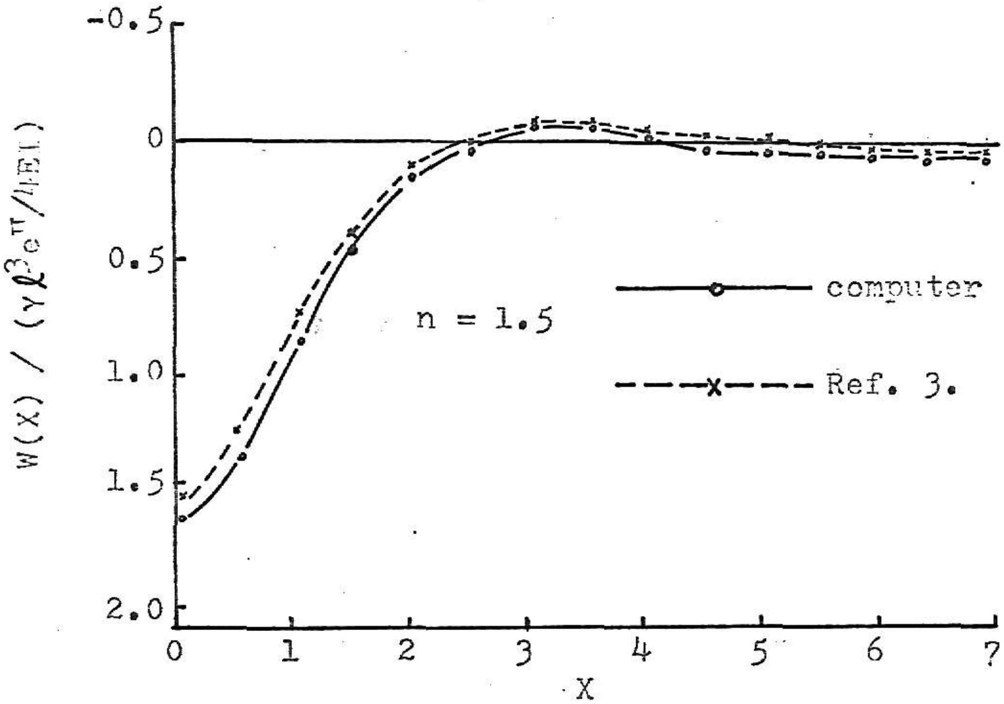


Fig. 15. - Dimensionless Deflection for Example 3 with Uniform Dead Load $\gamma = 31\ ^{lbs}/_{ft}$ & $Q = 12.9^{k}$

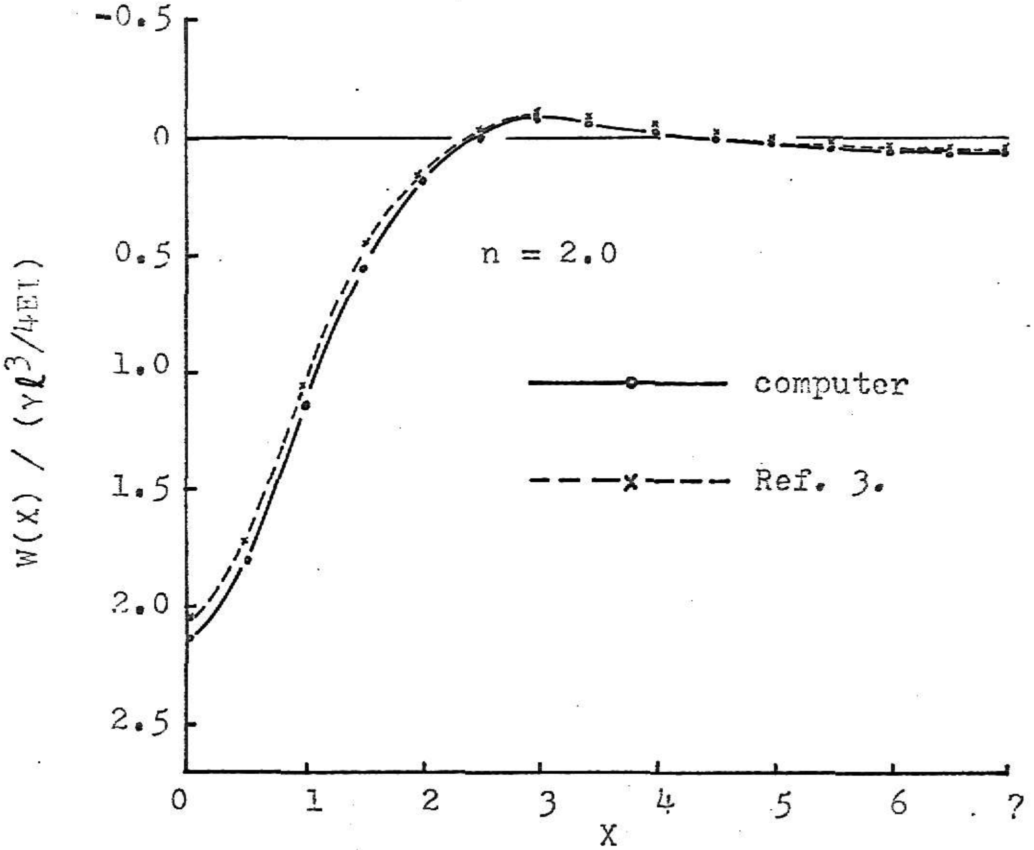


Fig. 16 - Dimensionless Deflection for Example 3 with Uniform Dead Load $\gamma = 31\ ^{lbs}/_{ft}$ & $Q = 17.2^{k}$

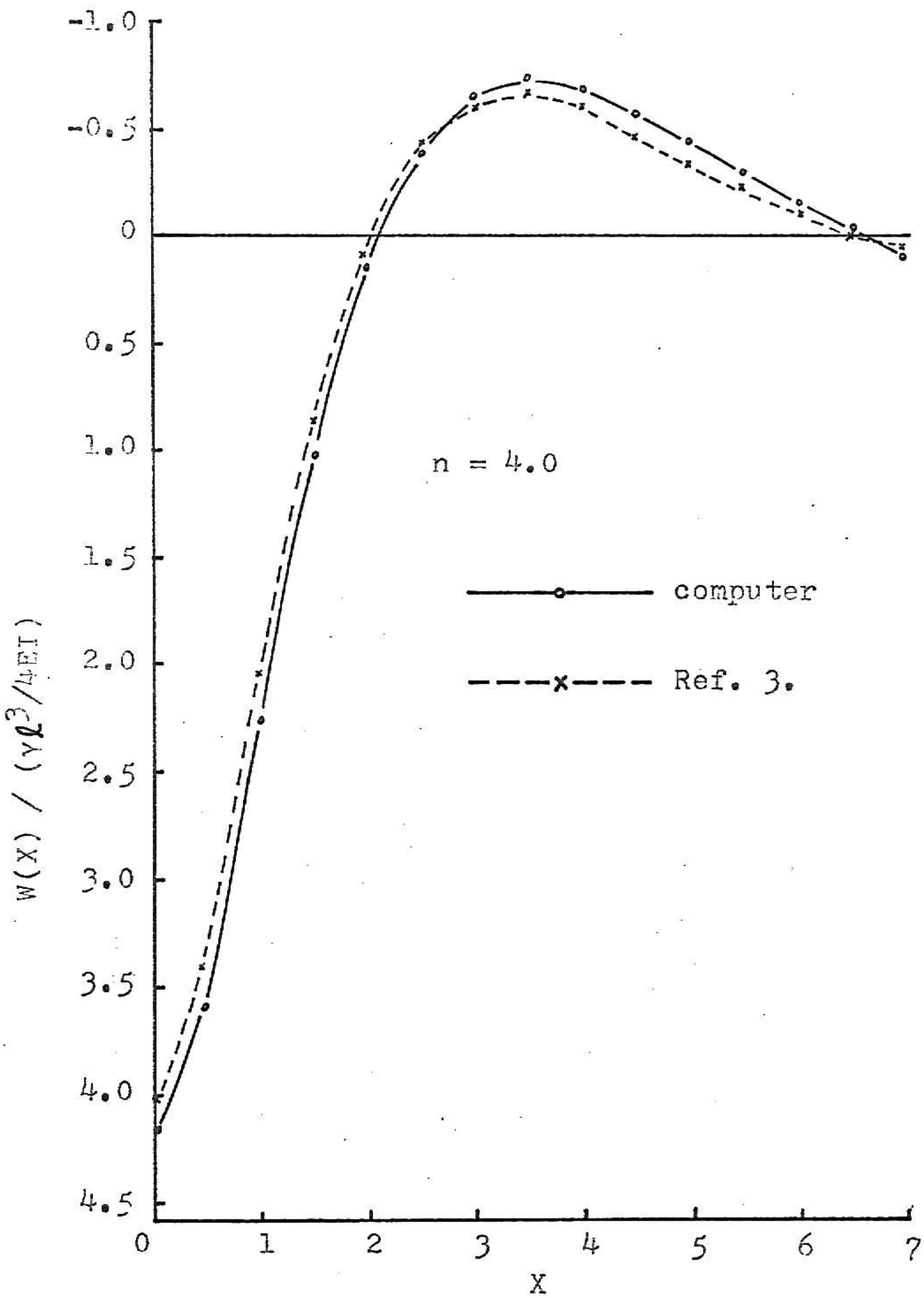


Fig. 17. - Dimensionless Deflection for Example 3 with Uniform Dead load $\gamma = 31\ ^{lbs}/_{ft}$ & $Q = 34.4^{k}$

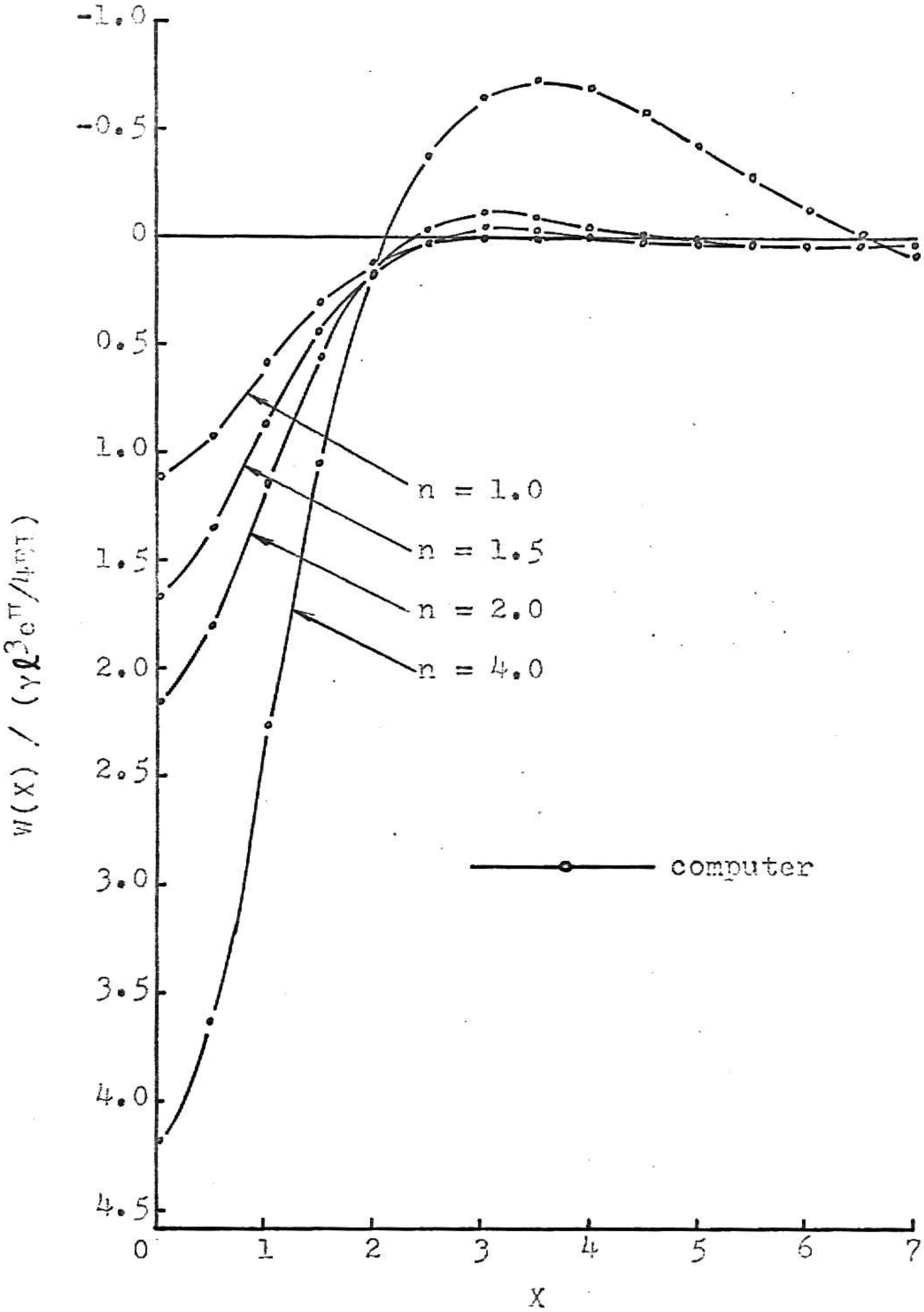


Fig. 18. - Dimensionless Deflection, Summary of Load Cases for Example 3.

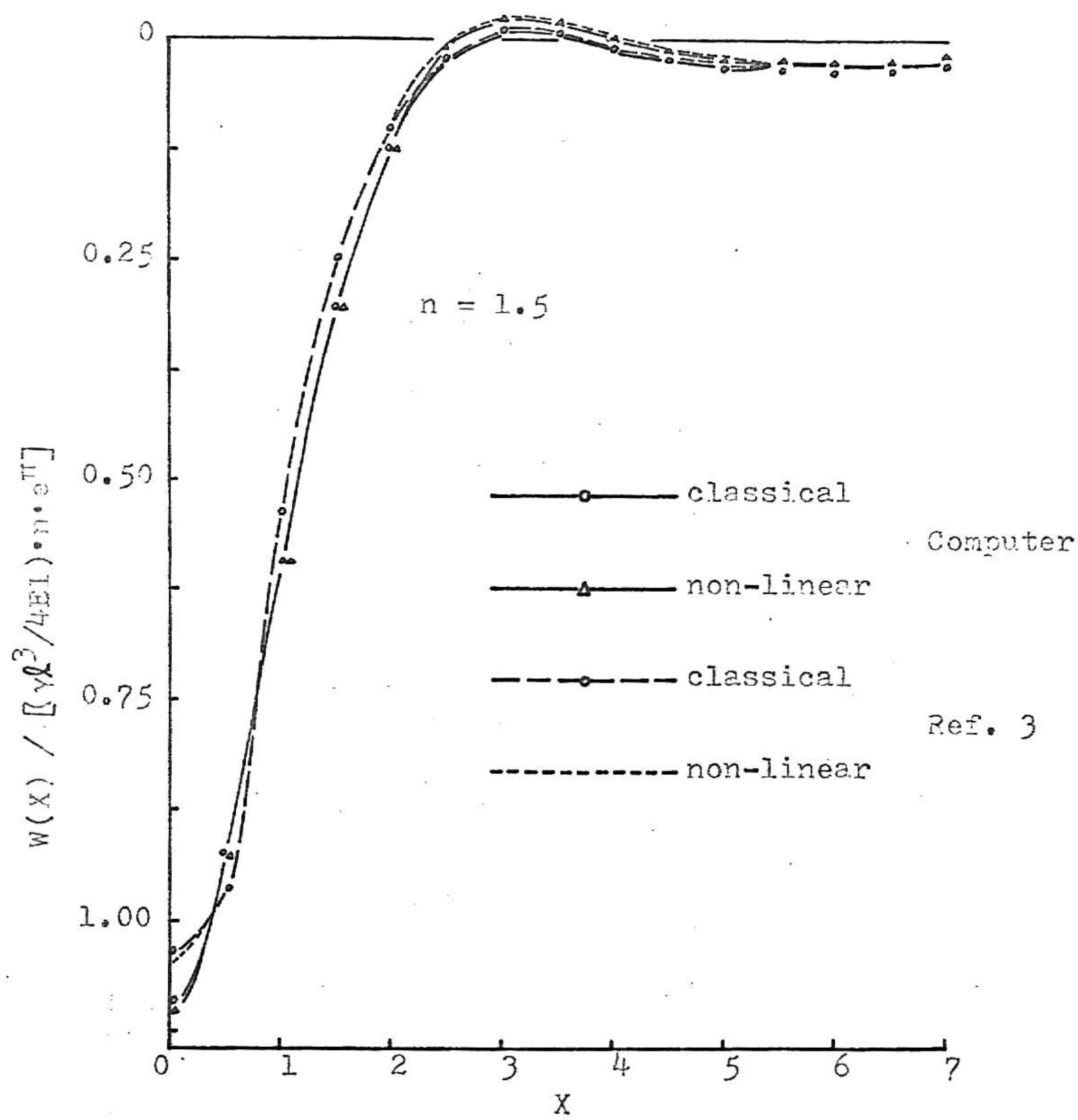


Fig. 19 - Comparison of Non-linear Solution with Classical Solution for Example 3, $Q = 12.9^k$

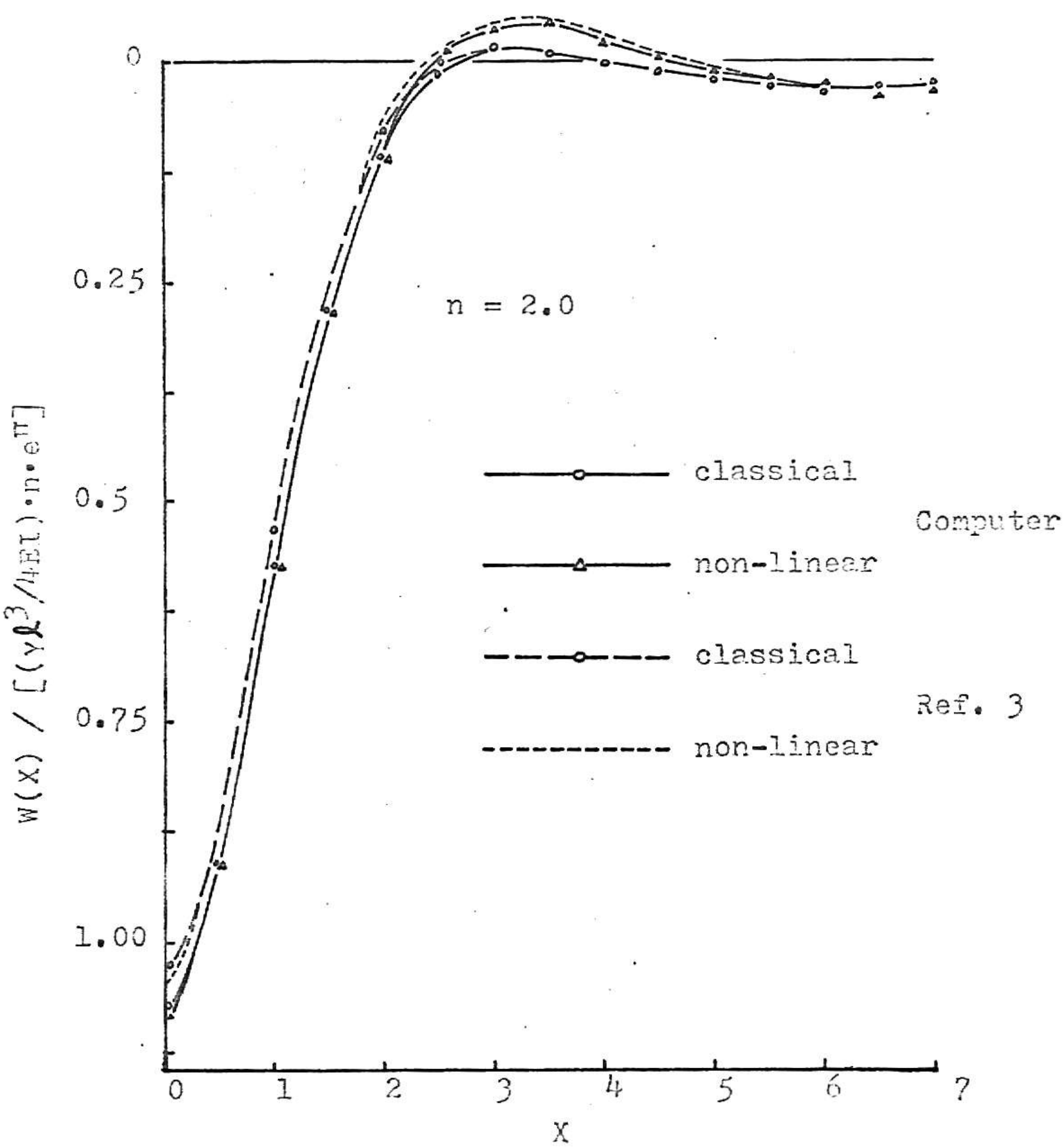


Fig. 20 - Comparison of Non-linear Solution with Classical Solution for Example 3, Q = 17.2 K

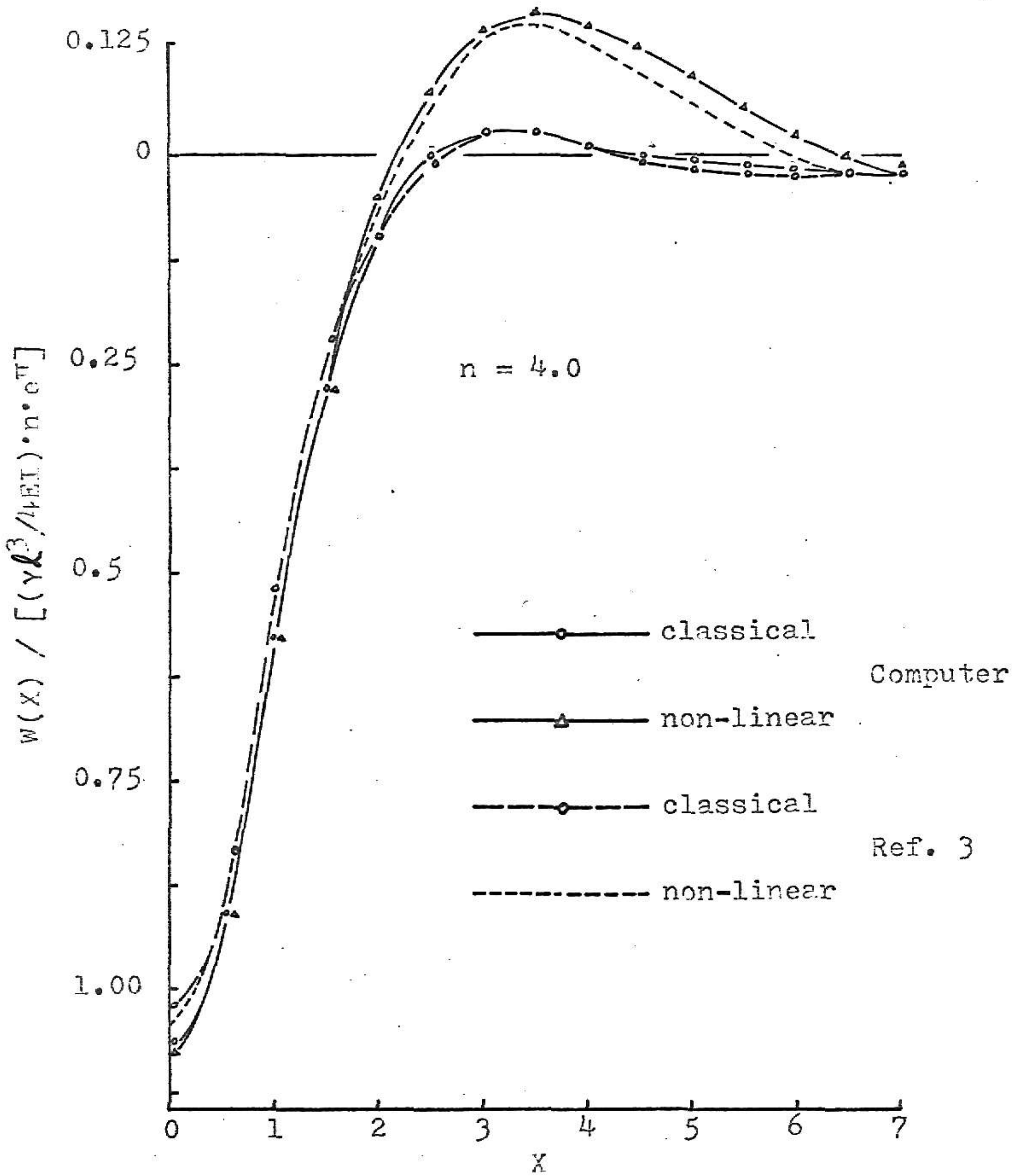


Fig. 21 - Comparison of Non-linear Solution with Classical Solution for Example 3, $Q = 34.4^k$

APPENDIX C - COMPUTER PROGRAM AND ITS FLOW CHART

Displacement method of beams on "one-way" elastic foundation analysis

(I) Program Explanation

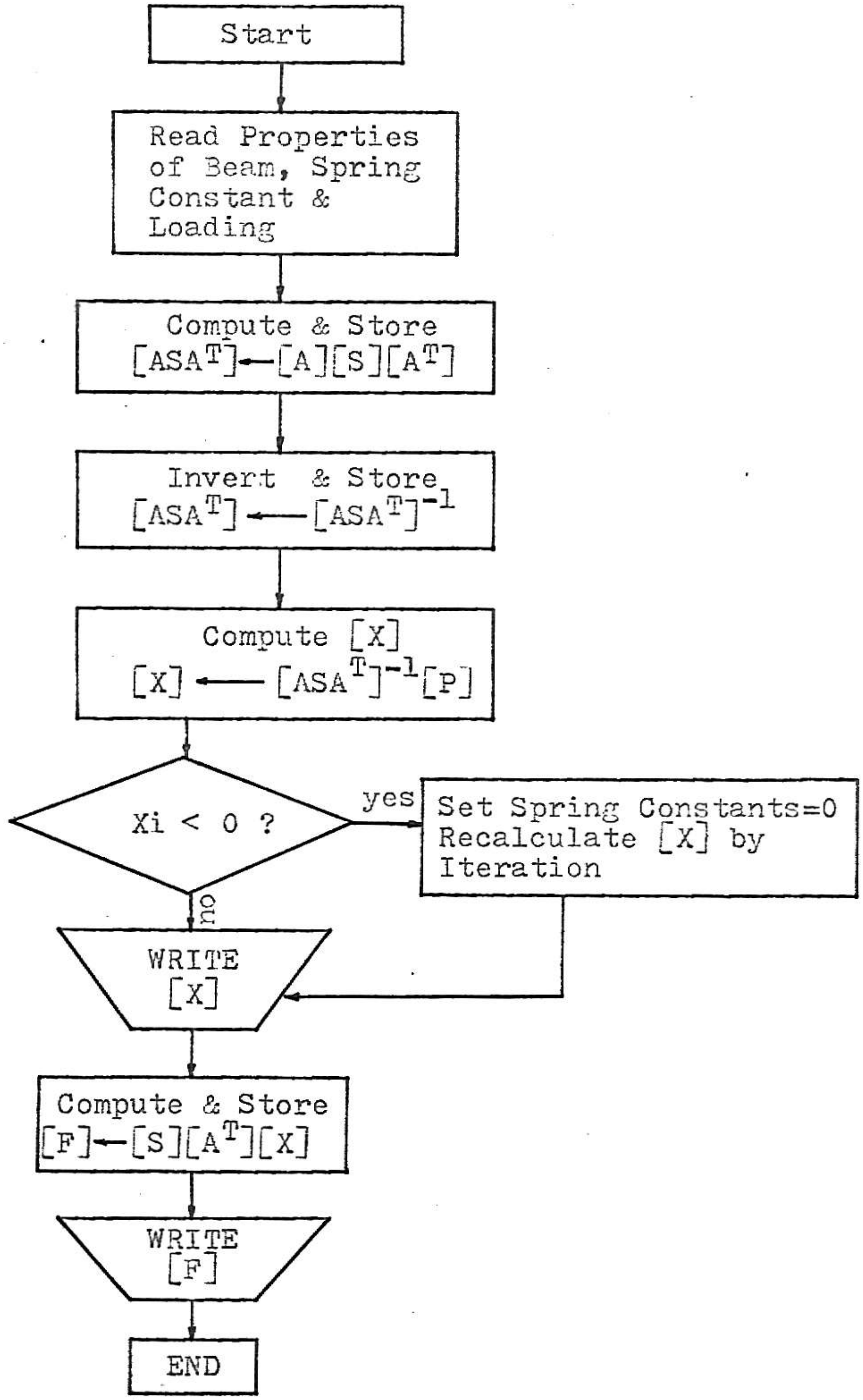
This program is written to solve the matrix equations of beams on elastic foundation by the displacement method and is a modification of one given by Wang(10), wherein the spring constant K, arising from upward deflections, is set equal to zero in the stiffness matrix[S], after the first iteration. The deflections are then recalculated. If the region of upward deflections expands, iteration is continued until all upward deflections are tested and their spring constants K are set equal to zero in [S] matrix. Then the iteration goes on to its last cycle and the final deflections are written out the tensionless foundation solutions. This is the essence of the program that follows.

(II) Fortran Name - The following symbols are used in this program.

Fortran Name	Quantity
[A]	Force - Load transformation matrix
[S]	Member stiffness matrix
[SAF]	Member stiffness matrix related to axial forces
[ASAT]	Transpose of [A] matrix
[P]	Load matrix
[X]	Displacement matrix
[F]	Force matrix

[e]	Deformation matrix
INDEX	Index of do loop taking on values from 1 to NP
IDONT	Index of do loop taking on values from 1 to NAF
NP	Degrees of freedom
NF	Total number of internal forces
NEM	Number of internal end moments
NAF	Number of internal axial forces
NLC	Load condition
NC	Number of cell length
NT	Index of tension or tensionless allowed for the foundation, taking values 1 or 0, respectively
W	Unit weight of beam (=γ)
Q	Concentrated center load
CL	Cell length of beam
XK	Spring constant
EIL	Flexural rigidity of beam
ISW	Test of upward deflection

(III) Flow Chart of Beams on One-Way Elastic Foundation Program



```
      DIMENSION A(80,120),S(160),SAF(40)
      DIMENSION ASAT(80,80), P(80,1), X(80,1), F(120,1), INDEX(80)
      DIMENSION IDONT(80)
  131 READ(1,101) NP,NF,NEM,NAF,NLC,NC,NT
  101 FORMAT(7I5)
      READ(1,102) W,Q,CL,XK,EIL
  102 FORMAT(5E14.6)
      WRITE(3,200) W,Q,CL,XK,EIL,NC
  200 FORMAT(1P5E14.6,5X,I3)
      DO 5 I=1,NP
      DO 5 J=1,NF
    5 A(I,J)=0.
      A(1,1)=1.
      A(NAF,NEM)=1.
      K=0
      DO 10 I=2,NC
      J=I+K
      A(I,J)=1.
      A(I,J+1)=1.
   10 K=K+1
      INAF=NAF+1
      A(INAF,1)=1./CL
      A(INAF,2)=1./CL
      A(INAF,NEM+1)=-1.
      A(NP,NEM-1)=-1./CL
      A(NP,NEM)=-1./CL
      A(NP,NF)=-1.
      J=1
      L=2
      DO 15 I=2,NC
      I1=I+NAF
      JJ=NEM+L
      A(I1,J)=-1./CL
      A(I1,J+1)=-1./CL
      A(I1,J+2)=+1./CL
```

```
      A(I1,J+3)=+1./CL
      A(I1,JJ)=-1.
      L=L+1
   15 J=J+2
      NEMT2=NEM*2
      DO 20 I=1,NEMT2
   20 S(I)=0.
      DO 25 I=1,NC
      I1=4*(I-1)+1
      I2=4*(I-1)+2
      I3=4*(I-1)+3
      I4=4*(I-1)+4
      S(I1)=(4.*EIL)/CL
      S(I2)=(2.*EIL)/CL
      S(I3)=(2.*EIL)/CL
   25 S(I4)=(4.*EIL)/CL
      DO 30 I=1,NAF
   30 SAF(I)=XK
      DO 35 I=1,NP
      DO 35 J=1,1
   35 P(I,J)=0.
      P(1,1)=W*CL*CL/12.
      P(NAF,1)=-P(1,1)
      P(NAF+1,1)=W*CL/2.
      P(2*NAF,1)=P(NAF+1,1)
      DO 50 K=2,NC
      K1=K+NAF
      MP=NAF+(NAF+1)/2.
      IF(K1-MP) 45,40,45
   45 P(K1,1)=W*CL
      GO TO 50
   40 P(K1,1)=W*CL+Q
   50 CONTINUE
      WRITE(3,103)
  103 FORMAT (52H1DISP METHOD OF BEAMS ON ELASTIC FOUNDATION ANALYSIS//)
      WRITE(3,104)
  104 FORMAT (13H0THE MATRIX A)
      DO 105 I=1,NP
  105 WRITE(3,106) I,(A(I,J), J=1,NF)
```

```
  106 FORMAT (4H ROW,I3,1X,1P4E16.7/(8X,1P4E16.7))
      WRITE(3,107)
  107 FORMAT (13HOTHE MATRIX S)
      DO 108 I=1,NEM
      I1=(I-1)/2*2+1
      I2=(I+1)/2*2
      I3=2*I-1
      I4=2*I
  108 WRITE(3,109) I,I1,S(I3),I2,S(I4)
  109 FORMAT (4H ROW,I3,5X,3HCOL,I3,1PE16.7,5X,3HCOL,I3,1PE16.7)
      DO 208 I=1,NAF
      I5=NEM+I
  208 WRITE(3,209) I5,I5,SAF(I)
  209 FORMAT (4H ROW,I3,5X,3HCOL,I3,1PE16.7)
      WRITE (3,110)
  110 FORMAT (13HOTHE MATRIX P)
      DO 111 I=1,NP
  111 WRITE(3,106) I,(P(I,J), J=1,NLC)
      DO 60 I=1,NAF
      I1=I+NAF
   60 IDONT(I1)=0
  301 DO 112 I=1,NP
      DO 112 J=1,NP
      ASAT(I,J)=0.
      DO 212 K=1,NEM
      K1=(K-1)/2*2+1
      K2=(K+1)/2*2
      K3=2*K-1
      K4=2*K
  212 ASAT(I,J)=ASAT(I,J)+A(I,K)*(S(K3)*A(J,K1)+S(K4)*A(J,K2))
      DO 213 K=1,NAF
  213 ASAT(I,J)=ASAT(I,J)+A(I,K+NEM)*SAF(K)*A(J,K+NEM)
  112 CONTINUE
      DO 113 I=1,NP
  113 INDEX(I)=0
  114 AMAX=-1.
      DO 115 I=1,NP
      IF (INDEX(I)) 115,116,115
  116 TEMP=ABS(ASAT(I,I))
```

```
      IF (TEMP-AMAX) 115,115,117
  117 ICOL=I
      AMAX=TEMP
  115 CONTINUE
      IF (AMAX) 118,1CO,119
  119 INDEX (ICOL)=1
      PIVOT=ASAT(ICOL,ICOL)
      ASAT (ICOL,ICOL)=1.0
      PIVOT=1./PIVOT
      DO 120 J=1,NP
  120 ASAT(ICOL,J)=ASAT(ICOL,J)*PIVOT
      CO 121 I=1,NP
      IF (I-ICOL) 122,121,122
  122 TEMP=ASAT(I,ICOL)
      ASAT (I,ICOL)=0.0
      CO 123 J=1,NP
  123 ASAT(I,J)=ASAT(I,J)-ASAT(ICOL,J)*TEMP
  121 CONTINUE
      GO TO 114
  118 CO 124 I=1,NP
      DO 124 J=1,NLC
      X(I,J)=0.
      DO 124 K=1,NP
  124 X(I,J)=X(I,J)+ASAT(I,K)*P(K,J)
      IF(NT) 70,70,65
   70 WRITE(3,125)
      WRITE(3,160) (I,X(I,1),I=1,NP)
  160 FORMAT(6(1X,2HX(I3,2H)=E12.5))
      ISW=0
      CO 61 I=1,NAF
      I1=I+NAF
      IF(X(I1,1)) 62,61,61
   62 IF(IDONT(I1)) 63,63,61
   63 IDONT(I1)=1
      SAF(I)=0.
      ISW=1
   61 CONTINUE
      IF(ISW) 65,65,301
   65 WRITE(3,125)
```

```
125 FORMAT (13HOTHE MATRIX X)
    DO 126 I=1,NP
126 WRITE(3,106) I,(X(I,J), J=1,NLC)
    DO 127 I=1,NEM
    I1=(I-1)/2*2+1
    I2=(I+1)/2*2
    I3=2*I-1
    I4=2*I
    DO 127 J=1,NLC
    F(I,J)=0.
    DO 127 K=1,NP
127 F(I,J)=F(I,J)+X(K,J)*(S(I3)*A(K,I1)+S(I4)*A(K,I2))
    DO 228 I=1,NAF
    I5=I+NEM
    F(I5,1)=0.
    DO 228 K=1,NP
228 F(I5,1)=F(I5,1)+SAF(I)*A(K,I5)*X(K,1)
    WRITE(3,128)
128 FORMAT (13HOTHE MATRIX F)
    DO 129 I=1,NF
129 WRITE(3,106) I,(F(I,J), J=1,NLC)
    GO TO 131
100 CONTINUE
    WRITE(3,130)
130 FORMAT (11HOZERO PIVOT)
    GO TO 131
    END
```

BEAMS ON ONE-WAY ELASTIC FOUNDATIONS

by

KOU-KWANG LIN

B.S., National Taiwan University, 1964

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Departmentment of Civil Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1970

ABSTRACT

The analysis of beams on one-way elastic foundations is based on Winkler's assumption that the continuous reaction of the foundation at every point is proportional to the deflection at that point. However, the tension property which is ordinarily assumed for a foundation is relaxed by assuming the foundation can take compression only. Under such conditions the foundation can be visualized as a set of closely spaced "one-way" springs. A matrix formulation is used to express the beam member deformations and forces in terms of spring joint displacements. Once the redundant displacements are known the elastic solution of this beam-foundation system can be obtained. A model of a typical steel beam (8WF31) on a micaceous silt subgrade was chosen to illustrate the numerical evaluation of the tensionless foundation solution. The numerical process was performed using a computer program written in Fortran IV. The beam-subgrade stiffness matrix was modified to take into account beam uplift by setting appropriate spring constants equal to zero in every cycle of iteration. The final joint displacements (deflections) were calculated following the last iteration. The results are in a good agreement with previous tensionless foundation solutions.