THE YIELD LINE ANALYSIS OF CONCRETE SLABS

by

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TABLE OF CONTENTS

LIST OF FIGURES

Figure		Page

LIST OF TABLES

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By Gurdial Singh Sandhu

SYNOPSIS

Yield line analysis for the prediction of ultimate flexural strength of reinforced concrete slab is presented in this report. The various aspects of modern yield line analysis, historical review of the subject, assumptions on which it is based, yield line criteria for slabs and methods of analysis are described in detail . The theoretical strength obtained by yield line analysis are in good agreement with the experimental results. The use of the technique is illustrated by numerical examples.

INTRODUCTION

An important aim of an engineer is to see that his structure has a suitable factor of safety against failure. The majority of structures all over the world have been designed using the elastic theory. An accurate indication of the factor of safety against failure is not obtained because the material may be acting plastically under the load it is designed to carry. Therefore, there is increasing desire among engineers to design the structure by the ultimate load technique.

In the case of slabs, elastic methods of design are very complicated, and for slabs of irregular shape, the elastic analysis is so complicated that without the help of a computer it is impossible to obtain an adequate solution.

An alternative technique, the yield line analysis is an ultimate load method in which the load at which a slab will fail is assessed. Its main virtue is that no matter how complex the slab shape or loading configuration, it is always possible to obtain a realistic value of collapse load. It is relatively simple to apply and many of the solutions predicted by this method have been substantiated experimentally. All these advantages make this method a design technique with which every structural engineer should be familiar.

The purpose of this report, therefore, is to describe the various aspects of modern yield line analysis in detail, beginning with the historical review of the subject and assumptions on which it is based, yield line criteria for slabs, and methods of analysis.

HISTORICAL REVIEW OF THE DEVELOPMENT OF YIELD LINE THEORY FOR SLABS

In order to appreciate what lies behind some of the most recent developments, it is important to recall how the yield line theory has evolved simultaneously with alternative methods for the plastic design of slabs. The pioneer of the yield line theory is Professor K.W. Johansen (1)*, who provided not merely the introductory theory but also a great number of practical examples. He assumed that failure of a slab occurs when lines along which the steel has yielded (which are idealized and called yield lines) have together formed a valid yield line mechanism. Once the valid mechanism has been determined the work method can be used to establish failure load by equating the expenditure of energy due to external loads at failure to the internal dissipation of energy on all the yield lines acting together. It is not certain that the failure load found is the smallest for the given value of bending moment. Hence, the most critical layout of yield lines is found by trial and error or by differentiation.

When the most critical layout has been found for a given pattern of yield lines, it will be found that if the overall equilibrium of the individual rigid regions between yield lines such as A and B in Fig. 1 is examined, using only the moment along the yield lines to uphold the rigid region; then the regions are not always in equilibrium. Hence Ingerslev (2) was not completely

*Number in parentheses refers to references listed in the bibliography.

right in presenting the idea that regions A and B were kept in equilibrium solely by the moments along the yield lines. This will not be true for unsymmetrical slabs (Fig. 2) because there will be nodal forces acting as junctions of yield lines.

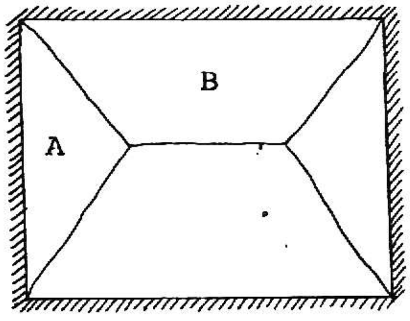


Fig. 1. Pattern where rigid regions are in equilibrium from moments alone.

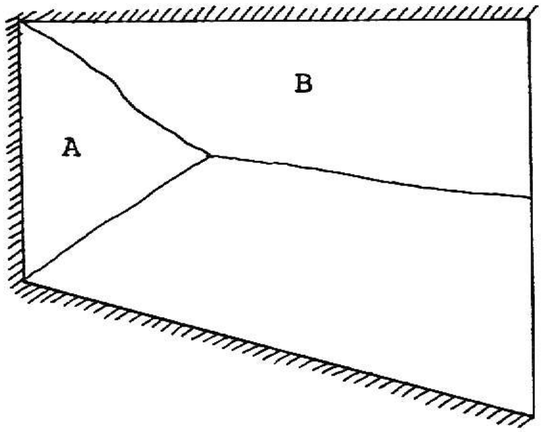


Fig. 2. Pattern where moments alone do not keep rigid regions in equilibrium.

Johansen (1) must be complimented on being the first to observe this. The concept of nodal forces is shown in Fig. 3. Johansen found that these nodal forces, which were statically equivalent to the unknown twisting moments and shears, had some special values when the worst (the moment across the yield line being the maximum value for a given load) pattern of yield lines was investigated.

Until about 1950 there still remained a certain disconcerting feature about both the equilibrium and work methods. This was that although both methods could be used to find the most critical layout of a particular pattern, it always seemed possible to discover a slightly more complicated pattern whose critical layout gave an even lower collapse load.

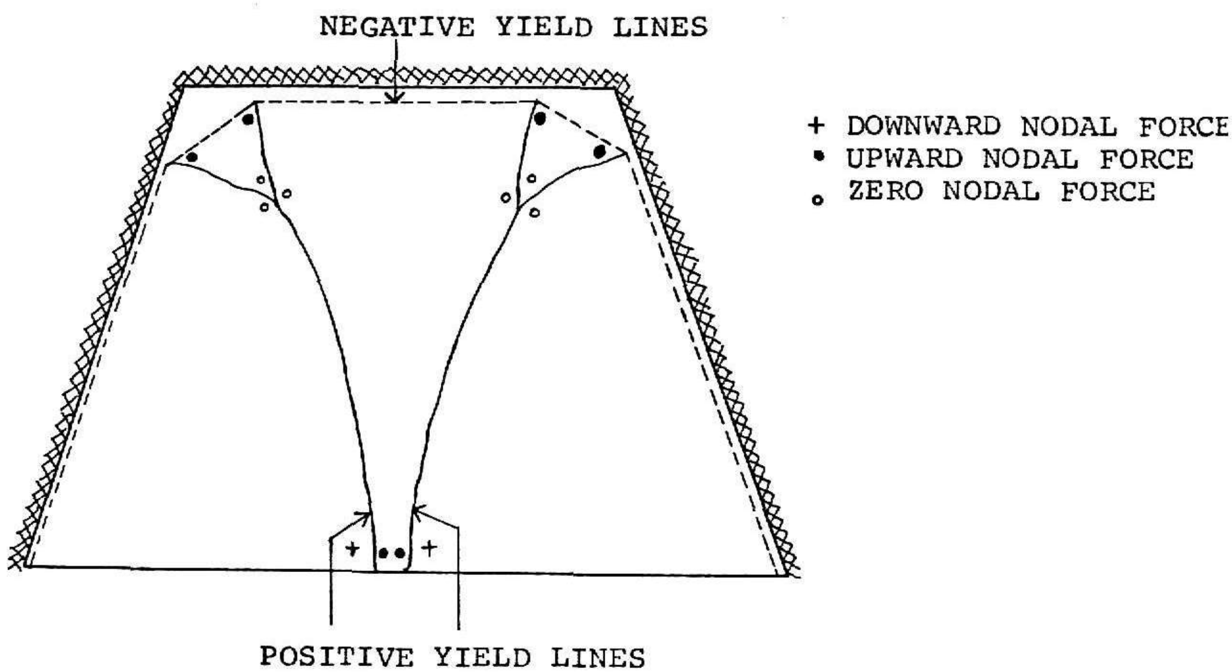


Fig. 3. Pattern where rigid regions are in equilibrium under the action of nodal forces.

This problem was answered when rules for limit analysis were stated by Professor W. Prager (3). These rules indicated that the lowest failure load had been reached if one could find coincidental "upper" and "lower" bound solutions. The conditions required to establish an upper- or lower-bound solution were essentially as follows.

(A) Upper Bound Solution (A solution, based on an assumed mechanism, which gives a correct or too high, value of the collapse load.)

(1) A valid mechanism, which satisfies the boundary conditions, should be found.

(2) Work equations should be satisfied.

(3) Either the material stays rigid or else it deforms plastically.

(a) Where deformation takes place the direction of strains is defined by a mechanism. The direction of strains must in turn define the yield stresses. (This is known as the yield line criterion.)

(B) Lower Bound Solution (Which gives oversafe or correct value of the collapse load)

(1) A complete stress field must be found everywhere satisfying the differential equations of equilibrium.

(2) The forces and moments at the edges must satisfy the boundary conditions.

(3) At no point can the principal stresses violate the yield line criterion.

Once the rules for limit analysis had been established, there were sustained efforts to find the coincidental "upper" and "lower" bound solutions for different boundary conditions. However, very few lower bound solutions were possible. The main difficulty encountered with lower bound technique lies in defining the stress fields between separate regions. Not even the simple case of a clamped isotropic square slab has as yet been solved rigorously.

In 1957, Mansfield (6) invented a method of finding the most critical layout for a system of yield lines involving noncircular "fans" of any shape. In 1961, Wood (4,5) presented a cyclic nodal formula for three yield lines, each governed by a different isotropic mesh of reinforcement. Many engineers like Jones (5), and Kwiecinski, (7) have contributed to the development of the yield line theory in recent years.

Recent tests have shown that the design of slabs by yield line theory is quite safe. Generally speaking, therefore, the yield line theory meets with growing acceptance, which is a tribute to Johansen, its founder.

ASSUMPTIONS MADE IN YIELD LINE THEORY

Before the technique of yield line analysis is described, it is important to appreciate that the method of analysis is a great simplification of the true behavior of a reinforced concrete slab. The actual force and moment systems which act on a small element in a slab under load are shown in Fig. 4 and Fig. 5.

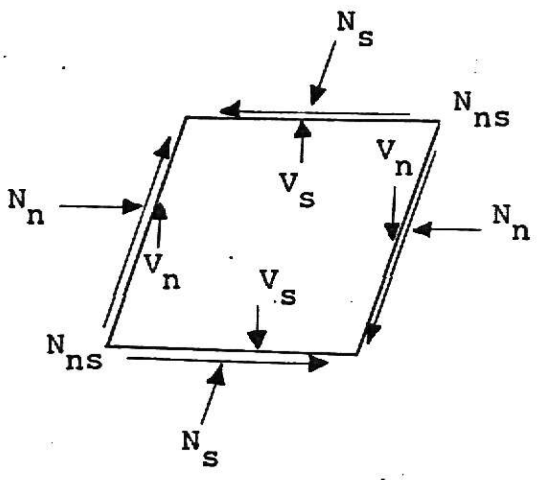


Fig. 4. Positive forces acting on a slab element.

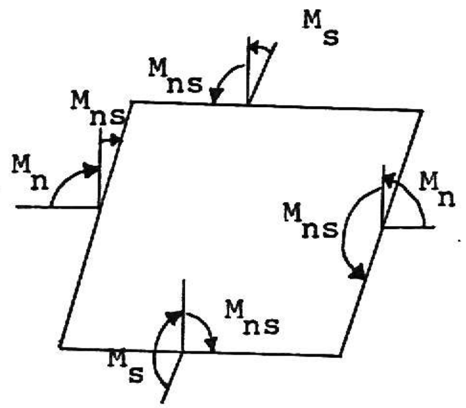


Fig. 5. Positive moments acting on a slab element.

If all the moments and forces acting on a slab element are considered and the equations compatible with the deformation system as well as satisfying conditions are written, it is anticipated that the resulting equations would be unsolvable. Instead it is possible only to develop techniques which involve simple expressions for the values of normal moments (Mn or Ms). Thus when studying the behavior of a slab at ultimate load, it has been assumed that collapse load could be arrived at by considering the bending action only.

It has also been assumed that there is no elastic deformation, therefore, the material either stays rigid or else it goes plastic. The idealized moment-rotation curve is shown in Fig. 6. The elastic deformations are negligible in comparison with the plastic ones This means that it is assumed that the parts of the slabs between yield lines remain plane, and that all the deformations take place at the yield lines. Yield lines must then be straight, as they form the intersections between inclined plates when the slab is in the deflected state. From the geometry of the deformed slab it can be seen that yield lines or yield lines produced must pass through the intersection point of the axes of rotation of the two slab parts adjacent to the yield line.

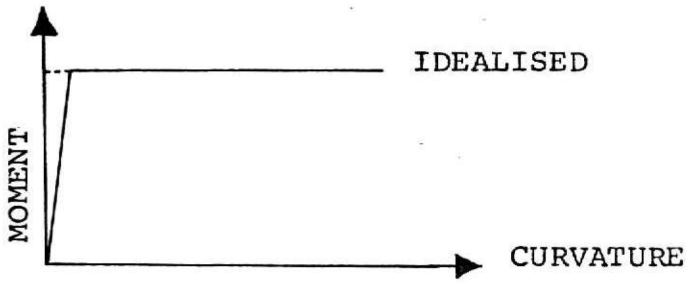
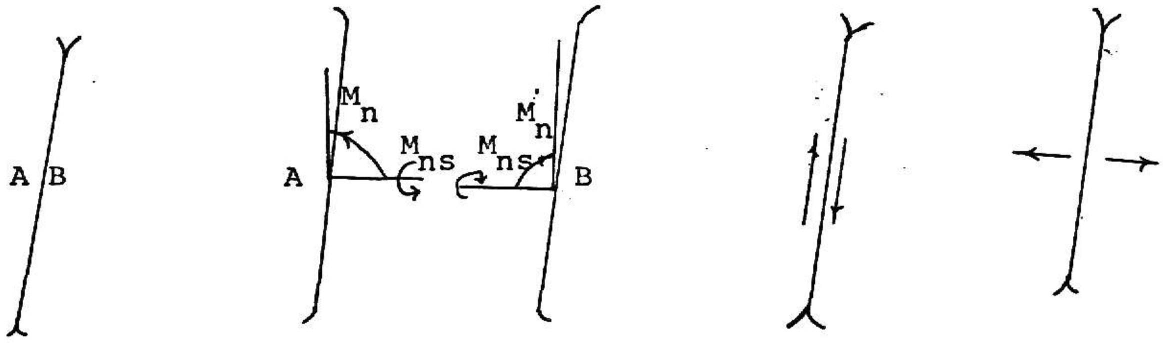


Fig. 6. Moment rotation relationship for slabs.

Axes of rotation of slab parts generally lie along linear supports, or pass over column heads. Typical yield line patterns are shown in Fig. 9.

SIGN CONVENTION FOR FORCES AND MOMENTS AND NOTATIONS

The sign convention that will be used for forces is that downward acting forces will be assumed positive and will be represented by +. Upward forces, which are assumed negative, are represented by a dot•. The sign convention for normal and twisting moment is as shown in Fig. 7-b. The vector notation to represent the moments along the yield line is shown in Fig. 7-c and 7-d.



Positive Yield Line (a)

Positive Moments on Each Side of Yield Line (b)

Positive Normal Moment Vectors (c)

Positive Twisting Moment Vectors (d)

Fig. 7. Sign convention and notations.

MOMENT KEY NOTATION

It is usual to express the ultimate bending strength of a slab in terms of moment per unit width of the slab. The moment key lines at the side of the slab are an abbreviated form of the statement "the normal moment/unit length on a yield line in this direction is the value given." Thus the moment key line marked m implies that if a yield line is in the direction of that moment key line, the normal moment/unit length along the yield line is m. If the key lines are drawn solid it implies positive bending strength; the key lines are shown broken for negative bending strength. Key lines are at right angles to the reinforcement as shown in Fig. 8.

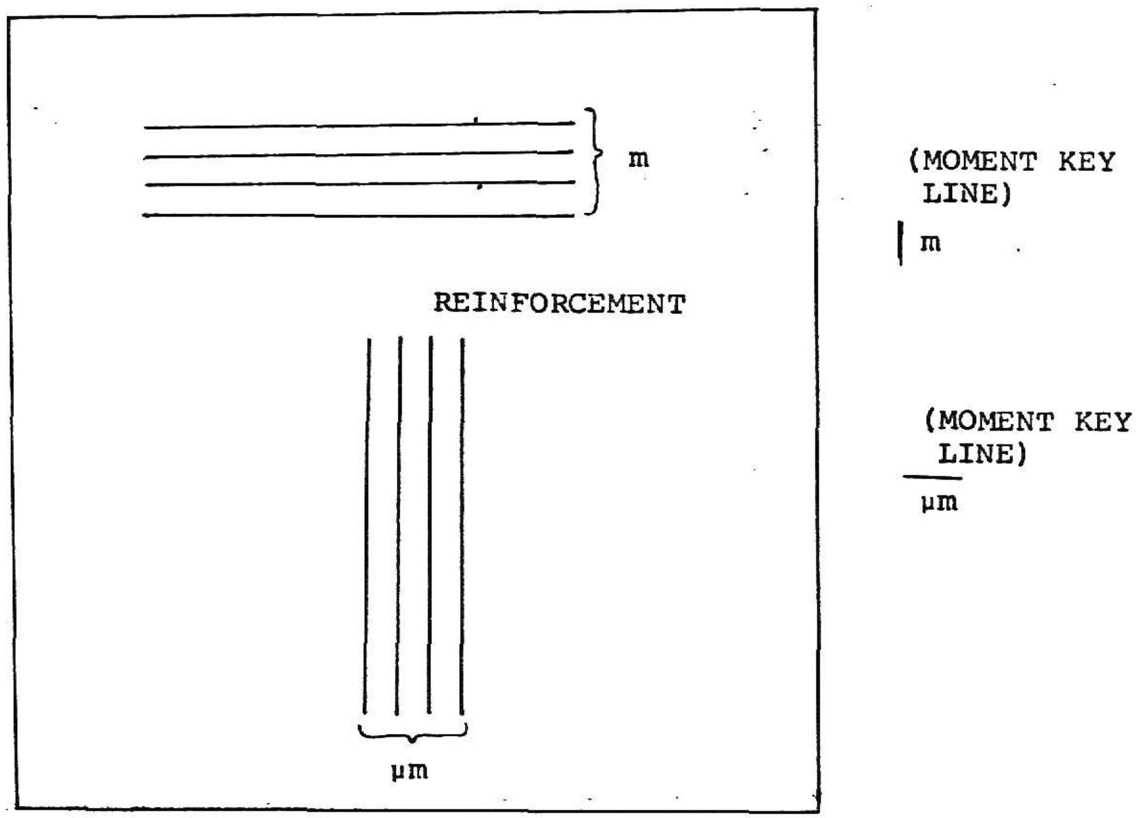


Fig. 8. Moment key line notation.

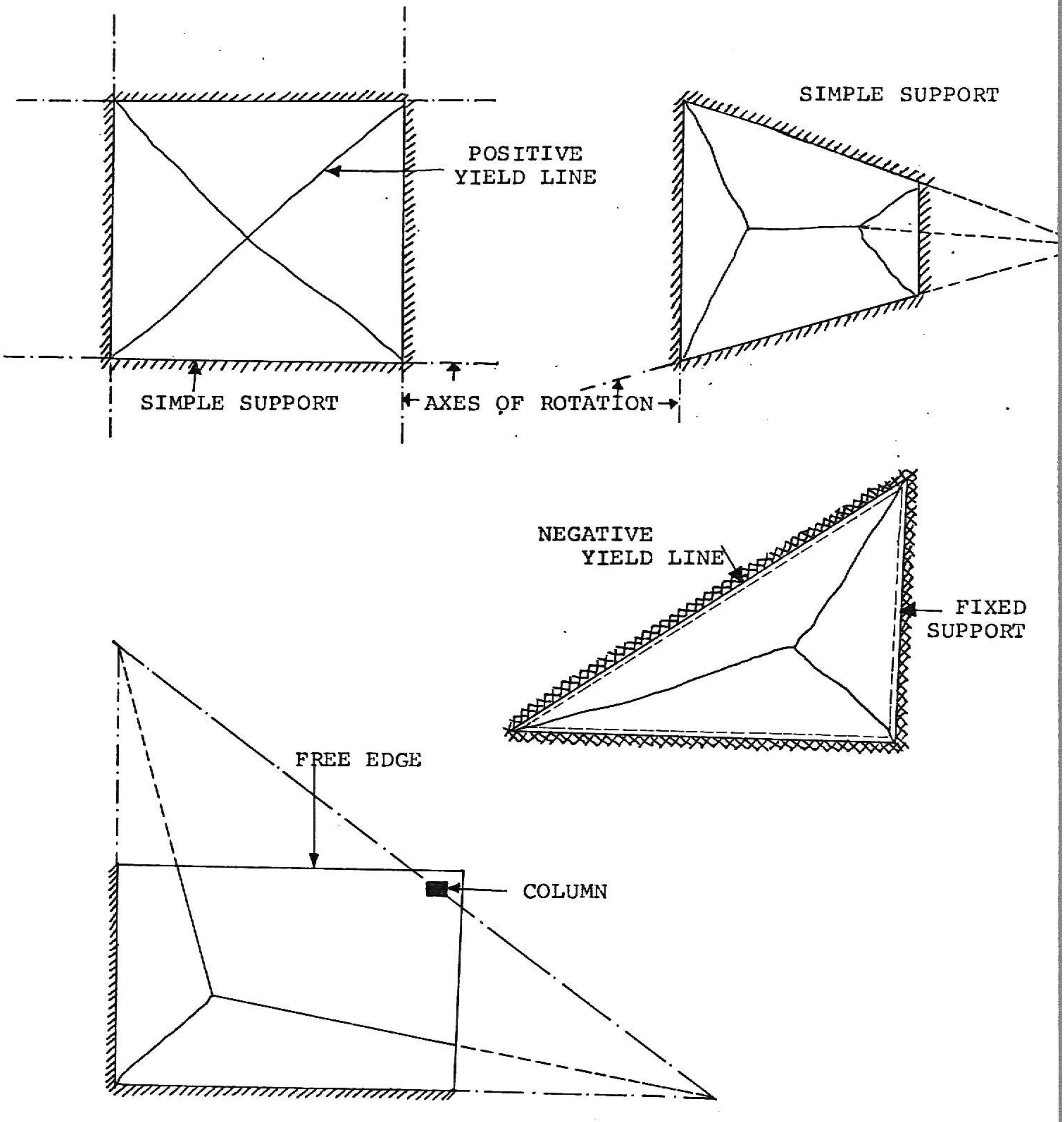


Fig. 9. Typical yield line crack patterns.

JOHANSEN'S STEPPED YIELD CRITERION

Although there are other possible yield criteria for evaluating the magnitude of the normal moments on the yield lines, this is still the yield criteria in common use. The basic assumptions underlying it are as follows.

(1) The normal and twisting moments on a yield line can be obtained by considering each band of reinforcement in turn, the total effect being the addition of the individual effects.

(2) For each band of reinforcement taken on its own the yield line may be considered to be divided into small steps parallel to, and at right angles to, the reinforcement, as shown in Fig. 10.

(3) All reinforcement crossing the yield line is assumed to yield.

(4) All reinforcement is assumed to stay in its original straight line when the steel yields, i.e., there is no "kinking" of the steel in crossing the yield line.

(5) When each band of reinforcement is considered on its own, on the small steps ar right angles to the reinforcement there is only a normal moment/unit length m while on the steps parallel to the reinforcement there is neither normal nor twisting moment.

(6) The values of normal and twisting moments on the yield line are such that they are equivalent to the components of the normal moments on the steps.

On the basis of the above assumptions, consider the yield line shown in Fig. 10 whose orientation to the m moment key line is ϕ, where ϕ is the angle measured clockwise from m to the yield line.

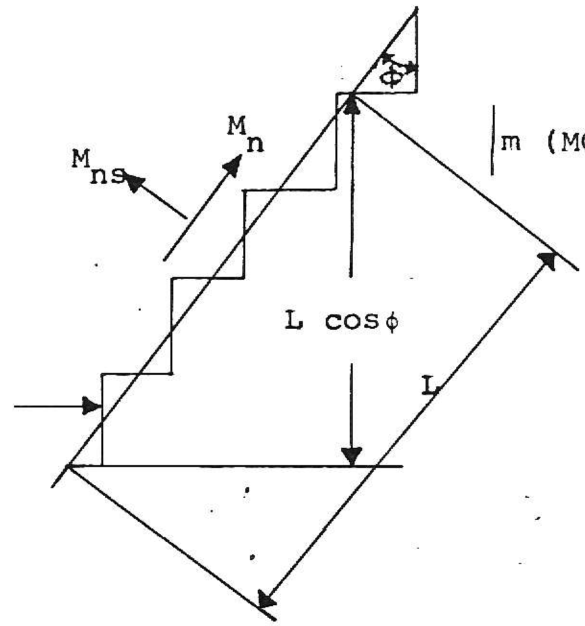


Fig. 10. Stepped yield line criterion.

If a length L of the yield line is considered, the normal moment on the steps parallel to the m moment key line is equivalent to mn/unit length and twisting moment mns/unit length acting along the yield line over a length L.

Now from the Fig. 10 it is clear that

$$m_n = m \cos^2\phi \tag{1}$$

$$m_{ns} = m \sin\phi \cos\phi \tag{2}$$

When there are several sets of reinforcement crossing the yield line, the total value of mn and mns will be the sum of the separate effects of the reinforcement so that in general

$$m_n = \sum_{i=1}^{n} m_i \cos^2\phi_i \tag{3}$$

$$m_{ns} = \sum_{i=1}^{n} m_i \cos\phi_i \sin\phi_i \tag{4}$$

where mi is the magnitude of a typical moment key line and ϕ_i is the angle measured clock-wise from the moment key line mi to the yield line.

Jones and Wood (6), after carefully studying other yield line criteria, have suggested the following yield line criteria. The value of the normal moment $m_n = \sum_{i=1}^{n} m_i \cos^2\phi_i$ will be accepted as being reliable for designers. However, the value indicated for the twisting moment on yield lines $m_{ns} = \sum_{i=1}^{n} m_i \sin\phi_i \cos\phi_i$ should be held suspect and not insisted upon.

YIELD LINE ANALYSIS BY WORK METHOD

The virtual work method is based on the principle that the external work done by the applied loads in causing a small virtual displacement is equal to the internal work done, or energy dissipated in rotation along the yield lines. Thus, having postulated a yield line pattern at failure, the first step is to give any convenient point in the slab a virtual deflection of Δ, in terms of which the corresponding deflection of all other parts of the slab may be calculated. Total work done by the loads to produce this deflection is given by

$$\text{External work done} = E_p = \sum \iint p\Delta dxdy \qquad (5)$$

where dx dy is the area of element (dxx dy). Consider p = distributed load on the slab at collapse, then the energy dissipated in rotation at a yield line is the total moment along the yield line multiplied by the rotation at the yield line. If θ_n is the normal rotation, which can be calculated in terms of the virtual deflection Δ, then the total internal work done is given by

$$\text{Internal work done } E_D = \quad (m_n \ \ell \ \theta_n) \tag{6}$$

where m_n = normal moment/unit length of yield line and ℓ = yield line length. The solution for the slab is obtained by equating the external work done to the internal energy absorbed, therefore,

$$\sum \iint p \ \Delta \ dx \ dy = \quad (m_n \ \ell \ \theta_n) \tag{7}$$

Equation (7) can be used to determine m_n for a given loading p, if the yield line pattern is known. However, equation (7) can also be used to determine the worst yield line pattern. The moment across the yield line being the maximum value, the worst yield pattern corresponding to a load p will give a maximum valud for m_n from equation (7) as compared to other patterns. If a type of pattern is assumed in accordance with the support conditions and characterized by a number of unknown parameters x_1, x_2, x_3,..., equation (7) may be written

$$m = f(x_1, x_2, x_3, \ldots., p) \tag{8}$$

The correct yield pattern then is formed by the maximum criteria

$$\frac{\partial f}{\partial x_1} = 0 \tag{9a}$$

$$\frac{\partial f}{\partial x_2} = 0 \tag{9b}$$

$$\vdots$$

$$\frac{\partial f}{\partial p} = 0 \tag{9c}$$

and the final moment m is determined by substituting the corresponding parameter values in equation (8).

<u>Example 1</u>: Find the collapse loads for a rectangular slab (as shown in Fig. 11) with isotropical reinforcement, simple supports and subjected to a uniformly distributed load p.

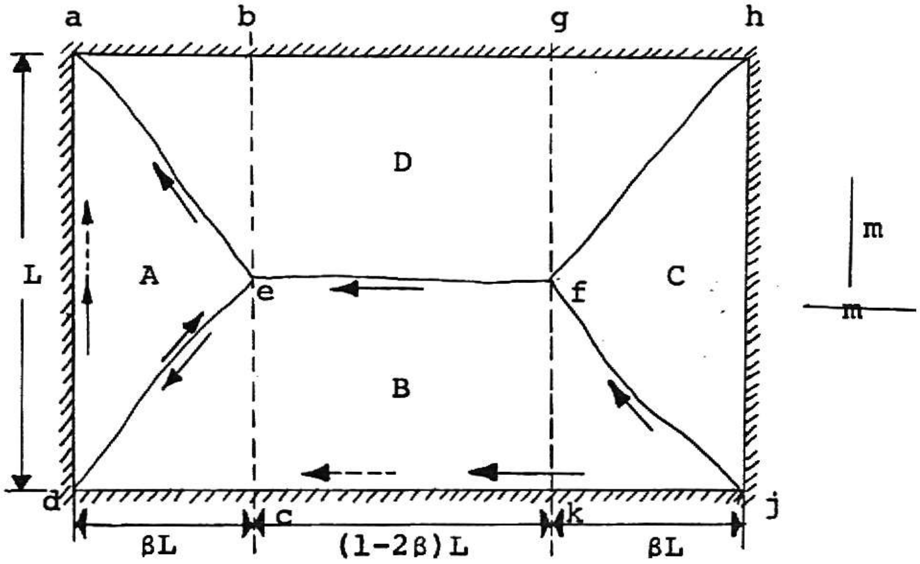


Fig. 11. Rectangular slab with uniform loading.

Line ef is given unit deflection. For the distributed load, instead of dealing with the whole of a rigid region we can conveniently subdivide the trapezoidal areas B and D each into two triangular areas and a rectangular area. Now the center of gravity of each of the three triangular areas abe, aed, and dec each deflect 1/3, as do the triangles within the area ghjk. The center of gravity of two rectangular areas bgef and ckfe deflect by 1/2. Hence

External work done $E_p = \frac{1}{3}\,p(2\alpha\beta L^2) + \frac{1}{2}\,p[d(1-2\beta)L^2]$

$$= \frac{1}{6}\,\alpha p L^2(3-2\beta) \qquad (10)$$

To calculate the internal dissipation of energy the x and y axes chosen will be the same for all regions. Since region A is similar to region C, and region B similar to D, the moment and rotation vectors need only be inserted on regions A and B.

The dissipation of work in the lines relative to region A is given by

$$D_A = [\ (m_x \ell_x \theta_x) + (m_y \ell_y \theta_y)] \tag{11}$$

but $\theta_y=0$ and moment vector cd is equivalent to lines cr, fe, and ed so that $\ell x = L$, $mx = m$ and $\theta_x = \frac{1}{\frac{1}{2}\alpha L} = \frac{2}{\alpha L}$. Since the moment vector cd and θ_x point in the same direction, the dissipation of work is positive and

$$E_{D_A} = m\ L \cdot \frac{2}{\alpha L} = \frac{2m}{\alpha} \tag{12}$$

For region B, the moment vector da is equivalent to the moments on the yield lines de and ea, and we therefore find $\theta_x=0$, $\theta_y = \frac{1}{\beta L}$, $\ell_y = \alpha L$ and $m_y = m$ Hence,

$$E_{D_B} = 0 + m\alpha L\ \frac{1}{\beta L} = \alpha m/\beta$$

$$\begin{aligned} \text{Total dissipation of energy} &= 2(E_{D_A} + E_{D_B}) \\ &= 2m\ (\frac{2}{\alpha} + \frac{\alpha}{\beta}) \\ &= \frac{2m}{\alpha\beta}\ (2\beta + \alpha^2) \end{aligned} \tag{13}$$

The work equation is, therefore, as follows

$$\frac{1}{6}\ \alpha p L^2 (3-2\beta) = \frac{2m}{\alpha\beta}(2\beta + \alpha^2)$$

$$\frac{m}{p} = \frac{1}{12}\ \alpha^2 L^2 (\frac{3\beta - 2\beta^2}{2\beta + \alpha^2}) \tag{14}$$

For m/p to be maximum $\qquad \dfrac{f_1(\beta)}{f_2(\beta)} = \dfrac{df_1(\beta)}{df_2(\beta)}$

Therefore

$$\frac{3\beta - 2\beta^2}{2\beta + \alpha^2} = \frac{3 - 4\beta}{2}$$

or $4\beta^2 + 4\alpha^2\beta - 3\alpha^2 = 0.$

The solution of the above equation is

$$\beta = \frac{1}{2}\alpha\left[\sqrt{(3 + \alpha^2)} - \alpha\right] \tag{15}$$

Example 2: Find the collapse load for a rectangular orthotropically reinforced slab as shown in Fig. 12.

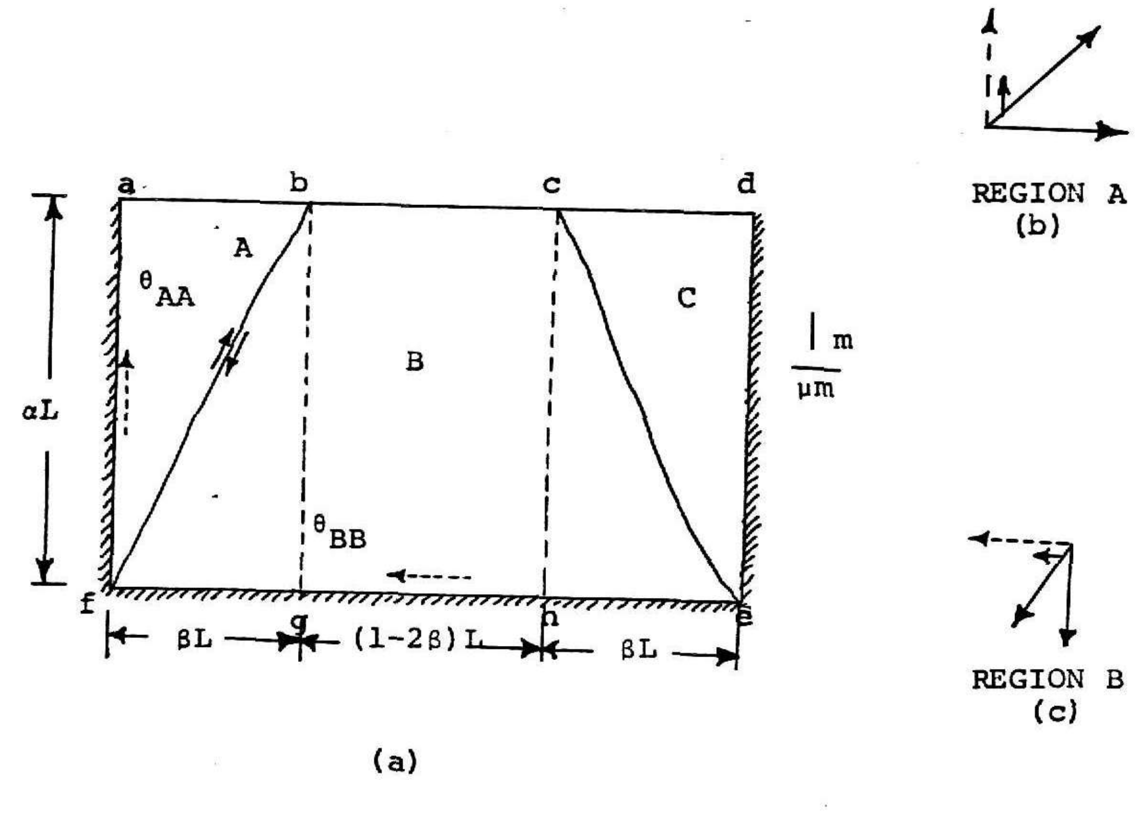


Fig. 12. Rectangular slab with one edge free.

Line bc is given a unit deflection and the expenditure of energy by external loads is easily calculated by noting that centers of gravity of the triangles within the areas abgf and cdeh each deflect by 1/3, and the center of gravity of the area bchg deflects by 1/2. Thus $E_p = pL^2 \frac{2\alpha}{3} + \frac{\alpha(1-2\beta)}{2}$

$$= \frac{1}{6}\,\alpha pL^2(3-2\ ) \qquad (16)$$

Since the pattern is symmetrical the moment vector arrows need only be inserted along line fb together with the rotation vector θ_{AA} and θ_{BB}. If the dissipation of work relative to region A is considered first, the rotation vector θ_{AA} and moment vector on region A along fb should be superimposed on the x and y axes to determine the sign of the vector components, as shown in Fig. 12b. From this it can be seen that there is no x component of the rotation θ_{AA}. Thus $\theta_x=0$, $\theta_y=\frac{1}{\beta L}$, $m_y=m$ and $p_y=\alpha L$. Hence

$$E_{D_A} = 0 + m\alpha L\,\frac{1}{\beta L} = \frac{m}{\beta}$$

For region B only line fb is considered since clearly the same amount of work is dissipated along line ce. The rotation vector θ_{BB} and the moment vector along line fb on region B have been superimposed on x and y axis in Fig. 12c, from which it can be seen that $\theta_y=0$, $m_x=\mu m$, $\ell_x=\beta L$ and $\theta_x=\frac{1}{\beta L}$. Hence work done in line fb relative to region B is $E_{DB} = \frac{\mu\beta M}{\alpha}$.

Total dissipation of work in line fb is $E_{D_{fb}} = \frac{\alpha m}{\beta} + \frac{\mu\beta m}{\alpha}$ and since line ce is similar to fb,

Total dissipation of work $= \frac{2m}{\alpha\beta}\,(\alpha^2 + \mu\beta^2)$ (17)

The work equation is found by equating Equation (16) to (17).

Hence $\frac{1}{6}\ \alpha p L^2(3-2\beta) = \frac{2m}{\alpha\beta}\ (\alpha^2+\mu\beta^2)$

or

$$\frac{m}{p} = \frac{1}{12}\ \alpha^2 L^2\ \frac{3\beta-2\beta^2}{\alpha^2+\mu\beta^2} \qquad (18)$$

For m/p to be maximum

$$\frac{3\beta - 2\beta^2}{\alpha+ \mu\beta^2} = \frac{3 - 4\beta}{2\mu\beta}$$

or

$$3\mu\ \beta^2 + 4\alpha^2\beta - 3\alpha^2 = 0$$

The solution of the above equation is

$$\beta = \frac{\alpha^2}{\mu}\left[\sqrt{\frac{4}{9} + \frac{\mu}{\alpha^2}} - \frac{2}{3}\right] \qquad (19)$$

It may be noted that maximum value of $\beta = 0.5$.

Example 3: Consider a skew slab with skew reinforcement, simply supported on two opposite edges and subjected to a uniform load p/unit area and a point load P at the center of the span.

The postulated yield line pattern is shown in Fig. 13 in which point e is given a ujit deflection. The directions of x and y axes are normal to the two faces of the slab. Since the pattern is symmetrical, only regions A and B need be considered in detail. Starting with Region A, θ_{AA}, can be inserted together with the moment vectors along the positive yield line. The external negative moment vector is in the same direction as θ_{AA}, as is the moment vector along da which is equivalent to the moment vectors along the lines de and ea. All the components of work are therefore positive.

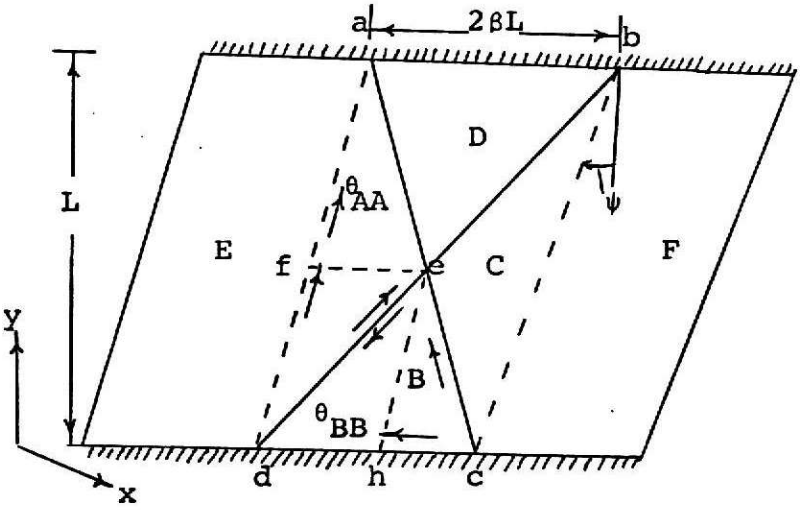


Fig. 13. Skew slab supported on two opposite sides.

The work done in the positive yield line is calculated first. It is clear that $\theta_x=0$, since a line normal to the X axis is parallel to the axis of rotation of region A and can be found by drawing a line normal to the Y axis and passing through point e, from which we find $\theta_y=1/\beta L$, $p_y=L$, and $m_y=\mu m$. For the negative yield line ad $\theta_x=0$, $\theta_y=\frac{1}{\beta L}$, $\ell_y=L$ and $m_y=i\mu m$, hence

$$E_{D_A} = (\mu mL \cdot \frac{1}{\beta L} + i\mu mL\,\frac{1}{\beta L})$$

For region B, the vector addition of the moments on ce and ed is a vector of length cd. This point is in the same direction as $\theta_{\beta\beta}$ so that work components are positive. $\theta_y=0$ and θ_x are found, by drawing a line normal to the X axis passing through e, to be $\frac{2\cos\psi}{L}$. The projection of the equivalent moment vector of length cd on to the X axis is $2\beta L\cos^2\psi$ and $m_x=m$ so that $E_{D_B}=4\beta m\cos^2\psi$ since regions A and C, and B and D are similar. Hence the total dissipation of energy

$$E_D = \frac{2\mu m(1+i)}{\beta} + 8\beta m\cos^2\psi \qquad (20)$$

The work done by external loads is easily found to be

$$E_p = p\,\frac{1}{3}\,2\beta L^2 + P$$

$$= \frac{2}{3}\,p\,\beta L^2 + P \tag{21}$$

Equating (19) to (21), we get

$$m[2\mu\frac{(1+i)}{\beta} + 8\beta\cos^2\psi] = \frac{2}{3}\,p\beta L^2 + P \tag{22}$$

No particular difficulty exists with this problem, but the main point of interest is that the loading is mixed. Hence, write this equation in the form

$$m = \frac{1}{3}\,pL^2\left[\frac{\left(\frac{3p}{pL^2} + 2\beta\right)\beta}{2\mu(1+i)+8\beta^2\cos^2\psi}\right] \tag{23}$$

To simplify put $\frac{3P}{pL^2} = P_o$, $\quad 2\mu(1+i) = \mu_o$
and $\cos^2\psi = \kappa$

Now m/p will be maximum when

$$\frac{(P_o + 2\beta)\beta}{\mu_o + \kappa\beta^2} = \frac{P_o + 4\beta}{2\kappa\beta} \tag{24}$$

or

$$\beta^2 - \beta\left(\frac{4\mu_o}{\kappa P_o}\right) - \frac{\mu_o}{\kappa} = 0 \tag{25}$$

The only real solution of (25) is

$$\beta = \frac{2\mu_o}{\kappa P_o}\left[1 + \sqrt{1 + \frac{\kappa P_o}{4\mu_o}}\right]$$

Substituting the value of B in (23)

$$m = \frac{pL^2}{24\cos^2\psi} \left[3 - \sqrt{1 + \frac{9\cos^2\psi}{\mu(1+i)p^2L^4}}\right] \quad (26)$$

For valid values of B, the only restriction is that $2\beta L$ shall not be wider than the slab.

YIELD LINE ANALYSIS BY THE EQUILIBRIUM METHOD

Unitl recently, the equilibrium method of solving yield line patterns has been called an alternative to the work method. Certainly the method seems different from the work method in that the equilibrium of each of the rigid regions is considered. It is quite clear, however, that the equilibrium method is the work method presented in another form. The equilibrium method accredited to Professor K.W. Johansen (1) can be stated as follows. When a layout of yield lines provides the stationary maximum moment for a particular pattern, certain forces termed nodal forces must be inserted at the junction of the yield lines or slab edges for the purpose of maintaining equilibrium.

Each rigid region considered on its own is then in equilibrium under the action of these forces, the moments along the yield lines and the externally applied loads. Since the magnitude of these nodal forces can often be predetermined from theorems given by Johansen (1), the stationary maximum condition can be found by obtaining equilibrium equations for the rigid regions. Before some examples are solved illustrating the use of equilibrium equations, it would be proper to know how the nodal forces are calculated.

Johansen's (1) first theorem states that the sum of the forces acting at a node is zero. Whatever system of statically equivalent forces acts on a region on one side of a yield line, equal and opposite forces act on the region on the other side of the line. The basic process by which Johansen (1) established the value of nodal forces, which is further extended by Wood and Jones (5), is as follows.

In order to calculate the nodal force between two yield lines it is first necessary to consider the equilibrium of a small triangular region abd marked A' in Fig. 14.

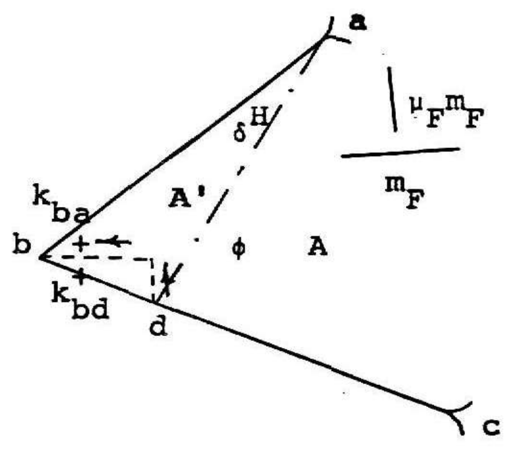


Fig. 14. Two yield line meeting at a point.

Let the first yield line ab be governed by the moment key lines mf and $\mu_F$mf and the second yield line bc by the moment key lines m_s and $\mu_s m_s$ which may have an orientation different from the mf and $\mu_F m_f$ moment key lines. The angle between first and second yield lines is measured clockwise from the first line to the second line. Let the statical equivalent of the twist and shears along ab, acting at b on region A', be k_{ba} and that due to the short length of yield line bd be k_{bd}. The "incomplete" nodal force (incomplete because the whole line bc is not considered) has the value

$$kA'_b = k_{bd} + k_{ba} \tag{26}$$

The equilibrium of region A' is now considered. If the steel which yields across yield line ab is also yielding across line ad, the path of ab and ad may be conceived as being stepped in the direction of m_f and $\mu_f m_f$ with the result that m_n and m_{ns} values which are assumed for convenience to act along these lines are equivalent to normal moments de$\mu_f m_f$ and eb m_f shown by the vector arrows. Since de is the net sum of the steps in the $\mu_f m_f$ direction and eb that in the m_f direction, the effect of the moments on ab and ad is the same as the normal (m_n)f and twisting moment $(m_{ns})_f$ along the direction and length of db due to the reinforcement mesh f. The sense in which they act externally on region A' is shown in Fig. 16. We still have to consider the normal moments m_n and twisting moments m_{ns} acting on bd due to reinforcing mesh, and these act externally on region A' in the sense given in Fig. 15. Thus the total effect of the normal and twisting moments along ab, ad, and bd is a normal moment/unit length $\left[(m_n)s - (m_n)f\right]_s$ and a twisting moment $\left[(m_{ns})_s - (m_{ns})_f\right]_s$ acting along bd. (mn_s) indicates the reinforcing mesh involved, while the second letter suffix, i.e., $\left[(m_{ns})s\right]_s$ indicates the direction along which the moment should be calculated.

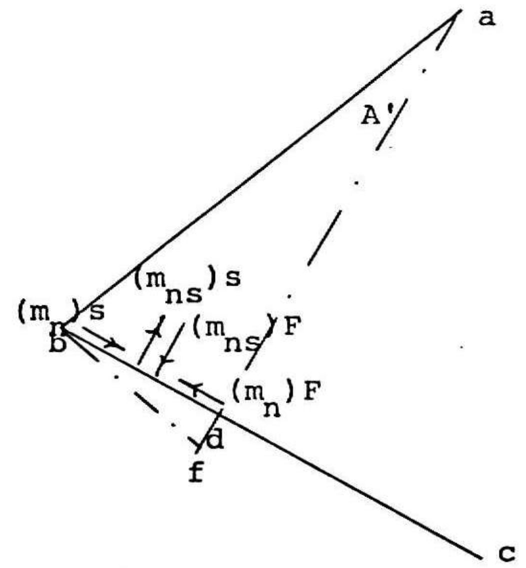


Fig. 15. Moments on yield lines.

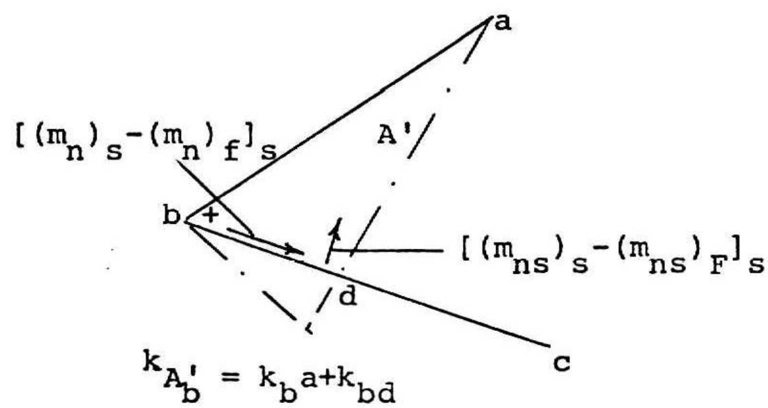


Fig. 16. Nodal forces.

When moments are taken about ad for the forces and moments shown in Fig. 16, for the equilibrium of region A', we get

$$(k_{bd} + k_{ba})\ bd\ \sin(\phi+\delta\phi) + [(m_n)_s - (m_n)_f]_s b\ d\cos(\phi+\delta\phi)$$
$$+ [(m_{ns})_s - (m_{ns})_f]_s b\ d\ \sin(\phi+\delta\phi) \quad (27)$$

If we divide by bd and when $\delta\phi\to 0$ it will be found that

$$k_{bd} + k_{ba} = [(m_n)_s - (m_n)_f]_s \cot\phi + [(m_{ns})_s - (m_{ns})_f]_s \quad (28)$$

Equation (28) does not give the value of the nodal forces but it may be used to evaluate it as follows. Consider the three lines meeting at a point as shown in Fig. 17a and let the moments on each be determined by three different meshes, 1, 2, and 3. For the equilibrium of small triangle A', we can use equation (28) by noting that the first line of its boundary relates to mesh 1 and the second to mesh 3. Thus changing 1 for f and 3 for s in equation (28).

$$k_{bd} + k_{ba} = [(m_n)_3 - (m_n)_1]_3 \cot\phi_{13} + [(m_{ns})_3 - (m_{ns})_1]_3 \qquad (29)$$

Similarly for region B'

$$k_{bd} + k_{be} = [(m_n)_3 - (m_n)_2]_3 \cot\phi_{23} + [(m_{ns})_3 - (m_{ns})_2]_3 \qquad (30)$$

where k_{be} is the statical equivalent of shears and twists along be acting at B on region B' if the statical equivalent of line be acting at b on region B is k_{be}. Then we have $-k_{be}$ acting on region A (Fig. 17b). Hence

$$k_{Ab} = k_{12} = k_{ba} - k_{be} \qquad (31)$$

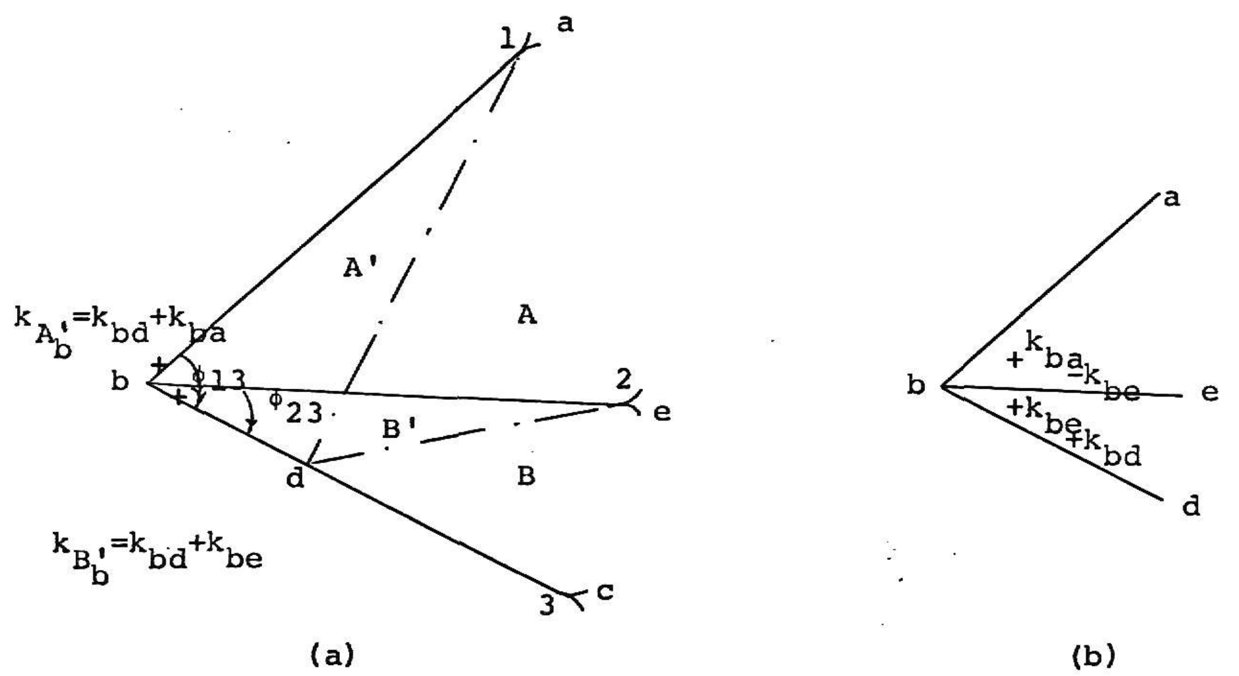


Fig. 17. Nodal forces acting at the junction of yield lines.

When we substitute for k_{ba} and k_{be} from (29) and (30), it will be found that

$$k_{12} = -k_{bd}-[(m_n)_3-(m_n)_1]_3\cot\phi_{13}-[(m_{ns})_3-(m_{ns})_1]_3$$

$$+ k_{bd} + [(m_n)_3-(m_n)_2]_3\cot\phi_{23}+[(m_{ns})_3-(m_{ns})_2]_3$$

or

$$k_{12}=[(m_n)_1-(m_n)_3]_3\cot\phi_{13}-[(m_n)_2-(m_n)_3]_3\cot\phi_{23}+[(m_{ns})_1-(m_{ns})_2] \tag{32}$$

The cyclic form for the nodal force equation (32) is considered to be most useful. Equation (32) need not, however, be used every time and there are certain types of nodes which occur so frequently that certain standard solutions derived from equation (32) should be known. These solutions are as follows.

(1) Node where the three yield lines have different isotropic reinforcement is shown in Fig. 18. Since the reinforcement is isotropic, $[(m_n)_1]_3 = m_1$, $[(m_n)_2]_3 = m_2$, $[(m_n)_3]_3 = m_3$

AND $\quad [(m_{ns})_1]_3 = [(m_{ns})_2]_3 = [(m_{ns})_3]_3 = 0$

Hence

$$k_{12} = (m_1 - m_3(\cot\phi_{13} - (m_2 - m_3)\cot\phi_{23} \qquad (33)$$

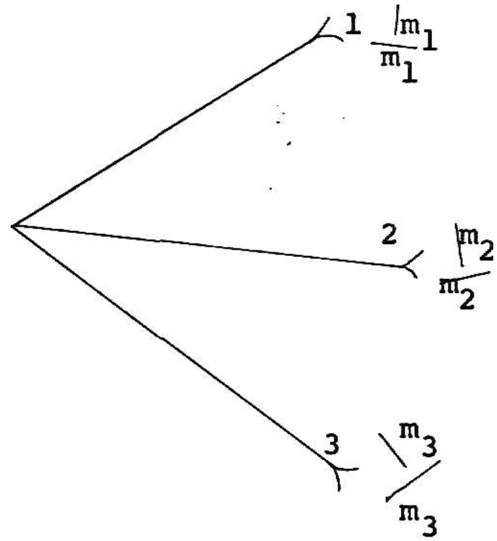


Fig. 18. Node where the three yield lines have different isotropic reinforcement.

(2) Where three lines have the same reinforcement, then $m_1 = m_2 = m_3$, hence, $k_{12} = 0$. (34)

(3) Yield line intersecting a free edge is shown in Fig. 19. Two lines will be regarded having zero moment value and a third having some specified value.

$$k_\psi = (m_n \cot\psi + m_{ns})_e \qquad (35)$$

If the reinforcement for the yield lines is isotropic and of value m, then $(m_{ns})_e = 0$. $(m_n)_e = m$, hence $k_\psi = m_2 \cot\psi$ (36)

With orthotropic reinforcement provided the reinforcement is perpendicular and parallel to an edge, then $(m_{ns})_e = 0$ and we get

$$k_\psi = (m_n)_e \cot\psi \tag{37}$$

where $(m_n)_e$ is the value of moment key line parallel to edge.

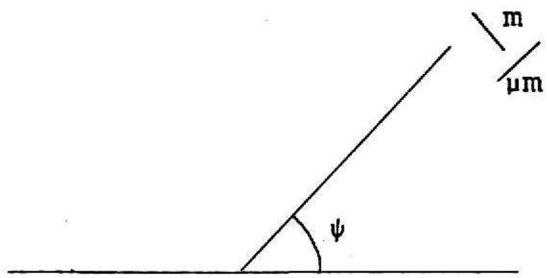


Fig. 19. Yield line intersecting a free edge.

Now some examples using the equilibrium method will be solved.

Example 1: Isotropically reinforced rectangular slab simply supported on four sides and subjected to a uniform load p/unit area.

The pattern is shown in Fig. 20 for the two equilibrium equations taken about the axes of rotation of region A and B. The nodal forces for e and f must be obtained since e and f are governed by the same mesh. It has been shown previously that all nodal forces at such nodes are zero.

If moments are taken about the axis ad for region A, the yield lines around region A give a moment $m\alpha L$ about the axis ad, so that $m\alpha L = \frac{1}{2}\, p\alpha\beta L^2 \cdot \frac{1}{3}\, \beta L = \frac{1}{6}\, p\alpha\beta^2 L^3$ or

$$m = \frac{1}{6}\, p\beta^2 L^2 \tag{38}$$

If moments are taken about axis ab for region B, it will be found that

$$mL = p\left[\frac{\beta\alpha^2 L^3}{12} + \frac{(1-2\beta)\alpha^2 L^3}{8}\right]$$

$$m = \frac{1}{24}\, p\alpha^2 L^2 (3-4\beta) \tag{39}$$

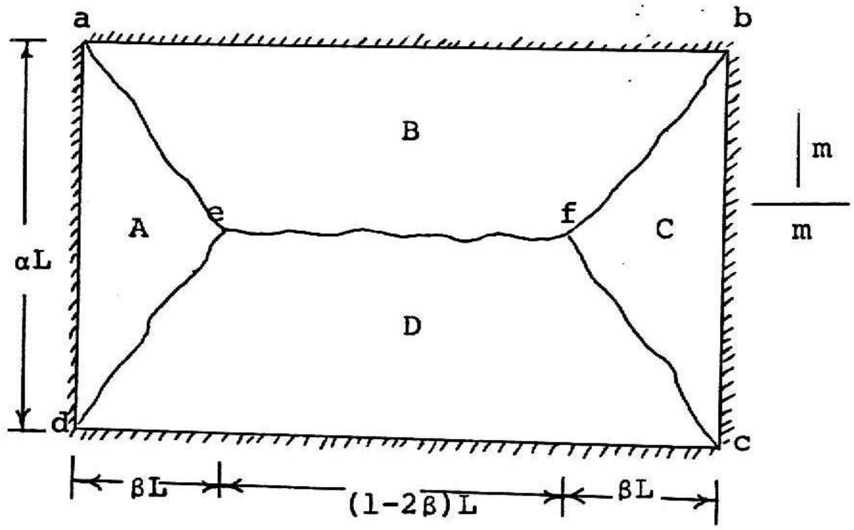


Fig. 20. A rectangular slab with uniform loading.

The moments about ab are equal to the moments about ad. This condition gives the critical value of β. Equating equations (38) and (39) we get

$$\alpha^2 \frac{(3-4\beta)}{24} = \beta^2/6$$

or

$$4\beta^2 + 4\alpha^2\beta - 3\alpha^2 = 0$$

$$\beta = \frac{1}{2}\left[\sqrt{3\alpha^2 - \alpha^4} - \alpha^2\right] \tag{40}$$

The critical value of m is obtained by substituting the value of

β into (38) or (39). Hence

$$m = \frac{1}{24} \; p\alpha^2 L^2 \left[\sqrt{3 + \alpha^2} - \alpha \right]^2 \tag{41}$$

As is seen, no differential process is required to establish the critical value of m as in the work method. In this example the nodal forces have been zero. In the next solution, however, the moment due to nodal forces will be considered.

<u>Example 2:</u> Orthotropically reinforced rectangular slab simply supported on three sides and free on the fourth, subjected to a uniform load p/unit area.

The yield line pattern is shown in Fig. 21a and the nodal forces at c and d are evaluated from equation (36). Considering point c, the angle ψ is the angle obtained when rotating clockwise from the yield line, thus

$$k\psi = k_{A_C} = (m_n)_e \cot\psi \;, \; (m_n)_e = \mu m$$

$$\cot\psi = -\beta/\alpha \qquad \text{Hence} \quad k_{A_C} = -\mu m \; \beta/\alpha \tag{42}$$

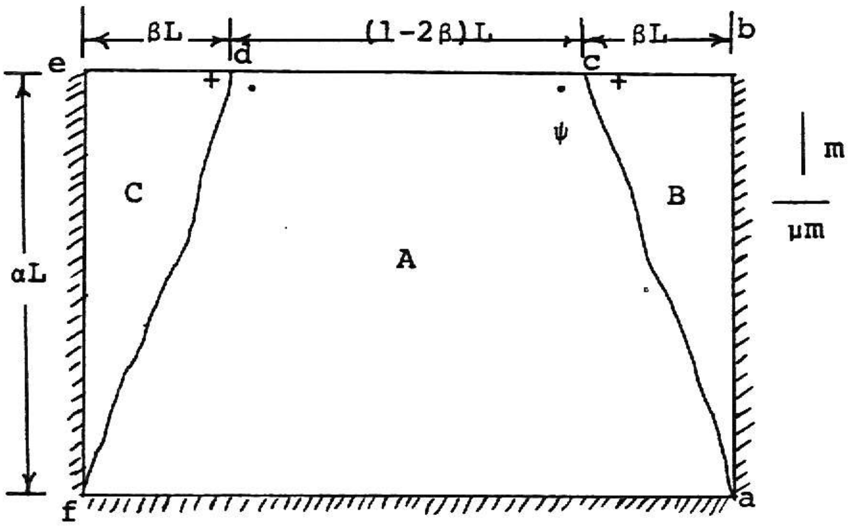


Fig. 21a. Slab supported on three edges

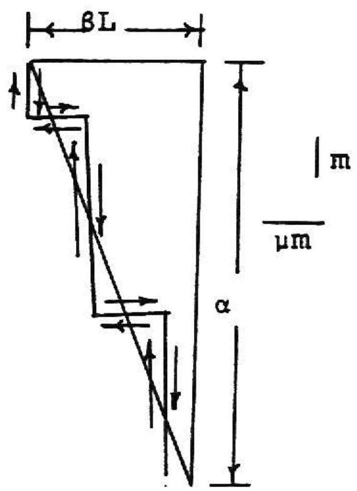


Fig. 21b. Stepped yield-line criterion.

As the sum of the forces at c is zero, $k_{B_C} = \mu m\ \beta/\alpha$ and by symmetry $k_{cd} = Um\beta/\alpha$ and $k_{A_d} = -\mu m\ \beta/\alpha$.

When taking moments for yield line ca about axis ab for region B only moments on the steps parallel to ab have any effect and the magnitude of the moment will be the sum of the lengths in that direction, which is αL multiplied by the normal moments on these steps, giving a moment of $m\alpha L$. When taking moments for region A about af for line ca only, it will be found the same way that the moment is $\mu m\beta L$. Line dg also gives $\mu m\beta L$ about axis ag.

If moments are taken about ag for region A, then

$$2\mu\beta L + \frac{2\beta\mu m}{\alpha} \cdot \alpha L = PL^3\left[\frac{\beta\alpha^2}{3} + \frac{(1-2\beta)}{2}\alpha^2\right]$$

or

$$4\mu\beta m = \frac{1}{6}\, p\alpha^2 L^2 (3-4\beta) \tag{43}$$

If moments are taken about ba for region B

$$m\alpha L - \frac{\mu m\beta}{\alpha} \cdot \beta L = \frac{p\alpha\beta L^2}{2} \cdot \frac{\beta L}{3}$$

or

$$m\left(1 - \frac{\mu\beta^2}{\alpha 2}\right) = \frac{1}{6}\, p\alpha\beta^2 L^2 \tag{44}$$

The critical value of β is found by equating equation (43) to equation (44) and we get

$$3\mu\beta^2 + 4\alpha^2\beta - 3\alpha^2 = 0 \tag{45}$$

This is the quadratic equation in β, the solution of which is

$$\beta = \frac{\alpha^2}{\mu}\left[\sqrt{\frac{4}{a} + \frac{\mu}{\alpha}} - \frac{2}{3}\right] \tag{46}$$

When this value is substituted into either equation (43) or (44), it will give

$$m = \frac{p\alpha^2 L^2}{24\mu}\left[\sqrt{\left(4 + \frac{9\mu}{\alpha^2}\right)} - 2\right] \tag{47}$$

It will be recalled that the cyclic nodal force equation is based on the assumption that we are dealing with a stationary maximum condition for a particular variable. If varying a particular parameter gives a non-stationary maximum solution, then we can still use nodal force theory but the nodal forces, except by symmetry, are unknown relative to that parameter. It is quite possible to have certain nodal forces in a pattern for which equation (32) is valid together with others where it is not valid. The next solution gives an interesting example of this relationship.

Example 3: A square slab simply supported on four sides having a hole with an irregular shape with a uniform load of p/unit area except over the hole.

The assumed pattern for the slab is shown in Fig. 22. Since yield lines tend to be drawn toward holes it is almost certain that the line passing through the corners a and f will be attracted to the corners of the holes at b and h. Thus having fixed the lines ab and fh, the nodal forces at b and h cannot be evaluated by equation (32) and their value is unknown. Since symmetry only exists about the center line marked, no conclusions about these values can be drawn. It will, therefore, be assumed that $k_{ah}=-k$. Hence, $k_{bh}=k$ and by symmetry $k_{ab}=-k$. The yield line passing through the corners d and g will be assumed to intersect the hole at a distance βL from gd. Since it is assumed that βL is a variable, the nodal forces at c and i can be calculated from equation (36), from which it will be found that $k_{\beta i} = -\frac{8\beta m}{3}$ AND $k_{Ci} = \frac{8\beta m}{3}$.

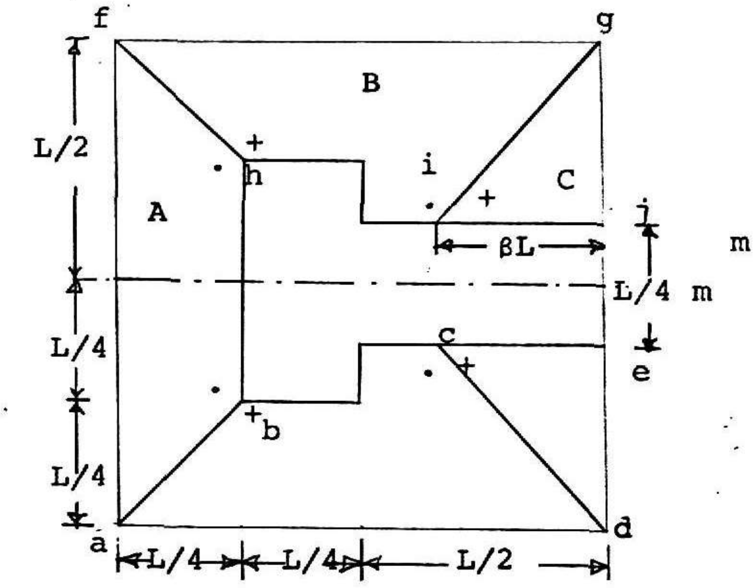


Fig. 22. Slab with an opening.

For region A, taking moments about af

$$\frac{1}{2}\,mL + 2\cdot\frac{1}{4}\,L\,k = pL^3\,[\,\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{12} + \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{8}]$$

or

$$m + k = \frac{1}{24}\,pL^2 \tag{48}$$

For region B, taking moments about fg

$$\frac{1}{4}\,mL + m\beta L + \frac{8}{3}\,L\cdot m\cdot\frac{3}{8} - \frac{1}{4}\,kL$$

$$= pL^3[\,\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{12} + \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{8} + (\frac{1}{2}-\beta)\frac{3}{8}\cdot\frac{3}{16} + \frac{1}{2}\cdot\frac{3}{8}\cdot\frac{\beta}{8}$$

or

$$m(1+8\beta) - k = \frac{pL^2}{192}\,[35 - 36\beta] \tag{49}$$

For region C taking moments about gj

$$\frac{3mL}{8} - \frac{8\beta^2 mL}{3} = p\beta^2 L^3 \cdot \frac{3}{8} \cdot \frac{1}{2} \cdot \frac{1}{3}$$

or

$$m = \frac{3}{2}\, pL^2 \left[\frac{\beta^2}{9-64\beta^2}\right] \tag{50}$$

Eliminating k from equations (48) and (49) gives

$$m(2 + 8\beta) = \frac{pL^2}{192}(43 - 36\beta) \tag{51}$$

Hence from equations (50) and (51), we obtain the quadratic equation

$$\beta^2 + 0.0973\beta - 0.116 = 0 \tag{52}$$

The solution of which is

$$\beta = 0.295$$

From the way the pattern is drawn the solution is valid provided $0 < \beta < 0.5$, and this condition is therefore satisfied since $\beta = .295$. When this value of β is substituted into either equation (50) or (51), it is found that

$$m = \frac{pL^2}{25.9}$$

The position of line ab and fh is correct because these are the lines of least resistance.

AFFINE SLABS

If the analysis of an orthotropic or skew reinforcement slab is compared with that of a similarly shaped isotropic slab, it will be found that there is a certain similarity between the expressions obtained at equivalent stages in the analysis. This similarity was first observed by Johansen (1) who invented the concept of "affine" slabs. An affine slab is an isotropic slab which, for the purpose of analysis, may be considered to be equivalent to an orthotropic or skew slab and is obtained by transforming it into an affine slab.

Unfortunately not all orthotropic or skew slabs can be transformed into simpler equivalent isotropic slabs, but when such a transformation is possible, the solution may be obtained more rapidly from the affine slab. Then from the original skew or orthotropic slab. Jones and Wood (5) gives the transformation rule as follows.

If a skew, or orthotropic, slab has reinforcement in two constant directions, such that the ratio of the ultimate moments due to each set of reinforcement taken separately is constant throughout the slab, then there exists an isotropic slab which gives a corresponding solution to the skew or orthotropic slab. The solution is such that the deflection at corresponding points in the affine slab are the same as in the skew slab. If at any point on the dkew or orthotropic slab, the ultimate moments due to the separate bands of steel m and μm, then the strength of the isotropic slab at the corresponding point is m in all directions.

The affine slab is drawn such that all distances measured in the direction of the m reinforcement remain the same, and this direction forms one coordinate axis for both slabs, which will be called the first coordinate axis. The second coordinate direction in the skew slab follows the μm reinforcement, but the second corresponding coordinate direction in the affine slab is taken at right angles to the first m reinforcement axis. All distances in the affine slab in this second coordinate direction are obtained by dividing corresponding lengths in the skew slab by $\sqrt{\mu}$. In addition all corresponding total loads in the affine slabs are obtained by dividing the original total loads by $\sqrt{\mu}\sin\gamma$ where γ is the angle between the two directions of reinforcement in the skew slab. It is important to establish the transformation rules for commonly occuring loads.

(A) Uniform Occuring Load of Intensity p. Consider a skew element x, y, then the area is $\sin\gamma \cdot dxdy$. The corresponding affine element is $\frac{dxdy}{\mu}$, hence by the rule stated above

$$\frac{p \sin\gamma dxdy}{\sqrt{\mu}\sin\gamma} = p' \frac{dxdy}{\sqrt{\mu}}$$

or

$$p' = p \tag{53}$$

(B) For a Point Load

$$p' = \frac{p}{\sqrt{\mu}\sin\gamma} \tag{54}$$

(C) Line Loads, For a line load of intensity $\dot{p}$ per unit length, as shown in Fig. 23, let x, y be the skew coordinate lengths of the line then the total load is $\bar{p}\sqrt{x^2+y^2+2xy\cos\gamma}$.

The corresponding total load on the affine slab will be

$$\bar{p}' \sqrt{x^2 + y^2/\mu}$$

so that

$$\frac{\bar{p}'}{\bar{p}} = \frac{1}{\sin\gamma} \sqrt{\frac{x^2+y^2+2xy\cos\gamma}{\mu x^2+y^2}}$$

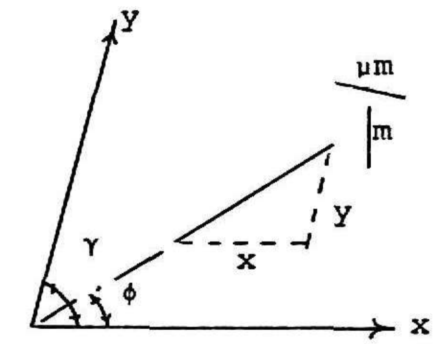


Fig. 23. Line Load.

When $\gamma = 90$, i.e., an orthotropic slab, the above equation reduces to

$$\bar{p}' = \bar{p} \Big/ \sqrt{\mu \cos^2\phi + \sin^2\phi} \tag{55}$$

where ϕ is the angle between the direction of the line load and the first coordinate axis. Now we shall solve some examples on affine transformation.

<u>Example</u>: Rectangular orthotropic slab simply supported on three sides with free edge.

The solution for a rectangular isotropic slab is

$$p = \frac{12m}{\alpha^2 L^2} \quad \frac{\alpha^2+\beta^2}{3\beta-2\beta^2} \tag{56}$$

where

$$\beta L = \alpha^2 L \left[\left(\frac{4}{9} + \frac{1}{\alpha 2}\right)^{1/2} - \frac{2}{3} \right] \tag{57}$$

βL denotes the distance along the edge to the point where the yield line meets the free edge.

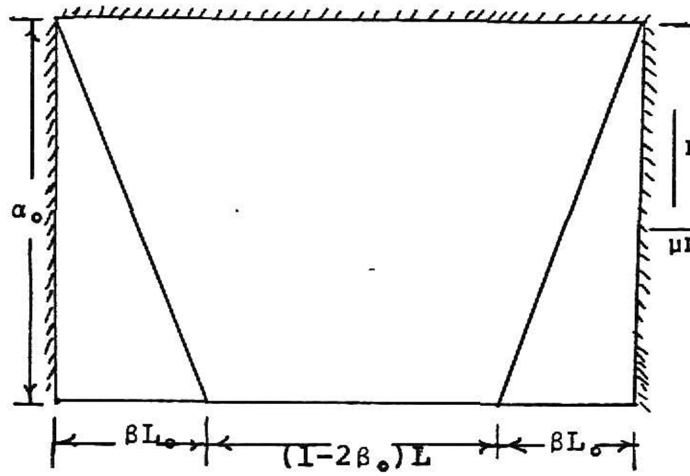


Fig. 24. Actual slab.

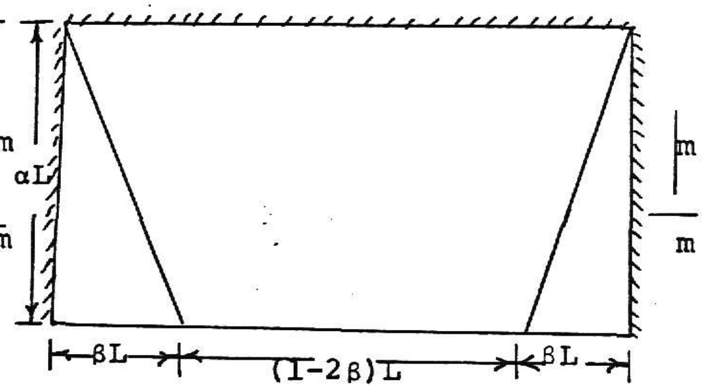


Fig. 25. Affine slab.

The length L, on the orthotropic slab, is arbitrarily placed in the m-reinforcement direction so that this is the same dimension on the affine slab. If $\alpha_o L$ is the length in the μm-reinforcement direction for the orthotropic slab, the affinity rules require $\alpha_o L = \sqrt{\mu}\alpha L$, so that the corresponding ratio of the sides is $\alpha_o = \alpha\sqrt{\mu}$. Also $\beta_o L = \beta L$, there being no change in that direction.

The solution for the orthotropic slab can therefore be written by substituting the value of α in equation (54), since the intensity of the distributed load does not change, thus

$$p = \frac{12\mu m}{\alpha_o^2 L^2}\left[\frac{\alpha_o^2/\mu + \beta_o^2}{3\beta_o - 2\beta_o^2}\right] \tag{58}$$

where
$$\beta_o L = \frac{\alpha_o^2 L}{\mu} \left[\left(\frac{4}{9} + \frac{\mu}{\alpha_o^2}\right)^{1/2} - \frac{2}{3} \right] \tag{59}$$

which is the same result as obtained by the work method.

<u>Example 2</u>: Skew slab simply supported on four sides, subjected to uniform load p/unit area (Fig. 26).

The skew slab will be transformed to an isotropic slab with $\alpha = \frac{\alpha_o}{\sqrt{\mu}}$, $\beta L = \beta_o L$ and $p' = p$. For the rectangular slab the solution is
$$m = \frac{1}{12}\, p\alpha^2 L^2 \left(\frac{3-4\beta}{2}\right) \tag{60}$$
where
$$\beta = \frac{1}{2}\, \alpha \left[\sqrt{(3+\alpha^2)} - \alpha \right] \tag{61}$$

The solution for the skew slab can therefore be written by inspection from equation (60) after substituting for α and β.

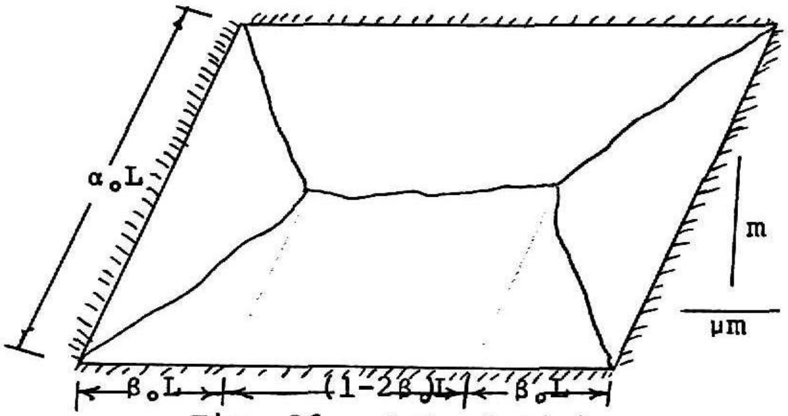


Fig. 26. Actual slab.

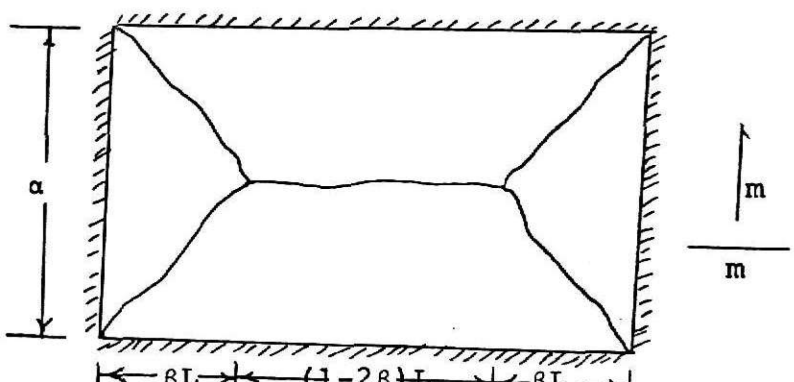


Fig. 27. Affine slab.

Thus

$$m = \frac{1}{12} p \frac{\alpha_o^2 L^2}{\mu} \quad (\frac{3-4\beta}{2}) \tag{62}$$

and

$$\beta_o = \frac{1}{2} \frac{\alpha_o}{\mu} \left[\sqrt{(3\mu + \alpha_o^2)} - \alpha_o \right] \tag{63}$$

These examples indicate that a "library" of solutions, collected for isotropic slabs, would have an extensive application.

POINT LOADS AND YIELD LINES FORMING CIRCULAR FANS

In the previous sections only straight yield line patterns were studied. However, there are certain circumstances in which the knowledge of what will be termed as fan mechanisms could be very useful.

In the corner of a slab, for example, it is quite common to see a set of yield lines radiating from a common focus and forming an elaborate cut-off, whereby the simple corner "lever" is replaced by a set of multiple triangles forming a "fan" as shown in Fig. 28.

As far as the designer is concerned, there is no compulsion to consider the effect of circular fans if there is only a distributed load. He will merely obtain slightly more accurate results by doing so. However, when heavy concentrated loads are present, fans centered on the point loads frequently form, and the failure load may be significantly reduced.

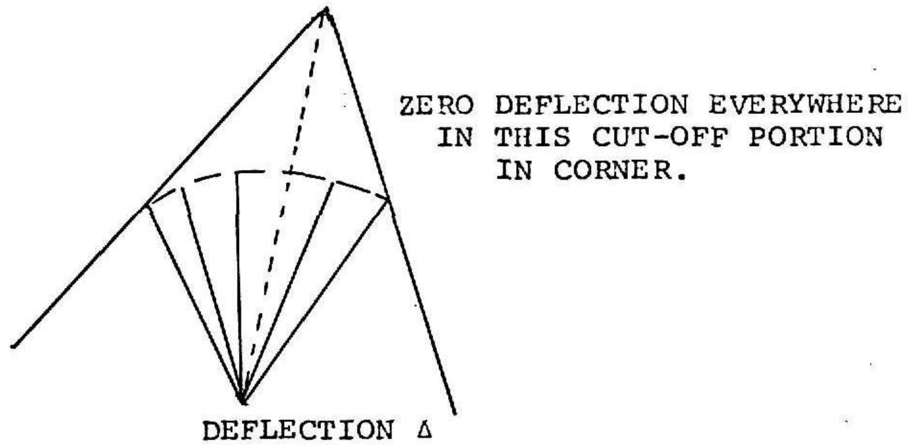


Fig. 28. Formation of a fan of yield lines.

DISSIPATION OF ENERGY IN CIRCULAR FANS OF YIELD LINES

To find the dissipation of energy in a fan of radius R, total angle in plane ϕ, and polar deflection Δ, consider any small component triangle of height R, and arc length R $d\phi$ as shown in Fig. 29. Using the directions r and ϕ, as in polar coordinates, then dissipation of energy by the vector method is given by

$$E_D = \sum (m_r \ell_r \theta_r + m_\phi \ell_\phi \theta_\phi) \tag{62}$$

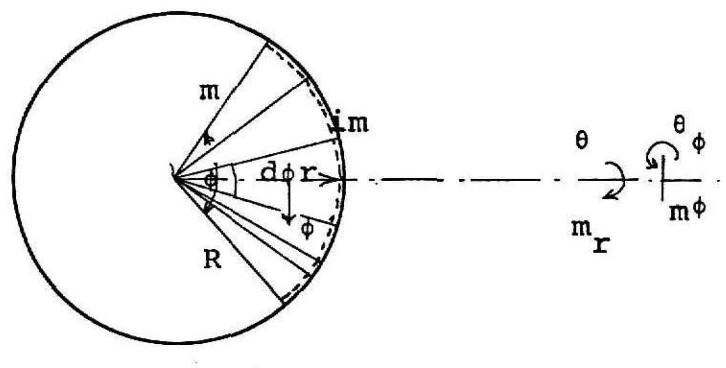


Fig. 29. Circular fan.

Axes r and ϕ are always at right angles (locally). The fact that there is zero displacement all around the boundary of the fan makes the rotation of the rigid element about the radius, θ_r, equal to zero. This leaves only $m\phi$, ℓ_ϕ, θ_ϕ to discuss. θ_ϕ is the rotation about the ϕ axis, so that $\theta_\phi = \frac{\Delta}{R}$. The vector quantity $m_\phi \ell_\phi$ is split into two parts. First there is energy dissipated by the negative yield line on the boundary for which m_ϕ denotes the circumferential moment key line -im and ℓ_ϕ is equal to $Rd\phi$. Secondly the two positive radial lines bounding the element together add up to the same vector length $Rd\phi$. The combined dissipation of energy is $m(1+i)\ Rd\phi \cdot \Delta/R$. Thus the total dissipation of energy is $E_D = \sum_{o}^{\phi} m(1+i)\Delta d\phi = m(1+i)\Delta\phi$ and for complete circular fan $\phi=2\pi$. Hence,

$$E_D = 2\pi m(1+i)\Delta \tag{65}$$

CONDITIONS UNDER WHICH EXTENSIVE FANS WILL DEVELOP

While no hard and fast rule can be given for the development of fan mechanisms, there are some definite tendencies which can be observed from a survey of many examples. The conditions likely to promote a fan mechanism can be summarized as follows:

(1) Heavy concentrated loads.

(2) Absence of top-reinforcement in corners of slab.

(3) Restraining moments on edges of slab.

(4) Acute angle corners.

(5) Free edges especially opposite corners.

Now some typical examples involving circular fans will be solved.

Example 1: Circular slab with distributed load p/unit area and with point load P at the center.

Consider the slab shown in Fig. 30 where the negative yield line has a radius R. The point under load P has a deflection Δ. The expenditure of energy by the external loads is

$$E = P \cdot \Delta + \frac{\pi R^2}{3} \cdot p \cdot \Delta \qquad (66)$$

The dissipation of energy is given by

$$ED = 2\pi m(1+i)m$$

Equating the dissipation of energy to its expenditure

$$m = \frac{P + \frac{1}{3}\pi R^2 p}{2\pi(1+i)} \qquad (67)$$

If there is no point load

$$m = \frac{pR^2}{2\pi(1+i)} \qquad (68)$$

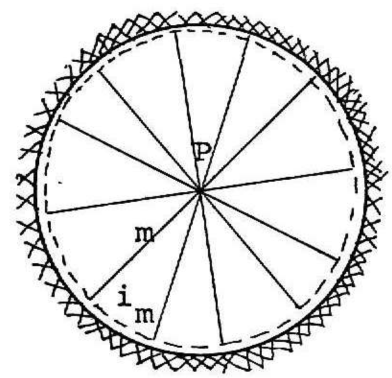


Fig. 30. Clamped circular slab.

If the distributed load is negligible compared to the point load, then

$$m = \frac{P}{2\pi(1+i)} \qquad (69)$$

In equation (69) all reference to the radius R has disappeared which means that with only a point load a complete fan of any radius could form with no change of collapse load, and also that $p = 2\pi m(1+i)$ is the collapse load for a clamped slab of any shape carrying only a point load P. The only necessary condition is that the boundary must be capable of a restraining moment of $-im$ at all points.

Example 2: Clamped square slab subjected to a uniform load.

If a complete inscribed circular fan, of radius $R = \frac{L}{2}$, is considered for distributed load then from equation (68), we get $m = \frac{pL^2}{48}$ when $i = 1$, but this complete circular fan gives the same solution as does the simple diagonal collapse mode (solution given in Ref. 5). A more critical mode occurs when fans of only limited extend form in the corners, as in the simple arrangement shown in Fig. 31. We have $R = \frac{L}{2}\,\mathrm{Sec}(\pi/4 - \phi/2)$, and the expenditure of energy by the external loads over the fans is equal to $\frac{1}{2} R^2 \cdot \phi \cdot \frac{p}{3}$, then the work equation is

$$4[2m\phi + 4m \cot(\pi/4 + \phi/2)] =$$

$$\frac{4P}{3} [\frac{L}{2} \cdot \frac{L}{2} \tan(\pi/4-\phi/2) + \frac{\phi}{2} \frac{L^2}{4} \mathrm{Sec}^2(\pi/4-\phi/2)]$$

from which

$$m = \frac{\frac{pL^2}{48} (\frac{\phi}{2}) \,\mathrm{Sec}^2(\pi/4-\phi/2) + \tan(\pi/4-\phi/2)}{\phi/2 + \cot(\pi/4 + \phi/2)} \qquad (70)$$

when $\phi = 30°$, we obtain

$$m = \frac{pL^2}{43.5} \qquad (71)$$

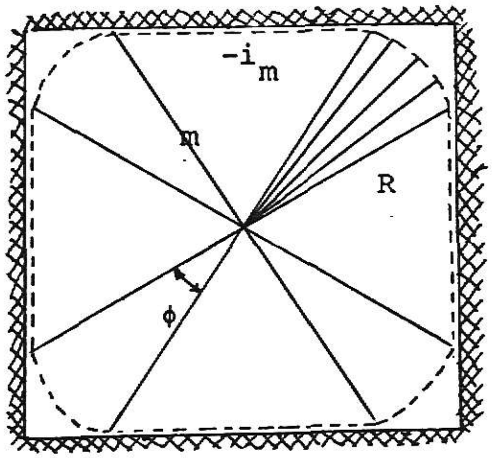


Fig. 31. Clamped square slab with unidorm load.

Example 3: Point load at the corner of a balcony.

This problem is solved by postulating a fan of limited extent shown in Fig. 32, with rigid regions each side of it bounded by yield lines. We have work equation as follows.

$$2i\ m\ \tan\phi + m(1+i)(\pi/2 - 2\phi) = P \tag{72}$$

and

$$\frac{dP}{d\phi} = 2i\ m\ \mathrm{Sec}^2\phi - 2m(1+i) = 0 \tag{73}$$

which gives

$$\tan\phi = \sqrt{i}$$

and

$$P = 2m\ \ i + m(1+i)(\pi/2 - 2\tan^{-1}\sqrt{\tfrac{1}{i}}) \qquad (74$$

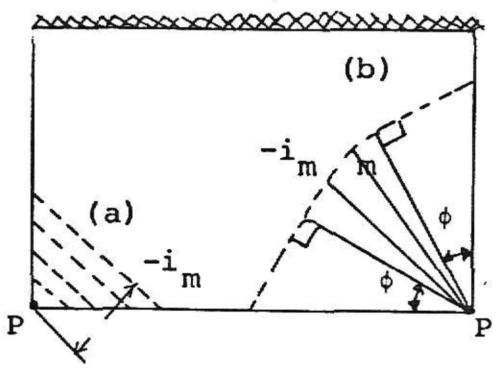


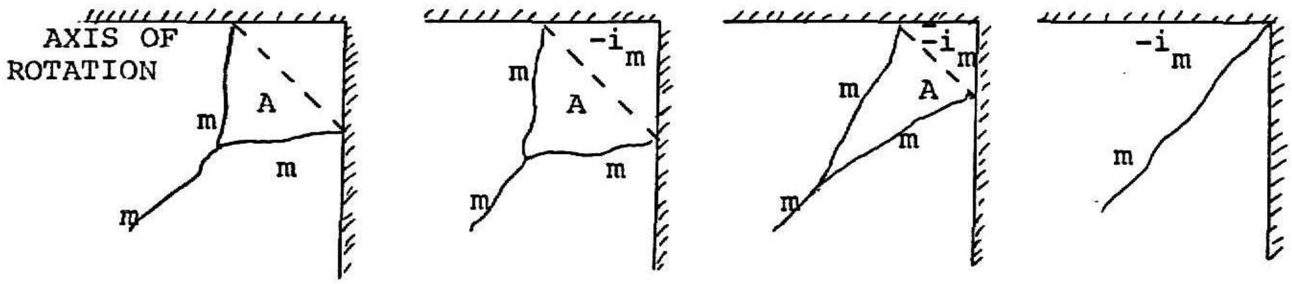
Fig. 32. Balcony with a point load at the corner.

when i = 1, indicating top reinforcement at full intensity covering the whole area locally, the fan disappears and is replaced by the single yield line system of mode a, which gives $P = 2im\frac{Z\cdot 1}{Z} = 2im$.

CORNER EFFECTS

As we have seen in the last section, positive yield lines entering corners tend to fork before reaching the corner producing fans and reducing the ultimate load on the slab.

The small slab segments marked A are known as corner levers and the factors which decide whether they form or not are the amount of top steel reinforcement provided at the corners and the degree of restraint at the corner. For example, at one corner of an isotropic square slab one of the three patterns may exist, as shown in Fig. 33.



Corners not held down (a) Corners held down: i = 0 (b) Corners held down: i = 1 (c) Corners held down: i = 1 (d)

Fig. 33. Corner levers.

If the corner is not held down, when load is applied it will rise off its supports, rotating about an axis as shown in Fig. 33a. At failure this must be accompanied, for geometrical reasons, by the formation of positive yield lines making a Y pattern. If the corner is held down the amount of top steel present will control the yield line pattern. If there is no top reinforcement the pattern will be similar to that for the case when corners are free to rise, but a negative yield line of zero strength will form across the corner, as shown in Fig. 33b. If there is some top reinforcement but not sufficient to give a yield line moment equal to positive yield line moment, then a longer, narrower corner lever will form, as shown in Fig. 33c. When top steel is such that a negative yield line moment is equal to positive yield moment then no corner lever will form, and a positive yield line will reach the corner, as shown in Fig. 33d.

EFFECT OF CORNER LEVERS ON COLLAPSE LOAD

Consider the square slab shown in Fig. 34, in which the axes of the main diagonals intersect the corners at 45°. A solution for the collapse load on the slab may be found by virtual work or by the equilibrium method. Full solutions are given by Johansen (1). The slab is assumed to be simply supported, with the corners held down. Top steel is provided in the corner only, to give a yield moment of im on the negative yield lines forming across the corner. The result of the analysis is tabulated in table 1 for various values of i. It can be seen that the lowest value of the ratio pL^2/m is 22 when no top reinforcement is provided, compared with 24 when no corner lever is formed. The values in the table also indicate that quite a small amount of top steel in the corner reduces the loss of ultimate load carrying capacity considerably.

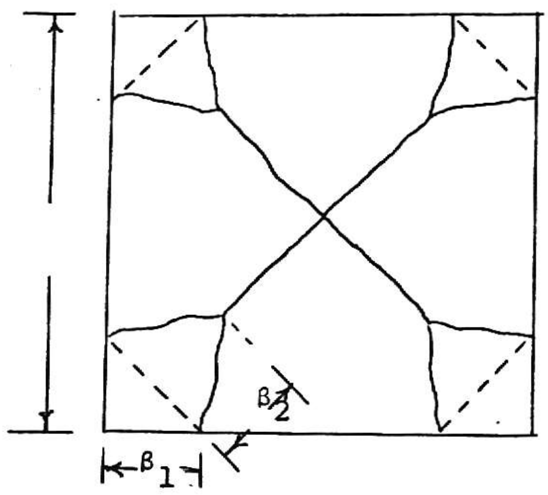


Fig. 34. Corner levers.

The data of Table 1 shows that, in practice, all that is necessary in the case of slabs with right angle corners without any top reinforcement is to reduce the sollapse load by 9 per cent compared with that calculated for full corner restraint.

Table 1. Values of pL^2/m
for square slabs with corner effect.

i	β_1/L	β_2/L	pL^2/m
0	0.159	0.52	22.0
0.25	0.11	0.57	23.0
0.50	0.069	0.62	23.6
1.00	0	0.707	24.0

A more dangerous situation exists in the case of acute angle corners. Fans are formed in case of such slabs and the slab should be analyzed for those fans.

CONCLUSIONS

Johansen's stepped yield criterion is considered adequate for evaluating the normal moments on yield lines. Vector component technique involves less work than the arithmetic technique for calculating the internal dissipation of work.

The equilibrium method is the work method presented in another arithmetic form, and the choice of which method should be used depends on the experience of the designer. The work method gives quick results in some typical slab boundary conditions whereas the equilibrium method may give quick results with some other more difficult boundary conditions.

There is no necessity for considering the fan mechanism when the slab is carrying a distributed load only. However, when heavy concentrated loads are involved, it is important to analyze the slab considering fan mechanisms.

For analysis of skew or orthotropic slabs, a library of solutions collected for isotropic slabs is very useful. Although yield line patterns with corner levers give lower values of collapse loads, they are often neglected in the analysis of slabs with right angle corners. However, as discussed in this report, the collapse load thus found is reduced by nine per cent.

APPENDIX

NUMERICAL SOLUTIONS

Find the ultimate moment for a triangular slab 20 feet by 10 feet, as shown in Fig. 36, supporting a uniform load of 250 p.s.f.

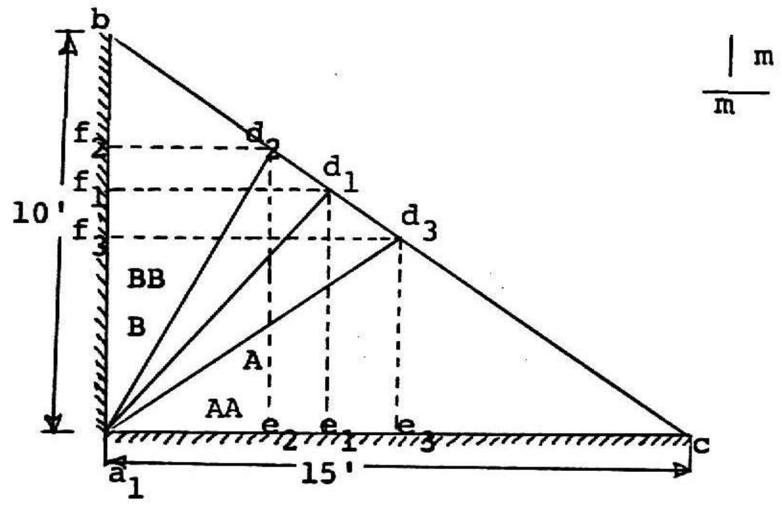


Fig. 35. Triangular slab with uniform loading.

Generally a trial layout will never be very far from the critical layout if the yield lines running into supported corners are such that they tend to bisect the corner angle, and this is the basis of the choice of layout shown in Fig. 34.

Point d is given a unit deflection and since ad is inclined at 45°, $a_1e_1 = e_1d_1$, $\theta_{AA} = \frac{1}{a_1e_1}$ and $\theta_{BB} = \frac{1}{d_1f_1}$.

Hence dissipation of energy in regions A and B is given by

$$Ed = ma_1e_1 \times \frac{1}{d_1e_1} + m \times a_1f_1 \frac{1}{d_1f_1} = 2m$$

Now expenditure of energy by loads is given by

$$Ep = 250 \times 1/3 \times 15 \times 10 \times 1/2 = 6250 \text{ lb. ft.}$$

Equating Ed to Ep, we get

$$m = 3125 \text{ lb. ft.}$$

The alternative layouts are now tried.

Trial 2: $a_1e_2 = 2.5'$ (The figure is drawn to scale and lengths are measured).

which give $Ed = m\left[\frac{4.5}{7} + \frac{7}{4.5}\right] = 2.2\ m.$

Trial 3: $a_1e_3 = 7.6'$

$d_Le_3 = 5'$ which gives

$$Ed = m\left[\frac{7.6}{5} + \frac{5}{7.6}\right] = m(1.52 + .67) = 2.19m$$

Each of these trials gives the value of m which is less than the first trial. Hence the first pattern is close to the maximum value.

Now the same problem will be solved by the equilibrium method. A numerical technique can also be used with the equilibrium method. Strictly speaking, the nodal force values given by equation (32) are valid only for the stationary maximum position for the parameters. If a trial layout is drawn which is not that corresponding to the stationary maximum position, then we may not use these nodal force values. If, however, we use the values and obtain equilibrium equations for each rigid region, we will obtain numerical values for m for each rigid region. The values of m for each region will be different, if the layout is not a stationary maximum layout. By observing the various m values, it is possible to see which way the layout should be changed in order to make the m values the same for each region. The numerical process is now described.

Now let us first consider trial 2 in which the value of ψ=90° with nodal forces being zero.

$$m_1 \times 4.5 = 250 \times 10 \times \frac{4.5}{2} \times \frac{4.5}{3}$$

or $$m_1 = 1840 \text{ lb. ft.}$$

$$m_2 \times 7 = 15 \times \frac{7}{2} \times 250 \times \frac{7}{3}$$

or $$m_2 = 4360 \text{ lb. ft.}$$

Considering trial 1, $k\psi = m \cot\psi$; hence

$$6m_1 - m_1 \cot\psi \times 6 = \frac{10 \times 6}{2} \times \frac{6}{3} \times 250$$

$$m_1 - m_1 \cot\psi = 2500$$

$$\psi = 78°, \ (1 - .212)m_1 = 2500$$

$$m_1 = \frac{2500}{.788} = 3100 \text{ ft. lbs.}$$

$$m_2 \times 6 + m_2 \cot 78° \times 6 = \frac{15 \times 6}{2} \times \frac{6}{3} \times 250$$

$$1{,}212\ m_2 = 3750$$

$$m_2 = 3100 \text{ ft. lbs.} \simeq 3125 \text{ ft. lbs.}$$

which is the value we obtained in the case of the work method.

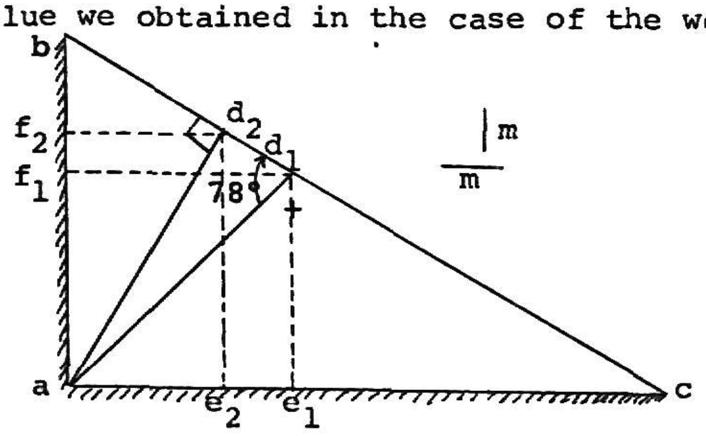


Fig. 36. Triangular slab with uniform loading.

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THE YIELD LINE ANALYSIS OF CONCRETE SLABS

by

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ABSTRACT

The yield line analysis for the prediction of ultimate flexural strength of a reinforced concrete slab is presented. The analysis is based on the formation of a yield line pattern, the location of which depends on loading and boundary conditions. The yield line theory is simplified, by making some assumptions, to analyze even complex slabs with limited mathematical effort. The results obtained by yield line analysis are in good agreement with the experimental results.

For the behavior of a slab at ultimate load it has been assumed that the collapse load could be arrived at by considering the bending action only. It has also been assumed that the elastic deformations are negligible in comparison with the plastic ones and that the slab will not fail until a valid mechanism is formed. The general crack pattern may be deduced logically from the position of the load and boundary configuration of the slab. Once the general crack pattern is known, collapse load may be calculated by the virtual work method or the equilibrium method.

A skew or orthotropic slab should be transformed into an isotropic slab. A "library" of solutions, collected for isotropic slabs, would be of much help.

There is no need for considering the fan mechanism when the slab is carrying a distributed load only. However, when heavy concentrated loads are involved, it is important to analyze the slab considering fan mechanism.

Although yield line patterns with corner levers give lower value of collapse loads, they are generally neglected in the analysis of slabs with right angle corners and the collapse load thus found is reduced by nine per cent. In the case of acute angle corners, the slab should be analyzed for fan mechanisms.