BUCKLING LOADS OF TENSIONED CIRCULAR PLATES SUBJECT TO CONCENTRATED IN-PLANE LOADING

by 4589

ROBERT PHILLIP DUBOIS

B. S., Kansas State University, 1969

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1970

Approved by:

F. C. Appl
Major Professor

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NOMENCLATURE

R_a	radius of center hole in disk
R_b	clamping radius
R	outer radius
R_c	radius at which tensioning is done
ρ	density
E	elastic coefficient
h	thickness of disk
$T(r)$	temperature as a function of r
V	potential energy of disk
D	plate rigidity = $E\, h^3/(12(1-\nu^2))$
$W(r,\theta)$	lateral deflection of disk
ν	Poisson's ratio
r	radial coordinate
θ	angular coordinate
σ_r	radial stress
σ_θ	hoop stress
$\tau_{r\theta}$	shear stress
∇^2	Laplacean Operator
∇^4	Biharmonic Operator
P	mignitude of in-plane load
$U(r,\theta)$	radial displacement function
$V(r,\theta)$	tangential displacement function
ϕ	stress function

p	equivalent compressive loading used in stress due to tensioning equation
α	thermal expansion coefficient
T_0, T_1, T_2	constants in $T(r)$
γ_1, γ_2	constants in $T(r)$
w_{ni}	eigenfunctions
c_{ni}	coefficients in series expansion of $W(r,\theta)$
R_{ni}	functions of the variable r
$J_n(x)$	Bessel function of the first kind of order n
$Y_n(x)$	Bessel function of the second kind of order n
$I_n(x)$	Modified Bessel function of the first kind of order n
$K_n(x)$	Modified Bessel function of the second kind of order n
B_{ni}	constant in R_{ni}
C_{ni}	constant in R_{ni}
D_{ni}	constant in R_{ni}
k_{ni}	constant in R_{ni}
i,j,k,m,n	integers
N, I	integers
S_{ni}	constants defined by equation (26)
$Q_{ni,mj}$	constants defined by equation (27)
$Q'_{ni,mj}$	constants defined by equation (37)
$Q''_{ni,mj}$	constants defined by equation (49)
$P\sigma'$	stress due to in-plane loading
σ''	stress not due to in-plane loading
ϵ_r	radial strain
ϵ_θ	hoop strain

$\gamma_{r\theta}$	shear strain
V	vector form of c_{ni}
SM	matrix form of $(k_{ni}R)^4 S_{ni}$
QM	matrix form of $Q'_{ni,mj}$
QTP	matrix form of PR $Q''_{ni,mj}/D$

INTRODUCTION

Many theoretical and experimental investigations have been made about the stability and vibration characteristics of a circular plate. Kirchhoff [10] in 1850 and Rayleigh [15] in 1877 considered the transverse vibration of an unconstrained thin circular disk. The more practical case of a thin circular disk clamped at the center was studied by Southwell [16] in 1921.

Bryan [2] in 1891 was one of the first to investigate the stability of circular plates but he and others who followed usually limited their investigations to constant thickness plates with clamped or simply supported boundaries and uniformly distributed compressive or shearing forces [20].

Other studies followed which included the effects of temperature gradients [4,8,9,13,14], of tensioning [5,6] and of concentrated in-plane loading [17] on the stability and transverse vibration characteristics of a thin circular disk. One important practical application of these studies is in the design of circular saw blades. However, a saw blade is affected simultaneously by several of the boundary conditions considered separately and the net effect of these boundary conditions on the stability and vibration characteristics of the disk are not linear combinations of the various boundary conditions. This report considers a combination of boundary conditions which more closely approximates the operating conditions of a circular saw blade. A constant thickness thin circular disk is clamped at a given radius; the disk is subjected to a concentrated in-plane radial loading; a radial temperature distribution is assumed which corresponds to experimentally measured temperature distributions in circular saw blades. The disk is also assumed to have been tensioned. A sketch of the disk is shown in Figure 1. Only the buckling loads were determined but the method of analysis could be easily extended to include calculation of the vibration characteristics.

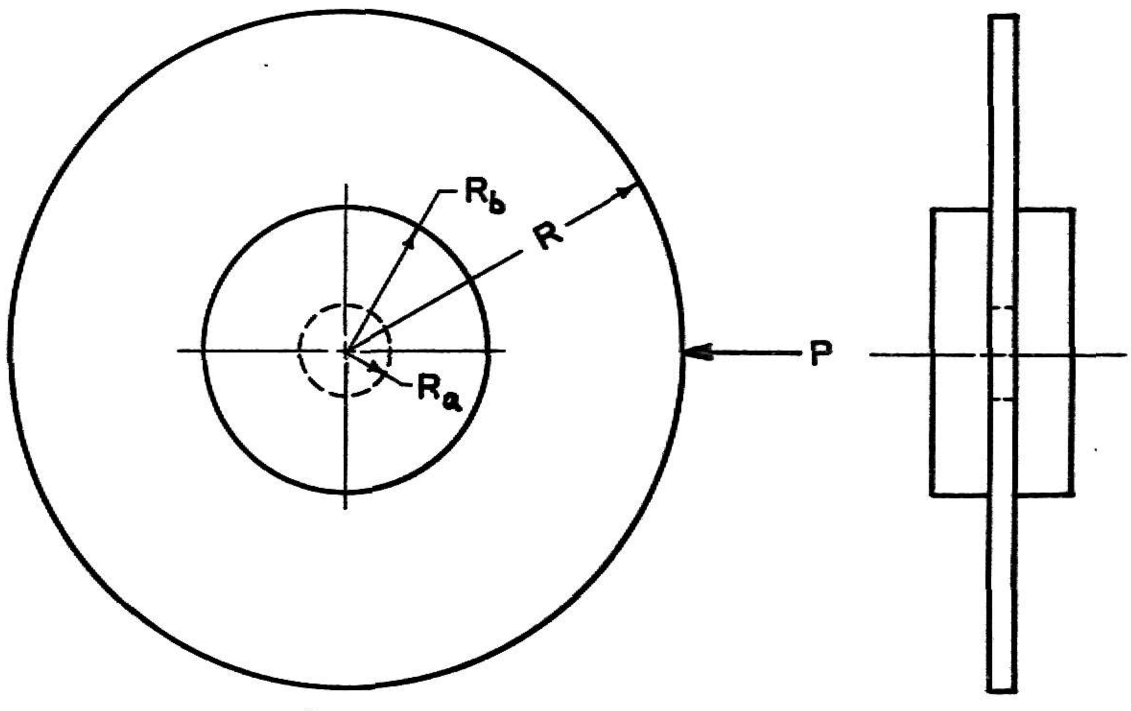

Figure 1

Disk clamped at the center with concentrated in-plane radial loading.

THEORETICAL ANALYSIS

It was assumed that the disk was made from a homogeneous isotropic linearly elastic material. The effects of rotary inertia and transverse shearing force were neglected and small deflections were assumed. These small deflections were also assumed to produce only higher order changes in the internal stress distributions. With these assumptions the potential energy of the disk becomes [18,19]

$$V = \frac{D}{2}\int_A [\nabla^2 W]^2 - 2(1-\nu)\left[\frac{\partial^2 W}{\partial r^2}\left(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}\right) - \left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial W}{\partial \theta}\right)\right)^2\right] dA$$

$$+ \frac{h}{2}\int_A \left[\sigma_r\left(\frac{\partial W}{\partial r}\right)^2 + \sigma_\theta\left(\frac{1}{r}\frac{\partial W}{\partial \theta}\right)^2 + 2\,\tau_{r\theta}\left(\frac{\partial W}{\partial r}\right)\left(\frac{1}{r}\frac{\partial W}{\partial \theta}\right)\right] dA \qquad (1)$$

$$+ \frac{h}{2E}\int_A [(\sigma_r + \sigma_\theta)^2 - 2(1-\nu)(\sigma_r\sigma_\theta - \tau_{r\theta}^2)]\, dA$$

The lateral deflection of the plate is $W = W(r,\theta)$ and σ_r, σ_θ, and $\tau_{r\theta}$ are the radial, hoop, and shear stresses acting in the plane of the plate, respectively. The stresses depend upon the temperature gradient, method of tensioning, and the in-plane loading. However, assuming that all quantities are fixed except for the in-plane load, the potential energy of the disk becomes a function of the loading and the lateral deflection. The loading which results in minimum potential energy of the disk is the desired buckling load [22].

The stress distribution in the disk is the sum of the stress distributions due to in-plane loading, to temperature gradients, and to tensioning plus any residual stresses resulting from the manufacturing process. These residual stresses were assumed to be zero.

To find the stress distribution due to in-plane loading a stress function in polar coordinates was assumed. If ϕ is a stress function, then $\nabla^4\phi$ must equal zero in order to satisfy the compatability equations. The resulting stresses would be

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \tag{2}$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} \tag{3}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) \tag{4}$$

The most general solution for a function in polar coordinates which satisfies the relationship $\nabla^4\phi = 0$ is

$$\begin{aligned}\phi = {} & a_o \ln(r) + b_o r^2 + c_o r^2 \ln(r) + d_o r^2\theta + a_o'\theta + b_o' r^2\theta \ln(r) \\ & + c_o'\theta \ln(r) - \frac{1}{2}c_1 r\theta \cos\theta + f_1 r\theta \ln(r) \cos\theta \\ & + f_1' r\theta \ln(r) \sin\theta + \frac{1}{2}a_1 r\theta \sin\theta + [\, b_1 r^3 + c_1' r^{-1} + \\ & b_1' r \ln(r)] \cos\theta + [d_1 r^3 + c_1' r^{-1} + d_1' r \ln(r)] \sin\theta + \\ & \sum_{n=2}^{\infty} [\, a_n r^n + b_n r^{n+2} + a_n' r^{-n} + b_n' r^{-n+2}] \cos n\theta \\ & \sum_{n=2}^{\infty} [\, c_n r^n + d_n r^{n+2} + c_n' r^{-n} + d_n' r^{-n+2}] \sin n\theta\end{aligned} \tag{5}$$

In order to be able to solve for the constants in the stress function the boundary conditions need to be given. The stress distribution at the edge of the disk is found by first considering the Fourier series expansion of a uniformly distributed loading over an arc of length 2ε and intensity such that the total load is P. Then letting $\varepsilon \to 0$ the resulting circumferential stress distribution due to the concentrated in-plane load becomes

$$\sigma_r\Big|_{r=R} = \frac{-P}{\pi Rh}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \cos n\theta\right] \tag{6}$$

if θ is measured from the radius along which the loading acts. Also at the outer edge the shear stress is zero, $\tau_{r\theta}(R,\theta) = 0$. If $U(r,\theta)$ is the radial displacement function and $V(r,\theta)$ is the tangential displacement function and the disk is clamped at a radius R_b, then $U(R_b,\theta) = V(R_b,\theta) = 0$. Using these boundary conditions the constants in the stress function can be found (See Appendix **A** for details). The resulting stress distribution is

$$\sigma_r = \frac{-P}{\pi Rh}\sum_{k=0}^{\infty}\left[a_k^1\left(\frac{r}{R}\right)^{k-2} + a_k^2\left(\frac{r}{R}\right)^{k} + a_k^3\left(\frac{r}{R}\right)^{-k-2} + a_k^4\left(\frac{r}{R}\right)^{-k}\right]\cos k\theta \tag{7}$$

$$\sigma_\theta = \frac{-P}{\pi Rh}\sum_{k=0}^{\infty}\left[b_k^1\left(\frac{r}{R}\right)^{k-2} + b_k^2\left(\frac{r}{R}\right)^{k} + b_k^3\left(\frac{r}{R}\right)^{-k-2} + b_k^4\left(\frac{r}{R}\right)^{-k}\right]\cos k\theta \tag{8}$$

$$\tau_{r\theta} = \frac{-P}{\pi Rh}\sum_{k=1}^{\infty}\left[c_k^1\left(\frac{r}{R}\right)^{k-2} + c_k^2\left(\frac{r}{R}\right)^{k} + c_k^3\left(\frac{r}{R}\right)^{-k-2} + c_k^4\left(\frac{r}{R}\right)^{-k}\right]\sin k\theta \tag{9}$$

where

$$a_0^1 = -b_0^1 = \rho^2(1/\gamma + \rho^2)/2$$

$$a_0^2 = b_0^2 = \gamma(1/\gamma + \rho^2)/2$$

$$a_0^3 = a_0^4 = b_0^3 = b_0^4 = 0$$

$$a_1^1 = 1$$

$$b_1^1 = c_1^1 = 0$$

$$a_1^2 = b_1^2/3 = c_1^2 = (1-\nu)(\gamma\rho^2+1)/(4(1+\beta\rho^4))$$

$$a_1^3 = -b_1^3 = c_1^3 = -\rho^2(1-\nu)(\gamma-\beta\rho^2)/(4(1+\beta\rho^4)$$

$$a_1^4 = b_1^4 = c_1^4 = -(1-\nu)/4$$

and for k> 1

$$a_k^1 = -b_k^1 = -c_k^1 = [k\beta\rho^{-2k} + (k^2-1+\beta^2)\rho^2 - k(k-1)]/d_k$$

$$k(k+2)a_k^2 = -k(k-2)b_k^2 = -(k^2-4)c_k^2 = -k(k^2-4)[\beta\rho^{-2k} + k + 1 - k\rho^{-2}]/d_k$$

$$a_k^3 = -b_k^3 = c_k^3 = [(k^2-1+\beta^2)\rho^2 - k(k+1) - k\beta\rho^{2k}]/d_k$$

$$k(k-2)a_k^4 = -k(k+2)b_k^4 = (k^2-4)c_k^4 = k(k^2-4)[k\rho^{-2} + \beta\rho^{2k} - k + 1]/d_k$$

$$d_k = 2[\beta(\rho^k + \rho^{-k})^2 + (k^2-1)(\rho-\rho^{-1})^2 + (\beta\rho-\rho^{-1})^2]$$

where $\rho = R_b/R$ $\qquad \beta=(3-\nu)/(1+\nu)$ $\qquad \gamma=(1+\nu)/(1-\nu)$

A common industrial practice is to tension saw blades in order to increase their stiffness. The tensioning process itself is not well understood but it does involve plastic deformation of the disk material at various locations. The stress distribution resulting from tensioning is highly dependent upon how the tensioning is done, the location of the plastic deformation and how much the material is plastically deformed. However, the stress distribution resulting from one particular tensioning process was measured experimentally and can be closely approximated by [12]

$$\left.\begin{aligned} \sigma_r &= \Phi_{3A} + \Phi_{4A} r^{-2} \\ \sigma_\theta &= \Phi_{3A} - \Phi_{4A} r^{-2} \end{aligned}\right\} \quad R_a < r < R_c$$

$$\left.\begin{aligned} \sigma_r &= \Phi_{3B} + \Phi_{4B} r^{-2} \\ \sigma_\theta &= \Phi_{3B} - \Phi_{4B} r^{-2} \end{aligned}\right\} \quad R_c < r < R \qquad (10)$$

$$\tau_{r\theta} = 0$$

This particular stress distribution was assumed for use in later calculations although other stress distributions would be equally applicable if the method of tensioning was changed. Two boundary conditions are $\sigma_r|_{r=R_a} = \sigma_r|_{r=R} = 0$ since the inner and outer edges of the disk are free. The third boundary condition is $\sigma_r|_{r=R_c} = -p$ where R_c is the radius at which tensioning is done and p is related to the amount of plastic deformation at the radius R_c. These boundary conditions are sufficient to solve for the constants in equation (10).

$$\Phi_{3A} = R_a^2\ R_c^2\ p/(R_c^2 - R_a^2) \qquad \Phi_{4A} = -p\ R_c^2/(R_c^2 - R_a^2)$$
$$\Phi_{3B} = R_c^2\ R^2\ p/(R_c^2 - R^2) \qquad \Phi_{4B} = -p\ R_c^2/(R_c^2 - R^2) \tag{11}$$

Temperature gradients within the circular disk also have an effect upon the stability of the disk due to the stresses induced by the temperature gradients. Assuming that temperature is a function of r alone, the radial and hoop stresses due to temperature for a constant thickness thin circular disk become [20]

$$\sigma_r = \frac{\alpha E}{r}\left[\frac{r^2 - R_a^2}{R^2 - R_a^2}\int_{R_a}^{R} r\ T(r)\ dr - \int_{R_a}^{r} r\ T(r)\ dr\ \right] \tag{12}$$

$$\sigma_\theta = \frac{\alpha E}{r}\left[\frac{r^2 + R_a^2}{R^2 - R^2}\int_{R_a}^{R} r\ T(r)\ dr - \int_{R_a}^{r} r\ T(r)\ dr\ \right] \tag{13}$$

Since temperature is a function of r only, $\tau_{r\theta} = 0$. For circular saws a temperature distribution which is a function of r alone is quite reasonable since the speed of the saw is so much greater than the rate of heat conductivity. The actual temperature distribution is a function of rotation speed, thermal conductivity, cooling fluid properties, and saw surface conditions but a convenient approximation to commonly encountered temperature distributions measured experimentally is

$$T(r) = T_0 + T_1(r/R)^{-\gamma 1} + T_2(r/R)^{\gamma 2} \tag{14}$$

With the temperature distribution given by equation (14) the radial and hoop stresses become

$$\begin{aligned}\sigma_r = \frac{\alpha E}{r^2}\Big[&\frac{r^2-R_a^2}{R^2-R_a^2}\Big\{\frac{T_1}{2-\gamma 1}\Big(R^2-R_a^2\Big(\frac{R}{R_a}\Big)^{\gamma 1}\Big) + \frac{T_2}{2+\gamma 2}\Big(R^2-R_a^2\Big(\frac{R_a}{R}\Big)^{\gamma 2}\Big)\Big\} \\ &+ \Big\{\frac{-T_1 R^{\gamma 1}}{2-\gamma 1}\left(r^{2-\gamma 1}-R_a^{2-\gamma 1}\right) - \frac{T_2 R^{-\gamma 2}}{2+\gamma 2}\left(r^{2+\gamma 2}-R_a^{2+\gamma 2}\right)\Big\}\Big]\end{aligned} \tag{15}$$

$$\begin{aligned}\sigma_\theta = \frac{\alpha E}{r^2}\Big[&\frac{r^2+R_a^2}{R^2-R_a^2}\Big\{\frac{T_1}{2-\gamma 1}\Big(R^2-R_a^2\Big(\frac{R}{R_a}\Big)^{\gamma 1}\Big) + \frac{T_2}{2+\gamma 2}\Big(R^2-R_a^2\Big(\frac{R_a}{R}\Big)^{\gamma 2}\Big)\Big\} \\ &- \frac{T_1 R^{\gamma 1}}{2-\gamma 1}\left(r^{2-\gamma 1}-R_a^{2-\gamma 1}\right) - \frac{T_2 R^{-\gamma 2}}{2+\gamma 2}\left(r^{2+\gamma 2}-R_a^{2+\gamma 2}\right) + T_0\left(R_a^2-r^2\right)\Big]\end{aligned} \tag{16}$$

Now that the stresses have been specified it is necessary to find $W(r,\theta)$.

The lateral deflection of the disk is assumed to be represented by the series

$$W(r,\theta) = \sum_{n=0}^{\infty}\sum_{i=0}^{\infty} c_{ni}\, w_{ni}(r,\theta) \tag{17}$$

where $w_{ni}(r,\theta) = R_{ni}(r)\cos n\theta$ (18)

and

$$R_{ni}(r) = J_n(k_{ni}r) + B_{ni}Y_n(k_{ni}r) + C_{ni}I_n(k_{ni}r) + D_{ni}K_n(k_{ni}r) \tag{19}$$

The $J_n(x)$, $Y_n(x)$, $I_n(x)$, and $K_n(x)$ are the Bessel functions of the first and second kind, standard and modified. This particular series representation was chosen because the w_{ni}'s satisfy the geometric boundary conditions of the disk,

$$w_{ni}(R_b,\theta) = \frac{\partial w_{ni}}{\partial r}\Big|_{r=R_b} = 0 \tag{20}$$

and the natural boundary conditions for the free boundary except for the point where the load is applied. The geometric boundary conditions must be satisfied by each term in the series representation if the Rayleigh-Ritz energy method is to be used.

The values of the k_{ni}'s, B_{ni}'s, C_{ni}'s, and D_{ni}'s for various clamping ratios, R_b/R, are given by St. Cyr [17]. These values can also be calculated by solving the following equations simultaneously.

$$\begin{aligned}
&\alpha\ [\ J_n(kR) - (1-\nu)\{\frac{n(n-1)}{k^2R^2} J_n(kR) + \frac{1}{kR} J_{n+1}(kR)\}] \\
&+\beta\ [\ Y_n(kR) - (1-\nu)\{\frac{n(n-1)}{k^2R^2} Y_n(kR) + \frac{1}{kR} Y_{n+1}(kR)\}] \\
&-\gamma\ [\ I_n(kR) + (1-\nu)\{\frac{n(n-1)}{k^2R^2} I_n(kR) - \frac{1}{kR} I_{n+1}(kR)\}] \\
&-\delta\ [\ K_n(kR) + (1-\nu)\{\frac{n(n-1)}{k^2R^2} K_n(kR) + \frac{1}{kR} K_{n+1}(kR)\}] = 0
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\alpha\ [\ nJ_n(kR) - kR\ J_{n+1}(kR) + \frac{(1-\nu)n^2}{k^2R^2}\{(n-1)J_n(kR) - kR\ J_{n+1}(kR)\}] \\
&+\beta\ [\ nY_n(kR) - kR\ Y_{n+1}(kR) + \frac{(1-\nu)n^2}{k^2R^2}\{(n-1)Y_n(kR) - kR\ Y_{n+1}(kR)\}] \\
&-\gamma\ [\ nI_n(kR) + kR\ I_{n+1}(kR) - \frac{(1-\nu)n^2}{k^2R^2}\{(n-1)I_n(kR) + kR\ I_{n+1}(kR)\}] \\
&-\delta\ [\ nK_n(kR) - kR\ K_{n+1}(kR) - \frac{(1-\nu)n^2}{k^2R^2}\{(n-1)K_n(kR) - kR\ K_{n+1}(kR)\}] = 0
\end{aligned}$$

$$\alpha\ J_n(kR_b) + \beta\ Y_n(kR_b) + \gamma\ I_n(kR_b) + \delta\ K_n(kR_b) = 0$$

$$\alpha \left[\frac{n}{kR_b} J_n(kR_b) - J_{n+1}(kR_b) \right] + \beta \left[\frac{n}{kR_b} Y_n(kR_b) - Y_{n+1}(kR_b) \right]$$
$$+ \gamma \left[\frac{n}{kR_b} I_n(kR_b) + I_{n+1}(kR_b) \right] + \delta \left[\frac{n}{kR_b} K_n(kR_b) - K_{n+1}(kR_b) \right] = 0 \qquad (21)$$

Since the system of equations is homogeneous the determinant of the coefficient matrix must be zero. From this condition the (kR) values can be determined. For any n, which corresponds to the number of nodal diameters, an infinite number of values of (kR) satisfy the requirement that the determinant of the coefficient matrix be zero due to the periodicity of the Bessel functions. The smallest (kR) value for a given n corresponds to the case of zero nodal circles. The next larger value corresponds to one nodal circle, etc. Let i be the number of nodal circles for the clamped disk subject to free vibration and k_{ni} be the k value for a given n and i. If the coefficient matrix has a determinant equal to zero, there are an infinite number of solutions for α, β, γ, and δ. Therefore α is set equal to 1. Then unique values for β, γ, and δ exist and $B_{ni} = \beta$, $C_{ni} = \gamma$, and $D_{ni} = \delta$ for a given n and i.

The linear equations result from the substitution of

$$W(r,\theta) = [\, \alpha\, J_n(kr) + \beta\, Y_n(kr) + \gamma\, I_n(kr) + \delta\, K_n(kr)\,] \cos n\theta \qquad (22)$$

which satisfies the differential equation of transverse displacement into the boundary conditions

$$\left.\begin{aligned} &\frac{\partial^2 W}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}\right) = 0 \\ &\frac{\partial}{\partial r}\left[\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}\right] + \frac{(1-\nu)}{r^2}\frac{\partial^2}{\partial \theta^2}\left[\frac{\partial W}{\partial r} - \frac{W}{r}\right] = 0 \end{aligned}\right\} r=R \qquad (23)$$

$$W(R_b,\theta) = \frac{\partial W}{\partial r} = 0 \qquad \} \quad r = R_b$$

The potential energy of the disk as given by equation (1), upon substitution of the deflection relations as given by equation (17), becomes a function of the c_{ni}'s and P, assuming the temperature distribution is given as well as the tensioning method. The orthogonality relationships of the w_{ni}'s as given by St. Cyr [17] are used to simplify the resulting potential energy equation. Necessary and sufficient conditions for the potential energy to have an extreme value are

$$\frac{\partial V}{\partial c_{ni}} = 0 \qquad n,i = 0,1,2,3,... \tag{24}$$

The following set of linear homogeneous equations for the c_{ni}'s and P results.

$$c_{ni}(k_{ni}R)^4 \, S_{ni} + \frac{PR}{D} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} c_{mj} Q_{ni,mj} = 0 \tag{25}$$

where

$$S_{ni} = \frac{1}{R^2} \int_A w_{ni}^2(r,\theta) \, dA \tag{26}$$

and $Q_{ni,mj} = Q_{mj,ni}$

$$= \frac{Rh}{P} \int_A \Big[\sigma_r \frac{\partial w_{mj}}{\partial r} \frac{\partial w_{ni}}{\partial r} + \sigma_\theta \left(\frac{1}{r} \frac{\partial w_{mj}}{\partial \theta}\right)\left(\frac{1}{r} \frac{\partial w_{ni}}{\partial \theta}\right)$$
$$+ \tau_{r\theta} \left(\frac{\partial w_{mj}}{\partial r} \frac{1}{r} \frac{\partial w_{ni}}{\partial \theta} + \frac{\partial w_{ni}}{\partial r} \frac{1}{r} \frac{\partial w_{mj}}{\partial \theta}\right) \Big] \, dA \tag{27}$$

An approximation to the deflection is obtained by terminating the infinite series expansion of $W(r,\theta)$ at n=N and i=I. Since the stresses are functions of P, r, and θ, it is advantageous to break the stresses up into two parts, one part a function of r and θ only and a second part which is a multiple of P and a function of r and θ. Say $\sigma = P\,\sigma' + \sigma''$ where σ' and σ'' are functions of r and θ only. Similarly break $Q_{ni,mj}$ up into two parts,

$$Q_{ni,mj} = Q'_{ni,mj} + P\,Q''_{ni,mj} \tag{28}$$

Equation (25) becomes

$$c_{ni}(k_{ni}R)^4\,S_{ni} + \frac{PR}{D}\sum_{m=0}^{N}\sum_{j=0}^{I} c_{mj}Q'_{ni,mj} + \frac{PR}{D}\sum_{m=0}^{N}\sum_{j=0}^{I} c_{mj}Q''_{ni,mj} = 0 \tag{29}$$

Letting

$$\begin{aligned}
&V(n*N+i+1) = c_{ni}\\
&SM(n*N+i+1,n*N+i+1) = (k_{ni}R)^4\,S_{ni}\\
&SM = \text{diagonal matrix}\\
&QM(n*N+i+1,m*N+j+1) = Q'_{ni,mj}\\
&QTP(n*N+i+1,m*N+j+1) = PR\,Q''_{ni,mj}/D
\end{aligned} \tag{30}$$

equation (29) can be written in matrix form as

$$[SM][V] + \frac{PR}{D}[QM][V] + [QTP][V] = [0]$$

or

$$-\,[QM]^{-1}[SM + QTP][V] = \frac{PR}{D}[V] \tag{31}$$

This is recognized as an eigenvalue problem with PR/D as an eigenvalue and [V] as an eigenvector. The smallest eigenvalue is the first buckling load and the corresponding eigenvector gives the coefficients in the series expansion of the deflection for the shape of the disk under this buckling load.

NUMERICAL CALCULATIONS

In order to find the lowest eigenvalue of the matrix equation given by equation (31) it is first necessary to calculate the elements of the SM, QM, and QTP matrices. The diagonal elements of the SM matrix are simply the S_{ni}'s given by equation (26) multiplied by $(k_{ni}R)^4$. Substitution of equation (18) into equation (26) gives

$$S_{ni} = \frac{1}{R^2}\int_A w_{ni}^2(r,\theta)\cdot dA = \frac{1}{R^2}\int_{R_b}^{R} R_{ni}^2(r)\ r \int_0^{2\pi} \cos^2 n\theta\ d\theta\ dr \tag{32}$$

$$\text{for } n\neq 0 \qquad \int_0^{2\pi} \cos^2 n\theta\ d\theta = \pi \tag{33}$$

$$\text{for } n=0 \qquad \int_0^{2\pi} \cos^2 n\theta\ d\theta = 2\pi \tag{34}$$

$$\text{Therefore } S_{0i} = \frac{2\pi}{R^2}\int_{R_b}^{R} R_{0i}^2(r)\ r\ dr \tag{35}$$

$$\text{and} \qquad S_{ni} = \frac{\pi}{R^2}\int_{R_b}^{R} R_{ni}^2(r)\ r\ dr \qquad \text{with } n\neq 0 \tag{36}$$

The remaining integration necessary in the calculation of S_{ni} was done numerically using Romberg Integration. It should be noted that the values of S_{ni} depend only upon R_b and R. Therefore, for a fixed R_b and R, the S_{ni} values are constant no matter what the temperature gradient, loading, and tensioning stresses might be.

To calculate the elements of the QM matrix, equation (27) is used with the substitution of $P\sigma'$ for σ. Thus

$$Q'_{ni,mj} = Q'_{mj,ni}$$

$$= \frac{Rh}{P} \int_A \Big[P\sigma'_r \frac{\partial w_{mj}}{\partial r} \frac{\partial w_{ni}}{\partial r} + \frac{P\sigma'_\theta}{r^2} \frac{\partial w_{mj}}{\partial \theta} \frac{\partial w_{ni}}{\partial \theta}$$

$$+ \frac{P\tau'_{r\theta}}{r} \Big(\frac{\partial w_{mj}}{\partial r} \frac{\partial w_{ni}}{\partial \theta} + \frac{\partial w_{mj}}{\partial \theta} \frac{\partial w_{ni}}{\partial r} \Big)\Big] \, dA \tag{37}$$

Letting

$$f_1(k,r) = [\, a_k^1 \, (\tfrac{r}{R})^{k-2} + a_k^2 \, (\tfrac{r}{R})^{k} + a_k^3 \, (\tfrac{r}{R})^{-k-2} + a_k^4 \, (\tfrac{r}{R})^{-k}] \tag{38}$$

$$f_2(k,r) = [\, b_k^1 \, (\tfrac{r}{R})^{k-2} + b_k^2 \, (\tfrac{r}{R})^{k} + b_k^3 \, (\tfrac{r}{R})^{-k-2} + b_k^4 \, (\tfrac{r}{R})^{-k}] \tag{39}$$

$$f_3(k,r) = [\, c_k^1 \, (\tfrac{r}{R})^{k-2} + c_k^2 \, (\tfrac{r}{R})^{k} + c_k^3 \, (\tfrac{r}{R})^{-k-2} + c_k^4 \, (\tfrac{r}{R})^{-k}] \tag{40}$$

the functions σ'_r, σ'_θ, and $\tau'_{r\theta}$ can be written

$$\sigma'_r = \frac{-1}{\pi Rh} \sum_{k=0}^{\infty} f_1(k,r) \cos k\theta \tag{41}$$

$$\sigma'_\theta = \frac{-1}{\pi Rh} \sum_{k=0}^{\infty} f_2(k,r) \cos k\theta \tag{42}$$

$$\tau'_{r\theta} = \frac{-1}{\pi Rh} \sum_{k=0}^{\infty} f_3(k,r) \sin k\theta \tag{43}$$

Therefore, using polar coordinates,

$$Q'_{ni,mj} = \frac{Rh}{P} \int_{R_b}^{R} \Big[\frac{-Pr}{\pi Rh} \frac{\partial R_{ni}}{\partial r} \frac{\partial R_{mj}}{\partial r} \sum_{k=0}^{\infty} f_1(k,r) \int_0^{2\pi} \cos n\theta \cos m\theta \cos k\theta \, d\theta$$

$$\frac{-Pmn}{r\pi Rh} R_{mj}(r) \, R_{ni}(r) \sum_{k=0}^{\infty} f_2(k,r) \int_0^{2\pi} \sin n\theta \sin m\theta \cos k\theta \, d\theta$$

$$+ \frac{nP}{\pi Rh} \frac{\partial R_{mj}}{\partial r} R_{ni}(r) \sum_{k=0}^{\infty} f_3(k,r) \int_0^{2\pi} \sin n\theta \cos m\theta \sin k\theta \, d\theta \tag{44}$$

$$+ \frac{mP}{\pi Rh} \frac{\partial R_{ni}}{\partial r} R_{mj}(r) \sum_{k=0}^{\infty} f_3(k,r) \int_0^{2\pi} \cos n\theta \sin m\theta \sin k\theta \, d\theta \Big] \, dr$$

But

$$\int_0^{2\pi} \cos n\theta \cos m\theta \cos k\theta \, d\theta = \frac{1}{4} \Big[\frac{\sin \theta(k+m+n)}{k+(m+n)} + \frac{\sin \theta(k+m-n)}{k+(m-n)}$$

$$+ \frac{\sin \theta(k-m+n)}{k-(m-n)} + \frac{\sin \theta(k-m-n)}{k-(m+n)} \Big]_{\theta=0}^{\theta=2\pi} \tag{45}$$

$$\begin{aligned} &= 2\pi && \text{if } k = m = n = 0 \\ &= \frac{\pi}{2} && \text{if } k = |m - n| \neq 0 \\ &= \pi && \text{if } k = |m - n| = 0,\ m \neq 0 \\ &= \frac{\pi}{2} && \text{if } k = m + n,\ m \neq 0 \neq n \\ &= 0 && \text{otherwise} \end{aligned}$$

Therefore, for any fixed m and n, there are at most two k values, say K_{11} and K_{12}, for which the integral of equation (45) is not equal to zero.

Also

$$\int_0^{2\pi} \sin n\theta \sin m\theta \cos k\theta \, d\theta = \frac{1}{4} \Big[\frac{\sin \theta(k+m-n)}{k+(m-n)} + \frac{\sin \theta(k-m+n)}{k-(m-n)}$$

$$- \frac{\sin \theta(k+m+n)}{K+(m+n)} - \frac{\sin \theta(k-m-n)}{k-(m+n)} \Big]_{\theta=0}^{\theta=2\pi} \tag{46}$$

$$\begin{aligned} &= \pi && \text{if } k = |m - n| = 0,\ m \neq 0 \\ &= \frac{\pi}{2} && \text{if } k = |m - n| \neq 0 \\ &= -\frac{\pi}{2} && \text{if } k = m + n,\ m \neq 0,\ n \neq 0 \\ &= 0 && \text{otherwise} \end{aligned}$$

For any fixed m and n, there are at most two k values, say K_{21} and K_{22}, for which the integral of equation (46) is not equal to zero.

And

$$\int_0^{2\pi} \cos m\theta \sin n\theta \sin k\theta \, d\theta = \frac{1}{4} \Big[\frac{\sin \theta(k+m-n)}{k+(m-n)} + \frac{\sin \theta(k-m-n)}{k-(m+n)}$$

$$- \frac{\sin \theta(k+m+n)}{k+(m+n)} - \frac{\sin \theta(k-m+n)}{k-(m-n)} \Big]_{\theta=0}^{\theta=2\pi} \tag{47}$$

$$\begin{aligned}
&= \pi \quad \text{if } k = n \neq 0,\ m = 0 \\
&= -\frac{\pi}{2} \text{ if } k = |m - n| \neq 0,\ m \neq 0,\ m>n \\
&= \frac{\pi}{2} \quad \text{if } k = |m - n| \neq 0,\ m \neq 0,\ n>m \\
&= \frac{\pi}{2} \quad \text{if } k = m + n,\ m \neq 0,\ n \neq 0 \\
&= 0 \quad \text{otherwise}
\end{aligned}$$

For any fixed m and n, there are at most two k values, say K_{31} and K_{32}, for which the integral of equation (47) is not equal to zero.

Also

$$\int_0^{2\pi} \sin k\theta \sin m\theta \cos n\theta \, d\theta = \frac{1}{4} \Big[\frac{\sin \theta(k+n-m)}{k+(n-m)} + \frac{\sin \theta(k-n-m)}{k-(n+m)}$$

$$- \frac{\sin \theta(k+n+m)}{k+(n+m)} - \frac{\sin \theta(k-n+m)}{k-(n-m)} \Big]_{\theta=0}^{\theta=2\pi}$$

$$\begin{aligned}
&= \pi \quad \text{if } k = m \neq 0,\ n = 0 \\
&= -\frac{\pi}{2} \text{ if } k = |m - n| \neq 0,\ n \neq 0,\ n \;\ m \\
&= \frac{\pi}{2} \quad \text{if } k = |m - n| \neq 0,\ n \neq 0,\ m \;\ n \\
&= \frac{\pi}{2} \quad \text{if } k = m + n,\ m \neq 0,\ n \neq 0 \\
&= 0 \quad \text{otherwise}
\end{aligned}$$

Therefore, for any fixed m and n, there are at most two values of k, say K_{41} and K_{42}, for which the integral of equation (48) is not equal to zero. Using the results of equations (45), (46), (47), and (48), each infinite sum reduces to the sum of at most two terms.

Thus $Q'_{ni,mj}$ can be found by numerically integrating equation (44) with respect to r. It should be noted that P, an unknown value, conveniently cancells out of the equation for $Q'_{ni,mj}$.

For the calculation of $P\,Q''_{ni,mj}$, equation (27) gives

$$P\,Q''_{ni,mj} = P\,Q''_{mj,ni}$$

$$= Rh \int_A \left[\sigma''_r \frac{\partial w_{mj}}{\partial r} \frac{\partial w_{ni}}{\partial r} + \frac{\sigma''_\theta}{r^2} \frac{\partial w_{mj}}{\partial \theta} \frac{\partial w_{ni}}{\partial \theta} + \frac{\tau''_{r\theta}}{r} \left(\frac{\partial w_{mj}}{\partial r} \frac{\partial w_{ni}}{\partial \theta} + \frac{\partial w_{mj}}{\partial \theta} \frac{\partial w_{ni}}{\partial r} \right) \right] dA \tag{49}$$

Now σ''_r is the sum of the radial stress due to temperature gradients and tensioning and is a function of r. And σ''_θ is the sum of the hoop stress due to temperature gradients and tensioning and is a function of r. For the temperature distribution and method of tensioning considered $\tau''_{r\theta} = 0$. Thus

$$P\,Q''_{ni,mj} = Rh \int_{R_b}^{R} \left[r\,\sigma''_r \frac{\partial R_{ni}}{\partial r} \frac{\partial R_{mj}}{\partial r} \int_0^{2\pi} \cos m\theta \cos n\theta \, d\theta + \frac{mn\sigma''_\theta}{r} R_{ni}(r)\, R_{mj}(r) \int_0^{2\pi} \sin m\theta \sin n\theta \, d\theta \right] dr \tag{50}$$

But

$$\int_0^{2\pi} \cos m\theta \cos n\theta \, d\theta \begin{cases} = 2\pi & \text{if } m = n = 0 \\ = \pi & \text{if } m = n \neq 0 \\ = 0 & \text{otherwise} \end{cases} \tag{51}$$

and

$$\int_0^{2\pi} \sin m\theta \sin n\theta \, d\theta \begin{cases} = \pi & \text{if } m = n \neq 0 \\ = 0 & \text{otherwise} \end{cases} \tag{52}$$

The equation for P $Q''_{ni,mj}$, equation (50), can now be integrated numerically with respect to r using the results of equations (51) and (52).

RESULTS AND CONCLUSIONS

Although the buckling load calculated in an upper bound to the true value due to the stiffening of the eigenvalue problem by termination of the infinite series expansion of $W(r,\theta)$ at finite values, the number of terms considered, $N = 4$ and $I = 4$, should make the difference between these values insignificant. Also, increasing N from 4 to 5 and I from 4 to 5 results in a 2.5 fold increase in the number of calculations required. The decrease in accuracy of the matrix calculations resulting from increasing N and I could also eliminate any increased accuracy in the approximation for $W(r,\theta)$ unless all elements of the matrices were calculated more accurately. Due to the numerous calculations involved in determining each element, the time involved for the calculation of each element would be increased significantly if more accuracy were required. For $N = 4$ and $I = 4$, calculation of the QM matrix alone required approximately 25 minutes on an IBM 360/50.

As a check upon the accuracy of the computer program (see Appendix B) used to calculate the elements of the SM, QM, and QTP matrices, two special cases were first considered. These cases were for a clamped circular disk with clamping ratio 0.1 and 0.3 and with no tensioning nor temperature gradients. These cases are identical to ones considered by St. Cyr [17]. The results calculated agree exactly for a clamping ratio of 0.1 and agree to within 1.4% for a 0.3 clamping ratio with those given by St. Cyr.

Due to time limitations the temperature gradient was neglected in all calculations although its effect upon internal stresses in the disk could easily be included in the computer program (see subprograms STRESR and STRHOP in Appendix B).

Four cases of tensioning were considered: two tensioning radii with two amounts of tensioning applied at each radius. From Table 1 it can be seen that tensioning at the smaller radius decreased the buckling load but by a rather small percent, 0.1% for p = 150 and 1.0% for p = 500. However, for tensioning at the larger radius the buckling load was increased by 6.8% for p = 150 and decreased by 29.6% for p = 500. From these results it is obvious that there is a p between zero and 500 such that the buckling load will be maximized if tensioning is done at the larger radius. Also, if the lesser amount of tensioning is used, there is a radius at which the tensioning should be applied in order to maximize the buckling load. The buckling load is seen to depend

R_b/R	PR/D	p (p.s.i.)	R_c/R
0.3	5.415	0	---
0.1	3.365	0	---
0.1	3.360	150	0.275
0.1	3.593	150	0.825
0.1	3.329	500	0.275
0.1	2.367	500	0.825

Table 1: Buckling loads of a clamped circular disk for several cases of tensioning.

upon both the amount of tensioning and its location. The determination of the proper combination of amount of tensioning and tensioning radius needed to maximize the buckling load will require further calculations.

It is hoped that the calculation of buckling loads for disks with different clamping ratios, different amounts of tensioning, different tensioning radii and temperature gradients can be accomplished in the near future with the purpose of finding methods of controlling the buckling load of a thin clamped circular disk. A method of predicting buckling loads and finding methods of maximizing buckling loads for circular disks with tensioning and temperature gradients should be quite useful in the design and manufacture of circular saw blades.

BIBLIOGRAPHY

[1] Berolzheimer, C. P., and C. H. Best
Thin Circular Saw Blades. Forest Products Journal, Vol. IX, No. 11, pp 404-412, 1959.

[2] Bryan, G. H.
On the Stability of a Plane under Thrusts in Its Own Plane, with Applications to the "Buckling" of the Sides of a Ship. Proc. London Math Soc., 22:54-67, 1891.

[3] Crandall, S. H.
Engineering Analysis. McGraw-Hill, New York, 1956.

[4] Cumming, J. D., and G. S. Gill
Temperature Gradients and Thermal Stresses in Circular Saws. Wood Science, Vol. 1, No. 2, Oct. 1968.

[5] Dugdale, D. S.
Theory of Circular Saw Tensioning. International Journal of Production Research, Vol. 4, No. 3, 1966.

[6] Dugdale, D. S.
Flexure of Thin Plates Containing Internal Stress. International Journal of Engineering Science, Vol. 6, pp 239-249,Pergamon Press, 1968.

[7] Filonenko and Borodich
Theory of Elasticity, Dover Publications, Inc., New York, 1965.

[8] Hsu, T. R.
Thermal Shock on a finite Disk Due to an Instantaneous Point Heat Source. Journal of Applied Mechanics, Paper No. 69-APM-3.

[9] Hsu, T. R.
Transient Thermal Stresses on a Finite Disk Due to a Continuous Point Heat Source. Journal of Engineering for Industry. Paper No. 69-WA/PVP-6.

[10] Kirchhoff, G.
Ueber die Schwingungen einer Elastischen Scheibe. Crelle's Journal, 40:51-88, 1850.

[11] Mote, C. D., Jr.
Effect of In-Plane Stresses on the Vibration Characteristics of Clamped-Free Discs. Unpublished Ph. D. dissertation, University of California, 1963.

BIBLIOGRAPHY (contd.)

[12] Mote, C. D., Jr.
Free Vibration of Initially Stressed Circular Disks. Journal of Engineering for Industry, Paper No. 64-WA/MD-4, 1964.

[13] Mote, C. D., Jr.
Unsymmetrical, Transient Heat Conduction: Rotating Disc Applications. Journal of Engineering for Industry, Paper No. 69-DE-B.

[14] Mote, C. D., Jr., and J. E. Rakowski
Thermally Induced Natural Frequency Variations in a Thin Disk. Experimental Mechanics, Jan., 1969.

[15] Lord Rayleigh
The Theory of Sound, Vol. I, Dover Publications, New York, 1945.

[16] Southwell, R. V.
On the Free Transverse Vibrations of a Uniform Circular Disc Clamped at Its Center; and on the Effects of Rotation. Proc. Royal Soc., London, 101:133-153, 1922.

[17] St. Cyr, W. W., II
Vibration and Stability of Circular Plates Subjected to Concentrated In-Plane Forces. Ph. D. dissertation, Kansas State University, 1965.

[18] Timoshenko, S. P., and J. M. Gere
Theory of Elastic Stability, McGraw-Hill, New York, 1961.

[19] Timoshenko, S. P., and S. Woinowsky-Krieger
Theory of Plates and Shells, McGraw-Hill, New York, 1959.

[20] Timoshenko, S. P., and J. N. Goodier
Theory of Elasticity, McGraw-Hill, New York, 1951.

[21] Vogel, S. M., and D. W. Skinner
Natural Frequencies of Transversely Vibrating Uniform Annular Plates. Journal of Applied Mechanics, Dec. 1965.

[22] Wang, C. T.
Applied Elasticity, McGraw-Hill, New York, 1953.

APPENDIX

APPENDIX A

Calculation of stresses due to in-plane loading.

From equations (2) and (5) the radial stress can be calculated.

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\begin{aligned}
&= a_0 r^{-2} + 2b_0 + c_0 + 2d_0\theta + 2c_0 \ln(r) + 2b_0'\theta \ln(r) + b_0'\theta \\
&\quad + c_0'\theta r^{-2} + f_1\theta r^{-1}\cos\theta + f_1'\theta r^{-1}\sin\theta - 2f_1 r^{-1}\ln(r)\sin\theta \\
&\quad + 2f_1' r^{-1}\ln(r)\cos\theta + [\, a_1 r^{-1} + 2b_1 r - 2a_1' r^{-3} + b_1' r^{-1}\,]\cos\theta \\
&\quad + [\, c_1 r^{-1} + 2d_1 r - 2c_1' r^{-3} + d_1' r^{-1}\,]\sin\theta \qquad (1A) \\
&\quad + \sum_{n=2}^{\infty} [n(1-n)a_n r^{n-2} + (n+2-n^2)b_n r^n - n(1+n)a_n' r^{-n-2} + (2-n-n^2)b_n' r^{-n}]\cos n\theta \\
&\quad + \sum_{n=2}^{\infty} [n(1-n)c_n r^{n-2} + (n+2-n^2)d_n r^n - n(1+n)c_n' r^{-n-2} + (2-n-n^2)d_n' r^{-n}]\sin n\theta
\end{aligned}$$

Since σ_r is an even function due to symmetry about the point of application of the in-plane load, then c_n, d_n, c_n', and d_n' are all equal to zero for $n = 1,2,3,\ldots$. Since σ_r is a single-valued function of θ, then

$$d_0 = b_0' = c_0' = f_1 = f_1' = 0.$$

Omitting terms which have already been shown to be equal to zero, σ_θ becomes

$$\begin{aligned}
\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} &= -a_o r^{-2} + 2b_0 + 3c_0 + 2c_0 \ln(r) \\
&\quad + [\, 6b_1 r + 2a_1' r^{-3} + b_1' r^{-1}\,]\cos\theta \qquad (2A) \\
&\quad + \sum_{n=2}^{\infty} [n(n-1)a_n r^{n-2} + (n+2)(n+1)b_n r^n + n(n+1)a_n' r^{-n-2} + (n-2)(n-1)b_n' r^{-n}]\cos\theta
\end{aligned}$$

and

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}$$

$$= a_0' r^{-2} + [\ 2b_1 r - 2a_1' r^{-3} + b_1' r^{-1}\ ] \sin\theta \tag{3A}$$

$$+ \sum_{n=2}^{\infty} [n(n-1)a_n r^{n-2} + n(n+1)b_n r^n - n(n+1)a_n' r^{-n-2} - n(n-1)b_n' r^{-n}] \sin n\theta$$

Since $\tau_{r\theta} = 0$ at $r = R$, then $a_0' = 0$. The radial displacement, $U(r,\theta)$, can be found by simply integrating the well known relationship between stress, strain, and displacement,

$$E\ \varepsilon_r = E\frac{\partial U}{\partial r} = \sigma_r - \nu\ \sigma_\theta \tag{4A}$$

to obtain

$$E\ U(r,\theta) = -(1+\nu)a_0 r^{-1} + 2(1-\nu)b_0 r + (1-3\nu)c_0 r + 2(1-\nu)(r\ \ln(r) - r)c_0$$

$$+ [\ \ln(r)\ a_1 + (1-3\nu)b_1 r^2 + (1+\nu)a_1' r^{-2} + (1-\nu)\ \ln(r)\ b_1'] \cos\theta$$

$$+ \sum_{n=2}^{\infty} [-n(1+\nu)a_n r^{n-1} + (2-2\nu-n-n\nu)b_n r^{n+1} + n(1+\nu)a_n' r^{-n-1} \tag{5A}$$

$$+ (2-2\nu+n+n\nu)b_n' r^{-n+1}\ ] \cos n\theta + F_1(\theta)$$

Similarly the tangential displacement $V(r,\theta)$ can be found by integrating the relationship between stress, strain, and displacement,

$$E\frac{\partial V}{\partial\theta} = E\ r\ \varepsilon_\theta - E\ U = r\ \sigma_\theta - r\ \nu\ \sigma_r - E\ U \tag{6A}$$

to obtain

$$E\ V(r,\theta) = 4r\theta c_0 + [\ -(\nu + \ln(r))a_1 + (5+\nu)b_1 r^2 + (1+\nu)a_1' r^{-2}$$

$$+ (1-\nu)(1 - \ln(r))b_1'\ ] \sin\theta$$

$$+ \sum_{n=2}^{\infty} [n(1+\nu)a_n r^{n-1} + (n+n\nu+4)b_n r^{n+1} + n(1+\nu)a_n' r^{-n-1} + \tag{7A}$$

$$(n-4+n\nu)b_n' r^{-n+1} \,]\, \sin n\theta \quad - \int F_1(\theta)\, d\theta \quad + F_2(r)$$

Since E $V(r,\theta)$ is a single valued function of θ, then $c_0 = 0$. And from the relationship

$$E\, \gamma_{r\theta} = 2(1+\nu)\, \tau_{r\theta} = \frac{E}{r}\frac{\partial U}{\partial \theta} + E\, \frac{\partial V}{\partial r} - \frac{EV}{r} \tag{8A}$$

the equation

$$2(1+\nu)b_1' + (1-\nu)a_1 + 2(1-\nu)b_1' = \int F_1(\theta)\, d\theta + \frac{dF_1(\theta)}{d\theta} + r\left[\frac{dF_2(r)}{dr} - \frac{F_2(r)}{r}\right] \tag{9A}$$

is obtained by substituting the results of equations (3A), (5A), and (7A) into equation (8A). Equation (9A) must hold for every value of θ and r. Therefore

$$\frac{dF_2(r)}{dr} - \frac{F_2(r)}{r} = 0 \tag{10A}$$

The solution of equation (10A) is

$$F_2(r) = H\, r \tag{11A}$$

The combination of equation (10A) and (9A) results in

$$\int F_1(\theta)\, d\theta + \frac{dF_1(\theta)}{d\theta} = 2(1+\nu)b_1' + (1-\nu)a_1 + 2(1-\nu)b_1' \tag{12A}$$

The solution of equation (12A) is

$$F_1(\theta) = K \cos\theta + L \sin\theta + [\,(1-\nu)a_1/2 + 2b_1'\,]\,\theta \sin\theta - [\,2b_1' + (1-\nu)a_1/2\,] \cos\theta \tag{13A}$$

Since $U(r,\theta)$ is a single-valued function of θ, then

$$(1-\nu)a_1/2 + 2b_1' = 0 \tag{14A}$$

Since $U(r,\theta)$ is an even function due to symmetry about the point of application of the in-plane load, $L = 0$. Substitution of equation (13A) into equation (7A) gives

$$\begin{aligned} E\,V(r,\theta) = {} & [\,-(\nu+\ln(r))a_1 + (5+\nu)b_1 r^2 + (1+\nu)a_1' r^{-2} + \\ & (1-\nu)(1-\ln(r))b_1'\,]\sin\theta \quad + \sum_{n=2}^{\infty} [\,n(1+\nu)a_n r^{n-1} + \\ & (n+n\nu+4)b_n r^{n+1} + n(1+\nu)a_n' r^{-n-1} + (n-4+n\nu)b_n' r^{-n+1}]\sin n\theta \\ & - K\sin\theta + H\,r \end{aligned} \tag{15A}$$

Since $V(R_b,\theta) = 0$ and $H\,R_b$ is independent of the other terms in equation (15A), then $H = 0$. The resulting equations for σ_r, σ_θ, $\tau_{r\theta}$, $E\,U(r,\theta)$ and $E\,V(r,\theta)$ then become

$$\begin{aligned} \sigma_r = {} & a_0 r^{-2} + 2b_0 + [a_1 r^{-1} + 2b_1 r - 2a_1' r^{-3} + b_1' r^{-1}\,]\cos\theta \\ & + \sum_{n=2}^{\infty} [n(1-n)a_n r^{n-2} + (n+2-n^2)b_n r^n - n(n+1)a_n' r^{-n-2} - (n-2+n^2)b_n' r^{-n}]\cos n\theta \end{aligned} \tag{16A}$$

$$\begin{aligned} \sigma_\theta = {} & -a_0 r^{-2} + 2b_0 + [\,6b_1 r + 2a_1' r^{-3} + b_1' r^{-1}\,]\cos\theta \\ & + \sum_{n=2}^{\infty} [n(n-1)a_n r^{n-2} + (n+2)(n+1)b_n r^n + n(n+1)a_n' r^{-n-2} + (n-2)(n-1)b_n' r^{-n}]\cos n\theta \end{aligned} \tag{17A}$$

$$\begin{aligned} \tau_{r\theta} = {} & [\,2b_1 r - 2a_1' r^{-3} + b_1' r^{-1}\,]\sin\theta \\ & + \sum_{n=2}^{\infty} [n(n-1)a_n r^{n-2} + n(n+1)b_n r^n - n(n+1)a_n' r^{-n-2} - n(n-1)b_n' r^{-n}\,]\sin n\theta \end{aligned} \tag{18A}$$

$$E\,U(r,\theta) = -(1+\nu)a_0 r^{-1} + 2(1-\nu)b_0 r + [\, K + a_1 \ln(r) + (1-3\nu)b_1 r^2 + (1+\nu)a_1' r^{-2} + (1-\nu)\ln(r)\, b_1' \,] \cos\theta + \sum_{n=2}^{\infty} [\, -n(1+\nu)a_n r^{n-1} + (2-2\nu-n-n\nu)b_n r^{n+1} + n(1+\nu)a_n' r^{-n-1} + (2-2\nu+n+n\nu)b_n' r^{-n+1}] \cos n\theta \tag{19A}$$

$$E\,V(r,\theta) = [\, -K - (\nu + \ln(r))a_1 + (5+\nu)b_1 r^2 + (1+\nu)a_1' r^{-2} + (1-\nu)(1-\ln(r))b_1' \,] \sin\theta + \sum_{n=2}^{\infty} [\, n(1+\nu)a_n r^{n-1} + (n+n\nu+4)b_n r^{n+1} + n(1+\nu)a_n' r^{-n-1} + (n+n\nu-4)b_n' r^{-n+1}] \sin n\theta \tag{20A}$$

From the boundary conditions

$$\sigma_r\Big|_{r=R} = \frac{-P}{\pi R h}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \cos n\theta\right] \tag{21A}$$

$$\tau_{r\theta}\Big|_{r=R} = 0 \tag{22A}$$

$$U(R_b,\theta) = 0 \tag{23A}$$

$$V(R_b,\theta) = 0 \tag{24A}$$

and the fact that $\sin m\theta$ is not a linear combination of $\sin n\theta$'s for $n \neq m$, the following sets of simultaneous linear equations result. The variable r has been normalized so that $R = 1$.

For $n = 0$

$$\begin{aligned} \frac{-P}{2\pi R h} = \frac{\lambda}{2} &= a_0 + 2b_0 \\ 0 &= -(1+\nu)a_0 + 2(1-\nu)b_0 \end{aligned} \tag{25A}$$

Therefore

$$a_0 = \lambda\rho^2/(2(\gamma+\rho^2)) \qquad\qquad b_0 = \lambda\gamma/(4(\gamma+\rho^2) \tag{26A}$$

where $\rho = R_b/R$ and $\gamma = (1+\nu)/(1-\nu)$

For n = 1

$$\frac{-P}{\pi Rh} = \lambda = a_1 + 2b_1 - 2a_1' + b_1'$$

$$0 = 2b_1 - 2a_1' + b_1' \qquad (27A)$$

$$0 = K + \ln(\rho)a_1 + (1-3\nu)\rho^2 b_1 + (1+\nu)\rho^{-2}a_1' + (1-\nu)\ln(\rho)b_1'$$

$$0 = -K - (\nu + \ln(\rho))a_1 + (5+\nu)\rho^2 b_1 + (1+\nu)\rho^{-2}a_1' + (1-\nu)(1-\ln(\rho))b_1'$$

$$0 = (1-\nu)a_1 + 4b_1' \qquad (14A)$$

Therefore

$$\begin{aligned} a_1 &= \lambda \\ b_1 &= \lambda(1-\nu)(\gamma\rho^2+1)/(8(\beta\rho^4+1)) \\ a_1' &= \lambda\rho^2(1-\nu)(\gamma-\beta\rho^2)/(8(1+\beta\rho^4)) \\ b_1' &= -\lambda(1-\nu)/4 \end{aligned} \qquad (28A)$$

where $\rho = R_b/R$ $\gamma = (1+\nu)/(1-\nu)$ $\beta = (3-\nu)/(1+\nu)$

For n>1

$$\begin{matrix} n(1-n) & -(n+1)(n-2) & -n(n+1) & -(n-1)(n+2) \\ n(n-1) & n(n+1) & -n(n+1) & -n(n-1) \\ -n(1+\nu)\rho^{n-1} & (2-2\nu-n-n\nu)\rho^{n+1} & n(1+\nu)\rho^{-n-1} & (2-2\nu+n+n\nu)\rho^{-n+1} \\ n(1+\nu)\rho^{n-1} & (n+n\nu+4)\rho^{n+1} & n(1+\nu)\rho^{-n-1} & (n+n\nu-4)\rho^{-n+1} \end{matrix} \quad \begin{matrix} a_n \\ b_n \\ a_n' \\ b_n' \end{matrix} = \begin{matrix} \lambda \\ 0 \\ 0 \\ 0 \end{matrix} \qquad (29A)$$

This set of linear equations can most easily be solved using Kramer's Rule to get

ρ^2

$$a_n = \frac{-\lambda[n\beta\rho^{-2n} + (n^2-1+\beta^2) - n(n-1)]}{2n(n-1)[\beta(\rho^n+\rho^{-n})^2 + (n^2-1)(\rho-\rho^{-1})^2 + (\beta\rho-\rho^{-1})^2]}$$

$$(30A)$$

$$b_n = \frac{\lambda[\beta\rho^{-2n} + (n+1) - n\rho^{-2}]}{2(n+1)[\beta(\rho^n+\rho^{-n})^2 + (n^2-1)(\rho-\rho^{-1})^2 + (\beta\rho-\rho^{-1})^2]}$$

$$a_n' = \frac{-\lambda[(n^2-1+\beta^2)\rho^2 - n(n+1) - n\beta\rho^{2n}]}{2n(n+1)[\beta(\rho^n+\rho^{-n})^2 + (n^2-1)(\rho-\rho^{-1})^2 + (\beta\rho-\rho^{-1})^2]}$$

$$b_n' = \frac{-\lambda[n\rho^{-2} + \beta\rho^{2n} - (n-1)]}{2(n-1)[\beta(\rho^n+\rho^{-n})^2 + (n^2-1)(\rho-\rho^{-1})^2 + (\beta\rho-\rho^{-1})^2]} \qquad (30A)$$

where $\lambda = -P/(\pi Rh)$, $\gamma = (1+\nu)/(1-\nu)$, $\beta = (3-\nu)/(1+\nu)$, $\rho = R_b/R$.

After substitution of equations (26A), (28A), and (30A) into equations (16A), (17A), and (18A), then equation (31A) results

$$\sigma_r = \frac{-P}{\pi Rh}\sum_{k=0}^{\infty}\left[a_k^1\left(\frac{r}{R}\right)^{k-2} + a_k^2\left(\frac{r}{R}\right)^{k} + a_k^3\left(\frac{r}{R}\right)^{-k-2} + a_k^4\left(\frac{r}{R}\right)^{-k}\right]\cos k\theta$$

$$\sigma_\theta = \frac{-P}{\pi Rh}\sum_{k=0}^{\infty}\left[b_k^1\left(\frac{r}{R}\right)^{k-2} + b_k^2\left(\frac{r}{R}\right)^{k} + b_k^3\left(\frac{r}{R}\right)^{-k-2} + b_k^4\left(\frac{r}{R}\right)^{-k}\right]\cos k\theta \qquad (31A)$$

$$\sigma_{r\theta} = \frac{-P}{\pi Rh}\sum_{k=1}^{\infty}\left[c_k^1\left(\frac{r}{R}\right)^{k-2} + c_k^2\left(\frac{r}{R}\right)^{k} + c_k^3\left(\frac{r}{R}\right)^{-k-2} + c_k^4\left(\frac{r}{R}\right)^{-k}\right]\sin k\theta$$

where

$$a_0^1 = -b_0^1 = \rho^2/(2(\gamma+\rho^2))$$
$$a_0^2 = b_0^2 = \gamma/(2(\gamma+\rho^2)) \qquad (32A)$$
$$a_0^3 = a_0^4 = b_0^3 = b_0^4 = 0$$
$$a_1^1 = 1 \qquad b_1^1 = c_1^1 = 0$$
$$a_1^2 = b_1^2/3 = c_1^2 = (1-\nu)(1+\gamma\rho^2)/(4(1+\beta\rho^4))$$
$$a_1^3 = -b_1^3 = c_1^3 = -\rho^2(1-\nu)(\gamma-\beta\rho^2)/(4(1+\beta\rho^4)) \qquad (33A)$$
$$a_1^4 = b_1^4 = c_1^4 = -(1-\nu)/4$$

and for k>1

$$a_k^1 = -b_k^1 = -c_k^1 = [\, k\beta\rho^{-2k} + (k^2-1+\beta^2)\rho^2 - k(k-1)\,]/d_k \qquad (34A)$$

$$k(k+2)a_k^2 = -k(k-2)b_k^2 = -(k^2-4)c_k^2 = -k(k^2-4)[\beta\rho^{-2k} + k + 1 - k\rho^{-2}]/d_k \qquad (35A)$$

$$a_k^3 = -b_k^3 = c_k^3 = [\ (k^2-1+\beta^2)\rho^2 - k(k+1) - k\beta\rho^{2k}\ ]/d_k \tag{36A}$$

$$k(k-2)\,a_k^4 = -k(k+2)\,b_k^4 = (k^2-4)\,c_k^4 = k(k^2-4)\,[k\rho^{-2} + \beta\rho^{2k} - (k-1)\,]/d_k \tag{37A}$$

$$d_k = 2[\ \beta(\rho^k+\rho^{-k})^2 + (k^2-1)(\rho-\rho^{-1})^2 + (\beta\rho-\rho^{-1})^2\ ] \tag{38A}$$

These recursion formulas agree with those given by St. Cyr [17] with certain exceptions. The numerator of the formula for a_k^2, b_k^2, and c_k^2 should be as given in equation (35A). With the numerator in the formula for a_k^2, b_k^2, and c_k^2 as given by St. Cyr, a_2^2 would be non-zero which it certainly isn't as shown by equation (35A). Any attempt to calculate b_2^2 and/or c_2^2 using St. Cyr's recursion formula would necessitate division by zero— yet b_2^2 and c_2^2 are both obviously finite. The formula for d_k is also slightly different from the one given by St. Cyr. As a check on the validity of the recursion formulas given by equation (30A), the calculated a_k's, b_k's, a_k''s and b_k''s were substituted back into the boundary conditions given by equations (21A), (22A), (23A), and (24A). All boundary conditions were satisfied ± 0.000002 for $\lambda = 1$ and k as large as 10.

APPENDIX B

Although this listing of the computer program used for the calculations described in this report is not essential to the report it is hoped that it would be helpful to anyone wishing to investigate a particular set of boundary conditions. Documentation of the program is hopefully sufficient to facilitate the understanding of the program logic so that the program can be easily used or modified to include other conditions or modified to meet the requirements of a particular computer installation. The subroutines BESJ, BESI, BESK, and BESY simply return the values of $J_n(z)$, $I_n(z)$, $K_n(z)$, and $Y_n(z)$, respectively. Subroutine QATR is simply the single-precision version of DQATR which is given.

```
C****
C****     CAPR       = CUTER RADIUS OF DISK
C****     RA         = INNER RADIUS OF DISK
C****     RB         = CLAMPING RADIUS
C****     A1         = VECTOR OF CONSTANTS IN RADIAL STRESS DUE TO LOADING EQU.
C****     A2         = VECTOR OF CONSTANTS IN RADIAL STRESS DUE TO LOADING EQU.
C****     A3         = VECTOR OF CONSTANTS IN RADIAL STRESS DUE TO LOADING EQU.
C****     A4         = VECTOR OF CONSTANTS IN RADIAL STRESS DUE TO LOADING EQU.
C****     B1         = VECTOR OF CONSTANTS IN HCOP STRESS DUE TO LOADING EQU.
C****     B2         = VECTOR OF CONSTANTS IN HCOP STRESS DUE TO LOADING EQU.
C****     B3         = VECTOR OF CONSTANTS IN HCOP STRESS DUE TO LOADING EQU.
C****     B4         = VECTOR OF CONSTANTS IN HCOP STRESS DUE TO LOADING EQU.
C****     C1         = VECTOR OF CONSTANTS IN SHEAR STRESS DUE TO LOADING EQU.
C****     C2         = VECTOR OF CONSTANTS IN SHEAR STRESS DUE TO LOADING EQU.
C****     C3         = VECTOR OF CONSTANTS IN SHEAR STRESS DUE TO LOADING EQU.
C****     C4         = VECTOR OF CONSTANTS IN SHEAR STRESS DUE TO LOADING EQU.
C****     TR         = VECTOR GIVING RADII AT WHICH TENSIONING IS DONE.
C****     TP         = VECTOR GIVING EQUIVALENT LOADING AT DIFFERENT RADII USED
C****                  IN CACULATING STRESS DUE TO TENSIONING
C****     RHO        = DENSITY OF DISK
C****     ELAS       = YOUNG'S MODULUS
C****     H          = THICKNESS OF DISK
C****     WO         = ANGULAR VELOCITY OF ROTATION
C****     TEMP       = TEMPERATURE AS A FUNCTION OF RADIUS
C****     ALP        = THERMAL EXPANSION COEFFICIENT
C****     ANU        = POISSON'S RATIO
C****     R          = RADIAL COMPONENT
C****     THETA      = ANGULAR COMPONENT
C****     RIG        = FLEXURAL RIGIDITY OF DISK
C****     B          = MATRIX OF COEFFICIENTS FOR R SUB N,I
C****     C          = MATRIX OF COEFFICIENTS FOR R SUB N,I
C****     D          = MATRIX OF COEFFICIENTS FOR R SUB N,I
C****     K          = MATRIX OF COEFFICIENTS FOR R SUB N,I
C****     N          = TERM NUMBER
C****     I          = TERM NUMBER
```

```
C****      M          = TERM NUMBER
C****      J          = TERM NUMBER
C****      NTERMS     = NUMBER OF N TERMS IN EXPANSION OF W(R,THETA)
C****      ITERMS     = NUMBER OF I TERMS IN EXPANSION OF W(R,THETA)
C****
C****************************************************************************
      REAL K
C**** IF TENSIONING IS DONE AT MORE THAN 10 RADII, THEN THE SIZE
C**** OF TR AND TP WILL HAVE TO BE INCREASED.
      COMMON /DATA/ELAS,RIG,TR(10),TP(10),WO,NTERMS,ITERMS
      COMMON /VARIA/R,RA,RB,RHO,ALP,ANU,THETA,CAPR,H
C**** IF NTERMS>5 OR ITERMS>5 THEN THE SIZE OF B, C, D, AND K WILL HAVE TO BE
C**** INCREASED.
      COMMON B(5,5),C(5,5),D(5,5),K(5,5)
      COMMON /TERM/N,I,M,J
      COMMON /PREC/ACC,WORD2,WORD3,WORD4,WORD5
      DATA REED,PUNCH/'READ','PUNC'/
C**** IF NTERMS*ITERMS>25 THEN THE SIZE OF QM, QMTP, AND SNIKNI WILL HAVE TO
C**** BE INCREASED.
      DIMENSION QM(25,25),QMTP(25,25),SNIKNI(25)
C**** READ IN DATA
      CALL DATAIN
      DO 10 IN=1,NTERMS
      DO 10 II=1,ITERMS
      N=IN-1
      I=II-1
      MM=4*N+I+1
C**** IF SNI VECTOR HAS ALREADY BEEN CALCULATED SKIP TO CALCULATION OF
C**** QM AND QMTP MATRICES.
      IF(WORD3.EQ.REED)GO TO 5
C**** CALCULATE FIRST COEFFICIENT S SUB N,I
      CALL SNI(CAPR,S,RB,CAPR)
      SNIKNI(MM)=(K(N+1,I+1)*CAPR)**4*S
      CALL INTIME(ITIME)
      WRITE(3,109)N,I,MM,SNIKNI(MM),ITIME
```

```
C**** IF QM AND QMTP MATRICES HAVE ALREADY BEEN CALCULATED, SKIP TO 10.
    5 IF(WORD4.EQ.REED.AND.WORD5.EQ.REED)GO TO 10
      DO 9 IM=1,NTERMS
      DO 9 IJ=1,ITERMS
      M=IM-1
      J=IJ-1
      NN=4*M+J+1
C**** THE QM AND QMTP MATRICES ARE SYMMETRIC. CALCULATE UPPER TRIANGULAR
C**** ELEMENTS ONLY.
      IF(MM.GT.NN)QM(MM,NN)=QM(NN,MM)
      IF(MM.GT.NN)QMTP(MM,NN)=QMTP(NN,MM)
      IF(MM.GT.NN)GO TO 8
C**** CALCULATE Q SUB N,I,M,J AND QTP SUB N,I,M,J
      CALL QNIMJ(RB,CAPR,Q,QTP)
      QM(MM,NN)=Q
      QMTP(MM,NN)=QTP*CAPR/RIG
    8 CONTINUE
      CALL INTIME(ITIME)
      WRITE(3,110)N,I,M,J,MM,NN,QM(MM,NN),MM,NN,QMTP(MM,NN),ITIME
    9 CONTINUE
   10 CONTINUE
      KK=NTERMS*ITERMS
      IF(WORD3.EQ.REED)READ(1,111)(SNIKNI(I),I=1,KK)
      WRITE(3,103)(SNIKNI(I),I=1,KK)
      IF(WORD3.EQ.PUNCH)WRITE(2,104)(I,SNIKNI(I),I=1,KK)
      IF(WORD4.EQ.REED)READ(1,112)((QM(I,J),J=1,KK),I=1,KK)
      WRITE(3,105)((I,J,QM(I,J),J=1,KK),I=1,KK)
      IF(WORD4.EQ.PUNCH)WRITE(2,106)((I,J,QM(I,J),J=1,KK),I=1,KK)
      IF(WORD5.EQ.REED)READ(1,113)((QMTP(I,J),J=1,KK),I=1,KK)
      WRITE(3,107)((I,J,QMTP(I,J),J=1,KK),I=1,KK)
      IF(WORD5.EQ.PUNCH)WRITE(2,108)((I,J,QMTP(I,J),J=1,KK),I=1,KK)
      CALL DUCLD(SNIKNI,QM,QMTP,KK)
  103 FORMAT('1SNIKNI MATRIX'//(5E20.6/))
  104 FORMAT('SNIKNI(',I2,')=',4X,E15.6)
  105 FORMAT('1QM MATRIX'//(1X,4('QM(',I2,',',I2,')=',E15.6,5X)/))
```

```
106 FORMAT(2('QM(',I2,',',I2,')=',E15.6,5X))
107 FORMAT('1QMTP MATRIX'//(1X,4('QMTP(',I2,',',I2,')=',E15.6,5X)/))
108 FORMAT(2('QMTP(',I2,',',I2,')=',E15.6,5X))
109 FORMAT('1 N= ',I3,3X,'I= ',I3,3X,'SNIKNI(',I2,')= ',E15.6,5X,
   ?'TIME= ',I10)
110 FORMAT(' N= ',I3,3X,'I= ',I3,3X,'M= ',I3,3X,'J= ',I3,3X,
   ?'QM(',I2,',',I2,')=',E15.6,5X,'QMTP(',I2,',',I2,')=',E15.6,5X,
   ?'TIME= ',I10)
111 FORMAT(15X,E15.6)
112 FORMAT(10X,E15.6,15X,E15.6)
113 FORMAT(12X,E15.6,17X,E15.6)
    STOP
    END
```

```
      SUBROUTINE DATAIN
      REAL K
      COMMON /PREC/ACC,WORD2,WORD3,WORD4,WORD5
C**** DIMENSIONS OF B, C, D, AND K MATRICES MUST CONFORM TO THOSE IN MAIN PROG
      COMMON B(5,5),C(5,5),D(5,5),K(5,5)
      COMMON / VARIA/R,RA,RB,RHO,ALP,ANU,THETA,CAPR,H
C**** DIMENSIONS OF VECTORS TP AND TR MUST CONFORM TO THOSE IN MAIN PROG.
      COMMON /DATA/ELAS,RIG,TR(10),TP(10),WO,NTERMS,ITERMS
C**** SIZE OF VECTORS IN COMMON/CONST/ MUST CONFORM TO SIZES IN MAIN PROGRAM.
      COMMON /CONST/A1(10),A2(10),A3(10),A4(10),B1(10),B2(10),B3(10),
     ? B4(10),C1(10),C2(10),C3(10),C4(10)
C**** SIZE OF VECTORS IN COMMON/TENS/ MUST CONFORM TO SIZES IN MAIN PROGRAM.
      COMMON /TENS/NTR,AT3(10),AT4(10),BT3(10),BT4(10)
      DATA RCRDS,PUNCH/'READ','PUNC'/
      READ(1,100)NTERMS,ITERMS,CAPR
C**** A SUBROUTINE COULD BE CALLED AT THIS POINT TO CALCULATE THE K, B, C, D
C**** MATRICES.
      READ(1,101)((K(J,L),L=1,ITERMS),J=1,NTERMS)
      READ(1,101)((B(J,L),L=1,ITERMS),J=1,NTERMS)
      READ(1,101)((C(J,L),L=1,ITERMS),J=1,NTERMS)
      READ(1,101)((D(J,L),L=1,ITERMS),J=1,NTERMS)
      DO 3 J=1,NTERMS
      DO 3 L=1,ITERMS
    3 K(J,L)=K(J,L)/CAPR
      READ(1,102)RA,RB,RHO,ALP,ANU,H
      READ(1,103)ELAS,WO,ACC
      READ(1,104)TR,TP,NTR
      RIG=ELAS*H*H*H/(12.0*(1.-ANU*ANU))
      WRITE(3,107)
      WRITE(3,108)((K(J,L),L=1,ITERMS),J=1,NTERMS)
      WRITE(3,109)((B(J,L),L=1,ITERMS),J=1,NTERMS)
      WRITE(3,1010)((C(J,L),L=1,ITERMS),J=1,NTERMS)
      WRITE(3,1011)((D(J,L),L=1,ITERMS),J=1,NTERMS)
      WRITE(3,1012)RA,RB,RHO,ALP,ANU,H,ELAS,WO,TR,TP,RIG
```

```
      GAM=(1.+ANU)/(1.-ANU)
      RAT=RB/CAPR
      RINV=CAPR/RB
      BET=(3.-ANU)/(1.+ANU)
      READ(1,1013)WORD,WORD2,WORD3,WORD4,WORD5
      IF(WORD.EQ.RCRDS)READ(1,1014)A1,A2,A3,A4,B1,B2,B3,B4,C1,C2,C3,C4
      IF(WORD.EQ.RCRDS)GO TO 30
C**** IF THE COEFFICIENTS IN THE STRESS EQUATION DUE TO LOADING ARE
C**** TO BE READ IN PUNCH 'READ CARDS' IN COL. 1-10.  IF THE COEFFICIENTS ARE
C**** BE PUNCHED OUT PUNCH 'PUNCH CARDS' IN COL. 1-11. IF THE COEFFICIENTS
C**** ARE TO BE CALCULATED AND NOT PUNCHED OUT PUT IN A BLANK CARD.
C**** IF THE TENSIONING COEFFICIENTS ARE TO BE READ IN PUNCH 'READ' IN COL. 1-
C**** OF THE SECOND CARD.  IF THE COEFFICIENTS ARE TO BE PUNCH OUT PUNCH 'PUNC
C**** IN COL. 1-5 OF THE SECOND CARD. OTHERWISE PUT IN A BLANK CARD.
C**** IF SNIKNI IS TO BE READ IN PUNCH 'READ SNIKNI' IN COL. 1-11 OF THE THIRD
C**** CARD. IF THE SNIKNI CALCULATED ARE TO BE PUNCH OUT PUNCH 'PUNCH SNIKNI'
C**** IN COL. 1-12 OF THE THIRD CARD. OTHERWISE PUT IN A BLANK CARD.
C**** IF THE Q MATRIX IS TO BE READ IN PUNCH 'READ QM' IN COL. 1-7 OF THE FOUR
C**** CARD. IF THE Q MATRIX IS TO BE PUNCH OUT PUNCH 'PUNCH QM' IN COL. 1-8 OF
C**** THE FOURTH CARD. OTHERWISE PUT IN A BLANK CARD.
C**** IF THE QTP MATRIX IS TO BE READ IN PUNCH 'READ QTP' IN COL. 1-8 OF THE
C**** FIFTH CARD. IF THE QTP MATRIX IS TO BE PUNCH OUT PUNCH 'PUNCH QTP' IN CO
C**** 1-9 OF THE FIFTH CARD. OTHERWISE PUT IN A BLANK CARD.
      A1(1)=0.5D0*RAT*RAT/(GAM+RAT*RAT)
      B1(1)=-A1(1)
      C1(1)=0.0D0
      A2(1)=0.5D0*GAM/(GAM+RAT*RAT)
      B2(1)=A2(1)
      C2(1)=0.0D0
      A3(1)=0.0D0
      B3(1)=0.0D0
      C3(1)=0.0D0
      A4(1)=0.0D0
      B4(1)=0.0D0
      C4(1)=0.0D0
```

```
 A1(2)=1.0DC
 B1(2)=C.0D0
 C1(2)=C.0D0
 A2(2)=C.25D0*(1.D0-ANU)*(1.DC+GAM*RAT*RAT)/(1.D0+BET*RAT*RAT*RAT*
? RAT)
 B2(2)=3.D0*A2(2)
 C2(2)=A2(2)
 A3(2)=-RAT*RAT*(1.D0-ANU)*0.25D0*(GAM-BET*RAT*RAT)/(1.D0+BET*RAT*
? RAT*RAT*RAT)
 B3(2)=-A3(2)
 C3(2)=A3(2)
 A4(2)=-0.25D0*(1.D0-ANU)
 B4(2)=A4(2)
 C4(2)=A4(2)
 DO 27 II=2,9
 DEN=BET*(RAT**II+RINV**II)*(RAT**II+RINV**II)+DFLOAT(II*II-1)*
? (RAT-RINV)*(RAT-RINV)+(BET*RAT-RINV)*(BET*RAT-RINV)
 AT=(-DFLOAT(II)*BET*RINV**(2*II)+(1.D0-DFLOAT(II*II)-BET*BET)*RAT*
?RAT+DFLOAT(II*II-II))/(2.D0*DFLOAT(II*II-II)*DEN)
 BT=(BET*RINV**(2*II)+DFLOAT(II+1)-DFLOAT(II)*RINV*RINV)/
? (2.DC*DFLOAT(II+1)*DEN)
 CT=(DFLOAT(II)*BET*RAT**(2*II)+DFLOAT(II*II+II)+(1.D0-DFLOAT(II*II
?)-BET*BET)*RAT*RAT)/(2.D0*DFLOAT(II*II+II)*DEN)
 DT=(DFLOAT(II)-1.D0-DFLOAT(II)*RINV*RINV-BET*RAT**(2*II))/
?(2.D0*DFLOAT(II-1)*DEN)
 A1(II+1)=DFLOAT(II-II*II)*AT
 B1(II+1)=-A1(II+1)
 C1(II+1)=-A1(II+1)
 A2(II+1)=DFLOAT(II+2-II*II)*BT
 B2(II+1)=DFLOAT(II*II+3*II+2)*BT
 C2(II+1)=DFLOAT(II*II+II)*BT
 A3(II+1)=-DFLOAT(II*II+II)*CT
 B3(II+1)=-A3(II+1)
 C3(II+1)=A3(II+1)
 A4(II+1)=DFLOAT(-II+2-II*II)*DT
```

```
      B4(II+1)=DFLOAT(II*II-3*II+2)*DT
      C4(II+1)=DFLCAT(II-II*II)*DT
      C4(II+1)=II*(1.-II)*DT
   27 CONTINLE
   30 WRITE(3,105)A1,A2,A3,A4,B1,B2,B3,B4,C1,C2,C3,C4
      IF(WCRC.EQ.PUNCH)GO TO 31
      GO TC 40
   31 WRITE(2,106)RAT,A1,RAT,A2,RAT,A3,RAT,A4,RAT,B1,RAT,B2,RAT,B3,RAT,
     ?B4,RAT,C1,RAT,C2,RAT,C3,RAT,C4
   40 IF(WORC2.EQ.RCRDS)GO TO 50
      DO 45 I=1,NTR
      TZ=TR(I)
      TPZ=TP(I)
      AT3(I)=RA*RA*TZ*TZ*TPZ/(TZ*TZ-RA*RA)
      AT4(I)=-TPZ*TZ*TZ/(TZ*TZ-RA*RA)
      BT3(I)=TPZ*CAPR*CAPR*TZ*TZ/(TZ*TZ-CAPR*CAPR)
   45 BT4(I)=-TPZ*TZ*TZ/(TZ*TZ-CAPR*CAPR)
      IF(WORD2.EQ.PUNCH)WRITE(2,1015)(I,AT3(I),I,AT4(I),I,BT3(I),I,
     ? BT4(I),I=1,NTR)
      GO TC 55
   50 READ (3,1016)(AT3(I),AT4(I),BT3(I),BT4(I),I=1,NTR)
   55 WRITE(3,1017)(AT3(I),AT4(I),BT3(I),BT4(I),I=1,NTR)
  100 FORMAT(2I5,F10.6)
  101 FORMAT(4F15.6)
  102 FORMAT(3F20.6)
  103 FORMAT(E20.6,2F20.6)
  104 FORMAT(5F10.5/5F10.5/5F10.5/5F10.5/I5)
  105 FORMAT('1 THE COEFFICIENTS USED IN THE STRESS FUNCTION DUE TO LOAD
     ?ING ARE'/12(5X,5D20.12//))
  106 FORMAT('A1',F3.1,5E15.6/5X,5E15.6/'A2',F3.1,5E15.6/5X,5E15.6/
     ?'A3',F3.1,5E15.6/5X,5E15.6/'A4',F3.1,5E15.6/5X,5E15.6/
     ?'B1',F3.1,5E15.6/5X,5E15.6/'B2',F3.1,5E15.6/5X,5E15.6/
     ?'B3',F3.1,5E15.6/5X,5E15.6/'B4',F3.1,5E15.6/5X,5E15.6/
     ?'C1',F3.1,5E15.6/5X,5E15.6/'C2',F3.1,5E15.6/5X,5E15.6/
     ?'C3',F3.1,5E15.6/5X,5E15.6/'C4',F3.1,5E15.6/5X,5E15.6)
```

```
 107 FORMAT('1ECHO CHECK OF DATA')
 108 FORMAT(//' MATRIX K(N,I)'/(14X,4D25.12))
 109 FORMAT(//' MATRIX B(N,I)'/(14X,4D25.12))
1010 FORMAT(//' MATRIX C(N,I)'/(14X,4D25.12))
1011 FORMAT(//' MATRIX D(N,I)'/(14X,4D25.12))
1012 FORMAT(//' INNER RADIUS=',F7.2,16X,'CLAMPING RADIUS=',F6.2//
    ?' DENSITY=',F14.4,14X,'THERMAL COEFF.= ',F8.4//
    ?' POISSON''S RATIO=',F5.3,15X,'THICKNESS=',F14.4//' ELASTICITY=',
    ?E14.3,11X,'ANGULAR VELOCITY=',F10.1//
    ?' TR=',10F12.2//' TP=',10F12.2//' RIGIDITY=',F15.3)
1013 FORMAT(A4)
1014 FORMAT(5X,5E15.6)
1015 FORMAT('AT3(',I2,')=',E12.5,'AT4(',I2,')=',E12.5,'BT3(',I2,')=',
    ?E12.5,'BT4(',I2,')=',E12.5)
1016 FORMAT(4(8X,E12.5))
1017 FORMAT(///' COEFFICIENTS IN THE STRESS DUE TO TENSIONING EQUATIONS
    ? ARE'/(4E25.5))
     RETURN
     END


    FUNCTION FCT5(R)

     COMMON / TERM/N,I,M,J
     COMMON / VARIA/DUMMY,RA,RB,RHO,ALP,ANU,THETA,CAPR,H
     RN=DRNIDR(R)
     RM=DRMJDR(R)
     STR=STRESR(R)
     FCT5=3.141592653589*R*RN*RM*STR
     IF(N.EQ.0)FCT5=2.*FCT5
     RETURN
     END
```

```
      FUNCTION FCT6(R)

       COMMON /TERM/ N,I,M,J
       COMMON / VARIA/DUMMY,RA,RB,RHO,ALP,ANU,THETA,CAPR,H
       IF(N.EQ.0.OR.M.EQ.0)GO TO 10
       RN=RNI(R)
       RM=RMJ(R)
       STR=STRHOP(R)
       FCT6=3.141592653589*RN*RM*STR*M*N/R
       RETURN
    10 FCT6=0.0
       RETURN
       END


      SUBROUTINE SNI(CR,S,XLL,AUU)

       REAL*8 FCT11,AC,AUX,CAPR,XL,XU,T9
       EXTERNAL FCT11
       COMMON /PREC/ACC,WORD2,WORD3,WORD4,WORD5
       COMMON /TERM/N,I,M,J
       DIMENSION AUX(50)
       XL=DBLE(XLL)
       XU=DBLE(AUU)
       CAPR=DBLE(CR)
       AC=DBLE(ACC)
       CALL DQATR(XL,XU,AC,20,FCT11,T9,IER,AUX)
       S=T9*3.141592653589793/(2.D0*CAPR*CAPR)
       IF(N.EQ.0)S=2.0*S
       IF(IER.NE.0)WRITE(3,101)IER,T9,AUX
   101 FORMAT(1X,88('*')//' INTEGRATION OF FCT11 GAVE A RETURN CODE OF',
      ? I3,5X,'T9 =',D20.12//' AUX MATRIX IS'/(1X,5D25.12))
       RETURN
       END
```

```
      SUBROUTINE QNIMJ(XL,XU,Q,QTP)

      EXTERNAL FCT1,FCT5,FCT6
      COMMON / VARIA/R,RA,RB,RHO,ALP,ANU,THETA,CAPR,H
      COMMON /PREC/ACC,WORD2,WORD3,WORD4,WORD5
      COMMON /TERM/N,I,M,J
      DATA REED/'READ'/
      DIMENSION AUX(20)
      IF(WORD4.EQ.REED)Q=0.0
      IF(WORD4.EQ.REED)GO TO 5
      CALL QATR(XL,XU,ACC,20,FCT1,Q,IER1,AUX)
      IF(IER1.NE.0)WRITE(3,101)IER1,Q,AUX
    5 IF(WORD5.EQ.REED)GO TO 10
      IF(M.NE.N)GO TO 10
      CALL QATR(XL,XU,ACC,08,FCT5,Y5,IER5,AUX)
      IF(IER5.NE.0)WRITE(3,105)IER5,Y5,AUX
      CALL QATR(XL,XU,ACC,08,FCT6,Y6,IER6,AUX)
      IF(IER6.NE.0)WRITE(3,106)IER6,Y6,AUX
      QTP=CAPR*H*(Y5+Y6)
      RETURN
   10 QTP=0.0
      RETURN
  101 FORMAT(1X,88('*')//' INTEGRATION OF FCT1 GAVE A RETURN CODE OF',
     ? I3,5X,'Y1 =',E15.6//1X,'AUX MATRIX IS'/(1X,10E13.5))
  105 FORMAT(1X,88('*')//' INTEGRATION OF FCT5 GAVE A RETURN CODE OF',
     ? I3,5X,'Y5 =',E15.6//1X,'AUX MATRIX IS'/(1X,10E13.5))
  106 FORMAT(1X,88('*')//' INTEGRATION OF FCT6 GAVE A RETURN CODE OF',
     ? I3,5X,'Y6 =',E15.6//1X,'AUX MATRIX IS'/(1X,10E13.5))
      END
```

```
      FUNCTION FCT1(R)

      COMMON / VARIA/DUMMY,RA,RB,RHO,ALP,ANU,THETA,CAPR,H
      COMMON / TERM/N,I,M,J
C**** DIMENSIONS OF VECTORS IN COMMON/CONST/ MUST CONFORM TO THEIR
C**** DIMENSIONS IN THE MAIN PROGRAM.
      COMMON /CONST/A1(10),A2(10),A3(10),A4(10),B1(10),B2(10),B3(10),
     ? B4(10),C1(10),C2(10),C3(10),C4(10)
      RAT=R/CAPR
      RINV=CAPR/R
      RN=RNI(R)
      RM=RMJ(R)
      DRNI=DRNIDR(R)
      DRMJ=DRMJDR(R)
      RESULT=0.0
      KK=IABS(M-N)
      GO TO (10,20,30,30,30,30),KK
      RESULT=(A1(1)+A3(1))*RINV*RINV+A2(1)+A4(1)
      GO TO 40
   10 RESULT=((A1(2)+A3(2)*RINV*RINV+A4(2))*RINV+A2(2)*RAT)*0.5D0
      GO TO 40
   20 RESULT=     (A1(3)+A2(3)*RAT*RAT+RINV*RINV*(A3(3)*RINV*RINV+A4(3)))
     ? *0.5D0
      GO TO 40
   30 KK=KK+1
      RESULT=     ((A1(KK)+A2(KK)*RAT*RAT)*RAT**(KK-3)+(A3(KK)*RINV*RINV+
     ?      A4(KK))*RINV**(KK-1))*0.5D0
   40 KK=M+N
      GO TO (50,60,70,70,70,70,70,70,70,70,70),KK
      RESULT=RESULT+(A1(1)+A3(1))*RINV*RINV+A2(1)+A4(1)
      GO TO 80
   50 RESULT=RESULT+((A1(2)+A3(2)*RINV*RINV+A4(2))*RINV+A2(2)*RAT)*0.5D0
      GO TO 80
   60 RESULT=RESULT+     (A1(3)+A2(3)*RAT*RAT+RINV*RINV*(A3(3)*RINV*RINV+
     ?      A4(3)))*0.5D0
```

```
      GC TC 80
   7C KK=KK+1
      RESULT=RESULT+    ((A1(KK)+A2(KK)*RAT*RAT)*RAT**(KK-3)+
     ?(A3(KK)*RINV*RINV+A4(KK))*RINV**(KK-1))*0.5DO
   8C CONTINLE
      FCT1=-R*DRMJ*DRNI*RESULT
      IF(M.EC.0.OR.N.EQ.0)GO TO 2CC
      KK=IABS(M-N)
      GO TC (110,120,130,130,130,130),KK
      RESULT=(B1(1)+B3(1))*RINV*RINV+B2(1)+B4(1)
      GO TO 140
  110 RESULT=((B1(2)+B4(2)+B3(2)*RINV*RINV)*RINV+RAT*B2(2))*0.5DO
      GO TC 140
  120 RESULT=    (B1(3)+B2(3)*RAT*RAT+RINV*RINV*(B3(3)*RINV*RINV+B4(3)))
     ? *0.5CC
      GO TO 140
  130 KK=KK+1
      RESULT=    ((B1(KK)+RAT*RAT*B2(KK))*RAT**(KK-3)+
     ?     (B3(KK)*RINV*RINV+B4(KK))*RINV**(KK-1))*0.5DO
  140 KK=M+N
      GO TO (180,160,170,170,170,170,170,170,170),KK
  160 RESULT=RESULT-    (B1(3)+B2(3)*RAT*RAT+RINV*RINV*(B3(3)*RINV*RINV+
     ?B4(3)))*0.5DO
      GO TO 180
  170 KK=KK+1
      RESULT=RESULT-    ((B1(KK)+RAT*RAT*B2(KK))*RAT**(KK-3)+
     ?(B3(KK)*RINV*RINV+B4(KK))*RINV**(KK-1))*0.5DO
  180 FCT1=-M*N*RM*RN*RESULT/R+FCT1
  200 IF(N.EC.0)GO TO 300
      IF(M.EQ.0)GO TO 290
      KK=IABS(M-N)
      GC TC (210,220,230,230,230,230),KK
      RESULT=0.DO
      GO TO 240
  210 RESULT=((C1(2)+C3(2)*RINV*RINV+C4(2))*RINV+C2(2)*RAT)*0.5DO
```

```
      IF(M.GT.N)RESULT=-RESULT
      GO TO 240
  220 RESULT=     (C1(3)+C2(3)*RAT*RAT+RINV*RINV*(C3(3)*RINV*RINV+C4(3)))
     ? *0.5DO
      IF(M.GT.N)RESULT=-RESULT
      GO TO 240
  230 KK=KK+1
      RESULT=     (RAT**(KK-3)*(C1(KK)+RAT*RAT*C2(KK))
     ?+RINV**(KK-1)*(C3(KK)*RINV*RINV+C4(KK)))*0.5DO
      IF(M.GT.N)RESULT=-RESULT
  240 KK=M+N
      GO TO (250,260,270,270,270,270,270,270,270,270),KK
  250 RESULT=RESULT+((C1(2)+C3(2)*RINV*RINV+C4(2))*RINV+C2(2)*RAT)*0.5DO
      GO TO 280
  260 RESULT=RESULT+     (C1(3)+C2(3)*RAT*RAT+RINV*RINV*(C3(3)*RINV*RINV+
     ?C4(3))) *0.5DO
      GO TO 280
  270 KK=KK+1
      RESULT=RESULT+     (RAT**(KK-3)*(C1(KK)+RAT*RAT*C2(KK))+
     ?RINV**(KK-1)*(C3(KK)*RINV*RINV+C4(KK)))*0.5DO
  280 FCT1=M*DRMJ*RN*RESULT+FCT1
      GO TO 300
  290 KK=N+1
      GO TO (292,292,294,296,296,296,296),KK
  292 RESULT=(C1(2)+C3(2)*RINV*RINV+C4(2))*RINV+C2(2)*RAT
      GO TO 280
  294 RESULT=C1(3)+C2(3)*RAT*RAT+RINV*RINV*(C3(3)*RINV*RINV+C4(3))
      GO TO 280
  296 RESULT=RAT**(KK-3)*(C1(KK)+RAT*RAT*C2(KK))+
     ? RINV**(KK-1)*(C3(KK)*RINV*RINV+C4(KK))
      GO TO 280
  300 IF(M.EQ.0)RETURN
      IF(N.EQ.0)GO TO 390
      KK=IABS(M-N)
      GO TO (310,320,330,330,330,330),KK
```

```
      RESULT=0.D0
      GO TO 340
  310 RESULT=((C1(2)+C3(2)*RINV*RINV+C4(2))*RINV+C2(2)*RAT)*0.5D0
      IF(N.GT.M)RESULT=-RESULT
      GO TO 340
  320 RESULT=    (C1(3)+C2(3)*RAT*RAT+RINV*RINV*(C3(3)*RINV*RINV+C4(3)))
     ? *0.5D0
      IF(N.GT.M)RESULT=-RESULT
      GO TO 340
  330 KK=KK+1
      RESULT=    (RAT**(KK-3)*(C1(KK)+RAT*RAT*C2(KK))
     ?+RINV**(KK-1)*(C3(KK)*RINV*RINV+C4(KK)))*0.5D0
      IF(N.GT.M)RESULT=-RESULT
  340 KK=M+N
      GO TO (350,360,370,370,370,370,370,370,370,370),KK
  350 RESULT=RESULT+((C1(2)+C3(2)*RINV*RINV+C4(2))*RINV+C2(2)*RAT)*0.5D0
      GO TO 380
  360 RESULT=RESULT+    (C1(3)+C2(3)*RAT*RAT+RINV*RINV*(C3(3)*RINV*RINV+
     ?C4(3)))*0.5D0
      GO TO 380
  370 KK=KK+1
      RESULT=RESULT+    (RAT**(KK-3)*(C1(KK)+RAT*RAT*C2(KK))+
     ?RINV**(KK-1)*(C3(KK)*RINV*RINV+C4(KK)))*0.5D0
  380 FCT1=M*DRNI*RM*RESULT+FCT1
      RETURN
  390 KK=M+1
      GO TO (392,392,394,396,396,396,396),KK
  392 RESULT=(C1(2)+C3(2)*RINV*RINV+C4(2))*RINV+C2(2)*RAT
      GO TO 380
  394 RESULT=C1(3)+C2(3)*RAT*RAT+RINV*RINV*(C3(3)*RINV*RINV+C4(3))
      GO TO 380
  396 RESULT=RAT**(KK-3)*(C1(KK)+RAT*RAT*C2(KK))+
     ? RINV**(KK-1)*(C3(KK)*RINV*RINV+C4(KK))
      GO TO 380
      END
```

```
      DOUBLE PRECISION FUNCTION FCT11(R)

      REAL*8 RJ,RY,RI,RK,BN,CN,DN,R,X
      REAL K
      COMMON /TERM/N,I,M,J
C**** DIMENSIONS OF B, C, D, AND K MATRICES MUST CONFORM TO THOSE IN MAIN PROG
      COMMON B(5,5),C(5,5),D(5,5),K(5,5)
      NP1=N+1
      IP1=I+1
      X=DBLE(K(NP1,IP1))*R
      XX=X
      CALL BESJ(XX,N,RRJ,1.0E-05,IERJ)
      CALL BESY(XX,N,RRY,IERY)
      CALL BESI(XX,N,RRI,IERI)
      CALL BESK(XX,N,RRK,IERK)
      BN=DBLE(B(NP1,IP1))
      CN=DBLE(C(NP1,IP1))
      DN=DBLE(D(NP1,IP1))
      RJ=DBLE(RRJ)
      RY=DBLE(RRY)
      RI=DBLE(RRI)
      RK=DBLE(RRK)
      FCT11=RJ+BN*RY+CN*RI+DN*RK
      FCT11=2.D0*FCT11*FCT11*R
      RETURN
      END
```

```
      FUNCTICN RNI(R)
C**** THIS FLNCTION CALCULATES R SLB N,I AT R
      REAL K
      COMMCN /TERM/N,I,M,J
C**** DIMENSIONS OF B, C, D, AND K MATRICES MUST CONFORM TO THOSE IN MAIN PROG
      COMMCN B(5,5),C(5,5),D(5,5),K(5,5)
      N1=N+1
      I1=I+1
      X=K(N1,I1)*R
      CALL BESJ(X,N,XJ,1.0E-05,IERJ)
      CALL BESI(X,N,XI,IERI)
      CALL BESK(X,N,XK,IERK)
      CALL BESY(X,N,XY,IERY)
      RNI=XJ+B(N1,I1)*XY+C(N1,I1)*XI+D(N1,I1)*XK
      RETURN
      END


      FUNCTICN RMJ(R)
C**** THIS FLNCTION CALCULATES R SLB M,J AT R
      REAL K
      COMMCN /TERM/N,I,M,J
C**** DIMENSIONS OF B, C, D, AND K MATRICES MUST CONFORM TO THOSE IN MAIN PROG
      COMMCN B(5,5),C(5,5),D(5,5),K(5,5)
      M1=M+1
      J1=J+1
      X=K(M1,J1)*R
      CALL BESJ(X,M,XJ,1.0E-05,IERJ)
      CALL BESI(X,M,XI,IERI)
      CALL BESY(X,M,XY,IERY)
      CALL BESK(X,M,XK,IERK)
      RMJ=XJ+B(M1,J1)*XY+C(M1,J1)*XI+D(M1,J1)*XK
      END
```

```
      FUNCTION DRNIDR(R)
C**** THIS FUNCTION CALCULATES THE DERIVATIVE OF R SUB N,I WITH
C**** RESPECT TO R EVALUATED AT R
      REAL K
      COMMON / TERM/N,I,M,J
C**** DIMENSIONS OF B, C, D, AND K MATRICES MUST CONFORM TO THOSE IN MAIN PROG
      COMMON B(5,5),C(5,5),D(5,5),K(5,5)
      NP1=N+1
      IP1=I+1
      X=K(NP1,IP1)*R
      CALL BESJ(X,N,XJ,1.0E-05,IERJ)
      CALL BESJ(X,NP1,XJ1,1.0E-05,IERJ1)
      CALL BESI(X,N,XI,IERI)
      CALL BESI(X,NP1,XI1,IERI1)
      CALL BESY(X,N,XY,IERY)
      CALL BESY(X,NP1,XY1,IERY1)
      CALL BESK(X,N,XK,IERK)
      CALL BESK(X,NP1,XK1,IERK1)
      DRNIDR=N*XJ/X-XJ1+B(NP1,IP1)*(N*XY/X-XY1)+C(NP1,IP1)*(N*XI/X+XI1)+
     ?         D(NP1,IP1)*(N*XK/X-XK1)
      DRNIDR=DRNIDR*K(NP1,IP1)
      RETURN
      END
```

```
      FUNCTION DRMJDR(R)
C**** THIS FUNCTION CALCULATES THE DERIVATIVE OF R SUB M,J WITH
C**** RESPECT TO R EVALUATED AT R
      REAL K
      COMMON / TERM/N,I,M,J
C**** DIMENSIONS OF B, C, D, AND K MATRICES MUST CONFORM TO THOSE IN MAIN PROG
      COMMON B(5,5),C(5,5),D(5,5),K(5,5)
      MP1=M+1
      JP1=J+1
      X=K(MP1,JP1)*R
      CALL BESJ(X,M,XJ,1.0E-05,IERJ)
      CALL BESJ(X,MP1,XJ1,1.0E-05,IERJ1)
      CALL BESI(X,M,XI,IERI)
      CALL BESI(X,MP1,XI1,IERI1)
      CALL BESY(X,M,XY,IERY)
      CALL BESY(X,MP1,XY1,IERY1)
      CALL BESK(X,M,XK,IERK)
      CALL BESK(X,MP1,XK1,IERK1)
      DRMJDR=M*XJ/X-XJ1+B(MP1,JP1)*(M*XY/X-XY1)+C(MP1,JP1)*(M*XI/X+XI1)+
     ?     D(MP1,JP1)*(M*XK/X-XK1)
      DRMJDR=K(MP1,JP1)*DRMJDR
      RETURN
      END
```

```
      FUNCTION STRHOP(R)
C**** THIS FUNCTION CALCULATES HOOP STRESS DUE TO TEMPERATURE
C**** GRADIENTS, TENSIONING, AND ROTATION AS A FUNCTION OF R
      COMMON /TENS/NTR,AT3(10),AT4(10),BT3(10),BT4(10)
      COMMON /DATA/ELAS,RIG,TR(10),TP(10),WO,NTERMS,ITERMS
      STRHOP=0.0
      DO 20 II=1,NTR
      IF(R.GT.TR(II))GO TO 10
      STRHOP=STRHOP-AT3(II)/(R*R)+AT4(II)
      GO TO 20
   10 STRHOP=STRHOP-BT3(II)/(R*R)+BT4(II)
   20 CONTINUE
      RETURN
       END

      FUNCTION STRESR(R)
C**** THIS FUNCTION CALCULATES RADIAL STRESS DUE TO TEMPERATURE
C**** GRADIENTS, TENSIONING, AND ROTATION AS A FUNCTION OF R
      COMMON /TENS/NTR,AT3(10),AT4(10),BT3(10),BT4(10)
      COMMON /DATA/ELAS,RIG,TR(10),TP(10),WO,NTERMS,ITERMS
      STRESR=0.0
      DO 20 II=1,NTR
      IF(R.GT.TR(II))GO TO 10
      STRESR=STRESR+AT3(II)/(R*R)+AT4(II)
      GO TO 20
   10 STRESR=STRESR+BT3(II)/(R*R)+BT4(II)
   20 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE BUCLD(S,W,X,KK)
      DIMENSION L(25),M(25),S(25),Q(625),QTP(625),C(25),A(625)
      DIMENSION W(25,25),X(25,25)
      DO 20 I=1,KK
   20 C(I)=1.0
      DO 1 I=1,KK
    1 X(I,I)=X(I,I)+S(I)
      CALL ARRAY(2,KK,KK,25,25,Q,W)
      CALL ARRAY(2,KK,KK,25,25,QTP,X)
      CALL MINV(QTP,KK,D,L,M)
      IF(D.EQ.0.0)WRITE(3,101)
      CALL GMPRD(QTP,Q,A,KK,KK,KK)
      SAVE1=0.0
      KSQ=KK*KK
      DO 5 I=1,KSQ
    5 QTP(I)=-A(I)
      DO 12 K=1,50
      CALL GMPRD(QTP,C,S,KK,KK,1)
      SMAX=0.0
      DO 10 I=1,KK
      IF(ABS(S(I)).GT.ABS(SMAX))SMAX=S(I)
   10 CONTINUE
      DO 11 I=1,KK
   11 C(I)=S(I)/SMAX
      EIGEN=SMAX
   12 CONTINUE
      EIGEN=1./EIGEN
      WRITE(3,102)(C(I),I=1,KK)
      WRITE(3,103)EIGEN
  101 FORMAT(' QTP + SNIKNI MATRIX IS SINGULAR')
  102 FORMAT(' THE EIGENVECTOR IS'/(5E25.6))
  103 FORMAT(' THE FACTOR R*P/RIG=',E20.6)
      RETURN
      END
```

```
      SUBROUTINE DQATR(XL,XU,EPS,NDIM,FCT,Y,IER,AUX)

      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*4 Y
      DIMENSION AUX(1)
      AUX(1)=0.5D0*(FCT(XL)+FCT(XU))
      H=XU-XL
      IF(NDIM-1)8,8,1
    1 IF(H)2,10,2
    2 HH=H
      E=EPS/DABS(H)
      DELT2=0.D0
      P=1.D0
      JJ=1
      DO 7 I=2,NDIM
      Y=AUX(1)
      DELT1=DELT2
      HD=HH
      HH=0.5D0*HH
      P=0.5D0*P
      X=XL+HH
      SM=0.D0
      DO 3 J=1,JJ
      SM=SM+FCT(X)
    3 X=X+HD
      AUX(I)=0.5D0*AUX(I-1)+P*SM
      Q=1.D0
      JI=I-1
      DO 4 J=1,JI
      II=I-J
      Q=Q+Q
      Q=Q+Q
    4 AUX(II)=AUX(II+1)+(AUX(II+1)-AUX(II))/(Q-1.D0)
      DELT2=DABS(Y-AUX(1))
      IF(I-5)7,5,5
```

```
 5 IF(DELT2-E)10,10,6
 6 IF(DELT2-DELT1)7,11,11
 7 JJ=JJ+JJ
 8 IER=2
 9 Y=H*AUX(1)
   RETURN
10 IER=0
   GO TO 9
11 IER=1
   Y=H*Y
   RETURN
   END
```

ACKNOWLEDGEMENTS

The author wishes to express his appreciation to Dr. F. C. Appl of the Department of Mechanical Engineering, Kansas State University, for his guidance and counsel in the preparation of this report and for the use of his extensive references on circular disks.

VITA

Robert Phillip DuBois

Candidate for the Degree of

Master of Science

Thesis: BUCKLING LOADS OF TENSIONED CIRCULAR PLATES SUBJECT TO CONCENTRATED IN-PLANE LOADING

Major Field: Mechanical Engineering

Biographical:

Personal Data: Born in Independence, Missouri, November 28, 1946, the son of Robert C. and Dorothy J. DuBois.

Education: Attended grade school in the Kansas City, Missouri, area until 1960; attended grade school and high school in Ogden, Utah, until 1964; graduated from Raytown High School, Raytown, Missouri in 1965; attended Kansas City Metropolitan Junior College evening school during school year of 1964; received a Bachelor of Science degree in Mathematics and a Bachelor of Science degree in Mechanical Engineering from Kansas State University in June, 1969.

Professional Experience: Worked for Bendix, Launch Support Division, at the John F. Kennedy Space Center for three summers while an undergraduate; worked at the Kansas State University Computing Center for three and one-half years while attending school.

BUCKLING LOADS OF TENSIONED CIRCULAR PLATES SUBJECT TO CONCENTRATED IN-PLANE LOADING

by

ROBERT PHILLIP DUBOIS

B. S., Kansas State University, 1969

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1970

Name: Robert Phillip DuBois Date of Degree: August, 1970

Institution: Kansas State University Location: Manhattan, Kansas

Title of Study: BUCKLING LOADS OF TENSIONED CIRCULAR PLATES SUBJECT TO CONCENTRATED IN-PLANE LOADING

Pages in Study: 56 Candidate for Degree of Master of Science

Major Field: Mechanical Engineering

Scope and Method of Study: A thin circular plate, subject to boundary conditions closely approximating the boundary conditions for a circular saw blade, is considered. The potential energy of the plate, which is a function of internal stresses and lateral deflection, is minimized using an infinite series representation for the lateral deflection. Internal stresses due to tensioning, temperature gradients and in-plane loading are determined with corrections to loading stress values previously published. The equations resulting from the minimization of the potential energy are then changed to an eigenvalue problem with the buckling loads as the eigenvalue and the eigenvector as a vector of coefficients which could be used to calculate the shape of the disk at the buckling load.

Findings and Conclusions: Buckling loads were calculated which agreed with previously published theoretical values and previously determined experimental values. Sufficient time was not available to complete calculations for a case including temperature gradients. However, several cases of tensioning were considered and tensioning does affect the buckling loads. More cases of tensioning need to be considered

before a method of controlling the buckling load can be determined. The method used seems quite useful but the number of calculations involved necessitates the use of a very fast digital computer.

MAJOR PROFESSOR'S APPROVAL F. C. Appl