PROBLEMS IN POTENTIAL PERTAINING TO A PARABOLA

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INTRODUCTION

Potential at a point P is defined as the work done in bringing a unit mass from infinity to the point P.* Consider a mass dm attracting a unit mass. By definition work is equal to the product of force and distance, and attraction between masses is equal to the product of the masses divided by the distance squared.

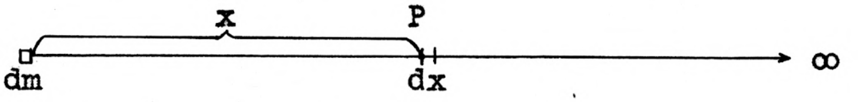

Fig. 1

In Fig. 1 the work done in moving the unit mass a distance dx is

$$W = -dm(1)\ dx/x^2.$$

The negative sign is used because x decreases. Also the work done in moving the unit mass from infinity to P is

$$W = -\int_{\infty}^{x} \frac{dm\ dx}{x^2} = \left[\frac{dm}{x}\right]_{\infty}^{x} = \frac{dm}{x}$$

Thus the potential V at a point P due to mass dm is

$$V = \frac{dm}{x}. \qquad (1)$$

The potential at a point P due to a mass M surrounding dm is

$$V = \int \frac{dm}{x} \qquad (2)$$

* Millikan and Miles, Electricity, Sound, and Light, p. 9.

integrated throughout the entire mass M.

In order to find V a set of rectangular axes is chosen, and then the potential at any point is found by equation (2). The potential at any point in space must satisfy Laplace's equation.* In spherical coordinates the equation is

$$rD_r^2(rV) + \frac{1}{\sin\theta} D_\theta(\sin\theta\ D_\theta V) + \frac{1}{\sin^2\theta} D_\phi^2 V = 0. \qquad (3)$$

It is seen that the potential at a point in space is independent of ϕ for masses which are symmetrical about the x-axis. When this is true Laplace's equation reduces to

$$rD_r^2(rV) + \frac{1}{\sin\theta} D_\theta(\sin\theta\ D_\theta V) = 0; \qquad (4)$$

therefore, the potential at a point not on the x-axis must satisfy (4) with the boundary condition that

$$V = \int \frac{dm}{x} \text{ when } \theta = 0.$$

To solve this equation assume $V - r^m P$ where P is a function of θ alone and m is a positive integer. The values for P are found to be $P_m(\cos\theta)$ which are the Legendre's Coefficients. Then

$$V = r^m P_m(\cos\theta)$$

is a particular solution of (4), and

* Byerly, Fourier's Series and Spherical Harmonics, p. 8.

$$V = \sum_{m=o}^{\infty} r^m P_m(\cos\theta) \qquad (5)$$

is also a particular solution. It is also found that

$$V = \frac{1}{r^{m+1}} P_m(\cos\theta)$$

is a particular solution of (4), and

$$V = \sum_{m=o}^{\infty} \frac{1}{r^{m+1}} P_m(\cos\theta) \qquad (6)$$

is also a particular solution.*

If the potential function at a point on the x-axis inside the mass M can be expressed in an infinite series of the form

$$V = A_o x^o + A_1 x^1 + A_2 x^2 + ----, \qquad (7)$$

then the potential function for any point inside the mass M is found by replacing x by r, and multiplying each term by the corresponding Legendre's Coefficient; therefore,

$$V = A_o r^o P_o(\cos\theta) + A_1 r^1 P_1(\cos\theta) + A_2 r^2 P_2(\cos\theta) + ----. \qquad (8)$$

This equation satisfies Laplace's equation and reduces to (7) when $\theta = 0$.

Likewise, if the potential function at a point on the x-axis outside the mass M can be expressed in an infinite

* Byerly, Fourier's Series and Spherical Harmonics, p. 17.

series of the form

$$V = \frac{A_o}{x^o} + \frac{A_1}{x^1} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \text{----}, \tag{9}$$

then by equation (6) the potential function for a point in space is

$$V = A_o + \frac{A_1}{r} P_o(\cos\theta) + \frac{A_2}{r^2} P_1(\cos\theta) + \frac{A_3}{r^3} P_2(\cos\theta) + \text{---}. \tag{10}$$

In Fourier's Series and Spherical Harmonics, Byerly finds the potential function at any point in space for rings, discs, spheres, and ellipsoids of revolution, but does not mention similar problems involving a parabola except in the historical summary.*

The purpose of this investigation is:

(1) To find the potential function for all points in space due to a solid generated by revolving the area bounded by the parabola $y^2 = 4a(a-x)$ and the y-axis about the x-axis;

(2) To find the potential function for all points on the x-axis due to the surface generated by revolving the arc of the parabola, $y^2 = 4a(a-x)$ between $x = 0$ and $x = a$, about the x-axis;

(3) To find the potential function on the x-axis due to the arc of the parabola, $y^2 = 4a(a-x)$, between $x = 0$ and $x = a$;

* Byerly, Fourier's Series and Spherical Harmonics, p. 272.

(4) To find the potential function on the z-axis due to the same arc;

(5) To find the potential function at a point on the x-axis due to a thin sheet bounded by the parabola $y^2 = 4a(a-x)$ and the y-axis;

(6) To find the solution of Laplace's equation in curvilinear coordinates, using two orthogonal paraboloids of revolution, $z^2 + y^2 = 4a(a-x)$ and $z^2 + y^2 = 4b(b+x)$, and the plane $y = cz$.

I. THE SOLID OF REVOLUTION

The potential at a point on the x-axis due to a disc is found by (2). In Fig. 2, $dm = K\ r d\theta dr$. The potential

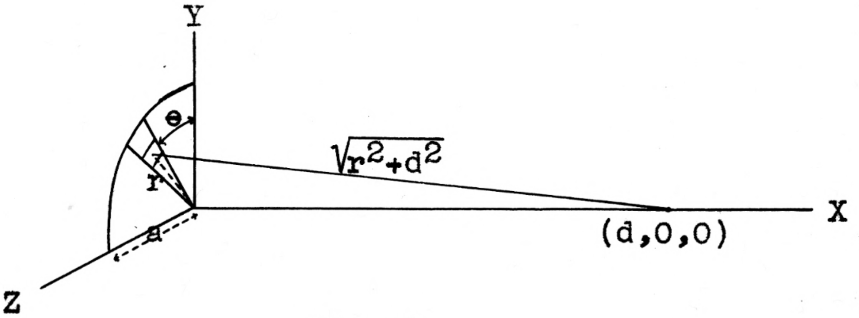

Fig. 2

at (d,0,0) due to the mass dm is

$$V = \frac{K\ r dr d\theta}{\sqrt{r^2+d^2}}.$$

The potential at (d,0,0) due to the entire disc is

$$V = K \int_{\theta=o}^{\theta=2\pi} \int_{r=o}^{r=a} \frac{r\,dr\,d\theta}{\sqrt{r^2+d^2}} = 2\pi K(\sqrt{a^2+d^2} - d). \qquad (11)$$

The constant K will be ommitted temporarily as it can be replaced whenever necessary.

The potential at the point (d,0,0) due to the volume generated by revolving the area in the first quadrant under the parabola $y^2 = 4a(a-x)$ about the x-axis is by equation (11) and Fig. 3

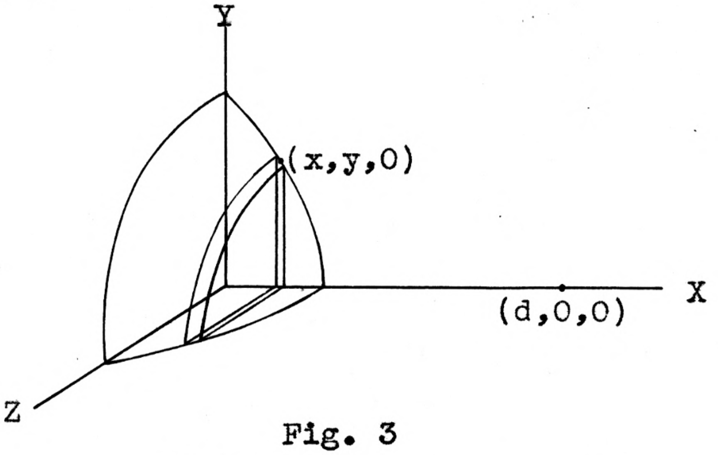

Fig. 3

$$V = 2\pi \int_0^a \left[\sqrt{y^2+(d-x)^2} - (d-x)\right] dx. \qquad (12)$$

Upon integration

$$V = \pi \left[2a^2-2ad-d^2+(2a+d)\sqrt{4a^2+d^2} + 4ad \log \frac{2a+d-\sqrt{4a^2+d^2}}{2a} \right]. \qquad (13)$$

Since d is any point on the x-axis, let d = x, and introduce a new constant c which is equal to 2a, then

$$V = \pi \left[\frac{c^2}{2} - cx - x^2 + (c+x)\sqrt{c^2+x^2} + 2cx \log \frac{c+x-\sqrt{c^2+x^2}}{c} \right]. \qquad (14)$$

This value of V must be expanded into an infinite series. By the binomial theorem

$$(c+x)\sqrt{c^2+x^2} = (c+x)x\left[1-(c/x)^2\right]^{\frac{1}{2}}$$

$$= x^2+cx+\frac{c^2}{2}+\frac{1}{2}\frac{c^3}{x}\frac{1\cdot 1}{2\cdot 4}-\frac{c^4}{x^2}-\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}-\frac{c^5}{x^3}+\text{----}.$$

In order to expand $\log(c+x-\sqrt{c^2+x^2})$, we differentiated and wrote it in the form

$$\frac{1}{2}\left[\frac{1}{x}+\frac{c}{x^2}(1-\frac{c^2}{x^2})^{-\frac{1}{2}}-\frac{1}{x}(1-\frac{c^2}{x^2})^{-\frac{1}{2}}\right].$$

After expansion and integration

$$2cx\log(c-x-\sqrt{c^2-x^2}) = -c^2-\frac{1\cdot 1}{2\cdot 2}\frac{c^3}{x}+\frac{1\cdot 1}{2\cdot 3}\frac{c^4}{x^2}+\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 4}\frac{c^5}{x^3}$$

$$-\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 5}\frac{c^6}{x^4}-\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}\frac{c^7}{x^5}+\text{----}kcx,$$

where k is the constant of integration. The sum of these series is

$$V = \pi\left[kcx-2cx\log c+\frac{1\cdot 1}{2\cdot 2}\frac{c^3}{x}+\frac{1\cdot 1}{2\cdot 3\cdot 4}\frac{c^4}{x^2}-\frac{1\cdot 1}{2\cdot 4\cdot 4}\frac{c^5}{x^3}-\text{--}\right]. \quad (15)$$

In order to evaluate the constant of integration k, we divided both members of (15) by πx and found the following limits:

$$\lim_{x\to\infty}\frac{V}{\pi x} = \lim_{x\to\infty}\left[kc-2c\log c+\frac{1\cdot 1}{2\cdot 2}\frac{c^3}{x^2}+\frac{1\cdot 1}{2\cdot 3\cdot 4}\frac{c^4}{x^3}-\text{---}\right].$$

The potential is always finite and approaches zero as $x\to\infty$; therefore, the limit of the left member is zero. The limit of the right member is easily seen to be $kc - 2c\log c$; therefore,

$$kc - 2c\log c = 0,$$

or $$k = 2 \log c.$$

After substituting this value of k in (15), the potential function for points on the x-axis became

$$V = \pi Kc^2 \left[\frac{1\cdot1}{2\cdot2}\frac{c}{x} + \frac{1\cdot1}{2\cdot3\cdot4}\frac{c^2}{x^2} - \frac{1\cdot1}{2\cdot4\cdot4}\frac{c^3}{x^3} - \frac{1\cdot1\cdot3}{2\cdot4\cdot5\cdot6}\frac{c^4}{x^4} + \frac{1\cdot1\cdot3}{2\cdot4\cdot6\cdot6}\frac{c^5}{x^5} + \frac{1\cdot1\cdot3\cdot5}{2\cdot4\cdot6\cdot7\cdot8}\frac{c^6}{x^6} - \text{----} \right] \tag{16}$$

when $x > c$ or $x > 2a$. And finally from (9) and (10)

$$V = \pi Kc^2 \left[\frac{1\cdot1}{2\ 2}\frac{c}{r} P_0(\cos\theta) + \frac{1\cdot1}{2\cdot3\cdot4}\frac{c^2}{r^2} P_1(\cos\theta) - \frac{1\cdot1}{2\cdot4\cdot4}\frac{c^3}{r^3} P_2(\cos\theta) - \frac{1\cdot1\cdot3}{2\cdot4\cdot5\cdot6}\frac{c^4}{r^4} P_3(\cos\theta) + \text{----} \right] \tag{17}$$

when $r > 2a$ and $\theta < \pi/2$.

<u>Potential Inside the Solid.</u> The potential on the x-axis due to a disc (11) was used in a similar manner as in the preceding section. In Fig. 4 the solid is divided into two parts by the plane $x = d$. The total potential is the sum of the potentials at the point due to each part separately; therefore,

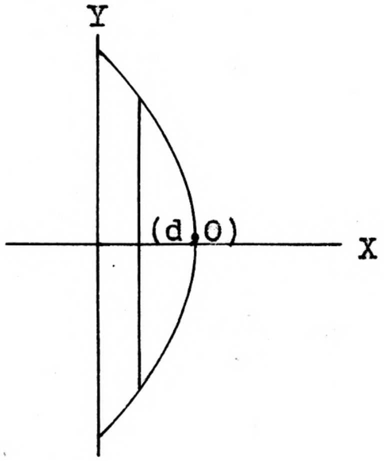

Fig. 4

$$V = 2\pi K \left\{ \int_0^d \left[\sqrt{y^2+(d-x)^2} - (d-x) \right] dx + \right.$$

$$+ \int_d^a \left[\sqrt{y^2+(x-d)^2} - (x-d) \right] \Bigg\} dx$$

where $y^2 = 4a(a-x)$.

Upon integration we have

$$V = \pi K \left[2ad-2a^2-d^2+(2a+d)\sqrt{4a^2+d^2} - 4ad \log \frac{2a+d+\sqrt{4a^2+d^2}}{2a} \right].$$

Using the same notation as in equation (14), we have

$$V = \pi K \left[cx- \frac{c^2}{2} -x^2+(c+x)\sqrt{c^2+x^2} -2cx \log \frac{c+x+\sqrt{c^2+x^2}}{c} \right]. \qquad (18)$$

The operation of expanding this function into a series was exactly the same as in the preceding section, except the method of evaluating the constant of integration k. The potential function before k was evaluated was

$$V = \pi K \left[c^2/2+2cx-2ckx+2cx \log c - 3/2\, x^2 + \frac{1}{2\cdot 2}\frac{x^3}{c} \right.$$

$$\left. + \frac{1\cdot 1}{2\cdot 3\cdot 4}\frac{x^4}{c^2} - \frac{1\cdot 1}{2\cdot 4\cdot 4}\frac{x^5}{c^3} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 5\cdot 6}\frac{x^6}{c^4} + \text{----} \right]. \qquad (19)$$

To evaluate k we found the following limits:

$$\lim_{x\to 0} \left[\frac{V/\pi K - c^2/2}{x} \right] = \lim_{x\to 0} \left[2c-2ck+2c \log c - 3/2\, x + \text{--} \right]. \qquad (20)$$

From (19) it is easily seen that $V = \pi K\, c^2/2$ when $x = 0$. Therefore, the left member of (20) is in the form of 0/0 when $x\to 0$. This is indeterminate so we differentiated the numerator and denominator with respect to x. dV/dx was taken from (18) and the left member of (20) became

$$\lim_{x\to 0} \left[c-2x+(c+x) \frac{x}{\sqrt{c^2+x^2}} + \sqrt{c^2+x^2} -2cx \left(\frac{1+x/\sqrt{c^2+x^2}}{c+x+\sqrt{c^2+x^2}} \right) - \right.$$

$$- 2c \log \frac{c+x+\sqrt{c^2+x^2}}{c} \Big]$$

or 2c-2c·log 2, and the limit of the right member is easily seen to 2c-2ck - 2c log c, and k = log 2c. After substituting this value of k in (19) the potential function for a point on the x-axis inside the solid became

$$V = \pi Kc^2 \left[\frac{1}{2} + 2(1-\log 2)\frac{x}{c} - \frac{3}{2}\frac{x^2}{c^2} + \frac{1\cdot 1}{2\cdot 2}\frac{x^3}{c^3} + \frac{1\cdot 1}{2\cdot 3\cdot 4}\frac{x^4}{c^4} - \frac{1\cdot 1}{2\cdot 4\cdot 4}\frac{r^5}{c^5} - \cdots\right] \quad (21)$$

when $x < a$. Finally from (7) and (8)

$$V = \pi Kc^2 \Big[\tfrac{1}{2}P_0(\cos\theta) + 2(1-\log 2)\, r/c\, P_1(\cos\theta) - \frac{3}{2}\frac{r^2}{c^2}P_2(\cos\theta) + \frac{1\cdot 1}{2\cdot 2}\frac{r^3}{c^3}P_3(\cos\theta) + \frac{1\cdot 1}{2\cdot 3\cdot 4}\frac{r^4}{c^4}P_4(\cos\theta) - \cdots\Big] \quad (22)$$

where $0 < r < a$, and $\theta < \pi/2$.

The series given in (16) is convergent for $x > 2a$ and the series in (21) is convergent for $x < a$. This leaves a region $a < x < 2a$ for which the potential is not given by (16) and (21).

<u>Potential in the Zone, $a < r < 2a$.</u> The value of V as given in (14) is

$$V = \pi K \left[c^2/2 - cx - x^2 + (c+x)\sqrt{c^2+x^2} + 2cx \log \frac{c+x-\sqrt{c^2+x^2}}{c}\right].$$

This may be expanded as follows:

$$\log \frac{c+x-\sqrt{c^2+x^2}}{c} = \log \frac{2x}{c+x+\sqrt{c^2+x^2}}$$

$$= \log 2x - \log (c+x+\sqrt{c^2+x^2})$$

$$= \log \frac{2x}{c+x} - \log (1+\frac{\sqrt{c^2+x^2}}{c+x}). \qquad (23)$$

Nor $\log (1+ \frac{\sqrt{c^2+x^2}}{c+x})$ may be expanded by the well known expression $\log (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ----$, when x 1, giving

$$\log (1+ \frac{\sqrt{c^2+x^2}}{c+x}) = \frac{\sqrt{c^2-x^2}}{c-x} - \frac{c^2+x^2}{2(c+x)^2} + \frac{(c^2+x^2)^{3/2}}{3(c+x)^3} - ----.$$

This is convergent for all positive values of x since $\sqrt{c^2+x^2} < c+x$. Finally,

$$V = \pi K \left\{ c^2/2 - cx - x^2 + (c+x)\sqrt{c^2+x^2} + 2cx \log 2x/(c+x) \right.$$
$$\left. -2cx \left[\frac{\sqrt{c^2+x^2}}{c+x} - \frac{c^2+x^2}{2(c+x)^2} + \frac{(c^2+x^2)^{3/2}}{3(c+x)^3} - ---- \right] \right\} \qquad (24)$$

when $x > 0$.

According to the limits of convergence (24) should give the potential at any point on the positive x-axis; however, because of the method of derivation of (16) it will only be expected to be true for $x > a$. For a further test let x = a in (24) and (14), and compare results. When x = a in (14) it becomes

$$V = \pi K a^2 (1.858492).$$

When x = a in (24) the sum of the first 17 terms of the series is $V = \pi K a^2(1.857004)$ and the sum of the first 18 terms is $V = \pi K a^2(1.859272)$.

When x = 2a in (14) it becomes $V = \pi K a^2(1.035240)$.

When $x = 2a$ in (24) the sum of the first 13 terms of the series is $V = \pi Ka^2(1.032619)$, and the sum of the first 14 terms is $V = \pi Ka^2(1.037083)$. We concluded that the series converges to the correct value in both cases.

II. THE SURFACE OF REVOLUTION

The potential at d due to the element of arc ds in

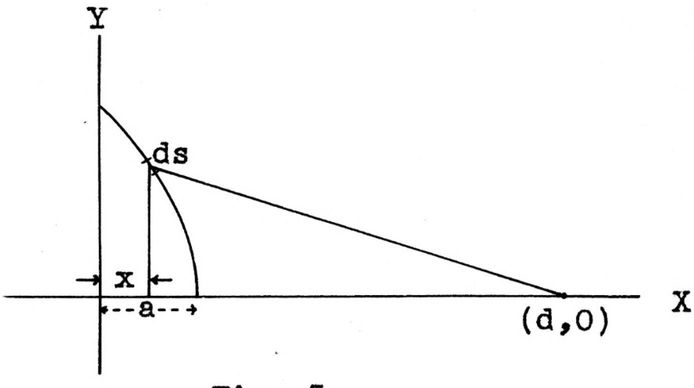

Fig. 5

Fig. 5 is

$$V = K\int_0^a \frac{ds}{\sqrt{y^2+(d-x)^2}} \text{;}$$

therefore, the potential at d due to the ring formed by rotating ds about the x-axis is

$$V = K\int_0^a \frac{2\pi y\ ds}{\sqrt{y^2+(d-x)^2}}$$

since all points on the ring are equidistant from (d,0). By substituting from equations $y^2 = 4a(a-x)$ and ds =

$\frac{2\sqrt{a}\ \sqrt{2a-x}\ dx}{y}$, V became

$$V - 4\sqrt{a}\pi K \int_0^a \frac{\sqrt{2a-x}\ dx}{\sqrt{4a(a-x)+(c-x)^2}}.$$

By substituting $2a-x = u^2$, the integral was reduced to

$$V = 8\sqrt{a}\pi K \int_{\sqrt{a}}^{\sqrt{2a}} \frac{u^2 du}{\sqrt{(u^2+c)^2-4ac}}.$$

In order to integrate, we expanded the integrand into an infinite series, thus

$$\frac{u^2}{\sqrt{(u^2+c)^2-4ac}} = \frac{u^2}{u^2+c}\left[1-\frac{4ac}{(u^2+c)^2}\right]^{-\frac{1}{2}}$$

$$= \frac{u^2}{u^2+c} + \sum_{m-3}^{\infty} \frac{1\cdot3\cdot5\cdots(m-2)}{2\cdot4\cdot6\cdots(m-1)}\,(4ac)^{\frac{m-1}{3}}\,\frac{u^2}{(u^2+c)^m}$$

where m = 3,5,7,----. Then

$$V = 8\sqrt{a}\pi K \int_{\sqrt{a}}^{\sqrt{2a}} \left[\frac{u^2}{u^2+c} + \sum_{m-3}^{\infty} \frac{1\cdot3\cdot5\cdots(m-2)}{2\cdot4\cdot6\cdots(m-1)}\,\frac{u^2}{(u^2+c)^m}\right] du. \quad (25)$$

We integrated the above series by the use of integral tables for m = 3,5,7, and 9, and collected the like terms. Since (d,0) was any point on the x-axis, we replaced d by x making the potential function a function of x.

In the following equations, let

$$Q = \tan^{-1}\sqrt{\frac{2a}{d}} - \tan^{-1}\sqrt{\frac{a}{d}}$$

and

$$R^n = \frac{\sqrt{2a}}{(2a+d)^n} - \frac{\sqrt{a}}{(a+d)^n}.$$

$$\int_{\sqrt{a}}^{\sqrt{2a}} \frac{u^2du}{(u^2+d)} = \sqrt{2a} - \sqrt{a} - \sqrt{d}\, Q$$

$$\frac{4ad}{2}\int_{\sqrt{a}}^{\sqrt{2a}} \frac{u^2du}{(u^2+d)^3} = \frac{4ad}{2}\left[\frac{1}{8}\,\frac{1}{d^{3/2}}\cdot Q + \frac{1}{8d}R^1 - \frac{1}{4}R^2\right]$$

$$\frac{1\cdot3}{2\cdot4}\,2^4a^2d^2\int_{\sqrt{a}}^{\sqrt{2a}} \frac{u^2du}{(u^2+d)^5} = \frac{1\cdot3}{2\cdot4}\,2^4a^2d^2\left[\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}\,\frac{1}{d^{7/2}}\,Q + \frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}\,\frac{R^1}{d^3} + \frac{5}{4\cdot6\cdot8}\,\frac{R^2}{d^2} + \frac{1}{6\cdot8}\,\frac{R^3}{d} - \frac{1}{8}R^4\right]$$

$$\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,2^6a^3d^3\int_{\sqrt{a}}^{\sqrt{2a}} \frac{u^2du}{(u^2+d)^7} = \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,2^6a^3d^3\left[\frac{3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot10\cdot12}\,\frac{1}{d^{11/2}}\,Q + \frac{3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot10\cdot12}\,\frac{R^1}{d^5} + \frac{5\cdot7\cdot9}{4\cdot6\cdot8\cdot10\cdot12}\,\frac{R^2}{d^4} + \frac{7\cdot9}{6\cdot8\cdot10\cdot12}\,\frac{R^3}{d^3} + \frac{9}{8\cdot10\cdot12}\,\frac{R^4}{d^2} + \frac{1}{10\cdot12}\,\frac{R^5}{d} - \frac{1}{12}R^6\right]$$

$$\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\,2^8a^4d^4\int_{\sqrt{a}}^{\sqrt{2a}} \frac{u^2du}{(u^2+d)^9} = \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\,2^8a^4d^4\left[\frac{3\cdot5\cdot7\cdot9\cdot11\cdot13}{2\cdot4\cdot6\cdot8\cdot10\cdot12\cdot14\cdot16}\,\frac{1}{d^{15/2}}\,Q + \frac{3\cdot5\cdot7\cdot9\cdot11\cdot13}{2\cdot4\cdot6\cdot8\cdot10\cdot12\cdot14\cdot16}\,\frac{R^1}{d^7} + \frac{5\cdot7\cdot9\cdot11\cdot13}{4\cdot6\cdot8\cdot10\cdot12\cdot14\cdot16}\,\frac{R^2}{d^6} + \frac{7\cdot9\cdot11\cdot13}{6\cdot8\cdot10\cdot12\cdot14\cdot16}\,\frac{R^3}{d^5} + \frac{9\cdot11\cdot13}{8\cdot10\cdot12\cdot14\cdot16}\,\frac{R^4}{d^4} + \frac{11\cdot13}{10\cdot12\cdot14\cdot16}\,\frac{R^5}{d^3} + \frac{13}{12\cdot14\cdot16}\,\frac{R^6}{d^2} + \frac{1}{14\cdot16}\,\frac{R^7}{d} - \frac{R^8}{18}\right].$$

Collecting all the like terms,

$$\sqrt{2a} - \sqrt{a} - \sqrt{d}\left[Q - \sqrt{d} + \frac{1}{2}\,\frac{1}{2\cdot4}\,2^2\,\frac{a}{d} + \frac{1\cdot3}{2\cdot4}\,\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{2^4}{8}\,\frac{a^2}{d^2} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot10\cdot12}\,2^6\,\frac{a^3}{d^3} + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\,\frac{3\cdot5\cdot7\cdot9\cdot11\cdot13}{2\cdot4\cdot6\cdot8\cdot10\cdot12\cdot14\cdot16}\,2^8\,\frac{a^4}{d^4} + ----\right]$$

$$R^1\left[\frac{1}{2}\,2^2\,\frac{1}{2\cdot4}\,a + \frac{1\cdot3}{2\cdot4}\,\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}\,2^4\,\frac{a^2}{d} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot10\cdot12}\,2^6\,\frac{a^3}{d^2} + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\,\frac{3\cdot5\cdot7\cdot9\cdot11\cdot13}{2\cdot4\cdot6\cdot8\cdot10\cdot12\cdot14\cdot16}\,2^8\,\frac{a^4}{d^3} + ----\right]$$

$$R^2\left[-\frac{1}{2}\,2^2\,\frac{1}{4}\,ad + \frac{1\cdot3}{2\cdot4}\,\frac{5}{4\cdot6\cdot8}\,2^4\,a^2 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{5\cdot7\cdot9}{4\cdot6\cdot8\cdot10\cdot12}\,2^6\,\frac{a^3}{d} + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\,\frac{5\cdot7\cdot9\cdot11\cdot13}{4\cdot6\cdot8\cdot10\cdot12\cdot14\cdot16}\,2^8\,\frac{a^4}{d^2} + ----\right]$$

$$R^3\left[\qquad + \frac{1\cdot3}{2\cdot4}\,\frac{1}{6\cdot8}\,2^4\,a^2d + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{7\cdot9}{6\cdot8\cdot10\cdot12}\,2^6\,a^3 + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\,\frac{7\cdot9\cdot11\cdot13}{6\cdot8\cdot10\cdot12\cdot14\cdot16}\,2^8\,\frac{a^4}{d} + ----\right] \tag{27}$$

$$R^4\left[\qquad - \frac{1\cdot3}{2\cdot4}\,\frac{1}{8}\,2^4\,a^2d^2 + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{9}{8\cdot10\cdot12}\,2^6a^3d + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\,\frac{9\cdot11\cdot13}{8\cdot10\cdot12\cdot14\cdot16}\,2^8a^4 + ----\right]$$

$$R^5\left[\qquad + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{1}{10\cdot12}\,2^6\,a^3d^2 + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\,\frac{11\cdot13}{10\cdot12\cdot14\cdot16}\,2^8\,a^4d + ----\right]$$

$$R^6\left[\qquad - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{1}{12}\,2^6\,a^3d^3 + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\,\frac{13}{12\cdot14\cdot16}\,2^8\,a^4d^2 + ----\right]$$

The resulting series were written in the following general form:

$$V = 8a\,\pi K\,(\sqrt{2}-1) + 8\pi K \Bigg\{ Q\,\sqrt{ax}\left[-1 + \sum_{m=3}^{\infty} \frac{1\cdot3\cdot5\cdots(m-2)}{2\cdot4\cdot6\cdots(m-1)}\,\frac{1\cdot3\cdot5\cdots(2m-5)}{2\cdot4\cdot6\cdots(2m-2)}\,2^{m-1}\left(\frac{a}{x}\right)^{\frac{m-1}{2}}\right] + R^{m+n-3}\left[\sum_{m=3}^{\infty}\sum_{n=1}^{n=2}(-1)^{n+1}\,\frac{1\cdot3\cdot5\cdots(m-2)}{2\cdot4\cdot6\cdots(m-1)}\,\frac{2^{m-1}\,a^{m/2}\,x^{\frac{m+2n-5}{2}}}{(2n+2m+6)\ldots(2m-4)(2m-2)}\right]$$

$$+ R^{n}\left[\sum_{m=5}^{\infty}\sum_{n=1}^{n=m-3}\frac{1\cdot3\cdot5\cdots(m-2)}{2\cdot4\cdot6\cdots(m-1)}\,\frac{(2n+1)(2n+3)\cdots(2m-7)(2m-5)}{2n(2n+2)\cdots\cdots(2m-4)(2m-2)}\,2^{m-1}\,a^{m/2}\,x^{\frac{2n-m+1}{2}}\right]\Bigg\}. \tag{28}$$

The first double summation contains all the leading terms of the series not including the first which is the coefficient of Q. The second double summation contains all the remaining terms in the doubly infinite series.

III. POTENTIAL ON THE X-AXIS DUE TO THE ARC

The equation in polar coordinates for the parabola $y^2 = 4a(a-x)$ is $r = a \sec^2 \theta/2$. In Fig. 6

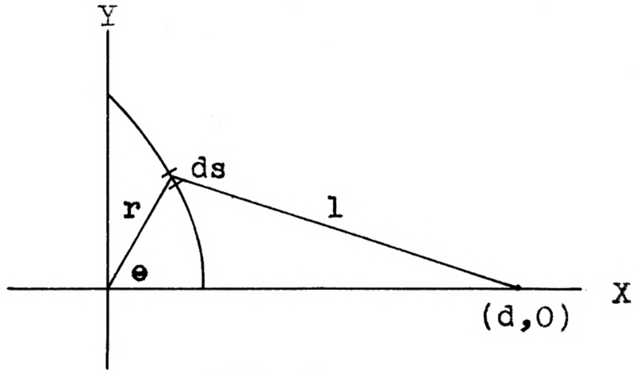

Fig. 6

$$ds = a \sec^3 \theta/2 \, d\theta,$$

and
$$l^2 = r^2 + d^2 - 2rd \cos\theta,$$

also,
$$l = \sqrt{(a \sec^2 \theta/2 + d)^2 - 4ad} \;;$$

therefore, the potential at (d,0) is

$$V = K \int_0^{\pi/2} \frac{a \sec^3 \theta/2 \, d\theta}{\sqrt{(a \sec^2 \theta/2 + d)^2 - 4ad}} \,.$$

Let $\sec \theta/2 = \sqrt{dx/a}$; then $d\theta = \sqrt{a}\, dx/x\sqrt{dx-a}$ and also let $a/d = k$ then

$$V = K \int_k^{2k} \frac{dx}{(x+1)\sqrt{1-k/x}\,\sqrt{1-4k/(x+1)^2}} \,. \tag{29}$$

By the binomial theorem

$$(1- k/x)^{-\frac{1}{2}} = 1 + \sum_{m=1}^{\infty} \frac{1\cdot3\cdot5\cdots(2m-1)}{2\cdot4\cdot6\cdots(2m)} (k/x)^m$$

and $$\frac{1}{x+1}\left[1- \frac{4k}{(x+1)}\right]^{-\frac{1}{2}} = \frac{1}{x+1} + \sum_{n=3}^{\infty} \frac{1\cdot3\cdot5\cdots(n-2)}{2\cdot4\cdot6\cdots(n-1)} \frac{(4k)^{\frac{n-1}{2}}}{(x+1)^n}$$

where $m = 1,2,3,$----, and $n = 3,5,7,$----.

After the two series were multiplied together (29) became

$$V = K\left[\int_k^{2k} \frac{dx}{x+1} + \sum_{n=3}^{\infty} \frac{1\cdot3\cdot5\cdots(n-2)}{2\cdot4\cdot6\cdots(n-1)} (4k)^{\frac{n-1}{2}} \int_k^{2k} \frac{dx}{(x+1)^n}\right.$$

$$+ \sum_{m=1}^{\infty} \frac{1\cdot3\cdot5\cdots(2m-1)}{2\cdot4\cdot6\cdots(2m)} k^m \int_k^{2k} \frac{dx}{x^m(x+1)}$$

$$+ \sum_{m=1}^{\infty}\sum_{n=3}^{\infty} \frac{1\cdot3\cdot5\cdots(2m-1)}{2\cdot4\cdot6\cdots(2m)} \frac{1\cdot3\cdot5\cdots(n-2)}{2\cdot4\cdot6\cdots(n-1)} 2^{n-1} k^{\frac{n-2m-1}{2}}$$

$$\left.\cdot\int_k^{2k} \frac{dx}{x^m(x+1)^n}\right]. \tag{30}$$

The first two terms were easily integrated, and the last two were taken from the integral tables. We substituted the value a/d back in place of k, and substituted x for d, because d is any point on the x-axis. Letting $R^n =$

$$\left(\frac{2a+x}{2a}\right)^n - \left(\frac{a+x}{a}\right)^n,$$

the complete function is

$$V = K\left\{ \log \frac{2a+x}{a+x}\right.$$

$$+ \sum_{n=3}^{\infty} \frac{1\cdot3\cdot5\cdots(n-2)}{2\cdot4\cdot6\cdots(n-1)} \frac{1}{n-1} \left(\frac{a}{x}\right)^{\frac{n-1}{2}} \left[\left(\frac{2x}{a+x}\right)^{n-1} - \left(\frac{2x}{2a+x}\right)^{n-1}\right] +$$

$$+\sum_{m=1}^{\infty}\sum_{s=o}^{m-2}\frac{1\cdot3\cdot5\cdots(2m-1)}{2\cdot4\cdot6\cdots(2m)}\;\frac{(m-1)!(-1)^{s-1}}{(m-s-1)!s!(m-s-1)}\left(\frac{a}{x}\right)^{m}R^{m-s-1}$$

$$\cdot(-1)^{m+1}\log\frac{2a+2x}{2a+x}$$

$$+\sum_{m=1}^{\infty}\sum_{n=3}^{\infty}\frac{1\cdot3\cdot5\cdots(2m-1)}{2\cdot4\cdot6\cdots(2m)}\;\frac{1\cdot3\cdot5\cdots(n-2)}{2\cdot4\cdot6\cdots(n-1)}\;2^{n-1}\left(\frac{a}{x}\right)^{\frac{n+2m-1}{2}}$$

$$\cdot\left[\sum_{s=o}^{m+n-2}\frac{(m+n-2)!(-1)^{s-1}}{(m+n-s-2)!s!(m-s-1)}R^{m-s-1}\right]\Bigg\} \qquad (31)$$

when $m-s-1 \neq 0$, and when $m-s-1 = 0$ the quantity in the last square bracket becomes

$$\left[\frac{(m+n-2)!}{(m-1)!(n-1)!}\;(-1)^{m-1}\log\frac{2a+2x}{2a+x}\right]$$

where $x > a$.

IV. POTENTIAL ON THE Z-AXIS DUE TO THE ARC

In Fig. 7 the element of arc ds in rectangular coordinates was found to be

$$ds = \sqrt{\frac{2a-x}{a-x}}\,dx.$$

It is easily seen that the distance l is given by $l = \sqrt{d^2+x^2+y^2}$; therefore, the potential at (0,0,d) is

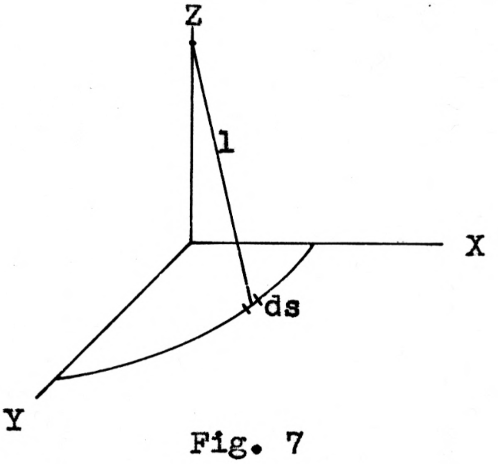

Fig. 7

$$V = K\int_0^a \frac{\sqrt{(2a-x)/(a-x)}}{\sqrt{d^2+x^2+y^2}}\,dx.$$

Where $y^2 = 4a(a-x)$

$$V = K\int_0^a \frac{\sqrt{2a-x}}{\sqrt{a-x}\;\sqrt{d^2+(2a-x)^2}}\,. \tag{32}$$

By making the substitution $a-x = u^2$, (32) became

$$V = \frac{2K\sqrt{a}}{d}\int_0^{\sqrt{a}} \frac{\sqrt{1+u^2/a}}{\sqrt{1+(u^2+a)^2/d^2}}\,du. \tag{33}$$

By the binomial theorem

$$(1+u^2/a)^{\frac{1}{2}} = 1+\tfrac{1}{2}u^2/a + \sum_{n=4}^{\infty}(-1)^{\frac{n-2}{2}}\frac{1\cdot1\cdot3\cdots(n-3)}{2\cdot4\cdot6\cdots(n)}\,\frac{1}{a^{n/2}}\,u^n$$

$$\left[1+\frac{(u^2+a)^2}{d^2}\right]^{-\frac{1}{2}} = 1+\sum_{m=2}^{\infty}(-1)^{m/2}\frac{1\cdot3\cdot5\cdots(m-1)}{2\cdot4\cdot6\cdots(m)}\,\frac{1}{d^m}\,(u^2+a)^m$$

where $m = 2,4,6,$----, and $n = 4,6,8,$----.

After we multiplied the two series together (33) became

$$V = \frac{2K\sqrt{a}}{d}\Bigg[\int_0^{\sqrt{a}} du$$

$$+\frac{1}{2a}\int_0^{\sqrt{a}} u^2du + \sum_{n=4}^{\infty}(-1)^{\frac{n-2}{2}}\frac{1\cdot1\cdot3\cdots(n-3)}{2\cdot4\cdot6\cdots(n)}\,\frac{1}{a^{n/2}}\int_0^{\sqrt{a}} u^n du$$

$$+\sum_{m=2}^{\infty}(-1)^{m/2}\frac{1\cdot3\cdot5\cdots(m-1)}{2\cdot4\cdot6\cdots(m)}\,\frac{1}{d^m}\int_0^{\sqrt{a}}(u^2+a)^m du \tag{34}$$

$$+\sum_{m=2}^{\infty}(-1)^{m/2}\frac{1\cdot3\cdot5\cdots(m-1)}{2\cdot4\cdot6\cdots(m)}\,\frac{1}{2ad^m}\int_0^{\sqrt{a}} u^2(u^2+a)^m\,du$$

$$+\sum_{n=4}^{\infty}\sum_{m=2}^{\infty}(-1)^{m-1}\frac{1\cdot1\cdot3\cdot5\cdots(n-3)}{2\cdot4\cdot6\cdots\cdots(n)}\;\frac{1\cdot3\cdot5\cdots(m-1)}{2\cdot4\cdot6\cdots(m)}\;\frac{1}{a^{n/2}d^m}\cdot\int_0^{\sqrt{a}}u^n(u^2+a)^m\,du\Bigg].$$

We expanded the integrands of the last three terms by the binomial theorem, and integrated term by term.

$$\int_0^{\sqrt{a}}du = \sqrt{a}$$

$$\int_0^{\sqrt{a}}u^2du = a\sqrt{a}/3$$

$$\int_0^{\sqrt{a}}u^n du = \left[\frac{u^{n+1}}{n+1}\right]_0^{\sqrt{a}} = \frac{a^{\frac{n+1}{2}}}{n+1}$$

$$\int_0^{\sqrt{a}}(u^2+a)^m\,du = \int_0^{\sqrt{a}}\sum_{s=0}^{m}\frac{m!\;u^{2m-2s}\;a^s}{(m-s)!\;s!}\,du$$

$$= \left[\sum_{s=0}^{m}\frac{m!\;u^{2m-2s-1}\;a^s}{(m-s)!\;s!\;(2m-2s-1)}\right]_0^{\sqrt{a}}$$

$$= \sum_{s=0}^{m}\frac{m!\;a^{m+\frac{1}{2}}}{(m-s)!\;s!\;(2m-2s+1)}$$

$$\int_0^{\sqrt{a}}u^2(u^2+a)^m\,du = \sum_{s=0}^{m}\frac{m!\;a^{m+3/2}}{(m-s)!\;s!\;(2m-2s+3)}$$

$$\int_0^{\sqrt{a}}u^n(u^2+a)^m\,du = \sum_{s=0}^{m}\frac{m!\;a^{\frac{m+n+1}{2}}}{(m-s)!\;s!\;(2m+n-2s+1)}. \tag{35}$$

When we multiplied each of these terms by the corresponding coefficient given in (34), we obtained

$$V = \frac{2K\sqrt{a}}{d}\left[\sqrt{a} + \frac{\sqrt{a}}{6} + \sum_{n=4}^{\infty}(-1)^{\frac{n-2}{2}}\frac{1\cdot1\cdot3\cdot5\cdots(n-3)}{2\cdot4\cdot6\cdots\cdots(n)}\,\frac{a}{n+1}\right.$$

$$+\sum_{m=2}^{\infty}(-1)^{m/2}\frac{1\cdot3\cdot5\cdots(m-1)}{2\cdot4\cdot6\cdots(m)}\,\frac{a^{m+\frac{1}{2}}}{d^m}\sum_{s=0}^{m}\frac{m!}{(m-s)!\;s!\;(2m-2s+1)}$$

$$+\sum_{m=2}^{\infty}(-1)^{m/2}\frac{1\cdot3\cdot5\cdots(m-1)}{2\cdot4\cdot6\cdots(m)}\,\frac{a^{m+\frac{1}{2}}}{2d^m}\sum_{s=0}^{m}\frac{m!}{(m-s)!\;s!\;(2m-2s+3)}$$

$$+\sum_{n=4}^{\infty}\sum_{m=2}^{\infty}(-1)^{m-1}\frac{1\cdot1\cdot3\cdot5\cdots(n-3)}{2\cdot4\cdot6\cdot8\cdots(n)}\,\frac{1\cdot3\cdot5\cdots(m-1)}{2\cdot4\cdot6\cdots(m)}\,\frac{a^{m+\frac{1}{2}}}{d^m}$$

$$\left.\cdot\sum_{s=0}^{m}\frac{m!}{(m-s)!\;s!\;(2m+n-2s+1)}\right]. \quad (36)$$

Finally, when we substituted z for d, we obatined the following complete function:

$$V = \left\{K\,\frac{a}{z}\left(\frac{7}{3} + \sum_{n=4}^{\infty}(-1)^{\frac{n-2}{2}}\frac{1\cdot3\cdot5\cdots(n-3)}{4\cdot6\cdot8\cdots(n)}\,\frac{1}{n+1}\right)\right.$$

$$+\left(\frac{a}{z}\right)^{m+1}\left[2\sum_{m=2}^{\infty}(-1)^{m/2}\frac{1\cdot3\cdot5\cdots(m-3)}{2\cdot4\cdot6\cdots(m)}\sum_{s=0}^{m}\frac{m!}{(m-s)!\,s!\,(2m-2s+1)}\right.$$

$$+\sum_{m=2}^{\infty}(-1)^{m/2}\frac{1\cdot3\cdot5\cdots(m-1)}{2\cdot4\cdot6\cdots(m)}\sum_{s=0}^{m}\frac{m!}{(m-s)!\;s!\;(2m-2s+3)}$$

$$+\sum_{n=4}^{\infty}\sum_{m=2}^{\infty}(-1)^{m-1}\frac{1\cdot3\cdot5\cdots(n-3)}{4\cdot6\cdot8\cdots(n)}\,\frac{1\cdot3\cdot5\cdots(m-1)}{2\cdot4\cdot6\cdots(m)}$$

$$\left.\left.\cdot\sum_{s=0}^{m}\frac{m!}{(m-s)!\;s!\;(2m+n-2s+1)}\right]\right\}. \qquad (37)$$

V. POTENTIAL AT POINTS ON THE X-AXIS DUE TO THE AREA

In Fig. 8 the element of mass dm is equal to dx dy,

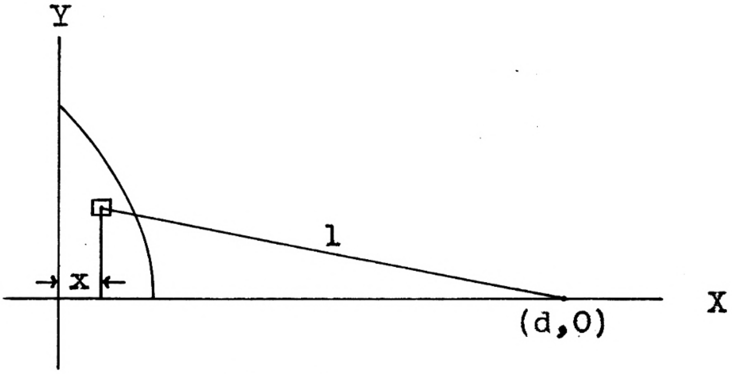

Fig. 8

and the distance $l = \sqrt{y^2+(d-x)^2}$; therefore, the potential at d is

$$V = K \int_0^a \int_0^{\sqrt{4a(a-x)}} \frac{dx\, dy}{\sqrt{y^2+(d-x)^2}} .$$

The constant K will be omitted temporarily as it can be replaced whenever necessary.

$$V = \int_0^a \left[\log\ (y^2+\sqrt{y^2+(d-x)^2})\right]_0^{\sqrt{4a(a-x)}} dx$$

$$= \int_0^a \log\left[\sqrt{4a(a-x)} + \sqrt{4a(a-x)+(d-x)^2}\right] dx - \int_0^a \log(d-x)dx$$

$$= \int_0^a \log\left[1+ \frac{\sqrt{4a(a-x)}}{\sqrt{4a(a-x)+(d-x)^2}}\right] dx$$

$$- \int_0^a \sqrt{\log\ 4a(a-x)+(d-x)^2}\ dx - \int_0^a \log(d-x)\ dx. \qquad (38)$$

We integrated the last term by parts and obtained

$$-\int_0^a \log(d-x)\,dx = a+d \log \frac{d-a}{d} - a \log(d-a). \qquad (39)$$

We also integrated the second term by parts and obtained

$$\int_0^a \log \sqrt{4a(a-x)+(d-x)^2}\,dx = \frac{2a+d}{2} \log(4a^2+d^2) - d \log(d-a)$$

$$- a - \sqrt{ad} \log \left[\frac{d^2+ad-2a^2+2a\sqrt{ad}}{d^2+ad-2a^2-2a\sqrt{ad}} \right]. \qquad (40)$$

We expanded the first term into the following infinite series:

$$\int_0^a \log \left[1+ \sqrt{\frac{4a(a-x)}{4a(a-x)+(d-x)^2}} \right] dx = \int_0^a \left\{ \left[\frac{4a(a-x)}{4a(a-x)+(d-x)^2} \right]^{\frac{1}{2}} \right.$$

$$- \frac{1}{2} \left[\frac{4a(a-x)}{4a(a-x)+(d-x)^2} \right] + \frac{1}{3} \left[\frac{4a(a-x)}{4a(a-x)+(d-x)^2} \right]^{3/2} \qquad (41)$$

$$\left. - \frac{1}{4} \left[\frac{4a(a-x)}{4a(a-x)+(d-x)^2} \right]^2 + \frac{1}{5} \left[\frac{4a(a-x)}{4a(a-x)+(d-x)^2} \right]^{5/2} --- \right\} dx.$$

Each term of the integrand of (41) is in the form of

$$(-1)^{n+1} \frac{1}{n} \left[\frac{4a(a-x)}{4a(a-x)+(d-x)^2} \right]^{n/2}$$

where $n = 1,2,3,----$.

If n takes on all the odd values, each term is positive and may be written as

$$\int_0^a \sum_{n=1}^{\infty} \frac{1}{n} \frac{(2a^{\frac{1}{2}})^n \, (a-x)^{n/2}}{(d-x)^n} \left[1 + \frac{4a(a-x)}{(d-x)^2} \right]^{-n/2}. \qquad (42)$$

After expanding, the expression

$$\left[1 + \frac{4a(a-x)}{(d-x)^2}\right]^{-n/2}$$

and performing the multiplication, we have for n = 1,3,5,7, and 9 the following series which may be combined:

$$
\begin{aligned}
\frac{2\sqrt{a}\sqrt{a-x}}{(d-x)}\left[1+\frac{4a\,(a-x)}{(d-x)^2}\right]^{-\frac{1}{2}} &= \frac{2\sqrt{a}\sqrt{a-x}}{d-x} - \frac{(2a^{\frac{1}{2}})^3(a-x)^{3/2}}{2(d-x)^3} + \frac{1\cdot 3}{2\cdot 4}\frac{(2a^{\frac{1}{2}})^5(a-x)^{5/2}}{(d-x)^5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{(2a^{\frac{1}{2}})^7(a-x)^{7/2}}{(d-x)^7} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\frac{(2a^{\frac{1}{2}})^9(a-x)^{9/2}}{(d-x)^9} - \cdots \\
\frac{1}{3}\frac{(2a^{\frac{1}{2}})^3(a-x)^{3/2}}{(d-x)^3}\left[1+\frac{4a(a-x)}{(d-x)^2}\right]^{-3/2} &= +\frac{(2a^{\frac{1}{2}})^3(a-x)^{3/2}}{3(d-x)^3} - \frac{1\cdot 3}{2\cdot 3}\frac{(2a^{\frac{1}{2}})^5(a-x)^{5/2}}{(d-x)^5} + \frac{1\cdot 3\cdot 5}{2\cdot 3\cdot 4}\frac{(2a^{\frac{1}{2}})^7(a-x)^{7/2}}{(d-x)^7} - \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 3\cdot 4\cdot 6}\frac{(2a^{\frac{1}{2}})^9(a-x)^{9/2}}{(d-x)^9} + \cdots \\
\frac{1}{5}\frac{(2a^{\frac{1}{2}})^5(a-x)^{5/2}}{(d-x)^5}\left[1+\frac{4a(a-x)}{(d-x)^2}\right]^{-5/2} &= +\frac{1}{5}\frac{(2a^{\frac{1}{2}})^5(a-x)^{5/2}}{(d-x)^5} - \frac{1\cdot 5}{2\cdot 5}\frac{(2a^{\frac{1}{2}})^7(a-x)^{7/2}}{(d-x)^7} + \frac{5\cdot 7}{2\cdot 4\cdot 5}\frac{(2a^{\frac{1}{2}})^9(a-x)^{9/2}}{(d-x)^9} - \cdots \\
\frac{1}{7}\frac{(2a^{\frac{1}{2}})^7(a-x)^{7/2}}{(d-x)^7}\left[1+\frac{4a(a-x)}{(d-x)^2}\right]^{-7/2} &= +\frac{1}{7}\frac{(2a^{\frac{1}{2}})^7(a-x)^{7/2}}{(d-x)^7} - \frac{7}{2\cdot 7}\frac{(2a^{\frac{1}{2}})^9(a-x)^{9/2}}{(d-x)^9} + \cdots \\
\frac{1}{9}\frac{(2a^{\frac{1}{2}})^9(a-x)^{9/2}}{(d-x)^9}\left[1+\frac{4a(a-x)}{(d-x)^2}\right]^{-9/2} &= +\frac{1}{9}\frac{(2a^{\frac{1}{2}})^9(a-x)^{9/2}}{(d-x)^9} - \cdots
\end{aligned}
\tag{43}
$$

Combining the above series, we obtained:

$$
\frac{2\sqrt{a}\sqrt{a-x}}{d-x} - \frac{(2a^{\frac{1}{2}})^3(a-x)^{3/2}}{6(d-x)^3} + \frac{3}{40}\frac{(2a^{\frac{1}{2}})^5(a-x)^{5/2}}{(d-x)^5} - \frac{5}{112}\frac{(2a^{\frac{1}{2}})^7(a-x)^{7/2}}{(d-x)^7} + \frac{35}{1152}\frac{(2a^{\frac{1}{2}})^9(a-x)^{9/2}}{(d-x)^9} - \cdots
$$

Therefore (42) became

$$
2\sqrt{a}\int_0^a \frac{\sqrt{a-x}}{d-x} - \frac{2a}{3}\frac{(a-x)^{3/2}}{(d-x)^3} + \frac{6a^2}{5}\frac{(a-x)^{5/2}}{(d-x)^5} - \frac{20a^3}{7}\frac{(a-x)^{7/2}}{(d-x)^7} + \frac{70a^4}{9}\frac{(a-x)^{9/2}}{(d-x)^9} - \cdots \tag{44}
$$

In order to integrate terms of the form $[\sqrt{a-x}/(d-x)]^n$, the following substitutions were made: $a-x = t^2$, $dx = -2t\,dt$, $d-x = t^2+k^2$, where $k^2 = d-a$. After integration we substituted $d-a$ for k^2 and then the first 4 terms of (44) became

$$
\begin{aligned}
&2\sqrt{a}\int_0^a \frac{\sqrt{a-x}}{d-x}\,dx = 4\sqrt{a}\int_0^a \frac{t^2}{t^2+k^2}\,dt = 4a - 4\sqrt{ad-a^2}\,\tan^{-1}\sqrt{a/(d-a)} \\
&-\frac{4a^{3/2}}{3}\int_0^a \frac{(a-x)^{3/2}}{(d-x)^3}\,dx = -\frac{8a^{3/2}}{3}\int_0^a \frac{t^4}{(t^2+k^2)^3}\,dt = +\frac{2}{3}\frac{a^3}{d^2} - \frac{2}{3}\frac{a^2}{d} + \frac{5}{3}\frac{a^2}{d} - \frac{a^{3/2}}{\sqrt{d-a}}\tan^{-1}\sqrt{a/(d-a)} \\
&+\frac{12a^{5/2}}{5}\int_0^a \frac{(a-x)^{5/2}}{(d-x)^3}\,dx = \frac{24a^{5/2}}{5}\int_0^a \frac{t^6}{(t^2+k^2)^5}\,dt = -\frac{3}{5}\frac{a^5}{d^4} - \frac{1}{2}\frac{a^4}{d^3} + \frac{1}{2}\frac{a^3}{d^2} - \frac{7}{8}\frac{a^3}{d^2} + \frac{3}{16}\frac{a^3}{d(d-a)} + \frac{5}{16}\frac{a^{5/2}}{(d-a)^{3/2}}\tan^{-1}\sqrt{a/(d-a)} \\
&-\frac{40a^{7/2}}{7}\int_0^a \frac{(a-x)^{7/2}}{(d-x)^7}\,dx = -\frac{80a^{7/2}}{7}\int_0^a \frac{t^8}{(t^2+k^2)^7}\,dt = \frac{20}{21}\frac{a^7}{d^6} + \frac{2}{3}\frac{a^6}{d^5} + \frac{5a^5}{12d^4} + \frac{5}{24}\frac{a^4}{d^3} - \frac{54}{96}\frac{a^4}{d^2(d-a)} - \frac{5}{64}\frac{a^4}{d(d-a)^2} - \frac{5}{64}\frac{a^{7/2}}{(d-a)^{5/2}}\tan^{-1}\sqrt{a/(d-a)}
\end{aligned}
\tag{45}
$$

The series in (43) are convergent for $d > 2a$; therefore, (44) and (45) are convergent for $d > 2a$.

When we combined the right hand members of (45) and substituted x for d, the equation was reduced to:

$$4a + \frac{a^2}{x} + \frac{7}{24}\frac{a^3}{x^2} - \frac{7}{24}\frac{a^4}{x^3} - \frac{11}{60}\frac{a^5}{x^4} + \frac{2}{3}\frac{a^6}{x^5} + \frac{20}{21}\frac{a^7}{x^6} + \text{----}$$

$$+ \frac{18a^3x-5a^4}{96x^2(x-a)} - \frac{5a^4}{64x(x-a)^2} + \text{----}$$

$$+ \sqrt{ax-a^2}\ (\tan^{-1}\sqrt{a/(x-a)}\ )\left[-4- \frac{a}{c-a} + \frac{5a^2}{16(c-a)^2} - \frac{5a^3}{64(c-a)^3} + \text{----}\right].$$

Next we considered the second and fourth terms of (41). We integrated the second term by the use of integral tables. Let $X = 4a(a-x) + (d$

and the second term becomes

$$-2a\int_0^a \frac{(a-x)dx}{X} = 2a\int_0^a \frac{xdx}{X} - 2a^2\int_0^a \frac{dx}{X} = a \log \frac{(d-a)^2}{4a^2+d^2} + \frac{a^2+ad}{2\sqrt{ad}} \log \frac{2a^2+d^2-ad+2a\sqrt{ad}}{2a^2+d^2-ad-2a\sqrt{ad}}.$$

We also integrated the fourth term by the use of integral tables and it became

$$- 4a^2\int_0^a \frac{(a-x)^2dx}{X^2} = -4a^4\int_0^a \frac{dx}{X^4} + 8a^3\int_p^a \frac{xdx}{X^2} - 4a^2\int_0^a \frac{x^2dx}{X^2} = \frac{9a^4}{2d(d-a)^2} - \frac{7a^3}{2(d-a)^2} + \frac{5a^2d}{2(d-a)^2} + \frac{ad^2}{2(d-a)^2} - \frac{7a^4}{d(4a^2+d^2)} - \frac{2a^3}{4a^2+d^2} - \frac{2a^2d}{4a^2+d^2} - \frac{7a^4+2a^3d}{8(ad}$$

We saw that the terms of (39), (40), (47), and (48) could be combined. We again substituted x for d and the sum of the four equations beca

$$\frac{9a^4}{2x(x-a)^2} - \frac{7a^3-5a^2x-ax^2}{2(x-a)^2} + \text{-----}$$

$$- \frac{7a^4}{x(4a^2+x^2)} - \frac{2a^3+2a^2x}{4a^2+x^2} + \text{-----}$$

$$+ \frac{x}{2} \log (4a^2+x^2) - x \log x + a \log (x+a) - (\sqrt{ax} - \frac{a^2+ax}{2\sqrt{ax}} + \frac{7a^4+2a^3x\pm a^2x^2}{8(ax)^{3/2}} - \text{------}) \log \frac{2a^2+x^2-ax+2a\sqrt{ax}}{2a^2-x^2-ax-2a\sqrt{ax}}.$$

Therefore $V = K\ [(48) + (49)]$.

When we combined the right hand members of (45) and substituted x for d, the equation was reduced to:

$$4a + \frac{a^2}{x} + \frac{7}{24}\frac{a^3}{x^2} - \frac{7}{24}\frac{a^4}{x^3} - \frac{11}{60}\frac{a^5}{x^4} + \frac{2}{3}\frac{a^6}{x^5} + \frac{20}{21}\frac{a^7}{x^6} + \text{----}$$

$$+ \frac{18a^3x-5a^4}{96x^2(x-a)} - \frac{5a^4}{64x(x-a)^2} + \text{----} \tag{46}$$

$$+ \sqrt{ax-a^2}\ (\tan^{-1}\sqrt{a/(x-a)}\ )\left[-4- \frac{a}{c-a} + \frac{5a^2}{16(c-a)^2} - \frac{5a^3}{64(c-a)^3} + \text{----}\right].$$

Next we considered the second and fourth terms of (41). We integrated the second term by the use of integral tables. Let $X = 4a(a-x) + (d-x)^2 = 4a^2+d^2-(4a+2d)x+x^2$ and the second term becomes

$$-2a\int_0^a \frac{(a-x)dx}{X} = 2a\int_0^a \frac{xdx}{X} - 2a^2\int_0^a \frac{dx}{X} = a \log \frac{(d-a)^2}{4a^2+d^2} + \frac{a^2+ad}{2\sqrt{ad}} \log \frac{2a^2+d^2-ad+2a\sqrt{ad}}{2a^2+d^2-ad-2a\sqrt{ad}}. \tag{47}$$

We also integrated the fourth term by the use of integral tables and it became

$$-4a^2\int_0^a \frac{(a-x)^2dx}{X^2} = -4a^4\int_0^a \frac{dx}{X^4} + 8a^3\int_p^a \frac{xdx}{X^2} - 4a^2\int_0^a \frac{x^2dx}{X^2} = \frac{9a^4}{2d(d-a)^2} - \frac{7a^3}{2(d-a)^2} + \frac{5a^2d}{2(d-a)^2} + \frac{ad^2}{2(d-a)^2} - \frac{7a^4}{d(4a^2+d^2)} - \frac{2a^3}{4a^2+d^2} - \frac{2a^2d}{4a^2+d^2} - \frac{7a^4+2a^3d+a^2d^2}{8(ad)^{3/2}} \log \frac{2a^2+d^2-ad+2a\sqrt{ad}}{2a^2+d^2-ad-2a\sqrt{ad}}. \tag{48}$$

We saw that the terms of (39), (40), (47), and (48) could be combined. We again substituted x for d and the sum of the four equations became

$$\frac{9a^4}{2x(x-a)^2} - \frac{7a^3-5a^2x-ax^2}{2(x-a)^2} + \text{-----}$$

$$- \frac{7a^4}{x(4a^2+x^2)} - \frac{2a^3+2a^2x}{4a^2+x^2} + \text{-----} \tag{49}$$

$$+ \frac{x}{2} \log (4a^2+x^2) - x \log x + a \log (x+a) - \left(\sqrt{ax} - \frac{a^2+ax}{2\sqrt{ax}} + \frac{7a^4+2a^3x\pm a^2x^2}{8(ax)^{3/2}} - \text{------}\right) \log \frac{2a^2+x^2-ax+2a\sqrt{ax}}{2a^2-x^2-ax-2a\sqrt{ax}}.$$

Therefore $V = K\left[(48) + (49)\right]$.

VI. LAPLACE'S EQUATION IN CURVILINEAR COORDINATES

In the xy-plane it is known that the orthogonal trajectory of the parabola $y^2 = 4a(a-x)$ is the parabola $y^2 = 4b(b+x)$. When these two parabolas are revolved about the x-axis, two surfaces are formed. The two surfaces are perpendicular at all points of intersection. Any plane containing the x-axis, $y = cz$, is seen to be perpendicular to both surfaces at all points of intersection; therefore, the three surfaces are mutually perpendicular for all positive values of a and b and all values of c. The parameters a, b, and c will now be regarded as a set of coordinates for a point of intersection of the three surfaces.

When we solved the equations of the three surfaces simultaneously, we obtained

$$x = a-b,$$

$$y = \frac{2c\sqrt{ab}}{\sqrt{c^2+1}}, \tag{50}$$

and

$$z = \frac{2\sqrt{ab}}{\sqrt{c^2-1}}.$$

Let

$$1/h_1^2 = (D_a x)^2 + (D_a y)^2 + (D_a z)^2,$$

$$1/h_2^2 = (D_b x)^2 + (D_b y)^2 + (D_b z)^2, \tag{51}$$

and

$$1/h_3^2 = (D_c x)^2 + (D_c y)^2 + (D_c z)^2.$$

Then Laplace's equation in this curvilinear system is*

$$D_a\left(\frac{h_1}{h_2h_3} D_aV\right) + D_b\left(\frac{h_2}{h_3h_1} D_bV\right) + D_c\left(\frac{h_3}{h_1h_2} D_cV\right) = 0. \quad (52)$$

When we substituted (50) in (51), we found

$$1/h_1^2 = (a+b)/a,$$
$$1/h_2^2 = (a+b)/b, \quad (53)$$

and

$$1/h_3^2 = 4ab/(c^2+1)^2.$$

When we substituted (53) in (52) Laplace's equation became

$$\frac{\partial}{\partial a}\left[\frac{2a}{c^2+1}\frac{\partial V}{\partial a}\right] + \frac{\partial}{\partial b}\left[\frac{2b}{c^2+1}\frac{\partial V}{\partial b}\right] + \frac{\partial}{\partial c}\left[\frac{(c^2+1)(a+b)}{2ab}\frac{\partial V}{\partial c}\right] = 0. \quad (54)$$

For cases where the attracting mass is symmetrical about the x-axis, $\partial V/\partial c$ is zero; therefore, (54) reduced to

$$a\frac{\partial^2 V}{\partial a^2} + \frac{\partial V}{\partial a} + b\frac{\partial^2 V}{\partial b^2} + \frac{\partial V}{\partial b} = 0. \quad (55)$$

In order to solve (55) we assumed that $V = A \cdot B$ where $A = f(a)$ and $B = \phi(b)$. After substituting this value of V in (55), the equation became

$$\frac{1}{A}\left(a\frac{d^2A}{da^2} + \frac{dA}{da}\right) = \frac{-1}{B}\left(b\frac{d^2B}{db^2} + \frac{dB}{db}\right). \quad (56)$$

Since A is a function of a alone and B is a function of b alone, we used the total derivative in (56). The left member of (56) is a function of a alone and the right member is a function of b alone; therefore, in order to be equivalent they must both be equal to a constant K. This led to the

* Byerly, Fourier's Series and Spherical Harmonics, p. 239.

two equations;

$$a \frac{d^2A}{da^2} + \frac{dA}{da} - AK = 0, \tag{57}$$

and

$$b \frac{d^2B}{db^2} + \frac{dB}{db} + BK = 0. \tag{58}$$

When we substituted a new variable $x = 2\sqrt{-Ka}$ in (57) it became

$$x^2 \frac{d^2V}{dx^2} + x \frac{dV}{dx} + x^2V = 0 \tag{59}$$

which is Bessel's equation when $n = 0$. The solution is, therefore,

$$A = J_o(x) = J_o(2\sqrt{-Ka}). \tag{60}$$

Also, when we substituted $y = 2\sqrt{Kb}$ in (58) it became

$$y^2 \frac{d^2V}{dy^2} + y \frac{dV}{dy} + y^2V = 0 \tag{61}$$

which is identical to (59) and the solution is

$$B = J_o(y) = J_o(2\sqrt{Kb}). \tag{62}$$

Finally, from the original assumption that $V = A \cdot B$, the solution of (55) became

$$V = J_o(2\sqrt{-Ka}) \cdot J_o(2\sqrt{Kb}). \tag{63}$$

CONCLUSION

(1) The potential at any point in space not in the zone $a < r < 2a$ due to the solid of revolution can be calculated by the use of equations (17) and (22). Equation

(24) gives the potential for points on the x-axis for $x > a$, but since it is not in the form of either (7) or (9) Legendre's coefficients cannot be used to make the function include the other points of the zone.

(2) The potential at points on the x-axis where $x > a$ due to the surface of revolution can be calculated by using (28).

(3) The potential at points on the x-axis where $x > a$ due to the arc can be calculated by using (31).

(4) The potential at any point on the z-axis due to the arc can be calculated by using (37).

(5) The potential at points on the x-axis where $x > 2a$ due to the area can be calculated by using (49).

(6) Assuming the mass is symmetrical about the x-axis a solution of Laplace's equation in the given set of curvilinear coordinates was the product of two particular Bessel's functions.

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