Applications of the Hardy-Littlewood method to polynomial congruences and Diophantine inequalities

by

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B.S., University of Patras, 2017

M.S., University of Patras, 2019

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AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
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Abstract

This manuscript revolves around two peer-reviewed results. First, we prove that for any positive integers k, q, n with $n > N(k)$, integer c, and polynomials $f_i(x)$ of degree k whose leading coefficients are relatively prime to q, there exists a solution $\underline{x}$ to the congruence

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that lies in a cube of side length at least $\max\{q^{1/k}, k\}$. Moreover, the result is best possible up to the determination of $N(k)$. The latter half of the manuscript is centred around Diophantine inequalities. Let $k \geq 2$, $s \geq \lceil k(\log k + 4.20032)\rceil$, and $\lambda_1, \ldots, \lambda_s, \omega \in \mathbb{R}$. Assume that the λ_i are non-zero, not all in rational ratio, and not all of the same sign in the case that k is even. Then, for any $\epsilon > 0$, the inequality

$$|\lambda_1 x_1^k + \lambda_2 x_2^k + \cdots + \lambda_s x_s^k + \omega| < \epsilon$$

has $\gg P^{s-k}$ integer solutions with $|x_i| \leq P$. Moreover the asymptotic formula for the number of smooth solutions is established assuming the same conditions hold.

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Chapter 1

Introduction

1.1 Overview of Waring's Problem and the Circle Method

Waring's Problem

Waring's problem, posed in 1770 by Edward Waring and later formalized by David Hilbert in 1909, asks if every natural number k has an associated number s, such that every natural number is the sum of at most s natural numbers raised to the power of k. That is, for a given k, one seeks the smallest s such that the equation

$$x_1^k + x_2^k + \cdots + x_s^k = N$$

has a solution $x_1, \ldots, x_s \in \mathbb{N} \cup \{0\}$ for every $N \in \mathbb{N}$.

Lagrange's 1770 proof showed that when $k = 2$, this holds with $s = 4$. With a non-constructive proof [15], Hilbert confirmed that such a value $s(k)$ exists for all k.

In the following century, multiple research papers appeared in the literature, focusing on determining explicit upper bounds for $s(k)$ for each k. Moreover, researchers also entertained the asymptotic behaviour of the number of representations of large integers as sums of kth powers [9, 14].

The Hardy–Littlewood Circle Method

Nowadays, the mainstream approach to Waring's problem with analytic methods is via the *circle method*, developed by Hardy, Littlewood and Ramanujan [13, 12]. The revision of this approach, pioneered by Vinogradov, analyses the exponential sum

$$f(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^k),$$

where $e(\beta) := e^{2\pi i \beta}$, and uses this to study the integral

$$R_{s,k}(N) = \int_0^1 f(\alpha)^s e(-\alpha N)\, d\alpha.$$

Notice, that by changing the order of summation and integration, one can see that $R_{s,k}(N)$ counts the number of s-tuples $\underline{x} \in \{1, \dots, P\}^s$ such that $x_1^k + \cdots + x_s^k = N$.

To evaluate the integral that defines $R_{s,k}(N)$, the unit interval is decomposed into the major arcs, regions where α is well-approximated by rational numbers with small denominators, and the minor arcs, defined as the compliment of the major arcs. Over those regions, the size of the integral is estimated separately.

The integral over the major arcs can be approximated using the singular series and the singular integral, which relate the local behaviour of the equation to the solvability over the integers. The asymptotic size of the main term stems from this section of the unit interval. On the minor arcs, non-trivial bounds are established to show that the contribution cannot overtake the main term.

In multiple variations of Waring's problem, when the circle method is deployed, the underlying structure of the approach is similar. The major arcs provide the arithmetic information and contribute the main term. In contrast, the minor arcs are harder to navigate, often demanding analytic estimates involving Weyl differencing [29], van der Corput's method [23], or modern refinements thereof.

Asymptotic Formula and Minimal Variables

One of the achievements of the circle method in the context of Waring's problem is the production of an asymptotic formula for $R_{s,k}(N)$ of the form

$$\frac{\Gamma(1+1/k)^s}{\Gamma(s/k)}\mathfrak{S}(N)N^{s/k-1}+o(N^{s/k-1}),$$

where $\mathfrak{S}(n)$ is the singular series mentioned earlier, and $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt$, for $\Re(z)>0$. This formula assumes that s is sufficiently large in terms of k and is meaningful for all sufficiently large values of N. To avoid the cases where for some small integer N the asymptotic formula does not provide a meaningful estimate, a new concept was introduced. $G(k)$ is defined as the Waring number, that is the minimum number of variables required such that every sufficiently large natural number is expressible as a sum of $G(k)$ kth powers of natural numbers [25, 36].

Vaughan [25], Wooley [31, 33], and others have managed to reduce the necessary number of variables required through advances in minor arc estimates.

Modern Developments and Motivation

Though originally developed for Waring-type problems, applications of the circle method have diversified. The fundamental advantage of this method is its ability to encompass the underlying structure of additive problems and arithmetic structures by utilizing exponential sums and integrals. It has been applied to solve the ternary Goldbach conjecture [28] as well as modern additive combinatorial problems.

This dissertation focus on various applications of the method related to problems inspired by Waring's problem. By employing improved exponential sum estimates and taking advantage of new structural insights of current research, this manuscript aims to extend the reach of the method and improve upon previous results.

1.2 Overview of Results

Utilizing the versatility of the circle method, one can tackle similar problems concerning polynomial congruences modulo an arbitrary non-negative integer q. Another application of the method is the approximation of real numbers by a polynomial. Diophantine inequalities are polynomial inequalities with integer solutions, and traditionally, these inequalities involve irrational coefficients in the polynomial. Chapters 2 and 3 each employ the circle method for a different variation of Waring's problem. The contents of those chapters are published work and are presented here roughly as they appear in the literature [5, 20].

1.2.1 Solutions to Polynomial Congruences with Variables Restricted to a Box

Chapter 2 investigates the solvability and distribution of solutions to congruences of the form

$$f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n) \equiv c \pmod{q}, \tag{1.1}$$

where each $f_i(x) \in \mathbb{Z}[x]$ is a univariate polynomial of fixed degree k, and the solution $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ is restricted to be within a box (or cube) $\mathcal{B} \subset \mathbb{Z}^n$, defined as

$$\mathcal{B} := \{\underline{x} \in \mathbb{Z}^n : c_i + 1 \leq x_i \leq c_i + B,\ 1 \leq i \leq n\}.$$

The primary goal is to understand when such congruences are solvable, with solution vectors within the confines of $\mathcal{B}$, and to quantify the number of such solutions when they exist.

Main Theorem

We prove the following existence result, which generalizes prior work on diagonal polynomial congruences:

Theorem. *Let q, n, k be positive integers, c be an integer, and $f_1(x), \ldots, f_n(x) \in \mathbb{Z}[x]$ be polynomials of degree k whose leading coefficients are relatively prime to q. Suppose*

that for $1 \le i \le n$ *and* $p \mid q$ *with* p *prime,* $f_i(x)$ *is not a constant function modulo* p*. There exists a constant* $N(k)$ *such that if* $n > N(k)$ *and* $\mathcal{B}$ *is a cube of side length* $B > \max\{q^{1/k}, k\}$*, then there is a solution of the congruence* (1.1) *in* $\mathcal{B}$*.*

This theorem extends Theorem 5.1 of [6], where each $f_i(x)$ is of the form $a_i x^k$ with $(a_i, q) = 1$. It also fits within a broader context of generalizations of Waring's problem, as studied by Kamke [19], Hua [16–18], Nechaev [22], Wooley [34], and Ford [10].

We also present a corollary that addresses the case of polynomials with mixed degrees:

Corollary. *Let* q, n, k *be positive integers,* c *be an integer, and* $f_1(x), \dots, f_n(x) \in \mathbb{Z}[x]$ *be polynomials of degree at most* k *whose leading coefficients are relatively prime to* q*. Suppose that for* $1 \le i \le n$ *and* $p \mid q$ *with* p *prime,* $f_i(x)$ *is not a constant function modulo* p*. For* $1 \le i \le n$*, let* k_i *denote the degree of* $f_i(x)$*, and suppose that* $B_i > \max\{q^{1/k_i}, k_i\}$*. There exists a constant* $N'(k)$ *such that if* $n > N'(k)$*, then there is a solution of the congruence* (1.1) *in any box of the form*

$$\{\underline{x} \in \mathbb{Z}^n : c_i + 1 \le x_i \le c_i + B_i, 1 \le i \le n\}.$$

Upper Bounds and Density of Solutions

In addition to existence results when $n \ge k^2 + k + 2$, we derive an upper bound for the number of solutions $N_q(\mathcal{B})$ to (1.1) in a cube $\mathcal{B}$ of side length $B \le q$:

$$N_q(\mathcal{B}) \ll_k \frac{B^n}{q} + B^{n-k},$$

where the implied constant depends only on k. This result quantifies the density of solutions in terms of the modulus q, the degree k, and B.

The bound extends to rectangular boxes with side lengths $B_1, \dots, B_n$, yielding:

$$N_q(\mathcal{B}) \ll_k \prod_{i=1}^{n} \left(\frac{B_i^n}{q} + B_i^{n-k} \right)^{1/n}.$$

Methodological Notes

Our approach combines exponential sum techniques and combinatorial estimates over residue class rings. The assumption, that the polynomials f_i are non-constant modulo p and that $B > k$, implies that $f_i(x) \bmod p$ is not constant on the edge $[c_i + 1, c_i + B]$.

Sharpness of Results

The conditions in the main theorem are essentially best possible. Examples are given to demonstrate that lowering the bound beyond $B \gg q^{1/k}$ can eliminate solvability.

1.2.2 Some Improvements on the Davenport-Heilbronn Method

In Chapter 3 refinements to the classical Davenport-Heilbronn method [8] are presented. Those refinements are aimed at improving the quantitative analysis of Diophantine inequalities of the form

$$|\lambda_1 x_1^k + \cdots + \lambda_s x_s^k + \omega| < \epsilon, \tag{1.2}$$

where $k \geq 20$, $s \geq \lceil k(\log k + 4.20032) \rceil$, and the real coefficients λ_i satisfy certain irrationality and sign conditions. The number of solutions, $N(P)$, to such inequalities, with the restriction that each $|x_i| \leq P$, is counted and the method guarantees a lower bound $N(P) \gg P^{s-k}$ for sufficiently large P.

The original Davenport-Heilbronn method provided this asymptotic only for certain admissible sequences of P going to infinity, due to dependencies on continued fraction expansions of irrational ratios of coefficients. Building upon the innovations of Freeman [11], who removed such restrictions using kernel-smoothing techniques inspired by Bentkus and Götze [1], this work aims to further improve the method's applicability and reduce the number of variables required.

Crucial to the adaptations are the following ideas:

- use of modern Weyl-type estimates to sharpen bounds on minor arc integrals,

- adaptation of the Hardy-Littlewood circle method via refined Weyl-type sums over numbers with only small prime divisors, leading to better control on error terms,
- employing amplification strategies in the spirit of Wooley [35] to achieve the required savings on the minor arcs.

The main result, Theorem 3.1.1, establishes that when certain conditions on k, s, and $\underline{\lambda}$ are met, there exist infinitely many integer solutions satisfying the inequality (1.2). The number of solutions with each $|x_i| \leq P$ grows like P^{s-k}. This expands the practical scope of the Davenport-Heilbronn method by:

- removing prior constraints on the construction of the major and minor arcs,
- bringing the threshold for the required number of variables closer to the conjectural optimum,
- introducing more accessible and generalizable techniques for related analytic problems.

Chapter 3 details the definitions, kernel functions, and estimates used to realize these improvements. Section 2 of Chapter 3 introduces admissible exponents and the function $T(P)$, which governs the structure of major and minor arcs. Section 3 of Chapter 3 enhances the Weyl-type estimates for smooth exponential sums. The subsequent sections of that chapter systematically implement the revised Davenport-Heilbronn method using this infrastructure.

These developments contribute to a broader project of refining additive number theoretic tools. They also serve as a prototype for extending classical techniques to a wider range of analytic and arithmetic problems.

Chapter 2

Solutions to Polynomial Congruences with Variables Restricted to a Box

2.1 Introduction

Let q, n, and k be positive integers, c be an integer, and $f_1(x), \ldots, f_n(x) \in \mathbb{Z}[x]$ be polynomials of degree k whose leading coefficients are relatively prime to q. Let $\mathcal{B}$ be the cube

$$\mathcal{B} := \{\underline{x} \in \mathbb{Z}^n : c_i + 1 \leq x_i \leq c_i + B, 1 \leq i \leq n\} \tag{2.1}$$

and $N_q(\mathcal{B})$ denote the number of solutions of

$$f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n) \equiv c \pmod{q} \tag{2.2}$$

with $\underline{x} \in \mathcal{B}$. The goal of this chapter is to obtain the following theorem on the distribution of solutions to (2.2).

Theorem 2.1.1. *Let q, n, k be positive integers, c be an integer, and $f_1(x), \ldots, f_n(x)$ $\in \mathbb{Z}[x]$ be polynomials of degree k whose leading coefficients are relatively prime to q. Suppose that for $1 \leq i \leq n$ and $p \mid q$ with p prime, $f_i(x)$ is not a constant function modulo p. There*

exists a constant $N(k)$ such that if $n > N(k)$ and $\mathcal{B}$ is a cube of side length $B > \max\{q^{1/k}, k\}$, then there is a solution of the congruence (2.2) in $\mathcal{B}$.

Theorem 2.1.1 is a generalization of Theorem 5.1 in [6], which handles the case where each $f_i(x)$ is of the form $a_i x^k$ with $(a_i, q) = 1$. This is similar to how various authors have studied a generalized Waring problem that allows for polynomial summands; see for instance the work of Kamke [19], Hua [16–18], Nechaev [22], Wooley [34], and Ford [10].

Theorem 2.1.1 is best possible up to the determination of $N(k)$. Indeed, consider the congruence

$$x_1^k + \cdots + x_n^k \equiv 0 \pmod{q},$$

and box $\mathcal{B}$ with $1 \leq x_i \leq B$, $1 \leq i \leq n$. Plainly with $n = N(k)$ we will need $B \gg_k q^{1/k}$ in order to solve this congruence. For $1 \leq i \leq n$ and a prime $p \mid q$, the requirements that $B > k$ and that $f_i(x)$ is not a constant function mod p imply that $f_i(x)$ mod p is not constant on the edge $[c_i + 1, c_i + B]$. In the statement of the theorem, the condition that $f_i(x)$ is not a constant function modulo p is equivalent to requiring that $(x^p - x) \nmid (f_i(x) - f_i(0))$ over $(\mathbb{Z}/p\mathbb{Z})[x]$.

One can also state a version of Theorem 2.1.1 that applies when the polynomials $f_i(x_i)$ are not all of the same degree.

Corollary 2.1.1. *Let q, n, k be positive integers, c be an integer, and $f_1(x), \ldots, f_n(x)$ $\in \mathbb{Z}[x]$ be polynomials of degree at most k whose leading coefficients are relatively prime to q. Suppose that for $1 \leq i \leq n$ and $p \mid q$ with p prime, $f_i(x)$ is not a constant function modulo p. For $1 \leq i \leq n$, let k_i denote the degree of $f_i(x)$, and suppose that $B_i > \max\{q^{1/k_i}, k_i\}$. There exists a constant $N'(k)$ such that if $n > N'(k)$, then there is a solution of the congruence (2.2) in any box of the form*

$$\{\underline{x} \in \mathbb{Z}^n : c_i + 1 \leq x_i \leq c_i + B_i, 1 \leq i \leq n\}.$$

Indeed, one can simply take $N'(k) = k \cdot \max\limits_{1 \leq d \leq k} N(d)$ with $N(d)$ the constant in Theorem 2.1.1. We note that in order to obtain a solution in any cube of side length $q^{1/k}$ for the case of

mixed degrees, it is necessary to have sufficiently many of the $f_i(x)$ of degree at least k.

2.2 A General Upper Bound on the Number of Solutions of (2.2)

Theorem 2.2.1. *Let q, n, k be positive integers, c be an integer, and $f_1(x), \dots, f_n(x) \in \mathbb{Z}[x]$ be degree-k polynomials whose leading coefficients are relatively prime to q. Suppose $n \geq k^2 + k + 2$ and $\mathcal{B}$ is a cube of edge length $B \leq q$ as in (2.1). Then*

$$N_q(\mathcal{B}) \ll_k \frac{B^n}{q} + B^{n-k}. \tag{2.3}$$

The upper bound in (2.3) can be generalized to a rectangular box with edges of different lengths as follows; when $B_i \in \mathbb{N}$ $(1 \leq i \leq n)$ and

$$\mathcal{B} := \{\underline{x} \in \mathbb{Z}^n : c_i + 1 \leq x_i \leq c_i + B_i, 1 \leq i \leq n\},$$

one can prove that

$$N_q(\mathcal{B}) \ll_k \prod_{i=1}^{n} \left(\frac{B_i^n}{q} + B_i^{n-k} \right)^{1/n}.$$

We begin the proof of the theorem with the following lemma. Let $\mathbb{Z}_q$ denote the residue class ring modulo q and $e_q(\cdot) := e^{2\pi i(\cdot)/q}$, an additive character on $\mathbb{Z}_q$. For any subsets $S_1, \dots, S_n$ of $\mathbb{Z}_q$, put $\mathcal{S} = S_1 \times \cdots \times S_n$, and define

$$I_{n,k,f}(\mathcal{S}) := \#\Big\{(\underline{x}, \underline{y}) \in \mathcal{S} \times \mathcal{S} : \sum_{i=1}^{n} f(x_i) \equiv \sum_{i=1}^{n} f(y_i) \pmod{q}\Big\}.$$

Lemma 2.2.1. *Let q, n, k be a positive integers, c be an integer, $S_1, \dots, S_{2n}$ be subsets of $\mathbb{Z}_q$, $\mathcal{T} = S_1 \times \cdots \times S_{2n}$, and $f_1(x), \dots, f_{2n}(x) \in \mathbb{Z}[x]$ be degree-k polynomials whose leading*

coefficients are relatively prime to q. Then, we have

$$\#\Big\{\underline{x} \in \mathcal{T} : \sum_{i=1}^{2n} f_i(x_i) \equiv c \pmod q\Big\} \leq \prod_{i=1}^{2n} I_{n,k,f_i}(\mathcal{S}_i^n)^{\frac{1}{2n}},$$

where $\mathcal{S}_i^n$ is the cartesian product of S_i with itself n times.

Proof. We have

$$\begin{aligned}\#\{\underline{x} \in \mathcal{T} : \sum_{i=1}^{2n} f_i(x_i) \equiv c \pmod q\} &= \tfrac{1}{q} \sum_{\underline{x} \in \mathcal{T}} \sum_{\lambda=1}^{q} e_q\left(\lambda\left(\sum_{i=1}^{2n} f_i(x_i) - c\right)\right) \\ &\leq \frac{1}{q} \sum_{\lambda=1}^{q} \left| e_q(-\lambda c) \prod_{i=1}^{2n} \sum_{x_i \in S_i} e_q\left(\lambda f_i(x_i)\right)\right| \\ &\leq \frac{1}{q}\left[\sum_{\lambda=1}^{q}\left|\sum_{x_1 \in S_1} e_q(\lambda f_1(x_1))\right|^{2n}\right]^{\frac{1}{2n}} \cdots \left[\sum_{\lambda=1}^{q}\left|\sum_{x_{2n} \in S_{2n}} e_q(\lambda f_{2n}(x_{2n}))\right|^{2n}\right]^{\frac{1}{2n}},\end{aligned}$$

by Hölder's inequality. Now, for $1 \leq i \leq n$, the sum

$$\sum_{\lambda=1}^{q}\left|\sum_{x_i \in S_i} e_q(\lambda f_i(x_i))\right|^{2n}$$

is q times the number of solutions of the congruence

$$f_i(x_1) + \cdots + f_i(x_n) \equiv f_i(y_1) + \cdots + f_i(y_n) \pmod q,$$

with variables restricted to S_i. Therefore,

$$\#\{\underline{x} \in \mathcal{T} : \sum_{i=1}^{2n} f_i(x_i) \equiv c \pmod q\} \leq \frac{1}{q} \prod_{i=1}^{2n} (q I_{n,k,f_i}(S_i^n))^{\frac{1}{2n}} = \prod_{i=1}^{2n} I_{n,k,f_i}(S_i^n)^{\frac{1}{2n}}.$$

□

2.3 Relating $I_{n,k,f}(\mathcal{B})$ to $J_{n,k}(B)$ and $J^*_{n,k}(B)$

In this section, for a degree-k polynomial $f(x)$ whose leading coefficients is relatively prime to q, we obtain an estimate for $I_{n,k,f}(\mathcal{B})$, the number of solutions of

$$\sum_{i=1}^{n} f(x_i) \equiv \sum_{i=1}^{n} f(y_i) \pmod{q}, \tag{2.4}$$

with $\underline{x}, \underline{y}$ in a cube of the type

$$\mathcal{B} = \mathcal{B}(c, B) := \{\underline{x} \in \mathbb{Z}^n : c+1 \le x_i \le c+B, 1 \le i \le n\}, \tag{2.5}$$

where $c \in \mathbb{Z}$, by first relating it to the number of solutions $J_{n,k}(B)$ to the system of congruences

$$\begin{aligned} x_1 + \cdots + x_n &\equiv y_1 + \cdots + y_n \pmod{q}, \\ x_1^2 + \cdots + x_n^2 &\equiv y_1^2 + \cdots + y_n^2 \pmod{q}, \\ &\vdots \\ x_1^k + \cdots + x_n^k &\equiv y_1^k + \cdots + y_n^k \pmod{q}, \end{aligned}$$

with $(\underline{x}, \underline{y}) \in \mathcal{B}(0, B) \times \mathcal{B}(0, B)$, and then relating it to the number of solutions of the corresponding system of equations (2.8).

Lemma 2.3.1. *For any positive integers B, n, k, q with $B \le q$, integer c, cube $\mathcal{B}(c, B)$ as in (2.5), and polynomial $f(x) = \alpha_k x^k + \cdots + \alpha_0 \in \mathbb{Z}[x]$ with $(\alpha_k, q) = 1$, we have*

$$I_{n,k,f}(\mathcal{B}) \le (2n)^{k-1} B^{\frac{k(k-1)}{2}} J_{n,k}(B). \tag{2.6}$$

Moreover, if for some j with $1 \le j < k$ we have $n(B^{j-1} - 1) < \frac{q-1}{2} \le n(B^j - 1)$, then

$$I_{n,k,f}(\mathcal{B}) \le (2n)^{j-1} B^{\frac{j(j-1)}{2}} q^{k-j} J_{n,k}(B).$$

Proof. First we observe that for any integers h_i, $1 \le i \le k-1$, the number $N(h_1, \dots, h_{k-1})$ of solutions of the system

$$\begin{aligned}
(x_1 - c) + \cdots + (x_n - c) &\equiv (y_1 - c) + \cdots + (y_n - c) + h_1 \pmod q \\
&\vdots \\
(x_1 - c)^{k-1} + \cdots + (x_n - c)^{k-1} &\equiv (y_1 - c)^{k-1} + \cdots + (y_n - c)^{k-1} + h_{k-1} \pmod q \\
f(x_1) + \cdots + f(x_n) &\equiv f(y_1) + \cdots + f(y_n) \pmod q
\end{aligned} \tag{2.7}$$

with $(\underline{x}, \underline{y}) \in \mathcal{B} \times \mathcal{B}$, is less than or equal to the number of solutions with all $h_j = 0$. This follows by applying the triangle inequality to the exponential sum representation for the number of solutions of the system,

$$\frac{1}{q^k} \sum_{\lambda_1=1}^{q} \cdots \sum_{\lambda_k=1}^{q} e_q(-\lambda_1 h_1 - \cdots - \lambda_{k-1} h_{k-1}) \sum_{\underline{x} \in \mathcal{B}} \sum_{\underline{y} \in \mathcal{B}} \\ e_q\left[\lambda_k \left(\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} f(y_i)\right) + \sum_{j=1}^{k-1} \lambda_j \left(\sum_{i=1}^{n} (x_i - c)^j - \sum_{i=1}^{n} (y_i - c)^j\right)\right].$$

By a change of variables $x_i \to x_i + c, y_i \to y_i + c$, $N(\underline{0})$ equals the number of solutions of the system of congruences

$$\begin{aligned}
x_1 + \cdots + x_n &\equiv y_1 + \cdots + y_n \pmod q \\
&\vdots \\
x_1^{k-1} + \cdots + x_n^{k-1} &\equiv y_1^{k-1} + \cdots + y_n^{k-1} \pmod q \\
f(x_1 + c) + \cdots + f(x_n + c) &\equiv f(y_1 + c) + \cdots + f(y_n + c) \pmod q,
\end{aligned}$$

with $(\underline{x}, \underline{y}) \in \mathcal{B}(0, B) \times \mathcal{B}(0, B)$. By the Binomial Theorem, provided that

$$x_1^j + \cdots + x_n^j \equiv y_1^j + \cdots + y_n^j \pmod q \qquad (1 \le j \le k-1),$$

the congruence

$$f(x_1 + c) + \cdots + f(x_n + c) \equiv f(y_1 + c) + \cdots + f(y_n + c) \pmod{q}$$

is equivalent to

$$\alpha_k x_1^k + \cdots + \alpha_k x_n^k \equiv \alpha_k y_1^k + \cdots + \alpha_k y_n^k \pmod{q},$$

and since $(\alpha_k, q) = 1$, this is equivalent to

$$x_1^k + \cdots + x_n^k \equiv y_1^k + \cdots + y_n^k \pmod{q}.$$

Therefore, $N(h_1, \ldots, h_{k-1}) \leq N(\underline{0}) \leq J_{n,k}(B)$ uniformly.

Now, any solution of congruence (2.4) with $\underline{x}, \underline{y} \in \mathcal{B}$, is a solution of a system of the type (2.7) for some h_i satisfying $|h_i| \leq n(B^i - 1)$, $1 \leq i \leq k-1$. Thus we obtain

$$\begin{aligned} I_{n,k}(\mathcal{B}) &= \sum_{|h_1| \leq n(B-1)} \cdots \sum_{|h_{k-1}| \leq n(B^{k-1}-1)} N(h_1, \ldots, h_{k-1}) \\ &\leq J_{n,k}(B) \prod_{i=1}^{k-1} (2n(B^i - 1) + 1), \end{aligned}$$

yielding the inequality in (2.6).

Suppose now that for some j with $1 \leq j < k$ we have $n(B^{j-1} - 1) < \frac{q-1}{2} \leq n(B^j - 1)$. Then the upper bound can be improved by simply allowing $h_j, \ldots, h_{k-1}$ to each run through a complete residue system modulo q. In this case, we see that there are at most $(2n)^{j-1} B^{1+2+\cdots+(j-1)} q^{k-j}$ choices for the h_i, yielding the second inequality in the lemma. $\square$

Next, we state Lemma 4.2 from [7], which relates $J_{n,k}(B)$ to $J^*_{n,k}(B)$, the number of

integer solutions to the system of equations

$$\begin{aligned} x_1 + \cdots + x_n &= y_1 + \cdots + y_n, \\ x_1^2 + \cdots + x_n^2 &= y_1^2 + \cdots + y_n^2, \\ &\vdots \\ x_1^k + \cdots + x_n^k &= y_1^k + \cdots + y_n^k, \end{aligned} \tag{2.8}$$

with $(\underline{x}, \underline{y}) \in \mathcal{B}(0, B) \times \mathcal{B}(0, B)$.

Lemma 2.3.2. *Let B, k, n, q be positive integers with $1 \leq B \leq q$.*
i) If $n(B^k - 1) < q$ then $J_{n,k}(B) = J^*_{n,k}(B)$.
ii) If $n(B^{k-1} - 1) < \frac{q-1}{2}$ then

$$J_{n,k}(B) \leq \left(2nB^k/q + 1\right) J^*_{n,k}(B).$$

iii) If for some j with $1 \leq j \leq k-1$ we have $n(B^{j-1} - 1) < \frac{q-1}{2} \leq n(B^j - 1)$, then

$$J_{n,k}(B) < 5(2n/q)^{k-j+1} B^{\frac{k(k+1)}{2} - \frac{j(j-1)}{2}} J^*_{n,k}(B).$$

Putting together the results of Lemma 2.3.1 and Lemma 2.3.2, we obtain a uniform upper bound for $I_{n,k,f}(\mathcal{B})$ in terms of $J^*_{n,k}(B)$.

Lemma 2.3.3. *For any positive integers B, n, k, q with $B \leq q$, integer c, cube $\mathcal{B}(c, B)$ as in (2.5), and polynomial $f(x) = \alpha_k x^k + \cdots + \alpha_0 \in \mathbb{Z}[x]$ with $(\alpha_k, q) = 1$, we have*

$$I_{n,k,f}(\mathcal{B}) \leq 5(2n)^k B^{\frac{1}{2}k(k-1)} \left(\frac{B^k}{q} + \frac{1}{2n}\right) J^*_{n,k}(B).$$

2.4 Estimation of $J^*_{n,k}(B)$

The task of estimating $J^*_{n,k}(B)$ has been a central problem in additive number theory since Vinogradov's seminal work [27] on Waring's problem. By the recent work of Bourgain, Demeter, and Guth [2] and Wooley [37], there exists a positive constant $c_1(n,k)$ such that for $n > \frac{1}{2}k(k+1)$, one has

$$J^*_{n,k}(B) \le c_1(n,k)B^{2n-\frac{1}{2}k(k+1)}. \tag{2.9}$$

Combining this result with Lemma 2.3.3, we obtain the following upper bound on $I_{n,k,f}(\mathcal{B})$.

Proposition 2.4.1. *For any positive integers B, n, k, q with $B \le q$ and $n > \frac{1}{2}k(k+1)$, integer c, cube $\mathcal{B}(c,B)$ as in (2.5), and polynomial $f(x) = \alpha_k x^k + \cdots + \alpha_0 \in \mathbb{Z}[x]$ with $(\alpha_k, q) = 1$, we have*

$$I_{n,k,f}(\mathcal{B}) \le 5\ c_1(n,k)(2n)^k \left(\frac{B^{2n}}{q} + \frac{1}{2n}B^{2n-k}\right),$$

where $c_1(n,k)$ is the positive constant appearing in (2.9).

2.5 Proof of Theorem 2.2.1

We may assume that $n = k^2+k+2$, by setting $n-(k^2+k+2)$ variables equal to arbitrary values in their specified intervals if $n > k^2+k+2$. Put $m = n/2$. For $1 \le i \le n$, let $S_i = \{x \in \mathbb{Z} : c_i + 1 \le x \le c_i + B\}$, and S_i^m be the cartesian product of S_i with itself m times, so that S_i^m is a cube of type (2.5). By Proposition 2.4.1, we have for each i that

$$I_{m,k,f_i}(S_i^m) \le 5\ c_1(m,k)\ n^k \left(\frac{B^n}{q} + \frac{1}{n}B^{n-k}\right),$$

and thus the theorem follows immediately from Lemma 2.2.1 (with n replaced by m). We observe that the constant in the upper bound of Theorem 2.2.1 may be taken to be

$$5\ c_1((k^2+k+2)/2,k)(k^2+k+2)^k.$$

2.6 The Value Set of $\sum_{i=1}^n f_i(x_i)$ over a Cube

For any cube $\mathcal{B}$ as in (2.1), we define $S_{\mathcal{B}}$ to be the set of values $\sum_{i=1}^n f_i(x_i)$ takes on modulo q as $\underline{x}$ runs through $\mathcal{B}$,

$$S_{\mathcal{B}} := \left\{ \sum_{i=1}^n f_i(x_i) \in \mathbb{Z}_q : \underline{x} \in \mathcal{B} \right\},$$

and put

$$N_{\mathcal{B}} := \#\left\{ (\underline{x},\underline{y}) \in \mathcal{B}\times\mathcal{B} : \sum_{i=1}^n f_i(x_i) \equiv \sum_{i=1}^n f_i(y_i) \pmod q \right\}.$$

By Theorem 2.2.1, we have for $n \geq \frac{1}{2}(k^2+k+2)$ and $f_1(x),\dots,f_n(x)$ degree-k polynomials whose leading coefficients are relatively prime to q, that

$$N_{\mathcal{B}} \ll_k \frac{B^{2n}}{q} + B^{2n-k},$$

and thus using the elementary relationship $|S_{\mathcal{B}}| \geq B^{2n}/N_{\mathcal{B}}$, we deduce the following lemma.

Lemma 2.6.1. *For any positive integers k,n,B,q with $n \geq \frac{1}{2}(k^2+k+2)$, cube $\mathcal{B}$ as in (2.1) with $B > q^{1/k}$, and polynomials $f_1(x),\dots,f_n(x) \in \mathbb{Z}[x]$ of degree k whose leading coefficients are relatively prime to q, we have $|S_{\mathcal{B}}| \gg_k q$.*

2.7 Proof of Theorem 2.1.1

Before proving Theorem 2.1.1, we recall a variant of the Cauchy-Davenport Theorem [6, Theorem 1.1] that is applicable to our situation.

Theorem 2.7.1. *Let $n \geq 1$ and $A_1,\dots,A_n$ be finite, nonempty subsets of an abelian group*

G, such that no A_i is contained in a coset of a proper subgroup of G. Then

$$|A_1 + \cdots + A_n| \geq \min\left\{|G|, \left(\tfrac{1}{2} + \tfrac{1}{2n}\right)\sum_{i=1}^{n}|A_i|\right\}.$$

Lemma 2.7.1. *For any positive integers k, n, B, q, cube $\mathcal{B}$ as in (2.1), and polynomials $f_1(x), \ldots, f_n(x) \in \mathbb{Z}[x]$ of degree k whose leading coefficients are relatively prime to q, the value set $S_{\mathcal{B}}$ is contained in a coset of a proper additive subgroup of $\mathbb{Z}_q$ if and only if there exists a prime $p|q$, such that for $1 \leq i \leq n$, $f_i(x_i)$ mod p is constant on the edge $[c_i+1, c_i+B]$ of the cube.*

Proof. Suppose that $S_{\mathcal{B}}$ is contained in a coset $d\mathbb{Z}_q + l$ for some $d|q$, $d \neq \pm 1$, and $l \in \mathbb{Z}$. We may assume that $d = p$, with p prime, by enlarging the subgroup if necessary. Thus, for all $\mathbf{x} \in \mathcal{B}$, we have

$$\sum_{i=1}^{n} f_i(x_i) \equiv l \pmod{p}.$$

In particular, for $1 \leq i \leq n$, $f(x_i)$ mod p must be constant on the interval $[c_i + 1, c_i + B]$. The converse is trivial. □

We now are in a position to prove Theorem 2.1.1.

Proof. Suppose that $B > \max\{q^{1/k}, k\}$ and that for $1 \leq i \leq n$, $f_i(x)$ is a polynomial with leading coefficient relatively prime to q such that for any prime $p \mid q$, $f_i(x)$ is not a constant function mod p. Let $A_1, A_2, \ldots, A_r$ be value sets of the type $S_{\mathcal{B}}$, with $n = \frac{1}{2}(k^2 + k + 2)$. By Lemma 2.6.1, since $B > q^{1/k}$, we have $|A_i| \geq \frac{q}{w_k}$, $1 \leq i \leq r$, for some positive constant w_k. We claim that for any prime divisor p of q, each $f_i(x_i)$ is not constant mod p on $[c_i+1, c_i+B]$. Indeed, if $p \leq B$ then each edge contains a full set of residues mod p and so $f_i(x_i)$ takes on at least two distinct values mod p; while if $p > B$, then for fixed a, the congruence $f_i(x_i) \equiv a \pmod{p}$ has at most $k < B$ solutions on the edge $[c_i+1, c_i+B]$, and so again $f_i(x_i)$ takes on at least two distinct values mod p on its corresponding edge. Thus by Lemma 2.7.1, none of the sets A_i are contained in a coset of a proper subgroup of $\mathbb{Z}_q$. By Theorem 2.7.1 it follows that for $r \geq 2w_k - 1$, we have $A_1 + \cdots + A_r = \mathbb{Z}_q$. Set $r = \lceil 2w_k \rceil - 1$. If we start with a

form in at least $\frac{1}{2}(k^2+k+2)r$ variables, we may partition the variables into r disjoint sets, each with at least $\frac{1}{2}(k^2+k+2)$ variables, and form r value sets $A_1, \ldots, A_r$ of the type $S_{\mathcal{B}}$, with $A_1 + \cdots + A_r = \mathbb{Z}_q$, completing the proof. $\square$

Chapter 3

Some Improvements on the Davenport-Heilbronn Method

3.1 Introduction

Davenport and Heilbronn [8] proved the following result using the method employed by this chapter. For positive integers k and s satisfying $k \geq 2$ and $s \geq 2^k + 1$, for any $\epsilon > 0$, and for a non-zero collection of real numbers $\lambda_1, \ldots, \lambda_s$, not all in rational ratio, there exist infinitely many integer solutions of

$$|\lambda_1 x_1^k + \cdots + \lambda_s x_s^k| < \epsilon, \tag{3.1}$$

with an additional natural restriction for the case of even k is that the coefficients λ_i do not all have the same sign. More specifically the Davenport-Heilbronn method provided a lower bound on the number of integer solutions in terms of the size of the variables. If one denotes by $N(P)$ the number of integer solutions of (3.1) for which the variables x_i are restricted to $[-P, P] \cap \mathbb{Z}$, then $N(P) \gg P^{s-k}$. The major limitation of the method was that the values that P is allowed to take, even though infinite, were based on the continued fraction expansion of the irrational quotient amongst the coefficients λ_i. That greatly restricted the eligible choices of P in the case that rapidly increasing terms appear in the continued fraction of the irrational quotient.

In the beginning of the century, more than half a century after the first deployment of the method, Freeman [11] overcame the aforementioned restrictions for the choice of P inspired by ideas of Bentkus and Götze [1]. Freeman's asymptotic formulas for $N(P)$ hold for all sufficiently large values of P.

Traditionally in the Davenport-Heilbronn method, when one applies Hölder's inequality over the minor arcs, sharp bounds for the integrals are used. The savings for the minor arc estimate are usually of minor importance. As long as such savings exist, in combination with sharp estimates for the majority of the integrals, the desired result is achieved. Improvements on the number of variables can be achieved with amplification methods (see [35]). Here those methods are applied for the integrals along with some adaptations of the Weyl-type bounds. The rest of the integral bounds are those of admissible exponents, which will be introduced in Section 2. The main result of this chapter is the following.

Theorem 3.1.1. *Let $k \geq 2$, $s \geq \lceil k(\log k + 4.20032)\rceil$, and λ_i $(1 \leq i \leq s)$ be non-zero real numbers, with $\lambda_j/\lambda_\kappa \notin \mathbb{Q}$ for some indices j and κ, and given that k is even, not all of the same sign. Then, for any $\epsilon > 0$ and $\omega \in \mathbb{R}$, there exist integers $x_1, \dots, x_s$ such that*

$$|\lambda_1 x_1^k + \lambda_2 x_2^k + \cdots + \lambda_s x_s^k + \omega| < \epsilon. \tag{3.2}$$

More specifically, the number of solutions with $|x_i| \leq P$ $(1 \leq i \leq s)$, where P is sufficiently large in terms of $\underline{\lambda}, \omega$, and ϵ, is of size $\gg_{\underline{\lambda},\omega,\epsilon} P^{s-k}$.

In previous work, Wooley [35, Theorem 1.2] has proven the cases where $3 \leq k \leq 20$, and Margulis [21] has proven the case $k = 2$. Hence in this chapter, unless otherwise specified, it is assumed that $k > 20$.

Without loss of generality, assume that $\lambda_1 > 0$ and for the case that k is even, $\lambda_m < 0$, for some $m \in \{2, \dots, s\}$. Note that whenever one of the exponents is odd, one can change the sign of the corresponding coefficient with no impact on the number of solutions. For that reason, it is assumed $\lambda_m < 0$ regardless of the parity of the exponents. Also, by reordering the terms if needed, there is no loss of generality in assuming that the first two coefficients

are involved in the irrational ratio. In the case that both λ_1 and λ_2 are positive, one of $\frac{\lambda_1}{\lambda_m}$ or $\frac{\lambda_2}{\lambda_m}$ must be irrational. Therefore, we may assume $\frac{\lambda_1}{\lambda_2} \notin \mathbb{Q}$, $\lambda_1 > 0$, and $\lambda_2 < 0$.

In the upcoming section, definitions and notation needed for the method are discussed. Section 3 adapts the current Weyl-type estimates to improve on the overall savings obtained by them. In Sections 4, 5, 6, and 7, the Davenport-Heilbronn method is applied with the underlying arcs of the typical Hardy-Littlewood method to be defined with the use of the function $T(P)$, which depends on the irrational quotient.

Throughout, the letter ε will denote a sufficiently small positive number, and P is assumed to be a large real number. The notation $\ll$ and $\gg$ of Vinogradov will be used frequently. The value of ε may change from statement to statement, and whenever ε appears in an expression, it is assumed that the statement holds, as long as ε is close enough to 0.

3.2 Davenport-Heilbronn Method

Consider the set of R-smooth numbers

$$\mathcal{A}(P,R) = \{n \in [1,P] \cap \mathbb{Z} : p \text{ prime and } p|n \text{ implies } p \leq R\}$$

and the smooth Weyl sum

$$f(\alpha; P, R) = \sum_{x \in \mathcal{A}(P,R)} e(\alpha x^k), \tag{3.3}$$

where $e(z)$ denotes $e^{2\pi iz}$. Whenever the choice of P and R are clear from context, the notation for the exponential sum in (3.3) is simplified to $f(\alpha)$. Fundamental to the Davenport-Heilbronn method is a suitable application for the typical Hardy-Littlewood method. The underlying dissection of the unit interval for the purposes of this chapter is as follows. The major arcs are defined to be the union of intervals

$$\mathfrak{M}_{HL}(q,p) = \{\alpha \in [0,1] : |q\alpha - p| \leq QP^{-k}\}$$

for which $0 \leq p \leq q \leq Q$ and $(p,q) = 1$. We use the notation $\mathfrak{M}_{HL}(Q)$ for the union of these major arcs. In the same spirit, the minor arcs of the circle method are defined to be

$$\mathfrak{m}_{HL}(Q) = [0,1] \setminus \mathfrak{M}_{HL}(Q).$$

Assume for the moment that the function $T(P)$, which will be defined in the next section, is increasing monotonically to infinity and depends only on the coefficients λ_i. Set $L(P) := T(P)^{1/2}$. Using a version of [11, Lemma 1], one has the following.

Lemma 3.2.1. *Let c and C be real numbers with $0 < c < C$. Then there is an even real function $K(\alpha) = K(\alpha; c, C)$ of real variable α, such that the function $\psi(\theta)$ defined by*

$$\psi(\theta) = \int_{-\infty}^{\infty} e(\alpha\theta) K(\alpha) d\alpha$$

satisfies the property that $0 \leq \psi(\theta) \leq 1$ for all real numbers θ, and additionally that

$$\psi(\theta) = \begin{cases} 0, & \text{when } |\theta| \geq C, \\ 1, & \text{when } |\theta| \leq c. \end{cases}$$

Lastly, the function $K(\alpha)$ satisfies the bound

$$K(\alpha) \ll \min\{C, |\alpha|^{-1}, (C-c)^{-1}|\alpha|^{-2}\}.$$

With the help of the above lemma, the following kernel functions are defined:

$$K_-(\alpha) := K(\alpha; \epsilon(1 - L(P)^{-1}), \epsilon) \tag{3.4}$$

and

$$K_+(\alpha) := K(\alpha; \epsilon, \epsilon(1 + L(P)^{-1})). \tag{3.5}$$

Moreover, define the indicator function

$$\mathcal{X}(\theta) = \begin{cases} 0, & \text{when } |\theta| \geq \epsilon, \\ 1, & \text{when } |\theta| < \epsilon. \end{cases} \tag{3.6}$$

It follows now that

$$\int_{-\infty}^{\infty} e(\theta\alpha) K_{-}(\alpha) d\alpha \leq \mathcal{X}(\theta) \leq \int_{-\infty}^{\infty} e(\theta\alpha) K_{+}(\alpha) d\alpha. \tag{3.7}$$

The difference of the indicator function and either integral above is zero, except possibly for the values of θ with $||\theta| - \epsilon| \leq \epsilon L(P)^{-1}$. The size of that difference is nonetheless absolutely bounded by 1.

Also define

$$R_{\pm}(P) := R_{\pm}(P; \eta) = \int_{-\infty}^{\infty} f(\lambda_1 \alpha) \cdots f(\lambda_s \alpha) e(\alpha\omega) K_{\pm}(\alpha) d\alpha, \tag{3.8}$$

and let $N(P) := N(P; \eta)$ be the number of integer solutions of (3.2) with $x_i \in \mathcal{A}(P, P^{\eta})$. Now from the Equations (3.4)-(3.8), it follows that

$$R_{-}(P) \leq N(P) \leq R_{+}(P). \tag{3.9}$$

In the following sections, $R_{-}(P)$ and $R_{+}(P)$ are shown to grow asymptotically at the same rate, and in extension an asymptotic formula for $N(P)$ is obtained. The latter serves as a lower bound for the total number of integer solutions of the inequality (3.2) with $1 \leq x_i \leq P$ $(1 \leq i \leq s)$.

Lastly, in order for the method to be applied one has to dissect the real numbers into subintervals. Traditionally three are sufficient: the major arc

$$\mathfrak{M}_{DH} = \{\alpha \in \mathbb{R} : |\alpha| \leq T(P) P^{-k}\}$$

which will provide the main term in the asymptotic formula, the minor arcs

$$\mathfrak{m}_{DH} = \{\alpha \in \mathbb{R} : T(P)P^{-k} < |\alpha| \leq T(P)\}$$

for which specially constructed tools are required to prove that its contribution cannot annihilate the main term, and the trivial arcs

$$\mathfrak{t}_{DH} = \{\alpha \in \mathbb{R} : T(P) < |\alpha|\}.$$

Finally, admissible exponents will be discussed briefly. A real number Δ_s is called an admissible exponent for a given k, if it satisfies

$$\int_0^1 |f(\alpha; P, R)|^s d\alpha \ll P^{s-k+\Delta_s+\varepsilon} \tag{3.10}$$

for all $\varepsilon > 0$, suitably small choice of η, $2 \leq R \leq P^{\eta}$, and P sufficiently large. If a statement involves $f(\alpha)$, either implicitly or explicitly, then it is asserted that for any $\varepsilon > 0$ there is a number $\eta > 0$, sufficiently small in terms of k, s, and ε, such that the statement holds uniformly. The arguments will only involve a finite number of statements, and consequently one may pass to the smallest of the numbers η that arise in this way, and then have all estimates in force with the same positive number η.

Without loss of generality, for all s, $\max\{0, k - s/2\} \leq \Delta_s \leq k$. By [32, Theorem 2.1], assuming that s is even and $k \geq 4$, the unique positive solution of the equation $\Delta e^{\Delta/k} = ke^{1-s/k}$ is an admissible exponent. Notice that if the solution of the above equation is an admissible exponent, $ke^{1-s/k} > \Delta$ is one as well. For the case that $s = 2n + 1$, by applying the trivial bound $|f(\alpha; P, R)| \leq P$ in (3.10) and using the admissible exponent $ke^{1-2n/k}$, one has

$$\int_0^1 |f(\alpha; P, R)|^s d\alpha \ll P \int_0^1 |f(\alpha; P, R)|^{2n} d\alpha \ll P^{s-k+ke^{1-2n/k}+\varepsilon} \ll P^{s-k+ke^{1-(s-2)/k}}.$$

In a similar fashion, one can ignore the ε in (3.10) for s even, using the admissible exponent

$ke^{1-(s-1)/k}$. For this reason, from here on the convention

$$\Delta_s = ke^{1-(s-2)/k} \tag{3.11}$$

is adopted.

3.3 Estimates of Weyl type

By the adjustments in the proof of Wooley [35, Lemma 8.1], and drawing inspiration from [3, Lemma 3.3], one acquires the following results. The proofs are included for completeness even though the bulk of the ideas come from the citations above. The following lemma is based on the underlying notion that if a Weyl sum is of significant size then good Diophantine approximations for the coefficient of the sum can be extrapolated.

Lemma 3.3.1. *Let $\varphi \in (0, \frac{1}{2})$ be fixed. There is a positive number μ, depending at most on k and φ, with the following property. Suppose that P is a real number sufficiently large in terms of k, and suppose that γ is a real number with $P^{-\frac{1}{2Dk^2}} \leq \gamma \leq 1$, where $D = 4.513951$. Then whenever $|f(\alpha, P, P^\eta)| \geq \gamma P$, there necessarily exist integers p and q with*

$$(p, q) = 1, \quad 1 \leq q \leq \mu\gamma^{-(2+\varphi)k} \quad \textit{and} \quad |q\alpha - p| \leq \mu\gamma^{-(2+\varphi)k}P^{-k}.$$

Proof. Let $|f(\alpha)| > \gamma P$ and $P^{-\frac{1}{2Dk^2}} \leq \gamma \leq 1$. By the Dirichlet Approximation Theorem, there exist $p, q \in \mathbb{Z}$ satisfying

$$(p, q) = 1,\ 1 \leq q \leq P^{k-\frac{1}{2}} \text{ and } |q\alpha - p| \leq P^{\frac{1}{2}-k}.$$

For the case where $q < P^{1/2}$, where $R = P^\eta$, the proof is similar to the one of [35, Lemma 8.1]. By [26, Lemma 7.2] with $M = P^{\frac{1}{2}+\varepsilon+\eta}$ and our assumption on q, one gets

$$|f(\alpha)| \ll (\log P)^3 q^\varepsilon (P(q + P^k|q\alpha - p|)^{-1/(2k)} + P^{\frac{7}{8}-\varepsilon}),$$

provided that η is small enough. The right hand side is dominated by $P(q+P^k|q\alpha-p|)^{-1/(2k+\varphi k)}$ when $q+P^k|q\alpha-p| \geq (\log P)^{\frac{15}{\varphi}k}$.

From Lemma 8.5 of [26] when $q+P^k|q\alpha-p| \leq (\log P)^{\frac{15}{\varphi}k}$, it is established that

$$|f(\alpha)| \ll q^{\varepsilon}P(q+P^k|q\alpha-p|)^{-1/k} + P(\log P)^{-\frac{15}{\varphi}k},$$

and again the right hand side is dominated by $P(q+P^k|q\alpha-p|)^{-1/(2k+\varphi k)}$.

Writing μ_1 for the implicit constant in $|f(\alpha)| \ll P(q+P^k|q\alpha-p|)^{-1/(2k+\varphi k)}$, it follows that

$$\gamma P \leq |f(\alpha)| \leq \mu_1 P(q+P^k|q\alpha-p|)^{-1/(2k+\varphi k)};$$

hence $(q+P^k|q\alpha-p|) \leq (\mu_1/\gamma)^{(2k+\varphi k)}$. This creates the pair of integers p, q along with their corresponding bounds mentioned in the statement of the lemma.

If one can find p', q' with $(p', q') = 1$, $1 \leq q' < P^{1/2}$ and $|q'\alpha - p'| \leq P^{1/2-k}$, then the previous case can be applied. Hence assume that if $|q\alpha - p| \leq P^{1/2-k}$ then $q \geq P^{1/2}$, and thus the conditions of [3, Lemma 3.4] are met. For $B = 4.51395$ and large enough P, one has

$$|f(\alpha)| < \frac{1}{2}P^{1-\frac{1}{2Bk^2}}.$$

Since $P^{1-\frac{1}{2Dk^2}} \leq |f(\alpha)|$, a contradiction about the size of $f(\alpha)$ arises. The desired result is achieved with $\mu = \mu_1^{(2+\varphi)k}$. □

Details of the above lemma were included to illustrate that improvements can be achieved on the range of γ as soon as better estimates over the minor arcs are achieved. For completeness, the proof of [35, Lemma 2.2] of Wooley will be repeated here. The only changes made are to incorporate the new bounds of Lemma 3.3.1. This is where the requirement for the irrational ratio amongst the coefficients arises.

Lemma 3.3.2. *Suppose that S and T are fixed real numbers with $0 < S \leq 1 \leq T$. Then*

$$\lim_{P\to\infty} \sup_{S\leq|\alpha|\leq T} P^{-2}|f(\lambda_1\alpha, P, P^{\eta})f(\lambda_2\alpha, P, P^{\eta})| = 0$$

given that $\lambda_1/\lambda_2 \notin \mathbb{Q}$.

Proof. Assume not. Then there exists $\varepsilon \in (0,1)$, an increasing sequence $\{P_m\}_{m=1}^{\infty}$ that goes to infinity, and a sequence $(\alpha_m) \subset [S,T]$ such that for all $m \in \mathbb{N}$,

$$|f(\lambda_1\alpha_m)f(\lambda_2\alpha_m)| \geq \varepsilon P_m^2. \tag{3.12}$$

Using the trivial bound $|f(\lambda_i\alpha_m)| \leq P_m$ one can infer from (3.12) that

$$|f(\lambda_i\alpha_m)| \geq \varepsilon P_m.$$

Let m be large enough that $P_m \geq \varepsilon^{-2Dk^2}$, and apply Lemma 3.3.1 with $\gamma = \varepsilon$. Let φ be a constant in $(0, \frac{1}{2})$. One now has for $i = 1, 2$ that there exist integers $p_{i,m}$ and $q_{i,m}$ such that $(p_{i,m}, q_{i,m}) = 1, \quad 1 \leq q_{i,m} \ll \varepsilon^{-(2+\varphi)k}$, and

$$|\lambda_i\alpha_m q_{i,m} - p_{i,m}| \ll \varepsilon^{-(2+\varphi)k} P_m^{-k}. \tag{3.13}$$

Observe that $p_{i,m} \neq 0$ for big enough m; otherwise

$$1 \ll |\lambda_i| S \leq |\lambda_i\alpha_m q_{i,m}| \ll \varepsilon^{-(2+\varphi)k} P_m^{-k} = o(1).$$

Since ε is fixed, the number of choices for $q_{i,m}$ is finite, and by the triangle inequality and (3.13), one can see that $p_{i,m}$ also can only have finitely many values since

$$|p_{i,m}| \ll \varepsilon^{-(2+\varphi)k} P_m^{-k} + |\lambda_i\alpha_m q_{i,m}| \ll 1.$$

Without loss of generality, one can assume that (p_1, q_1, p_2, q_2) is repeated infinitely often amongst the $(p_{1,m}, q_{1,m}, p_{2,m}, q_{2,m})$. This creates a contradiction because it implies that

$$\frac{\lambda_1}{\lambda_2} = \frac{\lambda_1\alpha_m}{\lambda_2\alpha_m} = \frac{\frac{p_1}{q_1}(1 + O(P_m^{-k}))}{\frac{p_2}{q_2}(1 + O(P_m^{-k}))} = \frac{p_1q_2}{p_2q_1}(1 + O(P_m^{-k})),$$

so $\dfrac{\lambda_1}{\lambda_2} = \dfrac{p_1 q_2}{p_2 q_1} \in \mathbb{Q}$. $\square$

The following lemma is a variation of Wooley's [35, Lemma 8.1], and its proof is presented for completeness. The main alteration is that the interval for $|\alpha|$ will only depend on $T(P)$, which helps define Q for the major arcs of the underlying typical method of Hardy and Littlewood. Moreover exponential savings over $T(P)$ are acquired from both sums.

Lemma 3.3.3. *Let $0 < \varphi < \frac{1}{2}$ be fixed. Suppose that $\lambda_1/\lambda_2 \notin \mathbb{Q}$. There exists a function $T(P)$ depending only on λ_1 and λ_2 with the property that $T(P)$ increases monotonically to infinity with $2 \leq T(P) \leq \log^{1/3} P$, such that*

$$\sup_{T(P)P^{-k} \leq |\alpha| \leq T(P)} |f(\lambda_1 \alpha, P, P^{\eta}) f(\lambda_2 \alpha, P, P^{\eta})| \ll P^2 T(P)^{-\frac{1-\varphi}{k}}.$$

Proof. By invoking the Lemma 3.3.2, one can create a sequence $\{P_n\}_{n=1}^{\infty}$ such that in the interval $1/n \leq |\alpha| \leq n$, one has

$$P^{-2} |f(\lambda_1 \alpha) f(\lambda_2 \alpha)| \leq 1/n$$

for all $P \geq P_n$. One can assume that the sequence is increasing and that $\log^{1/3} P_n \geq n$. Setting $T(P) = n$ for $P_n \leq P < P_{n+1}$, one gets the bound

$$\sup_{T^{-1}(P) \leq |\alpha| \leq T(P)} |f(\lambda_1 \alpha) f(\lambda_2 \alpha)| \leq P^2 T(P)^{-1}.$$

Suppose that $T(P)P^{-k} \leq |\alpha| \leq T^{-1}(P)$ and

$$|f(\lambda_1 \alpha)| \geq T(P)^{-\frac{1-\varphi}{2k}} P.$$

Since $T(P) \leq \log^{1/3} P$, by applying Lemma 3.3.1 with $\gamma = T(P)^{-\frac{1-\varphi}{2k}}$ and P sufficiently large, there exist integers p and q with $(p, q) = 1$,

$$1 \leq q \ll \gamma^{-(2+\varphi)k} = T(P)^{(2+\varphi)k \frac{1-\varphi}{2k}} \ll T(P)^{1-\varphi/2}$$

and

$$|\lambda_1 q\alpha - p| \ll \gamma^{-(2+\varphi)k} P^{-k} \ll T(P)^{1-\varphi/2} P^{-k}.$$

One can see that $|p| \leq |\lambda_1 \alpha q| + o(1) \ll T(P)^{-\varphi/2}$, and hence for large enough values of P, one has $p = 0$. This implies $|\alpha| < T(P)P^{-k}$, for sufficiently large P, which contradicts the hypothesis. It follows that for $T(P)P^{-k} \leq |\alpha| \leq T^{-1}(P)$ and P sufficiently large, one has $|f_1(\lambda_1\alpha)| \leq T(P)^{-\frac{1-\varphi}{2k}} P$, and similarly $|f(\lambda_2\alpha)| \leq T(P)^{-\frac{1-\varphi}{2k}} P$. Hence

$$\sup_{T(P)P^{-k} \leq |\alpha| \leq T(P)} |f(\lambda_1\alpha) f(\lambda_2\alpha)| \ll P^2 T(P)^{-\frac{1-\varphi}{2}(\frac{1}{k}+\frac{1}{k})} = P^2 T(P)^{-\frac{1-\varphi}{k}}. \qquad \square$$

3.4 Minor Arcs

Recent work of Brüdern and Wooley provides the major and minor arc bounds that are fundamental to this work. Two of their results, [3, Lemma 3.2] and [4, Theorem 6.1] are restated here.

Lemma 3.4.1. *Suppose that s is a real number with $s \geq 4k$ and Δ_s is an admissible exponent. Then whenever Q is a real number with $1 \leq Q \leq P^{k/2}$, one has the uniform bound*

$$\int_{\mathfrak{m}_{HL}(Q)} |f(\alpha; P, R)|^s d\alpha \ll P^{s-k} Q^{\varepsilon + 2\Delta_s/k}.$$

For reasons similar to the discussion leading to (3.11), the above lemma will be used without the ε for the convention that $\Delta_s = ke^{1-(s-2)/k}$. The next result uses admissible exponents for minor arcs; for a discussion see [4, Section 5].

Lemma 3.4.2. *Suppose that $k \geq 3$, $s \geq 2k+3$, and Δ_s^* is an admissible exponent for minor arcs with $\Delta_s^* < 0$. Let ν be any positive number with*

$$\nu < \min\left\{\frac{2|\Delta_s^*|}{k}, \frac{1}{6k}\right\}.$$

Then, when $1 \leq Q \leq P^{k/2}$, *one has the uniform bound*

$$\int_{\mathfrak{m}_{HL}(Q)} |f(\alpha; P, R)|^s d\alpha \ll P^{s-k} Q^{-\nu}.$$

In Brüden and Wooley's proof of [4, Theorem 6.2], it is demonstrated that $\Delta_s^* < 0$ for $s \geq \max\{\lfloor G_0(k) \rfloor + 1, 2k + 3\}$, where [4, Equation 7.8] shows that $G_0(k) \leq \lceil k(\log k + 4.20032) \rceil - 1$ for $k \geq 20$. Now by recalling the definition of Δ_s (3.11) and by applying the bounds of Lemmata 3.4.1 and 3.4.2 with Q chosen to be a fixed constant, one gets for $s \geq k\lceil (\log k + 4.20032) \rceil$ that

$$\int_0^1 |f(\beta)|^s d\beta \ll P^{s-k}. \tag{3.14}$$

In the remainder of the section, a bound for the minor arcs of the Davenport-Heilbronn method is explored.

Lemma 3.4.3. *Let* $\mathfrak{i}_n$ *be a unit interval contained in* $\mathfrak{m}_{DH}$. *Let* $k \geq 20$ *and* $s \geq \lceil k(\log k + 4.20032) \rceil$. *Then*

$$\int_{\mathfrak{i}_n} \prod_{i=1}^{s} |f(\lambda_i \alpha)| d\alpha \ll P^{s-k} (\log T(P))^{-2}. \tag{3.15}$$

Proof. Let $\mathfrak{d}$ be the set of real numbers α for which $\lambda_1 \alpha \pmod 1$ is contained in $\mathfrak{m}_{HL}(\log^{2s/\nu} T(P))$, and $\mathfrak{D} = \mathbb{R} \setminus \mathfrak{d}$. One can verify that that $\mathfrak{D}$ is the set of real numbers α for which $\lambda_1 \alpha \pmod 1$ is contained in $\mathfrak{M}_{HL}(\log^{2s/\nu} T(P))$.

Using Hölder's inequality reveals that

$$\int_{\mathfrak{d} \cap \mathfrak{i}_n} \prod_{i=1}^{s} |f(\lambda_i \alpha)| d\alpha \ll \Big(\int_{\mathfrak{d} \cap \mathfrak{i}_n} |f(\lambda_1 \alpha)|^s d\alpha \Big)^{1/s} \prod_{i=2}^{s} \Big(\int_{\mathfrak{i}_n} |f(\lambda_i \alpha)|^s d\alpha \Big)^{1/s}.$$

A change of variables allows for the use of Lemma 3.4.2 with $Q = \log^{2s/\nu} T(P)$ for the first integral. In the same spirit, another change of variables provides an upper bound, via (3.14), for the remaining integrals. The above formula now transforms to

$$\int_{\mathfrak{d} \cap \mathfrak{i}_n} \prod_{i=1}^{s} |f(\lambda_i \alpha)| d\alpha \ll P^{(s-k)/s} (\log^{2s/\nu} T(P))^{-\nu/s} P^{(s-k)\frac{s-1}{s}} \ll P^{s-k} (\log T(P))^{-2}, \tag{3.16}$$

where ν satisfies the conditions of Lemma 3.4.2.

Recall that $\lambda_1/\lambda_2 \notin \mathbb{Q}$. By an application of Holder's inequality, one can see that

$$\int_{\mathfrak{D}\cap \mathfrak{i}_n} \prod_{i=1}^{s} |f(\lambda_i \alpha)| d\alpha \leq \Big(\sup_{\alpha \in \mathfrak{i}_n} |f(\lambda_1\alpha) f(\lambda_2 \alpha)| \Big)^{\frac{\mu}{s+\mu}} \Big(\int_{\mathfrak{D}\cap \mathfrak{i}_n} |f(\lambda_1 \alpha)|^{\frac{s^2}{(s+2\mu)}} d\alpha \Big)^{\frac{s+2\mu}{s(s+\mu)}} \times \Big(\int_{\mathfrak{i}_n} |f(\lambda_2 \alpha)|^s d\alpha \Big)^{\frac{1}{s+\mu}} \prod_{i=3}^{s} \Big(\int_{\mathfrak{i}_n} |f(\lambda_i \alpha)|^s d\alpha \Big)^{1/s}.$$

Choosing $\mu = (s-4k)/4$ allows for the application of Lemma 3.4.1 to the first integral with $Q = \log^{2s/\nu} T(P)$ since

$$\frac{s^2}{s+2\mu} = 4k \frac{k + 2\mu + \frac{\mu^2}{k}}{k + \frac{3}{2}\mu} > 4k.$$

For the rest of the integrals, the bound (3.14) can be applied, while for the supremum Lemma 3.3.3 can be used. Hence,

$$\begin{aligned} &\int_{\mathfrak{D}\cap \mathfrak{i}_n} \prod_{i=1}^{s} |f(\lambda_i \alpha)| d\alpha \\ &\qquad \ll \Big(P^2 T(P)^{-\frac{1-\varphi}{k}} \Big)^{\frac{\mu}{s+\mu}} \Big(P^{\frac{s^2}{(s+2\mu)}-k} (\log^{2s/\nu} T(P))^{2e^{1+\frac{2}{k}-\frac{2s^2}{k(s+2\mu)}}} \Big)^{\frac{s+2\mu}{s(s+\mu)}} P^{(s-k)(\frac{1}{s+\mu}+\frac{s-2}{s})} \\ &\qquad = P^{s-k} T(P)^{-\frac{1-\varphi}{k}\frac{\mu}{s+\mu}+\varepsilon}. \end{aligned} \tag{3.17}$$

The proof is now concluded by combining (3.16) and (3.17) and observing that

$$\int_{\mathfrak{i}_n} \prod_{i=1}^{s} |f(\lambda_i \alpha)| d\alpha \ll \int_{\mathfrak{d}\cap \mathfrak{i}_n} \prod_{i=1}^{s} |f(\lambda_i \alpha)| d\alpha + \int_{\mathfrak{D}\cap \mathfrak{i}_n} \prod_{i=1}^{s} |f(\lambda_i \alpha)| d\alpha \ll P^{s-k} (\log T(P))^{-2}. \quad \square$$

In the above proof, $Q = \log T(P)$ was used in the definition of $\mathfrak{M}_{HL}$. Lemma 3.4.1 provides bounds for the terms of the product in (3.15) in terms of Q. The Weyl-type bound is based on the function $T(P)$, and should the size of $T(P)$ be asymptotically smaller than Q, then the process would not have worked. $T(P)$ can be allowed to grow arbitrarily slowly, but there are no obvious means of forcing its growth rate to increase. Hence, the Hardy-Littlewood arcs were defined to be based on $T(P)$ as well.

By eliminating the need for a secondary function $S(P)$ in the definition of $\mathfrak{m}_{DH}$ (see [35])

and only using the function $T(P)$, whose size can be made to increase arbitrarily slowly, the author sacrificed any hope of understanding how large the variables x_i need to be in order for a non-trivial solution to exist for (3.2). For a given set of coefficients, knowledge of the continued fraction expansion of the irrational quotient can provide insight on the size of the variables.

Now the contribution of the minor arcs is finalized. Recall that

$$\mathfrak{m}_{DH} = \{\alpha \in \mathbb{R} : T(P)P^{-k} < |\alpha| \leq T(P)\}.$$

Lemma 3.4.4. *Let $k \geq 20$ and $s \geq \lceil k(\log k + 4.20032)\rceil$. Then*

$$\int_{\mathfrak{m}_{DH}} \Big|\Big(\prod_{i=1}^{s} f(\lambda_i\alpha)\Big)e(\alpha\omega)K_{\pm}(\alpha)\Big|d\alpha \ll P^{s-k}(\log T(P))^{-1}.$$

Proof. Making use of the Equations (3.4) and (3.5) with $L(P) = T(P)^{1/2}$ in the definition of $K_{\pm}(\alpha)$, along with Lemmata 3.2.1 and 3.4.3,

$$\begin{aligned}
&\int_{\mathfrak{m}_{DH}} \Big|\Big(\prod_{i=1}^{s} f(\lambda_i\alpha)\Big)e(\alpha\omega)K_{\pm}(\alpha)\Big|d\alpha \\
&\qquad \ll \int_{T(P)P^{-k}}^{1+T(P)P^{-k}} \Big|\prod_{i=1}^{s} f(\lambda_i\alpha)\Big|d\alpha + \sum_{1\leq n\leq T(P)} n^{-1}\int_{n}^{n+1} \Big|\prod_{i=1}^{s} f(\lambda_i\alpha)\Big|d\alpha \\
&\qquad \ll (1+\log(T(P))P^{s-k}(\log T(P))^{-2} \\
&\qquad \ll P^{s-k}(\log T(P))^{-1}.
\end{aligned}$$

□

3.5 Trivial Arcs

As usual with the Davenport-Heilbronn method and its applications, the trivial arcs are handled in a routine manner. Recall that

$$\mathfrak{t}_{DH} = \{\alpha \in \mathbb{R} : T(P) < |\alpha|\}.$$

Lemma 3.5.1. *For $k \geq 20$ and $s \geq \lceil k(\log k + 4.20032)\rceil$, then*

$$\int_{\mathfrak{t}_{DH}} |f(\lambda_1\alpha)\cdots f(\lambda_s\alpha)e(\alpha\omega)K_{\pm}(\alpha)|d\alpha \ll P^{s-k}T(P)^{-1/2}.$$

Proof. By the upper bound for $K_{\pm}(\alpha)$ from Lemma 3.2.1, and by making use of Equations (3.4) and (3.5) with $L(P) = T(P)^{1/2}$, one has

$$\begin{aligned}\int_{\mathfrak{t}_{DH}} |f(\lambda_1\alpha)\cdots f(\lambda_s\alpha)e(\alpha\omega)K_{\pm}(\alpha)|d\alpha& \\ \ll T(P)^{1/2}\sum_{n=0}^{\infty}(n+T(P))^{-2}\int_{n+T(P)}^{n+1+T(P)}\prod_{i=1}^{s}|f(\lambda_i\alpha)|d\alpha.&\end{aligned} \tag{3.18}$$

For the latter integral, by utilizing (3.14) and applying Hölder's inequality, one gets

$$\begin{aligned}T(P)^{1/2}\sum_{n=0}^{\infty}(n+T(P))^{-2}\int_{n+T(P)}^{n+1+T(P)}|f(\lambda_1\alpha)\cdots f(\lambda_s\alpha)|d\alpha& \\ \ll T(P)^{-1/2}\prod_{i=1}^{s}\Big(\int_0^1|f(\lambda_i\alpha)|^s d\alpha\Big)^{1/s} \ll P^{s-k}T(P)^{-1/2}.&\end{aligned} \tag{3.19}$$

By combining (3.18) and (3.19), the proof of the lemma is complete. □

3.6 Major Arcs

Let

$$u_i(\alpha) = \int_0^P e(\lambda_i\alpha\gamma^k)d\gamma.$$

By integration by parts, one has that $u_i(\alpha) \ll P(1+P^k|\alpha|)^{-1/k}$. By [30, Lemma 8.5] (see also [24, Lemma 5.4]), for $1 \leq i \leq s$,

$$\sup_{\alpha\in\mathfrak{M}_{DH}}|f(\lambda_i\alpha) - \xi' u_i(\alpha)| \ll \frac{P}{\log P}T(P) \ll PT(P)^{-2},$$

where ξ' is a positive constant, since $T(P) \leq \log^{1/3} P$. One infers that

$$\sup_{\alpha \in \mathfrak{M}_{DH}} |f(\lambda_1 \alpha) \cdots f(\lambda_s \alpha) - \xi u_1(\alpha) \cdots u_s(\alpha)| \ll P^s T(P)^{-2},$$

with $\xi = \xi'^s$, and from the fact that $|K_{\pm}(\alpha)| \ll 1$ for $\alpha \in \mathfrak{M}_{DH}$ and $\text{meas}(\mathfrak{M}_{DH}) \ll T(P)P^{-k}$, one sees that

$$\int_{\mathfrak{M}_{DH}} |(f(\lambda_1 \alpha) \cdots f(\lambda_s \alpha) - \xi u_1(\alpha) \cdots u_s(\alpha)) e(\alpha\omega) K_{\pm}(\alpha)| d\alpha \ll P^{s-k} T(P)^{-1}. \tag{3.20}$$

Using the indicator function from (3.6) and (3.7), then in the current setting

$$\Big| \int_{-\infty}^{\infty} e(\Omega\alpha) K_{\pm}(\alpha) d\alpha - \mathcal{X}(\Omega) \Big| \neq 0 \tag{3.21}$$

only if $|\epsilon - |\Omega|| \leq \epsilon T(P)^{-1/2}$.

Lemma 3.6.1. *Let $k \geq 20$ and $s \geq k+1$. Then for P sufficiently large,*

$$\int_{-\infty}^{\infty} \Big(\prod_{i=1}^{s} u_i(\alpha) \Big) e(\alpha\omega) K_{\pm}(\alpha) d\alpha = 2\epsilon \mathfrak{P} \mathcal{I} P^{s-k} + O(P^{s-k} T(P)^{-1/2}),$$

where $\mathfrak{P} = \prod_{i=1}^{s} k^{-1} |\lambda_i|^{-1/k}$,

$$\mathcal{I} = \int_{\mathcal{S}} y_2^{1/k-1} \cdots y_s^{1/k-1} \Big(y_2 \mp y_3 \mp \cdots \mp y_s - \frac{\omega}{P^k} \Big)^{1/k-1} d\underline{y} > 0,$$

and $\mathcal{S}$ denotes the set of points $(y_2, \ldots, y_s)$ in $[0, |\lambda_2|] \times \cdots \times [0, |\lambda_s|]$ satisfying the condition that

$$y_2 \mp y_3 \mp \cdots \mp y_s - \frac{\omega}{P^k} \in [0, \lambda_1].$$

Proof. Let $\mathcal{B} = [0, \lambda_1] \times \cdots \times [0, |\lambda_s|]$. Using the change of variables $y_i = |\lambda_i| \gamma_i^k P^{-k}$, the integral transforms to

$$\mathfrak{P} P^s \int_{\mathcal{B}} (y_1 \cdots y_s)^{\frac{1}{k}-1} \int_{-\infty}^{\infty} e\Big(\alpha P^k \Big(y_1 - y_2 \pm y_3 \pm \cdots \pm y_s + \frac{\omega}{P^k} \Big) \Big) K_{\pm}(\alpha) d\alpha d\underline{y}. \tag{3.22}$$

The innermost integral is equal to $\mathcal{X}(P^k|y_1 - y_2 \pm y_3 \pm \cdots \pm y_s + \frac{\omega}{P^k}|)$ unless

$$\Big|\epsilon - P^k|y_1 - y_2 \pm y_3 \pm \cdots \pm y_s + \frac{\omega}{P^k}|\Big| \le \epsilon T(P)^{-1/2}$$

by setting $\Omega = P^k|y_1 - y_2 \pm y_3 \pm \cdots \pm y_s + \frac{\omega}{P^k}|$ in the formula (3.21). One starts by focusing on the set of values for which the characteristic function is not equal to the innermost integral. The value of their difference is bounded by one, and they differ only on a subset of the values of $\underline{y} \in \mathcal{B}$ for which

$$\epsilon P^{-k}(1 - T(P)^{-1/2}) < \Big|y_1 - y_2 \pm y_3 \pm \cdots \pm y_s + \frac{\omega}{P^k}\Big| < \epsilon P^{-k}(1 + T(P)^{-1/2}).$$

The volume of this set is $\ll \epsilon P^{-k} T(P)^{-1/2}$, and hence the contribution of the integral over this set is $\ll P^{s-k} T(P)^{-1/2}$. This will be proved to be an acceptable error.

As in [35, Section 6], a volume calculation shows that

$$\int_{\mathcal{B}} (y_1 \cdots y_s)^{\frac{1}{k}-1} \mathcal{X}\Big(P^k\Big|y_1 - y_2 \pm y_3 \pm \cdots \pm y_s + \frac{\omega}{P^k}\Big|\Big) d\underline{y} = 2\varepsilon P^{-k}\mathcal{I} + O(P^{-k-1}). \tag{3.23}$$

One now focuses on showing that the set $\mathcal{S}$ is not empty and that $\mathcal{I} > 0$. Let σ be a positive constant that satisfies $\sigma < \frac{1}{2s-1} \min_{1 \le i \le s} |\lambda_i|$. For the moment assume that each of the variables $y_3, y_4, \ldots, y_s$ is limited to the interval $(\frac{1}{2}\sigma, \sigma)$, with the exception that y_2 is restricted to $\big((s-1)\sigma + \frac{\omega}{P^k}, s\sigma + \frac{\omega}{P^k}\big)$ instead, where P is large enough in terms of ω. Note that for such a choice of $(y_3, \ldots, y_s)$, one has

$$-2(s-1)\sigma < -y_2 \pm y_3 \pm \cdots \pm y_s + \frac{\omega}{P^k} < -\sigma.$$

The reason for this endeavour is that with the above restrictions on $(y_2, \ldots, y_s)$, there is a y_1 which satisfies both $0 \le y_1 \le \lambda_1$ and

$$y_1 - y_2 \pm y_3 \pm \cdots \pm y_s + \frac{\omega}{P^k} = 0. \tag{3.24}$$

Let $\underline{v}$ be a solution of (3.24) that satisfies the above parameters, and set

$$\mathcal{L} = \frac{1}{2}\min\{2(s-1)\sigma - v_1, v_1 - \sigma\}.$$

Now, if $y_i \in [v_i - \frac{\mathcal{L}}{s-1}, v_i + \frac{\mathcal{L}}{s-1}]$ for $2 \le i \le s$, there is a $y_1 \in [0, \lambda_1]$ for which (3.24) holds. Hence, the $(s-1)$-dimensional volume of $\mathcal{S}$ is bounded from below by a positive constant. Now notice that the integrand of $\mathcal{I}$ is a product where each term is non-negative. Thus, one has that $\mathcal{I}$ is bounded from below by a positive constant.

Combining (3.21), (3.22), and (3.23) completes the proof of the lemma. □

The main contribution of the integral in Lemma 3.6.1 is from $\alpha \in \mathfrak{M}_{DH}$ since for $s \ge 2$,

$$\begin{aligned}\int_{\mathbb{R}\setminus\mathfrak{M}_{DH}} \Big(\prod_{i=1}^{s} u_i(a)\Big) e(\alpha\omega) K_{\pm}(\alpha) d\alpha &\ll \int_{|\alpha|>T(P)P^{-k}} P^s(1+P^k|\alpha|)^{-s} d\alpha \\ &\ll \int_{|\alpha|>T(P)P^{-k}} P^s(1+P^k|\alpha|)^{-2} d\alpha \ll P^{s-k}T(P)^{-1},\end{aligned}$$

which implies that

$$\int_{\mathfrak{M}_{DH}} \Big(\prod_{i=1}^{s} u_i(\alpha)\Big) e(\alpha\omega) K_{\pm}(\alpha) d\alpha = 2e\mathfrak{P}\mathcal{I}P^{s-k} + O(P^{s-k}T(P)^{-1/2}).$$

The conclusion of the discussion above is summarized in the following lemma.

Lemma 3.6.2. *Assume that the conditions of Lemma 3.6.1 are satisfied. Then*

$$\int_{\mathfrak{M}_{DH}} \Big(\prod_{i=1}^{s} f_i(\alpha)\Big) e(\alpha\omega) K_{\pm}(\alpha) d\alpha = 2\xi e\mathfrak{P}\mathcal{I}P^{s-k} + O(P^{s-k}T(P)^{-1/2}).$$

3.7 The Proof of Theorem 3.1.1

Theorem 3.1.1 is an immediate implication of the following result.

Theorem 3.7.1. *Let $k \ge 20$, $s > \lceil k(\log k + 4.20032)\rceil$, $\lambda_1/\lambda_2 \notin \mathbb{Q}$, $\lambda_1 > 0$, and $\lambda_2 < 0$. The*

number of integer solutions of inequality (3.2) *with* $x_i \in \mathcal{A}(P, P^\eta)$ $(1 \le i \le s)$ *is*

$$2\xi\epsilon\mathfrak{P}\mathcal{I}P^{s-k} + O(P^{s-k}(\log T(P))^{-1}),$$

where $\mathfrak{P}$ *and* $\mathcal{I}$ *satisfy their respective definitions in Lemma* 3.6.1 *and* ξ *is a positive constant depending on* η.

Proof. Using the conclusions of Sections 4, 5, and 6, the proof of Theorem 3.7.1 can be assembled. From the definition of $R_\pm(P)$ in (3.8), it is obvious that

$$\Big|R_\pm(P) - \int_{\mathfrak{M}_{DH}} \Big(\prod_{i=1}^{s} f(\lambda_i\alpha)\Big)e(\alpha\omega)K_\pm(\alpha)d\alpha\Big| \le \int_{\mathfrak{m}_{DH}\cup\mathfrak{t}_{DH}} \Big|\prod_{i=1}^{s} f(\lambda_i\alpha)\Big|K_\pm(\alpha)d\alpha,$$

and whence from Lemmata 3.4.4 and 3.5.1, it follows that

$$\Big|R_\pm(P) - \int_{\mathfrak{M}_{DH}} \Big(\prod_{i=1}^{s} f(\lambda_i\alpha)\Big)e(\alpha\omega)K_\pm(\alpha)d\alpha\Big| = O(P^{s-k}(\log T(P))^{-1}). \tag{3.25}$$

By Lemma 3.6.2, formula (3.25) implies

$$R_\pm(P) = 2\xi\epsilon\mathfrak{P}\mathcal{I}P^{s-k} + O(P^{s-k}(\log T(P))^{-1}),$$

and hence (3.9) completes the proof of Theorem 3.7.1. □

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