OPTIMAL CONTROL AND ANALOG COMPUTER SIMULATION

OF SOME INDUSTRIAL MANAGEMENT PROBLEMS

by 45

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INTRODUCTION

The development of management science may be said to consist of the application of mathematical techniques to the solution of problems arising in business. During the past decade, there has been a remarkable growth of interest in problems of dynamic optimization. This has given rise to a number of methods useful for rendering systems optimum. One such method is Pontryagin's maximum principle. It was developed in 1956 for continuous processes and has been mainly applied in the field of optimum systems control. Its application to dynamic operations research models is very limited.

Pioneering work on the application of the maximum principle to dynamic operations research models has been done by Connors and Teichroew in their recent book [1]. Part I of this report has resolved all thirteen examples presented in the book. The purpose of resolving all the examples considered by Connors and Teichroew [1] is two-fold. First, Connors and Teichroew solve some of the examples by the classical calculus of variations and some by the maximum principle. In this report, a maximum principle approach is adopted for all the examples. Second, the solution procedure used by Connors and Teichroew is very tedious for some of their examples, which can be solved more simply by the approach taken in this report. Moreover, the solutions obtained by them for some of the examples are different from those obtained in this report.

Pioneering work on digital simulation of management models has been done by Forrester [3]. Part II of this report deals with optimal solution and analog computer simulation of some industrial management problems. In analog simulation a set of differential equations is considered, whereas in the digital simulation done by Forrester [3], the differential equations are reduced to difference equations or the continuous nature of the model is reduced to a discrete one.

Example 1 of Part II deals with optimum production scheduling where the system of equations derived from the maximum principle is simulated after performing a one dimensional search. Example 2 is a modification of Example 1 where, instead of a single period process, a periodic process is assumed. In this example, a two dimensional search is carried out and then the model is simulated. Examples 1 and 2 yield the optimal solution because the set of equations simulated is derived by the maximum principle. Example 3 deals with an inventory control model [Example 8, Part I] and the model is simulated with a number of production functions for a fixed demand rate. The optimal solution obtained by simulation is the same one as obtained by the maximum principle.

PART I

OPTIMAL CONTROL OF SOME INVENTORY AND PRODUCTION SCHEDULING PROBLEMS

1. STATEMENT OF THE ALGORITHM OF THE CONTINUOUS MAXIMUM PRINCIPLE [2]

Pontryagin's maximum principle is concerned, in general, with solving problems of the following type:

Suppose the performance equations of a control process have the form

$$\frac{dx_i}{dt} = f_i(x_1(t),\ x_2(t),\ \ldots,\ x_s(t);\ \theta_1(t),\ \ldots,\ \theta_r(t)),\ t_0 \leq t \leq T$$

$$x_i(t_0) = \alpha_i, \qquad i = 1, 2, \ldots, s,$$

or the vector form

$$\frac{dx}{dt} = f(x(t);\ \theta(t)),\quad x(t_0) = \alpha, \tag{1}$$

where $x(t)$ is an s-dimensional vector function representing the state of the process at time t and $\theta(t)$ is an r dimensional vector function representing the decision at time t.

A typical optimization problem associated with such a process is to find a piecewise continuous decision vector function, $\theta(t)$, subject to the constraints

$$\psi_i(\theta_1(t),\ \theta_2(t),\ \ldots,\ \theta_r(t)) \leq 0, \qquad i = 1, 2, \ldots, m, \tag{2}$$

which makes a function of the final values of the state

$$S = \sum_{i=1}^{s} c_i x_i(T), \qquad c_i = \text{constant}, \tag{3}$$

an extremum when the initial condition $x(t_0) = \alpha$ is given. The function, S, which is to be extremized, is the objective function of the process.

The procedure for solving the problem is to introduce an s dimensional adjoint vector $z(t)$ and a Hamiltonian function H which satisfy the following relations:

$$H(z(t), x(t), \theta(t)) = \sum_{i=1}^{s} z_i f_i(x(t), \theta(t)) \tag{4}$$

$$\frac{dz_i}{dt} = -\frac{\partial H}{\partial x_i} = -\sum_{j=1}^{s} z_j \frac{\partial f_j}{\partial x_i}, \qquad i = 1, 2, \ldots, s \tag{5}$$

$$z_i(T) = c_i, \qquad i = 1, 2, \ldots, s \tag{6}$$

The optimal decision vector function $\theta^*(t)$, which makes S an extremum (maximum or minimum), is the decision vector function $\theta(t)$, which renders the Hamiltonian function H an extremum for $t_0 \leq t \leq T$. If the optimal decision vector function $\theta^*(t)$ is interior to the set of admissible decisions $\theta(t)$, a necessary condition for S to be an extremum with respect to $\theta(t)$ is

$$\frac{\partial H}{\partial \theta} = 0 \tag{7}$$

If $\theta(t)$ is constrained, the optimal vector function $\theta^*(t)$ is determined either by solving equation (7) for $\theta(t)$ or by searching the boundary of the set.

Once the decision vector function $\theta(t)$ is chosen, the adjoint vector function $z(t)$ is uniquely determined by equations (5) and (6), and the initial condition at $t = t_0$, $x(t_0) = \alpha$.

Theorem: Let $\theta(t)$, $t_0 \le t \le T$ be a piecewise continuous vector function satisfying the constraints given in equation (2). In order that the scalar function S given in equation (3) may be an extremum for a process described by equation (1), with the initial condition at $t = t_0$, $x(t_0) = \alpha$, given, it is necessary that there exist a non-zero continuous vector function $z(t)$ satisfying equations (5) and (6) and that the vector function $\theta(t)$ be so chosen that $H(z(t), x(t), \theta(t))$ is a extremum for every t, $t_0 \le t \le T$. Furthermore, the extreme value of H is a constant for every t.

The algorithm presented can be extended [2] to handle a variety of problems usually encountered in practice, for example, processes with fixed end points, processes with choice of initial values, nonautonomous systems, processes with memory in decision, processes described by a second order differential equation with constant coefficients, processes with constraints imposed on state variables and so on. A simplified derivation of the algorithm is also presented in [2].

2. OPTIMAL CONTROL [2]

Control problems can be classified, in general, according to the forms of the functional objective function, the specifications of the initial and final state, and the types of controller. Some typical forms are

a. Functional objective function

1. Minimal time control, in which $S = \int_{t_0}^{T} dt$

2. Minimal integrated error, in which $S = \int_{t_0}^{T} f(x_i)\, dt$,

   $f(x_i)$ is of the form, $|x_i|$, $(x_i)^2$, ...

3. Minimal integrated effort, in which $S = \int_{t_0}^{T} f(\theta_i)\, dt$,

   $f(\theta_i)$ is of the form, $|\theta_i|$, $(\theta_i)^2$, ...

4. Minimal terminal error in which

$$S = \left[x_1(T)\right]^2 + \left[x_2(T)\right]^2 + \ldots + \left[x_s(T)\right]^2$$

b. Specifications of the Initial and Final State

1. End point fixed; that is, $x_1(0)$, $x_2(0)$, ..., $x_s(0)$ and $x_1(T)$, $x_2(T)$, ..., $x_s(T)$ are completely specified.

2. End points partly fixed; that is, only some of the $x_i(0)$ and $x_i(T)$, $i = 1, 2, \ldots, s$ are specified.

3. Final point not fixed; that is, $x_1(T)$, $x_2(T)$, ..., $x_s(T)$ are not specified.

   Different combinations of the functional and the terminal state are possible in practical control problems.

c. Types of Control.

In extremizing the Hamiltonian, H, with respect to θ, the terms that do not involve θ can be disregarded, and therefore H can be divided into two parts as

$$H = (\text{group of terms in which } \theta \text{ appears explicitly}) + (\text{other terms which do not involve } \theta)$$
$$= H^* + \text{remainder}$$

Then for a constrained control variable, $\theta_{min} \leq \theta \leq \theta_{max}$, the following three types of optimal control can be considered:

1. Bang-Bang. This is the case in which the objective function does not include θ or has a term that is proportional to θ, and H^* is of the form

$$H^* = h\theta, \tag{8}$$

where h may be a function of time. The conditions for H^* to be minimum are

$$\theta = \theta_{min} \qquad \text{if} \quad h > 0$$

$$\theta = \theta_{max} \qquad \text{if} \quad h < 0$$

2. Bang-Bang with coasting (bang-off bang). This is the case in which the objective function involves $|\theta|$ and H^* is of the form

$$H^* = w|\theta| + h\theta, \tag{9}$$

where w is a positive constant and h involves $z_i(t)$. The function H^* attains its minimum when

$$\theta = \theta_{min} \qquad \text{if} \quad w < h$$

$$\theta = 0 \qquad \text{if} \ -w < h < +w$$

$$\theta = \theta_{max} \qquad \text{if} \quad h < -w$$

If the objective function is to be maximized, the inequalities are reversed for the above two types of control.

3. Continuous with or without saturation. This is the case in which the objective function involves $(\theta)^2$. If H^* is of the form

$$H^* = w(\theta)^2 + h\theta \ , \tag{10}$$

then H^* attains its extreme value only if

$$\frac{\partial H^*}{\partial \theta} = 0$$

or

$$\theta = -\frac{h}{2w}$$

which is the optimal control if it is in the range of admissible control. Therefore θ can take any value between the two limits, θ_{min} and θ_{max}.

3. CASE STUDIES

Example 1

This example deals with production scheduling and the objective is to minimize the total cost comprised of mechanical changes and personnel costs associated with changing the production rate. The cost associated with the mechanical changes due to a change in the production rate, p(t), is assumed to vary as the square of the rate of change of the production rate, $\dot{p}(t)$, represented by

$$C_1 = k_1 \, [\dot{p}(t)]^2 \tag{1}$$

The personnel cost is comprised of the cost for hiring new workers if the production rate is increased and savings obtained due to the reduction in labor force if the production rate is decreased. The cost or the saving is assumed to be proportional to the rate of change of the production rate and varies linearly with time as represented by

$$C_2 = k_2 \, t \, \dot{p}(t) \tag{2}$$

The total cost equation over a time interval $[t_0, T]$ can then be written as

$$C = \int_{t_0}^{T} [C_1 + C_2]\, dt$$

$$= \int_{t_0}^{T} \{k_1 [\dot{p}(t)]^2 + k_2\, t\, \dot{p}(t)\}\, dt \tag{3}$$

where k_1, and k_2 are cost coefficients.

The problem is to choose a rate of production $p(t)$ to minimize equation (3) subject to the boundary conditions

$$p(t_0) = p_0; \quad p(T) = p_1 \tag{4}$$

Solution:

Let

$$x_1(t) = p(t)$$

$$\frac{dx_1(t)}{dt} = \frac{dp(t)}{dt} = \theta(t) \tag{5}$$

where

$$x_1(t_0) = p_0, \quad x_1(T) = p_1 \tag{5a}$$

To solve this problem we introduce additional state variables $x_2(t)$ and $x_3(t)$ such as

$$x_2(t) = \int_{t_0}^{t} \{k_1[\theta(t)]^2 + k_2 t\theta(t)\}\, dt$$

$$x_3(t) = t$$

It follows that

$$\frac{dx_2(t)}{dt} = k_1\ [\theta(t)]^2 + k_2 t\ \theta(t), \qquad x_2(t_0) = 0 \tag{6}$$

$$\frac{dx_3(t)}{dt} = 1, \qquad x_3(t_0) = t_0 \tag{7}$$

The problem is thus transformed into that of minimizing $x_2(T)$; then we have

$$c_1 = 0, \quad c_2 = 1 \text{ and } c_3 = 0$$

The Hamiltonian function is

$$H = z_1\theta + z_2[k_1\theta^2 + k_2 x_3 \theta] + z_3(1) \tag{8}$$

According to the definition of the adjoint vector, we have

$$\frac{dz_1}{dt} = -\frac{\partial H}{\partial x_1} = 0 \tag{9}$$

$$\frac{dz_2}{dt} = -\frac{\partial H}{\partial x_2} = 0, \quad z_2(T) = c_2 = 1 \tag{10}$$

$$\frac{dz_3}{dt} = -\frac{\partial H}{\partial x_3} = -z_2\ k_2\theta \tag{11}$$

Solution of equations (9) and (10) for $z_1(t)$ and $z_2(t)$ for $t_0 \leq t \leq T$ gives

$$z_1(t) = \text{constant} \tag{12}$$

$$z_2(t) = 1 \tag{13}$$

Hence the Hamiltonian can be rewritten as

$$H = z_1\theta + [k_1\theta^2 + k_2x_3\theta] + z_3(1)$$

Therefore H^*, the portion of H which depends on θ is as follows:

$$H^* = (z_1 + k_2x_3)\ \theta + k_1\theta^2 \tag{14}$$

It is worth mentioning that in minimizing the objective function the Hamiltonian must be minimized. Inspection of H^* shows that the optimal controller should be a continuous type. The optimal control for this type is found from the condition

$$\frac{\partial H^*}{\partial \theta} = 0 = z_1 + k_2x_3 + 2k_1\theta \tag{15}$$

or

$$\theta = -\frac{(z_1 + k_2x_3)}{2k_1} \tag{16}$$

Replacing x_3 by t in equation (16) and using this result with equation (12) in equation (5) yields

$$\frac{dx_1}{dt} = -\frac{(z_1 + k_2t)}{2k_1} \tag{17}$$

or

$$x_1(t) = -\frac{k_2}{4k_1}t^2 + A_1t + A_2 \tag{18}$$

where A_1 and A_2 are constants of integration. From the boundary conditions of equation (5a) A_1 and A_2 are obtained as

$$A_1 = \frac{p_0 - p_1}{t_0 - T} + \frac{k_2}{4k_1}(t_0 + T)$$

$$A_2 = p_0 - t_0 \left(\frac{p_0 - p_1}{t_0 - T} + \frac{k_2 T}{4k_1}\right) \qquad (19)$$

Since k_1 is a positive constant, the sufficient condition for minimizing the cost function

$$\frac{\partial^2 H^*}{\partial \theta^2} = 2k_1 > 0$$

is satisfied.

Therefore the optimal policy is to adjust the production rate so that it follows a parabolic arc connecting the end points (t_0, p_0) and (T, p_1) such as

$$p(t) = -\frac{k_2}{4k_1} t^2 + \left(\frac{p_0 - p_1}{t_0 - T} + \frac{k_2}{4k_1}(t_0 + T)\right) t$$

$$+ p_0 - t_0 \left(\frac{p_0 - p_1}{t_0 - T} + \frac{k_2 T}{4k_1}\right) \qquad (20)$$

Differentiating equation (17) with respect to t we have

$$\ddot{x}_1(t) = \ddot{p}(t) = -\frac{k_2}{2k_1} \qquad (21)$$

From the fact that $\ddot{p}(t) \leq 0$ it follows that the optimal production policy is concave. It is of interest to determine whether the optimal policy requires producing up to a peak rate and then dropping down, as in Figure 1 (a), or if the optimal policy is monotone, as in Figure 1 (b).

The condition leading to a monotone policy can be determined analytically. Differentiating equation (20) and setting the result to zero gives the point t^* where p(t) attains its maximum value as

$$t^* = \frac{2k_1}{k_2} \left(\frac{p_0 - p_1}{t_0 - T}\right) + \frac{1}{2} (t_0 + T) \tag{22}$$

Then

$$t_0 \leq t^* \leq T \tag{23}$$

if and only if,

$$t_0 \leq 2 \frac{k_1}{k_2} \left(\frac{p_0 - p_1}{t_0 - T}\right) + \frac{1}{2} (T + t_0) \leq T \tag{24}$$

or

$$-\left(\frac{T - t_0}{2}\right) \leq \frac{2k_1}{k_2} \left(\frac{p_0 - p_1}{t_0 - T}\right) \leq \frac{T - t_0}{2} \tag{25}$$

But $s = (p_0 - p_1) / (t_0 - T)$ is the slope of the secant connecting the end points of the policy. Hence, the policy is monotone, if and only if, the product of the slope s of the secant connecting the end points of the policy and the quotient of cost coefficients $2k_1/k_2$ is greater than $-\bar{t}$ or less than $\bar{t}$ where $\bar{t} = \frac{1}{2} (T - t_0)$.

The same solution was obtained by Connors and Teichrow (1) by the method of the classical calculus of variation.

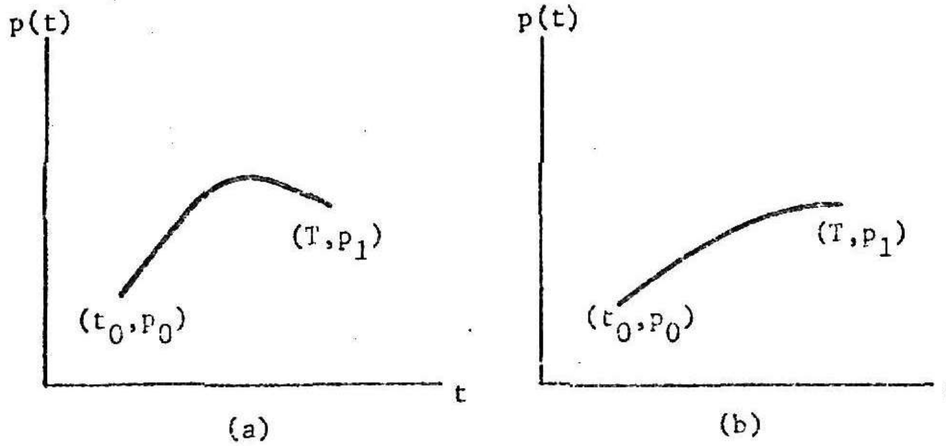

Fig. 1. a) Nonmonotone policy for Example 1
b) Monotone policy for Example 1

Example 2

This example is a modification of the production scheduling problem discussed in the previous example. Here the cost function is comprised only of mechanical changes associated with a change in the production rate, $p(t)$, represented by

$$C_1 = k_1 \; [p(t)]^2 \tag{1}$$

The problem is to choose a rate of production, $p(t)$, to minimize the cost function represented by

$$C = \int_{t_0}^{T} C_1 \; dt$$

$$= \int_{t_0}^{T} k_1[p(t)]^2 \; dt \tag{2}$$

subject to the boundary conditions

$$p(t_0) = p_0; \quad p(T) = p_1 \tag{3}$$

Solution:

Let

$$p(t) = \theta(t)$$

where

$$\theta(t_0) = p_0, \quad \theta(T) = p_1$$

By introducing a state variable $x_1(t)$ in which

$$x_1(t) = \int_{t_0}^{t} k_1[\theta(t)]^2 \; dt \tag{4}$$

we have

$$\frac{dx_1(t)}{dt} = k_1 \, [\theta(t)]^2, \; x_1(t_0) = t_0 \tag{5}$$

and

$$S = x_1(T) \tag{6}$$

The Hamiltonian function is

$$H = z_1 k_1 \theta^2 \tag{7}$$

The component of the adjoint vector is defined by

$$\frac{dz_1}{dt} = - \frac{\partial H}{\partial x_1} = 0, \quad z_1(T) = 1 \tag{8}$$

Solving for $z_1(t)$ from equation (8) yields

$$z_1(t) = 1, \quad t_0 \leq t \leq T \tag{9}$$

Hence the portion of the Hamiltonian which depends on θ becomes

$$H^* = k_1 \theta^2 \tag{10}$$

This shows that the optimal control is continuous as in the previous example.

The optimal control for this type is found from the condition

$$\frac{\partial H^*}{\partial \theta} = 0 = 2k_1 \theta \tag{11}$$

which has as its only solution

$$\theta^*(t) = p^*(t) = 0 \tag{12}$$

The solution represented by equation (12) satisfies the boundary conditions if and only if $p_0 = p_1 = 0$. For the optimal policy to satisfy the boundary constraints represented by equation (3), there must exist a sequence of functions, $p^n(t) \varepsilon A$, which approach the true minimizing function $p^*(t)$ as n tends to ∞.

Therefore the optimal solution is

$$\begin{aligned} p^*(t) &= p_0 && \text{if } t = t_0 \\ &= 0 && \text{if } t_0 \leq t \leq T \\ &= p_1 && \text{if } t = T \end{aligned}$$

Example 3

This example deals with a production smoothing problem involving two products for which the production rates are $p_1(t)$ and $p_2(t)$ respectively. It is possible to change a flow of capital between the processes of these products to vary either production rate. The total cost is comprised of the cost associated with changing the production rates and the cost of re-allocating capital from one production process to another. The cost of changing the production rates is assumed to be proportional to the sum of the square of the production rates represented by

$$C_1 = k_1 \, [\dot{p}_1(t)]^2 + k_2 \, [\dot{p}_2(t)]^2 \tag{1}$$

The cost of reallocating capital from one production process to another is taken to be proportional to the product of the rate of change of production rates represented by

$$C_2 = k_3\ \dot{p}_1(t)\ \dot{p}_2(t) \tag{2}$$

It is assumed that the labor force is not changed; consequently, there are no personnel costs involved as these are merely reassigned between the production processes.

The problem is to choose rates of production $p_1(t)$ and $p_2(t)$ which minimize the total cost represented by

$$C = \int_{t_0}^{T} [C_1 + C_2]\ dt$$

$$= \int_{t_0}^{T} \{k_1\ [\dot{p}_1(t)]^2 + k_2\ [\dot{p}_2(t)]^2 + k_3\ \dot{p}_1(t)\ \dot{p}_2(t)\}\ dt \tag{3}$$

and satisfies the boundary conditions

$$\begin{aligned} p_1(t_0) &= p_1^0, \quad p_1(T) = p_1^1 \\ p_2(t_0) &= p_2^0, \quad p_2(T) = p_2^1 \end{aligned} \tag{3a}$$

Solution:

Let

$$x_1(t) = p_1(t)$$

$$x_2(t) = p_2(t)$$

$$\frac{dx_1(t)}{dt} = \frac{dp_1(t)}{dt} = \theta_1(t) \tag{4}$$

$$\frac{dx_2(t)}{dt} = \frac{dp_2(t)}{dt} = \theta_2(t) \tag{5}$$

where

$$x_1(t_0) = p_1^0, \quad x_1(T) = p_1^1$$

$$x_2(t_0) = p_2^0, \quad x_2(T) = p_2^1$$

By introducing an additional variable $x_3(t)$ in which

$$x_3(t) = \int_{t_0}^{t} \{k_1[\theta_1(t)]^2 + k_2[\theta_2(t)]^2 + k_3\ \theta_1(t)\ \theta_2(t)\}\ dt \tag{6}$$

we have

$$\frac{dx_3}{dt} = k_1\theta_1{}^2 + k_2\theta_2{}^2 + k_3\theta_1\theta_2, \quad x_3(t_0) = 0 \tag{7}$$

and

$$S = x_3(T) \tag{8}$$

The Hamiltonian is

$$H = z_1\theta_1 + z_2\theta_2 + z_3(k_1\theta_1{}^2 + k_2\theta_2{}^2 + k_3\theta_1\theta_2) \tag{9}$$

The components of the adjoint vector are defined by

$$\frac{dz_1}{dt} = -\frac{\partial H}{\partial x_1} = 0 \tag{10}$$

$$\frac{dz_2}{dt} = -\frac{\partial H}{\partial x_2} = 0 \tag{11}$$

$$\frac{dz_3}{dt} = -\frac{\partial H}{\partial x_3} = 0, \quad z_3(T) = 1 \tag{12}$$

Solving equations (10), (11) and (12) for $z_1(t)$, $z_2(t)$ and $z_3(t)$ for $t_0 \leq t \leq T$, we obtain

$$z_1(t) = \text{constant} \tag{13}$$

$$z_2(t) = \text{constant} \tag{14}$$

$$z_3(t) = 1 \tag{15}$$

Hence

$$H^* = z_1\theta_1 + z_2\theta_2 + [k_1\theta_1{}^2 + k_2\theta_2{}^2 + k_3\theta_1\theta_2] \tag{16}$$

The optimal control for this type is found from the condition

$$\frac{\partial H^*}{\partial \theta_1} = 0 = z_1 + 2k_1\theta_1 + k_3\theta_2 \tag{17}$$

$$\frac{\partial H^*}{\partial \theta_2} = 0 = z_2 + 2k_2\theta_2 + k_3\theta_1 \tag{18}$$

Substitution of equations (4) and (5) into equations (17) and (18) yields

$$z_1 + 2k_1\dot{x}_1 + k_3\dot{x}_2 = 0 \tag{19}$$

$$z_1 + 2k_2\dot{x}_2 + k_3\dot{x}_1 = 0 \tag{20}$$

Differentiating equations (19) and (20) with respect to t, we have

$$2k_1\ddot{x}_1 + k_3\ddot{x}_2 = 0 \tag{21}$$

$$2k_2\ddot{x}_2 + k_3\ddot{x}_1 = 0 \tag{22}$$

Equation (22) yields

$$\ddot{x}_2(t) = -\frac{k_3}{2k_2}\ddot{x}_1(t) \tag{23}$$

Substituting the result of equation (23) into equation (21) we obtain

$$\ddot{x}_1(t)\ [2k_1 - \frac{k_3^{\ 2}}{2k_2}] = 0 \tag{24}$$

or

$$x_1(t) = A_1\ t + A_2 \tag{25}$$

and

$$x_2(t) = A_3\ t + A_4 \tag{26}$$

where A_i, $i = 1, 2, 3, 4$ are integration constants. Application of the boundary conditions from equation (3a) into equations (25) and (26) yields

$$\begin{aligned} p_1(t) &= [\frac{p_1^0 - p_1^1}{t_0 - T}]\ t + p_1^0 - t_0\ [\frac{p_1^0 - p_1^1}{t_0 - T}] \\ \\ p_2(t) &= [\frac{p_2^0 - p_2^1}{t_0 - T}]\ t + p_2^0 - t_0\ [\frac{p_2^0 - p_2^1}{t_0 - T}] \end{aligned} \tag{26}$$

as optimal production rates for this problem.

The same results are obtained by the classical calculus of variation [1].

Example 4

This example illustrates optimal allocation of advertising expenditure where the objective is to maximize total sales in a known interval of time represented by

$$S_T = \int_{t_0}^{T} s(t)\ dt \tag{1}$$

It is assumed that the sales rate, $\dot{s}(t)$, is proportional to the square root of the quantity one plus the square of the rate of change of advertising rate, $a(t)$,

$$s(t) = k\sqrt{1 + \dot{a}(t)^2} \tag{2}$$

where k, an unknown proportionality factor, is a function of time t and the advertising rate, $a(t)$, as represented by

$$k = A[t, a(t)] \tag{3}$$

The problem is to determine the value of the advertising rate, $a(t)$, so as to maximize the objective function

$$S_T = \int_{t_0}^{T} A[t, a(t)]\sqrt{1 + \dot{a}(t)^2}\ dt \tag{4}$$

with the given initial condition $a(t_0) = a_0$. The present advertising policy is to phase into another advertising campaign starting at time T and therefore the final condition $a(T) = \phi$ must be satisfied.

Solution:

Let

$$x_1(t) = a(t),\ x_1(t_0) = a_0,\quad x_1(T) = \phi$$

$$\frac{dx_1(t)}{dt} = \frac{da(t)}{dt} = \theta(t) \tag{5}$$

By introducing an additional variable, $x_2(t)$, in which

$$x_2(t) = \int_{t_0}^{t} A[t,\ x_1(t)]\sqrt{1 + \theta(t)^2}\ dt$$

we have

$$\frac{dx_2}{dt} = A[t,\ x_1]\sqrt{1 + \theta^2}\ ,\quad x_2(t_0) = 0 \tag{6}$$

and

$$S = x_2(T) \tag{7}$$

Let A represent $A[t,\ x_1]$.

The Hamiltonian is

$$H = z_1\theta + z_2\ A\sqrt{1 + \theta^2} \tag{8}$$

The components of the adjoint vector are defined by

$$\frac{dz_1}{dt} = -\frac{\partial H}{\partial x_1} = -\ z_2\frac{\partial A}{\partial x_1}\sqrt{1 + \theta^2} \tag{9}$$

$$\frac{dz_2}{dt} = - \frac{\partial H}{\partial x_2} = 0, \quad z_2(T) = 1 \tag{10}$$

It follows from equation (10) that

$$z_2(t) = 1, \qquad t_0 \leq t \leq T \tag{11}$$

$$\frac{dz_1}{dt} = - \frac{\partial A}{\partial x_1} \sqrt{1 + \theta^2} \tag{12}$$

The Hamiltonian then becomes

$$H^* = z_1\theta + A\,[t,\, x_1]\,\sqrt{1 + \theta^2} \tag{13}$$

The necessary condition for optimal control

$$\frac{\partial H^*}{\partial \theta} = 0 = z_1 + \frac{A\theta}{\sqrt{1 + \theta^2}} \tag{14}$$

yields

$$z_1 = - \frac{A\theta}{\sqrt{1 + \theta^2}} \tag{15}$$

Rewriting equation (12) as

$$\frac{dz_1}{dt} = - \frac{\partial A/\partial t}{\partial x_1/\partial t} \sqrt{1 + \theta^2} \tag{16}$$

we have from the definition of equation (5)

$$\frac{dz_1}{dt} = - \frac{\partial A}{\partial t} \frac{\sqrt{1 + \theta^2}}{\theta} \tag{17}$$

Eliminating $z_1(t)$ from equations (15) and (17) we get

$$\frac{1}{A}\frac{dA}{dt} = \frac{\theta}{(1+\theta^2)}\frac{d\theta}{dt} \tag{18}$$

Integration of equation (18) yields

$$A = A_1(1+\theta^2)^{\frac{1}{2}}$$

where A_1 is an integration constant.

A numerical solution of equations (5) and (12) or (17) with boundary conditions $x_1(t_0) = a_0$ and $x_1(T) = \phi$ can be obtained. However, an analytical solution was not obtainable.

In reference [1], an analytical solution was not obtained, but the final condition was arrived at by using the transversality condition of the classical calculus of variations. However, the same end condition could not be obtained by the maximum principle.

Example 5

This example deals with production scheduling and inventory control. The rate of change in finish goods inventories is equal to the difference between the production and sales rate represented by

$$\frac{dI(t)}{dt} = P(t) - S(t) \tag{1}$$

subject to

$$P(0) = P_0,\ P(T) = P_1,\ I(0) = I_0,\ I(T) = I_1 \tag{2}$$

We assume that the rate of the costs from holding inventories or stockouts and production is approximated by the quadratic functions of the inventory level and that of the production rate represented by

$$c_T = \int_0^T [c_I\ I(t)^2 + c_P\ P(t)^2]\ dt \tag{3}$$

The problem is to find the optimum production planning to minimize the cost function represented by equation (3) which is subject to the constraint given by equation (1).

Solution:

Let

$$x_1(t) = I(t)$$

$$\theta(t) = P(t)$$

The performance equation of the system is given by

$$\frac{dx_1(t)}{dt} = \theta(t) - S(t) \tag{4}$$

Let

$$\frac{dx_2(t)}{dt} = c_I\ [x_1(t)]^2 + c_P[\theta(t)]^2, \quad x_2(0) = 0 \tag{5}$$

Then we have

$$H = z_1\ [\theta - S] + z_2\ [c_I x_1^{\ 2} + c_P \theta^2] \tag{6}$$

$$\frac{dz_1}{dt} = -2z_2\ c_I x_1 \tag{7}$$

$$\frac{dz_2}{dt} = 0, \quad z_2(T) = 1 \tag{8}$$

It follows from equation (8) that

$$z_2(t) = 1, \quad 0 \leq t \leq T \tag{9}$$

$$\frac{dz_1}{dt} = -2c_I x_1 \tag{10}$$

Thus the Hamiltonian becomes

$$H = z_1[\theta - S] + [c_I x_1^{\,2} + c_P \theta^2] \tag{11}$$

and

$$H^* = z_1\theta + c_P\theta^2 \tag{12}$$

The necessary condition of optimal control

$$\frac{\partial H^*}{\partial \theta} = 0 = z_1 + 2c_p\theta \tag{13}$$

yields

$$2c_P\theta = -z_1 \tag{14}$$

Differentiating equation (14) with respect to t and substituting this result into equation (10) gives

$$\frac{c_P}{c_I}\frac{d\theta}{dt} = x_1 \tag{15}$$

Equation (15) together with equation (4) constitute a pair of differential equations in two unknown functions $x_1(=I)$ and $\theta(=P)$. Combining equations (15) and (4) we have

$$\frac{c_P}{c_I}\ddot{\theta}(t) - \theta(t) = -S(t) \tag{16}$$

By solving the above differential equation, as shown in Section 1 of the Appendix, the optimum production rate is found as

$$p(t) = \frac{P_1 - P_0\, e^{-\rho T} + \frac{1}{2}\rho \int_0^T [e^{\rho(T-x)} - e^{-\rho(T-x)}]\, S(x)\, dx}{e^{\rho T} - e^{-\rho T}} \cdot e^{-\rho t}$$

$$- \frac{P_1 - P_0\, e^{\rho T} + \frac{1}{2}\rho \int_0^T [e^{\rho(T-x)} - e^{-\rho(T-x)}]\, S(x)\, dx}{e^{\rho T} - e^{-\rho T}} \cdot e^{-\rho t}$$

$$- \frac{1}{2}\rho\, [e^{\rho t} \int e^{-\rho t}\, S(t)dt - e^{-\rho t} \int e^{\rho t}\, S(t)\, dt] \tag{17}$$

where

$$\rho = \sqrt{\frac{c_I}{c_P}}$$

The above example is solved by Connors and Teichroew [1] by the calculus of variations technique and a slightly different result is obtained.

Example 6

This example deals with optimal control of a chemical reaction. The rate of change of the amount of waste product, $\dot{x}(t)$, is linearly related to the production of waste product, x(t), and the temperature of reaction, u(t), as represented by

$$\frac{dx(t)}{dt} = ax(t) - bu(t) \tag{1}$$

where a, b > 0 are constants.

The cost of waste product is taken to be proportional to the square of the amount of waste product produced and the cost associated with the temperature policy is assumed to be proportional to the square of the temperature applied to the reaction.

The problem is to determine an optimum temperature policy so as to minimize the total cost represented by

$$c_T = \frac{1}{2} \int_0^T [qx(t)^2 + ru(t)^2] \, dt \tag{2}$$

where q is the cost coefficient for waste product and r is the cost coefficient for temperature.

Solution:

Let

$$x_1(t) = x(t)$$

$$\theta(t) = u(t)$$

The performance equation of the system is given by

$$\frac{dx_1(t)}{dt} = ax_1(t) - b\theta(t), \quad x_1(0) = \alpha \tag{3}$$

Let

$$\frac{dx_2(t)}{dt} = \frac{1}{2} \{q[x_1(t)]^2 + r[\theta(t)]^2\} , \quad x_2(0) = 0 \tag{4}$$

Then we have

$$H = z_1 [ax_1 - b\theta] + z_2[\frac{1}{2} (qx_1^2 + r\theta^2)] \tag{5}$$

$$\frac{dz_1}{dt} = - [az_1 + z_2qx_1] \tag{6}$$

$$\frac{dz_2}{dt} = 0, \quad z_2(T) = 1 \tag{7}$$

It follows from equation (7) that

$$z_2(t) = 1, \quad 0 \leq t \leq T \tag{8}$$

$$\frac{dz_1}{dt} = - az_1 - qx_1 \tag{9}$$

Thus the Hamiltonian becomes

$$H = z_1 [ax_1 - b\theta] + \frac{1}{2} [qx_1^2 + r\theta^2] \tag{10}$$

and

$$H^* = - bz_1\theta + \frac{1}{2} r\theta^2 \tag{11}$$

From the necessary condition of optimal control we have

$$\frac{\partial H^*}{\partial \theta} = 0 = -bz_1 + r\theta \tag{12}$$

or

$$\theta = \frac{b}{r} z_1 \tag{13}$$

Substitution of the result of equation (13) into equation (3) gives

$$\frac{dx_1}{dt} = ax_1 - \frac{b^2}{r} z_1 \tag{14}$$

or

$$z_1 = \frac{r}{b^2} \left(ax_1 - \frac{dx_1}{dt} \right) \tag{15}$$

Differentiating equation (15) with respect to t we get

$$\frac{dz_1}{dt} = \frac{r}{b^2} \left(a \frac{dx_1}{dt} - \frac{d^2x_1}{dt^2} \right) \tag{16}$$

Substituting the result of equations (16) and (15) into equation (9) we obtain

$$\frac{d^2x_1}{dt^2} = \left(a^2r^2 + b^2rq\right) \frac{1}{r^2} x_1 \tag{17}$$

or

$$\frac{d^2x_1}{dt^2} - \lambda^2 x_1 = 0 \tag{18}$$

where

$$\lambda = \frac{1}{r}\sqrt{a^2r^2 + b^2rq}$$

The solution of equation (18) is

$$x_1(t) = A_1e^{\lambda t} + A_2e^{-\lambda t} \tag{19}$$

Differentiation of equation (19) with respect to t and substitution of this result together with that of equation (19) into equation (15) yields

$$z_1(t) = \frac{r}{b^2}\left[A_1e^{\lambda t}\,(a-\lambda) + A_2e^{-\lambda t}(a+\lambda)\right] \tag{20}$$

Application of the conditions $x_1(0) = \alpha$ and $z_1(T) = 0$ to equation (19) and (20) respectively yields

$$A_1 = \frac{\alpha(a+\lambda)}{(a+\lambda) - (a-\lambda)e^{2\lambda T}} \tag{21}$$

$$A_2 = -\frac{\alpha(a-\lambda)e^{2\lambda T}}{(a+\lambda) - (a-\lambda)e^{2\lambda T}} \tag{22}$$

Equations (19) and (20) can then be rewritten as

$$x_1(t) = \alpha\left[\frac{(a+\lambda)e^{\lambda t} - (a-\lambda)e^{\lambda(2T-t)}}{(a+\lambda) - (a-\lambda)e^{2\lambda T}}\right] \tag{23}$$

$$z_1(t) = \frac{r\alpha}{b^2}\left(a^2 - \lambda^2\right)\left\{\frac{e^{\lambda t} - e^{\lambda(2T-t)}}{(a+\lambda) - (a-\lambda)e^{2\lambda T}}\right\} \tag{24}$$

Equation (13) can be rewritten as

$$\theta(t) = \frac{\alpha}{b}\left(a^2-\lambda^2\right)\left\{\frac{e^{\lambda t} - e^{\lambda(2T-t)}}{(a+\lambda) - (a-\lambda)e^{2\lambda T}}\right\} \tag{25}$$

or

$$\theta(t) = \frac{1}{b}\left(a^2-\lambda^2\right)\left\{\frac{1 - e^{2\lambda(T-t)}}{(a+\lambda) - (a-\lambda)e^{2\lambda(T-t)}}\right\} x_1(t) \tag{26}$$

where

$$\lambda = \frac{\sqrt{a^2r^2 + b^2rq}}{r}$$

Hence the optimal temperature policy is represented by

$$u(t) = \left\{\frac{e^{\frac{2\sqrt{a^2r^2+b^2rq}}{r}(T-t)} - 1}{\left(a+\frac{\sqrt{a^2r^2+b^2rq}}{r}\right) - \left(a - \frac{\sqrt{a^2r^2+b^2rq}}{r}\right)e^{\frac{2\sqrt{a^2r^2+b^2rq}}{r}(T-t)}}\right\}\cdot\frac{bqx_1(t)}{r} \tag{27}$$

The solution given by equation (27) is different from that given in reference [1].

Example 7

This example deals with an inventory control model. The rate of change of finish goods inventory is taken to be equal to the difference of the production and sales rate as represented by

$$\frac{dI(t)}{dt} = P(t) - Q(t), \quad I(0) = I_0 \tag{1}$$

where I(t) is the level of inventory at time t, Q(t), the rate of sales at time t, and P(t), the rate of production at time t which is constrained within a fixed maximum and minimum level.

$$P_{min} \leq P(t) \leq P_{max}$$

It is further assumed that the rate of change of sales declines at a rate proportional to the production as represented by

$$\frac{dQ(t)}{dt} = -\lambda\ P(t), \quad Q(0) = Q_0 \tag{2}$$

where λ is a positive constant.

The problem is to find an optimum production rate which minimizes the sum of production cost and inventory cost in a given time period, $0 \leq t \leq T$, such as

$$S = \int_0^T [c\ P(t) + h\ I(t)]\ dt \tag{3}$$

where

c = production cost per unit,

h = inventory cost per unit per unit time

Solution:

Let

$$x_1(t) = I(t)$$

$$x_2(t) = Q(t)$$

$$\theta(t) = P(t)$$

The performance equations of the system are

$$\frac{dx_1(t)}{dt} = \theta(t) - x_2(t), \quad x_1(0) = I_0 \tag{4}$$

$$\frac{dx_2(t)}{dt} = -\lambda\theta(t), \quad x_2(0) = Q_0 \tag{5}$$

Let

$$\frac{dx_3(t)}{dt} = c\theta(t) + hx_1(t), \quad x_3(0) = 0 \tag{6}$$

Then we have

$$H = z_1(\theta-x_2) - z_2\lambda\theta + z_3(c\theta+hx_1) \tag{7}$$

$$\frac{dz_1}{dt} = -hz_3 \tag{8}$$

$$\frac{dz_2}{dt} = z_1 \tag{9}$$

$$\frac{dz_3}{dt} = 0, \quad z_3(T) = 1 \tag{10}$$

It follows from equation (10) that

$$z_3(t) = 1, \quad 0 \leq t \leq T \tag{11}$$

Equations (7), (8) and (9) can then be written as

$$H = z_1 (\theta - x_2) - z_2 \lambda\theta + (c\theta + hx_1) \tag{12}$$

$$\frac{dz_1}{dt} = -h \tag{13}$$

$$\frac{dz_2}{dt} = z_1 \tag{14}$$

and

$$H^* = (z_1 - \lambda z_2 + c)\ \theta \tag{15}$$

Solution of equation (13) and application of the condition $z_1(T) = 0$ yields

$$z_1(t) = h(T-t) \tag{16}$$

Substituting equation (16) into equation (14) and solving for $z_2(t)$ with the condition $z_2(T) = 0$, we get

$$z_2(t) = -\frac{1}{2} h\ (T-t)^2 \tag{17}$$

Substituting the result of equations (16) and (17) into equation (15), we obtain

$$H^* = q\theta \tag{18}$$

where

$$q = h(T-t) + \frac{\lambda h}{2}(T-t)^2 + c$$

The Hamiltonian function represented by equation (18) is a linear function of θ and hence represents a bang-bang type of control.

Since $c,h>0$, q can therefore be positive or negative according to the value of λ such as

(a) If $\lambda>0$, then $q>0$

and the optimal production policy is

$$P(t) = P_{min}$$

(b) If $\lambda<0$, then q can be either positive or negative depending upon the relative magnitudes of c, h and λ.

The optimal production policy, then, is

$$\begin{aligned} P(t) &= P_{min} \quad \text{if } q>0 \\ &= P_{max} \quad \text{if } q<0 \end{aligned}$$

A much shorter and clearer solution than that in reference [1] is obtained here. This is possible because equation (18) is put in the square form and since c, $h > 0$, and therefore the optimal policy is dependent only on λ. In [1], the quadratic function is not put in the square form and consequently a more general solution is discussed.

Example 8

This example deals with a more realistic extension of the inventory model discussed in the previous example.

The rate of change of actual inventory at the factory warehouse, $\dot{IAF}(t)$, is taken to be equal to the difference between the production rate,

$P(t)$, and shipment rate from the factory, $SSF(t)$, as represented by

$$\frac{d}{dt}\,[IAF(t)] = P(t) - SSF(t),\ IAF(0) = IAF^0 \tag{1}$$

The rate of change of the level of unfilled orders, $\dot{UOF}(t)$, is taken to be the difference between the demand received at the factory, $DRF(t)$, and shipment sent from the factory, $SSF(t)$, as represented by

$$\frac{d}{dt}\,[UOF(t)] = DRF(t) - SSF(t),\quad UOF(0) = UOF^0 \tag{2}$$

It is assumed that the shipment rate at the factory is related to the level of unfilled orders, $UOF(t)$, by a first order exponential lag with time constant α as represented by

$$SSF(t) = \frac{1}{\alpha}\int_0^{\infty} UOF(t-\tau)\ e^{-\tau/\alpha}\ d\tau \tag{3}$$

The problem is to find an optimal production rate so as to minimize the cost function represented by

$$S = c_I\int_0^T IAF(t)dt + c_P\int_0^T P(t) + c_S\int_0^T [DRF(t) - P(t)]\ dt \tag{4}$$

where

c_I = unit inventory holding cost

c_P = unit production cost

c_S = unit shortage cost.

Solution:

Let

$$x_1(t) = IAF(t)$$

$$x_2(t) = SSF(t)$$

$$x_3(t) = UOF(t)$$

$$\theta(t) = P(t)$$

By employing the change of variables, $x = t-\tau$, equation (3) can be modified as (see Section 2 of Appendix)

$$\frac{d}{dt}[SSF(t)] = -\frac{1}{\alpha}[SSF(t) - UOF(t)],\ SSF(0) = SSF^0 \tag{5}$$

The performance equations of the system are

$$\frac{dx_1(t)}{dt} = \theta(t) - x_2(t),\quad x_1(0) = IAF^0 \tag{6}$$

$$\frac{dx_2(t)}{dt} = -\frac{1}{\alpha}x_2(t) + \frac{1}{\alpha}x_3(t),\quad x_2(0) = SSF^0 \tag{7}$$

$$\frac{dx_3(t)}{dt} = DRF(t) - x_2(t),\quad x_3(0) = UOF^0 \tag{8}$$

Let

$$x_4(t) = \int_0^t \{c_I x_1(t) + c_P\theta(t) + c_S\ (DRF(t)-\theta(t))\}\ dt \tag{9}$$

Then

$$\frac{dx_4}{dt} = c_I x_1 + c_P\theta + c_S\ (DRF-\theta),\quad x_4(0) = 0 \tag{10}$$

Hence we have

$$H = z_1(\theta-x_2) + \frac{1}{\alpha} z_2 (x_3-x_2) + z_3(DRF-x_2)$$

$$+ z_4 [c_I x_1 + c_P\theta + c_S (DRF-\theta)] \tag{11}$$

$$\frac{dz_1}{dt} = -z_4 c_I \tag{12}$$

$$\frac{dz_2}{dt} = z_1 + \frac{1}{\alpha} z_2 + z_3 \tag{13}$$

$$\frac{dz_3}{dt} = -\frac{1}{\alpha} z_2 \tag{14}$$

$$\frac{dz_4}{dt} = 0, \quad z_4(T) = 1 \tag{15}$$

It follows from equation (15) that

$$z_4(t) = 1 \tag{16}$$

Equations (11) and (12) can then be rewritten as

$$H = z_1(\theta-x_2) + \frac{1}{\alpha} z_2(x_3-x_2) + z_3(DRF-x_2)$$

$$+ c_I x_1 + c_P\theta + c_S (DRF-\theta) \tag{17}$$

$$\frac{dz_1}{dt} = -c_I \tag{18}$$

and

$$H^* = (z_1+c_P-c_S)\theta \tag{19}$$

Integrating equation (18) with respect to t and applying the condition $z_1(T) = 0$ we get

$$z_1 = c_I(T-t) \tag{20}$$

Substitution of the result of equation (20) into equation (19) yields

$$H^* = q\theta \tag{21}$$

where

$$q = c_I(T-t) + c_P-c_S$$

Equation (21) represents the bang-bang type of control. The optimal production policy is represented by

$$\begin{aligned} P(t) &= P_{min} && \text{if } q > 0 \\ &= P_{max} && \text{if } q < 0 \\ &= \text{Unspecified} && \text{if } q = 0 \end{aligned}$$

When demand is less than production, the term representing shortage cost drops out and equation (21) is modified as

$$H^* = q'\theta \tag{22}$$

where

$$q' = c_I(T-t) + c_P$$

Since c_I, $c_P > 0$, the optimal policy is

$$P(t) = P_{min}.$$

A solution similar to that in [1] is obtained. However the solution obtained here is short and straightforward.

Example 9

This example deals with an advertising model in which the objective is to maximize revenue for a given period of time. This model deals with only a part of the market in which the purchasers existence is independent of advertising and where the effect of advertising in creating purchasers is not considered. A pool of prospective purchasers is assumed to exist and the outflow from this into actual buying is affected by advertising which in turn is affected by the level of sales at the factory. In this model, time delays existing in the advertising flow are included. These comprise the delay in reaching a decision on advertising expenditure, the delays in advertising agencies and media and the delay in building up consumer awareness of the existence of an advertising campaign.

It is assumed that the advertising presented by the media to the consumer, VMC(t), is related to the advertising decision at the factory, VDF(t), by a third order exponential lag with time constant, α, as represented by

$$VMC(t) = \int_0^\infty \int_0^\infty \int_0^\infty \frac{1}{\alpha^3} VDF(t-\eta-\varepsilon-\tau) e^{-(\eta+\varepsilon+\tau)/\alpha} \, d\tau d\varepsilon d\eta \qquad (1)$$

The rate of change in advertising awareness at the consumer level, $V\dot{A}C(t)$, is assumed to be related to the difference between the advertising presented to the consumer by media, VMC(t), and the advertising awareness at the consumer, VAC(t), by a first order exponential lag with time constant, β, as represented by

$$\dot{VAC}(t) = \frac{1}{\beta} \int_0^\infty [VMC(t-\tau) - VAC(t-\tau)]e^{-\tau/\beta}\, d\tau \tag{2}$$

It is further assumed that the rate of change of the demand rate, $\dot{RRR}(t)$, declines in proportion to the demand rate, RRR(t), if no advertising is done. The rate of change of the demand rate is also directly related to the consumer advertising awareness by a first order exponential lag with time constant equal to 1, as represented by

$$\dot{RRR}(t) = -\rho RRR(t) + \gamma \int_0^\infty VAC(t-\tau)\, e^{-\tau}\, d\tau \tag{3}$$

where ρ, γ are proportionality constants.

The problem is to determine the optimal amount of capital to allocate to advertising in order to maximize earnings over a given time period T, as represented by

$$S = \int_0^T [R \cdot RRR(t) - VMC(t)]\, dt \tag{4}$$

where R is the revenue per unit

Solution:

Let

$$x_1(t) = VMC(t)$$

$$x_2(t) = \frac{dx_1(t)}{dt}$$

$$x_3(t) = \frac{dx_2(t)}{dt}$$

$$x_4(t) = VAC(t)$$

$$x_5(t) = \frac{dx_4(t)}{dt}$$

$$x_6(t) = RRR(t)$$

$$x_7(t) = \frac{dx_6(t)}{dt}$$

$$\theta(t) = VDF(t)$$

Equations (1), (2) and (3) can be rewritten as (See Section 3 of Appendix)

$$\dddot{VMC}(t) + \frac{3}{\alpha}\ddot{VMC}(t) + \frac{3}{\alpha^2}\dot{VMC}(t) + \frac{1}{\alpha^3}VMC(t) = \frac{1}{\alpha^3}VDF(t) \tag{5}$$

$$\ddot{VAC}(t) + \frac{1}{\beta}\dot{VAC}(t) + \frac{1}{\beta}VAC(t) = \frac{1}{\beta}VMC(t) \tag{6}$$

$$\ddot{RRR}(t) + (1+\rho)\dot{RRR}(t) + \rho RRR(t) = \gamma VAC(t) \tag{7}$$

The performance equations are

$$\frac{dx_1}{dt} = x_2 \tag{8}$$

$$\frac{dx_2}{dt} = x_3 \tag{9}$$

$$\frac{dx_3}{dt} = \frac{1}{\alpha^3}\,\theta - \frac{1}{\alpha^3}\,x_1 - \frac{3}{\alpha^2}\,x_2 - \frac{3}{\alpha}\,x_3 \tag{10}$$

$$\frac{dx_4}{dt} = x_5 \tag{11}$$

$$\frac{dx_5}{dt} = \frac{1}{\beta}\,(x_1 - x_4 - x_5) \tag{12}$$

$$\frac{dx_6}{dt} = x_7 \tag{13}$$

$$\frac{dx_7}{dt} = \gamma x_4 - \rho x_6 - (1+\rho)\,x_7 \tag{14}$$

Let

$$x_8(t) = \int_0^t (Rx_6 - x_1)\,dt$$

Then we have

$$\frac{dx_8}{dt} = Rx_6 - x_1, \quad x_8(0) = 0 \tag{15}$$

and

$$H = z_1x_2 + z_2x_3 + z_3\,\left(\frac{1}{\alpha^3}\,\theta - \frac{1}{\alpha^3}\,x_1 - \frac{3}{\alpha^2}\,x_2 - \frac{3}{\alpha}\,x_3\right)$$

$$+ z_4x_5 + z_5\,\frac{1}{\beta}\,(x_1 - x_4 - x_5) + z_6x_7 + z_7(\gamma x_4 - \rho x_6 - (1+\rho)x_7)$$

$$+ z_8\,(Rx_6 - x_1) \tag{16}$$

$$\frac{dz_1}{dt} = \frac{1}{\alpha^3} z_3 - \frac{1}{\beta} z_5 + z_8 \tag{17}$$

$$\frac{dz_2}{dt} = -z_1 + \frac{3}{\alpha^2} z_3 \tag{18}$$

$$\frac{dz_3}{dt} = -z_2 + \frac{3}{\alpha} z_3 \tag{19}$$

$$\frac{dz_4}{dt} = \frac{1}{\beta} z_5 - \gamma z_7 \tag{20}$$

$$\frac{dz_5}{dt} = - z_4 + \frac{1}{\beta} z_5 \tag{21}$$

$$\frac{dz_6}{dt} = \rho z_7 - Rx_8 \tag{22}$$

$$\frac{dz_7}{dt} = - z_6 + (1+\rho)\, z_7 \tag{23}$$

$$\frac{dz_8}{dt} = 0, \quad z_8(T) = 1 \tag{24}$$

Equations (16) to (23), rewritten by using the solution of equation (24), $z_8(t) = 1$, yield

$$\begin{aligned} H = {} & z_1x_2 + z_2x_3 + z_3 \left(\frac{1}{\alpha^3}\,\theta - \frac{1}{\alpha^3}\,x_1 - \frac{3}{\alpha^2}\,x_2 - \frac{3}{\alpha}\,x_3\right) \\ & + z_4x_5 + z_5\,\frac{1}{\beta}\,(x_1-x_4-x_5) + z_6x_7 + z_7\,(\gamma x_4 - \rho x_6 - (1+\rho)x_7) \\ & + (Rx_6 - x_1) \end{aligned} \tag{25}$$

and

$$H^* = \frac{1}{\alpha^3} z_3 \theta \tag{26}$$

$$\frac{dz_1}{dt} = \frac{1}{\alpha^3} z_3 - \frac{1}{\beta} z_5 + 1, \qquad z_1(T) = 0 \tag{27}$$

$$\frac{dz_2}{dt} = - z_1 + \frac{3}{\alpha^2} z_3, \qquad z_2(T) = 0 \tag{28}$$

$$\frac{dz_3}{dt} = - z_2 + \frac{3}{\alpha} z_3, \qquad z_3(T) = 0 \tag{29}$$

$$\frac{dz_4}{dt} = \frac{1}{\beta} z_5 - \gamma z_7, \qquad z_4(T) = 0 \tag{30}$$

$$\frac{dz_5}{dt} = - z_4 + \frac{1}{\beta} z_5, \qquad z_5(T) = 0 \tag{31}$$

$$\frac{dz_6}{dt} = \rho z_7 - R, \qquad z_6(T) = 0 \tag{32}$$

$$\frac{dz_7}{dt} = - z_6 + (1+\rho) z_7, \qquad z_7(T) = 0 \tag{33}$$

Since the Hamiltonian, represented by equation (26), is a linear function of the decision vector, $\theta(t)$, we have a bang-bang type of optimal control.

Hence

$$
\begin{aligned}
VDF(t) &= VDF_{max} && \text{if} \quad z_3 > 0 \\
&= VDF_{min} && \text{if} \quad z_3 < 0 \qquad (34) \\
&= \text{Unspecified} && \text{if} \quad z_3 = 0
\end{aligned}
$$

Example 10

This example deals with an inventory-control production scheduling model with fixed time and fixed right hand end points. The problem is to find a production rate, $P(t)$, which will change the inventory level from $I(0) = I^0$ to $I(T) = I^1$ and the sales rate from $S(0) = S^0$ to $S(T) = S^1$ in such a way that

$$c_T = \int_0^T c_P \, P^2(t) \, dt \tag{1}$$

is a minimum subject to

$$\frac{dI(t)}{dt} = P(t) - S(t) \tag{2}$$

$$\frac{dS(t)}{dt} = - \alpha \, P(t) \tag{3}$$

where $I(t)$ is the inventory level at time t and $S(t)$ is the sales rate at time t.

Solution:

Let

$$x_1(t) = I(t)$$

$$x_2(t) = S(t)$$

$$\theta(t) = P(t)$$

The performance equations are

$$\frac{dx_1}{dt} = \theta - x_2, \quad x_1(0) = I^0, \quad x_1(T) = I^1 \tag{4}$$

$$\frac{dx_2}{dt} = -\alpha\theta, \quad x_2(0) = S^0, \quad x_2(T) = S^1 \tag{5}$$

Let

$$x_3(t) = \int_0^t c_P\ [\theta(t)]^2\ dt$$

Then we have

$$\frac{dx_3}{dt} = c_P\ \theta^2, \quad x_3(0) = 0 \tag{6}$$

and

$$H = z_1(\theta - x_2) + z_2\ (-\alpha\theta) + z_3\ (c_P\theta^2) \tag{7}$$

$$\frac{dz_1}{dt} = 0 \tag{8}$$

$$\frac{dz_2}{dt} = z_1 \tag{9}$$

$$\frac{dz_3}{dt} = 0, \quad z_3(T) = 1 \tag{10}$$

Equation (7), rewritten by using the solution of equation (10), $z_3(t) = 1$, yields

$$H = z_1(\theta - x_2) + z_2(-\alpha\theta) + c_p\theta^2 \tag{11}$$

The optimal control is found from the condition that

$$\frac{\partial H}{\partial \theta} = 0 = z_1 - \alpha z_2 + 2c_p\theta \tag{12}$$

that is,

$$\theta = \frac{1}{2c_p}\,(\alpha z_2 - z_1) \tag{13}$$

From equations (8) and (9) we have

$$z_1 = A_1 \tag{14}$$

$$z_2 = A_1 t + A_2 \tag{15}$$

where A_1, A_2 are constants of integration.

Equation (13) can be rewritten as

$$\theta = \frac{1}{2c_p}\,(\alpha A_1 t + \alpha A_2 - A_1) \tag{16}$$

Substituting the extremal control (equation 16) into equations (4) and (5) and integrating to obtain the solution in terms of the initial conditions $x_1(0) = I^0$ and $x_2(0) = S^0$, we get

$$x_1(t) = I^0 - S^0 t + \frac{t}{12c_p}\,(\alpha^2 t^2 - 6)\,A_1 + \frac{\alpha t}{4c_p}\,(2+\alpha t)A_2 \tag{17}$$

$$x_2(t) = S^0 - \frac{\alpha t}{4c_p}\,(\alpha t - 2)\,A_1 - \frac{1}{2c_p}\,(\alpha^2 t)A_2 \tag{18}$$

Making use of the fact that $x_1(T) = I^1$ and $x_2(T) = S^1$, equations (17) and (18) may be solved for A_1 and A_2 in terms of S^0, S^1, I^0 and I^1 as

$$A_1 = -\frac{12c_P}{\alpha^3 T^3}\,[2\alpha(I^1 - I^0 + S^0 T) + (2+\alpha T)(S^1 - S^0)] \tag{19}$$

$$A_2 = -\frac{6c_P}{\alpha^4 T^3}\,(2-\alpha T)\;[2\alpha(I^1 - I^0 + S^0 T) + (2+\alpha T)(S^1 - S^0)] - \frac{2c_P}{\alpha^2 T}\,(S^1 - S^0) \tag{20}$$

Substituting these values of A_1 and A_2 into equation (16) and rearranging terms yields the optimal control

$$P(t) = \frac{3}{\alpha^2 T^3}\,[2\alpha(I^1 - I^0 + S^0 T) + (2+\alpha T)(S^1 - S^0)]\,(T-2t) - \frac{1}{\alpha T}\,(S^1 - S^0) \tag{21}$$

The optimal policy represented by equation (21) is different from that obtained in reference [1].

Example 11

This example deals with process inventory in which bounds are imposed on the rate of change of inventory level as well as on the production rate. It is given that the second time derivative of the inventory level, $\ddot{I}(t)$, is directly proportional to the production rate, $P(t)$, as represented by

$$\frac{d^2 I(t)}{dt^2} = P(t), \quad I(0) = I_0, \quad \frac{dI(0)}{dt} = I_1 \tag{1}$$

Furthermore, the rate of change of process inventory level, $\dot{I}(t)$, is constrained to be greater than or equal to a certain minimum inventory level and the production rate, $P(t)$, is constrained to lie between P_{min} and P_{max}.

$$\frac{dI(t)}{dt} \geq I_{min}$$

$$P_{min} \leq P(t) \leq P_{max}$$

The problem is to find a production rate, P(t), so as to minimize the total cost represented by

$$C_T = \int_0^T [I(t)^2 + k_1\dot{I}(t)^2 + k_2P(t)^2]\ dt \tag{2}$$

where k_1, k_2 are positive constants.

It should be noted that the units are not consistent in the formulation of the problem.

Solution:

Let

$$x_1(t) = I(t)$$

$$x_2(t) = \frac{dx_1(t)}{dt}$$

$$\theta(t) = P(t)$$

The performance equations are

$$\frac{dx_1}{dt} = x_2, \quad x_1(0) = I_0 \tag{3}$$

$$\frac{dx_2}{dt} = \theta, \quad x_2(0) = I_1 \tag{4}$$

Let

$$x_3(t) = \int_0^t [I(t)^2 + k_1\dot{I}(t)^2 + k_2P(t)^2]\, dt$$

then we have

$$\frac{dx_3}{dt} = \dot{x}_1^2 + k_1x_2^2 + k_2\theta^2, \quad x_3(0) = 0 \tag{5}$$

The Hamiltonian and adjoint system for this problem are

$$H = z_1x_2 + z_2\theta + z_3(x_1^2 + k_1x_2^2 + k_2\theta^2) \tag{6}$$

$$\frac{dz_1}{dt} = - 2z_3x_1 \tag{7}$$

$$\frac{dz_2}{dt} = -z_1 - 2z_3k_1x_2 \tag{8}$$

$$\frac{dz_3}{dt} = 0, \quad z_3(T) = 1 \tag{9}$$

Rewriting equations (6), (7) and (8) using the solution of equation (9), $z_3(t) = 1$, we get

$$H = z_1x_1 + z_2\theta + (x_1^2 + k_1x_2^2 + k_2\theta^2) \tag{10}$$

$$\frac{dz_1}{dt} = -2x_1 \tag{11}$$

$$\frac{dz_2}{dt} = -z_1 - 2k_1x_2 \tag{12}$$

From the stational necessary condition for optimal control, $\frac{\partial H}{\partial \theta} = 0$, we obtain

$$\theta = - \frac{z_2}{2k_2} \tag{13}$$

Since the state and control variables are bounded, the value of $z_2(t)$ is found from the following set of constraints

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) && \text{if } I_{min} \le x_2(t), \quad x_1(0) = I_0 \\
\dot{x}_2(t) &= - \frac{z_2(t)}{2k_2} && \text{if } 2k_2P_{min} \le - z_2(t) \le 2k_2P_{max}, \; x_2(0) = I_1 \\
&= P_{max} && \text{if } - z_2(t) > 2k_2P_{max} \\
&= P_{min} && \text{if } - z_2(t) < 2k_2P_{min} \\
\dot{z}_1(t) &= - 2x_1(t), \quad z_1(T) = 0 \\
\dot{z}_2(t) &= - z_1(t) - 2k_1x_2(t) && \text{if } I_{min} < x_2(t), \quad z_2(T) = 0 \\
&= - z_1(t) - 2k_1I_{min} && \text{if } x_2(t) < I_{min}
\end{aligned} \tag{14}$$

The solution of $x_1(t)$, $x_2(t)$, $z_1(t)$, $z_2(t)$ and $\theta(t)$ can be obtained from the above equation (see Section 4 of the Appendix).

The above problem is a four point boundary value (or split value) problem, and a two dimensional search for $z_1(0)$ and $z_2(0)$ to satisfy the end conditions $z_1(T) = 0$ and $z_2(T) = 0$ must be carried out. A numerical solution by the analog computer can also be obtained.

Example 12

This example deals with a model for determining advertising expenditures for a firm producing a single product. The market capacity is M dollars, and if the firm does no advertising, its rate of change of sales at any time t, $\dot{S}(t)$, will decrease at a rate proportional to the rate of sales, S(t), at that time. If the firm advertises, the rate of change of sales will increase at a rate proportional to the rate of advertising, A(t), but this increase affects only that part of the market that is not already purchasing the product. Therefore, the rate of change of sales due to a combined effect of these two factors is represented by

$$\frac{dS(t)}{dt} = -\lambda S(t) + \gamma A(t) \left[1 - \frac{S(t)}{M}\right] , \quad S(0) = S_0 \tag{1}$$

where λ, γ are positive constants.

The problem is to choose an advertising rate, A(t), which will maximize the total sales represented by

$$S_T = \int_0^t S(t)\, dt \tag{2}$$

Solution:

Let

$$x_1(t) = S(t), \quad x_1(0) = S_0$$

$$x_2(t) = \int_0^t x_2(t)dt, \quad x_2(0) = 0$$

$$\theta(t) = A(t)$$

The performance equations are

$$\frac{dx_1(t)}{dt} = -\lambda x_1(t) + \gamma\theta(t)\left(1 - \frac{x_1(t)}{M}\right) \tag{3}$$

$$\frac{dx_2(t)}{dt} = x_1(t), \quad x_2(0) = 0 \tag{4}$$

The Hamiltonian and the adjoint system are

$$H = z_1\left(-\lambda x_1 + \gamma\theta(1 - \frac{x_1}{M})\right) + z_2 x_1 \tag{5}$$

$$\frac{dz_1}{dt} = \lambda z_1 + \frac{\gamma}{M}\theta z_1 - z_2, \quad z_1(T) = 0 \tag{6}$$

$$\frac{dz_2}{dt} = 0, \quad z_2(T) = 1 \tag{7}$$

Equations (5) and (6), rewritten by using the solution of equation (7),

$z_2(t) = 1$, yield

$$H = z_1\left(-\lambda x_1 + \gamma\theta(1 - \frac{x_1}{M})\right) + x_1 \tag{8}$$

and

$$H^* = \left(1 - \frac{x_1}{M}\right)\gamma z_1\theta \tag{9}$$

$$\frac{dz_1}{dt} = \lambda z_1 + \frac{\gamma}{M}\theta z_1 - 1 \tag{10}$$

Since equation (9) is a linear function of the decision vector, $\theta(t)$, the optimal control should be the bang-bang type. Thus the conditions for optimal control are

$$A(t) = A_{max} \qquad \text{if } \left(1 - \frac{x_1}{M}\right) \gamma z_1 > 0$$

$$= 0 \qquad \text{if } \left(1 - \frac{x_1}{M}\right) \gamma z_1 < 0 \tag{11}$$

$$= \text{unspecified} \qquad \text{if } \left(1 - \frac{x_1}{M}\right) \gamma z_1 = 0 \ .$$

Case (a):

When $A(t) = A_{max}$

$$\frac{dx_1}{dt} = -\left(\lambda + \frac{\gamma}{M} A_{max}\right) x_1 + \gamma A_{max} \tag{12}$$

$$\frac{dz_1}{dt} = z_1\left(\lambda + \frac{\gamma}{M} A_{max}\right) - 1 \tag{13}$$

Solving equations (12) and (13) using the conditions $x_1(0) = S_0$ and $z_1(T) = 0$, we get

$$x_1(t) = \left(S_0 - \frac{\gamma A_{max}}{\lambda + \frac{\gamma A_{max}}{M}}\right) e^{-\left(\lambda + \frac{\gamma}{M} A_{max}\right)t} + \frac{\gamma A_{max}}{\lambda + \frac{\gamma A_{max}}{M}} \tag{14}$$

$$z_1(t) = \frac{1}{\lambda + \frac{\gamma A_{max}}{M}} \left[1 - e^{\left(\lambda + \frac{\gamma A_{max}}{M}\right)(t-T)}\right] \tag{15}$$

$$\left(1 - \frac{x_1}{M}\right)\gamma z_1 = \gamma\left\{1 - \frac{1}{M}\left[\left(S_0 - \frac{A_{max}}{\lambda + \frac{\gamma A_{max}}{M}}\right) e^{-\left(\lambda + \frac{\gamma}{M} A_{max}\right) t}\right.\right.$$

$$\left.\left. + \frac{\gamma A_{max}}{\lambda + \frac{\gamma}{M} A_{max}}\right]\right\} \frac{1}{\lambda + \frac{\gamma}{M} A_{max}} \left[1 - e^{\left(\lambda + \frac{\gamma}{M} A_{max}\right)(t-T)}\right] \tag{16}$$

Let

$$A = \frac{\gamma A_{max}}{M\lambda + \gamma A_{max}}$$

$$B = \lambda + \frac{\gamma}{M} A_{max}$$

Equation (16) can be rewritten as

$$\left(1 - \frac{x_1}{M}\right)\gamma z_1 = \{1 - \left(\frac{S_0}{M} - A\right) e^{-Bt} - A\} \frac{\gamma}{B} \left(1 - e^{B(t-T)}\right)$$

Since $\gamma, \lambda > 0$

therefore

$$1 > A > 0$$

$$B > 0$$

$$1 > \frac{S_0}{M} > 0$$

$$\left(\frac{S_0}{M} - A\right) e^{-Bt} < 1$$

$$\left[1 - \left(\frac{S_0}{M} - A\right) e^{-Bt} - A\right] > 0$$

$$\frac{\gamma}{B}\left(1 - e^{B(t-T)}\right) > 0$$

Hence

$$\left(1 - \frac{x_1}{M}\right)\gamma z_1 = \gamma \left\{1 - \frac{1}{M}\left[\left(S_0 - \frac{\gamma A_{max}}{\lambda + \frac{\gamma A_{max}}{M}}\right) e^{-\left(\lambda + \frac{\gamma}{M} A_{max}\right)t}\right.\right.$$

$$\left.\left. + \frac{\gamma A_{max}}{\lambda + \frac{\gamma}{M} A_{max}}\right\} \frac{1}{\lambda + \frac{\gamma}{M} A_{max}} \left[1 - e^{\left(\lambda + \frac{\gamma}{M} A_{max}\right)(t-T)}\right] > 0 \quad (17)$$

Case (b):

When $A(t) = 0$

$$\frac{dx_1}{dt} = -\lambda x_1 \quad (18)$$

$$\frac{dz_1}{dt} = \lambda z_1 - 1 \quad (19)$$

Equations (18) and (19) can be solved by using the conditions, $x_1(0) = S_0$ and $z_1(T) = 0$ as

$$x_1(t) = S_0 e^{-\lambda t} \tag{20}$$

$$z_1(t) = \frac{1}{\lambda}\left[1 - e^{\lambda(t-T)}\right]$$

$$\left[1 - \frac{x_1}{M}\right]\gamma z_1 = \frac{\gamma}{\lambda}\left[1 - \frac{1}{M}(S_0\ e^{-\lambda t})\right]\left[1 - e^{\lambda(t-T)}\right] \tag{22}$$

Since $\gamma, \lambda > 0$

$$1 > \left[1 - e^{\lambda(t-T)}\right] > 0$$

$$1 > \left[1 - \frac{1}{M}(S_0\ e^{-\lambda t})\right] > 0$$

Hence

$$\left[1 - \frac{x_1}{M}\right]\gamma z_1 = \frac{\gamma}{\lambda}\left[1 - \frac{1}{M}(S_0\ e^{-\lambda t})\right]\left[1 - e^{\lambda(t-T)}\right] > 0 \tag{23}$$

It can be observed that equation (17) satisfies the condition of equation (11) when $A(t) = A_{max}$ whereas equation (23) violates the condition of equation (11) when $A(t) = 0$. Therefore, the optimal advertising policy is to allocate all available advertising capital. The result could have been intuitively arrived at, because sales increase in proportion to the advertising rate and our objective function is to maximize total sales, $S_T = \int_0^T S(t)\,dt$. A more realistic objective function would be to maximize the net profit during a given interval of time T, represented by $\int_0^T [S(t) - A(t)]\, dt$.

A solution similar to that in reference [1] is obtained.

Example 13

This example is a slight modification of the advertising model discussed in the previous example. If the firm does no advertising, then its rate of change of sales will decrease at a rate proportional to the rate of sales at that time. However, if the firm advertises, its rate of change of sales at any time increases in proportion to the advertising done in the previous time period. Therefore, the rate of change of sales due to a combined effect of these two factors is represented by

$$\frac{dS(t)}{dt} = -\lambda S(t) + \gamma \int_0^\infty A(t-\tau)e^{-\tau}d\tau \tag{1}$$

The problem is to find an advertising rate which maximizes the total sales represented by

$$S_T = \int_0^t S(t)\, dt \tag{2}$$

Solution:

Let

$$x_1(t) = S(t), \quad x_1(0) = S_0$$

$$x_2(t) = \frac{dx_1(t)}{dt}, \quad x_2(0) = S_1$$

$$x_3(t) = \int_0^t x_1(t)\, dt, \quad x_3(0) = 0$$

$$\theta(t) = A(t)$$

By employing a change of variable, $x = t-\tau$, (see Section 5 of Appendix) equation (1) can be written as

$$\frac{dx_2(t)}{dt} = \gamma\theta(t) - (1+\lambda)\, x_2(t) - \lambda x_1(t) \tag{3}$$

The performance equations are

$$\frac{dx_1}{dt} = x_2, \quad x_1(0) = S_0 \tag{4}$$

$$\frac{dx_2}{dt} = \gamma\theta - (1+\lambda)\, x_2 - \lambda x_1, \quad x_2(0) = S_1 \tag{5}$$

$$\frac{dx_3}{dt} = x_1, \quad x_3(0) = 0 \tag{6}$$

The Hamiltonian and the adjoint system are

$$H = z_1 x_2 + z_2 \left(\gamma\theta - (1+\lambda)\, x_2 - \lambda x_1\right) + z_3 x_1 \tag{7}$$

$$\frac{dz_1}{dt} = \lambda z_2 - z_3, \quad z_1(T) = 0 \tag{8}$$

$$\frac{dz_2}{dt} = -z_1 + (1+\lambda) z_2, \quad z_2(T) = 0 \tag{9}$$

$$\frac{dz_3}{dt} = 0, \quad z_3(T) = 1 \tag{10}$$

Rewriting equations (7), (8) and (9) and using the result of equation (10) we have

$$H = z_1 x_2 + z_2 \left(\gamma\theta - (1+\lambda)\ x_2 - \lambda x_1\right) + x_1 \tag{11}$$

and

$$H^* = z_2 \gamma\theta \tag{12}$$

$$\frac{dz_1}{dt} = \lambda z_2 - 1 \tag{13}$$

$$\frac{dz_2}{dt} = -z_1 + (1+\lambda)\ z_2 \tag{14}$$

Solution of equations (13) and (14), together with conditions $z_1(T) = 0$, $z_2(T) = 0$ yields (see Section 6 of Appendix)

$$z_2(t) = \frac{1}{\lambda} - \frac{1}{\lambda(1-\lambda)}\ e^{\lambda(t-T)} + \frac{1}{(1-\lambda)}\ e^{(t-T)} \tag{15}$$

Since equation (12) is a linear function of the decision vector, $\theta(t)$, it represents a bang-bang type of optimal control. Therefore the optimal advertising rate is represented by

$$\begin{aligned} A(t) &= A_{max} && \text{if } z_2(t) > 0 \\ &= 0 && \text{if } z_2(t) < 0 \\ &= \text{unspecified} && \text{if } z_2(t) = 0 \end{aligned} \tag{16}$$

A solution similar to that in reference [1] is obtained.

PART II

SIMULATION AND OPTIMAL SOLUTION OF SOME INDUSTRIAL MANAGEMENT PROBLEMS BY AN ANALOG COMPUTER

1. INTRODUCTION

The electronic analog computer (EAC) consists basically of high-gain, direct-current, negative-feedback, electronic amplifiers used in conjunction with simple resistor - capacitor (R-C) circuits. In addition, special devices, such as function multipliers, function generators, diodes, and relays of various types, are used as auxiliary equipment for the simulation of more complex problems. Problem solution by the EAC is accomplished by analogy; that is, the computer is programmed so that its circuit equations have the same mathematical form as the equations of the problem. In this type of computer, voltages represent various physical quantities, such as inventory, production, sales, cost and so on. The magnitudes of the voltages are related to the numerical values of the physical quantities by the use of scale factors. The analog computer performs its operations in a continuous process with respect to time so that there is a relationship between the computing time of the analog computer and the variables of the problem being solved. By arbitrarily varying this relationship, the total computing time required to obtain a solution on the EAC may be varied, within limits, at the discretion of the operator. This process is known as time scaling.

2. BASIC ELEMENTS OF THE ANALOG COMPUTER

The basic elements of an EAC are relatively few in number. First, and most important, are the operational amplifiers; these, when used in conjunction with appropriate input and feedback impedances, supply the important operations of (i) addition and subtraction (ii) multiplication and division by a constant and (iii) integration. Second, it is necessary to provide for the problem coefficients in the simulation of the problem by electrical circuits. This is accomplished by ratios of feedback and input impedances in the R-C network employed by the use of coefficient potentiometer. Third, some type of control system must be provided to start, hold, and stop the computation and to allow for proper application of the initial condition voltages required by the problem. These control functions are performed by a system of automatic relays in the computer referred to as operate-reset relays. Fourth, as the problems being solved grow more complex, units capable of multiplying and dividing variable quantities must be provided. The representation of nonlinear phenomenon is facilated by the use of equipment such as diodes, operational relays, and differential relays. Other complex problems will require the use of arbitrary - function - generation equipment for representing functions which are difficult to describe mathematically.

3. CASE STUDIES

Example 1 Optimum Production Scheduling

In a manufacturing company, forecasting is used in desiguing production rules which anticipate and prepare for sales fluctuations. A buffer inventory is maintained so that errors in sales forecasts will not cause runouts or will not force rapid changes in the rate of plant operation. In general, the rate of change in finished goods inventories is equal to the difference between the production and sales rate, that is

$$\frac{d[I(t)]}{dt} = P(t) - S(t) \tag{1}$$

where $I(t)$, $P(t)$ and $S(t)$ represent the inventories, production and sales at time t, respectively.

We assume that the rate of the costs from holding inventories and stockouts can be approximated by the quadratic $c_I(I(t) - I^*)^2$, and the rate at which manufacturing costs are incurred can be approximated by the quadratic $c_P(P(t) - P^*)^2$, where c_I and c_P are constant, and I^* and P^* represent the desired inventory and production level of the plant respectively.

Therefore the total cost, c_T, incurred from time 0 to T is

$$c_T = \int_0^T \{c_I(I(t) - I^*)^2 + c_P(P(t) - P^*)^2\}\, dt \tag{2}$$

The problem is to find the optimum production to minimize the cost function represented by equation (2) which is subject to the constraint given by equation (1).

Solution:

Pontryagin's maximum principle is used to arrive at a system of equations [4] which are then simulated on the EAI TR-48 Analog computer.

Let us define

$$x_1(t) = I(t)$$

$$\theta(t) = P(t)$$

Then equation (1) becomes

$$\frac{dx_1(t)}{dt} = \theta(t) - S(t), \quad x_1(0) = I(0) \tag{3}$$

We introduce $x_2(t)$ such that

$$x_2(t) = \int_0^t \{c_I(x_1(t) - I^*)^2 + c_P(\theta(t) - P^*)^2\}\, dt$$

or

$$\frac{dx_2}{dt} = c_I(x_1-I^*)^2 + c_P(\theta-P^*)^2, \quad x_2(0) = 0 \tag{4}$$

The problem is thus transformed into that of minimizing $x_2(T)$.

The Hamiltonian is

$$H = z_1(\theta-S) + z_2\left[c_I(x_1-I^*)^2 + c_P(\theta-P^*)^2\right] \tag{5}$$

The components of the adjoint vector are

$$\frac{dz_1}{dt} = - 2\, z_2\, c_I(x_1-I^*), \quad z_1(T) = 0 \tag{6}$$

$$\frac{dz_2}{dt} = 0, \quad z_2(T) = 1 \tag{7}$$

Solving equation (7) for z_2 gives

$$z_2(t) = 1 \tag{8}$$

The optimal control for this type is found from the condition

$$\frac{\partial H}{\partial \theta} = 0 = z_1 + 2c_P(\theta-P^*) \tag{9}$$

or

$$z_1(t) = - 2c_P\,(\theta-P^*) \tag{10}$$

Combination of equations (10) and (6) yields $\theta(T) = P^*$ and

$$- 2c_P \frac{d\theta}{dt} = - 2c_I\,(x_1 - I^*) \tag{11}$$

$$\frac{d\theta}{dt} = \frac{c_I}{c_P}\,[x_1(t) - I^*], \; \theta(t) = P^* \tag{12}$$

Equations (12) and (3) constitute a pair of differential equations in two unknown functions $x_1(=I)$ and $\theta(=P)$. The solution of these two equations yield the optimum inventory and production planning.

The solution of equations (3) and (12) are

$$x_1(t) = A_1 e^{\lambda t} + A_2 e^{-\lambda t} + (x_1)_p \tag{13}$$

$$\theta(t) = A_1 \ e^{\lambda t} - A_2 \lambda e^{-\lambda t} + \frac{d(x_1)_p}{dt} + S(t) \tag{14}$$

where

$$\lambda^2 = \frac{c_I}{c_P}$$

A_1 and A_2 are constants to be determined by the initial conditions and $(x_1)_p$ is the particular solution of the equation to be decided by the forms or values of I^* and S.

Analog Computer Solution

A solution by the analog computer is presented as follows:

(i) Summary of equations:

$$\frac{dx_1(t)}{dt} = \theta(t) - S(t), \quad x_1(0) = I(0)$$

$$\frac{d\theta(t)}{dt} = \frac{c_I}{c_P} (x_1(t) - I^*), \quad \theta(T) = P^*$$

$$\frac{dx_2(t)}{dt} = c_I \ (x_1(t) - I^*)^2 + c_P \ (\theta(t) - P^*)^2, \quad x_2(0) = 0$$

An unscaled computer diagram is given in Figure 1.

(ii) Numerical Example:

We assume that the sales function is a sinusoidal function of one cycle duration with an amplitude of 300 about a mean of 700.

$$S(t) = 700 + 300 \text{ Sin } \frac{2\Pi}{T} t \text{ units/week} \tag{15}$$

The other parameter values and initial conditions are given as

$I^* = 1000$ units

$P^* = 700$ units/week

$c_I = \$1/(\text{unit})^2$ week

$c_P = \$100$ week$/(\text{unit})^2$

$T = 52$ weeks (=1 year)

$\theta(T) = P^* = 700$ units/week

$x_1(0) = 1200$ units

$x_2(0) = 0$

The scaling table for the analog computer is given in Table 1.

The scaled equations become:

$$\frac{d[.0001x_1]}{dt} = \frac{.0001}{.001} [.001\,\theta - .001S]$$

$$[.0001\, x_1(0)] = .0001\, x_1(0), \quad x_1(0) = 1200$$

or

$$\frac{d[.0001x_1]}{d\tau} = .1\, [.001\,\theta - .001S]$$

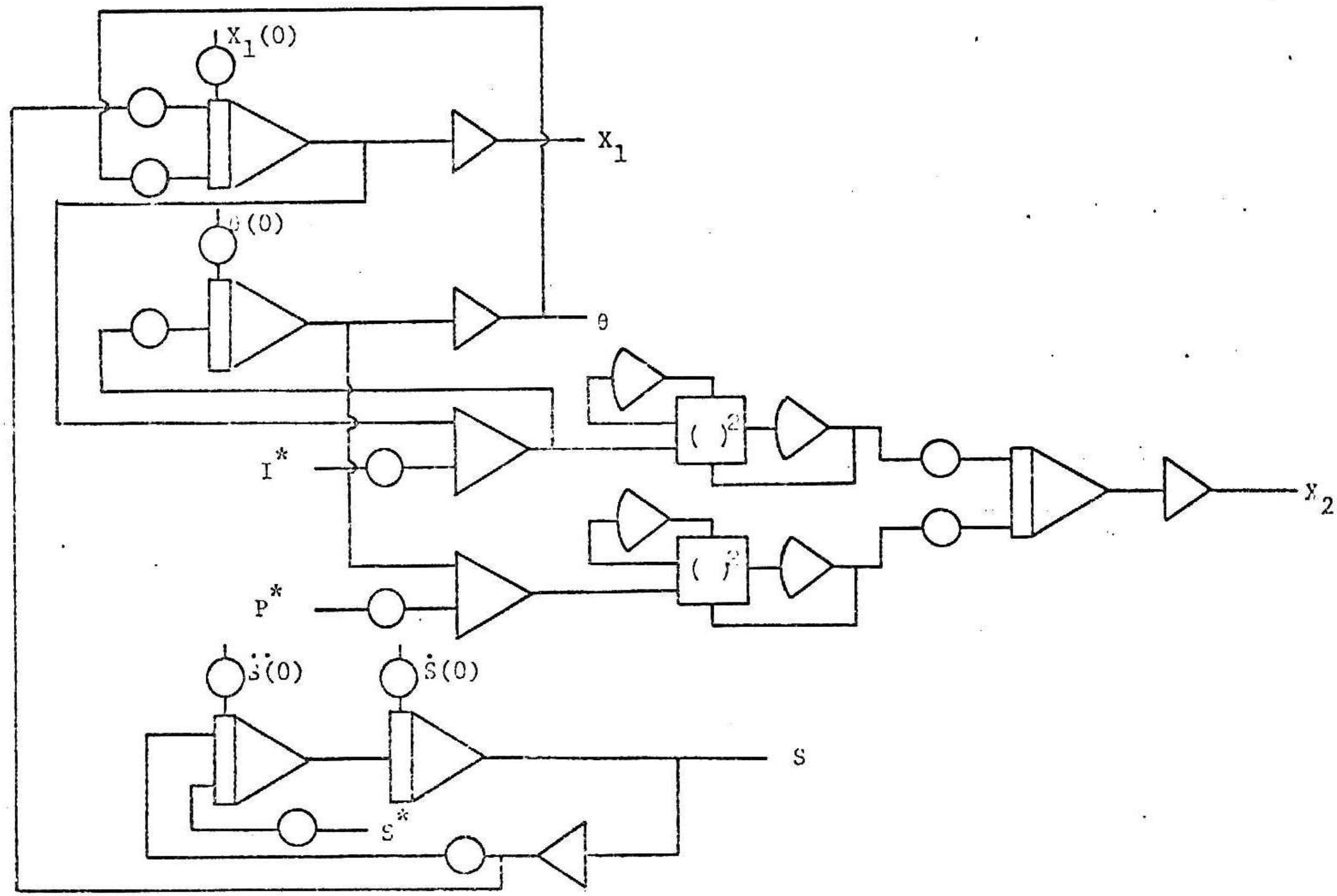

Fig. 1. Unscaled analog computer diagram

Table 1

Scaling Table

Variable	Maximum Estimated Value	Scale factor = 1/Max.Est. Val	Computer Variable
$x_1(t)$	10,000	0.0001	$[.0001\ x_1(t)]$
$\theta(t)$	1,000	0.001	$[.001\ \theta(t)]$
$S(t)$	1,000	0.001	$[.001\ S(t)]$
$x_2(t)$	10^9	10^{-9}	$[10^{-9}\ x_2(t)]$

$$[.0001\ x_1(0)] = .0001\ x_1(0)$$

$$\frac{d[.001\theta]}{dt} = .01 \times \frac{.001}{.0001} [.0001\ x_1 - .0001\ I^*], \quad I^* = 1000$$

$$[.001\ \theta(T)] = 0.001\ P^*, \quad P^* = 700$$

or

$$\frac{d[.001\theta]}{dt} = .1\ [.0001\ x_1 - .0001\ I^*]$$

$$[.001\ \theta(T)] = 0.001\ P^*$$

$$\frac{d[10^{-9}x_2]}{dt} = \frac{1\mathrm{x}10^{-9}}{10^{-8}} [.0001\ x_1 - .0001\ I^*]^2 + \frac{100\mathrm{x}10^{-9}}{10^{-6}} [.001\theta - .001P^*]^2$$

$$[10^{-9}\ x_2(0)] = 0$$

or

$$\frac{d[10^{-9}x_2]}{dt} = .1[.0001\ x_1 - .0001\ I^*]^2 + .1[.001\ \theta - .001\ P^*]^2$$

$$[10^{-9}\ x_2(0)] = 0$$

The scaled computer diagram based on these modified equations is given in Figure 2.

(iii) Iteration procedure:

The system of equations to be solved is a two point (or split) boundary value problem because the initial inventory level, $x_1(0)$, and final production rate, $\theta(T)$, are given (see equations 3 and 12). In order to arrive at the final production rate, a trial and error procedure was adopted.

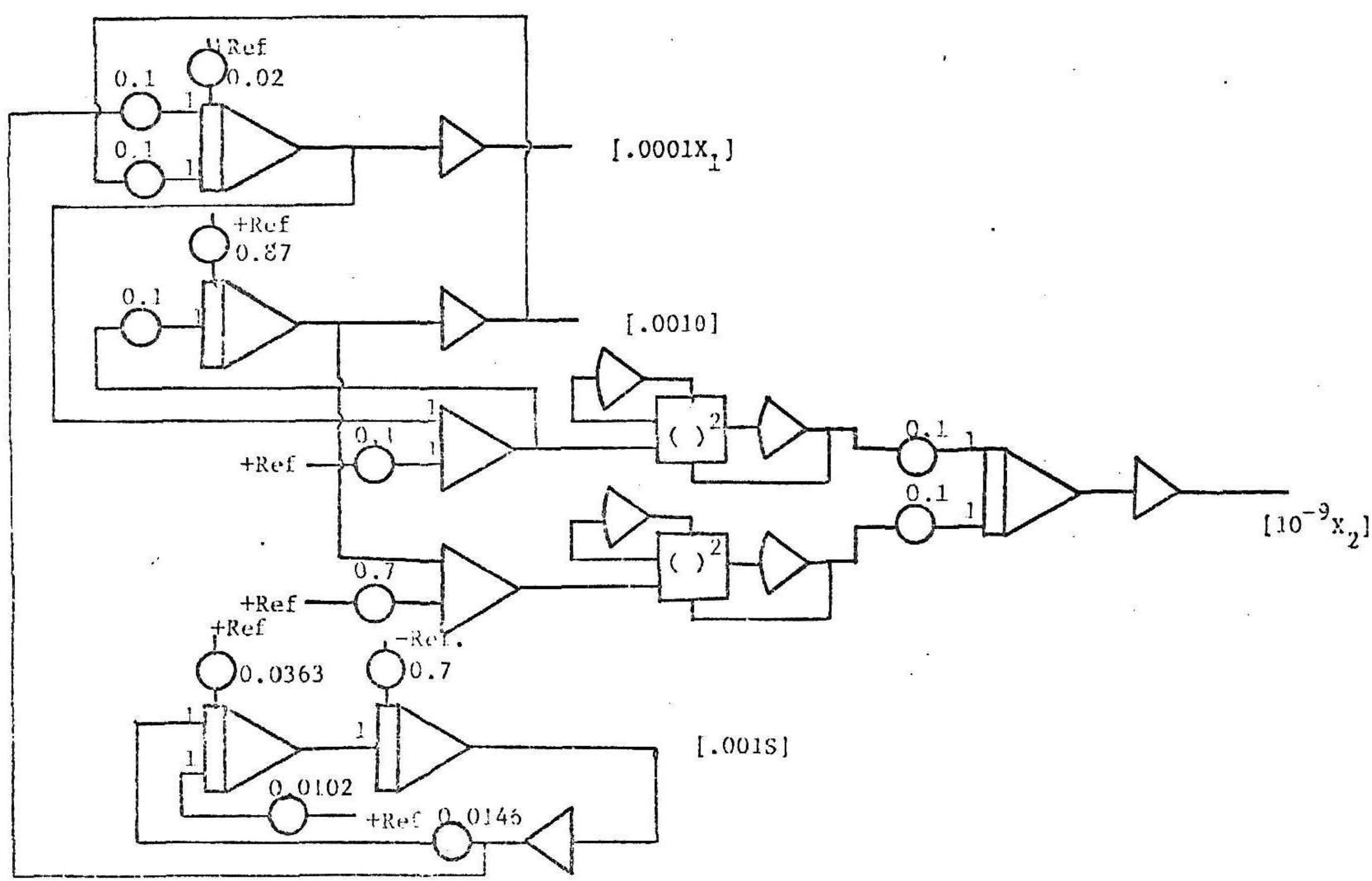

Fig. 2. Scaled computer diagram

Two initial production rates, 500 and 900 units/week, on either side of the final production rate were chosen and their end values observed. On the basis of observed end values, the interval of trial was systematically reduced until an initial value of 826 units/week was found to satisfy the desired final production rate.

(iv) Results and discussion:

The behaviour of the sales rate, production rate, inventory level and total cost at optimal condition is presented in Figure 3. Figure 4 represents the variation of total cost with different initial production rates. Figures 5 to 8 show the behaviour of the system parameters at the optimal initial production rate, P(0) = 826 units/week, whereas Figures 9 and 10 compare the behaviour of inventory level at optimal and non-optimal initial production rates.

Analytical Solution

For a sinusoidal sales rate, given by equation (15), equations (13) and (14) become

$$x_1(t) = A_1 e^{\lambda t} + A_2 e^{-\lambda t} + \frac{a\,\frac{2\Pi}{T}}{\lambda^2 + \frac{4\Pi^2}{T^2}} \cos \frac{2\Pi}{T} t + I^* \tag{16}$$

$$\theta(t) = A_1 \lambda e^{\lambda t} - A_2 \lambda e^{-\lambda t} - \frac{a\,\frac{4\Pi^2}{T^2}}{\lambda^2 + \frac{4\Pi^2}{T^2}} \text{Sin} \frac{2\Pi}{T} t + (S_0 + a \,\text{Sin} \frac{2\Pi}{T} t) \tag{17}$$

Using the boundary conditions $x_1(0) = I(0)$ and $\theta(T) = P^*$, we obtain

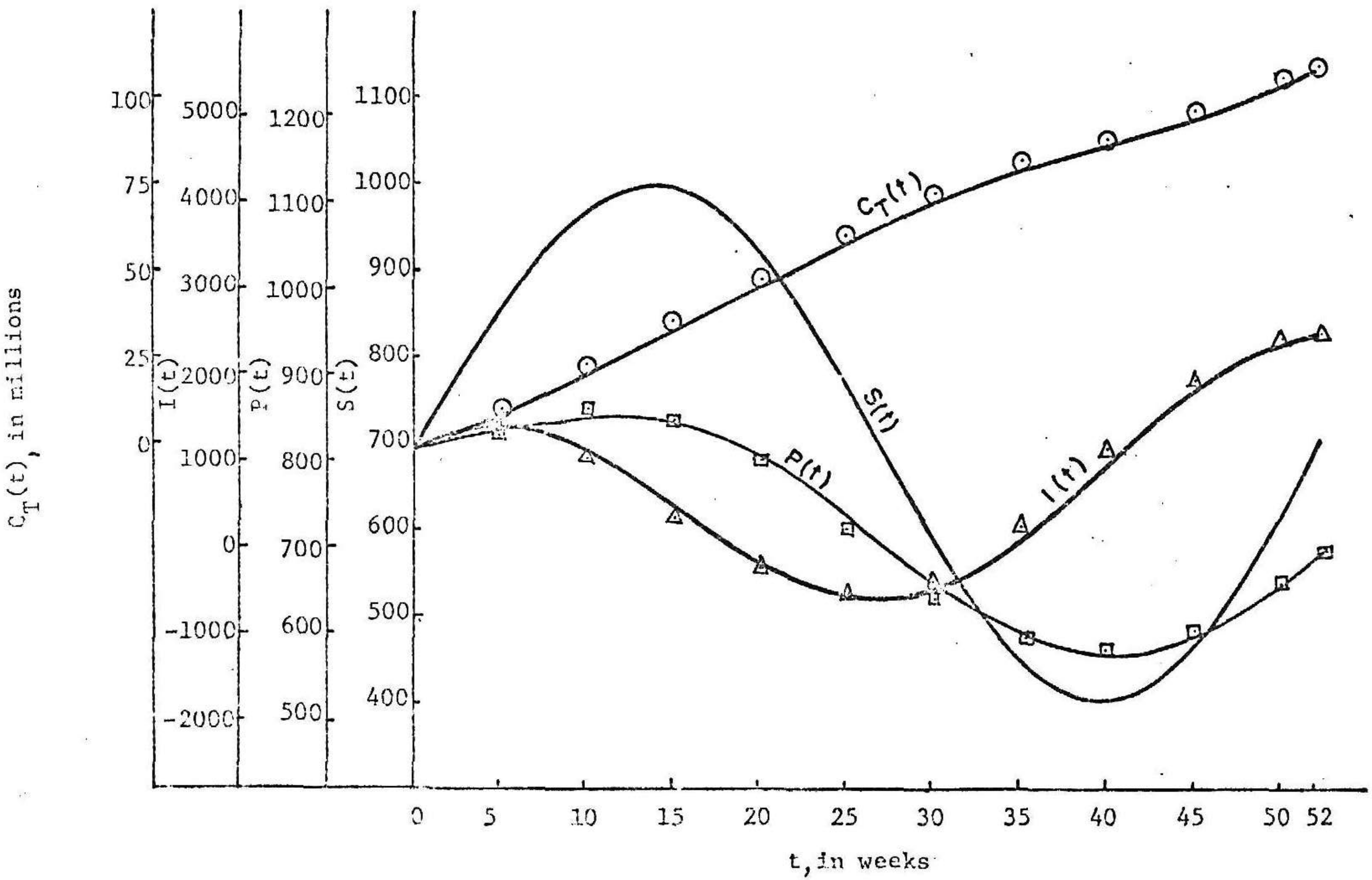

Fig. 3. Behaviour of sales rate, production rate, inventory level and total cost with respect to time given I(0) = 1200 units and 0(T) = 700 units/week

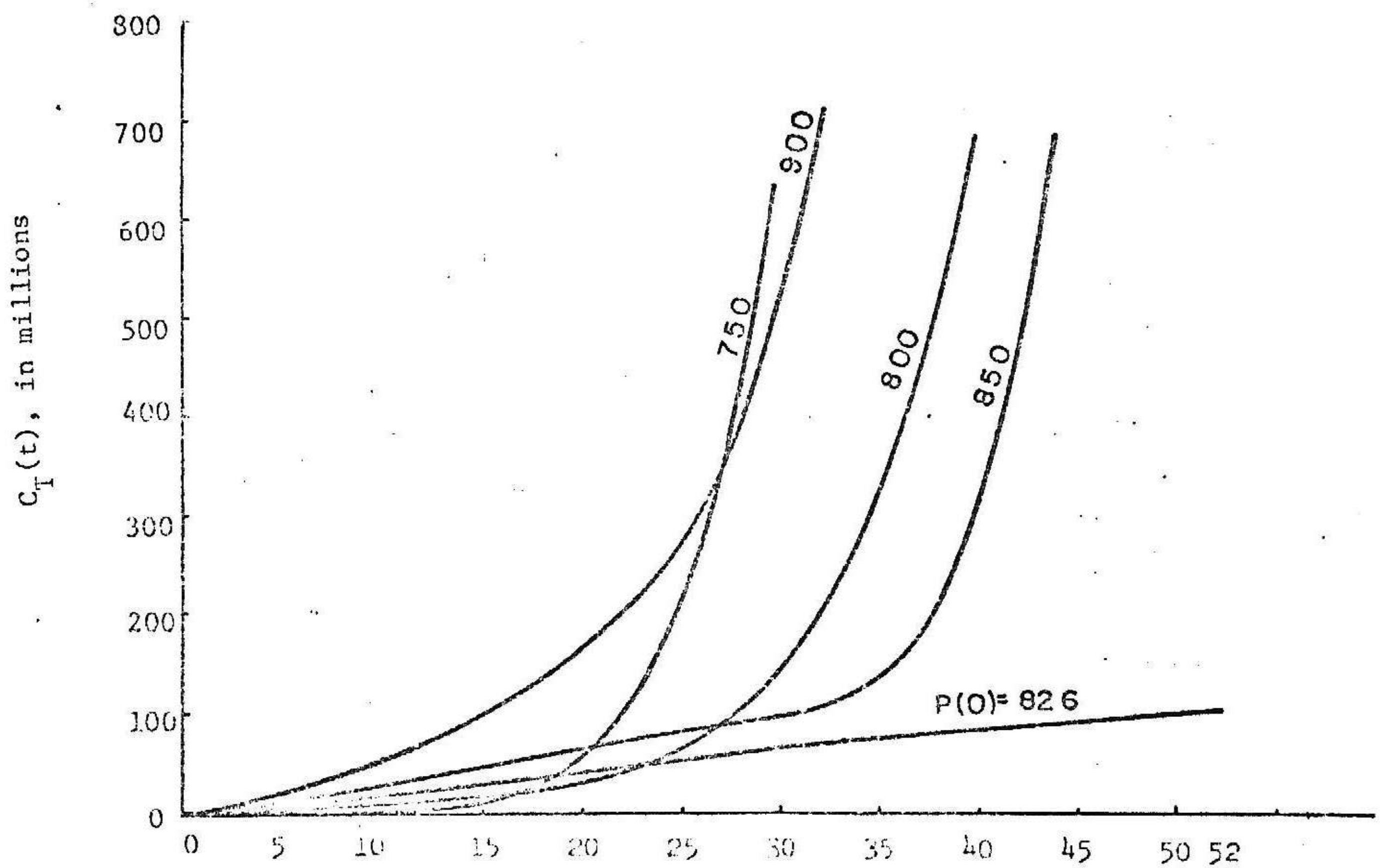

Fig. 4. Variation of total cost with different initial production rates, P(0) with I(0) = 1200 units

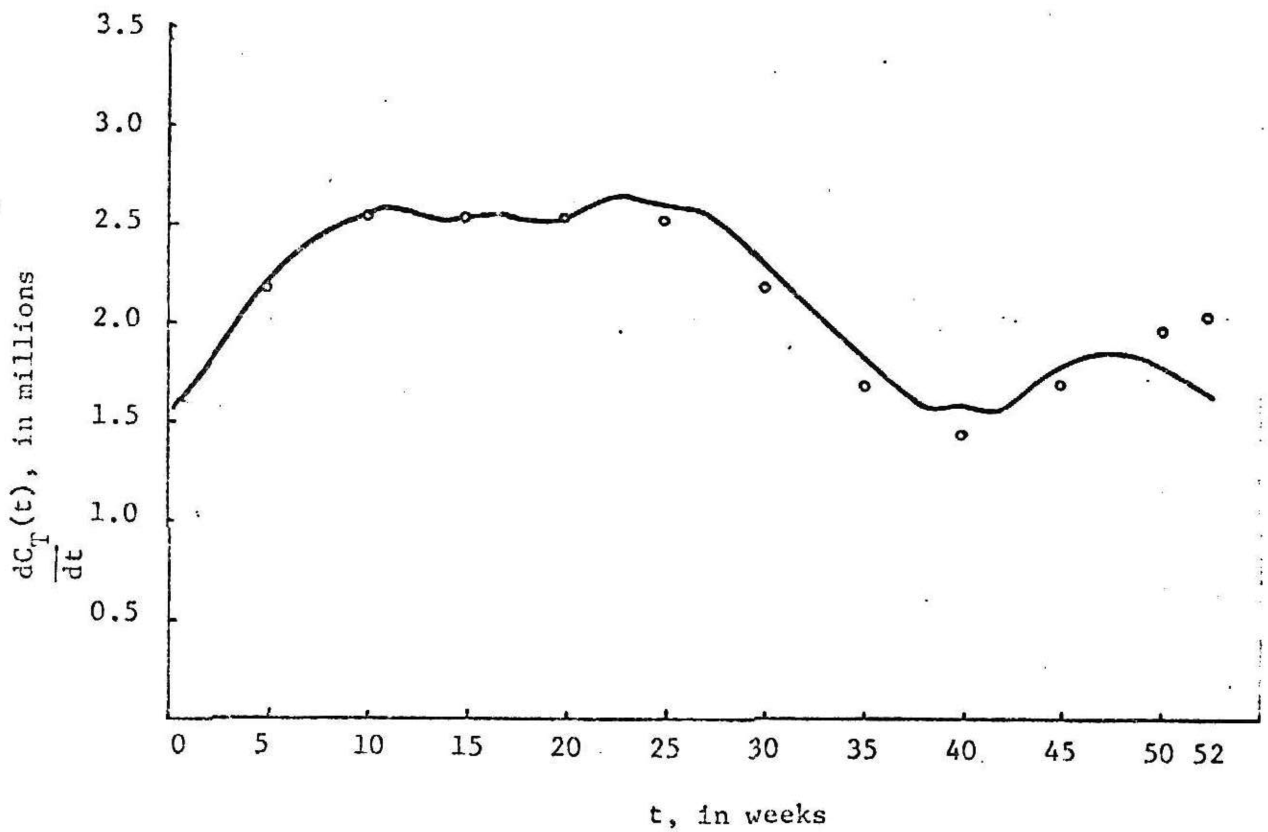

Fig. 5. Behaviour of cost with respect to time

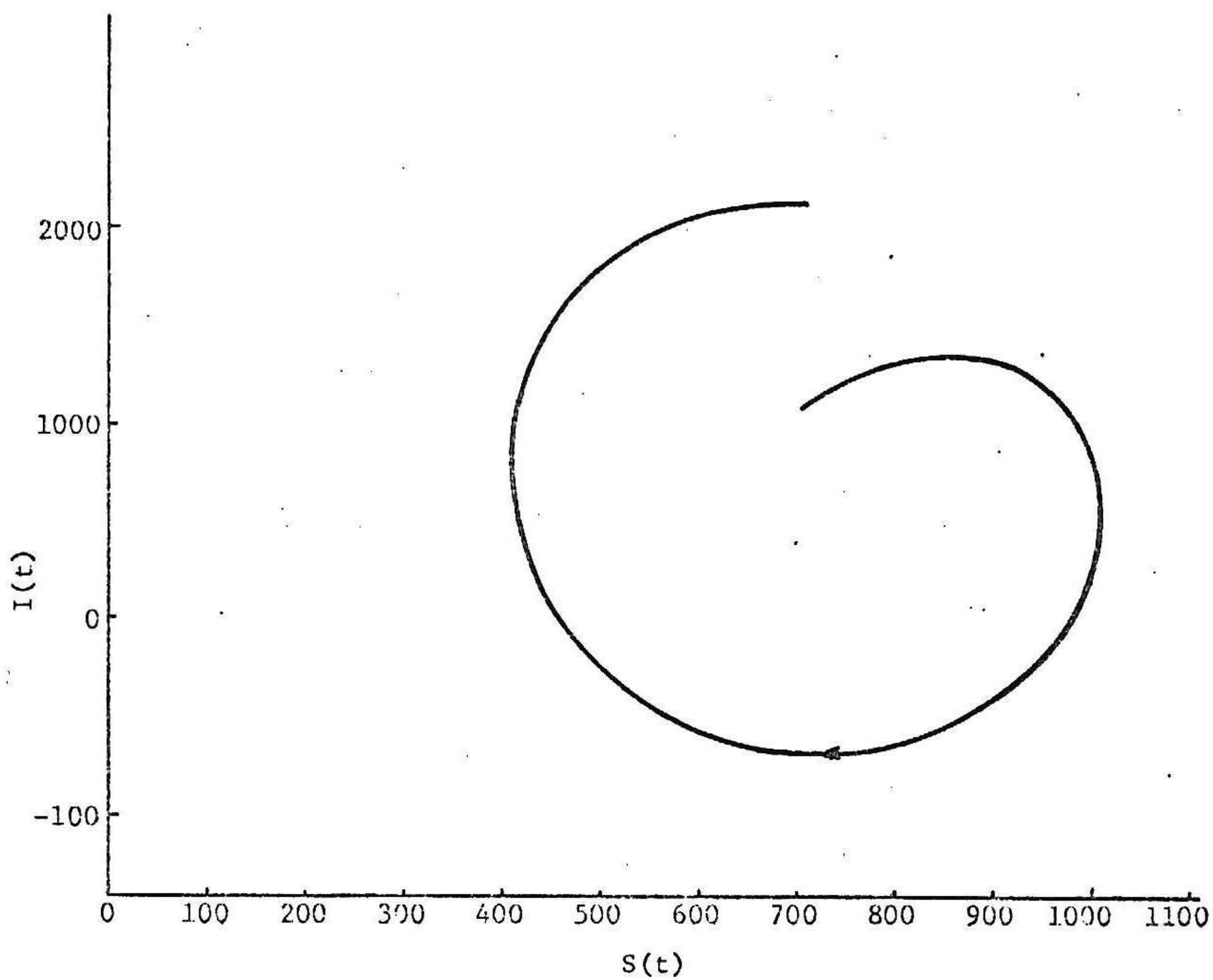

Fig. 6. Behaviour of inventory level with respect to sales rate

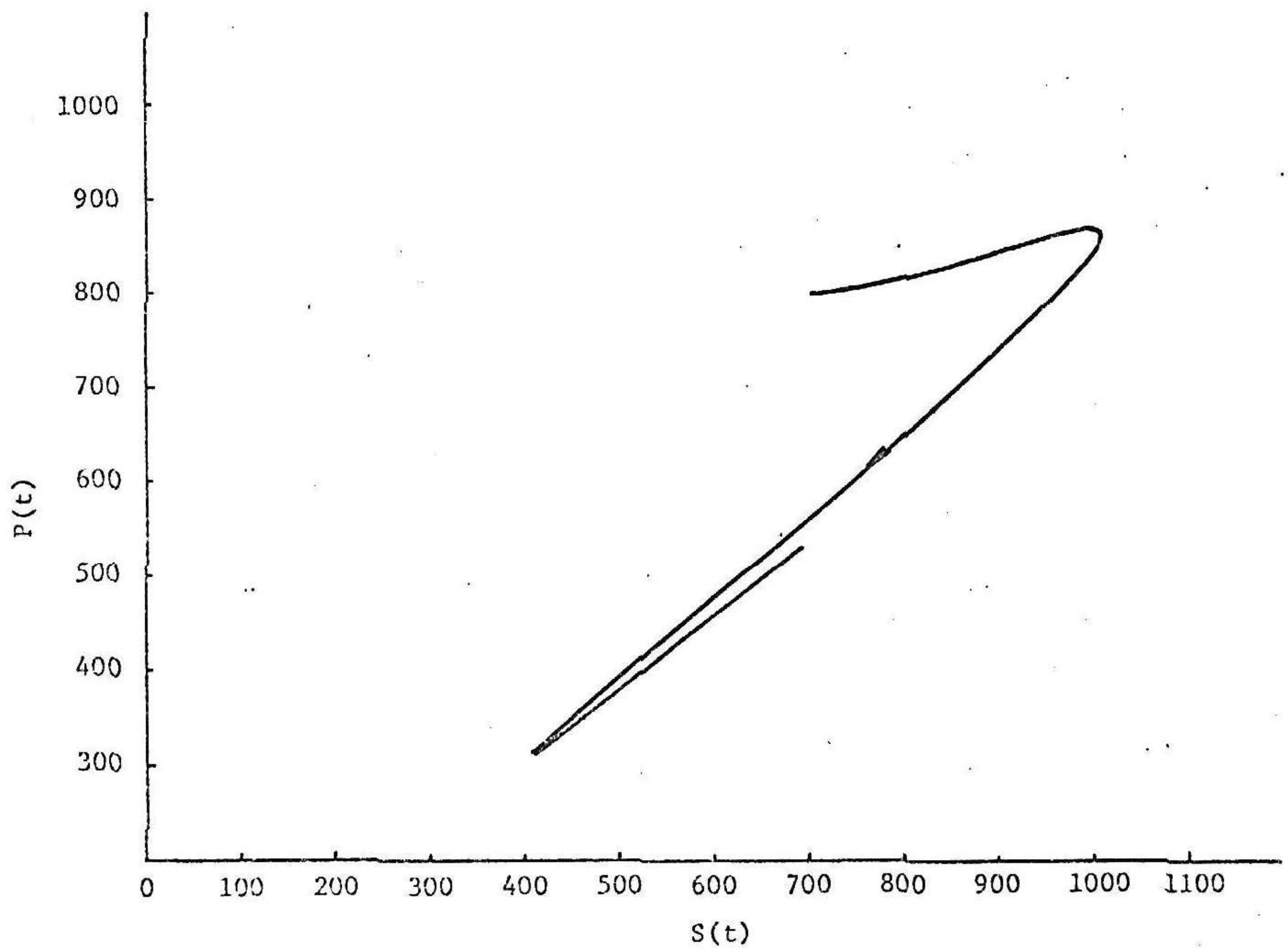

Fig. 7. Behaviour of production rate with respect to sales rate

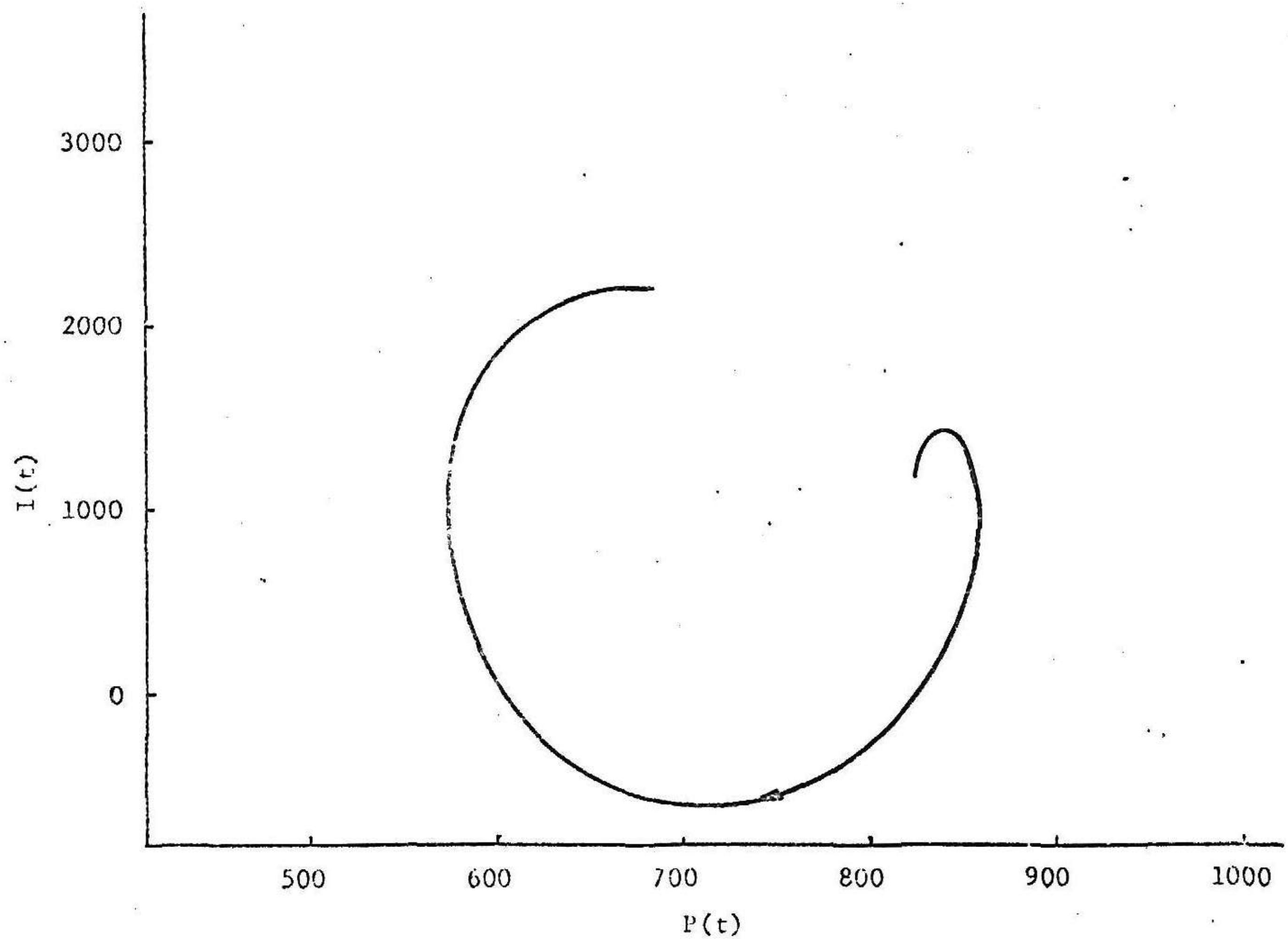

Fig. 8. Behaviour of inventory level with respect to sales rate

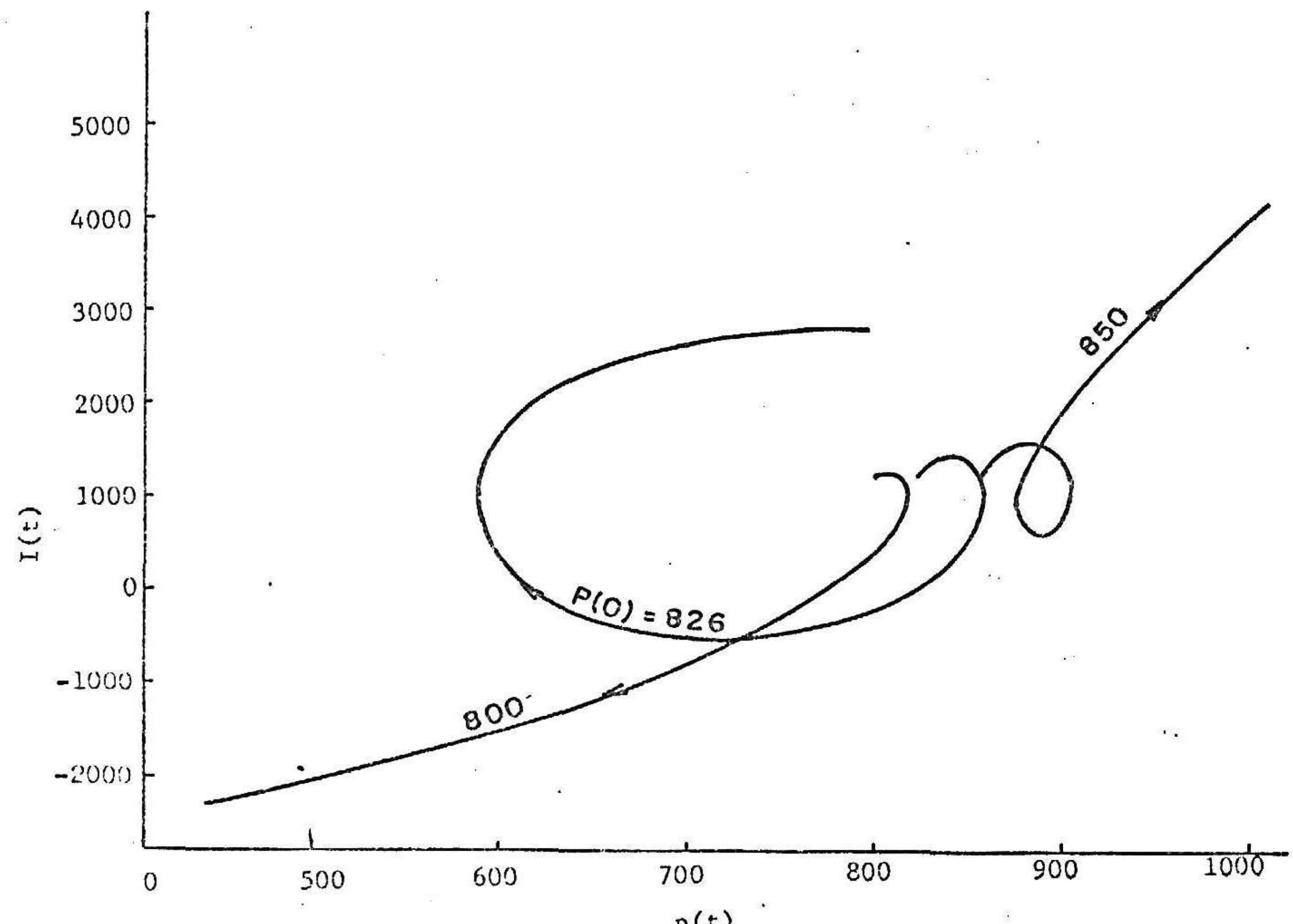

Fig. 9. Behaviour of inventory level with different initial production rates, P(0), with respect to production rate

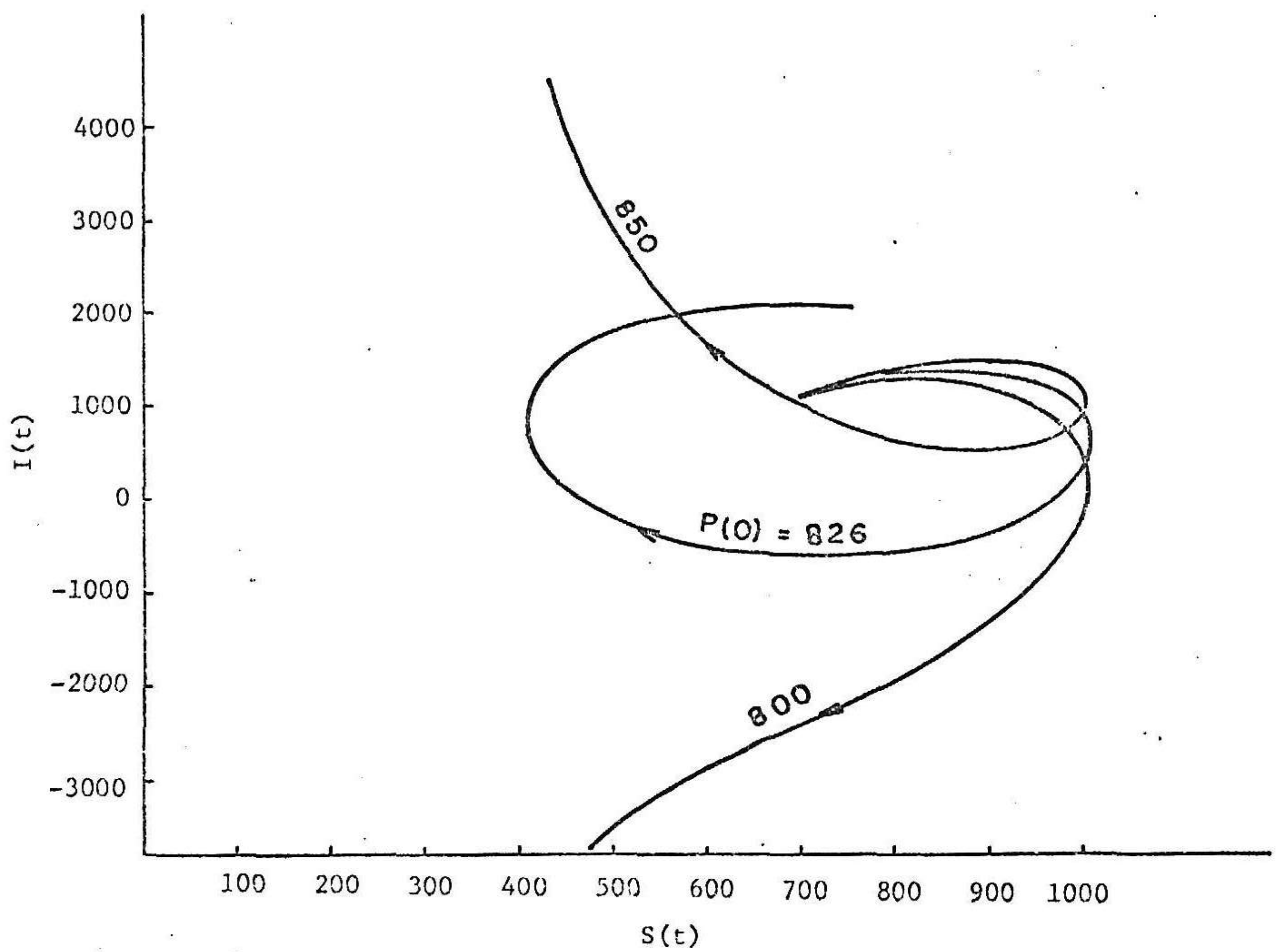

Fig. 10. Behaviour of inventory level with different initial production rates, P(0), with respect to sales rate

$$A_1 = \frac{\frac{1}{\lambda}(P^* - S_0) + \left\{ I(0) - I^* - \frac{a\,\frac{2\Pi}{T}}{\lambda^2 + \frac{4\Pi^2}{T^2}} \right\} e^{-\lambda T}}{e^{\lambda T} + e^{-\lambda T}} \tag{18}$$

$$A_2 = \frac{\left\{ I(0) - I^* - \frac{a\,\frac{2\Pi}{T}}{\lambda^2 + \frac{4\Pi^2}{T^2}} \right\} e^{\lambda T} - \frac{1}{\lambda}(P^* - S_0)}{e^{\lambda T} + e^{-\lambda T}} \tag{19}$$

Tables 2, 3 and 4 compare the values of the production rate, inventory level and total cost as obtained on the analog computer to that obtained by analytical solution. It can be observed from Figure 3 that there is less than 2% difference between the values obtained by the two methods.

The cummulative total cost with respect to time for different initial rates of production is represented in Figure 4 and it is observed that an initial production rate, P(0) = 826 units/week, yields a minimum total cost. All other initial production rates result in a much higher total cost.

Figure 5 shows the cost incurred at each interval of time. It can be observed that the cost increases initially, then remains constant for a period of time and finally decreases. This can be explained by the fact that the cost parameter for production rate is 100 times greater than that for inventory level and the cost equation (4) is more affected by a deviation from the desired production rate than a deviation from the desired inventory level. Hence whenever the production rate approaches P^*, the cost is relatively smaller. Figure 6 shows a phase plane plot between the sales

Table 2

Comparison of production rate as obtained on the analog computer and by analytical solution

Time (weeks)	Production rate (units/week)	
	Analog value	Analytical value
0	826	827.3
5	850	846.5
10	855	860.8
15	847	846.8
20	805	798.0
25	740	725.1
30	666	649.5
35	600	595.7
40	580	581.0
45	600	609.7
50	666	671.1
52	700	700.0

Table 3

Comparison of inventory level as obtained on the analog computer and by analytical solution

Time (weeks)	Inventory level (units)	
	Analog value	Analytical value
0	1200	1200.0
5	1400	1440.2
10	1100	1053.9
15	400	363.0
20	-200	-275.5
25	-550	-567.7
30	-500	-368.9
35	100	275.5
40	1000	1152.2
45	1800	1959.5
50	2400	2416.4
52	2480	2459.4

Table 4

Comparison of total cost as obtained on the analog computer and by analytical solution

Time (weeks)	Total cost (in millions of dollars)	
	Analog value	Analytical value
0	0	0
5	8	10.0
10	20	22.5
15	32.5	35.4
20	45	48.3
25	58	61.1
30	70	72.9
35	80	82.2
40	87.5	89.6
45	95	97.4
50	105	107.1
52	109	111.3

rate and inventory level. We call this a phase plane plot because we are plotting one variable against another. It can be observed from the figure that an increase in sales rate above the mean level reduces the inventory level, where as a decrease in sales rate below the mean level increases the inventory level. Figure 7 shows that the production rate increases, for an increase in sales rate, and decreases for a decrease in sales rate. Figure 8 shows the behaviour of the inventory level with respect to production rate.

It can be observed from Figure 9 that for a higher initial production rate the inventory level increases rapidly whereas for a lower initial production rate the number of stockouts increases. Figure 10 shows the behaviour of inventory level with respect to sales rate for different initial production rates. For a higher initial production rate, the inventory level increases, whereas for lower initial production rate, stockouts increase, in correspondence with that shown in Figure 9. When the initial inventory level was changed from 1200 units to 1400 units, the optimum policy yielded a lower initial production rate of 810 units/week. It can be observed from Figure 11 that the behaviour of the sales rate, production rate, inventory level and total cost correspond to that obtained with initial inventory I(0) = 1200 units (Figure 3). Figure 12 shows the variation of total cost with different initial production rates, P(0), and initial inventory level I(0) = 1400 units. It can be concluded that the inventory model is very sensitive to a variation in the initial production rate, P(0), but remains unaffected to a variation in the initial inventory level, I(0).

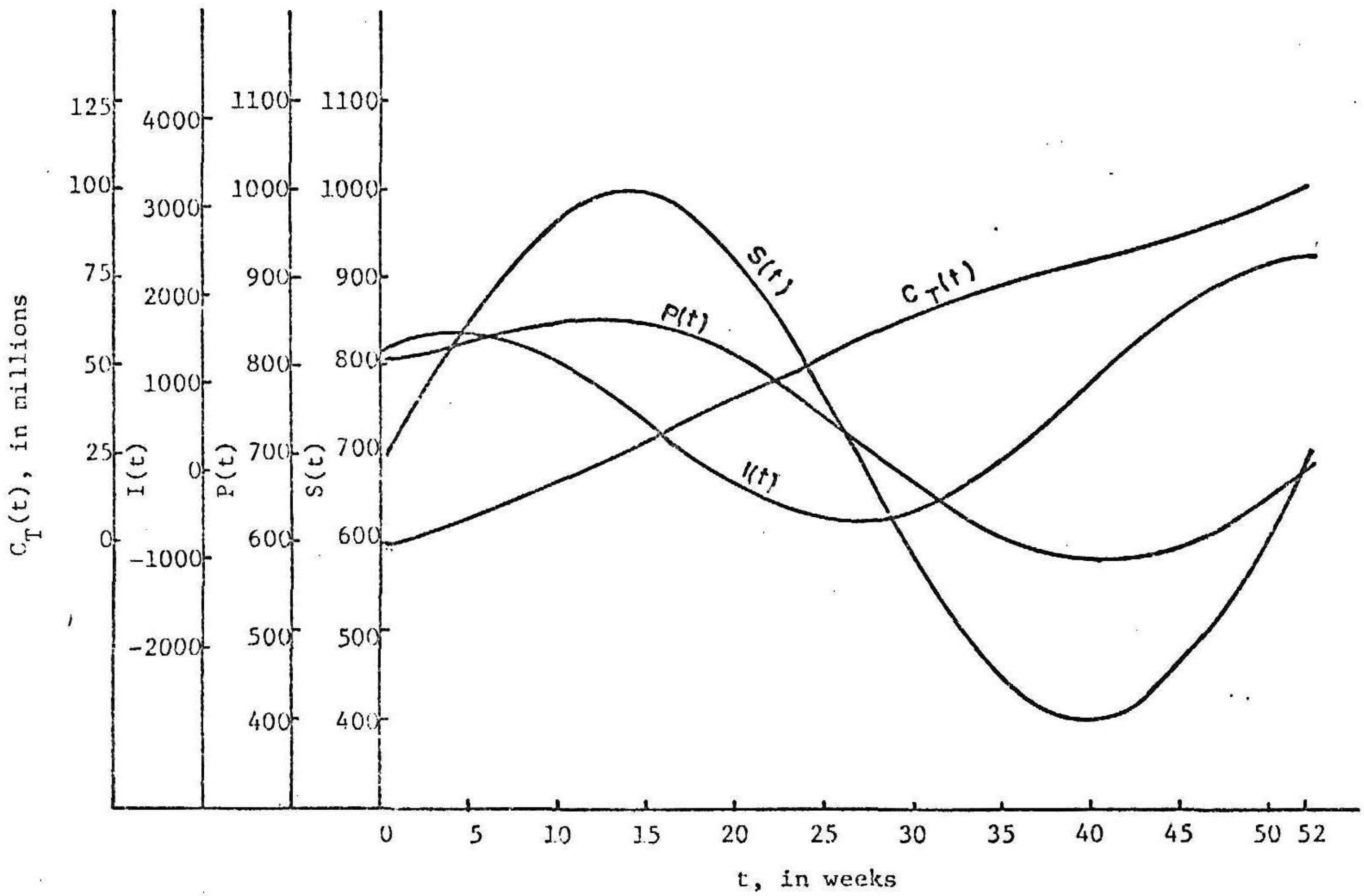

Fig. 11. Behaviour of sales rate, production rate, inventory level and total cost with respect to time, given I(0) = 1400 units and 0(T) = 700 units/week

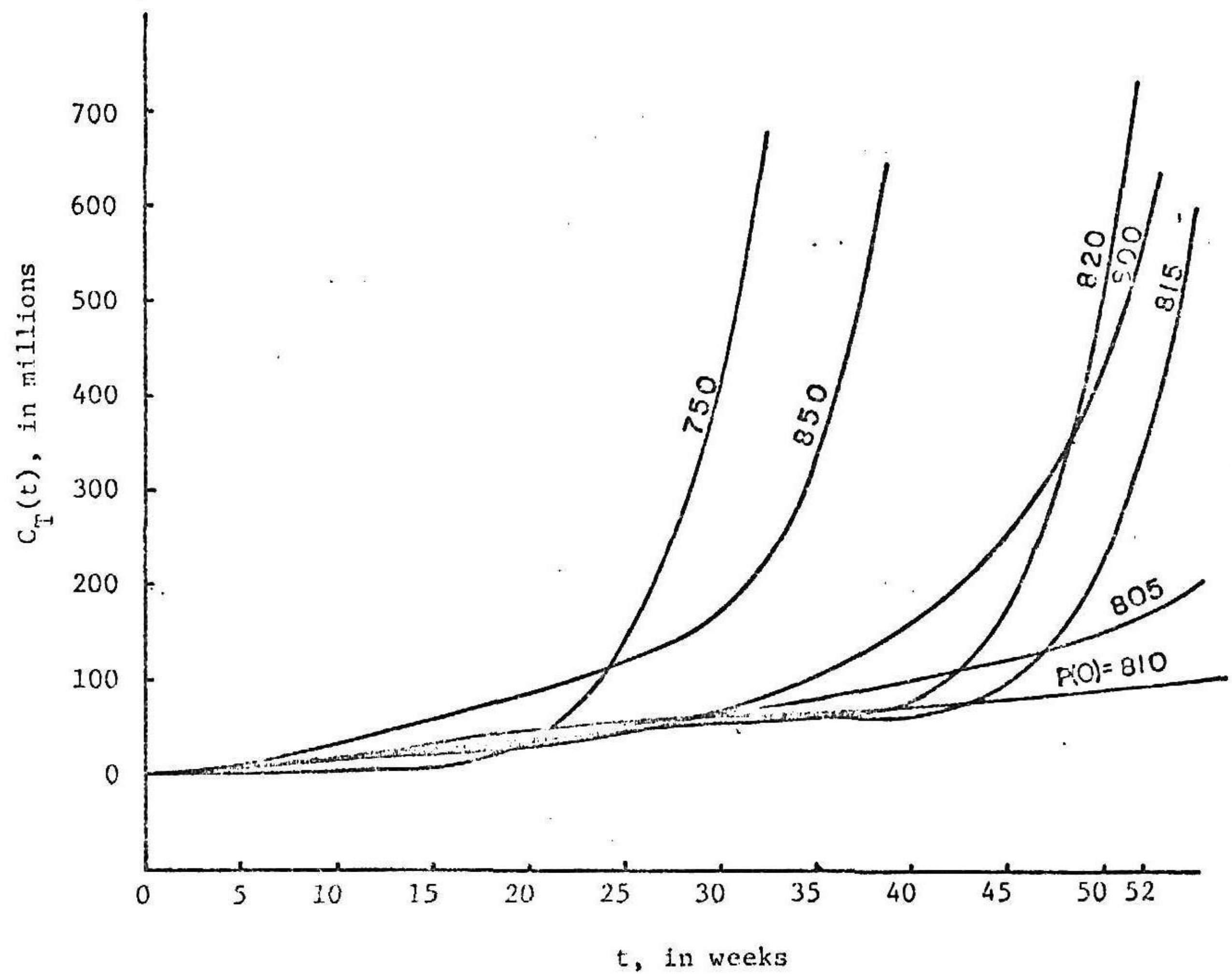

Fig. 12. Variation of total cost with different initial production rate, P(0), with I(0) = 1400 units

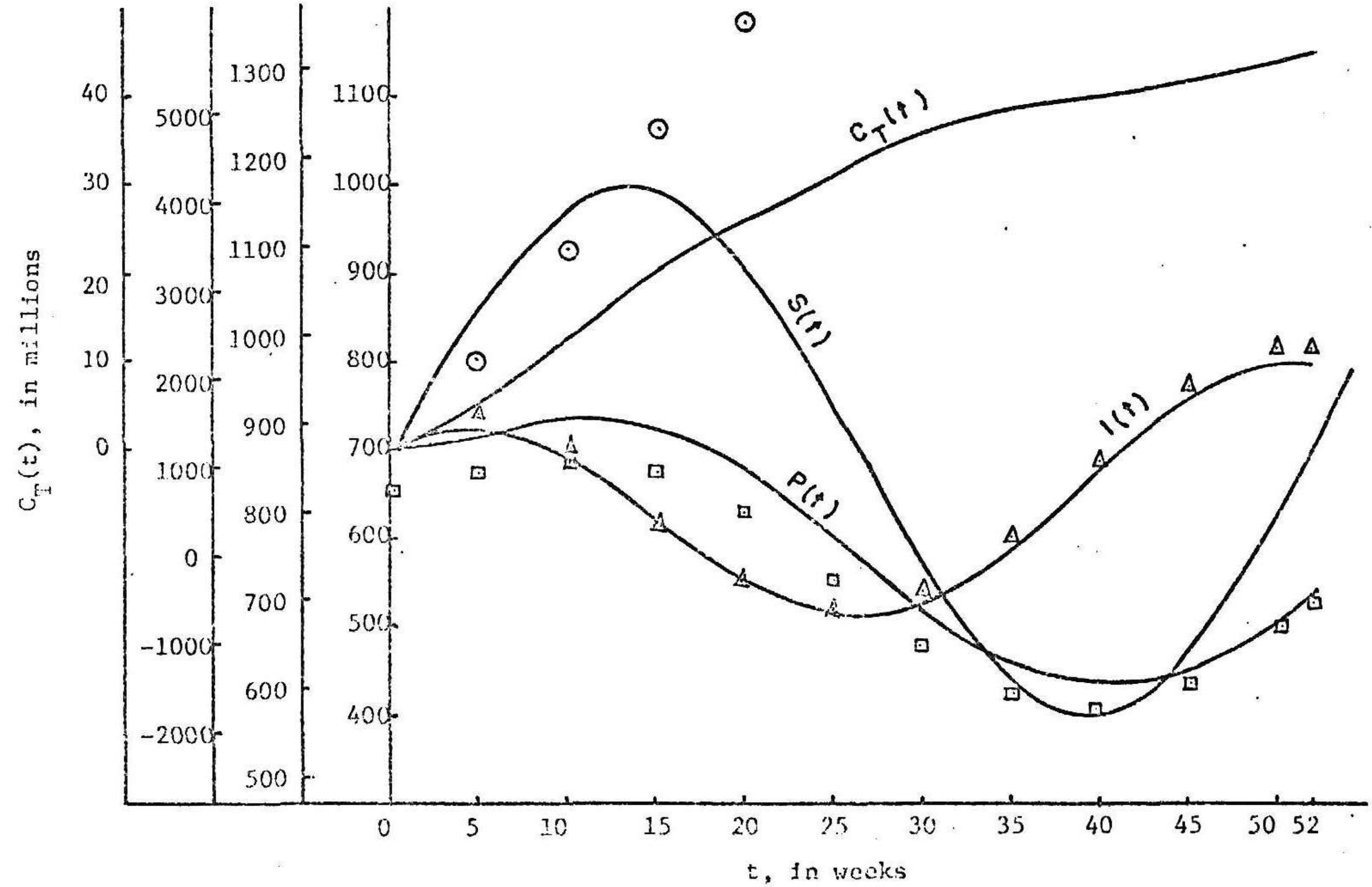

Fig. 13. Behaviour of sales rate, production rate, inventory level and total cost with respect to time

When the inventory and production scale factors were reduced by one tenth, an error of 5-6% was observed on the inventory level and production rate, whereas a large error was noticeable in the total cost as shown in Figure 13. This can be explained by the fact that the accuracy of the analog computer is affected when the voltage across the amplifiers goes below 0.001 V.

In this example the analytical solution was available [4]. But in cases where the analytical solution is difficult or unavailable, an approximate solution can be easily obtained on the analog computer. Moreover, a system of boundary value linear or non-linear differential equations which are difficult to solve by the digital computer can be easily solved on the analog computer. The utility of the analog computer is increasingly felt as the number of these boundary valued differential equations increases. Furthermore the analog computer is useful in obtaining graphic output at any point in the circuit; thus, variables can be very easily plotted against each other and with respect to time.

Example 2 Periodic (or Recyclic) Optimum Prduction Scheduling

This example is an extension of the previous example in that instead of a single time period, a periodic demand is assumed, i.e., the same demand is expected to occur for a number of time periods. The maximum principle for periodic processes is used to arrive at a system of equations to be solved.

The Maximum Principle for Periodic Processes [2]

Suppose that performance equations of a control process have the form

$$\frac{dx_i}{dt} = f_i(x_1(t),\ x_2(t), \ldots,\ x_s(t);\ \theta_1(t), \ldots,\ \theta_r(t)),$$

$$t_0 \leq t \leq T,$$

$$x_i(t_0) = x_i(T),$$

$$i = 1, 2, \ldots,\ s,$$

or the vector form

$$\frac{dx}{dt} = f(x(t);\ \theta(t)), \quad x(t_0) = x(T) \tag{1}$$

where $x(t)$ is an s-dimensional vector function representing the state of the process at time t and $\theta(t)$ is an r-dimensional vector function representing the decision at time t.

A typical optimization problem associated with such a process is to find a piecewise continuous decision vector function, $\theta(t)$, subject to the constraints

$$\psi_i\big(\theta_i(t),\ \theta_2(t), \ldots,\ \theta_r(t)\big) \leq 0, \quad i = 1,\ 2,\ \ldots,\ m, \tag{2}$$

which makes a function of the final values of the state

$$S = \sum_{i=1}^{s} c_i x_i(T), \quad c_i = \text{constant} \tag{3}$$

an extremum for the periodic condition $x(t_0) = x(T)$. The function, S, which is to be maximized (or minimized), is the objective function of the process.

The procedure for solving the problem to obtain the optimum control $\theta^*(t)$ and the corresponding trajectory $x^*(t)$, $t_0 \leq t \leq T$, is to introduce an s-dimensional adjoint vector $z(t)$ and a Hamiltonian function H which satisfy the following relations:

$$H(z(t), x(t), \theta(t)) = \sum_{i=1}^{s} z_i f_i(x(t); \theta(t)), \tag{4}$$

$$\frac{dz_i}{dt} = -\frac{\partial H}{\partial x_i} = -\sum_{j=1}^{s} z_j \frac{\partial f_j}{\partial x_i}, \quad i = 1, 2, \ldots, s, \tag{5}$$

$$z_i(T) = \sum_{j=1}^{s} z_j(t_0) \frac{\partial x_j(t_0)}{\partial x_i(T)} + c_i, \ i = 1, 2, \ldots, s . \tag{6}$$

The optimal decision vector function $\theta^*(t)$, which makes S maximum (or minimum), is the decision vector function $\theta(t)$, which renders the Hamiltonian function H maximum (or minimum) for $t_0 \leq t \leq T$. If the optimal decision vector function $\theta^*(t)$ is interior to the set of admissible decisions $\theta(t)$ {the set given by equation (2)}, a necessary condition for S to be an extremum with respect to $\theta(t)$ is

$$\frac{\partial H}{\partial \theta} = 0 . \tag{7}$$

If $\theta(t)$ is constrained, the optimal decision vector function $\theta^*(t)$ is determined either by solving equation (7) for $\theta(t)$ or by searching the boundary of the set.

Once the decision vector function $\theta(t)$ is chosen the trajectory $x(t)$ is fixed and the adjoint vector function $z(t)$ is uniquely determined by equation (5) and (6).

Analog Computer Solution

(i) Summary of equations:

Using the above algorithm, equations (3), (4) and (12) of Example 1 can be modified as

$$\frac{dx_1(t)}{dt} = \theta(t) - S(t), \quad x_1(0) = x_1(T)$$

$$\frac{d\theta(t)}{dt} = c_I(x_1(t) - I^*), \; \theta(0) = \theta(T)$$

$$\frac{dx_2(t)}{dt} = c_I(x_1(t) - I^*)^2 + c_P(\theta(t) - P^*)^2, \quad x_2(0) = 0$$

The same set of parameter values is given and the periodic nature of the demand is assumed. The scaling table for the analog computer solution is given in Table 1 and the scaled computer diagram is presented in Figure 1.

(ii) Iteration procedure:

The initial conditions of the system of equations to be solved were not known except that they were periodic for a base period of 52 weeks. The problem, therefore, was one which involved a two dimensional search for the two variables, P(0), and I(0). A trial and error procedure was adopted and the interval of error was systematically reduced till a best set of initial points was obtained, to satisfy the periodic nature of the system, at initial inventory level, I(0) = 2460 units, and initial production rate, P(0) = 700 units/week.

Table 1

Scaling Table

Variable	Maximum Estimated Value	Scale factor = 1/Max.Est. Value	Computer Variable
$x_1(t)$	10,000	.0001	$[.0001\ x_1(t)]$
$\theta(t)$	1,000	.001	$[.001\ \theta(t)]$
$S(t)$	1,000	.001	$[.001\ S(t)]$
$x_2(t)$	10^9	$[10^{-9}]$	$[10^{-9} x_2(t)]$

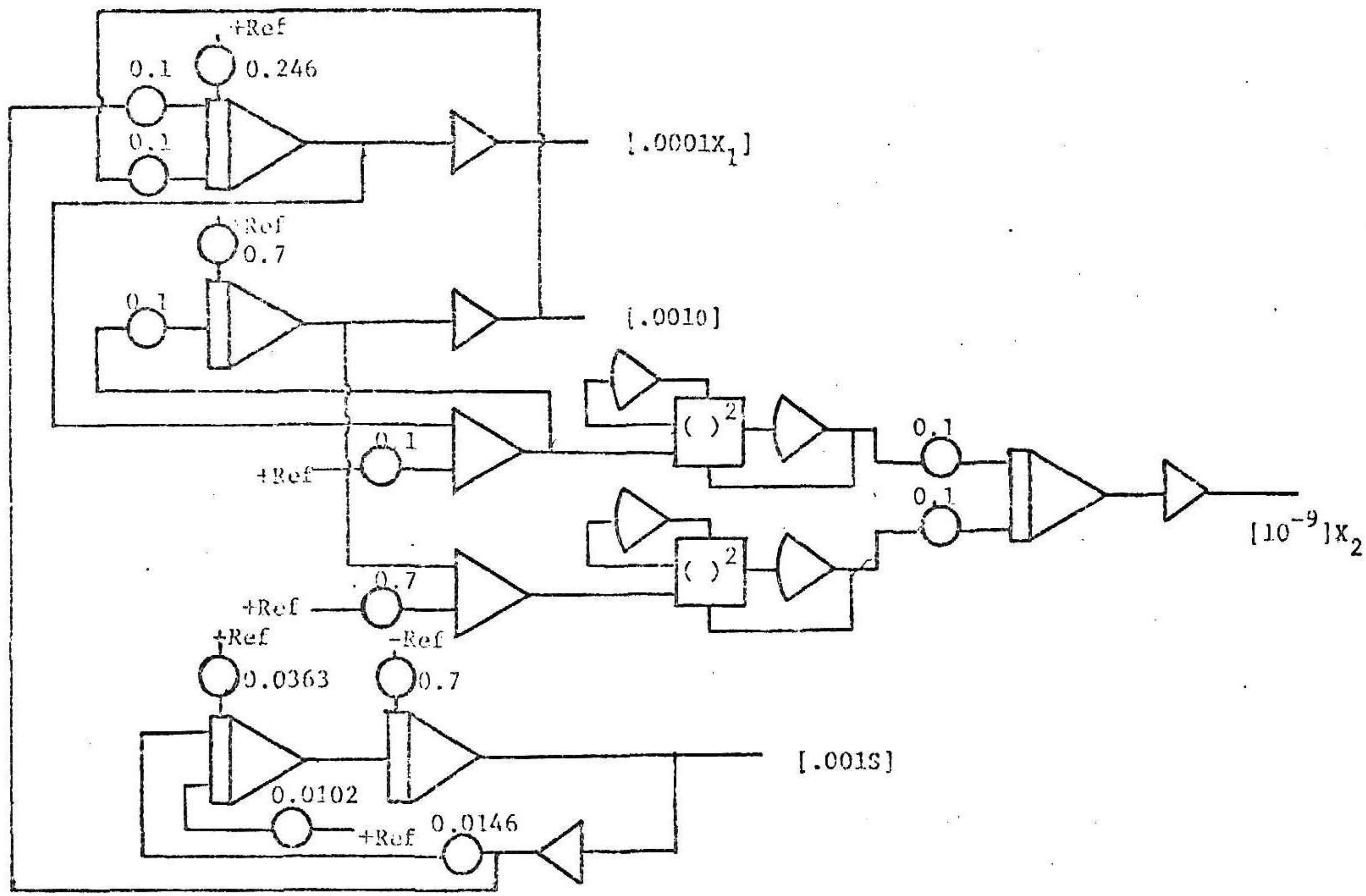

Fig. 1. Scaled computer diagram

(iii) Results and Discussion:

The behaviour of the sales rate, production rate, inventory level and total cost is presented in Figure 2.

Analytical Solution:

Combining the equations representing the rate of change of inventory level and production rate we get

$$\frac{d^2x_1}{dt^2} = \lambda^2 (x_1 - I^*) - \frac{dS}{dt} \tag{8}$$

where

$$\lambda^2 = \frac{c_I}{c_P}$$

The solution of equation (8) is

$$x_1(t) = A_1 e^{\lambda t} + A_2 e^{-\lambda t} + \frac{a \frac{2\Pi}{T}}{\lambda^2 + \frac{4\Pi^2}{T^2}} \cos \frac{2\Pi}{T} t + I^* \tag{9}$$

$$\theta(t) = A_1\lambda e^{\lambda t} - A_2\lambda e^{-\lambda t} - \frac{a \frac{4\Pi^2}{T^2}}{\lambda^2 + \frac{4\Pi^2}{T^2}} \sin \frac{2\Pi}{T} t + (S_0 + a \operatorname{Sin} \frac{2\Pi}{T} t) \tag{10}$$

where A_1, A_2 are integration constants obtained by combining the complementary function and particular solution of equation (8). Application of boundary conditions $x_1(0) = x_1(T)$ and $\theta(0) = \theta(T)$ yields the optimum inventory level and production rate as

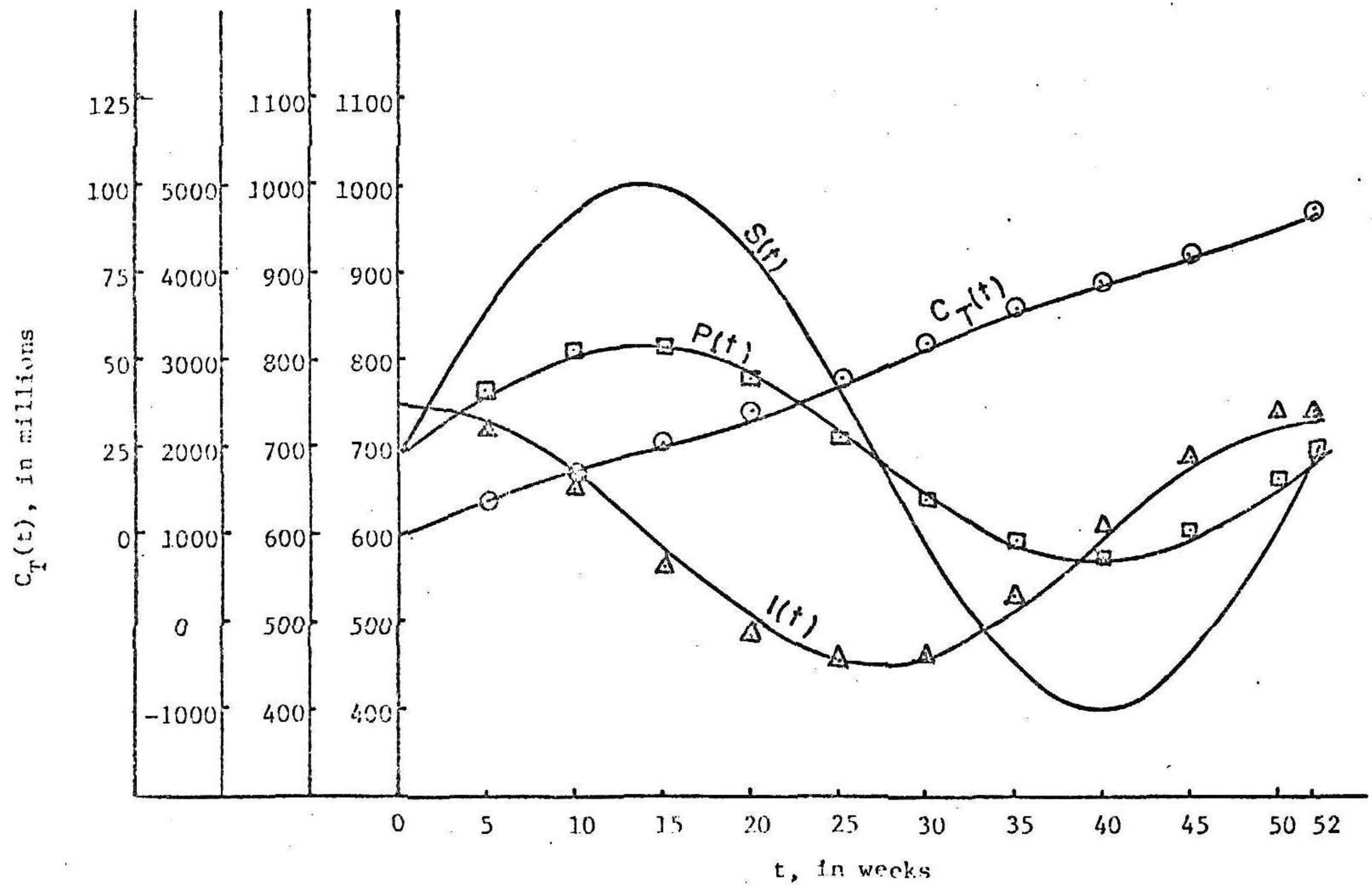

Fig. 2. Behaviour of sales rate, production rate, inventory level and total cost with respect to time

Table 2

Comparison of production rate as obtained on the analog computer and by analytical solution

Time (weeks)	Production rate (units/week)	
	Analog Value	Analytical Value
0	700	700.00
5	760	769.28
10	805	814.03
15	818	818.41
20	780	780.87
25	720	714.70
30	645	643.33
35	590	592.02
40	570	578.94
45	595	608.72
50	655	670.82
52	695	700.00

Table 3

Comparison of inventory level as obtained on the analog computer and by analytical solution

Time (weeks)	Inventory level (units)	
	Analog value	Analytical value
0	2460	2473.5
5	2300	2212.7
10	1650	1522.5
15	850	647.3
20	50	-102.9
25	-400	-462.8
30	-380	-304.7
35	120	315.2
40	950	1177.6
45	1800	1977.2
50	2250	2430.7
52	2300	2473.5

Table 4

Comparison of total cost as obtained on the analog computer and by analytical solution

Time (weeks)	Total cost (millions of dollars) Analog value	Analytical value
0	0	0
5	10	10.47
10	19	19.23
15	25	26.78
20	33	36.12
25	43	45.39
30	54	56.04
35	63	65.18
40	71	72.83
45	80	80.91
50	89	90.80
52	93	95.12

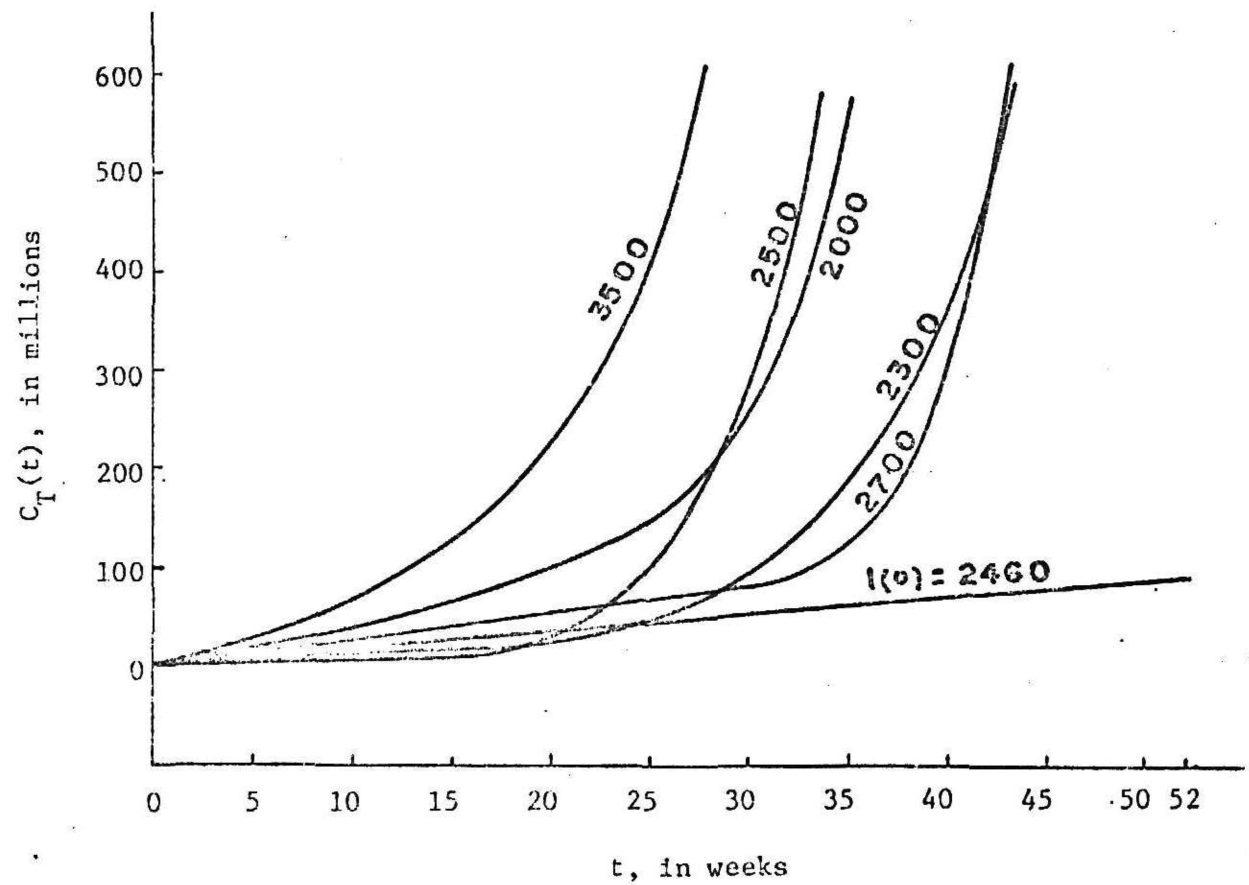

Fig. 3. Variation of total cost for different initial inventory level, I_0 and fixed optimum initial production rate

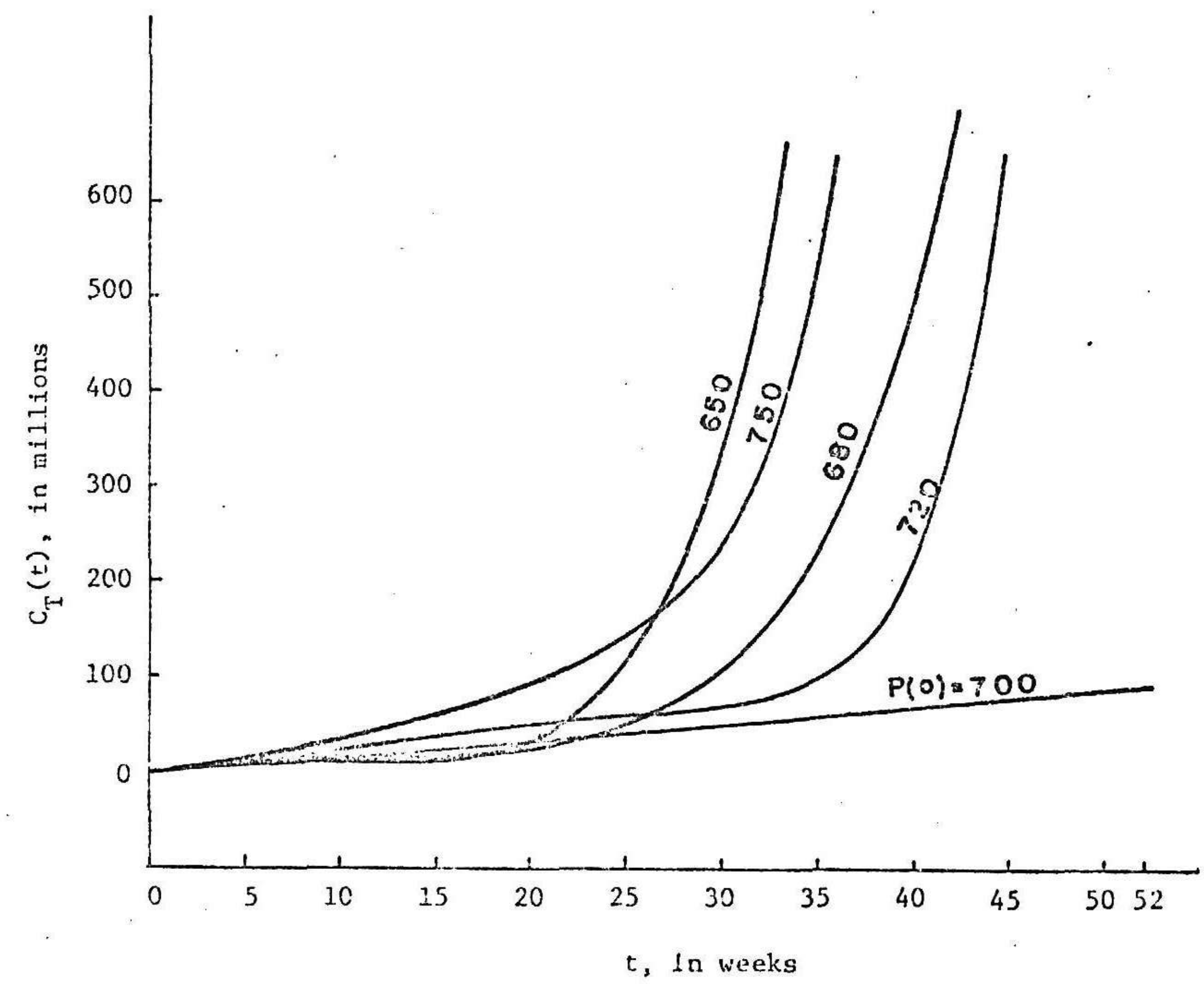

Fig. 4. Variation of total cost for different initial production rates, P(0), and fixed optimum initial inventory level

$$x_1(t) = \frac{a \frac{2\Pi}{T}}{\lambda^2 + \frac{4\Pi^2}{T^2}} \cos \frac{2}{T} t + I^* \tag{11}$$

$$\theta(t) = - \frac{a \frac{4\Pi^2}{T^2}}{\lambda^2 + \frac{4\Pi^2}{T^2}} \text{Sin} \frac{2\Pi}{T} t + (S_0 + a \text{ Sin} \frac{2\Pi}{T} t) \tag{12}$$

These solutions are periodic functions, and they are valid beyond $t = T$.

Tables 2, 3 and 4 compare the values of the production rate, inventory level and total cost as obtained on the analog computer to those obtained by analytical solution. It can be observed from Figure 2 that there is less than 2% difference between the values obtained by the two methods.

The cummulative total cost with respect to time for different initial levels of inventory, I(0), with optimum production rate, P(0) = 700 units/week, and for different initial production rates, P(0), with optimum initial inventory level, I(0) = 2460 units, is presented in Figures 3 and 4 respectively. It is observed that a minimum total cost is obtained for optimum initial values and all other inventory levels and production rates result in a much higher total cost.

Example 3 Inventory Control Model

This example deals with an analog computer simulation of the inventory model considered in Example 8 of Part I. It is assumed that the rate of change of actual inventory at factory warehouse at time t, $\dot{I}AF(t)$, is equal to the production rate at time t, P(t), minus the shipment rate at the factory at time t, SSF(t), as represented by

$$\dot{IAF}(t) = \mathring{P}(t) - SSF(t), \quad IAF(0) = IAF^0 \tag{1}$$

It is assumed that the shipment rate at the factory, SSF(t), is related to the level of unfilled orders at time t, UOF(t), by a first order exponential lag with time constant α, as represented by

$$SSF(t) = \frac{1}{\alpha} \int_0^\infty UOF(t-\tau) e^{-\frac{\tau}{\alpha}} d\tau \tag{2}$$

The rate of change of the level of unfilled orders, UOF(t), is assumed to be equal to the demand received at the factory, DRF(t), minus the shipments sent from the factory, SSF(t), as represented by

$$\dot{UOF}(t) = DRF(t) - SSF(t), \quad UOF(0) = UOF^0 \tag{3}$$

The objective of this simulation is to find a production rate which minimizes the sum of inventory holding cost, production cost and shortage cost over the interval [0,T].

$$c_T = c_I \int_0^T IAF(t)dt + c_P \int_0^T P(t)dt + c_S \int_0^T [DRF(t) - P(t)]dt \tag{4}$$

where

c_I = unit inventory cost

c_P = unit production cost

c_S = unit shortage cost

Analog Computer Solution

By employing the change of variables $x = t - \tau$, (see Section 2 of Appendix) equation (2) can be modified as

$$\dot{SSF}(t) = -\frac{1}{\alpha} SSF(t) + \frac{1}{\alpha} UOF(t)$$

(i) Summary of equations:

$$\frac{d[IAF(t)]}{dt} = P(t) - SSF(t), \quad IAF(0) = IAF^0$$

$$\frac{d[SSF(t)]}{dt} = -\frac{1}{\alpha} SSF(t) + \frac{1}{\alpha} UOF(t), \; SSF(0) = SSF^0$$

$$\frac{d[UOF(t)]}{dt} = DRF(t) - SSF(t), \quad UOF(0) = UOF^0$$

$$\frac{d[c_T(t)]}{dt} = c_I \; [IAF(t)] + (c_P - c_S) \; [P(t)] + c_S \; [DRF(t)], \; c_T(0) = 0$$

(ii) Initial conditions:

$SSF^0 = 600$ units/week

$UOF^0 = 100$ units

$IAF^0 = 50$ units

$c_T(0) = 0$

The scaling table for the analog computer solution is given in Table 1.

The scaled equations become:

$$\frac{d[.0001 \; IAF(t)]}{dt} = \frac{.0001}{.001} [.001 \; P(t)] - \frac{.0001}{.001} [.001 \; SSF(t)]$$

$$[.0001 \; IAF(0)] = .0001 \; IAF^0$$

Table 1

Scaling Table

Variable	Maximum Estimated Value	Scale factor = 1/Max.Est. Value	Computer Variable
IAF(t)	10,000	0.0001	[.0001 IAF(t)]
P(t)	1,000	0.001	[.001 P(t)]
SSF(t)	1,000	0.001	[.001 SSF(t)]
UOF(t)	1,000	0.001	[.001 UOF(t)]
DRF(t)	1,000	0.001	[.001 DRF(t)]
$c_T(t)$	10^6	10^{-6}	$[10^{-6} c_T(t)]$

$$\frac{d[.0001\ IAF(t)]}{dt} = .1[.001\ P(t)] - .1[.001\ SSF(t)]$$

$$[.0001\ IAF(0)] = .0001\ IAF^0, \quad IAF^0 = 50$$

$$\frac{d[.001\ SSF(t)]}{dt} = -\frac{.001}{.001}[.001\ SSF(t)] + \frac{.001}{.001}[.001\ UOF(t)]$$

$$[.001\ SSF(0)] = .001\ SSF^0$$

or

$$\frac{d[.001\ SSF(t)]}{dt} = -[.001\ SSF(t)] + [.001\ UOF(t)]$$

$$[.001\ SSF(0)] = .001\ SSF^0, \quad SSF^0 = 600$$

$$\frac{d[.001\ UOF(t)]}{dt} = \frac{.001}{.001}[.001\ DRF(t)] - \frac{.001}{.001}[.001\ SSF(t)]$$

$$[.001\ UOF(0) = .001\ UOF^0$$

or

$$\frac{d[.001\ UOF(t)]}{dt} = [.001\ DRF(t)] - [.001\ SSF(t)]$$

$$[.001\ UOF(0)] = .001\ UOF^0, \quad UOF^0 = 100$$

$$\frac{d[10^{-6}\ c_T(t)]}{dt} = \frac{10^{-6}}{.0001} c_I\ [.0001\ IAF(t)] - \frac{10^{-6}}{.001}(c_S - c_P)[.001P(t)]$$

$$+ \frac{10^{-6}}{.001} c_S[.001\ DRF(t)]$$

$$[10^{-6}\ c_T(0)] = 0$$

or

$$\frac{d[10^{-6}\ c_T(t)]}{dt} = .01c_I[.0001\ IAF(t)] - .001\ (c_S - c_P)[.001P(t)]$$

$$+\ .001\ c_S\ [.001\ DRF(t)]$$

$$[10^{-6}\ c_T(0)] = 0$$

(iii) Input functions:

The inventory model was simulated using various production functions. The mathematical representation for some of these functions for a time period T = 52 weeks is as follows.

(a) Sinusoidal function: The equation for a sinusoidal function with an amplitude of 100 about a mean of 900 is

$$P(t) = 100 \sin \frac{2\Pi}{T} t + 900, \quad P(0) = 900, \quad T = 52$$

$$\dot{P}(t) = 100 \left(\frac{2\Pi}{T}\right) \cos \frac{2\Pi}{T} t, \quad \dot{P}(0) = 100 \left(\frac{2\Pi}{T}\right)$$

$$\ddot{P}(t) = -\ 100 \left(\frac{2\Pi}{T}\right)^2 \sin \frac{2\Pi}{T} t$$

or

$$\ddot{P}(t) = -\left(\frac{2\Pi}{T}\right)^2 P(t) + \left(\frac{2\Pi}{T}\right)^2 900$$

The scaled computer diagram for the function is given in Figure 1.

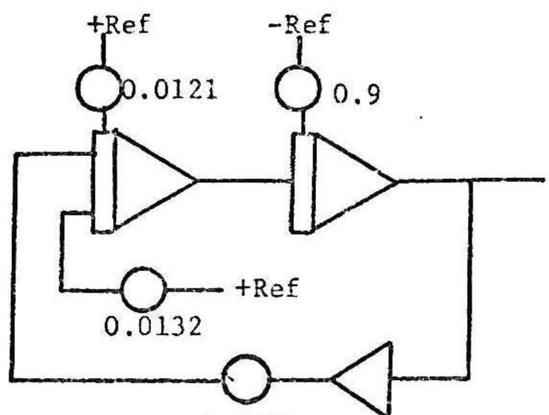

Fig. 1. Sinusoidal function

(b) Linear function: The equation for a linearly increasing function is

$$P(t) = mt + P_{min}, \qquad P(0) = P_{min} = 800$$

$$P(T) = P_{max} = 1000$$

$$\dot{P}(t) = 3.84$$

The equation for a linearly decreasing function is

$$P(t) = -mt + P_{max}, \quad P(0) = P_{max} = 1000$$

$$P(T) = P_{min} = 800$$

$$\dot{P}(t) = -3.84$$

The scaled computer diagram for the above two functions is represented in Figures 2(a) and 2(b) respectively

(c) Step function: The equation for an increasing step function is

$$P(t) = P_{min} \qquad t \leq t_1, \qquad P_{min} = 800$$

$$P(t) = P_{max} \qquad t_1 < t \leq T, \quad P_{max} = 1000$$

and similarly the equation for a decreasing step function is

$$P(t) = P_{max} \qquad t \leq t_1$$

$$P(t) = P_{min} \qquad t_1 < t \leq T$$

The scaled computer diagram for this step function is represented in Figure 3

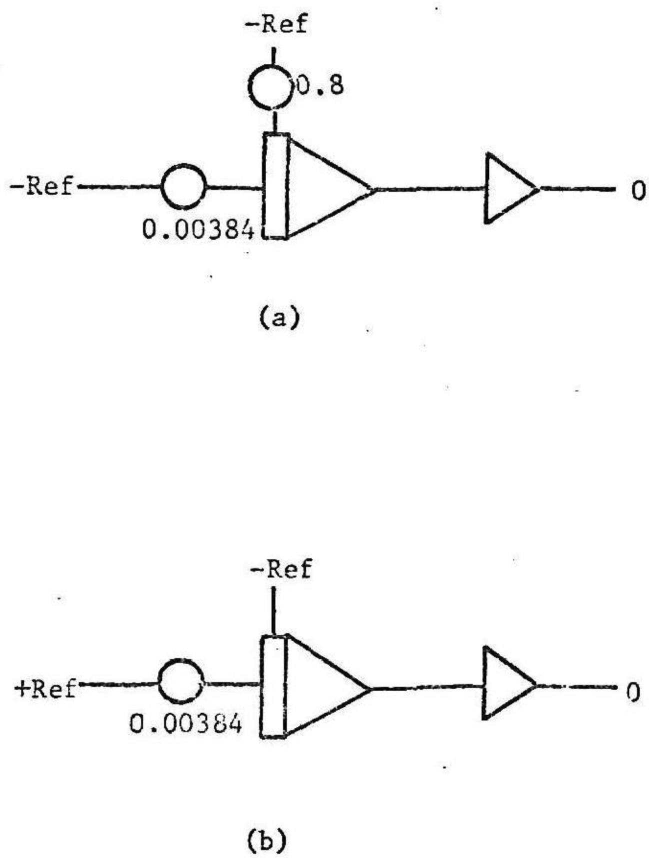

Fig. 2. Linear function

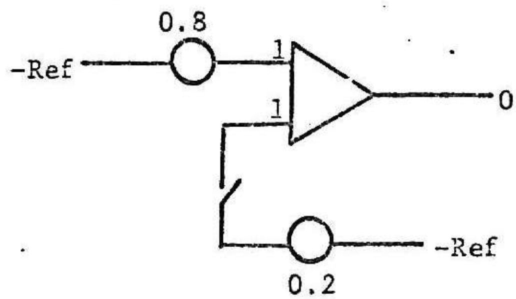

Fig. 3. Step function

(iv) Parameter Values:

Case I

α = 1 week

DRF(t) = 1000 units/week

c_I = \$0.5 /(unit)(week)

c_P = \$5/(unit)

c_S = \$50/(unit)

Case II

α = 1 week

DRF(t) = 1000 units/week

c_I = \$0.5/(unit)(week)

c_P = \$5/(unit)

c_S = 0

The scaled computer diagram for Cases I and II is represented in Figures 4 and 5 respectively.

(v) Results

The behaviour of the different production rates with which the model was simulated is presented in Figure 6. The abbreviations used are

A - constant production rate at P_{max} = 1000 units/week

B - constant production rate at P_{min} = 800 units/week

C - linearly increasing production rate from P_{min} to P_{max}

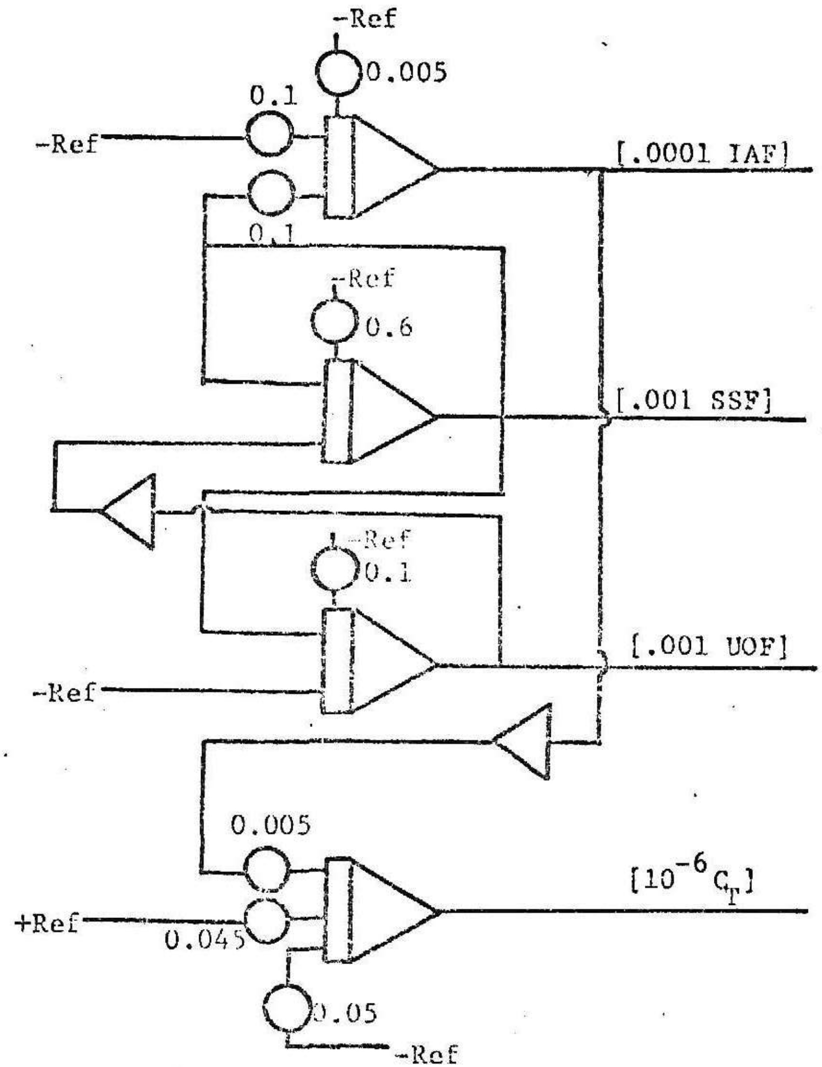

Fig. 4. Scaled computer diagram for case I

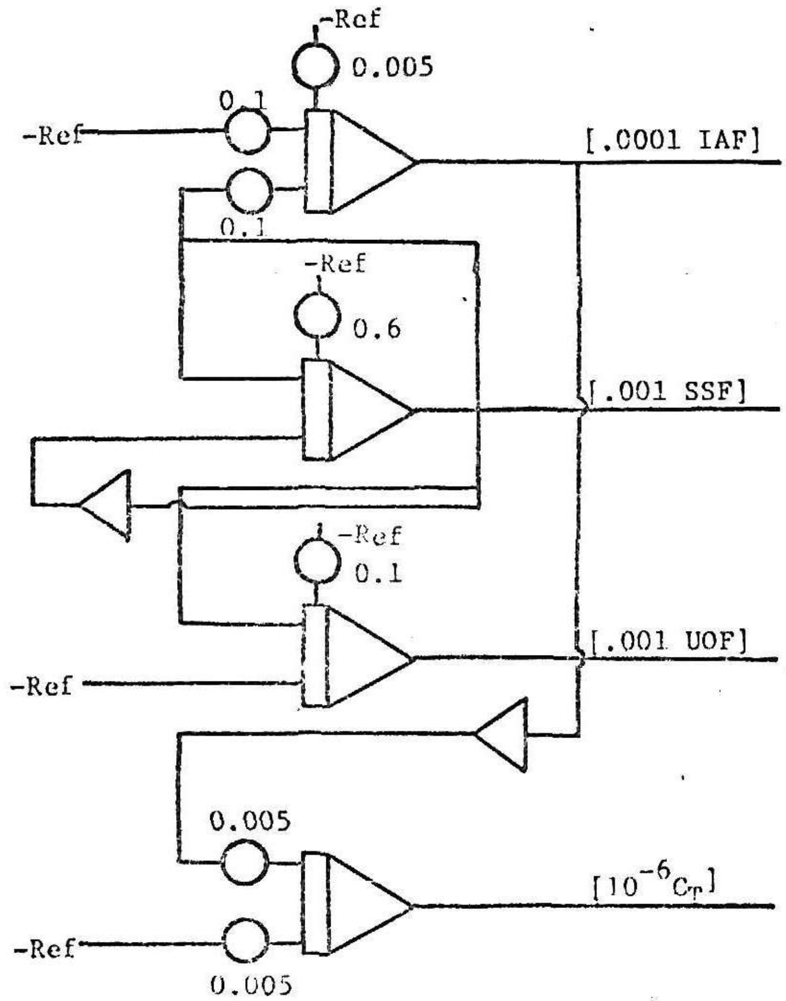

Fig. 5. Scaled computer diagram for case II

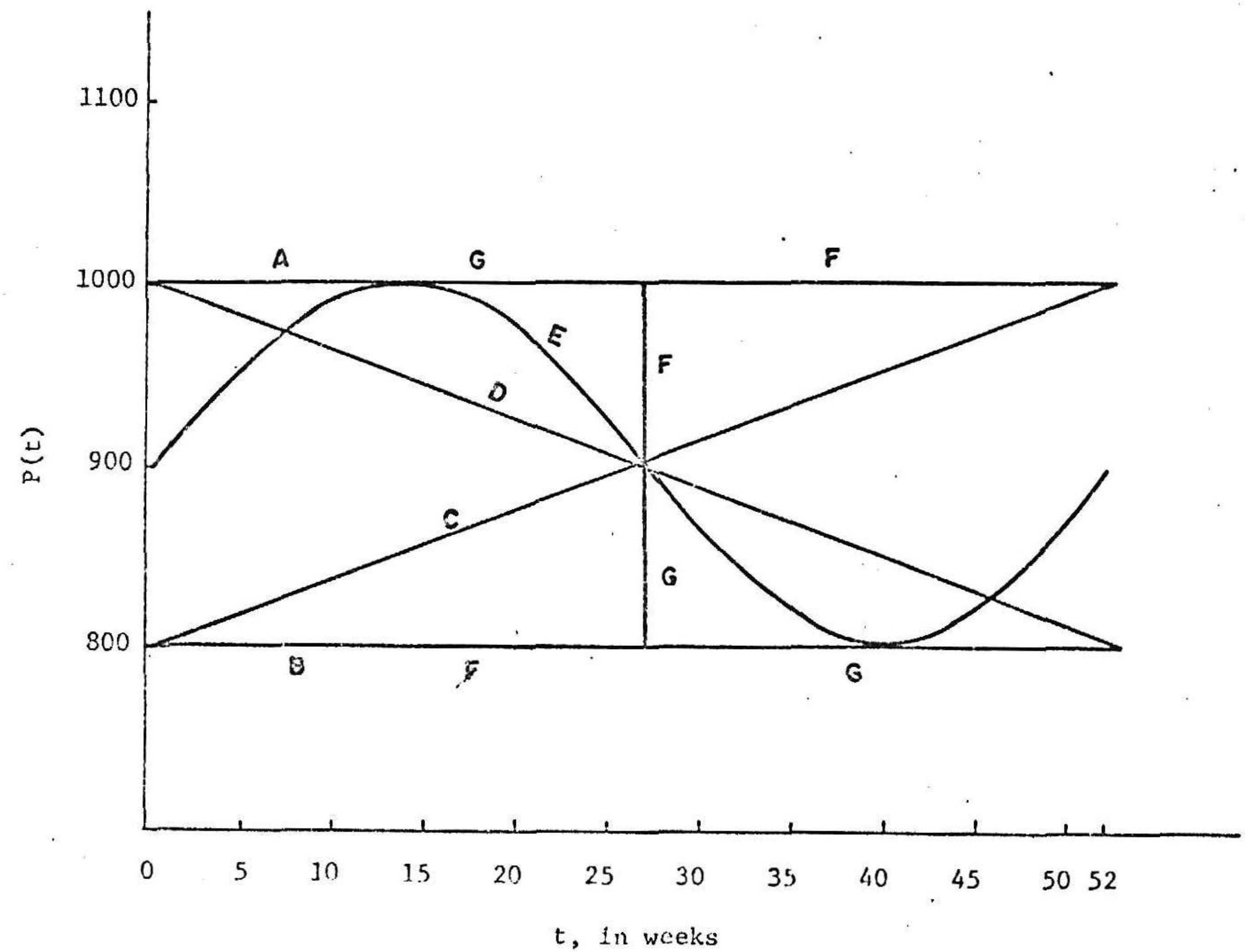

Fig. 6. Behaviour of different production rates for case I & II

D - linearly decreasing production rate from P_{max} to P_{min}

E - sinusoidal production rate with an amplitude of 100 units/week about a mean of 900 units/week

F - symmetrical step increase from P_{min} to P_{max}

G - symmetrical step decrease from P_{max} to P_{min}

H - step increase from P_{min} to P_{max} at $t = 22$ weeks

I - step decrease from P_{max} to P_{min} at $t = 22$ weeks

The cummulative total cost with respect to time for different production rates is presented in Figure 7 for Case I and in Figure 8 for Case II. Figures 9, 10 and 11 represent the behaviour of actual inventory at factory, shipment rate at factory and unfilled orders at factory, for different production rates, with respect to time.

(vi) Discussion:

Case I

It can be observed from Figure 7 that minimum total cost is incurred with maximum production and thus the optimal policy for this set of *parameter values is to schedule maximum production.*

The inventory level at the factory (see Figure 9) is observed to be constant at 1000 units for the optimal production rate whereas all other production rates result in large stockouts. This can be explained by the fact that the optimal production rate represents the demand rate also and consequently at steady state the inventory level remains unaffected. All other production rates are lower than the demand rate and hence the number of stockouts is expected to increase with time.

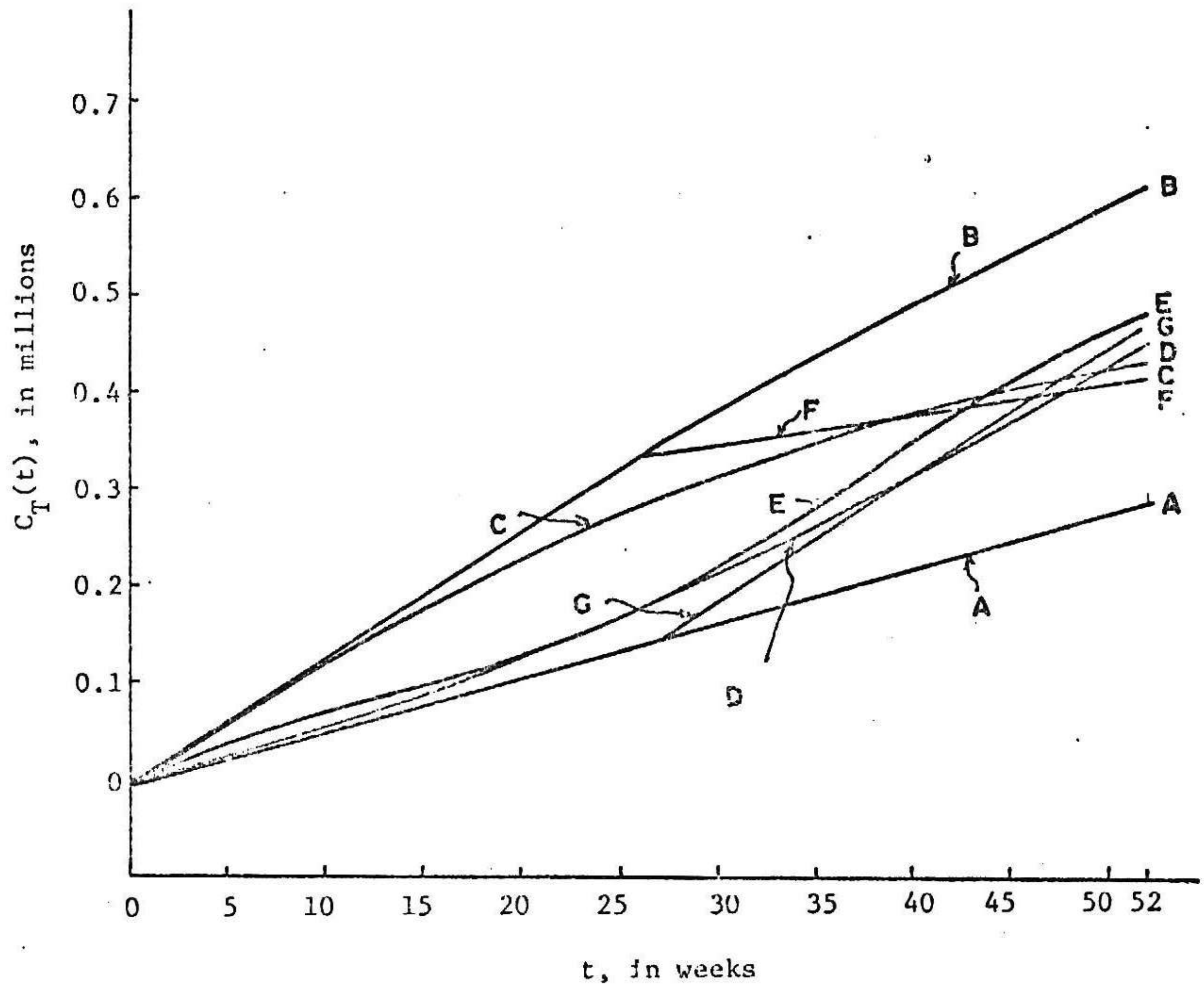

Fig. 7. Behaviour of total cost with different production rate for case I

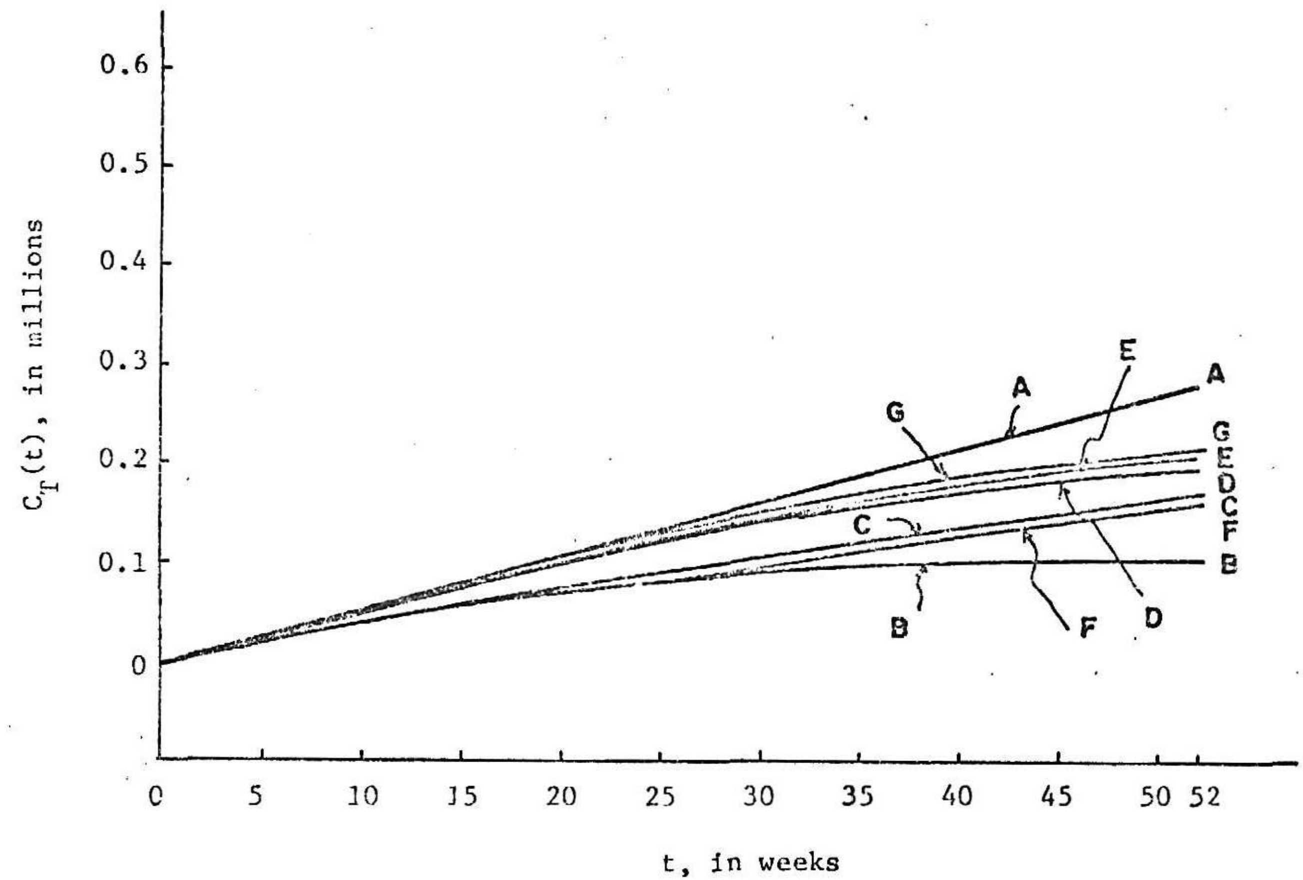

Fig. 8. Behaviour of total cost with different production rates for case II

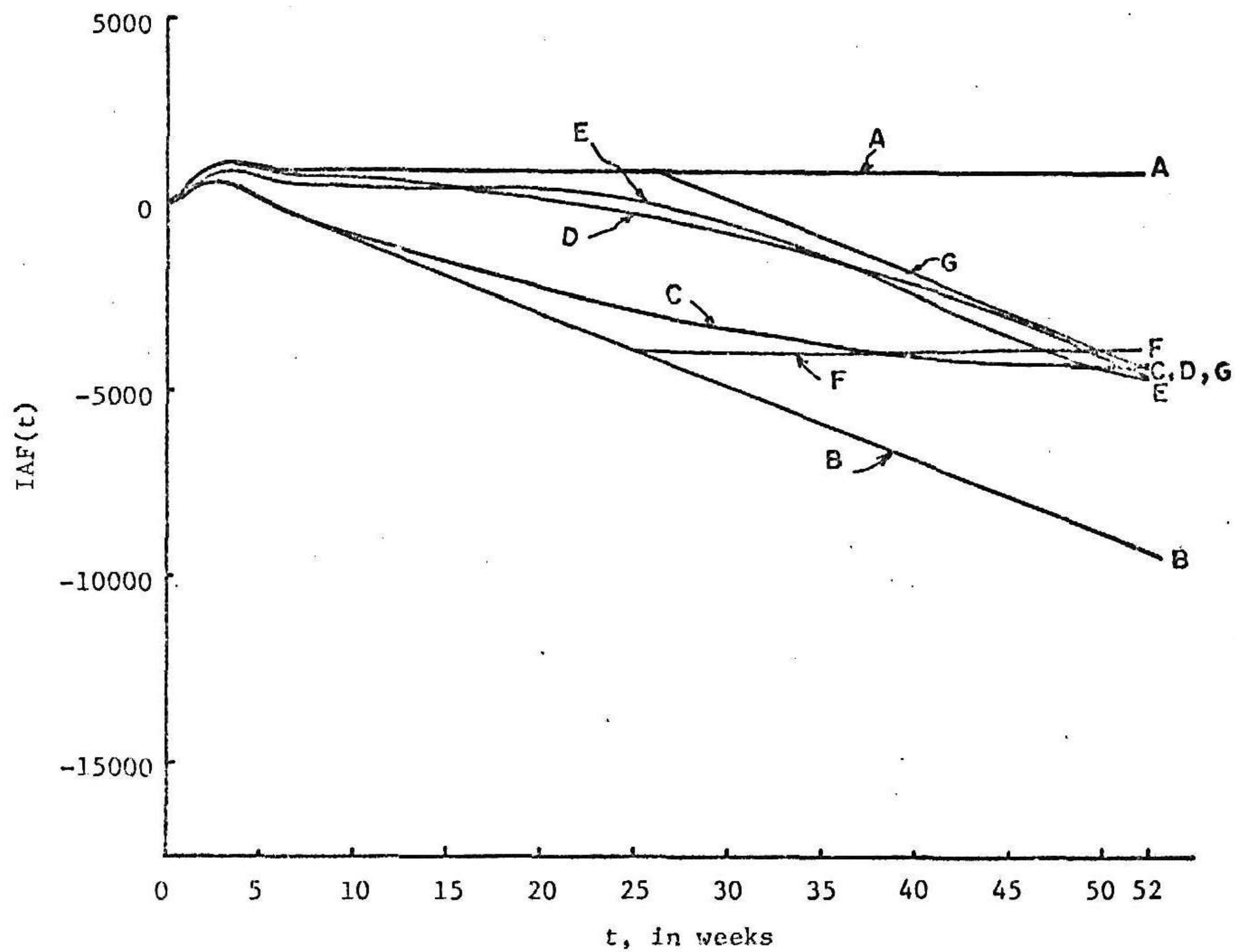

Fig. 9. Behaviour of inventory level with different production rate for cases I, II and III

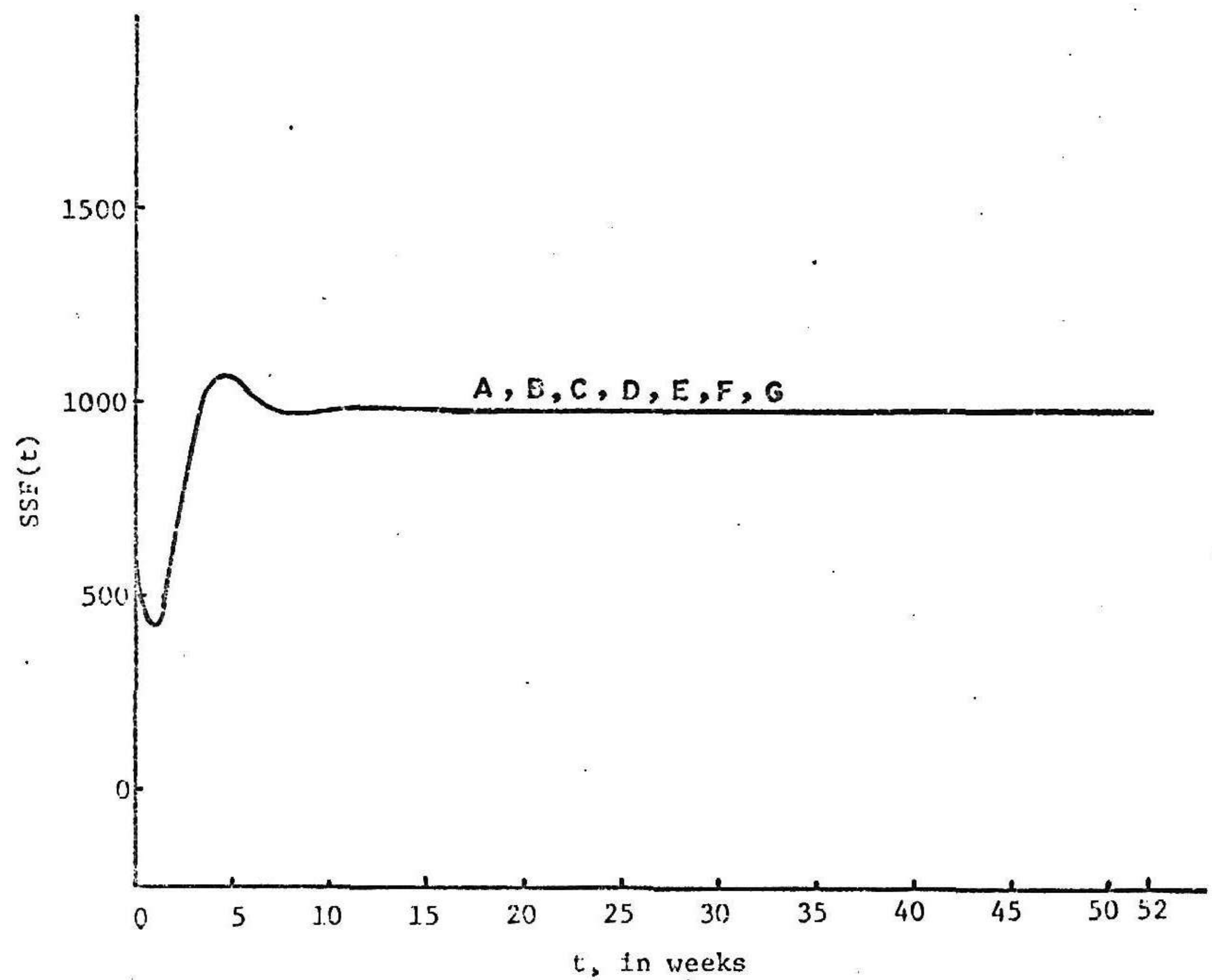

Fig. 10. Behaviour of shipment rate with different production rates for cases I, II and III

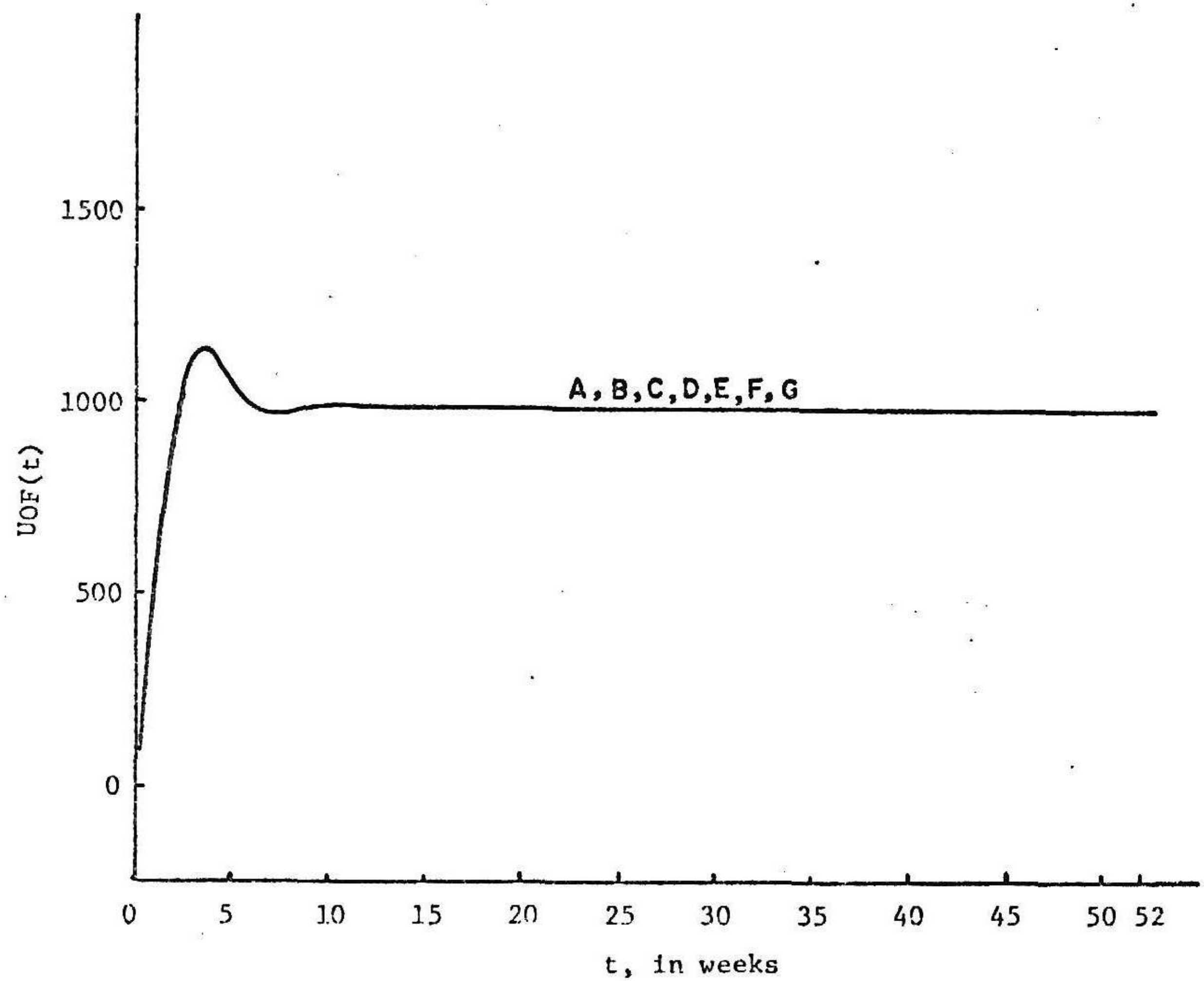

Fig. 11. Behaviour of unfilled orders with different production rates for cases I, II and III

Figures 10 and 11 represent the behaviour of the shipment rate and unfilled orders at factory with respect to time. It can be observed that these are not affected by the behaviour of the production rate and reach a steady state value equal to that of the demand rate. This can be explained by the fact that the mathematical equations representing the shipment rate and unfilled orders at factory are independent of the production rate and dependent on the demand rate only. Consequently, at steady state, they stabilize at a magnitude equal to the demand rate.

Case II

The production rates with which the model was simulated in this case were kept the same as in Case I (see Figure 6).

The cummulative total cost with respect to time for different production rates is presented in Figure 8 and for the set of parameter values represented by this case, the optimal policy is to schedule minimum production.

The behaviour of the inventory level, shipment rate and unfilled orders at factory for different production rates with respect to time were observed to be the same as in Case I (see Figures 9, 10 and 11). This observation is not a coincidence because for a change of parameter values, the performance equations remain unaffected and only the objective function is modified. Consequently, we can expect that only the optimal production policy will vary whereas the behaviour of the other system variables will remain unaffected.

The optimal production policy arrived at, for the above set of parameter values, is in agreement with that observed by analytical solution (Example 8, Part I).

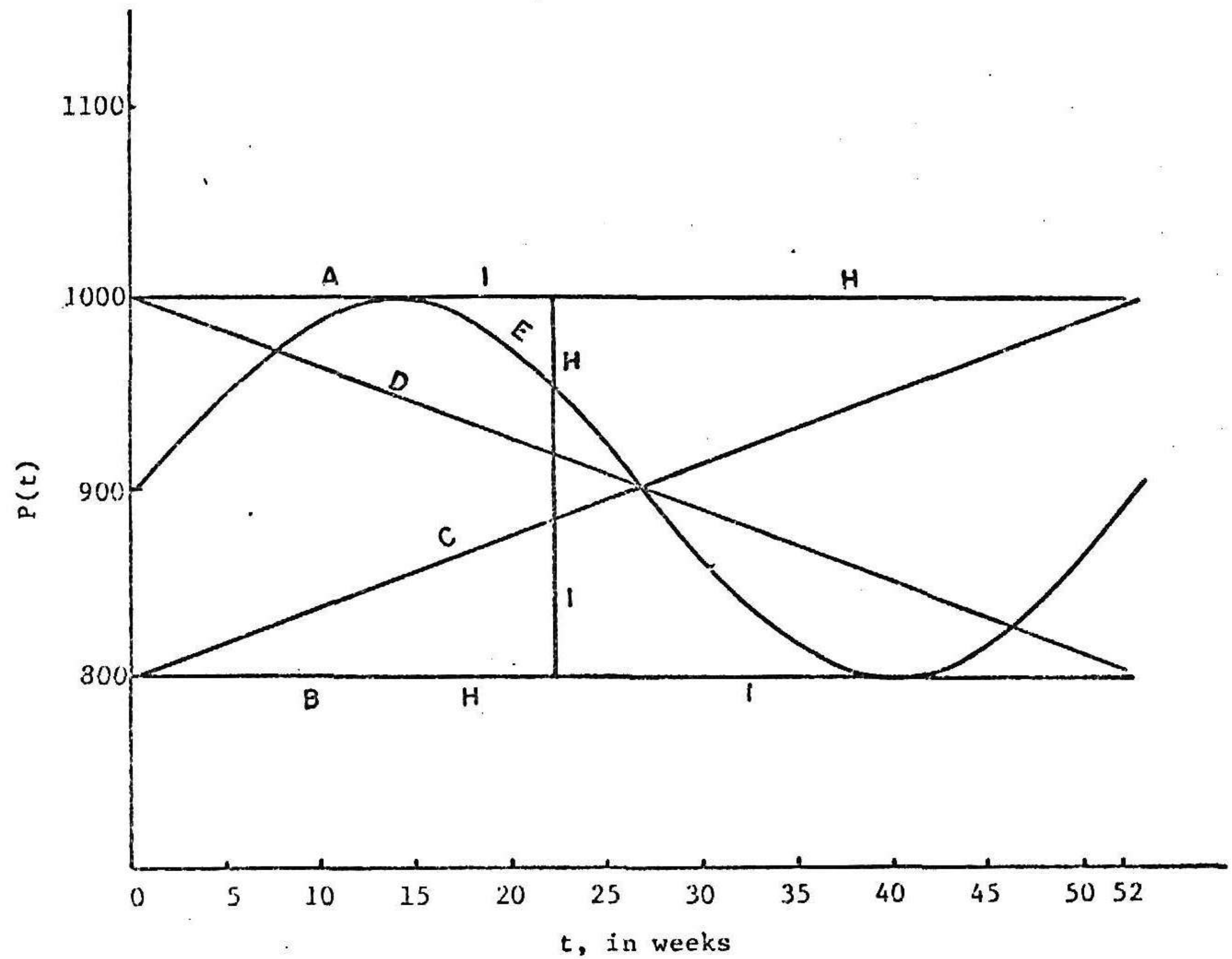

Fig. 12. Behaviour of different production rates for case III

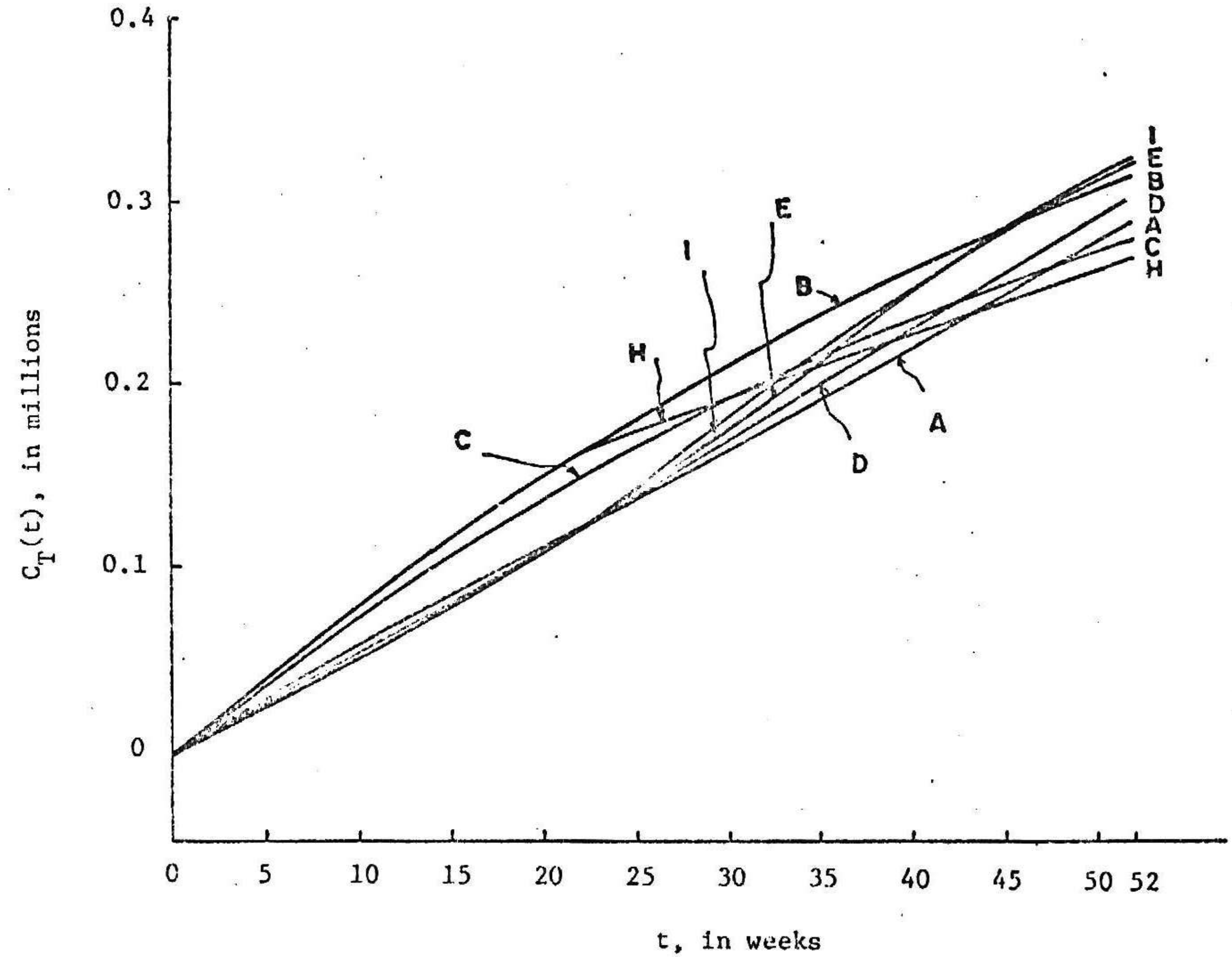

Fig. 13. Behaviour of total cost with different production rates for Case III

Based on the optimal production policy represented by equation (21) of Example 8 of Part I, a third set of parameter values were chosen as

Case III

α = 1 week

DRF(t) = 1000 units/week

c_I = \$0.5/(unit)(week)

c_P = \$5/(unit)

c_S = \$20/(unit)

The optimal policy for this set of parameters is a step increase from minimum to maximum production rate at time t = 22 weeks.

Figure 12 represents the behaviour of the different production rates and Figure 13 represents the variation of total cost with respect to time with these production rates.

It is observed that the optimal production policy arrived at by the maximum principle yields minimum total cost.

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APPENDIX

Section 1

In this section the differential equation (16) of Example 5 is solved.

$$\frac{c_P}{c_I}\,\theta(t) - \theta(t) = -S(t)$$

Let

$$\rho = \sqrt{\frac{c_I}{c_P}}$$

The above differential equation can be written as

$$(D^2 - \rho^2)\ P(t) = -\rho^2\ S(t)$$

or

$$\left(\frac{d}{dt} + \rho\right)\left(\frac{d}{dt} - \rho\right)\ P(t) = -\rho^2\ S(t)$$

Let

$$P(t) = Q(t) \quad \left(\frac{d}{dt} - \rho\right)$$

so that

$$\left(\frac{d}{dt} + \rho\right)\ Q(t) = -\rho^2\ S(t)$$

$$\frac{dQ(t)}{dt} + \rho\ Q(t) = -\rho^2\ S(t)$$

The integrating factor of the above equation is $e^{\rho t}$.

Hence

$$Qe^{\rho t} = -\rho^2 \int e^{\rho t} S(t)\, dt + A_1$$

$$Q = -\rho^2 e^{-\rho t} \int e^{\rho t} S(t)\, dt + A_1 e^{-\rho t}$$

or

$$\frac{dP(t)}{dt} - \rho P(t) = -\rho^2 e^{-\rho t} \int e^{\rho t} S(t)\, dt + A_1 e^{-\rho t}$$

The integrating factor of the above equation, similarily, is $e^{-\rho t}$.

Hence

$$P(t)\, e^{-\rho t} = -\rho^2 \int \{ e^{-\rho t} \cdot e^{-\rho t} \int e^{\rho t} S(t)\, dt\} dt$$

$$+ \int A_1 e^{-2\rho t} dt + A_2$$

$$P(t)\, e^{-\rho t} = -\rho^2 \{\frac{-1}{2\rho} e^{-2\rho t} \int e^{\rho t} S(t)\, dt + \frac{1}{2} \int e^{-\rho t} S(t)\, dt\}$$

$$- \frac{1}{2\rho} A_1 e^{-2\rho t} + A_2$$

or

$$P(t) = \frac{\rho}{2} e^{-\rho t} \int e^{\rho t} S(t)\, dt - \frac{\rho}{2} e^{\rho t} \int e^{-\rho t} S(t)\, dt$$

$$- \frac{1}{2\rho} A_1 e^{-\rho t} + A_2 e^{\rho t}$$

Application of the boundary conditions $P(0) = P_0$ and $P(T) = P_1$ gives

$$P_0 = -\frac{1}{2\rho} A_1 + A_2$$

$$P_1 = \frac{\rho}{2} e^{-\rho T} \int_0^T e^{\rho t} S(t)\, dt - \frac{\rho}{2} e^{\rho T} \int_0^T e^{-\rho t} S(t)\, dt$$

$$- \frac{1}{2\rho} A_1 e^{-\rho T} + A_2 e^{\rho T}$$

Now

$$A_2 = P_0 + \frac{1}{2\rho} A_1$$

Therefore

$$P_1 = \frac{\rho}{2} e^{-\rho T} \int_0^T e^{\rho t} S(t)\, dt - \frac{\rho}{2} e^{\rho T} \int_0^T e^{-\rho t} S(t)\, dt$$

$$- \frac{1}{2\rho} A_1 e^{-\rho T} + (P_0 + \frac{1}{2\rho} A_1) e^{\rho T}$$

or

$$A_1 = \frac{P_1 - P_0 e^{\rho T} + \frac{\rho}{2} e^{\rho T} \int_0^T e^{-\rho t} S(t)\, dt - \frac{\rho}{2} e^{-\rho T} \int_0^T e^{\rho t} S(t)\, dt}{\frac{1}{2\rho} (e^{\rho T} - e^{-\rho T})}$$

and

$$A_2 = \frac{P_1 - P_0 e^{-\rho T} + \frac{\rho}{2} e^{\rho T} \int_0^T e^{-\rho t} S(t)\, dt - \frac{\rho}{2} e^{-\rho T} \int_0^T e^{\rho t} S(t)\, dt}{e^{\rho T} - e^{-\rho T}}$$

Substitution of the values of A_1 and A_2 in the expression for P(t) yields

$$P(t) = \frac{P_1 - P_0\ e^{-\rho T} + \frac{\rho}{2}\int_0^T [e^{\rho(T-t)} - e^{-\rho(T-t)}]\ S(t)\ dt}{e^{\rho T} - e^{-\rho T}} \cdot e^{\rho t}$$

$$- \frac{P_1 - P_0 e^{\rho T} + \frac{\rho}{2}\int_0^T [e^{\rho(T-t)} - e^{-\rho(T-t)}]\ S(t)\ dt}{e^{\rho T} - e^{-\rho T}} \cdot e^{-\rho t}$$

$$+ \frac{1}{2}\rho\ [e^{-\rho t} \int e^{\rho t}\ S(t)\ dt - e^{\rho t} \int e^{-\rho t}\ S(t)\ dt]$$

By changing the variable under the integral sign we have

$$P(t) = \frac{P_1 - P_0\ e^{-\rho T} + \frac{1}{2}\rho \int_0^T [e^{\rho(T-x)} - e^{-\rho(T-x)}]\ S(x)\ dx}{e^{\rho T} - e^{-\rho T}} \cdot e^{\rho t}$$

$$- \frac{P_1 - P_0\ e^{\rho T} + \frac{1}{2}\rho \int_0^T [e^{\rho(T-x)} - e^{-\rho(T-x)}]\ S(x)\ dx}{e^{\rho T} - e^{-\rho T}} \cdot e^{-\rho t}$$

$$- \frac{1}{2}\rho\ [e^{\rho t} \int e^{-\rho t}\ S(t)\ dt - e^{-\rho t} \int e^{\rho t}\ S(t)\ dt]$$

Section 2

Equation (3) of Example 8 represents a first order exponential lag with time constant, α.

$$SSF\ (t) = \frac{1}{\alpha} \int_0^{\infty} UOF\ (t-\tau)\ e^{-\tau/\alpha}\ d\tau$$

Employing a change of variable, $x = t - \tau$

$$SSF\ (t) = \frac{1}{\alpha} \int_{-\infty}^{t} UOF\ (x)\ e^{\frac{x-t}{\alpha}}\ dx$$

Using the Leibnitz theorem

$$\frac{dg[x,t]}{dt} = \int_{h(t)}^{k(t)} \frac{\partial g[x,t]}{\partial t} + g\ [k(t),\ t]\ \frac{dk(t)}{dt} - g\ [h(t),\ t]\ \frac{dh(t)}{dt}$$

where

$$g[x,t] = UOF(x)\ e^{\frac{x-t}{\alpha}}$$

$$h(t) = -\infty$$

$$k(t) = t$$

we have

$$\frac{d[SSF(t)]}{dt} = \frac{1}{\alpha}\ [\int_{-\infty}^{t} - \frac{1}{\alpha}\ UOF(x)\ e^{\frac{x-t}{\alpha}}\ dx + UOF(t)]$$

or

$$\frac{d[SSF(t)]}{dt} = -\frac{1}{\alpha^2} \int_0^{\infty} UOF(t-\tau)\ e^{\frac{-\tau}{\alpha}}\ d\tau + \frac{1}{\alpha}\ UOF(t)$$

Using the original expression for SSF(t), the resulting differential equation is

$$\frac{d[SSF(t)]}{dt} = -\frac{1}{\alpha}\,[SSF\ (t) - UOF(t)]$$

Section 3

In this section equations (1), (2) and (3) of Example 9 are modified.

(i) Equation (1)

$$VMC(t) = \int_0^\infty \int_0^\infty \int_0^\infty \frac{1}{\alpha^3}\, VDF\ (t-\eta-\varepsilon-\tau)\, e^{-\frac{(\eta+\varepsilon+\tau)}{\alpha}}\, d\tau\, d\eta\, d\varepsilon$$

Let

$$x = t - \tau$$

The above equation can then be modified as

$$VMC(t) = \int_0^\infty \int_0^\infty \frac{1}{\alpha^3} \left(\int_{-\infty}^{t} VDF(x-\eta-\varepsilon)\, e^{\frac{(x-t-\eta-\varepsilon)}{\alpha}}\, dx \right) d\eta\, d\varepsilon$$

Using the Leibnitz theorem

$$\frac{dg(x,t)}{dt} = \int_{h(t)}^{k(t)} \frac{\partial g(x,t)}{\partial t}\, dx + g[k(t),t]\,\frac{dk(t)}{dt} - g[h(t),t]\,\frac{dh(t)}{dt}$$

$$g(x,t) = VDF\ (x-\eta-\varepsilon)\, e^{\frac{(x-t-\eta-\varepsilon)}{\alpha}}$$

$$\frac{\partial g(x,t)}{\partial t} = -\frac{1}{\alpha}\, VDF\ (x-\eta-\varepsilon)\, e^{\frac{(x-t-\eta-\varepsilon)}{\alpha}}$$

$$g[k(t),t] = VDF\ (t-\eta-\varepsilon)\, e^{-\frac{(\eta+\varepsilon)}{\alpha}}$$

$$\frac{dk(t)}{dt} = 1$$

$$\frac{dh(t)}{dt} = 0$$

we have

$$\dot{VMC}(t) = \int_0^\infty \int_0^\infty \frac{1}{\alpha^3} \left(\int_{-\infty}^{t} -\frac{1}{\alpha} \text{VDF } (x-\eta-\varepsilon) e^{\frac{(x-t-\eta-\varepsilon)}{\alpha}} dx \right.$$

$$\left. + \text{VDF } (t-\eta-\varepsilon) e^{-\frac{(\eta+\varepsilon)}{\alpha}} \cdot 1 \right) d\eta d\varepsilon$$

or

$$\dot{VMC}(t) + \frac{1}{\alpha} \text{VMC}(t) = \int_0^\infty \int_0^\infty \frac{1}{\alpha^3} \text{VDF } (t-\eta-\varepsilon) e^{-\frac{(\eta+\varepsilon)}{\alpha}} d\eta d\varepsilon$$

Let

$$y = t - \eta$$

Application of the Leibnitz theorem yields,

$$\ddot{VMC}(t) + \frac{1}{\alpha} \dot{VMC}(t) = \int_0^\infty \frac{1}{\alpha^3} \left(\int_0^\infty -\frac{1}{\alpha} \text{VDF } (t-\eta-\varepsilon) e^{-\frac{(\eta+\varepsilon)}{\alpha}} d\eta + \text{VDF } (t-\varepsilon) e^{-\frac{\varepsilon}{\alpha}} \right) d\varepsilon$$

Replacing the first term of the right hand side in terms of VMC(t) and its derivative we get

$$\ddot{VMC}(t) + \frac{2}{\alpha} \dot{VMC}(t) + \frac{1}{\alpha^2} \text{VMC}(t) = \frac{1}{\alpha^3} \int^\infty \text{VDF}(t-\varepsilon) e^{-\frac{\varepsilon}{\alpha}} d\varepsilon$$

Let

$z = t - \varepsilon$

Applying the Leibnitz theorem again and rearranging the result in terms of VMC(t) and its derivatives we get

$$\dddot{VMC}(t) + \frac{3}{\alpha}\ddot{VMC}(t) + \frac{3}{\alpha^2}\dot{VMC}(t) + \frac{1}{\alpha^3}VMC(t) = \frac{1}{\alpha^3}VDF(t)$$

(ii) Equation (2)

$$\dot{VAC}(t) = \frac{1}{\beta}\int_0^\infty [VMC(t-\tau) - VAC(t-\tau)]e^{-\frac{\tau}{\beta}}\, d\tau$$

By employing a change of variable, x = t-τ, the above equation becomes

$$\dot{VAC}(t) = \frac{1}{\beta}\int_{-\infty}^{t} VMC(x)\, e^{\frac{x-t}{\beta}}\, dx - \frac{1}{\beta}\int_{-\infty}^{t} VAC(x)\, e^{\frac{x-t}{\beta}}\, dx$$

Using the Leibnitz theorem we obtain

$$\ddot{VAC}(t) = -\frac{1}{\beta}\left\{\frac{1}{\beta}\int_0^\infty [VMC(t-\tau) - VAC(t-\tau)]\, e^{-\frac{\tau}{\beta}}\, d\tau]\right\}$$

Rearranging terms we get

$$\ddot{VAC}(t) + \frac{1}{\beta}\dot{VAC}(t) + \frac{1}{\beta}VAC(t) = \frac{1}{\beta}VMC(t)$$

(iii) Equation (3)

$$\dot{RRR}(t) = -\rho RRR(t) + \gamma\int^\infty VAC(t-\tau)e^{-\tau}\, d\tau$$

By employing a change of variable, $x = t-\tau$, the above equation becomes

$$\dot{RRR}(t) + \rho RRR(t) = \gamma \int_{-\infty}^{t} VAC(x)\, e^{x-t}\, dx$$

Using Leibnitz theorem we obtain

$$\ddot{RRR}(t) + \dot{RRR}(t) = -\gamma \int_{0}^{\infty} VAC(t-\tau) e^{-\tau}\, d\tau + \gamma VAC(t)$$

Rearranging terms we get

$$\ddot{RRR}(t) + (1+\rho)\, \dot{RRR}(t) + \rho RRR(t) = \gamma VAC(t)$$

Section 4

In this section equation (14) of example 11 is solved.

$$\frac{dx_1}{dt} = x_2, \quad x_1(0) = I_0$$

$$\frac{dx_2}{dt} = -\frac{z_2}{2k_2}, \quad x_2(0) = I_1$$

$$\frac{dz_1}{dt} = -2x_1, \quad z_1(T) = 0$$

$$\frac{dz_2}{dt} = -z_1 - 2k_1x_2, \quad z_2(T) = 0$$

Eliminating each variable in turn, the above system of equations can be reduced to

$$\frac{d^4x_2}{dt^4} = -\frac{1}{k_2} x_2 + \frac{k_1}{k_2} \frac{d^2x_2}{dt^2}$$

or

$$\left(D^4 - \frac{k_1}{k_2} D^2 + \frac{1}{k_2}\right) x_2 = 0$$

The solution to the above fourth order differential equation is

$$x_2(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + A_3 e^{s_3 t} + A_4 e^{s_4 t}$$

where

$$s_1 = \frac{1}{\sqrt{2k_2}} \sqrt{k_1 + \sqrt{k_1^2 - 4k_2}}$$

$$s_2 = -\frac{1}{\sqrt{2k_2}} \sqrt{k_1 + \sqrt{k_1^2 - 4k_2}}$$

$$s_3 = \frac{1}{\sqrt{2k_2}} \sqrt{k_1 - \sqrt{k_1^2 - 4k_2}}$$

$$s_4 = -\frac{1}{\sqrt{2k_2}} \sqrt{k_1 - \sqrt{k_1^2 - 4k_2}}$$

and A_1, A_2, A_3, A_4 are constants.

Substituting the value of $x_2(t)$ in the system of equation we get

$$x_1(t) = k_1[A_1s_1e^{s_1t} + A_2s_2e^{s_2t} + A_3s_3e^{s_3t} + A_4s_4e^{s_4t}]$$

$$- k_2[A_1s_1^3e^{s_1t} + A_2s_2^3e^{s_2t} + A_3s_3^3e^{s_3t} + A_4s_4^3e^{s_4t}]$$

$$x_2(t) = A_1e^{s_1t} + A_2e^{s_2t} + A_3e^{s_3t} + A_4e^{s_4t}$$

$$z_1(t) = - 2k_1 \; [A_1e^{s_1t} + A_2e^{s_2t} + A_3e^{s_3t} + A_4e^{s_4t}]$$

$$+ 2k_2 \; [A_1s_1^2e^{s_1t} + A_2s_2^2e^{s_2t} + A_3s_3^2e^{s_3t} + A_4s_4^2e^{s_4t}]$$

$$z_2(t) = - 2k_2 \; [A_1s_1e^{s_1t} + A_2s_2e^{s_2t} + A_3s_3e^{s_3t} + A_4s_4e^{s_4t}]$$

Application of the boundary conditions yields

$$A_1[k_1s_1 - k_2s_1^3] + A_2[k_1s_2 - k_2s_2^3] + A_3[k_1s_3 - k_2s_3^3]$$

$$+ A_4 \; [k_1s_4 - k_2s_4^3] = I_0$$

$$A_1 + A_2 + A_3 + A_4 = I_1$$

$$A_1 e^{s_1 T}[k_1 - k_2 s_1^2] + A_2 e^{s_2 T}[k_1 - k_2 s_2^2] + A_3 e^{s_3 T}[k_1 - k_2 s_3^2]$$

$$+ A_4 e^{s_4 T}[k_1 - k_2 s_4^2]$$

$$A_1 s_1 e^{s_1 T} + A_2 s_2 e^{s_2 T} + A_3 s_3 e^{s_3 T} + A_4 s_4 e^{s_4 T} = 0$$

The constants A_1, A_2, A_3 and A_4 can be determined by solving the above four simultaneous equations and hence $z_2(t)$ can be determined.

Section 5

This section deals with equation (1) of Example 13.

$$\frac{dS(t)}{dt} = -\lambda S(t) + \gamma \int_0^\infty A(t-\tau)e^{-\tau}\, d\tau$$

By employing a change of variables $x = t-\tau$,

$$\frac{dS(t)}{dt} = -\lambda S(t) + \gamma \int_{-\infty}^{t} A(x)\ e^{x-t}\ dx$$

Using the Leibnitz theorem

$$\frac{d^2S(t)}{dt^2} = -\lambda\frac{dS(t)}{dt} + \gamma\left(-\int_{-\infty}^{t} A(x)\ e^{x-t}\ dx + A(t)\right)$$

or

$$\frac{d^2S(t)}{dt^2} = -\lambda\frac{dS(t)}{dt} + \gamma A(t) - \gamma\int_0^\infty A(t-\tau)e^{-\tau}\, d\tau$$

Substituting the original expression we obtain

$$\frac{d^2S(t)}{dt^2} = -\lambda\frac{dS(t)}{dt} + \gamma A(t) - \frac{dS(t)}{dt} - \lambda S(t)$$

or

$$\frac{d^2S(t)}{dt^2} + (1+\lambda)\frac{dS(t)}{dt} + \lambda S(t) = \gamma A(t)$$

Section 6

This section deals with solution of equations (13) and (14) of Example 13.

$$\frac{dz_1}{dt} = \lambda z_2 - 1, \quad z_1(T) = 0$$

$$\frac{dz_2}{dt} = -z_1 + (1+\lambda)z_2, \quad z_2(T) = 0$$

Differentiating the latter equation with respect to t

$$\frac{d^2z_2}{dt^2} = -\frac{dz_1}{dt} + (1+\lambda)\frac{dz_2}{dt}$$

or

$$\frac{d^2z_2}{dt^2} = -(\lambda z_2 - 1) + (1+\lambda)\frac{dz_2}{dt}$$

or

$$[D^2 - (1+\lambda)D + \lambda]\, z_2 = 1$$

$$(D-\lambda)(D-1)\ z_2 = 1$$

$$(z_2)_c = A_1\ e^{\lambda t} + A_2\ e^t$$

$$(z_2)_p = \frac{1}{\lambda}$$

Therefore

$$z_2(t) = A_1\ e^{\lambda t} + A_2\ e^t + \frac{1}{\lambda}$$

$$z_1(t) = A_1\ e^{\lambda t} + \lambda A_2\ e^t + \frac{(1+\lambda)}{\lambda}$$

Applying the boundary conditions $z_1(T) = 0$, $z_2(T) = 0$ we obtain

$$A_1 = -\frac{1}{\lambda(1-\lambda)}\ e^{-\lambda T}$$

$$A_2 = \frac{1}{1-\lambda}\ e^{-T}$$

$$z_2(t) = \frac{1}{\lambda} - \frac{1}{\lambda(1-\lambda)}\ e^{\lambda(t-T)} + \frac{1}{(1-\lambda)}\ e^{(t-T)}$$

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OPTIMAL CONTROL AND ANALOG COMPUTER SIMULATION OF SOME INDUSTRIAL MANAGEMENT PROBLEMS

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The objective of this report is two-fold: first, to determine the optimal control for some inventory-production scheduling problems by the continuous maximum principle, and second, to simulate and find the optimal solution of some industrial management problems on the analog computer.

The algorithm for the continuous maximum principle is stated first, and then the types of optimal control are discussed. In part I of this report, all thirteen examples presented in the book Optimal Control of Dynamic Operations Research Models by Connors and Teichroew have been reworked by the maximum principle. All their problems are resolved because their approach is very tedious. A simpler, clearer approach is adopted in this report. Moreover, their solution to some examples is different from that obtained in this report. All thirteen examples solved were in the categories of bang-bang or continuous types of control.

In part II of this report, analog simulations of a production scheduling model, both for a single demand period and for periodic demand, and an inventory control model are carried out. An optimal solution is also obtained for the above models by applying the maximum principle and arriving at a system of equations which are then simulated.