

A STUDY IN TORSION OF RECTANGULAR REINFORCED CONCRETE BEAMS

by

PING-HSING PAI

B.S., Taiwan Provincial Taipei Institute of Technology, 1960

3735

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Civil Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1970

Approved by:


Major Professor

LD
2668
R4
1970
P338
C.2

CONTENTS

i

INTRODUCTION	Page 1
REVIEW OF LITERATURE	
TORSIONAL STRESS AND TORSIONAL STIFFNESS OF HOMOGENEOUS SECTION	4
REINFORCED CONCRETE BEAM SUBJECTED TO PURE TORSION	10
RECTANGULAR BEAM UNDER COMBINED BENDING, SHEAR, AND TORSION	21
PRACTICAL DESIGN USING DIFFERENT METHODS	37
CONCLUSIONS	49
ACKNOWLEDGMENT	51
APPENDICES	
BIBLIOGRAPHY	53
NOTATIONS	55

LIST OF FIGURES

	Page
Fig. 1, 2, 3.	5
Fig. 3a	8
Fig. 4, 5, 6.	15
Fig. 7, 8.	22
Fig. 9, 10, 11.	24
Fig. 12, 13.	34
Fig. 15, 16.	38

INTRODUCTION

STATEMENT OF THE PROBLEM

Although torsion is often a secondary effect in reinforced concrete buildings, it is predominant in many structural members as in the case of corner lintels, bow girders in buildings, curved beams in water towers, and so on. Torsional stress and rigidity of reinforced concrete sections, by nature, are not susceptible to mathematical derivations, although these torsional properties of homogeneous sections have been well established in textbooks on strength of materials.¹ Many experimental investigations of reinforced concrete beams subjected to torsion are on record. From some of these experimental results, a library review of the literature on torsion of rectangular reinforced concrete beams is presented in this report.

PURPOSE OF THE STUDY

The purpose of this report is to investigate rectangular reinforced concrete beams subjected to pure torsion and torsion combined with direct shear and bending moment in order to get a better understanding of this specific problem. Meanwhile, for the purpose of comparison, this report presents several theories from papers concerning reinforced concrete beams subjected to torsion, and three different methods of designing the required reinforcement for a rectangular beam subjected to torsion with flexure.

1. C. K. Wang and C. G. Salmon, "Reinforced Concrete Design" McGraw-Hill Book Company, Inc., P. 529.

SCOPE OF THE STUDY

First, a brief review of the torsional stress and torsional stiffness of homogeneous sections is introduced. In the main portion of this study, the test results for rectangular reinforced concrete beams under pure torsion, combined torsion, shear and bending moment are described from papers. Finally, a practical design example solved by three different methods is presented.

As mentioned above, this report is limited to a library research of the important literature. The problem has been treated broadly rather than in depth, and the conclusions are those of the investigators whose work has been abstracted which are the most significant in the opinion of the writer.

REVIEW OF LITERATURE

TORSIONAL STRESS AND TORSIONAL STIFFNESS OF HOMOGENEOUS SECTION²

SAINT-VENANT'S THEORY

In Fig. (1), x , y are coordinates in the plane of a normal cross section with their origin in the center of twist and z is the coordinate along the longitudinal center line. The body is supposed to turn about the origin while fixed at $z=0$. Then Saint-Venant's assumptions are

$$\begin{aligned} u &= \theta z \cdot y \\ v &= -\theta z \cdot x \\ w &= f(x, y) \end{aligned} \tag{1-1}$$

where u , v , and w are the displacements of a point x , y , z from the untwisted state A to the twisted B, in the x , y , and z directions, respectively, and θ is the angle of twist of the shaft per unit length. From these assumptions we get

$$\epsilon_x = \epsilon_y = \epsilon_z = \gamma_{xy} = 0$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

Then from Fig. 1-2, setting the net upward force equal to zero and dividing by the volume element $dx \, dy \, dz$ gives the

-
2. This chapter is based on "Advanced Strength of Materials", J. P. Den Hartog, McGraw-Hill Book Company, Inc., New York, 1952, P. 3-16.

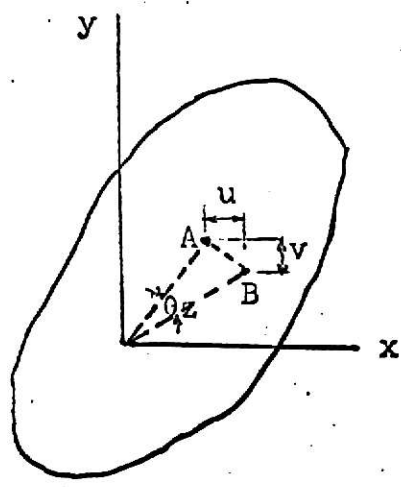


Fig. 1. The cross section, assumed by Saint-Venant, turning bodily about a center without distortion.

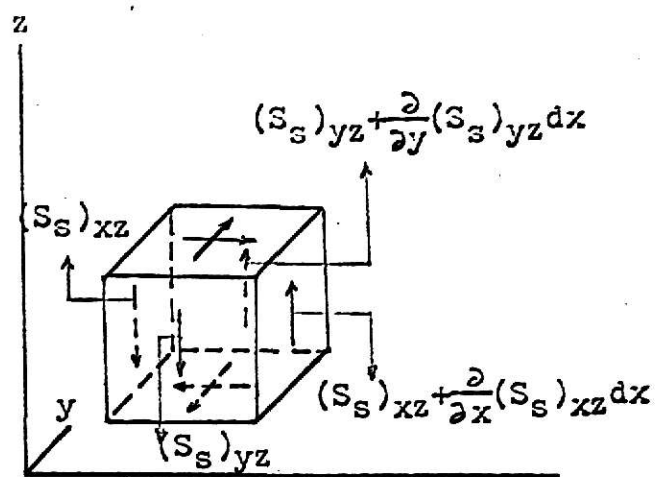


Fig. 2. Derivation of the equilibrium equation.

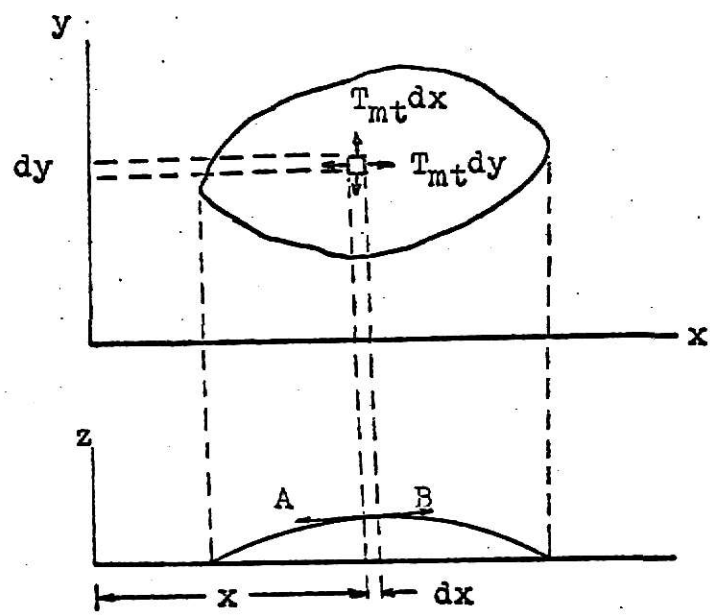


Fig. 3. Prandtl's Membrane Analogy

equilibrium equation

$$\frac{\partial}{\partial x} (S_s)_{xz} + \frac{\partial}{\partial y} (S_s)_{yz} = 0 \quad (1-2)$$

But Saint-Venant assumed that there is a function $\phi(x,y)$, such that the stresses can be found from it by differentiation, thus:

$$\begin{aligned} (S_s)_{yz} &= + \frac{\partial \phi}{\partial x} \\ (S_s)_{xz} &= - \frac{\partial \phi}{\partial y} \end{aligned} \quad (1-3)$$

The function ϕ is called the "stress function" of the problem; or "Saint-Venant's torsion stress function."

Finally he gets

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta \quad (1-4)$$

$$T = 2 \int_A \int \phi \, dA \quad (1-5)$$

Eq. 1-5 is the formula for the torque transmitted by the shaft.

PRANDTL'S MEMBRANE ANALOGY

Prandtl observed that the differential equation 1-4 for the stress function is the same as the differential equation for the shape of a stretched membrane, originally flat, which is then blown up by air pressure from the bottom. Let such a membrane be shown (Fig. 3), lying originally flat in the xy plane and then having air pressure P_a blow it up to ordinates z .

Then

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = - \frac{P_a}{T_{mt}} \quad (1-6)$$

If we adjust the membrane tension T_{mt} or the air pressure P_a so that P_a/T_{mt} becomes numerically equal to $2G\theta$, then Eq. 1-6 of the membrane is identical with Eq. 1-4 of the torsional stress function. If, moreover, we arrange the membrane so that its heights z remain zero at the boundary contour of the section, the heights z of the membrane are equal to the stress function ϕ ; the slopes of the membrane are equal to the shear stresses, and the twisting moment is numerically equal to twice the volume under the membrane.

NUMERICAL RESULTS

Circular Section

For a circular section, on account of radial symmetry, the height z of the membrane is not a function of the two coordinates x and y but can be said to depend on the single coordinate r . In fact, the slope of the membrane is proportional to the distance r from the center, and hence shear stress at any point is proportional to its radial distance from the center. Based on these facts, we get the torsional stress and torsional stiffness as follows:

$$\text{Stress} \quad S_s = \frac{T_r}{I_p} = \frac{T_r}{R^4/2}$$

$$\text{Stiffness} \quad C = \frac{T}{\theta} = GI_p = \frac{G\pi R^4}{2}$$

where R is the outside radius of the shaft, r is the radius at which the stress is measured, and T is the twisting moment on the shaft.

Rectangular Section

The torsional shear stress distribution over a rectangular section cannot be as easily derived as for a circular section. The stress in a rectangular section, according to Saint - Venant's stress distribution, varies from the maximum at the midpoint of the long side to zero at the corners. Calculating the torsional stress and torsional stiffness of rectangular section is tedious even using Prandtl's Membrane Analogy.

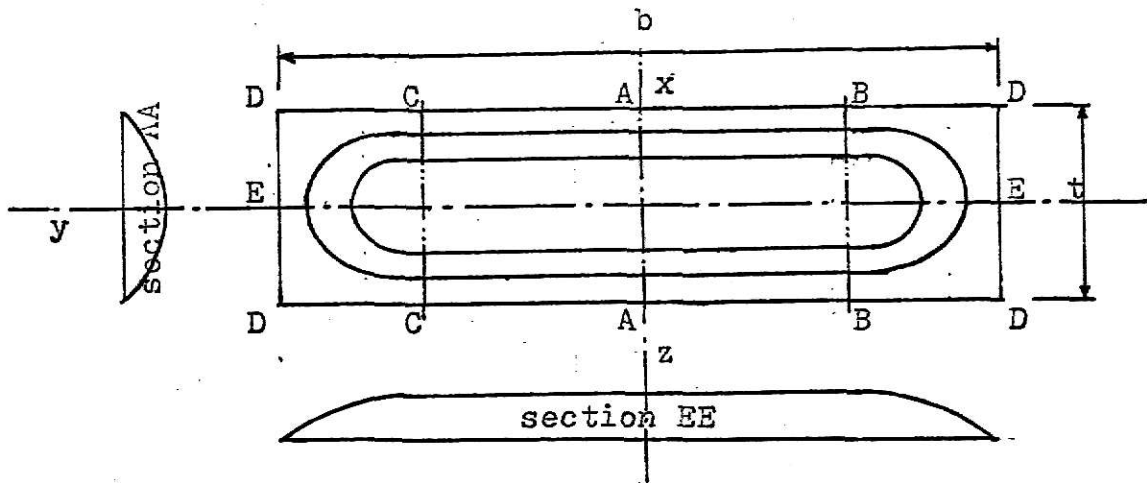


Fig. 3a. Membrane contour lines of a narrow rectangular cross section of breadth b and thickness t

Let us consider a narrow rectangular cross section as shown in Fig. 3a. If b is very much larger than t , we see by intuition

that the displacement of the membrane across AA, BB, or CC are all the same and that only near the ends DD does the membrane flatten down to zero. In this central portion, there being no curvature parallel to the y axis, the membrane is held down by vertical tension components in the x direction only. If we apply the membrane analogy method and neglect the flattening out of the membrane near the edges $y = b/2$, we can get the following equations

$$\text{Stress } S_s = \frac{3T}{bt^2}$$

$$\text{Stiffness } C = \frac{T}{\theta} = \frac{Gbt^3}{3}$$

The formulas mentioned above hold only if $b \gg t$. For less narrow cross sections, Saint-Venant found the solution as in the table below by a much more complicated method.

b/t	∞	10	5	3	2.5	2	1.5	1
$(S_s)_{\max}$								
$\frac{T}{bt^2}$	3.00	3.20	3.44	3.74	3.86	4.06	4.33	4.80
$\frac{T}{\theta}$								
$\frac{Gbt^2}{3}$	0.333	0.312	0.291	0.263	0.249	0.229	0.196	0.141

REINFORCED CONCRETE BEAM SUBJECTED TO PURE TORSION

INTRODUCTION

In general, there are three methods on record for predicting the ultimate resistance to torque of reinforced concrete members subjected to pure torsion. The first was developed by Rausch in 1929 and later modified by Andersen and Cowan. The second method, for predicting ultimate torque, was proposed by Lessig in 1958. The third one was presented by Hsu in 1968. The first and the third methods will be discussed in this chapter while the second one will be covered in the next chapter.

The general equation for ultimate torque in an underreinforced beam is

$$T_u = T_c + T_r \quad (2-1)$$

$$T_r = K A_e \frac{A_s f_{sy}}{s} \quad (2-2)$$

in which

T_u = ultimate torque of reinforced concrete beam;

T_c = the torque taken by plain concrete;

T_r = the torque taken by reinforcement;

K = coefficient, the slope of a T_u versus $b d A_s f_{sy} / s$ curve;

A_e = the effective cross sectional area of the section,

defined as the area contained within the centre lines
of the stirrup;

A_s = cross-sectional area of one stirrup leg;

f_{sy} = yield strength of stirrup;

S = spacing of stirrups in the direction parallel to the longitudinal axis of beam.

RAUSCH-ANDERSEN-COWAN THEORY

Rausch assumed: (i) both steel and concrete are elastic, (ii) and tensile stresses are taken by the reinforcement, (iii) all the bars in the section reach their yield stress. To evaluate the contribution of reinforcement in a reinforced concrete beam, Rausch, devised a network of bars to represent the action of a reinforced concrete member. In this model the concrete is represented by compression bars and the reinforcement by tension bars. For the case of 45° spiral reinforcement, Rausch used $K = 2\sqrt{2}$ in the torque equation 2-2 for all shapes of cross section. For beams with longitudinal and transverse reinforcement, Rausch's theory requires an equal volume of longitudinal and transverse reinforcement. With this arrangement, each part of the reinforcement resists a 45° component of the diagonal force so $K = 2$ for all shapes of cross section. Rausch's formulas are based on the circular section theory of torsion, and are therefore accurate only when applied to circular sections. In applying his theory to noncircular sections, he assumes that the stress in the spiral reinforcement at any point is directly proportional to the distance of that point from the axis of the beam. But from Saint-Venant's theory for rectangular sections, the maximum stress occurs at the middle of the longer sides, with the minimum stress at the corners.

As to the T_c term, Rausch considers that plain concrete's

ability to carry torque is greatly reduced and should be assumed to be zero since the concrete is badly cracked near ultimate load.

The Andersen-Cowan theory is based on Saint-Venant's classical elastic theory. It assumes that a plain concrete member fails under torsion when the maximum principal tensile stress reaches the tensile strength of the concrete. In reinforced concrete members, the resistance is taken as the strength of the plain concrete member plus the resistance contributed by the reinforcement.

In developing the Andersen-Cowan theory, a circular section with 45° spiral reinforcement as shown in Fig. 4 is examined. Because the member is subjected to pure torsion, the principal tension and compression are numerically equal on an element which is inclined at 45° to the longitudinal axis. By taking moments of the components of the reinforcement strength and the compression of the diagonal concrete elements perpendicular to the reinforcing bars, a twisting moment equation, of the same form as Rausch's equation, is obtained.

$$T_r = 2N A_s f_{sy} r_e \cos 45^\circ = 2\sqrt{2} \frac{A_e A_s f_{sy}}{s} \quad (2-3)$$

where N is the number of bars on an angle of 45° with the axis cut by a plane perpendicular to the axis of the member.

Because Rausch's assumptions are correct only for circular sections, Andersen, based on Saint-Venant's stress distribution, assumed the shear stress to vary parabolically along each face of the rectangular section, with a maximum at the center and

zero at the corners. Using this reasoning, he obtained the radius of the equivalent circular section. That is to say, he modified Rausch's formula by an efficiency factor which is a function of the arrangement of reinforcement and the height to width ratio of the cross section. Evaluation of this coefficient is difficult, especially in beams having a large number of bars.

For the purpose of obtaining a more accurate value for the coefficient K in Eq. 2-2, Cowan used energy methods, equating the work done by the torsion moment to the strain energy stored in the beam. If the tensile strength of the concrete is taken as zero, the strain energy in the beam due to pure torsion is divided equally between the tensile strain energy in the reinforcement and the compression strain energy in the concrete. In other words, the total strain energy is twice the strain energy stored in the reinforcement. Then, by integrating the energy along the spiral reinforcement and equating it to one-half of the work done by the external torque, an efficiency coefficient, K , is obtained. For circular sections with 45° reinforcement, Cowan got $K = 2\sqrt{2}$, similar to the results obtained by Rausch and Andersen. For rectangular sections with 45° spiral reinforcement, Cowan derived equations based on Saint-Venant's stress distribution, varying from zero at the corner to a maximum at the center of each face. He got $K = 1.59\sqrt{2} - 1.689\sqrt{2}$ within the range of $d'/b' = 1.0 - 3.0$. For convenience, he suggested $K = 1.60$ for all values of d'/b' , in which b' is the smaller center to center dimension of the rec-

tangular spiral reinforcement, and d' is the larger center to center dimension of the rectangular spiral reinforcement. For beams with longitudinal and transverse reinforcement, the Andersen-Cowan theory coincides with Rausch's theory, and requires an equal volume of longitudinal and transverse reinforcement, so each part of the reinforcement resists a 45° component of the diagonal force. Consequently, $K = 1.6$ in Cowan's theory.

For the term T_c in Eq. 2-1 Andersen and Cowan, as mentioned above, suggested that the concrete carries a torque equal to that of an unreinforced beam.

HSU'S THEORY

By observing the internal cracking and measuring the stresses in the reinforcement in many beam tests, Hsu proposed a new failure surface, a plane perpendicular to the wider face and inclined at 45° to the axis of the beam as shown in Fig. 5. The difference between this failure surface and that of Lessig in the next chapter is that this failure surface intersects the shorter faces of the cross section by lines at 90° to the axis of the beams, instead of about 45° as assumed by Lessig. This failure surface does not intersect the shorter legs of the stirrups, so it tends to give a conservative ultimate torque.

From the failure surface shown in Fig. 5, Hsu derived an equation for ultimate torque by considering the equilibrium of moments with respect to the axis of twist. First, assume all stirrups on the tension side of the beam to yield. These forces provide a torque

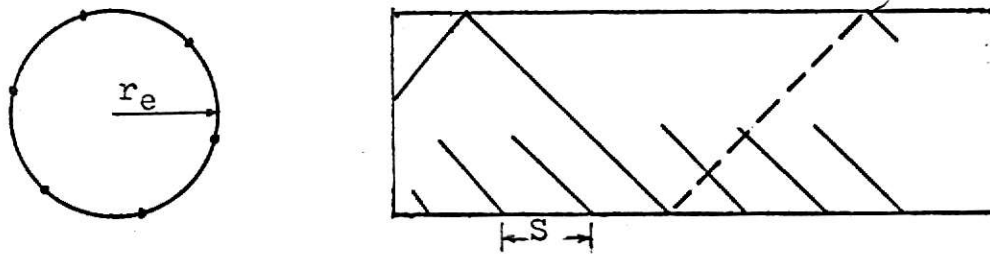


Fig. 4. Circular beam with spiral reinforcement

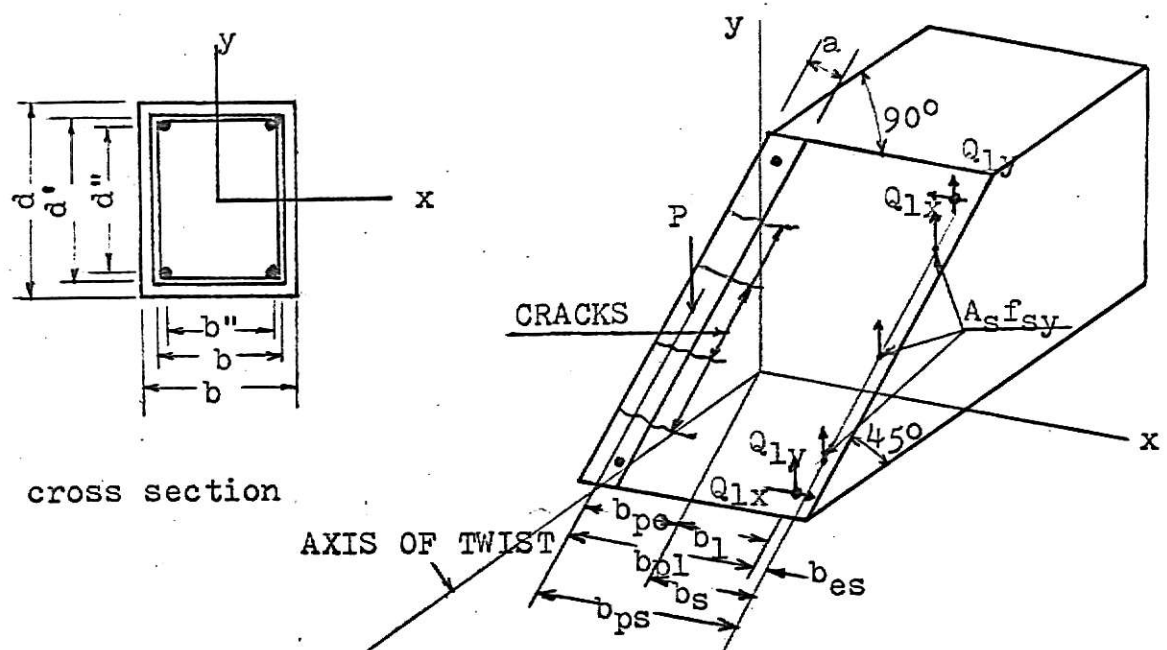


Fig. 5. Failure surface proposed by Hsu

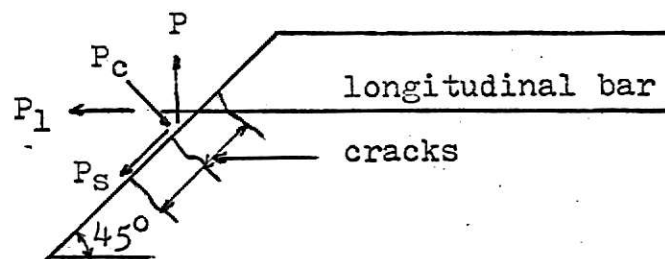


Fig. 6. The composition force P

$$T_{us} = \frac{d'}{s} A_s f_{sy} b_s \quad (2-4)$$

where T_{us} = ultimate moment contributed by stirrups; b_s = distance from the center of the longer legs of stirrups outside the shear-compression zone to the axis of twist.

To balance the force in the stirrups there must be a vertical force, p , in the concrete compression zone of the failure plane. This force cannot be provided solely by the concrete in compression in a direction parallel to the y -axis because 45° cracks separate this region into a series of inclined elements. This condition is shown in Fig. 5. The concrete in compression can furnish only a force perpendicular to the chosen failure surface. Consequently, both the shear strength of the concrete elements between cracks in the shear-compression zone and the resistance of the longitudinal steel must enter into the mechanism.⁸

From Fig. 6,

$$P = \sqrt{2} P_s + P_l \quad (2-5)$$

$$\text{but } P_l = \alpha A_1' \cdot f_{ly} \quad (2-6)$$

$$\text{so } P = \sqrt{2} P_s + \alpha A_1' \cdot f_{ly} \quad (2-7)$$

in which A_1' = cross-sectional area of longitudinal bars within the shear-compression zone (Hsu suggested that A_1' equals one half the total cross-sectional area of the longitudinal bars); f_{ly} = yield stress of longitudinal bars; α = an efficiency

8. Hsu, T. T. C., "Ultimate Torque of Reinforced Rectangular Beams", Journal of the American Concrete Institute, February, 1968, P. 494.

coefficient for longitudinal bars which should be less than unity.

Thus, the ultimate torque contributed by the shear-compression zone, is

$$T_{uc} = (\sqrt{2}P_s + \alpha A_1' f_{ly}) b_{pc} \quad (2-8)$$

in which T_{uc} = ultimate torque contributed by the shear-compression zone; b_{pc} = the distance from the center of shear compression zone to the axis of twist.

Then consider the dowel forces in the longitudinal bars outside the shear-compression zone. Fig. 5 shows that each of these dowel forces has two components, Q_{1x} and Q_{1y} in the x and y directions, respectively. If there are four corner longitudinal reinforcing bars as shown in Fig. 5, the moment of these forces with respect to the axis of twist is

$$T_{uq} = 2Q_{1x}d''/2 + 2Q_{1y}b_1 \quad (2-9)$$

in which T_{uq} = ultimate torque contributed by dowel action of longitudinal bars; Q_{1x} = dowel force in the x direction of one longitudinal corner bar outside the shear-compression zone; Q_{1y} = dowel force in the y direction of one longitudinal corner bar outside the shear-compression zone; d'' = larger distance between the center of longitudinal corner bars; b_1 = projection on the x-axis of the distance from the center of the longitudinal corner bars outside the shear-compression zone to the axis of twist.

The torsional resistance due to dowel action of tensile stirrups outside the shear-compression zone is neglected because it is small.

An external torque T_u , is found by combining T_{us} , T_{uc} and T_{uq} as follows:

$$T_u = \sqrt{2} P_s b_{pc} + A_l' f_{ly} b_{pc} + \frac{d'}{s} A_s f_{sy} b_s + Q_{lx} d'' + 2Q_{ly} b_l$$

Simplifying,

$$T_u = T' + K b' d' \frac{A_s f_{sy}}{s} \quad (2-11)$$

$$T' = \sqrt{2} P_s b_{pl} \quad (2-12)$$

$$K = \frac{b_{es}}{b'} + \alpha m \left(\frac{f_{ly}}{f_{sy}} \right) \left(1 + \frac{b'}{d'} \right) \left(\frac{b_{pl}}{b'} \right) + \frac{1}{4} \left(\frac{q \theta'}{f_{sy}} \right) d'' \left(\frac{d''}{d'} \right) \left(1 + \frac{d'}{b'} \right) m \quad (2-13)$$

in which q = dowel stresses per unit displacement; θ' = relative angle of twist of the two surfaces constituting the failure crack.

For evaluation of T' , there are two assumptions to be made for P_s , the shear force: First, the shear-compression zone is assumed to be rectangular, and second, the effect of stirrups within this range is neglected. Then

$$P_s = \sqrt{2} v_{av} da = \sqrt{2} K_v K_a b d v_c' \quad (2-14)$$

in which K_v = a coefficient relating the shear strength in

combined shear and compression to the pure shear strength;
 K_a = a coefficient relating to the depth of the compression zone to the smaller dimension of the rectangular section; v'_c = the pure shearing strength of concrete.

Hsu suggested $v'_c = 4.5 \sqrt{f'_c}$, $b_{pl} = 0.8b$. Substituting Eq. 2-14 into Eq. 2-12 obtains

$$T' = 7.2 K_v K_a b^2 d \sqrt{f'_c} \quad (2-15)$$

In general, the experimental value of T_c in Eq. 1 is

$$T_c = \frac{2.4}{\sqrt{b}} b^2 d \sqrt{f'_c} \quad (2-16)$$

In comparing T' and T_c , Hsu, after a series of beam tests, concluded that T' and T_c not only have the same parameters but are of approximately the same magnitude. He suggested that T' can be expressed by Eq. 2-16 for design purposes, although the natures of these two terms are different. The term T' is obtained theoretically and represents the torsional resistance provided by the shear strength of the shear-compression zone, and T_c is obtained experimentally as an intercept of a test curve with the T_u axis.

In regard to a value for K in Eq. 2-13, it is difficult to get reasonable analyses for α and $q\theta'$, but Hsu evaluated these two terms from test results and suggested

$$K = 0.66m + 0.33d'/b' \quad (2-17)$$

in which d'/b' should be taken as 2.6 whenever $d'/b' > 2.6$.

summing the above equations, the torque equation can be expressed by similar equations as follows:

$$T_u = \frac{2.4}{\sqrt{b}} b^2 d \sqrt{f'_c} + (0.66m \frac{f_{ly}}{f_{sy}} + 0.33 \frac{d'}{b'}) \frac{b'd'A_s f_{sy}}{s} \quad (2-18)$$

$$T_u = \frac{2.4}{\sqrt{b}} b^2 d \sqrt{f'_c} + (0.66m + 0.33 \frac{d'}{b'}) \frac{b'd'A_s f_{sy}}{s} \quad (2-19)$$

Eq. 2-19 is the design equation for rectangular beams reinforced with symmetrically located longitudinal bars and closed stirrups having the same yield strength, and Eq. 2-18 is the design equation for similar beams except the yield strengths of longitudinal bars and stirrups are different.

The equations are derived according to the assumption that both the longitudinal bars and the stirrups yield prior to crushing of concrete, so Hsu gives the following two limitations to assure this failure condition.

1. The total reinforcement provided must be less than the balanced total volume percentage which equals $2400\sqrt{f'_c}/f_{sy}$.

2. The ratio of volume of longitudinal bars to volume of stirrups must be such that both longitudinal bars and stirrups yield prior to failure. From test results this ratio should be within the range of 0.7 - 1.5.

RECTANGULAR BEAM UNDER COMBINED BENDING, SHEAR, AND TORSION

INTRODUCTION

The problem of combined bending, shear, and torsion is the most important one in the entire subject of torsion in concrete members from the point of view of the designer, but it is the most difficult to analyze. Any one of these applied stress resultants could control the behavior and especially the strength of a member. So far, insufficient experimental data are available to develop an accurate interaction surface formed by torsion, bending and shear, so it is easier to speak of the relationship of the variables in pairs. This section will review the theories proposed by Lessig, Mattock, Gesund and others.

LESSIG'S THEORY

From a series of beam tests, Lessig concluded that there are two possible modes of failure for reinforced concrete beams subjected to torsion, shear and bending moment. In the first mode (Fig. 7), the neutral axis of the failure surface intersects both vertical sides of the beam, which would indicate that the given beam section is subjected to torsion and bending moment, while shear is absent or has a relatively low value. In the second mode (Fig. 8), the neutral axis of the failure surface intersects both horizontal sides of the beam, which would indicate that the bending moment in the given beam section is relatively low compared to torsion and shear.

In deriving Lessig's equations, the following assumptions should be made.

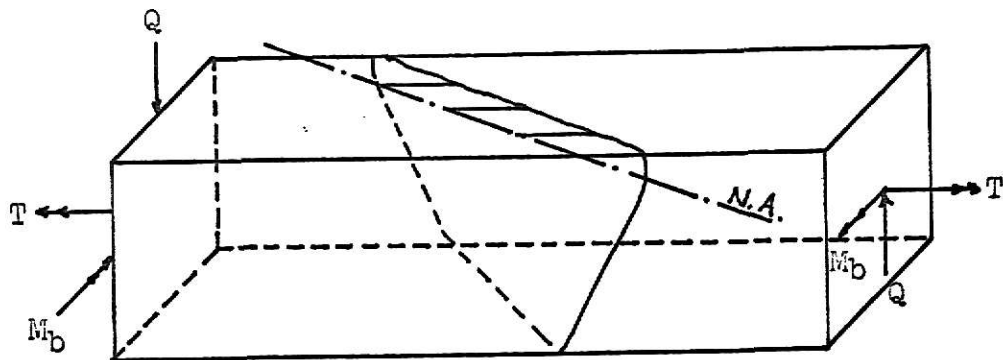


Fig. 7, First mode of failure of reinforced concrete beams under combined torsion and flexure.

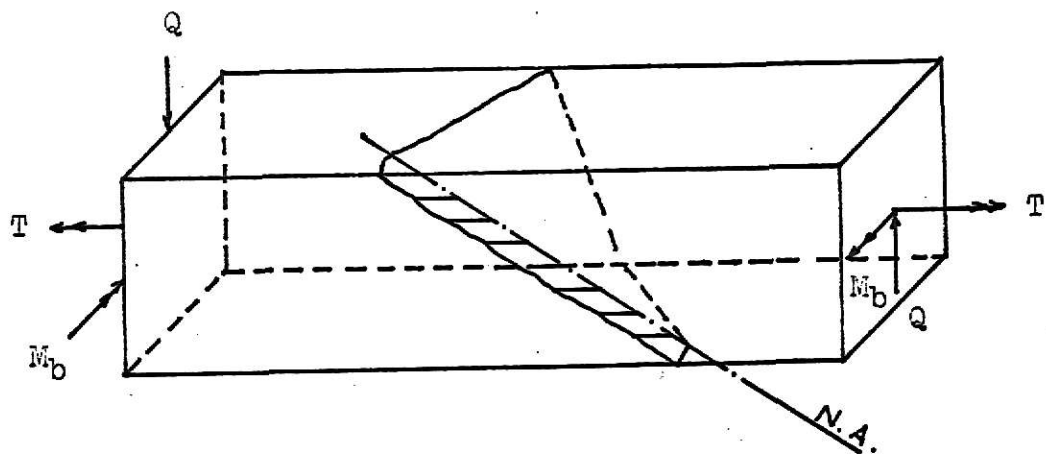


Fig. 8, Second mode of failure of reinforced concrete beams under combined torsion and flexure.

- 1). At the moment when a plastic hinge is formed, all the reinforcing steel intersecting the tension part of the failure surface reaches its yield point.
- 2). The tension capacity of the concrete in the failure surface is neglected.
- 3). The transverse reinforcement is uniformly distributed over the beam.
- 4). No external loads are applied within the section in which the element falls.
- 5). The concrete stress in the compression zone of the failure surface reaches the ultimate strength as in flexure compression.

Based on the assumptions and failure modes mentioned above, the design formulas can be derived by using an equilibrium equation, setting all moments due to external and internal forces equal to zero with respect to the neutral axis, acting on a plane perpendicular to the neutral axis.

FIRST MODE (Fig. 9, 10)

The moment, M_{ex} , due to external forces is

$$M_{ex} = M_b \frac{b}{L} + T \frac{c}{L} \quad (3-1)$$

in which M_b and T are bending moment and torsion, respectively. The direct shear, Q , is absent in the equation, because it lies in the plane of the axis of rotation.

The resisting moment due to internal forces consists of four parts, M_c , M_l , M_{hs} and M_{vs} .

1). M_c , due to normal compression forces in the concrete, is

$$M_c = f_c Z$$

where f_c = compressive strength of concrete, can be assumed as approximately equal to the ultimate strength of concrete in flexure, $0.85f_c'$; Z = static moment of compression zone about the neutral axis.

Referring to Fig. 11,

$$Z = \int_A x \, dy \left(\frac{x}{2} \cdot \frac{\sqrt{c^2 + b^2}}{L} \right)$$

but

$$x = x_2 + (x_1 - x_2) \frac{y}{\sqrt{c^2 + b^2}} = \frac{x_2 \sqrt{c^2 + b^2} + (x_1 - x_2)y}{\sqrt{c^2 + b^2}} \quad (3-2)$$

$$\begin{aligned} Z &= \frac{1}{2L\sqrt{c^2 + b^2}} \int_0^{\sqrt{c^2 + b^2}} [x_2 \sqrt{c^2 + b^2} + (x_1 - x_2)y]^2 dy \\ &= \frac{c^2 + b^2}{6L} (x_1^2 + x_1 x_2 + x_2^2) \end{aligned}$$

thus

$$M_c = \frac{f_c (c^2 + b^2)}{6L} (x_1^2 + x_1 x_2 + x_2^2) \quad (3-3)$$

2). M_l , due to bottom longitudinal reinforcement, is

$$M_l = f_{ly} A_{bl} \left(d_o - \frac{x_1 + x_2}{2} \right) \frac{b}{L} \quad (3-4)$$

where A_{bl} = the cross sectional area of all bottom longitudinal steel.

f_{ly} = tensile yielding stress of the longitudinal steel.

3). M_{hs} , due to horizontal stirrups, is

$$M_{hs} = f_{sy} A_s \frac{c \theta_1}{s} (d - d_3 - \frac{x_1 + x_2}{2}) \frac{c}{L} \quad (3-5)$$

where f_{sy} = yielding stress of stirrup.

A_s = cross-sectional area of one leg.

s = spacing of stirrups in the direction parallel to the longitudinal axis of beam.

4). M_{vs} , due to vertical stirrups.

Assume the angles β between the cracks on both vertical faces of the beam with respect to the vertical axis are equal, so

$$\cot \beta = \frac{c(1-\theta_1)}{2d - (x_1 + x_2)} \quad \frac{H}{V}$$

The lever arm of the total stirrups on the front face is

$$\left[\left(\frac{d - x_2}{2} \cot \beta \right) \frac{b}{c} - b_1 \right] \frac{c}{L}$$

then the resisting moment, M_{vs}^f , due to the total stirrups on the front face is

$$M_{vs}^f = F_{sy} \frac{A_s}{s} (d - x_2) \cot \beta \left(\frac{d - x_2}{2} \frac{b}{c} \cot \beta - b_1 \right) \frac{c}{L}$$

Similarly, the resisting moment, M_{vs}^r , due to the total stirrups on the rear face is

$$M_{vs}^r = F_{sy} \frac{A_s}{s} (d - x_1) \cot \beta \left(\frac{d - x_1}{2} \frac{b}{c} \cot \beta - b_1 \right) \frac{c}{L}$$

Adding up and substituting $\cot \beta$ into the equation, gives

$$M_{vs} = F_{sy} \frac{A_s}{S} \cdot \frac{c^2(1-\theta_1)}{L} \left[\frac{b}{2} (1-\theta_1) \frac{(d-x_1)^2 + (d-x_2)^2}{(2d-x_1-x_2)^2} - b_1 \right] \quad (3-6)$$

Summing up equations 3-1, 3-2, 3-3, 3-4, 3-5, 3-6 yields

$$\begin{aligned} M_b \cdot b + T \cdot c = & \frac{f_c(c^2 + b^2)}{6} (x_1^2 + x_1 x_2 + x_2^2) + F_{ly} A_{bl} (d_0 - \frac{x_1 + x_2}{2}) b \\ & + F_{sy} \frac{A_s}{S} \frac{c^2(1-\theta_1)}{L} \left[\frac{b}{2} (1-\theta_1) \frac{(d-x_1)^2 + (d-x_2)^2}{(2d-x_1-x_2)^2} - b_1 \right] \\ & + F_{sy} A_s \frac{c^2 \theta_1}{S} (d - d_3 - \frac{x_1 + x_2}{2}) \end{aligned} \quad (3-7)$$

The parameters x_1 and x_2 are symmetrical in Eq. 3-7, therefore differentiating it first for x_1 and then for x_2 yields identical results if $x_1 = x_2$. It is necessary, then, that $x_1 = x_2$. Substituting x for x_1 and x_2 into Eq. 3-7, gives

$$\begin{aligned} M_b \cdot b + T \cdot c = & f_c(c^2 + b^2) \frac{x^2}{2} + f_{ly} A_{bl} (d_0 - x) b \\ & - f_{sy} A_s \frac{c^2}{S} \left\{ \theta_1 (d - d_3 - x) + (1 - \theta_1) \left[\frac{b}{4} (1 - \theta_1) - b_1 \right] \right\} \end{aligned} \quad (3-8)$$

Differentiating Eq. 3-8 with respect to x yields

$$f_c(c^2 + b^2)x - f_{ly} A_{bl} b - f_{sy} A_s \frac{c^2 \theta_1}{S} = 0 \quad (3-9)$$

Eq. 3-9 represents the equation of the projections of all

forces acting in the direction normal to the plane of the compression zone. From this equation we can determine x as follows:

$$x = \frac{f_{ly} A_{bl}}{f_c (c^2 + b^2)} (b + p \theta_1 \frac{c^2}{d}) \quad (3-10)$$

in which

$$p = \frac{f_{sy} A_s d}{f_{ly} A_{bl} S}$$

Let

$$\phi = \frac{T}{M_b}$$

and substituting p , ϕ and f_c (Eq. 3-10) into Eq. 3-8 yields

$$M(1 + \frac{c}{b} \phi) = T(\frac{1}{\phi} + \frac{c}{b}) = f_{ly} A_{bl} \left\{ d_o - \frac{x}{2} + \frac{pc^2}{bd} [\theta_1 (d - d_1 - \frac{x}{2}) + \frac{b}{4} (1 - \theta_1)(1 - \theta_1 - \frac{4b_1}{b})] \right\} \quad (3-11)$$

Based on tests, Lessig suggests that the failure crack on three sides of beam can have the same angle of inclination, so θ_1 can be

$$\theta_1 = \frac{b}{2d + b} \quad (3-12)$$

Introducing the designations

$$d_j = d_o - x/2$$

$$J = \theta_1 (d - d_1 - \frac{x}{2}) + \frac{b}{4} (1 - \theta_1)(1 - \theta_1 - \frac{4b_1}{b}) \quad (3-13)$$

and substituting into Eq. 3-11 to get a simpler equation.

$$T = \phi M_b = f_{ly} A_{bl} \frac{d_j + pJc^2/bd}{1/\phi + c/b} \quad \text{How?} \quad (3-14)$$

Theoretically the least favorable direction of the neutral axis must correspond to minimum values of T or M_b for a given ratio ϕ . The value c , corresponding to the theoretical minimum value of T or M_b , can be derived by setting dT/dc or dM_b/dc equal to zero, while d_j and J , which are affected relatively little by x , are assumed constant. The ratio of c to b is given as

$$\frac{c}{b} = -\frac{1}{\phi} + \sqrt{\frac{1}{\phi^2} + \frac{d_j}{pJ} \frac{d}{b}} \quad (3-15)$$

The values of c from this formula are higher than the actual values determined by measurement in many cases, because there is considerable difference between the assumptions used in deriving the equations and the actual condition of failure surface. Through experimental observation, Lessig suggests the following formulas to determine c_{max} and J . c_{max} is based on the fact that the minimum crack inclination with respect to beam axis is 45° , and

$$c_{max} \leq 2d + b \quad (3-16)$$

$$\text{and } J = \theta_1(d - d_1 - x/2) \approx \theta_1 d_j \quad (3-17)$$

Substituting Eq. 3-17 into Eqs. 3-14 and 3-15, we get

$$T = \phi M_b = f_{ly} A_{b1} d_j \frac{1 + p\theta_1 c^2/bd}{1/\phi + c/b} \quad (3-18)$$

$$\frac{c}{b} = -\frac{1}{\phi} + \sqrt{\frac{1}{\phi^2} + \frac{1}{p\theta_1} \frac{d}{b}} \quad (3-19)$$

In solving practical problems Lessig suggested the following procedures.

- A). Calculate θ_1 and c from Eq. 3-12 and Eqs. 3-16 or 3-19.
- B). Determine x and d_j from Eq. 3-10 and Eq. 3-13.
- C). Compute values of M_b and T from Eq. 3-18.

SECOND MODE

Using the same procedures as were used in deriving equations for the first mode of failure, we can get the final equation for the second mode of failure as follows:

$$T = \frac{Qb}{2\phi} = f_{ly} A_{v1} b_j \frac{d}{c_2} \frac{1 + p_2\theta_2 c_2^2/bd}{1 + \phi} \quad (3-20)$$

wherein

Q = Transverse Shear;

$\phi = Qb/2T$;

A_{v1} = the area of longitudinal steel located near one vertical side of the beam;

$b_j = b - b_2 - x/2$;

$c_2 = d \sqrt{\frac{1}{p_2\theta_2} \frac{b}{d}} \leq 2b + d$!

$\theta_2 = \frac{d}{2b + d}$

$$p_2 = \frac{f_{sy} A_s b}{f_{ly} A_{vl} S} ;$$

$$x = \frac{f_{ly} A_{vl} (d + \psi \theta_2 c_2^2 / b)}{f_c (c_2^2 + d^2)}$$

$$f_c = 0.85 f'_c$$

A special case of this mode of failure is pure torsion ($Q = 0$, $\psi = 0$).

When it is difficult to determine which one predominates in the section of a beam subjected to torsion with flexure, we can compare the torsional ultimate capacity carried by both modes of failure under the given values of ϕ and ψ , i.e., the mode of failure which has the smaller capacity will be the predominant one. In the second mode of failure, if $x > 0$, the failure of the beam would be due to combined torsion and shear; if $x < 0$, failure of the beam is more probably due to the action of shear with bending moment.

In practical design, Lessig suggested that the yield stress of stirrups in Eqs. 3-18 and 3-20 should be multiplied by a coefficient, 0.8, which takes into account the fact that transverse bars are placed at a certain distance from each other while Eqs. 3-18 and 3-20 are based on the assumption of constant intensity of transverse reinforcement along the entire length of the beam.

MATTOCK'S THEORY FOR INTERACTION OF SHEAR AND TORSION

Based on the test results from Birkeland, Hamilton and on

his own test results for reinforced concrete beams subjected to combined torsion, bending and shear, Mattock concluded that the effect of shear and torsion acting simultaneously in a beam can be represented by an interaction diagram for shear and torsion as shown in Fig. 12. He assumed the interaction curve between shear and torsion to be a circular arc.

$$\left[\frac{Q_u}{Q_o} \right]^2 + \left[\frac{T_u}{T_o} \right]^2 = 1 \quad (3-21)$$

wherein T_u and Q_u are the ultimate torsional moment and ultimate shear resisted simultaneously by a beam. T_o is the strength of the beam in pure torsion and Q_o is the shear strength of the beam when subjected to shear and bending moment.

If we assume:

A). under combined shear and bending,

Q_c , ultimate shear carried by the concrete compression zone, equals the shear which causes diagonal tension cracking to occur, i.e., $Q_c = c'Q_o$;

B). under pure torsion,

T_c , ultimate torsion carried by the concrete compression zone, equals one half the torsional moment which causes diagonal tension cracking to occur, i.e., $2T_c = c''T_o$;

C). under combined shear and torsion,

Q_{ca} , ultimate shear carried by the concrete compression zone, equals the shear at diagonal tension cracking, i.e., $Q_{ca} = c'Q_u$; T_{ca} , ultimate torsion carried by the concrete compression zone, equals one half the torsion at

diagonal tension cracking, i.e., $2T_{ca} = c''T_u$;

then Eq. 3-21 becomes

$$\left[\frac{Q_{ca}}{Q_c} \right]^2 + \left[\frac{T_{ca}}{T_c} \right]^2 = 1 \quad (3-22)$$

Let v_c , v_{ca} , γ_c , γ_{ca} be the shear stresses due to Q_c , Q_{ca} , T_c , and T_{ca} , respectively. The shear stresses are proportional to the forces concerned, so from Eq. 3-22 we get

$$\left[\frac{v_{ca}}{v_c} \right]^2 + \left[\frac{\gamma_{ca}}{\gamma_c} \right]^2 = 1 \quad (3-23)$$

In solving for v_{ca} and γ_{ca} from Eq. 3-23, Mattock assumes that the ratio of torsional shear stress, γ , to shear stress, v , remains constant at a particular section during the load history of the beam, that is

$$\frac{\gamma}{v} = \frac{\gamma_u}{v_u} = \frac{2\gamma_{ca}}{v_{ca}}$$

Substituting this assumed equation into Eq. 3-23, one obtains

$$v_{ca} = v_c / \sqrt{1 + (\gamma/2\lambda)^2} \quad (3-24)$$

$$\gamma_{ca} = \gamma_c / \sqrt{1 + (2\lambda/\gamma)^2} \quad (3-25)$$

in which

$$\lambda = \gamma_c / v_c, \quad \gamma = \gamma_u / v_u.$$

Mattock suggests

$$\gamma_u = \frac{T_u}{b^2(d-b/3)/2} \leq \frac{11\sqrt{f_c'}}{\sqrt{1 + (1.1/\gamma)^2}}$$

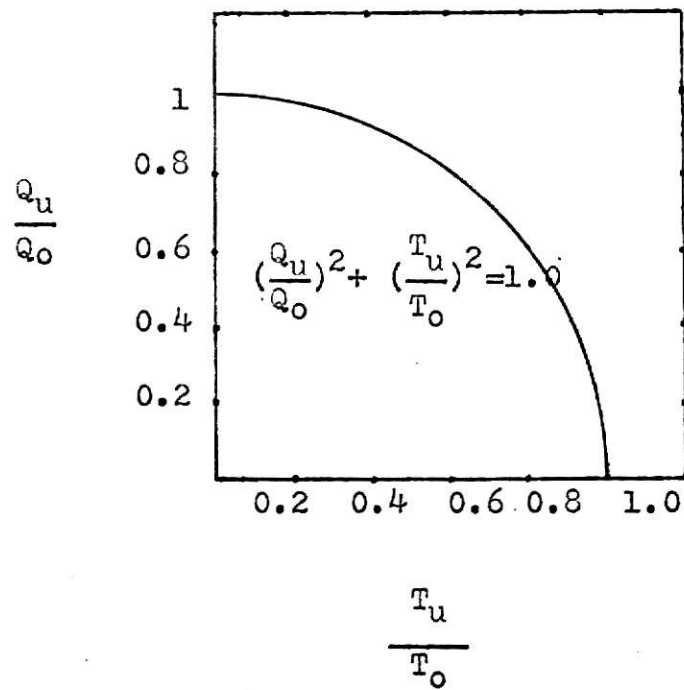


Fig. 12, Interaction of torsion and shear

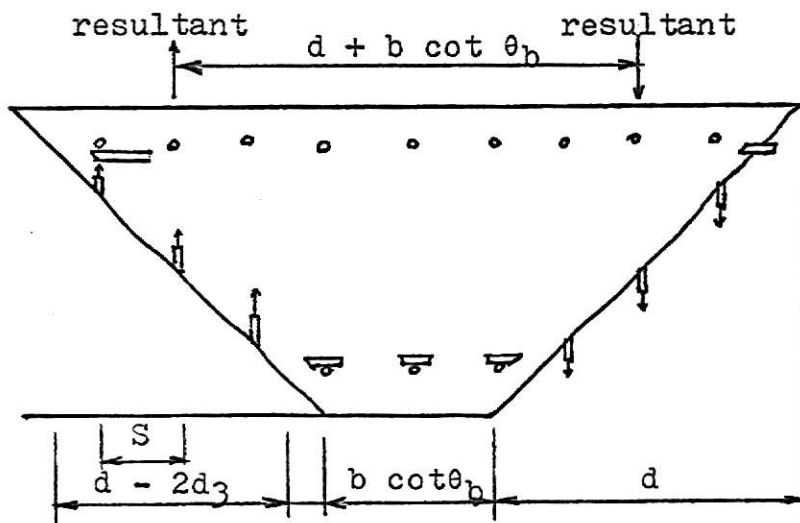


Fig. 13. Idealized side view of failure surface

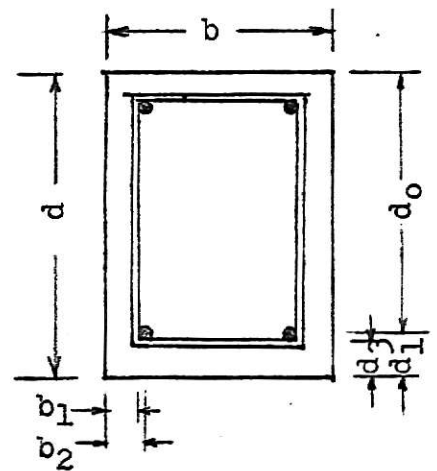


Fig. 14. Beam cross section

$$v_u = \frac{Q_u}{bd_o} \leq \frac{10f_c'}{\sqrt{1 + (\gamma/1.1)^2}}$$

ULTIMATE STRENGTH IN COMBINED BENDING AND TORSION PROPOSED BY GESUND, SCHUETTE, BUCHANAN AND GRAY

The equation for predicting the ultimate strength of rectangular reinforced concrete beams under combined bending and torsion in which bending moment predominates is based upon the shape of the failure surface shown in Fig. 13. The intersection of the failure surface with the sides of the beam and with the vertical plane containing the sides of the stirrups are straight lines making a 45° angle with the beam axis. The intersection of the failure surface with the bottom of the beam and with the horizontal planes containing the bottom steel and the bottom stirrups are straight lines making an angle θ_b with the beam axis.

From Fig. 13, it may be seen that the additional bending moment M_a arising from the effect of the stirrups caused by the torque may be expressed by

$$M_a = \frac{A_s f_{sy}}{s} (d - 2d_3)(d + b \cot \theta_b) \quad (3-26)$$

If we assume that the applied torque T at ultimate load is resisted solely by the stirrups and that the stress in the stirrups is constant over their vertical and bottom portions, we get the following moment equilibrium equation about a longitudinal axis along the top of the beam at midwidth:

$$T = \frac{A_s f_{sy}}{S} [(d - 2d_3)(b - 2b_1) + d_o(b - 2b_1) \cot \theta_b] \quad (3-27)$$

Let

$$C_t = \frac{M_a}{T} = \frac{(d - 2d_3)(d + b \cot \theta_b)}{(d - 2d_3)(b - 2b_1) + d_o(b - 2b_1) \cot \theta_b} \quad (3-28)$$

M_a is added to the applied bending moment of a beam, so if torsion is present, the ordinary ultimate moment, $M_u = \phi_{cr}[A_s f_y (d - a/2)]$, must be greater than the actual failure moment, M_f , or

$$M_f = M_u - M_a = M_u - C_t T$$

Letting ϕ = the ratio of applied torque to applied moment at failure, then

$$T = \phi M_f$$

$$M_f = M_u - C_t \phi M_f$$

$$M_f = \frac{M_u}{1 + C_t} \quad (3-29)$$

Therefore the actual bending moment acting at the beam is

$$M_{act} = M_b (1 + C_t \phi) \quad (3-30)$$

in which M_b = applied bending moment; the value of $\cot \theta_b$ can be assumed to be $1/2$ for $\phi \geq 0.25$, and to be zero for $\phi < 0.25$. However, this equation only holds for low values of ϕ . In practical problems, b_1 can be equal to b_2 , and $d_1 = d_3$.

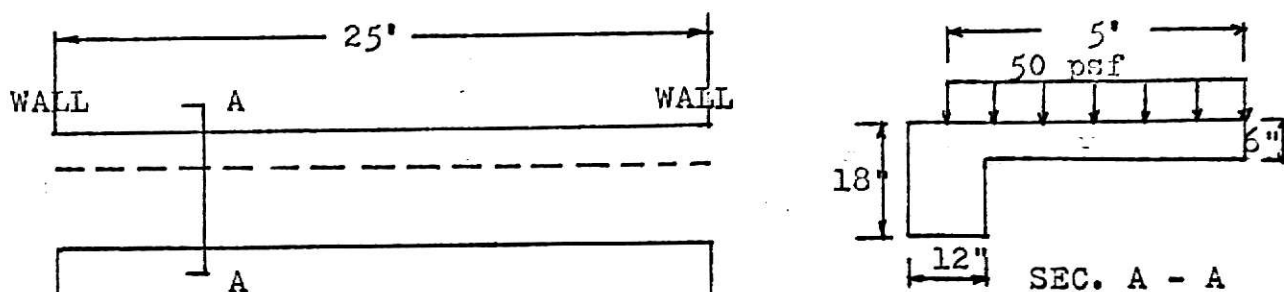
PRACTICAL DESIGN USING DIFFERENT METHODS

The minimum requirements for ultimate strength design of reinforced concrete members subjected to bending moment and shear have been provided by the ACI Building Code, ACI 318-63, while the requirements for design of reinforced concrete members subjected to torsion have not yet been fully developed and codified. It seems appropriate to use the same value of capacity reduction factor for torsion as for shear, i.e., $\Phi_{cr}=0.85$.

The example concerns the beam shown in Fig. 15, from which is cantilevered a slab supporting a live load of 50 psf. The bending moments, torsional moments and shears used in the design are shown in Fig. 16. (The beam is fixed at the ends and the design ultimate loads are as specified in section 1506 of ACI Code 318-63.)

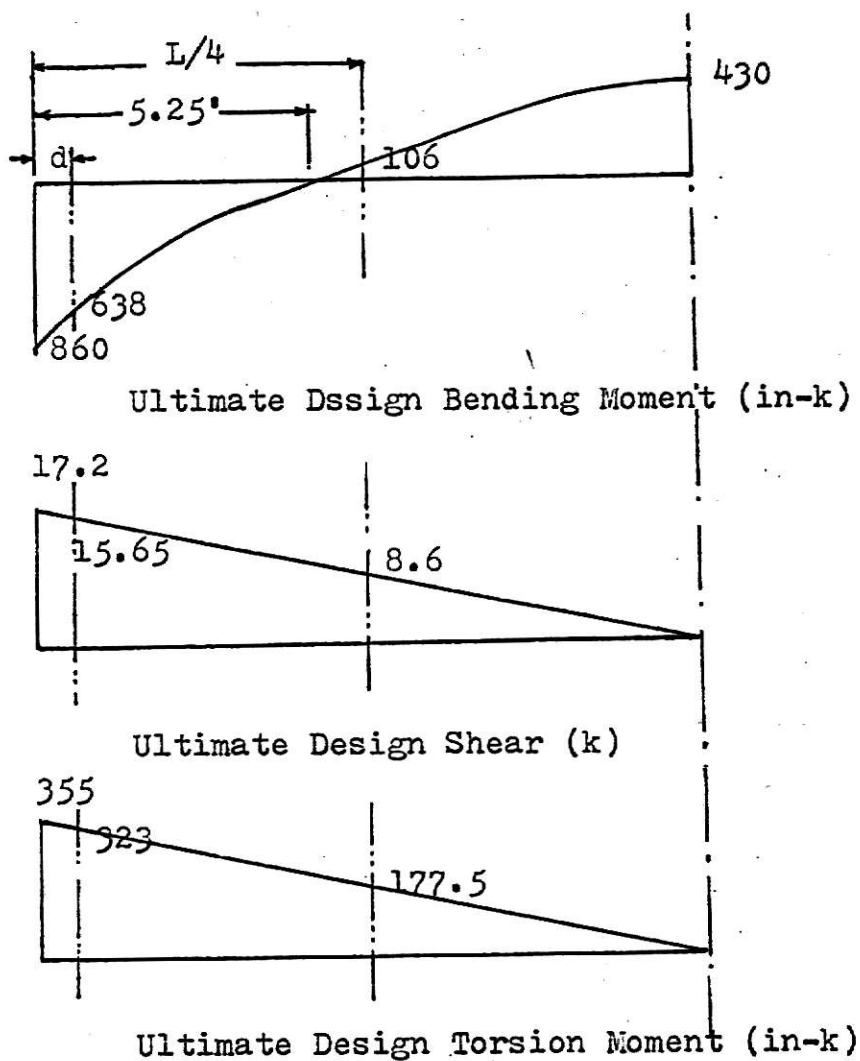
If $f'_c = 3000$ psi, $f_y = 50,000$ psi for both longitudinal and transverse reinforcement, design the reinforcement for the 12" x 18" rectangular section beam (i.e., neglecting the effect of the flange), using the three different methods as follows:

- A). Separated Method: Assume bending moment, torsional moment and shear acting independently. Design the reinforcement due to bending moment and shear according to ACI Code 318-63, and the reinforcement due to torsion by Hsu's Formula. Assume concrete does not take the shear due to torsion.
- B). Mattock's Method: Assume bending moment acts independently, while shear and torsion interact, i.e., taking the interaction of shear and torsion into account in design. Design the rein-



Slab loads: L.L = 50 psf, D.L. = 90 psf.

Fig. 15, Beam considere in design example



1.5 D + 1.8 LL

Fig. 16. Moments and Shears Used in Design Example

forcement due to bending moment according to ACI Code 318-63; calculate shear and torsional moment carried by the concrete by Mattock's equations; compute web reinforcement for shear by ACI Code 318-63, and reinforcement for torsion by Hsu's formula.

C). Lessig's Method: Design the reinforcement according to Lessig's theory, first mode of failure or second mode of failure.

A). SEPARATED METHOD

1. Bending moment $M_b = 638$ in-k

$$P_{max} = 0.75(0.85)^2(3/50)(87/87 + 50) = 0.0207$$

Assume $a = 1.9$ in

$$M_u = A_{lt}f_{ly}(d - a/2)$$

$$A_{lt} = \frac{638}{0.9 \times 50 (15.5 - 0.95)} = 0.98 \text{ in}^2$$

$$p = 0.98 / 12 \times 18 = 0.0045 < P_{max} \quad \text{O.K.}$$

$$A_{lt}f_{ly} = 0.85f_c'ba$$

$$a = \frac{0.98 \times 50}{0.85 \times 3 \times 10} = 1.94 \text{ in} \quad \text{O.K.}$$

2. Shear $Q = 15.65$ k

$$\text{Shear stress } v_c = 2 \phi_c \sqrt{f_c'} = 2 \times 0.85 \sqrt{3,000} = 93 \text{ psi}$$

Shear carried by concrete

$$Q_{cc} = 93 \times 12 \times 15.5 = 17.30 \text{ k} > 15.65 \text{ k}$$

3. Torsional moment $T_u = 323$ k

$$T_u = \phi_{cr} (0.66m + 0.33 \frac{d'}{b'}) \frac{b'd'A_s f_{sy}}{S}$$

use $m = 1$, A_s = one leg area.

$$b' = 12 - 2 \times 1.75 = 8.5 \text{ in}; \quad d' = 18 - 2 \times 1.75 = 14.5 \text{ in}$$

$$\begin{aligned} \frac{A_s}{S} &= \frac{T_u}{\phi_{cr} (0.66m + 0.33d'/b') b'd' f_{sy}} \\ &= \frac{323,000}{0.85(0.66 + 0.33 \times 14.5/8.5) 8.5 \times 14.5 \times 50,000} \\ &= 0.0505 \text{ in}^2/\text{in} \end{aligned}$$

4. Area of longitudinal reinforcement to resist torsion

$$m = \frac{A_{lt} S}{2A_s(b' + d')} = 1$$

$$\begin{aligned} A_{lt} &= \frac{2A_s}{S} (\text{torsion})(b' + d') = 2 \times 0.0505(8.5 + 14.5) \\ &= 2.32 \text{ in}^2 \end{aligned}$$

Total longitudinal reinforcement to resist torsion and bending

$$A_{lt}(\text{total}) = 0.98 + 2.32 = 3.30 \text{ in}^2$$

$$(\text{Top: } 0.98 + 2.32/2 = 2.14 \text{ in}^2; \text{ Bottom: } 2.32/2 = 1.16 \text{ in}^2)$$

Total web reinforcement

$$\frac{A_s}{S}(\text{total}) = 0.0505 \text{ in}^2/\text{in}$$

But Hsu suggests that p_t , the total volume percentage of reinforcement (including longitudinal bars and stirrups), should be less than p_{tb} , $2400\sqrt{f'_c}/f_{sy}$ (in percentage).

$$p_{tb} = 2400\sqrt{3000}/50,000 = 2.63\%$$

$$p_t = (3.30 + 2.32)/12 \times 18 = 2.60\% < p_{tb} \quad \text{O.K.}$$

The reinforcement required at the other section is summarized in Table 2.

B). MATTOCK'S METHOD

At distance "d" from support

1. Reinforcement due to bending moment same as the result by Separated Method.
2. Interaction of shear and torsion

Nominal shear stress due to ultimate shear

$$v_u = \frac{Q_u}{bd_o} = \frac{15,650}{10 \times 15.5} = 84 \text{ psi}$$

Nominal shear stress due to ultimate torsion

$$\tau_u = \frac{T_u}{b^2(d - b/3)/2} = \frac{323,000}{12^2(18 - 12/3)/2} = 333 \text{ psi}$$

$$\gamma = \tau_u/v_u = 333/84 = 3.96$$

$$\begin{aligned} v_{u(\max)} &= 10\sqrt{f'_c} / \sqrt{1 + (\gamma/1.1)^2} \\ &= 10\sqrt{3,000} / \sqrt{1 + (3.96/1.1)^2} = 146 \text{ psi} > 84 \text{ psi} \end{aligned}$$

O.K.

$$\begin{aligned} \tau_{u(\max)} &= 11\sqrt{f'_c} / \sqrt{1 + (1.1/\gamma)^2} \\ &= 11\sqrt{3,000} / \sqrt{1 + (1.1/3.96)^2} = 535 \text{ psi} > 333 \text{ psi} \end{aligned}$$

$$v_c = 2 \phi_{cr} \sqrt{f'_c} ; \quad \tau_c = 2.4 \phi_{cr} \sqrt{f'_c}$$

$$\lambda = \tau_c/v_c = 1.2$$

Torsional shear stress carried by the concrete

$$\tau_{ca} = \tau_c / \sqrt{1 + (2\lambda/\gamma)^2} = 0.85 \times 2.4\sqrt{3,000} / \sqrt{1 + (2.4/3.96)^2}$$

$$= 95.5 \text{ psi}$$

Torsional moment carried by concrete

$$T_{ca} = \gamma_{ca} d^2 (b - d/3)/2 = 95.5 \times 18^2 (12 - 18/3)/2 = 93,500 \text{ in-lb}$$

Shear stress carried by concrete

$$\begin{aligned} v_{ca} &= v_c / \sqrt{1 + (\gamma/2\lambda)^2} = 0.85 \times 2\sqrt{3,000} / \sqrt{1 + (3.96/2.4)^2} \\ &= 48.3 \text{ psi} \end{aligned}$$

Shear carried by concrete

$$Q_{ca} = v_{ca} b d_o = 48.3 \times 12 \times 15.5 = 9,000 \text{ lb}$$

Torsional moment to be resisted by reinforcement

$$T_u' = T_u - T_{ca} = 323,000 - 93,500 = 229,500 \text{ in-lb}$$

Web reinforcement for torsion

use $m = 1$; $A_s = \text{one leg area.}$

$$\begin{aligned} \frac{A_s}{S} &= \frac{T_u'}{\phi_c (0.66m + 0.33d'/b') b' d' f_{sy}} \\ &= \frac{229,500}{0.85(0.66 + 0.23 \times 14.5/8.5) 8.5 \times 14.5 \times 50,000} = 0.0359 \text{ in}^2/\text{in} \end{aligned}$$

Shear to be resisted by web reinforcement

$$Q_u' = Q_u - Q_{ca} = 15,650 - 9,000 = 6,650 \text{ lb}$$

Web reinforcement for shear ($A_s = \text{one leg area}$)

$$\frac{A_s}{S} = \frac{Q_u'}{2\phi_{cr}d_o f_y} = \frac{6,650}{2 \times 0.85 \times 15.5 \times 50,000} = 0.0051 \text{ in}^2/\text{in}$$

Total web reinforcement

$$\begin{aligned} \frac{A_s}{S} (\text{total}) &= \frac{A_s}{S} (\text{torsion}) + \frac{A_s}{S} (\text{shear}) \\ &= 0.0359 + 0.0051 = 0.041 \text{ in}^2/\text{in} \end{aligned}$$

Longitudinal reinforcement to resist torsion

$$\begin{aligned} A_{lt} &= \frac{2A_s}{S} (\text{torsion}) (b' + d') \\ &= 2 \times 0.0359 (8.5 + 14.5) = 1.66 \text{ in}^2 \end{aligned}$$

Total longitudinal reinforcement to resist torsion and bending moment

$$\begin{aligned} A_{lt} &= 0.98 + 1.66 = 2.64 \\ (\text{Top: } 0.98 + 1.66/2 &= 1.81 \text{ in}^2; \quad \text{Bottom: } 0.83 \text{ in}^2). \end{aligned}$$

The reinforcement required at the other section is summarized in Table 2.

C). LESSIG'S METHOD

1. Use the ratio of transverse to longitudinal reinforcement same as that from Mattock's method.
2. Design the reinforcements at distance "d" from support
3. First Mode

$$\theta_1 = \frac{b}{2d + b} = \frac{12}{2 \times 18 + 12} = 0.25$$

$$\phi = \frac{T}{M_b} = \frac{323}{638} = 0.506$$

$$p = \frac{0.8f_{sy}A_s d}{f_{ly}A_{bl}S} = \frac{0.8 \times 0.41 \times 18}{1.81} = 0.324$$

$$c = b \sqrt{\frac{1}{\phi^2} + \frac{1}{p\theta_1} \frac{d}{b}} - \frac{b}{\phi}$$

$$= 12 \sqrt{\frac{1}{(0.506)^2} + \frac{1}{0.324 \times 0.25} \frac{18}{12}} - \frac{12}{0.506} = 35 \text{ in}$$

$$c_{\max} = 2d + b = 36 + 12 = 48 \text{ in}$$

So $c = 35$ is acceptable

Assume $A_{bl} = 1.65$

$$x = \frac{f_{ly}A_{bl}}{f_c(c^2 + b^2)} (b + p\theta_1 \frac{c^2}{d})$$

$$= \frac{50,000 \times 1.65}{0.85 \times 3,000 (35^2 + 10^2)} (12 + 0.324 \times 0.25 \frac{35^2}{18}) = 0.42 \text{ in}$$

$$d_j = 15.50 - 0.42/2 = 15.29 \text{ in}$$

$$A_{bl} = \frac{T(1/\phi + c/b)}{\phi_{cr}f_{ly}d_j(1 + p\theta_1 c^2/bd)}$$

$$= \frac{323,000(1/0.506 + 35/12)}{0.85 \times 50,000 \times 15.29(1 + 0.324 \times 0.25 \times 35^2/12 \times 18)}$$

$$= 1.67 \text{ in}^2 \quad \text{O.K.}$$

$$\frac{A_s}{S} = \frac{p f_{ly}}{0.8d} = \frac{0.324 \times 1.67}{0.8 \times 18} = 0.0376 \text{ in}^2/\text{in}$$

4. Second mode

Using the value of A_s/S calculated by first mode, find A_{v1} required.

$$\theta_2 = \frac{d}{2b + d} = \frac{18}{2 \times 12 + 18} = 0.43$$

$$\psi = \frac{Qb}{2T} = \frac{15.60 \times 12}{2 \times 323} = 0.291$$

Assume $A_{v1} = 1.18 \text{ in}^2$

$$p_2 = \frac{0.8 f_{sy} A_s b}{f_{ly} A_{v1} S} = \frac{0.8 \times 0.0376 \times 12}{1.18} = 0.306$$

$$c_2 = d \sqrt{\frac{1}{p_2 \theta_2} \frac{b}{d}} = 18 \sqrt{\frac{1}{0.306 \times 0.43} \frac{12}{18}} = 40.5 \text{ in}$$

$$c_{2(\max)} = 2b + d = 2 \times 12 + 18 = 42 \text{ in}$$

So $c_2 = 40.5$ is acceptable

$$\begin{aligned} x &= \frac{f_{ly} A_{v1} (d + p \theta_2 c_2^2 / b)}{f_c (c_2^2 + d^2)} \\ &= \frac{50,000 \times 1.18 (18 + 0.314 \times 0.43 \times 40.5^2 / 12)}{0.85 \times 3,000 (40.5^2 + 18^2)} = 0.42 \text{ in} \end{aligned}$$

$$b_j = 9.5 - 0.42/2 = 9.29 \text{ in}$$

$$\begin{aligned}
 A_{v1} &= \frac{T}{\phi_{cr} f_{ly} b_j} \frac{c_2}{d} \frac{1 + \psi}{1 + p_2 \theta_2 c_2^2 / bd} \\
 &= \frac{323,000}{0.85 \times 50,000 \times 9.29} \frac{40.5}{18} \frac{1 + 0.291}{1 + 0.306 \times 0.43 (40.5)^2 / 12 \times 18} \\
 &= 1.18 \text{ in}^2 \quad \text{O.K.}
 \end{aligned}$$

5. Total longitudinal reinforcement

$$\begin{aligned}
 A_{lt} &= 2.85 \text{ in}^2 \\
 &\quad (\text{Top: } 1.67 \text{ in}^2; \text{ Bottom: } 1.18 \text{ in}^2)
 \end{aligned}$$

Total web reinforcement

$$\frac{A_s}{S} = 0.0376 \text{ in}^2$$

6. Using the economical values of p and p_2 suggested by Lessig

$$\text{first mode } p = \frac{1}{1 + 2\sqrt{\theta_1}/4} \frac{d}{b} = \frac{1}{1 + 2\sqrt{0.25}/0.506} \frac{18}{12} = 0.505$$

$$\text{second mode } p_2 = 0.555$$

obtains

At distance "d" from support

$$A_{b1} = 1.49 \text{ in}^2; A_{v1} = 0.87 \text{ in}^2; A_{lt} = 2.36 \text{ in}^2$$

$$\frac{A_s}{S} = 0.052 \text{ in}^2/\text{in}$$

At $L/4$ from support

$$A_{b1} = 0.42 \text{ in}^2; A_{v1} = 0.42 \text{ in}^2; A_{lt} = 0.84 \text{ in}^2$$

$$\frac{A_s}{S} = 0.0325 \text{ in}^2/\text{in}$$

TABLE 2. SUMMARY OF MOMENTS, FORCES AND REINFORCEMENT AREAS

ITEM			UNITS	DISTANCE "d" FROM SUPPORT	AT L/4 FROM SUPPORT	
M_b			in-k	638	106	
$T_u - T_{ca}$			in-k	323	177.50	
$Q_u - Q_{ca}$			k	15.65	8.60	
SEPARATED METHOD	$A_s/S(\text{torsion})$		in^2/in	0.0505	0.0278	
	$A_s/S(\text{shear})$		in^2/in	-	-	
	$A_s/S(\text{total})$		in^2/in	0.0505	0.0278	
	$A_{lt}(\text{bending})$		in^2	0.98	0.16	
	$A_{lt}(\text{torsion})$		in^2	2.32	1.28	
	A_{lt}	total	in^2	3.30	1.44	
		top	in^2	2.14	0.80	
		bottom	in^2	1.16	0.64	
MATTOCK'S METHOD	$A_s/S(\text{torsion})$		in^2/in	0.0359	0.0132	
	$A_s/S(\text{shear})$		in^2/in	0.0051	-	
	$A_s/S(\text{total})$		in^2/in	0.041	0.0132	
	$A_{lt}(\text{bending})$		in^2	0.98	0.16	
	$A_{lt}(\text{torsion})$		in^2	1.66	0.62	
	A_{lt}	total	in^2	2.64	0.78	
		top	in^2	1.81	0.31	
		bottom	in^2	0.83	0.47	
LESSIG'S METHOD	*	$A_s/S(\text{total})$	in^2/in	0.0376	0.0214	
		A_{lt}	total	in^2	2.85	1.30
			top	in^2	1.67	0.65
			bottom	in^2	1.18	0.65
	**	$A_s/S(\text{total})$	in^2/in	0.052	0.0325	
		A_{lt}	total	in^2	2.36	0.84
			top	in^2	1.49	0.42
			bottom	in^2	0.87	0.42

* Use the same value of p as in Mattock's method.

** Use the value of p suggested by Lessig.

Table 3. Comparison of three different methods for designing a rectangular fixed ended beam, of width = 12", depth = 18", span = 20', with $f_c' = 3,000$ psi and $f_y = 50,000$ psi.

ITEM	SEPARATED METHOD		MATTOCK'S METHOD		LESSIG'S METHOD	
	DISTANCE "d" FROM SUPPORT	AT L/4 FROM SUPPORT	DISTANCE "d" FROM SUPPORT	AT L/4 FROM SUPPORT	DISTANCE "d" FROM SUPPORT	AT L/4 FROM SUPPORT
LONGITUDINAL REINFORCEMENT	AREA REQUIRED	2.14	0.80	1.81	0.31	0.65
	top					
	bottom	1.16	0.64	0.83	0.47	0.65
	STEEL	4-#7	2-#7	3-#7	2-#7	2-#7
	USED	2-#7	2-#7	2-#6	2-#6	2-#7
	ACTUAL	2.40	1.20	1.80	1.20	1.20
TRANSVERSE REINFORCEMENT	AREA REQUIRED $A_{g/s}$ in ² /in	0.0505	0.0278	0.041	0.0132	0.0214
	STIRRUPS USED	#4@4"	#4@7"	#4@4.5"	#3@8"	#3@5"
	ACTUAL AREA $A_{g/s}$ in ² /in	0.050	0.0285	0.0445	0.0138	0.022

CONCLUSIONS

1. The Rausch-Andersen-Cowan theory is based on elastic theory, while the other theories are based on the different mechanisms supported by their own beam tests.

2. The general equation for predicting the ultimate torque of rectangular reinforced concrete beams subjected to pure torsion is

$$T_u = T_c + K A_e A_s f_{sy} / S$$

in which

T_c , the torque taken by concrete, equals zero in Rausch's equation, and equals the torque taken by an unreinforced beam in Andersen's, Cowan's and Hsu's equations.

K equals 2.0, 1.6, $0.66 f_{ly} / f_{sy} + 0.33 d' / b'$ (varying from about 1.0 to 1.5) in Rausch's, Cowan's and Hsu's equations, respectively. Andersen did not determine a general equation for K .

m , the ratio of volume of longitudinal bars to volume of stirrups, should be unity in the Rausch-Andersen-Cowan theory, and should be within the range of 0.7 - 1.5 in Hsu's theory.

3. Insufficient experimental data are available to develop an accurate interaction surface formed by torsion, bending and shear. Lessig's theory is based on two assumed failure surfaces. The first mode indicates that the given beam section is subjected to torsion and bending moment while shear is absent or has a

relatively low value. The second mode indicates that the bending moment in the given beam section is relatively low compared to torsion and shear. Mattock's theory is based on the assumption that bending moment acts independently, while shear and torsion interact and form a circular arc interaction curve. Gesund's equation, which only holds for low values of the ratio of applied torque to applied moment, implies the conclusion that transverse reinforcement will transform torque on a reinforced concrete beam into additional bending moment.

4. The practical example of designing the required reinforcement for a rectangular beam subjected to torsion with flexure shows:

- 1). The required area of reinforcing steel calculated by the Separated method is greater than that calculated by other methods.
- 2). The resisting capacity of the pure concrete members is zero according to Lessig's method, while the pure concrete members can resist a certain amount of shear and torsion according to Mattock's method. The increasing rate of resisting capacity of the reinforced concrete members due to increasing reinforcement calculated by Lessig's method is greater than that calculated by Mattock's method.

ACKNOWLEDGMENTS

The writer wishes to express his sincere appreciation to his major advisor Professor Vernon H. Rosebraugh for his advice and guidance throughout the study.

APPENDICES

BIBLIOGRAPHY

1. C. K. Wang and C. G. Salmon, "Reinforced Concrete Design" P. 527-544. McGraw-Hill Book Company, Inc., New York.
2. J. P. Den Hartog, "Advanced Strength of Materials," McGraw-Hill Book Company, Inc., New York, 1952, P. 3-16.
3. Paul Andersen, "Experiments with Concrete in Torsion," Transactions, ASCE, Vol. 100, Paper No. 1912, 1935, P. 949-983.
4. H. J. Cowan, "Elastic Theory for Torsional Strength of Rectangular Reinforced Concrete Beams," Magazine of Concrete Research, Vol. 2, No. 4, London, England, July, 1950, P. 3-8.
5. H. J. Cowan, "Design of Beams Subjected to Torsion Related to the New Australian Code," Journal of the American Concrete Institute, Vol. 31, No. 7, Jan., 1960, P. 591-618.
6. N. N. Lessig, "Determination of the Load-Carrying Capacity of Reinforced Concrete Elements with Rectangular Cross-Section under Simultaneous Action of Flexure and Torsion," Beton i Zhelezobeton, Moscow, USSR, No. 3, 1959, Foreign Literature Study No. 348, Portland Cement Association, Research and Development Library, Skokie, Ill.
7. N. N. Lessig, "Determination of the Load-Bearing Capacity of Reinforced Concrete Elements with Rectangular Cross-Section Subjected to Flexure with Torsion," Beton i Zhelezobeton, Moscow, USSR, No. 5, 1959, Foreign Literature Study No. 371, Portland Cement Association, Research and Development Library, Skokie, Ill.
8. T. T. C. Hsu, "Ultimate Torque of Reinforced Rectangular

Beams," Journal of the Structural Division, ASCE, February, 1968, P. 485-510.

9. A. H. Mattock, "How to Design for Torsion," Symposium on Torsion, SP No. 18, American Concrete Institute, Detroit, 1968, P. 469-495.

10. Hans Gesund, Frederick J. Schuette, George R. Buchanan, and George A. Gray, "Ultimate Strength in Combined Bending and Torsion of Concrete Beams Containing Both Longitudinal and Transverse Reinforcement," Journal of the American Concrete Institute, V. 61, No. 12, Dec. 1964, P. 1509-1522.

11. "Building Code Requirements for Reinforced Concrete ACI 318-63," American Concrete Institute, Detroit, June, 1963.

NOTATIONS

- a depth of equivalent rectangular stress block; in in.
- ✓ A_e the effective cross sectional area of the section, defined as the area contained within the centre lines of the stirrup;
- ✓ A_{bl} the cross section area of all bottom longitudinal steel;
- ✓ A_{lt} the cross section area of total longitudinal steel;
- A_{vl} the cross section area of longitudinal steel located near one vertical side of the beam;
- A_s the cross section area of one stirrup leg;
- b width of rectangular beam, in in.;
- ✓ b' smaller center-to-center dimension of a closed rectangular stirrup, in in.;
- b'' smaller distance between the centers of longitudinal corner bars, in in.;
- b_1 $(b - b')/2$;
- b_2 $(b - b'')/2$;
- b_{pc} the distance from the center of shear compression zone to the axis of twist, in in.;
- b_{ps} the distance from the center of the longer legs of stirrups outside the shear-compression zone to the axis of twist, in in.;
- C Torsional stiffness;
- c_t M_a/T ;
- d total depth of rectangular beam, in in.;
- ✓ d' larger center to center dimension of a closed rectangular stirrup, in in.;

- d'' larger distance between the centers of longitudinal corner bars, in in.;
- d_o distance from extreme compression fiber to centroid of tension reinforcement, in in.;
- d_1 $(d - d')/2$;
- f_c compressive strength of concrete in flexure with torsion, can be assumed as $0.85f'_c$, in psi ;
- ✓ f'_c compressive strength of concrete, in psi ;
- f_{ly} yield strength of longitudinal bar, in psi ;
- ✓ f_{sy} yield strength of stirrup, in psi ;
- ✓ K coefficient, the slope of a T_u versus $bdA_S f_{sy}/S$ curve;
- k_a a coefficient relating the depth of the compression zone to the smaller dimension of the rectangular section;
- k_v a coefficient relating the shear strength in combined shear and compression to the pure shear strength;
- L length of the axis of rotation in failure surface, in in.;
- M_a bending moment induced in beam due to torque in in-lb;
- M_b longitudinal bending moment applied to a beam, in in-lb;
- M_f $M_u - M_a$;
- M_u $A_{bl} f_{ly} j d$;
- m ratio of volume of longitudinal bars to volume of stirrups; defined as $m = A_{lt} S / 2 [A_S (b' + d')] ;$
- P_a air pressure in membrane analogy;
- P_s shear force on the shear-compression zone, in lb;
- p $0.8 f_{sy} A_S d / f_{ly} A_{bl} S$;
- p_2 $0.8 f_{sy} A_S b / f_{ly} A_{vl} S$;

- Q_{ca} ultimate shear carried by concrete compression zone (combined shear and torsion), in lb.;
 Q_u total ultimate shear; in lb;
 S spacing of stirrups in the direction parallel to the longitudinal axis of beam, in in.;
 T_{ca} ultimate torsional moment carried by compression zone (combined torsion and shear), in in-lb;
 T_{mt} membrane tension;
 T_u total ultimate torsional moment, in in-lb;
 u the displacement in x direction;
 v the displacement in y direction;
 v_c shear stress due to shear carried by concrete at ultimate, when member is not subjected to torsion, in psi;
 $\checkmark v_{ca}$ shear stress due to shear carried by concrete at ultimate, when member is subject to torsion and shear, in psi;
 v_u nominal ultimate shear stress due to shear, $= Q_u/bd_o$, in psi;
 w the displacement in z direction;
 θ angle of twist, in radian per in.;
 θ' relative angle of twist of two surfaces constituting the failure crack, in radian;
 α an efficiency coefficient of longitudinal bar;
 β the angle between the crack on vertical face of the beam with respect to the vertical axis;
 γ τ_u/v_u ;
 λ τ_c/v_c
 τ_c shear stress due to torsion carried by concrete at ultimate, when member is subject to torsion only, in psi;

- τ_{ca} shear stress due to torsion carried by concrete at ultimate,
when member is subject to torsion and shear, in psi;
- τ_u nominal ultimate shear stress due to torsion, in psi;
- ϕ T/M_b ;
- ϕ_{cr} capacity reduction factor;
- Φ Saint-Venant's torsion stress function;
- ψ $Qb/2T$.

A STUDY IN TORSION OF RECTANGULAR REINFORCED CONCRETE BEAMS

by

PING-HSING PAI

B.S., Taiwan Provincial Paipei Institute of Technology, 1960

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Civil Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1970

The purpose of this report is to investigate rectangular reinforced concrete beams subjected to pure torsion and torsion combined with direct shear and bending moment in order to get a better understanding of this specific problem. The report can be abstracted as follow:

1. Torsional Stress and Torsional Stiffness of Homogeneous Section — Review Saint-Venant's Theory, Prandtl's Membrane Analogy, and torsional stress and torsional stiffness of rectangular sections.

2. Reinforced Concrete Beams Subjected to Pure Torsion — Review Rausch-Andersen-Cowan Theory, developed by Rausch in 1929 and later modified by Andersen and Cowan. Review the theory presented by Hsu in 1968

3. Rectangular Reinforced Concrete Beams under Combined Bending, Shear and Torsion — Review the theories presented by Lessig in 1959, by Mattock in 1968, by Gesund and others in 1964.

4. Practical Design Using Different Methods —

- A). Separated Method: Assume bending moment, torsional moment and shear acting independently. Design the reinforcement due to bending moment and shear according to ACI Code 318-63, and the reinforcement due to torsion by Hsu's Formula. Assume concrete does not take the shear due to torsion.
- B). Mattock's Method: Assumed bending moment acts independently, while shear and torsion interact, i.e., taking the interaction of shear and torsion into account in design.

Design the reinforcement due to bending moment according to ACI Code 318-63; calculate shear and torsional moment carried by the concrete by Mattock's equation; compute web reinforcement for shear by ACI Code 318-63, and reinforcement for torsion by Hsu's Formula.

- C). Lessig's Method: Design the reinforcement according to Lessig's theory, first mode of failure or second mode of failure.