SOLUTION OF LAPLACE'S EQUATION
BY OPERATIONAL METHODS

by

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INTRODUCTION

The study of operational techniques, especially as applied to the solution of differential equations, was initiated by the work of Oliver Heaviside (1850-1925). At the time of its appearance Heaviside's method invoked considerable criticism from mathematicians, who were appalled by the lack of rigor in his procedures. However, within a few years, several investigators, among them Carson and Bromwich, found that the operations of Heaviside's calculus could be systematically derived through the use of certain integral equations. The subject of the operational calculus has now become a study of the properties and manipulations of these integral equations. The Laplace transform represents one such integral which is applied extensively in deriving the characteristics of the operational calculus and several other fields of mathematics and engineering.

There is today a great amount of work done on and with transformations of functions. The result of a transformation is usually called a "transform". The first part of the report deals with the Laplace transformation. Details about definitions, existence, and inverse transformation are explained. Various properties of the Laplace transform are discussed in the beginning. This is one of the most interesting and useful parts of the study of the transform method, and each property is illustrated with examples. Included here are discussions of partial differential equations, periodic functions, and many other special applications of the transform method.

The partial differential equations appearing most frequently in engineering are of the second order. Here a partial differential equation of the second order, in electrostatics, is derived. This equation is known as Laplace's equation, $\nabla^2\phi = 0$, which belongs to the elliptic type of partial differential equations. Another equation, hyperbolic type, $\nabla^2\phi = 1/c^2 \dfrac{\partial^2\phi}{\partial t^2}$, has been derived. The Laplace transform is used to solve each for specific boundary conditions. The method follows the following steps in the solution of Laplace's equations and wave equations:

1. Apply the transform twice to the differential equation. The first application will result in an ordinary differential equation, and the second transformation will yield an algebraic expression.

2. The algebraic expression is solved for the second transform.

3. The inverse transformation is performed twice to give the desired solution.

This report is concerned with the development of a few of the properties of the Laplace transformation and the application of the Laplace transformation to the solution of simple Laplace's equation and wave equations.

A Table of Operations and a Table of Transforms are included at the end of the work.

LAPLACE TRANSFORMATION

Definitions

The Laplace transformation is a mathematical operation indicated symbolically by $L\left[f(t)\right]$ and the operation is indicated by the equation (1).

$$L\left[f(t)\right] \equiv \int_0^{\infty} f(t)\ e^{-st}\ dt \equiv F(s) \tag{1}$$

For the present, s is assumed to be a real variable. The restrictions on the character of the function f(t) and on the range of the variable s, such that the integral in equation (1) converges are discussed. In the Laplace transformation equation, f(t) is usually called the "object" function and F(s) is called the "result" function. Equation (1) usually is written in the form

$$F(s) = L\left[f(t)\right] \tag{2}$$

where L denotes "the Laplace transformation of".

Inverse Transformation

The transformation back to the time domain may be accomplished by evaluating

$$f(t) = \frac{1}{2\pi j} \oint F(s)e^{st}\ ds \tag{3}$$

where $\oint$ denotes integration about a closed contour, or symbolically

$$f(t) = L^{-1}F(s) \tag{4}$$

where L^{-1} denotes the inverse transformation process.

The Laplace inversion formula (3) can be derived from the definition of f(t):

$$L\left[f(t)\right] = F(s) = \int_0^\infty f(t)e^{-st}\,dt \tag{5}$$

where $s = \sigma + jw$.

For convergence it is required that the real part of s be greater than some number σ_1:

$$\sigma > \sigma_1 \tag{6}$$

where σ_1 is such that the integral

$$\int_0^\infty \left|f(t)\right| e^{-\sigma_1 t}\,dt \quad \text{exists} \tag{7}$$

Now let us multiply both sides of equation (5) by $\dfrac{e^{st}}{2\pi j}$ and integrate:

$$\frac{1}{2\pi j}\int_{\sigma_2-j\infty}^{\sigma_2+j\infty} F(s)e^{st}ds = \frac{1}{2\pi j}\int_{\sigma_2-j\infty}^{\sigma_2+j\infty} e^{st}\,ds \int_0^\infty f(\tau)\,d\tau\, e^{-s\tau} \tag{8}$$

where $\sigma_2 > \sigma_1$ to ensure convergence. Interchanging the order of integration, results in

$$\frac{1}{2\pi j}\int_{\sigma_2-j\infty}^{\sigma_2+j\infty} F(s)\,e^{st}ds = \int_0^\infty f(\tau)d\tau\,\frac{1}{2\pi j}\int_{\sigma_2-j\infty}^{\sigma_2+j\infty} e^{s(t-\tau)}\,ds$$

$$= \int_0^\infty f(\tau)\,\delta(t-\tau)d\tau$$

$$\frac{1}{2\pi j}\int_{\sigma_2-j\infty}^{\sigma_2+j\infty} F(s)\,e^{st}\,ds = f(t) \tag{9}$$

which proves the inversion formula.

Existence of Laplace Transforms

The object function $f(t)$ is said to be of "exponential order" as t becomes infinite provided some constant α exists such that the product $e^{-\alpha t}|f(t)|$ is bounded for all t greater than some finite number T. Thus $|f(t)| < Me^{\alpha t}$ for $t > T$, where M is some constant.

The Laplace transform of a function $f(t)$ exists if $f(t)$ is sectionally continuous in every finite interval in the range $t \geqq 0$ and if $f(t)$ is of exponential order as $t \longrightarrow \infty$. This follows from the fact that the integrand of the Laplace transformation is integrable over the finite interval $0 \leqq t \leqq T$ for every positive number T, and the inequality (10) holds for some constant M.

$$\left|e^{-st} f(t)\right| < Me^{-(s-\alpha)t} \tag{10}$$

If $s > \alpha$ the integral from zero to infinity of the function on the right of the inequality in equation (10) exists. Thus the restrictions placed on $f(t)$ and s for the existence of a transform of the object function $f(t)$ have been established.

In short, the function $f(t)$ may be Laplace transformed if the following integral exists:

$$\int_0^{\infty} e^{-\alpha t} f(t)\, dt \tag{11}$$

where α is finite real number. This is equivalent to saying that the following must be true;

$$\lim_{t \to \infty} e^{-\alpha t} f(t) = 0 \tag{12}$$

OPERATIONAL PROPERTIES OF LAPLACE TRANSFORMATION

The usefulness of the Laplace transformation in the solution of Laplace's equation and the wave equations is based on some important theorems which follow as a consequence of the fundamental equation (1). In this section interest will be centered on those relations which are most applicable in this work. The following objectives will be kept in mind:

1. Obtaining the transforms of functions, which are met with frequently in Laplace's equation and the wave equations.

2. Establishing operations and relations by which may be extended a table of transforms and its utility.

3. Developing certain operation properties of the Laplace transform which will be applied to the solution of Laplace's equation and the wave equations.

Linearity Theorem

The Laplace transformation is a linear transformation for which superposition holds. For instance, when a function of time is multiplied by a constant C, the transform is multiplied by the same constant.

$$L\, C\left[f(t)\right] = C\, F(s) \tag{13}$$

Similarly, two time functions may be added as follows:

$$L\left[f_1(t) + f_2(t)\right] = F_1(s) + F_2(s) \tag{14}$$

Shifting Theorems

In considering the frequency of occurrence of exponential functions, the following theorems seem to be of great importance.

Theorem 1. If $F(s)$ is the Laplace transform of $f(t)$, then $F(s + a)$ is the Laplace transform of $e^{-at}f(t)$. This indicates that the substitution of $(s + a)$ for the variable s in the transform $F(s)$ of $f(t)$ corresponds to the multiplication of the object function $f(t)$ by e^{-at}, that is

$$F(s + a) = L\left[e^{-at} f(t)\right] \tag{15}$$

The proof of this theorem follows immediately from the definition of the Laplace transform (1).

$$\begin{aligned} L\left[e^{-at} f(t)\right] &= \int_0^\infty e^{-st} \left[e^{-at} f(t)\right] dt \\ &= \int_0^\infty e^{-(s+a)} f(t)\, dt = F(s + a) \end{aligned} \tag{16}$$

Theorem 2. If $F(s) = Lf(t)$, then for any positive constant a, $L\left[f_a(t)\right] = e^{-as} F(s)$, where $\left[f_a(t)\right] = 0$ when $0 < t < a$, and $\left[f_a(t)\right] = f(t - a)$ when $t > a$.

Let us find

$$L\left[f_a(t)\right] = \int_0^\infty f(t - a)\, e^{-st}\, dt \tag{17}$$

Make a change of variable, letting $t - a = x$, becomes

$$\begin{aligned} L\left[f_a(t)\right] &= \int_0^\infty f(x)\, e^{-s(a+x)}\, dx \\ &= e^{-as} \int_0^\infty f(x)\, e^{-sx}\, dx \end{aligned}$$

$$L\left[f(t - a)\right] = e^{-as} F(s) \qquad a \geqslant 0 \tag{18}$$

where F(s) is the transform of the time function without delay, and f(t - a) indicates that the function is zero until t = a.

Theorem 3. If f(t) is periodic with the period T and is sectionally continuous over this period, then

$$F(s) = \frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} f(t)\, dt \tag{19}$$

If any function f(t) is periodic of period T, then by definition f(t) = f(t + T), so that

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$F(s) = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t)\, dt \tag{20}$$

now let $\quad x = t - nT$

then $\quad t = x + nT$

and $\quad f(t) = f(x + nT)$

$\quad f(t) = f(x)$

These substituted in equation (20)

$$F(s) = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-s(x+nT)} f(x)\, dx$$

$$= \sum_{n=0}^{\infty} e^{-snT} \int_0^T e^{-sx} f(x)\, dx$$

$$F(s) = \frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} f(t)\, dt \tag{21}$$

$$F(s) = \frac{1}{1 - e^{-Ts}} F_1(s)$$

where $F_1(s) = \int_0^T e^{-st} f(t)\, dt$.

Change of Scale

If $L\left[f(t)\right] = F(s)$, then $L\left[f(at)\right] = \frac{1}{a} F(\frac{s}{a})$

Let us find

$$L\left[f(at)\right] = \int_0^{\infty} f(at)\, e^{-st}\, dt$$

$$= \frac{1}{a} \int_0^{\infty} f(at)\, e^{-(s/a)at}\, d(at)$$

$$= \frac{1}{a} \int_0^{\infty} e^{-(s/a)x}\, f(x)\, dx \qquad (22)$$

The integral is the Laplace transform of $f(x)$ with s replaced at s/a.

$$L\left[f(at)\right] = \frac{1}{a}\; L\left[f(t)\right]$$

If $L\left[f(t)\right] = F(s)$, then from the above equation

$$L\left[f(at)\right] = \frac{1}{a} F(s/a) \qquad (23)$$

Transforms of Real and Imaginary Parts

If a time function $z(t)$ is complex, it can be written

$$z(t) = x(t) + jy(t) \qquad (24)$$

where $x(t)$ is the real part of $z(t)$ and $y(t)$ is the imaginary part of $z(t)$, which may be denoted by

$$x(t) = \mathrm{Re}\left[z(t)\right] \qquad (25)$$

$$y(t) = \mathrm{Im}\left[z(t)\right] \qquad (26)$$

Now the transform of $z(t)$ is

$$L\left[z(t)\right] = \int_0^{\infty} z(t)\ e^{-st}\ dt$$

$$= \int_0^{\infty} x(t)\ e^{-st}\ dt + j\int_0^{\infty} y(t)\ e^{-st}\ dt \quad (27)$$

from these

$$\text{ReL}\left[z(t)\right] = L\left[x(t)\right] = L\left\{\text{Re}\left[z(t)\right]\right\} \quad (28)$$

and

$$\text{ImL}\left[z(t)\right] = L\left[y(t)\right] = L\left\{\text{Im}\left[z(t)\right]\right\} \quad (29)$$

Transformation of Derivatives

The Laplace transform of the derivative of a function is given by

$$L\left[\frac{d}{dt} f(t)\right] = s\ F(s) - f(0+) \quad (30)$$

where s = transform variable

$F(s)$ = Laplace transform of $f(t)$

$f(0+)$ = initial value of $f(t)$ evaluated as $t \longrightarrow 0$ from positive values.

Let the chosen time function be

$$\frac{d}{dt}\left[f(t)\right] = f'(t)$$

then

$$L\left[f'(t)\right] = \int_0^{\infty} f'(t)\ e^{-st}\ dt \quad (31)$$

Using the form for integrating by parts, let

$$f(t) = u \quad \text{and} \quad e^{-st}\ dt = dv$$

then

$f'(t)dt = du$ and $e^{-st}/-s = v$

$$\int_0^\infty f(t)\ e^{-st}dt = f(t)\ e^{-st}/-s \Big|_0^\infty - 1/-s \int_0^\infty e^{-st}\ f'(t)dt$$

from which

$$\int_0^\infty f'(t)\ e^{-st}dt = s\left[\int_0^\infty f(t)\ e^{-st}dt - \frac{f(t)\ e^{-st}}{-s}\Big|_0\right]$$

$$L\left[\frac{df(t)}{dt}\right] = s\left[F(s) + \frac{0}{s} - \frac{f(0+)}{s}\right]$$

therefore

$$L\left[f'(t)\right] = s\ F(s) - f(0+)$$

Similarly, the general expression for an n^{th} order derivative is

$$L\left[\frac{d^n f(t)}{dt^n}\right] = s^n\ F(s) - s^{n-1}f(0+) - s^{n-2}\ \frac{df}{dt}\ (0+)$$

$$- \ldots - \frac{d^{n-1}}{dt^{n-1}}\ f(0+) \qquad (32)$$

Transforms of Integrals

The Laplace transform of the integral of a function is given by

$$L\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0+)}{s} \qquad (33)$$

where $f^{-1}(0+) = \int f(t)\ dt$, evaluated as $t \longrightarrow \infty$ from positive values.

Let the chosen time function be $\int f(t)dt$.

$$L\left[\int f(t)dt\right] = \int_0^{\infty} e^{-st}\left[\int f(t)\ dt\right]dt \tag{34}$$

Using forms for integrating by parts, let

$$\int f(t)\,dt = u \quad \text{and} \quad e^{-st}dt = dv$$

then $f(t)dt = du$ and $e^{-st}/-s = v$. Therefore

$$\int_0^{\infty}\left[\int f(t)dt\right]e^{-st}dt = \int f(t)dt\ \frac{e^{-st}}{-s}\Bigg|_0^{\infty} - \int_0^{\infty}\frac{e^{-st}}{-s}\ f(t)dt$$

$$= \left[0 + \int_{+}\frac{f(t)dt}{s}\right] + \frac{1}{s}\int_0^{\infty} f(t)e^{-st}\ dt$$

$$L\left[f(t)dt\right] = \frac{f^{-1}(0+)}{s} + \frac{F(s)}{s} \tag{33}$$

Convolution Integral

If two time functions $f_1(t)$ and $f_2(t)$ are given, then they may be combined in the following way to give a new time function $g(t)$ which is called the convolution of $f_1(t)$ and $f_2(t)$, often written $f_1(t) * f_2(t)$.

$$g(t) = f_1(t) * f_2(t) = \int_0^t f_1(\tau)\ f_2(t - \tau)d\tau \tag{35}$$

By putting $\lambda = t - \lambda$, and noting that a definite integral is a function of its limits, and not of the variable of integration, it can be seen that the convolution integral may be written in either of two ways:

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau)\ f_2(t - \tau)d\tau$$

$$= \int_0^t f_1(t - \tau)\ f_2(\tau)d\tau \tag{36}$$

The importance of this operation to us will be apparent from the following theorem concerning the product of two transforms.

Transforms of Product

If $y_1(t)$ and $y_2(t)$ have Laplace transforms $F_1(s)$ and $F_2(s)$ respectively, then the inverse transform of the product $F_1(s) \cdot F_2(s)$ is given by the convolution of $y_1(t)$ and $y_2(t)$. Therefore

$$L\left[y_1(t) * y_2(t)\right] = F_1(s) \cdot F_2(s) \tag{37}$$

Alternatively, the transform of the convolution of two functions is equal to the product of their transforms.

A formal derivation is given below.

$$\begin{aligned} L\left[y_1(t) * y_2(t)\right] &= L\left[\int_0^t y_1(t - \tau)y_2(\tau)d\tau\right] \\ &= L\left[\int_0^t y_1(t - \tau)y_2(\tau)H(t - \tau)\right]d\tau \\ &= \int_0^\infty e^{-st}\left[y_1(t - \tau)y_2(\tau)H(t - \tau)d\tau\right]dt \\ &= \int_0^\infty y_2(\tau)\left[y_1(t - \tau)H(t - \tau)e^{-st}\ dt\right]d\tau \end{aligned}$$

Putting $t - \tau = \lambda$,

$$L\left[y_1(t) * y_2(t)\right] = \int_0^\infty y_2(\tau)\left[\int_{-\tau}^\infty y_1(\lambda)H(\lambda)e^{-s\lambda}e^{-s\tau}d\lambda\right]d\tau$$

$$= \int_0^\infty y_2(\tau)e^{-st}\left[\int_0^\infty y_1(\lambda)e^{-s\lambda}\,d\lambda\right]d\tau$$

$$= \int_0^\infty y_2(\tau)e^{-s\tau}\left[F_1(s)\right]\,d\tau$$

$$= F_1(s)\int_0^\infty y_2(\tau)\;e^{-s\tau}\,d\tau$$

$$L\left[y_1(t) * y_2(t)\right] = F_1(s)\;F_2(s) \qquad (37)$$

LAPLACE'S EQUATION AND WAVE EQUATION DERIVATIONS

The partial differential equations appearing most frequently in engineering are of the second order. These could be grouped into three main types, elliptic, hyperbolic, and parabolic. Laplace's equation, $\nabla^2\phi = 0$, belongs to the elliptic class. The wave equation, $\nabla^2\phi = (1/c^2)(\partial^2u/\partial t^2)$, belongs to the hyperbolic class. Heat conduction equations belong to parabolic class. These types of equations arising in physical situations are derived here and their solutions are done by using Laplace transformation in the last section of this report.

Laplace's Equation

As a first example let us consider a simple partial differential equation of the second order, arising in electrostatics. By Gauss' law of electrostatics we know that the flux of the electric vector E out of a surface s bounding an arbitrary volume V is 4π times the charge contained in V. Thus if ρ is the density of electric charge, then

$$\int_s E \cdot ds = 4\pi \int_V \rho \, d\tau \tag{38}$$

Using Green's theorem in the form

$$\int_s E \cdot ds = \int_V \text{div } E \, d\tau \tag{39}$$

and remembering that the volume V is arbitrary, we see that Gauss' law is equivalent to the equation

$$\text{div } E = 4 \pi \rho \tag{40}$$

Now it is readily shown that the electrostatic field is characterized by the fact that the vector E is derivable from a potential function ϕ by the equation

$$E = -\text{grad } \phi \tag{41}$$

Eliminating E between equations (40) and (41), we find that ϕ satisfies the equation

$$\nabla^2\phi + 4 \pi \rho = 0 \tag{42}$$

Equation (42) is known as Poisson's equation. In the absence of charges, ρ is zero, and equation (42) reduces to the simple form

$$\nabla^2\phi = 0 \tag{43}$$

This equation is known as Laplace's equation. Where ∇^2 is the Laplacian operator, which in cartesian coordinates takes the form

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{44}$$

If we are dealing with a problem in which the potential function $\emptyset$ does not vary with z, we then find that ∇^2 is replaced by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{45}$$

and that Laplace's equation becomes

$$\nabla^2{}_1 = 0 \tag{46}$$

a form which we shall refer to as the two-dimensional Laplace equation.

Wave Equation

As a second example we consider the flow of electricity in a long insulated cable. We shall suppose that the flow is one-dimensional so that the current i and the voltage $\emptyset$ at any point in the cable can be completely specified by one spatial coordinate x and a time variable t. If we consider the fall of potential in a linear element of length Δx situated at the point x, we find that

$$-\Delta\emptyset = i\, R\Delta x + L\Delta x \frac{\partial i}{\partial t} \tag{47}$$

where R is the series resistance per unit length and L is the inductance per unit length. If there is a capacitance to ground of C per unit length and a conductance G per unit length, then

$$-\Delta i = G\emptyset\Delta x + C\Delta x \frac{\partial \emptyset}{\partial t} \tag{48}$$

The relations (47) and (48) are equivalent to the pair of partial differential equations

$$\frac{\partial \phi}{\partial x} + R\, i + L \frac{\partial i}{\partial t} = 0 \tag{49}$$

$$\frac{\partial i}{\partial x} + G\, \phi + C \frac{\partial \phi}{\partial t} = 0 \tag{50}$$

Differentiating equation (49) with respect to x, we obtain

$$\frac{\partial^2 \phi}{\partial x^2} + R \frac{\partial i}{\partial x} + L \frac{\partial^2 i}{\partial x \partial t} = 0 \tag{51}$$

and similarly differentiating equation (50) with respect to t, obtains

$$\frac{\partial^2 i}{\partial x \partial t} + G \frac{\partial \phi}{\partial t} + C \frac{\partial^2 \phi}{\partial t^2} = 0 \tag{52}$$

Eliminating $\partial i/\partial x$ and $\partial^2 i/\partial x \partial t$ from equations (50), (51), and (52), it is found that ϕ satisfies the second-order partial differential equation

$$\frac{\partial^2 \phi}{\partial x^2} = LC \frac{\partial^2 \phi}{\partial t^2} + (RC + LG) \frac{\partial \phi}{\partial t} + RG\phi \tag{53}$$

Similarly, if we differentiate (49) with respect to t, (50) with respect to x, and eliminate $\partial^2 \phi/\partial x \partial t$ and $\partial \phi/\partial x$ from the resulting equations and equation (49), it is found that i is also a solution of equation (53).

Equation (53), which is called the telegraphy equation, reduces to a simple form in two special cases. If the leakage to ground is small, so that G and L may be taken to be zero, equation (53), reduces to the form

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{k} \frac{\partial \phi}{\partial t} \tag{54}$$

where $k = 1/RC$ is a constant. This equation is known as the one-dimensional diffusion equation.

When dealing with high-frequency phenomena in a cable, the terms involving the time derivations predominate. From equation (49) and equation (50) it is seen that this is equivalent to taking G and R to be zero in equation (53), in which case it reduces to

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} \tag{55}$$

where $c = \sqrt{1/LC}$.

The equation (55) is known as the one-dimensional wave equation.

GENERAL PRINCIPLE IN THE APPLICATION OF THE LAPLACE TRANSFORMATION TO PARTIAL DIFFERENTIAL EQUATIONS

The Laplace transformation is a very powerful tool for the solution of transient problems. It is also useful in many other problems involving solution of linear differential equations. Here it will be shown how the Laplace method can be applied to partial differential equations.

In a partial differential equation the unknown is a function of two or more independent variables. In case of two variables x and t, the unknown function is denoted by $u(x, t)$. In a partial differential equation a certain domain in the xt-plane is given at the outset, and it is within this domain that the unknown is to be determined. In the determination of the solution of partial differential equations, the boundary conditions and initial conditions are essential, so that a unique solution can be found.

When a partial differential equation is to be solved, the Laplace transform may be used for one variable, leaving an ordinary or partial differential equation to be solved by other means for the other variables, or the inverse transformation process may be employed sequentially so that solutions in all independent variables are obtained by transformation.

In general, the following steps are essential to solve partial differential equations by using the Laplace transformation technique.

First apply the Laplace transformation to the function u(x, t) and to the derivatives which occur. Because the transformation represents an integration with respect to a single variable, so it is essential to undertake the Laplace transformation relative to one of the variables in u, while the other is not involved. If the function u(x, t) transforms with respect to t, then by application of Laplace transformation definition, the transform form will be

$$L_t\left[u(x,\ t)\right] = \int_0^{\infty} e^{-st}\, u(x,\ t)\,dt$$

$$L_t\left[u(x,\ t)\right] \doteq U(x,\ s) \qquad (56)$$

where L_t denotes the Laplace transform with respect to t for a given function u(x, t).

By using the transforms of derivatives theorem $\partial u(x,\ t)/\partial t$ can be obtained by considering x constant.

$$L_t\left[\frac{\partial u(x,t)}{\partial t}\right] = sU(x,\ s) - u(x,\ 0^+) \tag{57}$$

and

$$L_t\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = s^2U(x,\ s) - su(x,\ 0^+) - u_t(x,\ 0^+) \tag{58}$$

where $u_t = \dfrac{\partial u}{\partial t}$.

To use this method for derivations with respect to x it must be assumed that the order of the differentiation and the integration in the Laplace integral can be interchanged. As an example,

$$\begin{aligned} L_t\left[\frac{\partial u(x,\ t)}{\partial x}\right] &= \frac{\partial}{\partial x}\int_0^\infty e^{-st}\,u(x,\ t)dt \\ &= \frac{\partial U(x,\ s)}{\partial x} \end{aligned} \tag{59}$$

and

$$\begin{aligned} L_t\left[\frac{\partial^2 u(x,\ t)}{\partial x\,\partial t}\right] &= \frac{\partial}{\partial x}\,L_t\left[\frac{\partial u(x,\ t)}{\partial t}\right] \\ &= \frac{\partial}{\partial x}\left[s\,U(x,\ s) - u(x,\ 0^+)\right] \end{aligned} \tag{60}$$

From these equations it is seen that the values $u(x,\ 0^+)$, $\partial u/\partial t\ (x,\ 0^+)$ are required when derivatives are transformed. These values may be obtained from given initial conditions and the given equation can be transformed by the Differentiation theorem, incorporating initial conditions to give an ordinary differential equation in the transform $U(x,\ s)$. Then the transformed equation can be solved for $U(x,\ s)$ by introducing the boundary conditions. As an example, consider a function $u(x, t) = A(t)$ having boundary conditions at $x = a$. Then,

$$u(a, t) = A(t)$$

i.e.,

$$\lim_{x \to a+} u(x, t) = A(t)$$

If it is assumed that the limit $x \to a+$ is interchangeable with the Laplace integral, then

$$\begin{aligned} \lim_{x \longrightarrow a+} U(x, s) &= \lim_{x \longrightarrow a+} L_t\left[u(x, t)\right] \\ &= L_t\left[\lim_{x \longrightarrow a+} u(x, t)\right] = L_t\left[A(t)\right] \qquad (61) \end{aligned}$$

and therefore

$$\lim_{x \longrightarrow a+} U(x, s) = A(s) \qquad (62)$$

Equations (61) and (62) indicate that the transform of the boundary value $A(t)$ is taken as the boundary value of the transform $U(x, s)$ at $x = a$.

It is seen that the transform of a partial differential equation has one less independent variable than the original. Equations with more than two independent variables (e.g., time and three space coordinates) can be reduced to ordinary differential equations by repeated transformation. It is important when successive transformations are made to choose a new transform variable each time. Choice of the order in which independent variables are transformed generally depends on the nature of the problem.

ILLUSTRATIVE PROBLEMS

Problem 1. Laplace's Equation

As an illustration of Laplace's equation, consider the electric potential distribution in a metal strip of uniform thickness and of constant width a, Fig. 1. The strip extends from $y = 0$ to $y \rightarrow \infty$. The sides are kept at ground potential, while an arbitrary distribution $\phi = f(x)$ is applied at the end $y = 0$. The thickness 1 of the strip does not affect the potential distribution. Thus Laplace's equation is

$$\nabla^2\phi_1 = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{63}$$

with the boundary conditions

1. $(0, y) = 0$
2. $(a, y) = 0$
3. $(x, \infty) = 0$
4. $(x, 0) = f(x)$

Now transform the equation (63) with respect to x. By an application of equation (58), the transform will be:

$$L_x\left[\frac{\partial^2\phi(x, y)}{\partial x^2}\right] + L_x\left[\frac{\partial^2\phi(x, y)}{\partial y^2}\right] = 0$$

$$\frac{d^2\,\Phi(s, y)}{dx^2} = -\left[s^2\,\Phi(s, y) - s\phi(x, 0) - \phi_x(0, y)\right] \tag{64}$$

where $\phi_x = \partial\phi/\partial x$.

By substituting the first boundary conditions into equation (64), it becomes

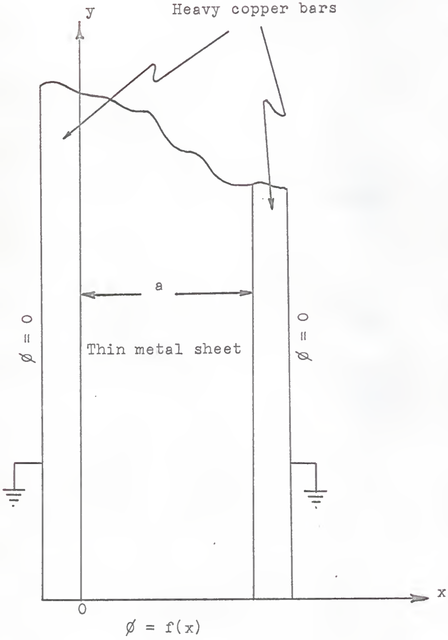

Fig. 1. Thin sheet of metal.

$$\frac{d^2\, \bar{\Phi}(s, y)}{dx^2} = -s^2\, \bar{\Phi}(s, y) + \phi_x(0, y) \tag{65}$$

Now transform equation (65) again, letting

$$L_y\left[\bar{\Phi}(s, y)\right] = \bar{\bar{\Phi}}(s, p)$$

then

$$L_y\left[\frac{d^2\, \phi(s, y)}{dx^2}\right] = p^2\, \bar{\bar{\Phi}}(s, p) - p\, \bar{\Phi}(s, 0) - \bar{\Phi}_y(s, 0)$$

and substituting the value of $L_y\left[d^2\, \bar{\Phi}(s, y)/dx^2\right]$ in equation (65),

$$p^2\bar{\bar{\Phi}}(s, p) - p\bar{\Phi}(s, 0) - \bar{\Phi}_y(s, 0) + s^2\bar{\bar{\Phi}}(s, p) = \phi_x(0, y) \tag{66}$$

then

$$\bar{\bar{\Phi}}(s, p) = \frac{p\, \bar{\bar{\Phi}}(s, 0) + \bar{\Phi}_y(s, 0) + \phi_x(0, y)}{(p^2 + s^2)} \tag{67}$$

By application of equation (62), the transform of boundary conditions (3) and (4) will be:

$$L_x\left[\phi(x, \infty)\right] = \bar{\Phi}(s, \infty) = 0$$

and $\quad L_x\left[\phi(x, 0)\right] = \bar{\Phi}(s, 0) = F(s)$

By substituting the values of $\bar{\Phi}(s, \infty)$ and $\bar{\Phi}(s, 0)$ in the equation (67) becomes

$$\bar{\bar{\Phi}}(s, p) = \frac{pF(s) + \bar{\Phi}_y(s, 0) + \phi_x(0, y)}{(p^2 + s^2)} \tag{68}$$

for this case, the partial fraction expansion is

$$\bar{\bar{\Phi}}(s, p) = \frac{A}{(p + js)} + \frac{A^*}{(p - js)} \tag{69}$$

where A^* is the complex conjugate of A. In other words, when the roots are conjugates, so are the partial fraction expansion coefficients, which may be evaluated as follows:

$$A\Big|_{p = -js} = \frac{-jsF(s) + \bar{\Phi}_y(s, 0) + \phi_x(0, y)}{-2js}$$

therefore

$$A = \frac{F(s)}{2} - \frac{\bar{\Phi}_y(s, 0)}{j2s} - \frac{\phi_x(0, y)}{j2s} \tag{70}$$

and

$$A^*\Big|_{p = +js} = \frac{jsF(s) + \bar{\Phi}_y(s, 0) + \phi_x(0, y)}{j2s}$$

therefore

$$A^* = \frac{F(s)}{2} + \frac{\bar{\Phi}_y(s, 0)}{j2s} + \frac{\phi_x(0, y)}{j2s} \tag{71}$$

Substituting value of A and A* in equation (69),

$$\bar{\Phi}(s,p) = \frac{\dfrac{F(s)}{2} - \dfrac{\bar{\Phi}_y(s,0)}{j2s} - \dfrac{\phi_x(0,y)}{j2s}}{(p + js)} + \frac{\dfrac{F(s)}{2} + \dfrac{\bar{\Phi}_y(s,0)}{j2s} + \dfrac{\phi_x(0,y)}{j2s}}{(p - js)} \tag{72}$$

then the inverse Laplace transformation of equation (72),

$$\bar{\Phi}(s,y) = \left[\frac{F(s)}{2} - \frac{\bar{\Phi}_y(s,0)}{j2s} - \frac{\phi_x(0,y)}{j2s}\right] e^{-jsy} + \left[\frac{F(s)}{2} + \frac{\bar{\Phi}_y(s,0)}{j2s} + \frac{\phi_x(0,y)}{j2s}\right] e^{+jsy} \tag{73}$$

Now from the boundary condition (3), $\phi(x, \infty) = 0$ and application of equation (62),

$$L_x\left[\phi(x, \infty)\right] = \bar{\Phi}(s, \infty) = 0$$

As $\bar{\Phi}(s, \infty) = 0$, so the coefficient of the second term of the equation (73) becomes zero; i.e.,

$$\frac{F(s)}{2} + \frac{\bar{\Phi}_y(s, 0)}{j2s} + \frac{\phi_x(0, y)}{j2s} = 0$$

Then
$$\frac{F(s)}{2} = - \frac{1}{j2s}\left[\Phi_y(s,\ 0) + \phi_x(0,\ y)\right] \tag{74}$$

By substituting the value of F(s)/2 in equation (73),

$$\phi(s,\ y) = \left[\frac{F(s)}{2} + \frac{F(s)}{2}\right] e^{-jys} = F(s)\ e^{-jys} \tag{75}$$

The inverse Laplace transformation of $L_x(s,\ y)$ by using shifting theorem (2),

$$\phi(x,\ y) = L_x^{-1}\ F(s)\ e^{-jys}$$

$$\phi(x,\ y) = f(x - jy)u(x - jy) \tag{76}$$

Also Laplace's equation (63) can be solved by the classical method known as separation of variables. We now assume that $\phi(x,\ y)$ is expressible as the product of two quantities X and Y,

$$\phi(x,\ y) = X_{(x)}\ Y_{(y)} \tag{77}$$

where X is a function of x alone and Y is a function of y alone.

Substituting equation (77) into equation (63) results in

$$X''Y + XY'' = 0$$

or
$$\frac{X''}{X} = - \frac{Y''}{Y}$$

where the primes on the functions X and Y represent differentiation with respect to the only variable present. Now the left-hand side of this equation is independent of y, and the right-hand side is independent of x; therefore their value is constant, say λ. Thus

$$X'' - \lambda X = 0$$

and
$$Y'' + \lambda Y = 0$$

We shall assume that $\lambda = -\beta^2 < 0$ and show that a negative value of λ allows the satisfaction of all the boundary conditions. With $\lambda = -\beta^2$, the general solution of

$$X'' + \beta^2 X = 0$$

is $$X_{(x)} = A\cos(\beta x) + B\sin(\beta x)$$

From the first boundary condition

$$\phi(0, y) = X_{(0)}\, Y_{(y)} = 0$$

and since $Y_{(y)} \neq 0$, it is found that

$$X_{(0)} = 0, \quad \text{or} \quad X_{(0)} = A = 0$$

Similarly from the second boundary condition

$$\phi(a, y) = 0 = X_{(a)}\, Y_{(y)}$$

implies

$$X_{(a)} = 0 \quad \text{or} \quad X_{(a)} = B\sin(\beta a) = 0$$

Since $B = 0$ would imply $\phi(x, y) \equiv 0$, one must have $\sin(\beta a) = 0$, or $\beta a = n\pi$, $\beta = n\pi/a$, $n = 1, 2, \ldots$. With these values of β,

$$X_{n(x)} = B_n \sin(n\pi/a)x$$

The solution of the equation for $Y_{(y)}$ yields

$$Y_{n(y)} = C_n e^{-(n\pi/a)y} + D_n e^{+(n\pi/a)y}$$

From the third boundary condition

$$\phi(x, \infty) = X_{(x)}\, Y_{(\infty)} = 0$$

and $X_{(x)} \not\equiv 0$, $Y_{(\infty)}$ must vanish.

$$Y_{(\infty)} = C_n e^{-(n\pi/a)\infty} + D_n e^{(n\pi/a)\infty} = 0$$

implies

$$Y_{(\infty)} = 0 \quad \text{or} \quad D_n = 0$$

and the functions

$$\phi_n(x, y) = X_{n(x)} Y_{n(y)}$$

$$\phi_n(x, y) = B_n \sin(n\pi/a)x \cdot C_n e^{-(n\pi/a)y}$$

$$\phi_n(x, y) = b_n \sin(n\pi/a)x \cdot e^{-(n\pi/a)y} \tag{78}$$

where $b_n = B_nC_n$. Equation (78) satisfies the partial differential equation of the wave equation (63) and the first three boundary conditions, whatever the constants b_n. Since equation (63) is linear, any linear combination of ϕ_n, such as

$$\phi(x, y) = \sum_{n=1}^{\infty} b_n \sin(n\pi/a)x\ e^{-(n\pi/a)y}$$

is also a formal solution of equation (63) satisfying three of the boundary conditions. In order that $\phi(x, y)$ satisfy the fourth boundary condition, one must choose the b_n such that

$$\phi(x, 0) = f(x) \sum_{n=1}^{\infty} b_n e^{-(n\pi/a)(0)} \sin(n\pi/a)x$$

and hence b_n are the Fourier sine coefficients of $f(x)$:

$$b_n = \frac{2}{a}\int_0^a f(x) \sin(n\pi/a)x\ dx$$

and the final solution of the problem is

$$\phi(x,y) = \frac{2}{a} \sum_{n=1}^{\infty} \left[\int_0^a f(x) \sin(n\pi/a)x\ dx\right] \sin(n\pi/a)x \cdot e^{-(n\pi/a)y}$$

Problem 2. Wave Equation

As an illustration of the wave equation, consider a transmission line of length ℓ with $R = G = 0$, is initially charged to a voltage ϕ_0 through its length ℓ at $t = 0$. The end $x = 0$ is short-circuited, and the voltage on the line at subsequent time t is required. As we are considering G and R to be zero, the voltage $\phi(x, t)$ in a transmission line satisfies the wave equation (55),

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{k^2} \frac{\partial^2\phi}{\partial t^2} \tag{55}$$

where $k^2 = 1/LC$

and the boundary conditions

1. $\phi(x, 0) = \phi_0$ (constant) $0 < x < \ell$
2. $\phi(0, t) = 0$

Now, letting

$L_t(\phi, t) = \Phi(x, s)$

Equation (55) becomes an ordinary differential equation

$$\frac{d^2\Phi(x,s)}{dx^2} = \frac{1}{k^2}\left[s^2\Phi(x,s) - s\phi(x,0) - \phi_t(x,0)\right] \tag{78a}$$

Introducing proper boundary conditions equation (78a) becomes

$$\frac{d^2\Phi(x, s)}{dx^2} - \frac{s^2}{k^2}\Phi(x, s) = -\frac{s\phi_0}{k^2} \tag{79}$$

The short circuit gives $\phi(x, t) = 0$, and the open circuit at $x = \ell$ means that $i(\ell, t) = 0$, for the boundary conditions. Applying equations (49) and (50) and by letting

$L_t\left[i(x, t)\right] = I(x, s)$, and $i(x, 0) = 0$

$$\frac{d\Phi(x, s)}{dx} = \Phi_x = -Ls\, I(x, s) \tag{80}$$

Thus the general solution of equation (80) is

$$\Phi(x, s) = A\, e^{(s/k)x} + B\, e^{-(s/k)x} + \frac{\phi_0}{s} \tag{81}$$

Now when $x = 0$, $\Phi(0, s) = 0$. Substituting into equation (81),

$$A + B + \frac{\phi_0}{s} = 0 \tag{82}$$

When $x = \ell$, $I(\ell, s) = 0$, and $\bar{O}_x = 0$. Differentiating equation (81) with respect to x and substituting $x = \ell$ in the result, one has

$$\frac{d\Phi(\ell, s)}{dx} = A \frac{s}{k} e^{(s/k)\ell} - B \frac{s}{k} e^{(s/k)\ell} = 0$$

Therefore

$$A\ e^{(s/k)\ell} = B\ e^{-(s/k)\ell} \tag{83}$$

Solving equations (82) and (83) as simultaneous equations in A and B, we have

$$A = \frac{-\phi_0\ e^{-(s\ell/k)}}{2s \cosh \frac{s\ell}{k}} \quad \text{and} \quad B = \frac{-\phi_0\ e^{(s\ell/)}}{2s \cosh \frac{s\ell}{k}}$$

Substituting these values in equation (81) gives

$$\Phi(x, s) \quad \frac{\phi_0}{s} - \frac{\phi_0 \cosh\left[s/k(1 - x)\right]}{s \cosh(s\ell/k)} \tag{84}$$

The inversion can be carried out by

$$\phi(x,t) = \phi_0 - \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} \frac{e^{st} \cosh\left[s/k(1 - x)\right]}{s \cosh(s\ell/k)}\, ds$$

the second term will have no simple poles at the origin due to the s in the denominator because the origin is a zero of the numerator, but will have a simple pole at each point which makes $\cosh(s\ell/k)$ equal to zero; i.e., at each root of

$$\operatorname{Cosh}(s\ell/k) = \cos(js\ell/k) = 0$$

or at $js\ell/k = \pm\pi/2, \pm 3\pi/2, \ldots, \pm (2n + 1)\pi/2$.

Simple poles are therefore located at

$$s = \pm j(2n + 1)\pi k/2\ell$$

therefore the sum of the residue of

$$\frac{e^{st}\,\phi_0 \cosh\left\{s/k(1-x)\right\}}{s\cosh(s\ell/k)}$$

at the points $s = \pm j(2n+1)\pi k/2\ell$, $n = 0, 1, 2, \ldots$. Thus

$$\sum \text{Residue} = \sum \left[\frac{\phi_0\, e^{st}\cosh\left[s/k(1-x)\right]}{\frac{d}{ds}\left\{s\cosh(s\ell/k)\right\}}\right]_{s=\pm j(2n+1)\pi k/2\ell}$$

$$= \sum \left[\frac{\phi_0\, e^{st}\left[\left\{\cosh(s\ell/k)\cdot\cosh(sx/k)\right\} - \left\{\sinh(s\ell/k)\cdot\sinh(sx/k)\right\}\right]}{s\ell/k\cdot\sinh(s\ell/k) + \cosh(s\ell/k)}\right]_{s=\pm j(2n+1)\pi k/2\ell}$$

Now $\cosh(s\ell/k) = 0$ at each of the poles, and

$$\operatorname{Sinh}(s\ell/k)\Big|_{s=+j\frac{(2n+1)\pi k}{2\ell}} = j\sin\left[(2n+1)\pi/2\right] = (-1)^n j$$

and

$$\operatorname{Sinh}(s\ell/k)\Big|_{s=-j\frac{(2n+1)\pi k}{2\ell}} = -j\sin\left[(2n+1)\pi/2\right] = -(-1)^n j$$

so that

$$\sum \text{Residue} = \sum \frac{\phi_0\cdot -e^{-j(2n+1)\,kt/2\ell}(-1)^n j\left\{\sinh\frac{-j(2n+1)\pi k}{2\ell}\cdot\frac{x}{k}\right\}}{\frac{j(2n+1)\pi k}{2\ell}\cdot\frac{1}{k}\cdot(-1)^n j}$$

$$+ \frac{\phi_0\cdot -e^{+j(2n+1)\pi kt/2\ell}-(-1)^n j\sinh\left\{\frac{j(2n+1)\pi k}{2\ell}\cdot\frac{x}{k}\right\}}{\frac{-j(2n+1)\pi k}{2\ell}\cdot\frac{1}{k}\cdot(-)(-1)^n j}$$

$$= \sum \frac{-\phi_0 \, e^{\frac{+j(2n+1)\pi k}{2\ell} t} \, j \sin\left\{\frac{(2n+1)\pi x}{2\ell}\right\}}{j(2n+1)\pi/2}$$

$$+ \sum \frac{-\phi_0 \, e^{\frac{-j(2n+1)\pi k}{2\ell} t} \, j \sin\left\{\frac{(2n+1)\pi x}{2\ell}\right\}}{j(2n+1)\pi/2}$$

$$= \frac{-2\phi_0}{\pi} \sum \frac{\sin\left\{\frac{(2n+1)\pi x}{2\ell}\right\}}{(2n+1)} \left[e^{\frac{j(2n+1)\pi k}{2\ell} t} + e^{\frac{-j(2n+1)\pi k}{2\ell} t} \right]$$

$$= \frac{-4\phi_0}{\pi} \sum \frac{\sin\left\{\frac{(2n+1)\pi x}{2\ell}\right\}}{(2n+1)} \cdot \cos\left\{\frac{(2n+1)\pi k}{2\ell} t\right\}$$

Substituting the value of $k = \sqrt{1/LC} = 1/\sqrt{LC}$, the final solution of equation (55) becomes

$$\phi(x,t) = \phi_0 \left[1 - \frac{4}{\pi} \left\{ \sum_{n=0,1,2...}^{\infty} \frac{\sin\left\{\frac{(2n+1)\pi x}{2\ell}\right\}}{(2n+1)} \cos(2n+1) \frac{\pi}{2\ell\sqrt{LC}} t \right\} \right]$$

Thus the solution is obtained as a Fourier series. However, this form is not the most informative from a physical point of view, and an alternative approach is to expand the transform in a series of exponentials, following Heaviside, obtaining the voltage as a series of step functions. This shows clearly the physical concept of successive reflections, and we will obtain this from the voltage whose transform is given by

$$\bar{\Phi}(x,\ s) = \frac{\phi_0}{s} - \frac{\phi_0 \cosh\left\{s/k(\ell - x)\right\}}{s \cosh(s\ell/k)}$$

$$= \frac{\phi_0}{s} - \frac{\phi_0}{s} \cdot \frac{e^{-sx/k}\left\{1 + e^{-2s(\ell - x)/k}\right\}}{1 + e^{-2s\ell/k}}$$

$$= \frac{\phi_0}{s} - \frac{\phi_0}{s}\left[e^{-sx/k}\left\{1 + e^{-2s(\ell - x)/k}\right\}\left\{\frac{1}{1 + e^{-2s\ell/k}}\right\}\right]$$

$$= \frac{\phi_0}{s} - \frac{\phi_0}{s}\left[e^{-sx/k}\left\{1 + e^{-2s(\ell - x)/k}\right\}\left\{1 - e^{-2s\ell/k} + e^{-4s\ell/k} - \dots\right\}\right]$$

$$= \frac{\phi_0}{s} - \frac{\phi_0}{s}\left[e^{-sx/k} - e^{-s(x+2\ell)/k} + e^{-s(x-2\ell)/k} + e^{-(x+4\ell)/k} - e^{-s(x-4\ell)/k} - \dots\right]$$

Inverting term by term using the shifting theorem

$$\begin{aligned}\phi(x,\ t) = \phi_0 &- \phi_0 u\left[t - x/k\right] + \phi_0 u\left[t - (x + 2\ell)/k\right] \\ &- \phi_0 u\left[t - (x - 2\ell)/k\right] - \phi_0 u\left[t - (x + 4\ell)/k\right] \\ &+ \phi_0 u\left[t - (x - 4\ell)/k\right] - \dots\end{aligned}$$

This form gives the voltage at any point as the sum of an infinite number of step waves, each of velocity $k = 1/\sqrt{LC}$.

Also the wave equation (55) can be solved by the classical method known as separation of variables. We now assume that $\phi(x,\ t)$ is expressible as the product of two quantities X and T,

$$\phi(x,\ t) = X_{n(x)}\ T_{n(t)} \tag{85}$$

where X is a function of x alone and Y is a function of y alone.

If we substitute equation (85) into equation (55) one obtains

$$X'' T = \frac{1}{k^2} T'' X \quad \text{or} \quad \frac{X''}{X} = \frac{T''}{k^2 T}$$

where the primes on the functions X and T represent differentiation with respect to the only variable present. Now the right-hand side of the above equation is independent of x and the left-hand side is independent of t. Since they are equal, their common value cannot be a function of t or x, and must therefore be a constant, say λ:

$$\frac{X''}{X} = \lambda \quad \text{and} \quad \frac{T''}{k^2 T} = \lambda$$

Hence $\quad X'' - \lambda X = 0, \quad T'' - \lambda k^2 T = 0$

The partial differential equation of equation (55) has thus been reduced to two ordinary differential equations.

Assume $\lambda = -\beta^2 < 0$, so that given boundary conditions can be satisfied, where β is a real number. In this case the general solution of

$$X'' + \beta^2 X = 0 \text{ is } X_{(x)} = A\cos(\beta x) + B\sin(\beta x)$$

and the general solution of the equation on $T_{(t)}$,

$$T'' + \beta^2 k^2 T = 0 \text{ is } T_{(t)} = C\cos(\beta kt) + D\sin(\beta kt)$$

From the given boundary conditions (2)

$$\phi(0, t) = 0 = X_{(0)} T_{(t)} = \left[A\cos(\beta 0) + B\sin(\beta 0)\right] T_{(t)}$$

$$0 = A$$

and differentiating (with respect to x and using proper boundary condition)

$$\begin{aligned} \frac{\partial \phi(\ell, t)}{\partial x} &= X'_{(\ell)} T_{(t)} \\ &= 0 = \left[-A\beta \sin(\beta\ell) + B\beta\cos(\beta\ell)\right] T_{(t)} \\ &= 0 = B\beta\cos(\beta\ell) \end{aligned}$$

Since $B \neq 0$, it is concluded that $\cos(\beta\ell)$ should be zero; i.e., $\cos(\beta\ell) = 0$, or

$$\beta\ell = (2n + 1)\pi/2, \ \beta = (2n + 1)\pi/2\ell, \ n = 0, 1, 2, 3, \ldots,$$

with these values of β

$$X_{n(x)} = B_n \sin\left\{\frac{(2n + 1)\pi}{2\ell} x\right\}$$

Again by using the proper boundary condition

$$\frac{\partial \phi(x, 0)}{\partial t} = 0 = X_{(x)}T'(0)$$

$$0 = -C\beta k \sin(0) + D \cos(0)$$

$$0 = D$$

Since $\beta = (2n + 1)\pi/2\ell$, we have a solution $T_{n(t)}$ for each integral value of n:

$$T_{n(t)} = C_n \cos\left\{\frac{(2n + 1)\pi}{2\ell} kt\right\}$$

and for every integral value of n, $n = 0, 1, 2, \ldots$.

$$X_{n(x)}T_{n(t)} = b_n \sin\left\{\frac{(2n+1)\pi}{2\ell} x\right\} \cos\left\{\frac{(2n+1)\pi}{2\ell} kt\right\},$$

where $b_n = C_nB_n$.

Now equation (55) is linear, and any linear combination of solutions is a solution represented by Fourier series expansion of $\phi(x, t)$ in the form

$$\phi(x,t) = \frac{a_0}{2} + \sum_{n=0}^{\infty} b_n \sin\left\{\frac{(2n+1)\pi}{2\ell} x\right\} \cos\left\{\frac{(2n+1)\pi}{2\ell} kt\right\}$$

by introducing boundary condition (1), $a_0 = 2\phi_0$, and

$b_n = -\dfrac{4\phi_0}{(2n + 1)\pi}$, and the final solution of the problem is

$$\phi(x, t) = \phi_0 - \frac{4\phi_0}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \sin\left\{\frac{(2n+1)\pi x}{2\ell}\right\} \cos\left\{\frac{(2n+1)}{2\ell} \pi k t\right\}$$

CONCLUSION

The problems which have been considered are intended to show that Laplace transformation provides a clear and standard method of dealing with boundary and initial value problems in certain partial differential equations. It reduces those which contain two independent variables with constant coefficients to boundary value problems of ordinary differential equations. In general, these are easy to solve. The most difficult part of the transformation method is in transforming the solution back to original space. Since direct inverse Laplace transformation operations tables are not available for transforming the solution back to original space in the case of wave equations, the auxiliary method for doing this has been by use of the complex inversion formula (9). Also along with operational methods, each problem is solved by the classical method known as the separation of variables method. By this method, the partial differential equations are broken down into ordinary differential equations, and the final solution is built up from particular solutions of these ordinary differential equations.

The Laplace transform, which has aroused so much interest in electric circuit theory, is applicable also to field problems. Here is a good field for further study.

TABULATION OF LAPLACE TRANSFORMS

Table 1 contains the operation pairs which are derived in the previous sections.

Table 1. Laplace transforms of operations and functions.

f(t)	: F(s)	
$f(t)$	$f(t)e^{-st}dt$	(1)
$cf(t)$	$cF(s)$	(13)
$f_1(t) + f_2(t)$	$F_1(s) + F_2(s)$	(14)
$f(t)\ e^{-at}$	$F(s + a)$	(15)
$f(t - a)$	$e^{-as}F(s) \qquad a \geqslant 0$	(18)
f(t) Periodic, with a period T	$\dfrac{F_1(s)}{1-e^{-Ts}}\ ,\ F_1(s) = \int_0^T e^{-st}f(t)dt$	(21)
$f(at)$	$1/a\ F(s/a)$	(23)
$\mathrm{Re}\ f(t)$	$\mathrm{Re}\ F(s)$	(28)
$\mathrm{Im}\ f(t)$	$\mathrm{Im}\ F(s)$	(29)
df/dt	$sF(s) - f(0^+)$	(30)
d^nf/dt^n	$s^nF(s)-s^{n-1}f(0^+) - \ldots - d^{n-1}/dt^{n-1}f(0^+)$	
$f(t)dt$	$F(s)/s + f^{-1}(0^+)/s$	(33)
$f_1(t) * f_2(t)$	$F_1(s)\ F_2(s)$	(37)
$u(x,\ t)$	$e^{-st}u(x,\ t)dt = U(x,\ s)$	(56)
$\left[\partial u(x,\ t)/\partial t\right]$	$sU(x,\ s) - u(x,\ 0^+)$	(57)
$\left[\partial^2 u(x,\ t)/\partial t^2\right]$	$s^2U(x,s) - su(x,0^+) - u_t(x,0^+)$	(58)
$\left[\partial u(x,\ t)/\partial x\right]$	$\partial/\partial x \int_0^\infty e^{-st}u(x,t)dt=U(x,s)/x$	(59)
$\left[\partial^2 u(x,t)/\partial x \partial t\right]$	$\partial/\partial x\ \left[sU(x,s) - u(x,0^+)\right]$	(60)
$\lim_{x \to a^+} u(x,t)=A(t)$	$\lim_{x \to a^+} U(x,s) = A(s)$	(62)

Table 2. Direct Laplace transforms of some specific functions.

$f(t)$:	$F(s)$
$\delta(t)$= Unit impulse	1
$u(t)$ or 1 unit step	$1/s$
$t^{n-1}/(n-1)!$ n = integer	$1/s^n$
e^{-at}	$1/(s + a)$
te^{-at}	$1/(s + a)^2$
$t^{n-1}/(n-1)!\ e^{-at}$	$1/(s + a)^n$
$1/\omega \sin(\omega t)$	$1/s^2 + \omega^2$
$\sin(\omega t)$	$\omega/s^2 + \omega^2$
$\cos(\omega t)$	$s/s^2 + \omega^2$
$1 - \cos(\omega t)$	$\omega^2/s(s^2 + \omega^2)$
$\sin(\omega t + \theta)$	$(s \sin\theta + \omega\cos\theta)/(s^2 + \omega^2)$
$\cos(\omega t + \theta)$	$(s \cos\theta - \omega\sin\theta)/(s^2 + \omega^2)$
$e^{-at}\sin(\omega t)$	$\omega/(s + a)^2 + \omega^2$
$e^{-at}\cos(\omega t)$	$(s + a)/(s + a)^2 + \omega^2$
$t\sin(\omega t)$	$2\omega s/(s^2 + \omega^2)^2$
$t\cos(\omega t)$	$s^2 - \omega^2/(s^2 + \omega^2)$
$t\, e^{-at}\sin(\omega t)$	$2\omega(s + a)/\left[(s + a)^2 + \omega^2\right]^2$
$t\, e^{-at}\cos(\omega t)$	$(s + a)^2 - \omega^2/\left[(s + a)^2 + \omega^2\right]^2$
$\sinh(at)$	$a/(s^2 - a^2)$
$1/a\left[\sinh(at)\right]$	$1/(s^2 - a^2)$
$\cosh(at)$	$s/(s^2 - a^2)$
$1/a^2\left[\cosh(at) - 1\right]$	$1/s(s^2 - a^2)$

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SOLUTION OF LAPLACE'S EQUATION
BY OPERATIONAL METHODS

by

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AN ABSTRACT OF
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The main purpose of the paper is the solution of the problems which have been considered to show the operational method of the Laplace transformation. The Laplace transform provides a clear and standard technique of dealing with boundary and initial value problems in certain partial differential equations.

The first part of the report deals with the Laplace transformation. Details about definitions, existence, and inverse transformation are explained via the exponential form of the Fourier integral theorem. In order to construct a Laplace transform for a given function of time f(t), first multiply f(t) by e^{-st}, where s is a complex number, $s = \sigma + jw$, and then the product is integrated with respect to time over the interval zero to infinity. The result is the Laplace transform of f(t), which is designated as F(s). Thus the Laplace transform of f(t) is given as

$$f(t) = \int_0^{\infty} f(t)e^{-st}\,dt = F(s)$$

where $f(t) = 0$, $t < 0$. The inverse Laplace transformation is given by the complex inversion integral

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_2 - j\infty}^{\sigma_2 + j\infty} F(s)e^{st}\,ds$$

where σ_2 is a constant. This integral is used in transforming the solution back to original space in the problem of the wave equation. Various properties of the Laplace transform are discussed in the second section.

Laplace's equation, $\nabla^2\phi = 0$, arising in electrostatics and the wave equation, $\nabla^2\phi = 1/k^2(\partial^2\phi/\partial t^2)$, arising in a long insulated cable have been derived. The Laplace transform is used to solve for specific boundary conditions. The method follows the following scheme in the solution of Laplace's equations and wave equation.

Scheme

Original space: [Partial differential equation + initial conditions + boundary conditions] → Solution

↓ L - transformation L^{-1}- transformation ↑

Image space: [Ordinary differential equation ↓ + boundary conditions] ⟶ Solution

Thus it reduces the partial differential equation which contains two independent variables with constant coefficients to boundary value problems of ordinary differential equations. The most difficult part of the transformation method is in transforming the solution back to original space. Also along with operational methods, each problem is solved by the classical method known as separation of variables. A table of operations and a table of transforms are included at the end of the work.