Estimation of Treatment Effects Under Combined Sampling and Experimental Designs

by

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B.S., Kansas State University, 1999
B.S., Kansas State University, 1999
M.S., Kansas State University, 2003

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Statistics
Collage of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas
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Abstract

Over the years sampling and experimental design have developed independently with little mutual compatibility. However, many studies do (or should) involve both a sampling design and an experimental design. For example, a polluted site may be exhaustively partitioned into area plots, a random sample of plots selected, and the selected plots randomly assigned to three clean-up regimens. In this research the relationship between sampling design and experimental design is discussed and a basic review of each is given. An estimator that combines sampling and experimental design is presented and it's development explained. Properties of this estimator will be derived and some applications of the estimator will be examined. Finally, a simulation study comparing this estimator with the traditional estimator will be presented.

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Acknowledgments

UnHee Kim	For her support and friendship
Dallas Johnson	For taking over when Dr. Pontius was unable to fulfil major professor duties
James Neill Paul Nelson Philip Gipson	For serving as my committee
Mark Smith	For cooking dinner
Irene Hobbs	For babysitting

Dedication

Dr. Jeffrey S. Pontius
1954-2006

Chapter 1

Introduction

1.1 Motivating Ideas

In observational studies, much attention is given to the sampling design (e.g., simple random sampling, stratified sampling, cluster sampling, etc.) so that the realized sample is as representative of the population as statistically possible. Then the selected sample is observed for certain characteristics, that is, response variables are observed. In designed experiments, usually little statistical attention is given to how the experimental units become candidates for the experiment, but much statistical attention is given to the assignment of treatments to these units (e.g., completely randomized, randomized complete block, split plot, etc.). Then certain response values are observed that are thought to be influenced by the treatments. Many studies involve both a sampling design and an experimental design. For example, a polluted site may be exhaustively partitioned into area plots, a random sample of plots selected, and the selected plots randomly assigned to one of three clean-up regimens.

In designed experiments, the primary objective is to make comparisons between treatments for the purpose of determining "causal" effects. Generally, experimenters use a stochastic approach to these statistical studies that treats the population of units as infinite and the response variables, $Y_1, Y_2, ...$, as stochastic with respect to some probability density function, such as, the normal distribution in classical ANOVA.

The finite sampling design based approach to a statistical study is based on the idea that the population of interest is finite and fixed, with N units uniquely labeled $i = 1, 2, ..., N$. The observed response values on the units in the population, $\mathbf{y} = y_1, ..., y_N$, are fixed (non-stochastic) but unknown. Often sampling is only thought to be associated with surveys, where interest is only in describing the population.

In both sampling and experiments, generally a response can only be observed for one treatment level for each unit at one time. That is, one cannot observe a response for both treatment and control on a single unit at the same time. For example, a unit cannot be exposed to a pollutant and not exposed simultaneously. In this case, treatment differences cannot be identified based on observing one unit. Thus, for the set of responses actually observed, there is a corresponding set of unobservable responses from treatments that might have been applied (Smith and Sugden, 1988, Thompson, 2002). Therefore, one may prefer to study the average difference between treatment means, $\mu_k - \mu_{k'}$, where k and k', $k \neq k'$, are any two of the treatment labels, $k = 1, ..., T$. Also, for both experiments and sampling, response values can only be observed for the units in the study. Thus, the researcher cannot be sure that treatment differences observed in the experiment would be observed in the population if the whole population could be observed.

Recently more interest has been generated in studying the unification of sampling and experimental design approaches to statistical studies. Depending on the type of study, the researcher may have limited control over the design aspects of the the study or may be able to completely specify the sampling and experimental designs. For example, when units are selected by simple random sampling and treatments are assigned to the resulting sample of units using a completely randomized design, the researcher has complete control of the sampling and experimental designs, and every treatment combination has equal probability of being observed (Thompson, 2002). When only a survey is conducted, such as in an observational study, the researcher has control over the sampling design, but may not have control over the treatment assignments. Often when designed experiments are conducted, the researcher controls the treatment assignments to units, but may not be able account for how the units were selected for inclusion into the experiment, for example when a sample of convenience is used. Table 1.1 is a reproduction from Smith and Sugden (1988) illustrating this relationship between sampling and experiments.

Table 1.1: Classification of Studies with Treatments

		Treatment Assignment	
		Control	No Control
Sample Selection	Control	Experiment within a survey, or survey within an experiment	Analytic survey
	No Control	Experiment	Uncontrolled observational study

As an illustration, consider a study to determine the effects of asthma treatments for asthma sufferers in a given city. In an uncontrolled observational study the researcher may visit clinics and interview patients as they leave the clinic to identify the type of asthma treatment they received. If the patient allows, the researcher could also conduct a follow-up interview to determine the patients' responses to the treatments. In this case, the researcher has no control over the unit selection nor the treatment assignment.

Alternatively, the researcher may choose to conduct an experiment by assigning an asthma treatment to each patient that enters the clinic with asthma symptoms in a completely randomized design and then monitor the patients' responses to the treatments. Here, the researcher has control over the treatment assignment but has no control over the selection of units.

Another alternative is for the researcher to conduct an analytic survey. In this case, the researcher must have a complete list of all asthma patients at all clinics in the city. Then the researcher could take a simple random sample without replacement from the list of patients, contact the patients and interview them regarding their asthma treatment and their responses to the treatments. Here, the researcher has control over the sampling design but has no control over the treatment assignments.

The final alterative is to combine the experiment and the survey. First, the researcher must have a complete list of all asthma patients at all clinics in the city. The researcher may take a simple random sample without replacement of the patients from the list. Then the researcher could assign an asthma treatment to each patient in the sample based on a completely randomized design and monitor the patients' responses to the treatments. In this case, the researcher has control over both the sampling and experimental designs.

1.2 Historical Perspective

The reason why sampling and experiments became such different and noninteracting forms of statistical study is the subject of some debate. Some statisticians believe that sampling and experiments are inherently different (Fienberg and Tanur, 1996). Fienberg and Tanur (1996) point to the progenitors of sampling and experimental designs as the culprits of this divergence. They argue that the conflict which arose between Jerzy Neyman and Ronald Aylmer Fisher is the root cause of the separation. The conflict between Neyman and Fisher revolved around testing, not experimentation. Nevertheless, the ensuing arguments about the validity of their statistical approaches prevented them from combining their ideologies, thus driving the two disciplines apart.

Fienberg and Tanur (1987) do suggest some other influences, though not the driving forces, on the lack of "cross-fertilization" between experimental and sampling methodologies. First, despite the fact that sampling and experiments have a common theoretical base, they developed in separate directions. The focus for designed experiments shifted more toward complex treatment structures (multiple complex factors) than toward complex control structures (e.g. blocking and Latin squares). In sampling, the development of control structures, such as, stratification and clustering, moved the focus to unequal unit inclusion probabilities. Second, the amount of knowledge required to be an expert in both areas is so large that most researchers do not master both disciplines any more. Third, the fields of applications that utilize sampling or experiments tend to be drawn to one or the other areas, thus reinforcing the separation.

Both Fienberg and Tanur (1987) and Smith and Sugden (1988) point out that, historically, sampling and experiments had different objectives. Sampling tended to be used for descriptive studies designed to estimate aggregates for generalization to a real finite population. Designed experiments have primarily been used to investigate "causal" relationships between treatments and outcomes for units in some hypothetical population. These differing objectives, may have lead to another point of contention between sampling and experiments, the use of multiple tests or, as Fienberg and Tanur (1987) put it, simultaneous inference. Making multiple comparisons is easily accessible and tempting for survey researchers, but is decried as taboo and "capitalizing on chance" by experimentalists, thus driving away samplers.

This section would be incomplete without some comment on the parallels between sampling and designed experiments. Namely, both methodologies use randomization to provide valid results (Fienberg and Tanur, 1987). In experiments, randomization is used to control nuisance factors so that comparisons can be made between treatment results. For sampling, randomization is the mechanism that allows generalization to the entire population. Also, homogeneous groups are used in both designed experiments and sampling to control error (variability) (Fienberg and Tanur, 1987). To minimize experimental error, experimentalists use homogeneous groups within blocks, and in sampling, homogeneous groups are used in stratified samples to minimize sampling error. A more complete list, as constructed by Fienberg and Tanur (1987), is given in Table 1.2.

Table 1.2: Parallels between sampling and experimental design concepts

Experiments	Sampling
Randomization	Random sampling
Blocking	Stratification
Latin squares	Lattice sampling
Split-plot designs	Cluster sampling
Covariance adjustments	Post-stratification

1.3 Purpose of Study

For the purpose of this study consider the case suggested by Smith and Sugden (1998) where the researcher has control over both the sampling design and the experimental design. Thompson (2002) proposed an unbiased estimator of the treatment mean for this case, similar to the Horvitz-Thompson estimator (HTE, see Chapter 2 for discussion of HTE). Thompson's estimator will be further explained in Chapter 3.

The objective of this dissertation is to evaluate and expound the estimator given by Thompson (2002). First, in Chapter 3 the variance and the estimator of the variance of Thompson's estimator will be derived, along with the variance and the estimator of the variance for the difference between treatment means based on Thompson's estimator. In Chapter 4, Thompson's estimator will be applied to some specific sampling and experimental designs. Finally, a simulation study comparing Thompson's estimator to the traditional estimator will be presented in Chapter 5.

There are two primary facets to the analysis of treatment effects, estimation and testing of hypotheses. Here, the focus will be on estimators of treatment effects, their variances and related inference procedures, primarily, for the estimation of treatment effects under common experimental designs where the experimental units have been selected from a finite population via some sampling design. Simulation will be used to evaluate these estimators under simple random sampling without replacement and a completely randomized experimental design.

Chapter 2

The Basics

In this chapter we will review some basics that will be a foundation for the development of Thompson's estimator in Chapter 3. The first section gives a concise review of finite population sampling. In the second section the Horvitz-Thompson estimator is developed and discussed. Horvitz-Thompson estimator gives a basis for the form of Thompson's estimator. Finally, this chapter concludes with a brief review of experimental design. All of the ideas in this chapter are incorporated to develop Thompson's estimator of treatment difference in Chapter 3.

2.1 Review of Finite Population Sampling

A **finite population**, $U = \{u_1, u_2, ..., u_N\}$, is a set of units, u_i, of fixed size N, where N may or may not be known, and where each unit in U is assigned a unique label, $i = 1, 2, ..., N$. Sometimes U is simply written as $U = \{1, 2, ..., N\}$. A sample, s, of size $n(s)$ is selected from the population, where $n(s)$ may be random or fixed, $n(s) = n$. Denote the observable response measured on unit i as y_i and form the response vector $\mathbf{y} = \{y_1, y_2, ..., y_N\}$. The y_i's are fixed for each unit and are only observed for the units in the sample.

For N known, each sample, s, of size $n(s)$, has a known, specified probability, $p(s)$, of being selected, based on the set of samples under consideration. The function $p(\cdot)$, which gives the probability of selecting s under a given selection procedure, is called the **sampling design** (Särndal, Swensson and Wretmen, 1992). The sampling design and the parameter of interest will, usually, indicate an appropriate estimator to be used. The combination of the sampling design and an estimator is called a **sampling strategy** (Särndal, Swensson and Wretmen, 1992).

The simplest way to visualize the selection of a sample is based on a draw-sequential sampling scheme. In a draw-sequential sampling scheme each unit drawn is based on a randomized experiment (Särndal, Swensson and Wretmen, 1992). The randomized experiment is applied for $n(s)$ draws. For example, consider simple random sampling without replacement for fixed sample size, n, using a draw-sequential scheme. Each unit is selected with equal probability from the units remaining in the population after the previous selection. That is, for all $n!$ permutations of the elements of s,

$$\begin{aligned} P(u_{i_1} \in s) &= \frac{1}{N} \\ P(u_{i_2} \in s) &= \frac{1}{N-1} \\ &\vdots \\ P(u_{i_n} \in s) &= \frac{1}{N-(n-1)} \end{aligned}$$

So, the sampling design under simple random sampling without replacement with fixed sample size is given by

$$p(s) = n! \prod_{j=1}^{n-1} \frac{1}{N-j} = \frac{1}{\binom{N}{n}} .$$

When each possible sample has a known probability of being selected, each unit in the population has a known probability of appearing in the selected samples (Lohr, 1999). The probability that unit u_i is in s is given by its **inclusion probability**,

$$P(u_i \in \text{ some } s) = \sum_{s \ni i} p(s) = \pi_i \ .$$

The notation here implies that the sum should be taken over all samples containing unit i. When $\pi_i > 0$ for all $i = 1, 2, .., N$, the sample is referred to as a **probability sample**. Probability sampling guarantees that each unit in the population has a positive chance of appearing in at least one sample and reduces the potential selection bias (Lohr, 1999). Also, the **joint inclusion probability** for units u_i and u_j is given by

$$P(u_i, u_j \in \text{ some } s) = \sum_{s \ni i,j} p(s) = \pi_{ij} \ .$$

For example, consider again the case of simple random sampling without replacement. Consider the selection of unit i into the sample. Then there are $N-1$ remaining units from which to select in order to fill the $n-1$ available spaces left in the sample. Thus, the probability for a given unit i being in some sample s is given by

$$\pi_i = \sum_{s \ni i} p(s) = \sum_{s \ni i} \frac{1}{\binom{N}{n}} = \frac{\binom{N-1}{n-1}}{\binom{N}{n}} = \frac{n}{N} \ ,$$

which is the inclusion probability for unit i. Similarly, if unit i and unit j, $i \neq j$, are selected to be in the sample there are $N-2$ remaining units from which to select in order to fill the $n-2$ available spaces left in the sample. Thus, the joint inclusion probability for units i and j, $i \neq j$, is given by

$$\pi_{ij} = \sum_{s \ni i,j} p(s) = \frac{\binom{N-2}{n-2}}{\binom{N}{n}} = \frac{n(n-1)}{N(N-1)} \ .$$

One way to obtain more accurate (i.e. unbiased) estimates is to utilize the inclusion probability for each unit based on the given sampling design (Särndal, Swensson and Wretmen, 1992). One family of estimators that incorporates the inclusion probabilities of units was developed by Horvitz and Thompson (1952). These estimators are based on the inverse of the inclusion probabilities so that a unit with a large inclusion probability is down weighted, whereas, a unit with a small inclusion probability is up weighted. These estimators are unbiased for all sampling designs (Hedayat and Sinha, 1991).

2.2 Review of Horvitz-Thompson Estimator

The original Horvitz-Thompson estimator (HTE) was developed for the finite population total $T(\mathbf{y}) = \sum_{i=1}^{N} y_i$ (Horvitz and Thompson, 1952). The general form of the HTE for $T(\mathbf{y})$, using the notation of Hedayat and Sinha (1991), is given by

$$\mathrm{HTE}(s, \mathbf{y}) = \sum_{i \in s} y_i/\pi_i = \sum_{i=1}^{N} Z_i y_i/\pi_i$$

where Z_i is an indicator variable that is one if unit i is in the sample and zero otherwise, that is, $Z_i \sim \mathit{Bernoulli}(\pi_i)$, $\pi_i > 0$ for all i, and $i \in s$ refers to distinct units in s. Note that π_i is known from the sampling design and that Z_i is a random variable whose value depends on the given sample, s, for each i.

It can be shown that the HTE is a homogeneous, linear (design) unbiased estimator (Hedayat and Sinha, 1991). It is homogeneous in the sense that the inclusion probabilities used to weight the observations are dependent on the sample selected (Hedayat and Sinha, 1991). Also, the HTE is unbiased (for a given design) for the population total since

$$E\left[\mathrm{HTE}(s, \mathbf{y})\right] = E\left[\sum_{i=1}^{N} Z_i \frac{y_i}{\pi_i}\right] = \sum_{i=1}^{N} \frac{y_i}{\pi_i} E(Z_i) = \sum_{i=1}^{N} y_i = T(\mathbf{y}) \ .$$

Note that in general, the HTE is not a uniform minimum variance unbiased estimator (umvue) since no such estimator exists for estimators of the form $t(s, \mathbf{y}) = \sum_{i \in s} a_{si} y_i$, where a_{si} depends on the sample and the unit drawn (Hedayat and Sinha, 1991). However, the HTE is the unique best (i.e. symmetrized) estimator for unicluster sampling designs, where unicluster means that any two samples taken by the given design are either disjoint or equivalent (Hedayat and Sinha, 1991).

The following derivations are standard results but are re-derived here for those who are unfamiliar with these procedures. The variance of the HTE given by Horvitz and Thompson (1952) is, using the notation of Lohr (1999),

$$\begin{aligned}
\mathrm{V_{HT}(HTE)} &= var\left(\sum_{i=1}^{N}\frac{Z_i y_i}{\pi_i}\right) \\
&= \sum_{i=1}^{N}\left(\frac{y_i}{\pi_i}\right)^2 var(Z_i) + \sum_{i\neq j}\left(\frac{y_i}{\pi_i}\right)\left(\frac{y_j}{\pi_j}\right) cov(Z_i, Z_j) \\
&= \sum_{i=1}^{N}\left(\frac{y_i}{\pi_i}\right)^2 \pi_i(1-\pi_i) + \sum_{i\neq j}\left(\frac{y_i}{\pi_i}\right)\left(\frac{y_j}{\pi_j}\right)(\pi_{ij}-\pi_i\pi_j) \\
&= \sum_{i=1}^{N}(y_i)^2\left(\frac{1-\pi_i}{\pi_i}\right) + \sum_{i\neq j} y_i y_j \left(\frac{\pi_{ij}-\pi_i\pi_j}{\pi_i\pi_j}\right) \\
&= \sum_{i=1}^{N}(y_i)^2\left(\frac{1}{\pi_i}-1\right) + \sum_{i\neq j} y_i y_j \left(\frac{\pi_{ij}}{\pi_i\pi_j}-1\right) .
\end{aligned}$$

$\mathrm{V_{HT}(HTE)}$ is appropriate for samples of fixed or variable size, n or $n(s)$, respectively. An alternative expression for the variance of HTE is given by

$$\mathrm{V_{SYG}(HTE)} = \sum_{i=1}^{N}\sum_{j>i}(\pi_i\pi_j - \pi_{ij})\left[\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j}\right]^2 ,$$

which was first supplied by Sen (1953) and by Yates and Grundy (1953) (Hedayat and Sinha, 1991). $\mathrm{V_{SYG}(HTE)}$ is only appropriate for fixed-size sampling designs.

$\mathrm{V_{SYG}(HTE)}$ can be derived from $\mathrm{V_{HT}(HTE)}$ in the following manner. First, note that for a fixed size sample, $n(s) = n$,

$$n = \sum_{i=1}^{N} Z_i = \sum_{i=1}^{N} E(Z_i) = \sum_{i=1}^{N}\pi_i ,$$

$$\sum_{j\neq i}\pi_j = \sum_{j=1}^{N}\pi_j - \pi_i = \sum_{j=1}^{N} E(Z_j) - \pi_i = n - \pi_i ,$$

and

$$\begin{aligned}
\sum_{j\neq i}\pi_{ij} &= \sum_{j\neq i} E(Z_i Z_j) \\
&= E\left[\sum_{j\neq i} Z_i Z_j\right]
\end{aligned}$$

$$
\begin{aligned}
&= E\left[Z_i \sum_{j\neq i} Z_j\right] \\
&= E\left[Z_i \left(\sum_{j=1}^{N} Z_j - Z_i\right)\right] \\
&= E[Z_i(n - Z_i)] \\
&= E[n(Z_i) - (Z_i)^2] \\
&= n\pi_i - \pi_i \\
&= \pi_i(n-1)
\end{aligned}
$$

Next, note that

$$
\begin{aligned}
\sum_{i\neq j}\left[\left(\frac{y_i}{\pi_i}\right)^2 + \left(\frac{y_j}{\pi_j}\right)^2\right](\pi_i\pi_j - \pi_{ij}) &= 2\sum_{i=1}^{N}\left(\frac{y_i}{\pi_i}\right)^2\left[\sum_{j\neq i}(\pi_i\pi_j - \pi_{ij})\right] \\
&= 2\sum_{i=1}^{N}\left(\frac{y_i}{\pi_i}\right)^2\left[\pi_i\sum_{j\neq i}\pi_j - \sum_{j\neq i}\pi_{ij}\right] \\
&= 2\sum_{i=1}^{N}\left(\frac{y_i}{\pi_i}\right)^2[\pi_i(n-\pi_i) - \pi_i(n-1)] \\
&= 2\sum_{i=1}^{N}(y_i)^2\left[\frac{\pi_i - (\pi_i)^2}{(\pi_i)^2}\right] \\
&= 2\sum_{i=1}^{N}(y_i)^2\left(\frac{1-\pi_i}{\pi_i}\right)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\sum_{i=1}^{N}(y_i)^2\left(\frac{1-\pi_i}{\pi_i}\right) + \sum_{i\neq j} y_i y_j\left(\frac{\pi_{ij} - \pi_i\pi_j}{\pi_i\pi_j}\right) \\
&= \frac{1}{2}\left[2\sum_{i=1}^{N}(y_i)^2\left(\frac{1-\pi_i}{\pi_i}\right)\right] + \sum_{i\neq j} y_i y_j\left(\frac{\pi_{ij} - \pi_i\pi_j}{\pi_i\pi_j}\right) \\
&= \frac{1}{2}\sum_{i\neq j}\left[\left(\frac{y_i}{\pi_i}\right)^2 + \left(\frac{y_j}{\pi_j}\right)^2\right](\pi_i\pi_j - \pi_{ij}) + \sum_{i\neq j} y_i y_j\left(\frac{\pi_{ij} - \pi_i\pi_j}{\pi_i\pi_j}\right) \\
&= \frac{1}{2}\sum_{i\neq j}\left(\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j}\right)^2(\pi_i\pi_j - \pi_{ij}) \\
&= \sum_{i=1}^{N}\sum_{j>i}\left(\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j}\right)^2(\pi_i\pi_j - \pi_{ij})\ .
\end{aligned}
$$

Estimates of both V_{HT}(HTE) and $\mathrm{V}_{\mathrm{SYG}}$(HTE) can be derived by introducing the indicator variable and the appropriate weight to make the estimate unbiased (provided $\pi_{ij} > 0$ for all i, j), as follows:

$$\begin{aligned}
\widehat{\mathrm{V}}_{\mathrm{HT}}(HTE) &= \sum_{i=1}^{N} \frac{(y_i)^2}{\pi_i}\left(\frac{1}{\pi_i} - 1\right) Z_i + \sum_{i \neq j} \frac{y_i y_j}{\pi_{ij}} \left(\frac{\pi_{ij} - \pi_i \pi_j}{\pi_i \pi_j}\right) Z_i Z_j \\
&= \sum_{i \in s} \frac{(y_i)^2}{\pi_i}\left(\frac{1}{\pi_i} - 1\right) + \sum_{\substack{i, j \in s \\ i \neq j}} \frac{y_i y_j}{\pi_{ij}} \left(\frac{\pi_{ij} - \pi_i \pi_j}{\pi_i \pi_j}\right),
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\mathrm{V}}_{\mathrm{SYG}}(HTE) &= \sum_{i=1}^{N} \sum_{j>i} \left(\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j}\right)^2 \left(\frac{\pi_i \pi_j - \pi_{ij}}{\pi_{ij}}\right) Z_i Z_j \\
&= \sum_{i \in s} \sum_{\substack{i, j \in s \\ i > j}} \left(\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j}\right)^2 \left(\frac{\pi_i \pi_j - \pi_{ij}}{\pi_{ij}}\right).
\end{aligned}$$

These variance estimates can be negative for both forms of the variance. However, $\widehat{\mathrm{V}}_{\mathrm{SYG}}$(HTE) does tend to be negative less often and gives less variable estimates. That is, $\widehat{\mathrm{V}}_{\mathrm{SYG}}$(HTE) is more stable than $\widehat{\mathrm{V}}_{\mathrm{HT}}$(HTE). This is, partly, because $\pi_i \pi_j - \pi_{ij} < 0$ is less frequent than $\pi_{ij} - \pi_i \pi_j < 0$. In some cases a careful choice of the underlying sampling design may ensure nonnegativity. For example, a fixed-size design, with $0 < \pi_{ij} \leq \pi_i \pi_j$ and $1 \leq i \neq j \leq N$, will always generate nonnegative estimates for V_{HT}(HTE) and $\mathrm{V}_{\mathrm{SYG}}$(HTE) (Hedayat and Sinha, 1991).

2.3 Review of Experimental Design

The objective of a designed experiment is to study the "causal" effect of some set of treatments. First note that the analysis of experiments is model based where units are assumed to be random observations from some hypothetical and infinite population. The members of this hypothetical population are considered to arise from some probability distribution, that is, $Y_i \sim f(Y_i, \theta)$, where $i = 1, 2, ...$

Now, consider the simple case of one treatment and a control, as presented by Holland (1986). The response from unit i after a treatment or control has been applied is $Y_i(t)$ or $Y_i(c)$, respectively. $Y_i(t)$ and $Y_i(c)$ cannot both be observed on the same unit at the same time. Thus, the **treatment effect** of t relative to c, $Y_i(t) - Y_i(c)$, cannot be estimated by observing a single observation on a single unit. The statistical solution to this problem is to consider the average effect

$$E(Y(t) - Y(c)) = E(Y(t)) - E(Y(c)) \ ,$$

which can be estimated. That is, information about the treatment effect, can be gained by observing different units. Then the exact mechanism, that is, the experimental design, that selects units for exposure to t or c is very important (Holland, 1986).

In order to evaluate $E(Y_i(t)) - E(Y_i(c))$, a set of experimental units must be available that is large enough to apply each treatment more than once (except for situations when non-replication is unavoidable). An **experimental unit** is a unit to which one treatment is applied independent of treatment application to other units. Ideally, experimental units are obtained from a population of units by some random mechanism, such as a random sampling design. However, such a mechanism may be too complex and require too much capital (time and money) to be practical. Thus, many times experimental units are selected by convenience. Here the case where a true random sample is available will be considered.

Designed experiments require two layers of control, using homogeneous groupings of sampling units (if possible) and randomizing treatments to the units selected. As mentioned in the introduction, homogeneous groupings of experimental units using blocking may allow experimentalists to control experimental error. That is, blocking helps to reduce the variability due to the units so that there can be a certain amount of confidence that apparent treatment difference are in fact due to differences between treatments and not differences between units.

Also, the randomization of treatment applications to the experimental units is used to ensure that the probability of any particular allocation of treatments to experimental units is equal for all possible allocations within a given homogeneous group (Mead, 1988). This randomization gives the experiment a sense of validity by reducing bias that could arise from a systematic assignment of treatments to units (Kuehl, 2000), and it facilitates generalization to some larger population. That is, if the experiment were repeated at some future time, it is expected to give similar results.

Randomization can be achieved either by allocating a treatment to a particular unit, or by allocating a unit to a particular treatment (Mead, 1988). The evaluation of $E(Y_i(t)) - E(Y_i(c))$ will depend on the allocation mechanism (i.e. the experimental design). Note that over control of the experimental design and lack of a true random selection of experimental units can make it difficult to identify the actual population of inference (Mead, 1988).

Chapter 3

Thompson's Estimator

Consider the case suggested by Smith and Sugden (1998) where the researcher has control over both the sampling design and the experimental design. Thompson (2002) proposed an unbiased estimator for the difference between two treatment means, similar to the HTE for this case. First, it is appropriate to develop the ideas behind this estimator. Then Thompson's estimator will be presented. Finally, some of it's properties will be derived and discussed.

3.1 Preliminary Ideas

Let π_i be the inclusion probability that unit i is in sample s, and let ${\alpha_i}_s^k$ be the probability that treatment k is assigned to unit i, given that $u_i \in s$. For example, if the sample size, $n(s)$, under the sampling design is random, ${\alpha_i}_s^k$, may be different for unit i depending if $n(s)$ is 'large' or 'small.' Given a fixed number of units for assignment to treatment k, unit i would have a greater chance of being assigned to treatment k if $n(s)$ was relatively 'small.'

If the assignment of treatment k does not depend on the selected sample, then $\alpha_{i_s}^k$ can be written as α_i^k (this is the case considered here). Note that π_i does not have to be the same for all units in the finite population, and α_i^k does not have to be the same for all treatments in the experiment.

Let y_i^k be the fixed response of unit i to treatment k. Let Z_i be an indicator variable that is one if $u_i \in s$ and zero otherwise. Let W_i^k be an indicator variable that is one if treatment k is assigned to unit i and zero otherwise. That is,

$$Z_i = \begin{cases} 1 & \text{if } u_i \in s \\ 0 & \text{otherwise} \end{cases} ,$$

and

$$W_i^k = \begin{cases} 1 & \text{if treatment k assigned to } u_i \\ 0 & \text{otherwise} \end{cases} .$$

Then $\pi_i = P(u_i \in s) = P(Z_i = 1)$, and $\alpha_i^k = P(\text{trt } k \text{ is assigned to unit } i) = P(W_i^k = 1)$, where $i = 1, ..., n$ and $k = 1, ..., T$.

Then, under a given sampling design,

$$Z_i \sim Bernoulli(\pi_i) .$$

Thus,

$$E(Z_i) = P(Z_i = 1) = \pi_i ,$$

so

$$E(Z_i^2) = (0)^2 P(Z_i = 0) + (1)^2 P(Z_i = 1) = \pi_i ,$$

and

$$var(Z_i) = E(Z_i^2) - [E(Z_i)]^2 = \pi_i - \pi_i^2 = \pi_i(1 - \pi_i) .$$

Also,

$$E(Z_i Z_j) = P(Z_i = 1, Z_j = 1) = \pi_{ij} ,$$

so

$$cov(Z_i, Z_j) = E(Z_i Z_j) - E(Z_i)E(Z_j) = \pi_{ij} - \pi_i \pi_j .$$

Likewise, under a given experimental design,

$$W_i^k \sim Bernoulli(\alpha_i^k) \ .$$

Thus,

$$E(W_i^k) = P(W_i^k = 1) = \alpha_i^k \ ,$$

so

$$E[(W_i^k)^2] = (0)^2 P(W_i^k = 0) + (1)^2 P(W_i^k = 1) = \alpha_i^k \ ,$$

and

$$var(W_i^k) = E[(W_i^k)]^2 - [E(W_i^k)]^2 = \alpha_i^k - (\alpha_i^k)^2 = \alpha_i^k(1 - \alpha_i^k) \ .$$

Also,

$$E(W_i^k W_j^k) = P(W_i^k = 1, W_j^k = 1) = \alpha_{ij}^k \ ,$$

so

$$cov(W_i^k, W_j^k) = E(W_i^k W_j^k) - E(W_i^k)E(W_j^k) = \alpha_{ij}^k - \alpha_i^k \alpha_j^k \ ,$$

where α_{ij}^k is the joint assignment probability of units i and j, $i \neq j$, to treatment k. For the current discussion assume that Z_i and W_i^k are independent.

To illustrate these ideas consider the sampling design *simple random sampling without replacement* (srswor) to which the experimental design, *completely randomized design* (CRD), is imposed on the srswor sample units. Then

$$\pi_i = P(Z_i = 1) = \frac{n}{N}$$

and

$$\pi_{ij} = P(Z_i = 1, Z_j = 1) = \frac{n(n-1)}{N(N-1)}$$

as demonstrated in Chapter 2. Similarly,

$$\alpha_i^k = P(W_i^k = 1) = \frac{n_k}{n}$$

and

$$\alpha_{ij}^k = P(W_i^k = 1, W_j^k = 1) = \frac{n_k(n_k - 1)}{n(n-1)} \ .$$

3.2 Development of Thompson's Estimator

As mentioned, Thompson (2002) focused on estimating the difference in treatment means, $\mu_k - \mu_{k'}$, where $\mu_k = \frac{1}{N}\sum_{i=1}^{N} y_i^k$ is the population mean under treatment k and $k \neq k'$. The conventional estimator for the average response from treatment k is

$$\bar{y}_k = \sum_{i \in s} \frac{y_i^k}{n_k} = \frac{1}{n_k}\sum_{i=1}^{N} y_i^k Z_i W_i^k ,$$

where t_i indicates the treatment assigned to unit i and $n_k = \sum_{i=1}^{N} Z_i W_i^k$ is the number of sample units assigned to treatment k (Thompson, 2002). However, under most sampling designs, this estimator is biased, since under a given sampling design, a given experimental design, and fixed treatment group size, n_k,

$$E(\bar{y}_k) = \sum_{i=1}^{N} \frac{1}{n_k} y_i^k E(Z_i) E(W_i^k) = \frac{1}{n_k}\sum_{i=1}^{N} y_i^k \pi_i \alpha_i^k \neq \mu_k ,$$

when Z_i and W_i^k are assumed to be independent. However,

$$E\left[\bar{y}_k \frac{n_k}{N\pi_i\alpha_i^k}\right] = \sum_{i=1}^{N} \frac{n_k}{N\pi_i\alpha_i^k} E(\bar{y}_k) = \mu_k .$$

Thus, an unbiased estimator of the mean population treatment effect, $\mu_k - \mu_{k'}$, is $\hat{\mu}_k - \hat{\mu}_{k'}$, where

$$\hat{\mu}_k = \frac{1}{N}\sum_{i \in s} \frac{y_i^k}{\pi_i \alpha_i^k} = \frac{1}{N}\sum_{i=1}^{N} \frac{y_i^k}{\pi_i\alpha_i^k} Z_i W_i^k$$

(Thompson, 2002). Note that Thompson (2002) did not provide variances or variance estimators for the mean estimator or the difference in means estimator. These variances and their estimators are derived in Section 3.3.

Returning to the example of simple random sampling without replacement (srswor) with a completely randomized design (CRD),

$$\begin{aligned}
\hat{\mu}_k &= \frac{1}{N}\sum_{i \in s} \frac{y_i^k}{\pi_i \alpha_i^k} \\
&= \frac{1}{N}\sum_{i \in s} y_i^k \left(\frac{N}{n}\right)\left(\frac{n}{n_k}\right) \\
&= \frac{1}{n_k}\sum_{i \in s} y_i^k .
\end{aligned}$$

This example will be discussed further in Chapter 4.

3.3 Properties of Thompson's Estimator

In Thompson's proposal and discussion of $\hat{\mu}_k$ there was no attempt to evaluate the variance of $\hat{\mu}_k$ or of $\hat{\mu}_k - \hat{\mu}_{k'}$. The purpose of the current research is to extend Thompson's initial proposal and discussion of $\hat{\mu}_k$. One important extension is to consider properties of Thompson's estimator such as the variances of $\hat{\mu}_k$ and $\hat{\mu}_k - \hat{\mu}_{k'}$. This section is dedicated to that pursuit and is the primary contribution of this research.

As shown in the previous section $\hat{\mu}_k$ is an unbiased estimator of μ_k. Similar to the HTE, $\hat{\mu}_k$ is homogeneous in the sense that the inclusion probabilities used to weight the observations depend on the sample selected. That is, $\hat{\mu}_k$ is a linear estimator of the form $t(s, \mathbf{y}) = \sum_{i \in s} a_{si} y_i$ where the a_{si}'s depend on s but not depend the y_i's. Thus, $\hat{\mu}_k$ is a homogeneous, unbiased linear estimator (HULE) of μ_k and $\hat{\mu}_k - \hat{\mu}_{k'}$ is a HULE of the difference in means (Hedayat and Sinha, 1991). Recall from Chapter 2 that no UMVUE exists for estimators of this form.

The variance of $\hat{\mu}_k$ is given by

$$\begin{aligned}
var(\hat{\mu}_k) &= \frac{1}{N^2}\sum_{i=1}^{N}\left(\frac{y_i^k}{\pi_i\alpha_i^k}\right)^2 var(Z_iW_i^k) + \frac{1}{N^2}\sum_{i\neq j}\left(\frac{y_i^k}{\pi_i\alpha_i^k}\right)\left(\frac{y_j^k}{\pi_j\alpha_j^k}\right) cov(Z_iW_i^k, Z_jW_j^k) \\
&= \frac{1}{N^2}\sum_{i=1}^{N}\left(\frac{y_i^k}{\pi_i\alpha_i^k}\right)^2 \pi_i\alpha_i^k(1-\pi_i\alpha_i^k) + \frac{1}{N^2}\sum_{i\neq j}\left(\frac{y_i^k}{\pi_i\alpha_i^k}\right)\left(\frac{y_j^k}{\pi_j\alpha_j^k}\right)(\pi_{ij}\alpha_{ij}^k - \pi_i\pi_j\alpha_i^k\alpha_j^k) \\
&= \frac{1}{N^2}\sum_{i=1}^{N}(y_i^k)^2\frac{\pi_i\alpha_i^k(1-\pi_i\alpha_i^k)}{(\pi_i\alpha_i^k)^2} + \frac{1}{N^2}\sum_{i\neq j}(y_i^k y_j^k)\frac{\pi_{ij}\alpha_{ij}^k - \pi_i\pi_j\alpha_i^k\alpha_j^k}{\pi_i\pi_j\alpha_i^k\alpha_j^k} \\
&= \frac{1}{N^2}\left[\sum_{i=1}^{N}(y_i^k)^2\frac{1-\pi_i\alpha_i^k}{\pi_i\alpha_i^k} + \sum_{i\neq j}(y_i^k y_j^k)\frac{\pi_{ij}\alpha_{ij}^k - \pi_i\pi_j\alpha_i^k\alpha_j^k}{\pi_i\pi_j\alpha_j^k\alpha_i^k}\right] ,
\end{aligned}$$

since

$$\begin{aligned}
var(Z_iW_i^k) &= E[(Z_iW_i^k)^2] - [E(Z_iW_i^k)]^2 \\
&= E[(Z_i)^2]E[(W_i^k)^2] - [E(Z_i)]^2[E(W_i^k)]^2 \\
&= \pi_i\alpha_i^k - (\pi_i)^2(\alpha_i^k)^2 \\
&= \pi_i\alpha_i^k(1-\pi_i\alpha_i^k) ,
\end{aligned}$$

and

$$\begin{aligned}
cov(Z_iW_i^k, Z_jW_j^k) &= E(Z_iZ_jW_i^kW_j^k) - E(Z_iW_i^k)E(Z_jW_j^k) \\
&= E(Z_iZ_j)E(W_i^kW_j^k) - E(Z_i)E(Z_j)E(W_i^k)E(W_j^k) \\
&= \pi_{ij}\alpha_{ij}^k - \pi_i\pi_j\alpha_i\alpha_j \ .
\end{aligned}$$

As with the HTE, $var(\hat{\mu}_k)$ can be rewritten in the Sen, Yates and Grundy form, for fixed sample and treatment group sizes, as follows

$$var_{alt}(\hat{\mu}_k) = \sum_{i=1}^{N}\sum_{j>i}(\pi_i\pi_j\alpha_i^k\alpha_j^k - \pi_{ij}\alpha_{ij}^k)\left[\frac{y_i}{\pi_i\alpha_i} - \frac{y_j}{\pi_j\alpha_j}\right]^2 \ .$$

$var_{alt}(\hat{\mu}_k)$ can be derived from $var(\hat{\mu}_k)$ in the following manner. First, note that

$$n_k = \sum_{i=1}^{N} Z_iW_i^k = \sum_{i=1}^{N} E(Z_i)E(W_i^k) = \sum_{i=1}^{N}\pi_i\alpha_i^k \ ,$$

$$\sum_{j\neq i}\pi_j\alpha_i^k = \sum_{j=1}^{N}\pi_j\alpha_j - \pi_i\alpha_i^k = \sum_{j=1}^{N}E(Z_j)E(W_j^k) - \pi_i\alpha_i^k = n_k - \pi_i\alpha_i^k \ ,$$

and

$$\begin{aligned}
\sum_{j\neq i}\pi_{ij}\alpha_{ij}^k &= \sum_{j\neq i}E(Z_iZ_j)E(W_i^kW_j^k) \\
&= E\left[\sum_{j\neq i}Z_iZ_jW_i^kW_j^k\right] \\
&= E\left[Z_iW_i^k\sum_{j\neq i}Z_jW_j^k\right] \\
&= E\left[Z_iW_i^k\left(\sum_{j=1}^{N}Z_jW_j^k - Z_iW_i^k\right)\right] \\
&= E[Z_iW_i^k(n_k - Z_iW_i^k)] \\
&= E[n_k(Z_iW_i^k) - (Z_iW_i^k)^2] \\
&= n_k\pi_i\alpha_i^k - \pi_i\alpha_i^k \\
&= \pi_i\alpha_i^k(n_k - 1)
\end{aligned}$$

Next, note that

$$
\begin{aligned}
&\sum_{i\neq j}\left[\left(\frac{y_i}{\pi_i\alpha_i^k}\right)^2+\left(\frac{y_j}{\pi_j\alpha_j^k}\right)^2\right](\pi_i\pi_j\alpha_i^k\alpha_j^k-\pi_{ij}\alpha_{ij}^k)\\
&=2\sum_{i=1}^{N}\left(\frac{y_i}{\pi_i\alpha_i^k}\right)^2\left[\sum_{j\neq i}(\pi_i\pi_j\alpha_i^k\alpha_j^k-\pi_{ij}\alpha_{ij}^k)\right]\\
&=2\sum_{i=1}^{N}\left(\frac{y_i}{\pi_i\alpha_i^k}\right)^2\left[\pi_i\alpha_i^k\sum_{j\neq i}\pi_j\alpha_j^k-\sum_{j\neq i}\pi_{ij}\alpha_{ij}^k\right]\\
&=2\sum_{i=1}^{N}\left(\frac{y_i}{\pi_i\alpha_i^k}\right)^2\left[\pi_i\alpha_i^k(n_k-\pi_i\alpha_i^k)-\pi_i\alpha_i^k(n_k-1)\right]\\
&=2\sum_{i=1}^{N}(y_i)^2\left[\frac{\pi_i\alpha_i^k-(\pi_i\alpha_i^k)^2}{(\pi_i\alpha_i^k)^2}\right]\\
&=2\sum_{i=1}^{N}(y_i)^2\left(\frac{1-\pi_i\alpha_i^k}{\pi_i\alpha_i^k}\right)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\sum_{i=1}^{N}(y_i)^2\left(\frac{1-\pi_i\alpha_i^k}{\pi_i\alpha_i^k}\right)+\sum_{i\neq j}y_iy_j\left(\frac{\pi_{ij}\alpha_{ij}^k-\pi_i\pi_j\alpha_i^k\alpha_j^k}{\pi_i\pi_j\alpha_i^k\alpha_j^k}\right)\\
&=\frac{1}{2}\left[2\sum_{i=1}^{N}(y_i)^2\left(\frac{1-\pi_i\alpha_i^k}{\pi_i\alpha_i^k}\right)\right]+\sum_{i\neq j}y_iy_j\left(\frac{\pi_{ij}\alpha_{ij}^k-\pi_i\pi_j\alpha_i^k\alpha_j^k}{\pi_i\pi_j\alpha_i^k\alpha_j^k}\right)\\
&=\frac{1}{2}\sum_{i\neq j}(\pi_i\pi_j\alpha_i^k\alpha_j^k-\pi_{ij}\alpha_{ij}^k)\left[\left(\frac{y_i}{\pi_i\alpha_i^k}\right)^2+\left(\frac{y_j}{\pi_j\alpha_j^k}\right)^2\right]+\sum_{i\neq j}y_iy_j\left(\frac{\pi_{ij}\alpha_{ij}^k-\pi_i\pi_j\alpha_i^k\alpha_j^k}{\pi_i\pi_j\alpha_i^k\alpha_j^k}\right)\\
&=\frac{1}{2}\sum_{i\neq j}(\pi_i\pi_j\alpha_i^k\alpha_j^k-\pi_{ij}\alpha_{ij}^k)\left[\frac{y_i}{\pi_i\alpha_i^k}-\frac{y_j}{\pi_j\alpha_j^k}\right]^2\\
&=\sum_{i=1}^{N}\sum_{j>i}(\pi_i\pi_j\alpha_i^k\alpha_j^k-\pi_{ij}\alpha_{ij}^k)\left[\frac{y_i}{\pi_i\alpha_i^k}-\frac{y_j}{\pi_j\alpha_j^k}\right]^2 .
\end{aligned}
$$

To identify an estimator of $var(\hat{\mu}_k)$ apply the necessary indicator variables and weights so that the estimator is unbiased. First, multiply each term by the appropriate indicators for the units included in the corresponding sum. The indicator variables in the first term are Z_i and W_i^k, and the indicator variables in the second term are Z_i, Z_j, W_i^k, and W_j^k. Next, multiply by the corresponding weights so that the estimator will be unbiased. The weights for the first term are $1/\pi_i$ and $1/\alpha_i^k$, since $E(Z_i) = \pi_i$ and $E(W_i^k) = \alpha_i^k$. The weights for the second term are $1/\pi_{ij}$ and $1/\alpha_{ij}^k$, since $E(Z_iZ_j) = \pi_{ij}$ and $E(W_i^kW_j^k) = \alpha_{ij}^k$.

Thus, an estimator of $var(\hat{\mu}_k)$ is given by

$$\begin{aligned}
\widehat{var}(\hat{\mu}_k) &= \frac{1}{N^2}\sum_{i=1}^{N}\frac{(y_i^k)^2}{\pi_i\alpha_i^k}\left(\frac{1-\pi_i\alpha_i^k}{\pi_i\alpha_i^k}\right)Z_iW_i^k \\
&\quad +\frac{1}{N^2}\sum_{i\neq j}\frac{(y_i^k y_j^k)}{\pi_{ij}\alpha_{ij}^k}\left(\frac{\pi_{ij}\alpha_{ij}^k-\pi_i\pi_j\alpha_i^k\alpha_j^k}{\pi_i\pi_j\alpha_j^k\alpha_i^k}\right)Z_iZ_jW_i^kW_j^k \\
&= \frac{1}{N^2}\sum_{i\in s}\frac{(y_i^k)^2}{\pi_i\alpha_i^k}\left(\frac{1-\pi_i\alpha_i^k}{\pi_i\alpha_i^k}\right) \\
&\quad +\frac{1}{N^2}\sum_{\substack{i,j\in s\\ i\neq j}}\frac{(y_i^k y_j^k)}{\pi_{ij}\alpha_{ij}^k}\left(\frac{\pi_{ij}\alpha_{ij}^k-\pi_i\pi_j\alpha_i^k\alpha_j^k}{\pi_i\pi_j\alpha_j^k\alpha_i^k}\right),
\end{aligned}$$

which is unbiased, since

$$\begin{aligned}
E\left[\widehat{var}(\hat{\mu}_k)\right] &= \frac{1}{N^2}\sum_{i=1}^{N}\frac{(y_i^k)^2}{\pi_i\alpha_i^k}\left(\frac{1-\pi_i\alpha_i^k}{\pi_i\alpha_i^k}\right)E(Z_i)E(W_i^k) \\
&\quad +\frac{1}{N^2}\sum_{i\neq j}\frac{(y_i^k y_j^k)}{\pi_{ij}\alpha_{ij}^k}\left(\frac{\pi_{ij}\alpha_{ij}^k-\pi_i\pi_j\alpha_i^k\alpha_j^k}{\pi_i\pi_j\alpha_j^k\alpha_i^k}\right)E(Z_iZ_j)E(W_i^kW_j^k) \\
&= \frac{1}{N^2}\sum_{i=1}^{N}\frac{(y_i^k)^2}{\pi_i\alpha_i^k}\left(\frac{1-\pi_i\alpha_i^k}{\pi_i\alpha_i^k}\right)\pi_i\alpha_i^k \\
&\quad +\frac{1}{N^2}\sum_{i\neq j}\frac{(y_i^k y_j^k)}{\pi_{ij}\alpha_{ij}^k}\left(\frac{\pi_{ij}\alpha_{ij}^k-\pi_i\pi_j\alpha_i^k\alpha_j^k}{\pi_i\pi_j\alpha_j^k\alpha_i^k}\right)\pi_{ij}\alpha_{ij}^k \\
&= \frac{1}{N^2}\left[\sum_{i=1}^{N}(y_i^k)^2\frac{1-\pi_i\alpha_i^k}{\pi_i\alpha_i^k}+\sum_{i\neq j}(y_i^ky_j^k)\frac{\pi_{ij}\alpha_{ij}^k-\pi_i\pi_j\alpha_i^k\alpha_j^k}{\pi_i\pi_j\alpha_j^k\alpha_i^k}\right].
\end{aligned}$$

The estimator $\widehat{var}(\hat{\mu}_k)$ is a homogeneous, unbiased quadratic estimator (HUQE). That is, the estimator is of the form $t(s,\mathbf{y})=\sum_{i\in s}a_{si}y_i^2+\sum_{i\neq j}a_{sij}y_i$ where the coefficients, a_{si} and a_{sij}, depend on the sample drawn but do not depend on y_i (Hedayat and Sinha, 1991).

Another look at the example of simple random sampling without replacement (srswor) with a completely randomized design (CRD), gives

$$\begin{aligned}
var(\hat{\mu}_k) &= \frac{1}{N^2}\sum_{i=1}^{N}(y_i^k)^2\frac{1-\pi_i\alpha_i^k}{\pi_i\alpha_i^k}+\frac{1}{N^2}\sum_{i\neq j}(y_i^ky_j^k)\frac{\pi_{ij}\alpha_{ij}^k-\pi_i\pi_j\alpha_i^k\alpha_j^k}{\pi_i\pi_j\alpha_j^k\alpha_i^k} \\
&= \frac{1}{N^2}\sum_{i=1}^{N}(y_i^k)^2\frac{1-(\frac{n}{N})(\frac{n_k}{n})}{(\frac{n}{N})(\frac{n_k}{n})}+\frac{1}{N^2}\sum_{i\neq j}(y_i^ky_j^k)\frac{\frac{n(n-1)}{N(N-1)}\cdot\frac{n_k(n_k-1)}{n(n-1)}-(\frac{n}{N})^2(\frac{n_k}{n})^2}{(\frac{n}{N})^2(\frac{n_k}{n})^2} \\
&= \frac{1}{N^2}\sum_{i=1}^{N}(y_i^k)^2\frac{1-\frac{n_k}{N}}{\frac{n_k}{N}}+\frac{1}{N^2}\sum_{i\neq j}(y_i^ky_j^k)\frac{\frac{n_k(n_k-1)}{N(N-1)}-(\frac{n_k}{N})^2}{(\frac{n_k}{N})^2}
\end{aligned}$$

$$= \frac{1}{N^2}\sum_{i=1}^{N}(y_i^k)^2\frac{N-n_k}{n_k} + \frac{1}{N^2}\sum_{i\neq j}(y_i^k y_j^k)\frac{n_k - N}{n_k(N-1)} \ ,$$

and

$$\begin{aligned}
\widehat{var}(\hat{\mu}_k) &= \frac{1}{N^2}\sum_{i=1}^{N}(y_i^k)^2\frac{1-\pi_i\alpha_i^k}{(\pi_i\alpha_i^k)^2}Z_iW_i^k + \frac{1}{N^2}\sum_{i\neq j}(y_i^k y_j^k)\frac{\pi_{ij}\alpha_{ij}^k - \pi_i\pi_j\alpha_i^k\alpha_j^k}{(\pi_{ij}\alpha_{ij}^k)\pi_i\pi_j\alpha_j^k\alpha_i^k}Z_iZ_jW_i^kW_j^k \\
&= \frac{1}{N^2}\sum_{i=1}^{N}(y_i^k)^2\frac{1-(\frac{n}{N})(\frac{n_k}{n})}{(\frac{n}{N})^2(\frac{n_k}{n})^2}Z_iW_i^k \\
&\quad + \frac{1}{N^2}\sum_{i\neq j}(y_i^k y_j^k)\frac{\frac{n(n-1)}{N(N-1)}\cdot\frac{n_k(n_k-1)}{n(n-1)} - (\frac{n}{N})^2(\frac{n_k}{n})^2}{\frac{n(n-1)}{N(N-1)}\cdot\frac{n_k(n_k-1)}{n(n-1)}(\frac{n}{N})^2(\frac{n_k}{n})^2}Z_iZ_jW_i^kW_j^k \\
&= \frac{1}{N^2}\sum_{i=1}^{N}(y_i^k)^2\frac{1-\frac{n_k}{N}}{(\frac{n_k}{N})^2}Z_iW_i^k + \frac{1}{N^2}\sum_{i\neq j}(y_i^k y_j^k)\frac{\frac{n_k(n_k-1)}{N(N-1)} - (\frac{n_k}{N})^2}{\frac{n_k(n_k-1)}{N(N-1)}(\frac{n_k}{N})^2}Z_iZ_jW_i^kW_j^k \\
&= \frac{1}{N^2}\sum_{i=1}^{N}(y_i^k)^2\frac{N(N-n_k)}{(n_k)^2}Z_iW_i^k + \frac{1}{N^2}\sum_{i\neq j}(y_i^k y_j^k)\frac{N(n_k-N)}{(n_k)^2(n_k-1)}Z_iZ_jW_i^kW_j^k \\
&= \frac{1}{N}\sum_{i=1}^{N}(y_i^k)^2\frac{N-n_k}{(n_k)^2}Z_iW_i^k + \frac{1}{N}\sum_{i\neq j}(y_i^k y_j^k)\frac{n_k-N}{(n_k)^2(n_k-1)}Z_iZ_jW_i^kW_j^k \ .
\end{aligned}$$

As mentioned in the previous chapter estimates of the variance of the HTE can be negative. The same problem exists for $\widehat{var}(\hat{\mu}_k)$. That is, $\widehat{var}(\hat{\mu}_k)$ can be negative, in particular, when $\pi_{ij}\alpha_{ij}^k - \pi_i\pi_j\alpha_i^k\alpha_j^k < 0$. As in the case of the HTE, the alternative form of the variance estimate is less likely to be negative if $\pi_i\pi_j\alpha_i^k\alpha_j^k - \pi_{ij}\alpha_{ij}^k > 0$. For the current example of simple random sampling without replacement (srswor) with a completely randomized design (CRD)

$$\pi_i\pi_j\alpha_i^k\alpha_j^k - \pi_{ij}\alpha_{ij}^k = \frac{n_k}{N}\left(\frac{N-n_k}{N(N-1)}\right) > 0 \ ,$$

since $N > n_k$.

Finally, consider the estimate of the treatment difference, $\hat{\mu}_k - \hat{\mu}_{k'}$. As mentioned, this estimate is unbiased for $\mu_k - \mu_{k'}$. The variance of $\hat{\mu}_k - \hat{\mu}_{k'}$ is given by

$$\begin{aligned}
var(\hat{\mu}_k - \hat{\mu}_{k'}) &= var(\hat{\mu}_k) + var(\hat{\mu}_{k'}) - 2cov(\hat{\mu}_k, \hat{\mu}_{k'}) \\
&= var(\hat{\mu}_k) + var(\hat{\mu}_{k'}) - 2[E(\hat{\mu}_k\hat{\mu}_{k'}) - E(\hat{\mu}_k)E(\hat{\mu}_{k'})]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{N^2}\sum_{i=1}^{N}(y_i^k)^2\frac{1-\pi_i\alpha_i^k}{\pi_i\alpha_i^k}+\frac{1}{N^2}\sum_{i\neq j}(y_i^k y_j^k)\frac{\pi_{ij}\alpha_{ij}^k-\pi_i\pi_j\alpha_i^k\alpha_j^k}{\pi_i\pi_j\alpha_j^k\alpha_i^k}\\
&\quad +\frac{1}{N^2}\sum_{i=1}^{N}(y_i^{k'})^2\frac{1-\pi_i\alpha_i^{k'}}{\pi_i\alpha_i^{k'}}+\frac{1}{N^2}\sum_{i\neq j}(y_i^{k'} y_j^{k'})\frac{\pi_{ij}\alpha_{ij}^{k'}-\pi_i\pi_j\alpha_i^{k'}\alpha_j^{k'}}{\pi_i\pi_j\alpha_j^{k'}\alpha_i^{k'}}\\
&\quad -2\left[\frac{1}{N^2}\sum_{i,j}y_i^k y_j^{k'}\frac{\pi_{ij}\alpha_{ij}^{kk'}}{\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}-\frac{1}{N^2}\sum_{i,j}y_i^k y_j^{k'}\right]\\
&= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\frac{1-\pi_i\alpha_i^t}{\pi_i\alpha_i^t}+\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^t y_j^t)\frac{\pi_{ij}\alpha_{ij}^t-\pi_i\pi_j\alpha_i^t\alpha_j^t}{\pi_i\pi_j\alpha_j^t\alpha_i^t}\\
&\quad -\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}y_i^k y_j^{k'}\left[\frac{\pi_{ij}\alpha_{ij}^{kk'}}{\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}-1\right] ,
\end{aligned}
$$

since

$$
\begin{aligned}
E(\hat{\mu}_k\hat{\mu}_{k'}) &= \frac{1}{N^2}\sum_{i,j}\frac{y_i^k y_j^{k'}}{\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}E(Z_iZ_jW_i^kW_j^{k'})\\
&= \frac{1}{N^2}\sum_{i,j}\frac{y_i^k y_j^{k'}}{\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}E(Z_iZ_j)E(W_i^kW_j^{k'})\\
&= \frac{1}{N^2}\sum_{i,j}\frac{y_i^k y_j^{k'}}{\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}\pi_{ij}P(W_i^k=1,W_j^{k'}=1)\\
&= \frac{1}{N^2}\sum_{i,j}\frac{y_i^k y_j^{k'}}{\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}\pi_{ij}\alpha_{ij}^{kk'} ,
\end{aligned}
$$

where $\alpha_{ij}^{kk'}=P(W_i^k=1,W_j^{k'}=1)$.

Now, to derive $\alpha_{ij}^{kk'}$, also define $\alpha_{ij}^{k'k}=P(W_i^{k'}=1,W_j^k=1)$. Note that $\alpha_{ij}^{kk'}=\alpha_{ij}^{k'k}$ since $E(\hat{\mu}_k\hat{\mu}_{k'})=E(\hat{\mu}_{k'}\hat{\mu}_k)$, which implies that $P(W_i^k=1,W_j^{k'}=1)=P(W_i^{k'}=1,W_j^k=1)$. Then

$$
\begin{aligned}
\alpha_{ij}^{kk'} &= \frac{1}{2}\left(\alpha_{ij}^{kk'}+\alpha_{ij}^{k'k}\right)\\
&= \frac{1}{2}\left(\alpha_i^k+\alpha_i^{k'}+\alpha_j^k+\alpha_j^{k'}-\alpha_{ij}^k-\alpha_{ij}^{k'}-1\right) .
\end{aligned}
$$

To understand the last step, first, consider assigning the treatments k, k', and not k or k', say, l. Then unit i can receive k, k', or l and similarly, unit j can receive k, k', or l. Probabilities for each case are depicted in Table 3.1. Note that,

$$
\alpha_i^k = P(\text{unit } i \text{ gets } k \text{ and unit } j \text{ gets } k,\, k',\, \text{or } l)
$$

$$
\begin{aligned}
&= P(\text{unit } i \text{ gets } k \text{ and unit } j \text{ gets } k) + P(\text{unit } i \text{ gets } k \text{ and unit } j \text{ gets } k') \\
&\qquad + P(\text{unit } i \text{ gets } k \text{ and unit } j \text{ gets } l) \\
&= \alpha_{ij}^{k} + \alpha_{ij}^{kk'} + \alpha_{ij}^{kl} \ .
\end{aligned}
$$

This implies that

$$\alpha_{ij}^{kl} = \alpha_i^k - \alpha_{ij}^k - \alpha_{ij}^{kk'} \ .$$

Now, all of the probabilities listed in Table 3.1 add to one. That is,

$$1 = \alpha_i^k + \alpha_i^{k'} + \alpha_j^k + \alpha_j^{k'} - \alpha_{ij}^k - \alpha_{ij}^{k'} - \alpha_{ij}^{kk'} - \alpha_{ij}^{k'k} \ .$$

Thus,

$$\alpha_{ij}^{kk'} + \alpha_{ij}^{k'k} = \alpha_i^k + \alpha_i^{k'} + \alpha_j^k + \alpha_j^{k'} - \alpha_{ij}^k - \alpha_{ij}^{k'} - 1 \ .$$

That is,

$$\alpha_{ij}^{kk'} = \frac{1}{2}\left(\alpha_i^k + \alpha_i^{k'} + \alpha_j^k + \alpha_j^{k'} - \alpha_{ij}^k - \alpha_{ij}^{k'} - 1\right) \ .$$

Table 3.1: Probabilities for treatments k, k', or l (not k or k')

i	j	Joint Assignment Probability
k	k	α_{ij}^k
k	k'	$\alpha_{ij}^{kk'}$
k'	k	$\alpha_{ij}^{k'k}$
k'	k'	$\alpha_{ij}^{k'}$
k	l	$\alpha_i^k - \alpha_{ij}^k - \alpha_{ij}^{kk'}$
k'	l	$\alpha_i^{k'} - \alpha_{ij}^{k'} - \alpha_{ij}^{k'k}$
l	k	$\alpha_j^k - \alpha_{ij}^k - \alpha_{ij}^{kk'}$
l	k'	$\alpha_j^{k'} - \alpha_{ij}^{k'} - \alpha_{ij}^{k'k}$

Then the variance of $\hat{\mu}_k - \hat{\mu}_{k'}$ is given by

$$var(\hat{\mu}_k - \hat{\mu}_{k'}) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\frac{1-\pi_i\alpha_i^t}{\pi_i\alpha_i^t} + \frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^t y_j^t)\frac{\pi_{ij}\alpha_{ij}^t - \pi_i\pi_j\alpha_i^t\alpha_j^t}{\pi_i\pi_j\alpha_j^t\alpha_i^t}$$

$$
\begin{aligned}
& -\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} y_i^k y_j^{k'} \left[\frac{\pi_{ij}\alpha_{ij}^{kk'}}{\pi_i\pi_j\alpha_i^k\alpha_j^{k'}} - 1\right] \\
= \; & \frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'} (y_i^t)^2 \frac{1-\pi_i\alpha_i^t}{\pi_i\alpha_i^t} + \frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'} (y_i^t y_j^t)\frac{\pi_{ij}\alpha_{ij}^t - \pi_i\pi_j\alpha_i^t\alpha_j^t}{\pi_i\pi_j\alpha_j^t\alpha_i^t} \\
& -\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} y_i^k y_j^{k'} \left[\frac{\pi_{ij}(\alpha_i^k + \alpha_i^{k'} + \alpha_j^k + \alpha_j^{k'} - \alpha_{ij}^k - \alpha_{ij}^{k'} - 1)}{2\pi_i\pi_j\alpha_i^k\alpha_j^{k'}} - 1\right] .
\end{aligned}
$$

Note that, for the case where only two treatments are applied $\alpha_{ij}^{kk'}$ simplifies to $\frac{1}{2}(1-\alpha_{ij}^k-\alpha_{ij}^{k'})$, since $\alpha_i^k + \alpha_i^{k'} = 1$ and $\alpha_j^k + \alpha_j^{k'} = 1$. Thus, when only two treatments are applied the estimate of the variance of the difference simplifies to

$$
\begin{aligned}
var(\hat{\mu}_k - \hat{\mu}_{k'}) = \; & \frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'} (y_i^t)^2 \frac{1-\pi_i\alpha_i^t}{\pi_i\alpha_i^t} + \frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'} (y_i^t y_j^t)\frac{\pi_{ij}\alpha_{ij}^t - \pi_i\pi_j\alpha_i^t\alpha_j^t}{\pi_i\pi_j\alpha_j^t\alpha_i^t} \\
& -\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} y_i^k y_j^{k'} \left[\frac{\pi_{ij}(1-\alpha_{ij}^k-\alpha_{ij}^{k'})}{2\pi_i\pi_j\alpha_i^k\alpha_j^{k'}} - 1\right] .
\end{aligned}
$$

To get an estimator of the variance of the difference, multiply by the appropriate indicator variables and corresponding weights as described before. An estimator of $\widehat{var}(\hat{\mu}_k - \hat{\mu}_{k'})$ is given by

$$
\begin{aligned}
\widehat{var}(\hat{\mu}_k - \hat{\mu}_{k'}) = \; & \frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'} (y_i^t)^2 \frac{1-\pi_i\alpha_i^t}{\pi_i\alpha_i^t}\cdot\frac{Z_iW_i^t}{\pi_i\alpha_i^t} \\
& +\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'} (y_i^t y_j^t)\frac{\pi_{ij}\alpha_{ij}^t - \pi_i\pi_j\alpha_i^t\alpha_j^t}{\pi_i\pi_j\alpha_j^t\alpha_i^t}\cdot\frac{Z_iZ_jW_i^tW_j^t}{\pi_{ij}\alpha_{ij}^t} \\
& -\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} y_i^k y_j^{k'} \left[\frac{\pi_{ij}(1-\alpha_{ij}^k-\alpha_{ij}^{k'})}{2\pi_i\pi_j\alpha_i^k\alpha_j^{k'}} - 1\right]\cdot\frac{Z_iZ_jW_i^kW_j^{k'}}{\pi_{ij}(1-\alpha_{ij}^k-\alpha_{ij}^{k'})} \\
= \; & \frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'} (y_i^t)^2 \left[\frac{1-\pi_i\alpha_i^t}{(\pi_i\alpha_i^t)^2}\right] Z_iW_i^t \\
& +\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'} (y_i^t y_j^t)\left[\frac{\pi_{ij}\alpha_{ij}^t - \pi_i\pi_j\alpha_i^t\alpha_j^t}{\pi_{ij}\alpha_{ij}^t(\pi_i\pi_j\alpha_j^t\alpha_i^t)}\right] Z_iZ_jW_i^tW_j^t \\
& -\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} y_i^k y_j^{k'} \left[\frac{\pi_{ij}(1-\alpha_{ij}^k-\alpha_{ij}^{k'}) - 2\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}{2\pi_{ij}(1-\alpha_{ij}^k-\alpha_{ij}^{k'})\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}\right] Z_iZ_jW_i^kW_j^{k'} .
\end{aligned}
$$

Clearly, this estimator is unbiased since

$$
E[\widehat{var}(\hat{\mu}_k - \hat{\mu}_{k'})]
$$

$$
\begin{aligned}
&= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\left[\frac{1-\pi_i\alpha_i^t}{(\pi_i\alpha_i^t)^2}\right]E(Z_i)E(W_i^t) \\
&\quad +\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^t y_j^t)\left[\frac{\pi_{ij}\alpha_{ij}^t-\pi_i\pi_j\alpha_i^t\alpha_j^t}{\pi_{ij}\alpha_{ij}^t(\pi_i\pi_j\alpha_j^t\alpha_i^t)}\right]E(Z_iZ_j)E(W_i^tW_j^t) \\
&\quad -\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}y_i^k y_j^{k'}\left[\frac{\pi_{ij}(1-\alpha_{ij}^k-\alpha_{ij}^{k'})-2\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}{2\pi_{ij}(1-\alpha_{ij}^k-\alpha_{ij}^{k'})\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}\right]E(Z_iZ_j)E(W_i^kW_j^{k'}) \\
&= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\left[\frac{1-\pi_i\alpha_i^t}{(\pi_i\alpha_i^t)^2}\right]\pi_i\alpha_i^t \\
&\quad +\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^t y_j^t)\left[\frac{\pi_{ij}\alpha_{ij}^t-\pi_i\pi_j\alpha_i^t\alpha_j^t}{\pi_{ij}\alpha_{ij}^t(\pi_i\pi_j\alpha_j^t\alpha_i^t)}\right]\pi_{ij}\alpha_{ij}^t \\
&\quad -\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}y_i^k y_j^{k'}\left[\frac{\pi_{ij}(1-\alpha_{ij}^k-\alpha_{ij}^{k'})-2\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}{2\pi_{ij}(1-\alpha_{ij}^k-\alpha_{ij}^{k'})\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}\right]\pi_{ij}(1-\alpha_{ij}^k-\alpha_{ij}^{k'}) \\
&= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\frac{1-\pi_i\alpha_i^t}{\pi_i\alpha_i^t}+\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^t y_j^t)\frac{\pi_{ij}\alpha_{ij}^t-\pi_i\pi_j\alpha_i^t\alpha_j^t}{\pi_i\pi_j\alpha_j^t\alpha_i^t} \\
&\quad -\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}y_i^k y_j^{k'}\left[\frac{\pi_{ij}(1-\alpha_{ij}^k-\alpha_{ij}^{k'})}{2\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}-1\right]\,.
\end{aligned}
$$

Note that this estimator can be generalized to the case where $T>2$ by replacing $1-\alpha_{ij}^k-\alpha_{ij}^{k'}$ with $\alpha_i^k+\alpha_i^{k'}+\alpha_j^k+\alpha_j^{k'}-\alpha_{ij}^k-\alpha_{ij}^{k'}-1$ for the probability $\alpha_{ij}^{kk'}$. Also, note that for probability sampling the variances described here and their estimators depend on the unit inclusion probabilities and the treatment assignment probabilities, π_i, π_{ij}, α_i^k, and α_{ij}^k, respectively.

Now, for the special case of srswor and a CRD, with only two treatments,

$$
\begin{aligned}
&var(\hat{\mu}_k-\hat{\mu}_{k'}) \\
&= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\left[\frac{1-\pi_i\alpha_i^t}{\pi_i\alpha_i^t}\right] \\
&\quad +\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^t y_j^t)\left[\frac{\pi_{ij}\alpha_{ij}^t-\pi_i\pi_j\alpha_i^t\alpha_j^t}{\pi_i\pi_j\alpha_j^t\alpha_i^t}\right] \\
&\quad -\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}y_i^k y_j^{k'}\left[\frac{\pi_{ij}(1-\alpha_{ij}^k-\alpha_{ij}^{k'})}{2\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}-1\right] \\
&= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\left[\frac{1-\frac{n}{N}\cdot\frac{n_t}{n}}{\frac{n}{N}\cdot\frac{n_t}{n}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^t y_j^t)\left[\frac{\frac{n(n-1)}{N(N-1)}\cdot\frac{n_t(n_t-1)}{n(n-1)}-(\frac{n}{N})^2(\frac{n_t}{n})^2}{(\frac{n}{N})^2(\frac{n_t}{n})^2}\right]\\
&-\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}y_i^k y_j^{k'}\left\{\frac{\frac{n(n-1)}{N(N-1)}\left[1-\frac{n_k(n_k-1)}{n(n-1)}-\frac{n_{k'}(n_{k'}-1)}{n(n-1)}\right]}{2(\frac{n}{N})^2\cdot\frac{n_k}{n}\cdot\frac{n_{k'}}{n}}-1\right\}\\
=&\ \frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\left[\frac{1-\frac{n_t}{N}}{\frac{n_t}{N}}\right]\\
&+\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^t y_j^t)\left[\frac{\frac{n_t(n_t-1)}{N(N-1)}-(\frac{n_t}{N})^2}{(\frac{n_t}{N})^2}\right]\\
&-\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}y_i^k y_j^{k'}\left[\frac{\frac{n(n-1)-n_k(n_k-1)-n_{k'}(n_{k'}-1)}{N(N-1)}}{2\frac{n_k n_{k'}}{N^2}}-1\right]\\
=&\ \frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\left[\frac{N}{n_t}-1\right]\\
&+\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^t y_j^t)\left[\frac{N(n_t-1)}{n_t(N-1)}-1\right]\\
&-\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}y_i^k y_j^{k'}\left[\frac{N^2}{2n_k n_{k'}}\cdot\frac{n(n-1)-n_k(n_k-1)-n_{k'}(n_{k'}-1)}{N(N-1)}-1\right]\\
=&\ \frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\left(\frac{N-n_t}{n_t}\right)\\
&+\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^t y_j^t)\left[\frac{n_t-N}{n_t(N-1)}\right]\\
&-\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}y_i^k y_j^{k'}\left\{\frac{N[n(n-1)-n_k(n_k-1)-n_{k'}(n_{k'}-1)]}{2n_k n_{k'}(N-1)}-1\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\widehat{var}(\hat{\mu}_k-\hat{\mu}_{k'})\\
=&\ \frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\left[\frac{1-\pi_i\alpha_i^t}{(\pi_i\alpha_i^t)^2}\right]Z_iW_i^t\\
&+\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^t y_j^t)\left[\frac{\pi_{ij}\alpha_{ij}^t-\pi_i\pi_j\alpha_i^t\alpha_j^t}{\pi_{ij}\alpha_{ij}^t(\pi_i\pi_j\alpha_j^t\alpha_i^t)}\right]Z_iZ_jW_i^tW_j^t\\
&-\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}y_i^k y_j^{k'}\left[\frac{\pi_{ij}(1-\alpha_{ij}^k-\alpha_{ij}^{k'})-2\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}{2\pi_{ij}(1-\alpha_{ij}^k-\alpha_{ij}^{k'})\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}\right]Z_iZ_jW_i^kW_j^{k'}\\
=&\ \frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\left[\frac{1-\frac{n}{N}\cdot\frac{n_t}{n}}{(\frac{n}{N}\cdot\frac{n_t}{n})^2}\right]Z_iW_i^t
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^t y_j^t)\left[\frac{\frac{n(n-1)}{N(N-1)}\cdot\frac{n_t(n_t-1)}{n(n-1)}-(\frac{n}{N})^2(\frac{n_t}{n})^2}{\frac{n(n-1)}{N(N-1)}\cdot\frac{n_t(n_t-1)}{n(n-1)}(\frac{n}{N})^2(\frac{n_t}{n})^2}\right]Z_iZ_jW_i^tW_j^t\\
&-\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}y_i^k y_j^{k'}\left\{\frac{\frac{n(n-1)}{N(N-1)}\left[1-\frac{n_k(n_k-1)}{n(n-1)}-\frac{n_{k'}(n_{k'}-1)}{n(n-1)}\right]-2(\frac{n}{N})^2\cdot\frac{n_k}{n}\cdot\frac{n_{k'}}{n}}{2\frac{n(n-1)}{N(N-1)}\left[1-\frac{n_k(n_k-1)}{n(n-1)}-\frac{n_{k'}(n_{k'}-1)}{n(n-1)}\right](\frac{n}{N})^2\cdot\frac{n_k}{n}\cdot\frac{n_{k'}}{n}}\right\}Z_iZ_jW_i^kW_j^{k'}\\
=&\frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\left[\frac{1-\frac{n_t}{N}}{(\frac{n_t}{N})^2}\right]Z_iW_i^t\\
&+\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^t y_j^t)\left[\frac{\frac{n_t(n_t-1)}{N(N-1)}-(\frac{n_t}{N})^2}{\frac{n_t(n_t-1)}{N(N-1)}(\frac{n_t}{N})^2}\right]Z_iZ_jW_i^tW_j^t\\
&-\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}y_i^k y_j^{k'}\left[\frac{\frac{n(n-1)-n_k(n_k-1)-n_{k'}(n_{k'}-1)}{N(N-1)}-2(\frac{n_kn_{k'}}{N^2})}{2\frac{n(n-1)-n_k(n_k-1)-n_{k'}(n_{k'}-1)}{N(N-1)}\cdot\frac{n_kn_{k'}}{N^2}}\right]Z_iZ_jW_i^kW_j^{k'}\\
=&\frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\left[\left(\frac{N}{n_t}\right)^2-\frac{N}{n_t}\right]Z_iW_i^t\\
&+\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^t y_j^t)\left[\left(\frac{N}{n_t}\right)^2-\frac{N(N-1)}{n_t(n_t-1)}\right]Z_iZ_jW_i^tW_j^t\\
&-\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}y_i^k y_j^{k'}\left[\frac{N^2}{2n_kn_{k'}}-\frac{N(N-1)}{n(n-1)-n_k(n_k-1)-n_{k'}(n_{k'}-1)}\right]Z_iZ_jW_i^kW_j^{k'}\\
=&\frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2N\left(\frac{N-n_t}{n_t^2}\right)Z_iW_i^t\\
&+\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^t y_j^t)N\left[\frac{n_t-N}{(n_t)^2(n_t-1)}\right]Z_iZ_jW_i^tW_j^t\\
&-\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}y_i^k y_j^{k'}N\left[\frac{N}{2n_kn_{k'}}-\frac{N-1}{n(n-1)-n_k(n_k-1)-n_{k'}(n_{k'}-1)}\right]Z_iZ_jW_i^kW_j^{k'}\\
=&\frac{1}{N}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\frac{N-n_t}{(n_t)^2}Z_iW_i^t\\
&+\frac{1}{N}\sum_{i\neq j}\sum_{t=k,k'}(y_i^t y_j^t)\frac{n_t-N}{(n_t)^2(n_t-1)}Z_iZ_jW_i^tW_j^t\\
&-\frac{2}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}y_i^k y_j^{k'}\left[\frac{N}{2n_kn_{k'}}-\frac{N-1}{n(n-1)-n_k(n_k-1)-n_{k'}(n_{k'}-1)}\right]Z_iZ_jW_i^kW_j^{k'}\ .
\end{aligned}
$$

Chapter 4

Applications of Thompson's Estimator to Specific Designs

The ideas previously discussed in developing $\hat{\mu}_k$ for simple random sampling without replacement (srswor) and completely randomized design (CRD) can be extended to other sampling and experimental designs. Thompson did not attempt to make any application of $\hat{\mu}_k$ in his 2002 proposal and discussion. Since the goal of the current research is to extend Thompson's proposal and discussion of $\hat{\mu}_k$, the application of $\hat{\mu}_k$ to a few basic sampling and experimental designs will be considered here. First, the inclusion and assignment probabilities will be developed for these sampling and experimental designs. Table 4.1 gives a summery of the inclusion and assignment probabilities for selected designs. Then $\hat{\mu}_k$ will be evaluated for some combinations of sampling and experimental designs. Table 4.2 contains a summary of these estimates for selected combinations of sampling and experimental designs.

4.1 Probabilities for Selected Experimental Designs

Assignment probabilities have not been presented elsewhere in the literature for any experimental design. However, only completely randomized and randomized complete block designs will be presented here. There is a multitude of interesting and useful experimental designs but these two seem to be the most basic and most commonly used by researchers. The study of other, more complex experimental designs, with regard to assignment probabilities is left to future research.

The CRD (completely randomized design) experimental design has already been discussed (see Chapter 3 Section 1). The assignment probabilities for CRD were shown to be

$$\alpha_i^k = \frac{n_k}{n} \qquad \text{and} \qquad \alpha_{ij}^k = \frac{n_k(n_k-1)}{n(n-1)} \ .$$

In a *randomized complete block* (RCB) experimental design, experimental units are assigned to B blocks. The idea is to group units that share some character trait, so the assignment to blocks is typically not random but deterministic. The units within each block are assigned to treatments in a completely randomized manner. Thus, since the assignment probability for a given unit is not influenced by the probability that the unit is within a given block,

$$\alpha_{i_b}^k = \frac{n_{bk}}{n_b} \qquad \text{and} \qquad \alpha_{i_b j_b}^k = \frac{n_{bk}(n_{bk}-1)}{n_b(n_b-1)} \ ,$$

where $b = 1, 2, ..., B$, n_{bk} is the number of units assigned to treatment k in block b and n_b is the number of units assigned to block b. Under a balanced RCB, where the number of units assigned to each treatment is equal across blocks, that is, $n_{bk} = n_{bk'}$ for $k \neq k'$,

$$\alpha_{i_b}^k = \alpha_i^k = \frac{n_k}{n} \qquad \text{and} \qquad \alpha_{i_b j_b}^k = \alpha_{ij}^k = \frac{n_k(n_k-1)}{n(n-1)} \ ,$$

where $n_k = \sum_{i=1}^B n_{bk}$ and $n = \sum_{i=1}^B n_b$. That is, the assignment probabilities in a balanced RCB simplify to the assignment probabilities in a CRD.

4.2 Probabilities for Selected Sampling Designs

The inclusion probabilities of many sampling designs have been derived in previous works, such as Särndal, Swensson and Wretmen (1992) and Lohr (1999). The inclusion probabilities for a few of those sampling designs will be restated here. There are some sampling designs for which the inclusion probabilities have not been derived, however, those will be left to future research for the time being.

The srswor (simple random sampling without replacement) sampling design was presented in previous sections (see Sections 2.1 and 3.1). The inclusion probabilities for srswor were shown to be

$$\pi_i = \frac{n}{N} \qquad \text{and} \qquad \pi_{ij} = \frac{n(n-1)}{N(N-1)} \ .$$

In *Poisson* sampling the sample is selected in the following manner. Let $Z_1, ..., Z_N$ be indicator variables, where Z_i is one if unit i is in the sample and zero otherwise. Then Z_i are independent $Bernoulli(\pi_i)$ random variables, with $0 < \pi_i \leq 1$. That is, π_i can be different for each unit in the population, the sample size, $n(s)$, $0 \leq n(s) \leq N$, is random, and $\pi_{ij} = \pi_i \pi_j$. *Bernouli* sampling is a special case of Poisson sampling where $\pi_i = \pi$, $\pi_{ij} = \pi^2$, and $n(s) = n$.

Under *stratified* sampling the population is divided into H mutually exclusive subgroups, called strata, of size N_h, $h = 1, 2, ..., H$, where $N = \sum_{i=1}^{H} N_h$. A sample, s_h, of size n_h, $0 \leq n_h \leq N_h$, is selected from each stratum. The set of stratum samples are combined to create the stratified sample, that is, $s = s_1 \cup s_2 \cup \cdots \cup s_H$.

The inclusion probabilities generated for stratified sampling depend on the sampling design used within each stratum. The sampling design and the sample size do not have to be the same for all strata, but they typically are. Consider, for example, a stratified sampling design where srswor is used within each stratum. Then

$$\pi_{i_h} = \frac{n_h}{N_h} \qquad \text{and} \qquad \pi_{i_h j_h} = \frac{n_h(n_h-1)}{N_h(N_h-1)} \ .$$

A balanced stratified sampling design, where $N_h = N_{h'}$ and $n_h = n_{h'}$, for $h \neq h'$, simplifies to a srswor design.

Table 4.1: Probabilities for Selected Designs

	Design	Individual Probability	Joint Probability
Sampling	SRSWOR	$\frac{n}{N}$	$\frac{n(n-1)}{N(N-1)}$
Designs	Poisson	π_i	$\pi_i\pi_j,\ i \neq j$
	Stratified (srswor)	$\frac{n_h}{N_h}$	$\frac{n_h(n_h-1)}{N_h(N_h-1)}$
Experimental	CRD	$\frac{n_k}{n}$	$\frac{n_k(n_k-1)}{n(n-1)}$
Designs	RCB	$\frac{n_{bk}}{n_b}$	$\frac{n_{bk}(n_{bk}-1)}{n_b(n_b-1)}$

4.3 Thompson's Estimator for Selected Designs

In some cases Thompson's estimator, $\hat{\mu}_k$, is the same as the standard estimator, $\bar{y}_k$. For example, recall that for the case where srswor and CRD are used we have

$$\pi_i = P(Z_i = 1) = \frac{n}{N} \qquad \text{and} \qquad \alpha_i^k = P(W_i^k = 1) = \frac{n_k}{n} \ .$$

Then

$$\begin{aligned} \hat{\mu}_k &= \frac{1}{N}\sum_{i\in s}\frac{y_i^k}{\pi_i\alpha_i^k} \\ &= \frac{1}{N}\sum_{i\in s} y_i^k \left(\frac{N}{n}\right)\left(\frac{n}{n_k}\right) \\ &= \frac{1}{n_k}\sum_{i\in s} y_i^k \\ &= \bar{y}_k \ , \end{aligned}$$

Recall that the inclusion probabilities for a balanced stratified sampling design with srswor in each strata simplify to the inclusion probabilities of a srswor and the assignment probabilities for a balanced RCD simplify to the assignment probabilities of a CRD. Whenever, a sampling design simplifies to srswor and the experimental design simplifies to CRD, $\hat{\mu}_k$ will simplify to $\bar{y}_k$. Example 1 demonstrates that $\hat{\mu}_k$ from a sampling design that simplifies to srswor combined with a experimental design that does not simplify to CRD does not simplify to $\bar{y}_k$. Example 2 demonstrates that $\hat{\mu}_k$ from a sampling design that does not simplify to srswor combined with a experimental design that does simplify to CRD does not simplify to $\bar{y}_k$. Note that all of the derivations in Section 4.3 represents original work.

Example 1 (srswor):

$$\begin{aligned}
\hat{\mu}_k &= \frac{1}{N}\sum_{i\in s}\frac{y_i^k}{\pi_i\alpha_i^k} \\
&= \frac{1}{N}\sum_{i\in s} y_i^k\left(\frac{N}{n}\right)\left(\frac{1}{\alpha_i^k}\right) \\
&= \frac{1}{n}\sum_{i\in s}\frac{y_i^k}{\alpha_i^k} \\
&\neq \bar{y}_k
\end{aligned}$$

Example 2 (CRD):

$$\begin{aligned}
\hat{\mu}_k &= \frac{1}{N}\sum_{i\in s}\frac{y_i^k}{\pi_i\alpha_i^k} \\
&= \frac{1}{N}\sum_{i\in s} y_i^k\left(\frac{1}{\pi_i}\right)\left(\frac{n}{n_k}\right) \\
&= \frac{1}{N}\left(\frac{n}{n_k}\right)\sum_{i\in s}\frac{y_i^k}{\pi_i} \\
&\neq \bar{y}_k
\end{aligned}$$

Now consider $\hat{\mu}_k$ for some other, more interesting, design combinations. Note that if these designs are "balanced" they will simplify to the less interesting cases just discussed.

Poisson Sampling and CRD:

$$\begin{aligned}
\hat{\mu}_k &= \frac{1}{N}\sum_{i\in s}\frac{y_i^k}{\pi_i\alpha_i^k} \\
&= \frac{1}{N}\sum_{i\in s} y_i^k\left(\frac{1}{\pi_i}\right)\left(\frac{n(s)}{n_k}\right) \\
&= \frac{n(s)}{N}\sum_{i\in s}\frac{y_i^k}{\pi_i n_k} \\
&= \frac{1}{N}\sum_{i\in s}\frac{y_i^k}{\pi_i}
\end{aligned}$$

Stratified Sampling and CRD:

$$
\begin{aligned}
\hat{\mu}_k &= \frac{1}{N}\sum_{i\in s}\frac{y_i^k}{\pi_i\alpha_i^k} \\
&= \frac{1}{N}\sum_{h=1}^{H}\left[\sum_{i\in s_h}\frac{y_{hi}^k}{\pi_{hi}\alpha_{hi}^k}\right] \\
&= \frac{1}{N}\sum_{h=1}^{H}\left[\sum_{i\in s_h} y_{hi}^k\left(\frac{N_h}{n_h}\right)\left(\frac{n_h}{n_{hk}}\right)\right] \\
&= \frac{1}{N}\sum_{h=1}^{H} N_h\sum_{i\in s_h}\frac{1}{n_{hk}}y_{hi}^k \\
&= \frac{1}{N}\sum_{i\in s}\frac{y_i^k}{\pi_i}
\end{aligned}
$$

Poisson Sampling and RCB:

$$
\begin{aligned}
\hat{\mu}_k &= \frac{1}{N}\sum_{i\in s}\frac{y_i^k}{\pi_i\alpha_i^k} \\
&= \frac{1}{N}\sum_{i\in s}\sum_{b=1}^{B} y_i^k\left(\frac{1}{\pi_i}\right)\left(\frac{n(s)}{n_k}\right)\left(\frac{n_{bk}}{n_b}\right) \\
&= \frac{n(s)}{N}\sum_{i\in s}\sum_{b=1}^{B}\frac{y_i^k}{\pi_i}\left(\frac{n_{bk}}{n_b n_k}\right)
\end{aligned}
$$

Stratified Sampling and RCB:

$$
\begin{aligned}
\hat{\mu}_k &= \frac{1}{N}\sum_{i\in s}\frac{y_i^k}{\pi_i\alpha_i^k} \\
&= \frac{1}{N}\sum_{h=1}^{H}\left[\sum_{i\in s_h}\sum_{b=1}^{B}\frac{y_{hi}^k}{\pi_{hi}\alpha_{bi}^k}\right] \\
&= \frac{1}{N}\sum_{h=1}^{H}\left[\sum_{i\in s_h}\sum_{b=1}^{B} y_{hi}^k\left(\frac{N_h}{n_h}\right)\left(\frac{n_h}{n_{hk}}\right)\left(\frac{n_{bk}}{n_b}\right)\right] \\
&= \frac{1}{N}\sum_{h=1}^{H} N_h\sum_{i\in s_h}\sum_{b=1}^{B} y_{hi}^k\left(\frac{n_{bk}}{n_{hk}n_k}\right)
\end{aligned}
$$

Table 4.2: Thompson's Estimator for Selected Designs

Sampling Design	Experimental Design	$\hat{\mu}_k$
SRSWOR	CRD	$\bar{y}_k$
SRSWOR	RCB	$\frac{1}{n}\sum_{i\in s}\sum_{b=1}^{B} y_{i_b}^k\left(\frac{n_{bk}}{n_{hk}n_k}\right)$
Poisson	CRD	$\frac{n(s)}{N}\sum_{i\in s}\frac{y_i^k}{\pi_i n_k}$
Poisson	RCB	$\frac{n(s)}{N}\sum_{i\in s}\sum_{b=1}^{B}\frac{y_i^k}{\pi_i}\left(\frac{n_{bk}}{n_b n_k}\right)$
Stratified	CRD	$\frac{1}{N}\sum_{h=1}^{H} N_h\sum_{i\in s_h}\frac{1}{n_{hk}}y_{hi}^k$
Stratified	RCB	$\frac{1}{N}\sum_{h=1}^{H} N_h\sum_{i\in s_h}\sum_{b=1}^{B} y_{i_{hb}}^k\left(\frac{n_{bk}}{n_{hk}n_k}\right)$

Also, note that for Poisson and CRD $\widehat{var}(\hat{\mu}_k)$ is given by

$$
\begin{aligned}
&\widehat{var}(\hat{\mu}_k)\\
&= \frac{1}{N^2}\sum_{i=1}^{N}(y_i^k)^2\frac{1-\pi_i\alpha_i^k}{(\pi_i\alpha_i^k)^2}Z_iW_i^k + \frac{1}{N^2}\sum_{i\neq j}(y_i^k y_j^k)\frac{\pi_{ij}\alpha_{ij}^k - \pi_i\pi_j\alpha_i^k\alpha_j^k}{(\pi_{ij}\alpha_{ij}^k)\pi_i\pi_j\alpha_j^k\alpha_i^k}Z_iZ_jW_i^kW_j^k\\
&= \frac{1}{N^2}\sum_{i=1}^{N}(y_i^k)^2\frac{1-\pi_i(\frac{n_k}{n})}{\pi_i^2(\frac{n_k}{n})^2}Z_iW_i^k + \frac{1}{N^2}\sum_{i\neq j}(y_i^k y_j^k)\frac{\pi_i\pi_j\frac{n_k(n_k-1)}{n(n-1)} - \pi_i\pi_j(\frac{n_k}{n})^2}{\pi_i\pi_j\frac{n_k(n_k-1)}{n(n-1)}\cdot\pi_i\pi_j(\frac{n_k}{n})^2}Z_iZ_jW_i^kW_j^k\\
&= \frac{1}{N^2}\sum_{i=1}^{N}(y_i^k)^2\frac{(\frac{n}{n_k})^2 - \pi_i(\frac{n}{n_k})}{\pi_i^2}Z_iW_i^k + \frac{1}{N^2}\sum_{i\neq j}(y_i^k y_j^k)\frac{(\frac{n}{n_k})^2 - \frac{n(n-1)}{n_k(n_k-1)}}{\pi_i\pi_j}Z_iZ_jW_i^kW_j^k\\
&= \frac{1}{N^2}\sum_{i=1}^{N}(y_i^k)^2\frac{\frac{n}{n_k}(\frac{n}{n_k} - \pi_i)}{\pi_i^2}Z_iW_i^k + \frac{1}{N^2}\sum_{i\neq j}(y_i^k y_j^k)\frac{\frac{n}{n_k}(\frac{n}{n_k} - \frac{n-1}{n_k-1})}{\pi_i\pi_j}Z_iZ_jW_i^kW_j^k\\
&= \frac{1}{N^2}\sum_{i=1}^{N}(y_i^k)^2\frac{\frac{n}{n_k}(\frac{n-\pi_i n_k}{n_k})}{\pi_i^2}Z_iW_i^k + \frac{1}{N^2}\sum_{i\neq j}(y_i^k y_j^k)\frac{\frac{n}{n_k}(\frac{n(n_k-1)-(n-1)n_k}{n(n_k-1)})}{\pi_i\pi_j}Z_iZ_jW_i^kW_j^k\\
&= \frac{1}{N^2}\sum_{i=1}^{N}(y_i^k)^2\frac{n(n-\pi_i n_k)}{\pi_i^2(n_k)^2}Z_iW_i^k + \frac{1}{N^2}\sum_{i\neq j}(y_i^k y_j^k)\frac{1}{\pi_i\pi_j}\cdot\frac{n_k-n}{n_k(n_k-1)})Z_iZ_jW_i^kW_j^k\ ,
\end{aligned}
$$

and $\widehat{var}(\hat{\mu}_k - \hat{\mu}_{k'})$ is given by

$$
\begin{aligned}
&\widehat{var}(\hat{\mu}_k - \hat{\mu}_{k'})\\
&= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\left[\frac{1-\pi_i\alpha_i^t}{(\pi_i\alpha_i^t)^2}\right]Z_iW_i^t\\
&\quad + \frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^t y_j^t)\left[\frac{\pi_{ij}\alpha_{ij}^t - \pi_i\pi_j\alpha_i^t\alpha_j^t}{\pi_{ij}\alpha_{ij}^t(\pi_i\pi_j\alpha_j^t\alpha_i^t)}\right]Z_iZ_jW_i^tW_j^t
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} y_i^k y_j^{k'}\left[\frac{\pi_{ij}(1-\alpha_{ij}^k-\alpha_{ij}^{k'})-2\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}{2\pi_{ij}(1-\alpha_{ij}^k-\alpha_{ij}^{k'})\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}\right] Z_iZ_jW_i^kW_j^{k'}\\
=&\frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\left[\frac{1-\pi_i\alpha_i^t}{(\pi_i\alpha_i^t)^2}\right] Z_iW_i^t\\
&+\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^ty_j^t)\left[\frac{\pi_i\pi_j\alpha_{ij}^t-\pi_i\pi_j\alpha_i^t\alpha_j^t}{\pi_i\pi_j\alpha_{ij}^t(\pi_i\pi_j\alpha_j^t\alpha_i^t)}\right] Z_iZ_jW_i^tW_j^t\\
&-\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} y_i^k y_j^{k'}\left[\frac{\pi_i\pi_j(1-\alpha_{ij}^k-\alpha_{ij}^{k'})-2\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}{2\pi_i\pi_j(1-\alpha_{ij}^k-\alpha_{ij}^{k'})\pi_i\pi_j\alpha_i^k\alpha_j^{k'}}\right] Z_iZ_jW_i^kW_j^{k'}\\
=&\frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\left[\frac{1-\pi_i\alpha_i^t}{(\pi_i\alpha_i^t)^2}\right] Z_iW_i^t\\
&+\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^ty_j^t)\left[\frac{\alpha_{ij}^t-\alpha_i^t\alpha_j^t}{\pi_i\pi_j\alpha_{ij}^t\alpha_j^t\alpha_i^t}\right] Z_iZ_jW_i^tW_j^t\\
&-\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} y_i^k y_j^{k'}\left[\frac{1-\alpha_{ij}^k-\alpha_{ij}^{k'}-2\alpha_i^k\alpha_j^{k'}}{2\pi_i\pi_j(1-\alpha_{ij}^k-\alpha_{ij}^{k'})\alpha_i^k\alpha_j^{k'}}\right] Z_iZ_jW_i^kW_j^{k'}\\
=&\frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\left[\frac{1-\pi_i\cdot\frac{n_t}{n}}{(\pi_i\cdot\frac{n_t}{n})^2}\right] Z_iW_i^t\\
&+\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^ty_j^t)\left[\frac{\frac{n_t(n_t-1)}{n(n-1)}-(\frac{n_t}{n})^2}{\pi_i\pi_j\frac{n_t(n_t-1)}{n(n-1)}(\frac{n_t}{n})^2}\right] Z_iZ_jW_i^tW_j^t\\
&-\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} y_i^k y_j^{k'}\left[\frac{1-\frac{n_k(n_k-1)}{n(n-1)}-\frac{n_{k'}(n_{k'}-1)}{n(n-1)}-2\frac{n_k}{n}\cdot\frac{n_{k'}}{n}}{2\pi_i\pi_j(1-\frac{n_k(n_k-1)}{n(n-1)}-\frac{n_{k'}(n_{k'}-1)}{n(n-1)})\frac{n_k}{n}\cdot\frac{n_{k'}}{n}}\right] Z_iZ_jW_i^kW_j^{k'}\\
=&\frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\left[\left(\frac{n}{\pi_in_t}\right)^2-\frac{n}{\pi_in_t}\right] Z_iW_i^t\\
&+\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^ty_j^t)\frac{1}{\pi_i\pi_j}\left[\left(\frac{n}{n_t}\right)^2-\frac{n(n-1)}{n_t(n_t-1)}\right] Z_iZ_jW_i^tW_j^t\\
&-\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} y_i^k y_j^{k'}\frac{1}{\pi_i\pi_j}\left[\frac{1/2}{\frac{n_k}{n}\cdot\frac{n_{k'}}{n}}\right] Z_iZ_jW_i^kW_j^{k'}\\
&-\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} y_i^k y_j^{k'}\frac{1}{\pi_i\pi_j}\left[\frac{-1}{1-\frac{n_k(n_k-1)}{n(n-1)}-\frac{n_{k'}(n_{k'}-1)}{n(n-1)}}\right] Z_iZ_jW_i^kW_j^{k'}\\
=&\frac{1}{N^2}\sum_{i=1}^{N}\sum_{t=k,k'}(y_i^t)^2\left[\frac{n(n-\pi_in_t)}{(\pi_in_t)^2}\right] Z_iW_i^t\\
&+\frac{1}{N^2}\sum_{i\neq j}\sum_{t=k,k'}(y_i^ty_j^t)\frac{1}{\pi_i\pi_j}\left[\frac{n(n_t-n)}{(n_t)^2(n_t-1)}\right] Z_iZ_jW_i^tW_j^t
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} y_i^k y_j^{k'} \frac{1}{\pi_i\pi_j}\left[\frac{n^2}{2n_k n_{k'}}\right] Z_i Z_j W_i^k W_j^{k'} \\
&-\frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} y_i^k y_j^{k'} \frac{1}{\pi_i\pi_j}\left[\frac{-n(n-1)}{n(n-1)-n_k(n_k-1)-n_{k'}(n_{k'}-1)}\right] Z_i Z_j W_i^k W_j^{k'} \, .
\end{aligned}
$$

Chapter 5

Simulation Study for Inference on Thompson's Estimator

One purpose of a simulation study is to evaluate the distributional properties of an estimator. In particular, one purpose of this simulation study is to evaluate properties of Thompson's estimator. Thus, the objective of the following simulation study is to evaluate $\hat{\mu}_k$, for which distributional properties are unknown.

5.1 Properties of Interest

As demonstrated in Chapter 3, $\hat{\mu}_k$ is an unbiased estimator of μ_k and $\widehat{var}(\hat{\mu}_k)$ is unbiased for $var(\hat{\mu}_k)$. Since $\hat{\mu}_k$ is an unbiased estimator of μ_k the "closeness" of $\hat{\mu}_k$ to μ_k can be evaluated by considering $var(\hat{\mu}_k)$. Thus, properties of $var(\hat{\mu}_k)$ and its estimator, $\widehat{var}(\hat{\mu}_k)$, will be considered. Properties of $\hat{\mu}_k - \hat{\mu}_{k'}$, $var(\hat{\mu}_k - \hat{\mu}_{k'})$, and $\widehat{var}(\hat{\mu}_k - \hat{\mu}_{k'})$ will also be considered.

No evaluation of an estimator is complete without some discussion of it's value for drawing conclusions about the population. Here confidence intervals will be proposed for μ_k and $\mu_k - \mu_{k'}$ and evaluated based on interval coverage and average width. For the intervals considered here, interval coverage is equivalent to the Type I error rate of testing the hypothesis that μ_k or $\mu_k - \mu_{k'}$ is in the respective interval. Therefore, the Type I error rate from these simulation results was not evaluated. However, to study the power of tests based on $\hat{\mu}_k$ additional simulation will be necessary. Such simulations are not done here but should be considered in future work.

Note that $\bar{y}_k$, as a rule, should not be compared for $\hat{\mu}_k$, since they are two different types of estimators. However, it is still of interest to observe the behavior of $\bar{y}_k$ versus the behavior of $\hat{\mu}_k$ under combined sampling and experimental design since $\bar{y}_k$ is the traditionally used estimator and $\hat{\mu}_k$ is the proposed estimator. Thus, the proposed confidence intervals for μ_k and $\mu_k - \mu_{k'}$ will be compared to traditional confidence intervals based on $\bar{y}_k$ and it's variance estimator.

5.2 Simulation Description

This simulation considered srswor with CRD. This is the simplest case of combined sampling and experimental design. Recall that for this case $\hat{\mu}_k = \bar{y}_k$. However, $\widehat{var}(\hat{\mu}_k) \neq S_k^2/n_k$, where $S_k^2 = \frac{1}{n_k-1}\sum_{i=1}^{n_k}(y_i^k - \bar{y}_k)^2$ is the traditional estimator of variance for $\bar{y}_k$. Only two treatments, where $n_k = n_{k'} = \frac{n}{2}$, were considered for this simulation. Henceforth, these treatments will be referred to as control (c) and treatment (t).

Data Generation:

It is reasonable to expect responses from the same unit to different treatments to be correlated with each other so populations of control and treatment responses were generated from a bivariate normal distribution. Responses were generated with the mean for control and treatment set to zero, variance set to one, and with the covariance taking values $\rho = (0,\ 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1)$. That is, $\begin{bmatrix} y_i^c \\ y_i^t \end{bmatrix} \sim BivariateNormal\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$ where $0 \leq \rho \leq 1$. Note that under srswor and a CRD, choosing population means equal to zero and variances equal to one is done without loss of generality. Populations of size 50, 100, 250, 500, 1000 and 5000 were generated. The simulated data were generated using R (2005) Version 2.1.1 software and the `mvrnorm` (multivariate normal) function in the *MASS* package.

Sample sizes considered here were $n = 10, 20, 30, 50, 100, 200$ depending on the population size (see Table 5.1). Keep in mind that half the units from each sample are assigned to the control group and half are assigned to the treatment group. That is, each treatment group receives only half, $\frac{n}{2}$, of the sample units. A total of 10000 populations were generated for each population size by sample size combination.

Table 5.1: Population and Sample Size

Population Size (N)	Sample sizes (n)					
50	10	20	30			
100	10	20	30	50		
250	10	20	30	50	100	
500	10	20	30	50	100	200
1000	10	20	30	50	100	200
5000	10	20	30	50	100	200

Sample Selection:

A simple random sample without replacement was selected from the set of identifiers for the N population units using the R function `sample`. The responses corresponding to the sampled identifiers comprised the srswor from the population of responses. Since, both y_i^c and y_i^t were generated, this sample contained responses for both control and treatment. However, a real sample can contain only one or the other but not both. Thus, to simulate random treatment assignment (i.e. a CRD), prior to sorting the sample the control responses were selected from the first half of the sample and treatment responses were selected from the remaining half of the sample.

Estimate Calculation:

The following estimates were calculated using R for each sample: $\bar{y}_c$, $\bar{y}_t$, S_c^2, S_t^2, $\hat{\mu}_c$, $\hat{\mu}_t$, $\widehat{var}(\hat{\mu}_c)$, $\widehat{var}(\hat{\mu}_t)$, $\bar{y}_t - \bar{y}_c$, $\hat{\mu}_t - \hat{\mu}_c$, and $\widehat{var}(\hat{\mu}_t - \hat{\mu}_c)$. See Appendix D for the R code that was used to generate these estimates. Note that the coding for this simulation was reasonably simple since srswor and a CRD were the only designs considered. Coding for the case where the π's are not equal and the α's are not equal is considerably more complex and requires matrix manipulations.

Inferential Evaluation:

From the above estimates confidence intervals for μ_c, μ_t, and $\mu_t - \mu_c$ were proposed. Specifically, the following proposed intervals were calculated, with $\alpha = 0.5$:

$$\bar{y}_c \pm t_{\frac{\alpha}{2}, n_c-1}\sqrt{S_c^2/n_c} = \bar{y}_c \pm t_{\frac{\alpha}{2}, \frac{n}{2}-1}\sqrt{\frac{2}{n}S_c^2} \ ,$$

$$\bar{y}_t \pm t_{\frac{\alpha}{2}, n_t-1}\sqrt{S_t^2/n_t} = \bar{y}_t \pm t_{\frac{\alpha}{2}, \frac{n}{2}-1}\sqrt{\frac{2}{n}S_t^2} \ ,$$

$$\bar{y}_t - \bar{y}_c \pm t_{\frac{\alpha}{2}, n_t+n_c-2}\sqrt{\left(\frac{S_t^2 + S_c^2}{2}\right)\left(\frac{1}{n_t} + \frac{1}{n_c}\right)} = \bar{y}_t - \bar{y}_c \pm t_{\frac{\alpha}{2}, n-2}\sqrt{\frac{2}{n}(S_t^2 + S_c^2)} \ ,$$

$$\hat{\mu}_c \pm t_{\frac{\alpha}{2}, n_c-1}\sqrt{\widehat{var}(\hat{\mu}_c)} = \hat{\mu}_c \pm t_{\frac{\alpha}{2}, \frac{n}{2}-1}\sqrt{\widehat{var}(\hat{\mu}_c)} \ ,$$

$$\hat{\mu}_t \pm t_{\frac{\alpha}{2}, n_t-1}\sqrt{\widehat{var}(\hat{\mu}_t)} = \hat{\mu}_c \pm t_{\frac{\alpha}{2}, \frac{n}{2}-1}\sqrt{\widehat{var}(\hat{\mu}_c)} \ ,$$

and

$$\hat{\mu}_t - \hat{\mu}_c \pm t_{\frac{\alpha}{2}, n_t+n_c-2}\sqrt{\widehat{var}(\hat{\mu}_t - \hat{\mu}_c)} = \hat{\mu}_t - \hat{\mu}_c \pm t_{\frac{\alpha}{2}, n-2}\sqrt{\widehat{var}(\hat{\mu}_t - \hat{\mu}_c)} \ .$$

Interval coverage and average width were estimated from the simulated results for each proposed interval. Note that an exploratory simulation conducted in preparation for this simulation indicated that $n_k - 1$ and $n_k + n_{k'} - 2$ were likely to be acceptable estimates for the degrees of freedom of the t critical points for the μ_k and $\mu_k - \mu_{k'}$ intervals, based on the variance estimates $\widehat{var}(\hat{\mu}_k)$ and $\widehat{var}(\hat{\mu}_k - \hat{\mu}_{k'})$, respectively. Also, note that the pooled variance estimator, $1/2(S_k^2 + S_{k'}^2)$, and corresponding degrees of freedom, $n_k + n_{k'} - 2$, were used for the traditional intervals since the data were generated with equal variances.

The performance of the Thompson based intervals versus the traditional intervals depends the behavior of the corresponding variance estimator. That is, since $\hat{\mu}_k = \bar{y}_k$, interval coverage and width depend on $\widehat{var}(\hat{\mu}_k)$ and S_k^2/n_k. It can be shown that $\widehat{var}(\hat{\mu}_k) < S_k^2/n_k$, since

$$
\begin{aligned}
S_k^2/n_k &= \frac{1}{n_k - 1}\sum_{i=1}^{n_k}(y_i^k - \bar{y}_k)^2 \\
&= \frac{1}{2n_k(n_k-1)}\sum_{i \neq j}(y_i^k - y_j^k)^2 \\
&= \frac{1}{n_k}\sum_{i=1}^{n_k}(y_i^k)^2 - \frac{1}{n_k(n_k-1)}\sum_{i \neq j} y_i^k y_j^k
\end{aligned}
$$

(Johnson, McGuire and Milliken, 1978). Thus intervals based on S_k^2/n_k should have more coverage and wider width than intervals based on $\widehat{var}(\hat{\mu}_k)$ (see Appendix B for histogram plots of both variance estimators).

5.3 Simulation Results

Distribution Results:

The distribution of $\hat{\mu}_t$, $\hat{\mu}_c$, $\hat{\mu}_t - \mu_t$, $\hat{\mu}_c - \mu_c$, $\hat{\mu}_t - \hat{\mu}_c$, and $(\hat{\mu}_t - \hat{\mu}_c) - (\mu_t - \mu_c)$ were symmetric about zero for all populations and sample sizes (see Appendix A for histogram plots). As expected the distributions tightened as sample sizes increased and as population sizes increased. The correlation between treatment responses appeared to have no effect on the behavior of these statistics.

The distribution of $var(\hat{\mu}_t)$, $var(\hat{\mu}_c)$, $\widehat{var}(\hat{\mu}_t)$, $\widehat{var}(\hat{\mu}_c)$, $var(\hat{\mu}_t - \hat{\mu}_c)$, and $\widehat{var}(\hat{\mu}_t - \hat{\mu}_c)$ tended to be right skewed as expected (see Appendix B for histogram plots). Also, as expected the distributions tightened as sample sizes increased and as population sizes increased. The correlation between treatment responses had no noticeable effect on the behavior of these statistics.

Inferential Results:

See Figures 5.1-5.12 for plots of interval coverage and average width estimated from the simulated results for each proposed interval. Note that the coverage plots include reference lines for 94.6%, 95% and 95.6%. Also, see Appendix C for tables of interval coverage and average width estimated from the simulated results for each proposed interval.

In general, for all population and sample sizes the interval coverage of μ_c and μ_t based on $\widehat{var}(\hat{\mu}_t)$ and $\widehat{var}(\hat{\mu}_c)$ was reasonable. That is, in general, the percent of coverage stayed between 94.6% and 95.6%, the standard normal interval for a proportion of 95% with 10000 observations (i.e. $.95 \pm 1.96\sqrt{(.95)(.05)/10000}$).

The coverage for $\mu_t - \mu_c$ based on traditional variance estimation and on $var(\hat{\mu}_t - \hat{\mu}_c)$ tended to decrease as the correlation between the control and treatment responses increased for all population sizes except N=5000. For the traditional variance estimator this downward trend generally stayed above 95% coverage, but for $var(\hat{\mu}_t - \hat{\mu}_c)$ this downward trend generally stayed below 95% coverage.

Coverage based on traditional variance estimators tended to be higher than coverage based on $\widehat{var}(\hat{\mu}_t)$ and $\widehat{var}(\hat{\mu}_c)$, and higher than the 95.6% standard normal upper limit. This difference in coverage increased as sample size increased, especially for small population sizes. Over all, as the population size increased all coverage values tended toward 95%, which was expected.

As would be expected the average interval width decreased as sample size increased for all populations. Average interval width was consistently smaller for intervals based on $\widehat{var}(\hat{\mu}_t)$ and $\widehat{var}(\hat{\mu}_c)$ than for intervals based on the traditional variance estimators for all population and sample sizes. This was the expected result since $\widehat{var}(\hat{\mu}_t)$ and $\widehat{var}(\hat{\mu}_c)$ were designed to account for the additional information from the sample selection and treatment design. See Table 5.2 for the average percent reduction in width of the Thompson based intervals versus the traditional intervals. The percent reduction was averaged over all values of ρ and over intervals for μ_t, μ_c, and $\mu_t - \mu_c$, since there was minimal variation in percent reduction over these values. For example, for a population of size 50 and a sample of size 20, the reduction in width for Thompson based intervals is consistent across values of ρ and is essentially the same for all three parameters (see Figure 5.2).

Table 5.2: Average Percent Reduction in Width

Population Size (N)	Sample sizes (n) 10	20	30	50	100	200
50	5.13	10.56	16.33			
100	2.53	5.13	7.80	13.40		
250	1.00	2.02	3.05	5.13	10.56	
500	0.50	1.01	1.51	2.53	5.13	10.56
1000	0.25	0.50	0.75	1.26	2.53	5.13
5000	0.05	0.10	0.15	0.25	0.50	1.01

The value of ρ did not appear to have any affect on average interval width for either the traditional intervals or the Thompson based intervals. However, ρ did appear to influence coverage rates for $\mu_k - \mu_{k'}$ based on both types of variance estimation. That is, interval coverage for $\mu_k - \mu_{k'}$ started to noticeably drop as the correlation between treatment and control increased. This drop tended to become more severe as sample size increased; yet the trend was less sever as population size increased. To illustrate this consider a population of size 50 with a sample of size 20. Form Figure 5.1, the coverage for $\mu_k - \mu_{k'}$ appears to begin to drop off at about $\rho = 0.4$. See Table 5.3 for a summary of the approximate coverage drop-off points based on ρ for the $\mu_k - \mu_{k'}$ intervals.

Table 5.3: $\mu_k - \mu_{k'}$ Coverage Drop-off vs. ρ

Population Size (N)	Sample sizes (n) 10	20	30	50	100	200
50	1.0	0.4	0.2			
100	1.0	0.6	0.4	0.2		
250	na	na	0.8	0.6	0.2	
500	na	na	1.0	0.8	0.6	0.4
1000	na	na	na	1.0	0.8	0.6
5000	na	na	na	na	na	na

Figure 5.1: Simulation Results for Interval Coverage N=50

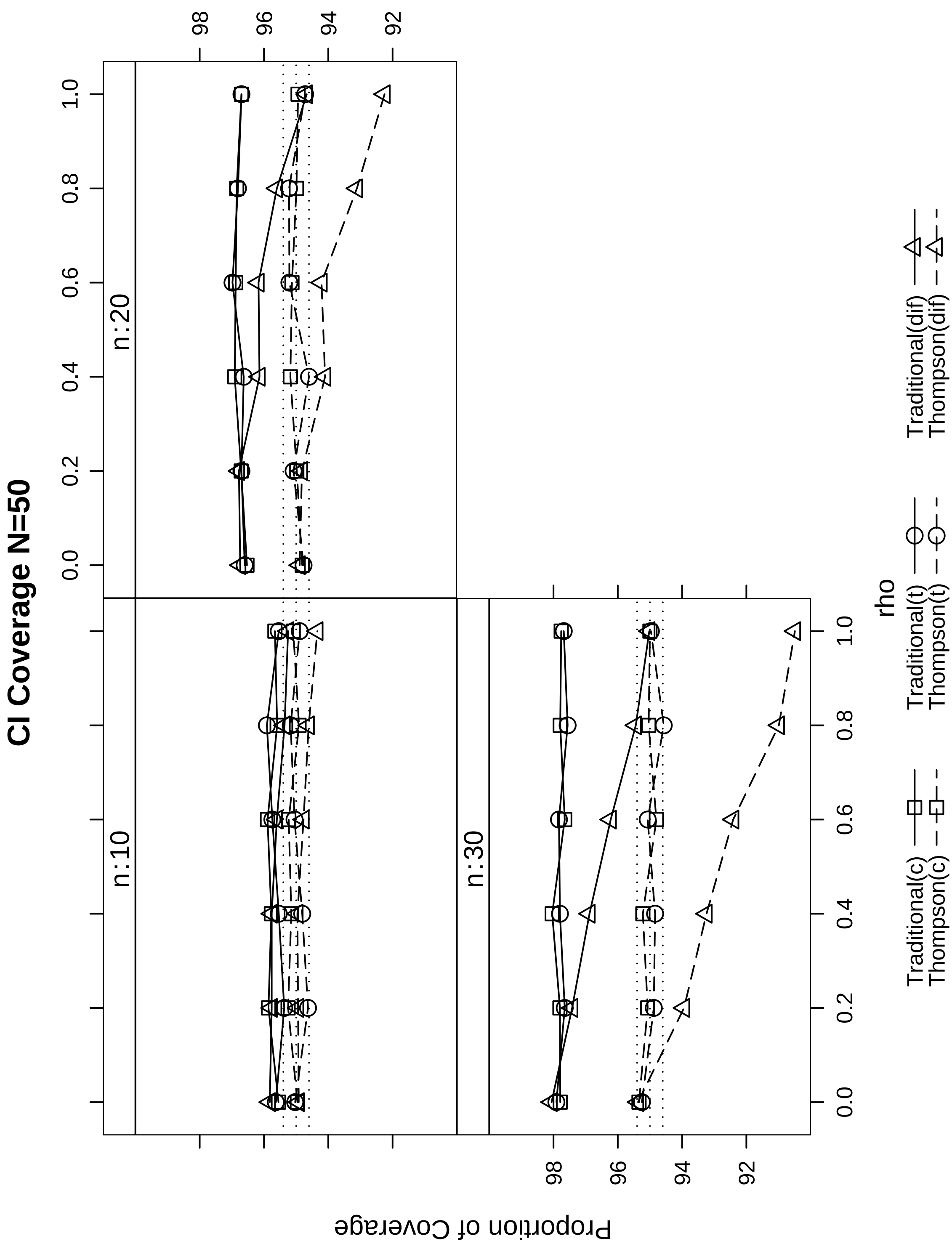


Figure 5.2: Simulation Results for Interval Width N=50

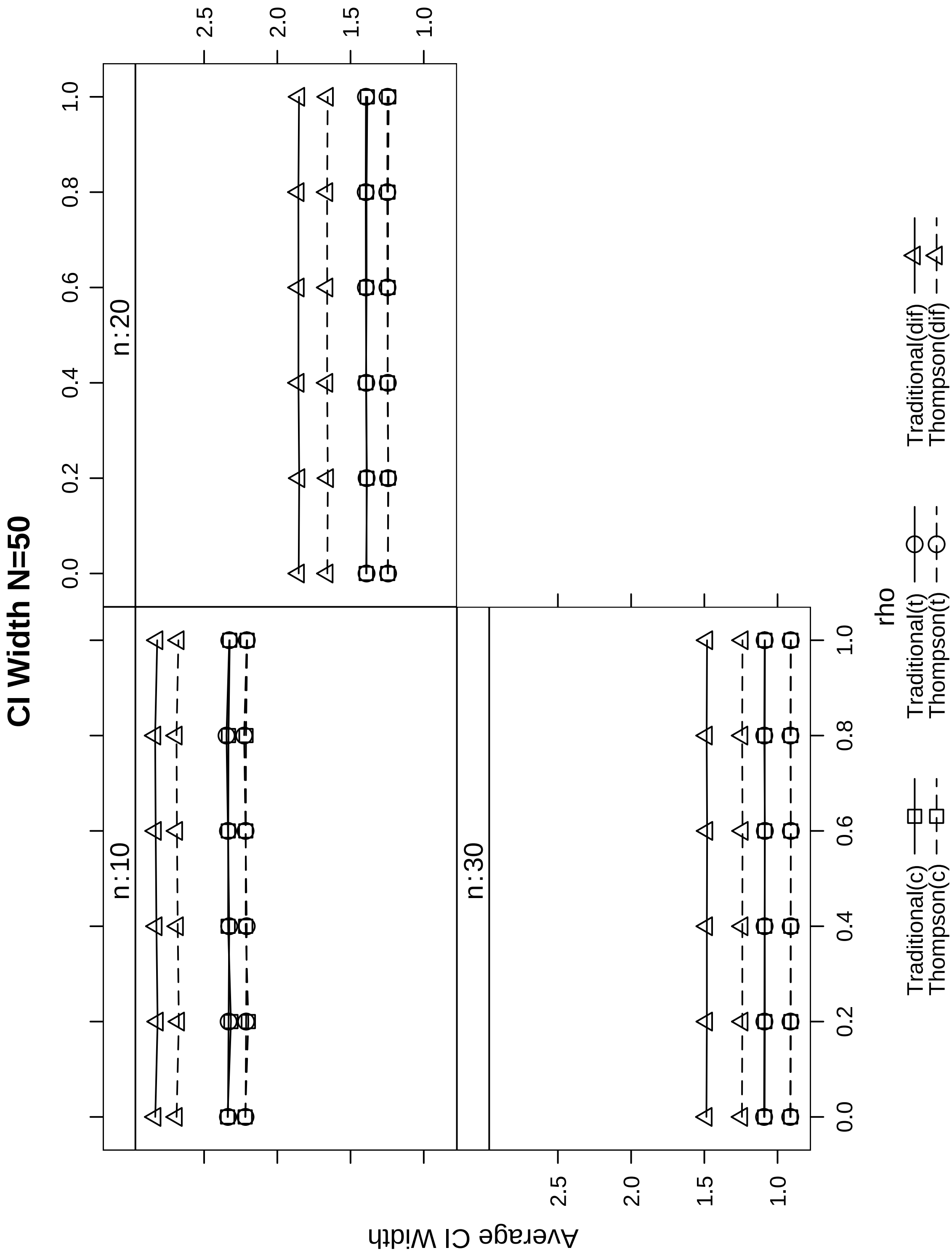


Figure 5.3: Simulation Results for Interval Coverage N=100

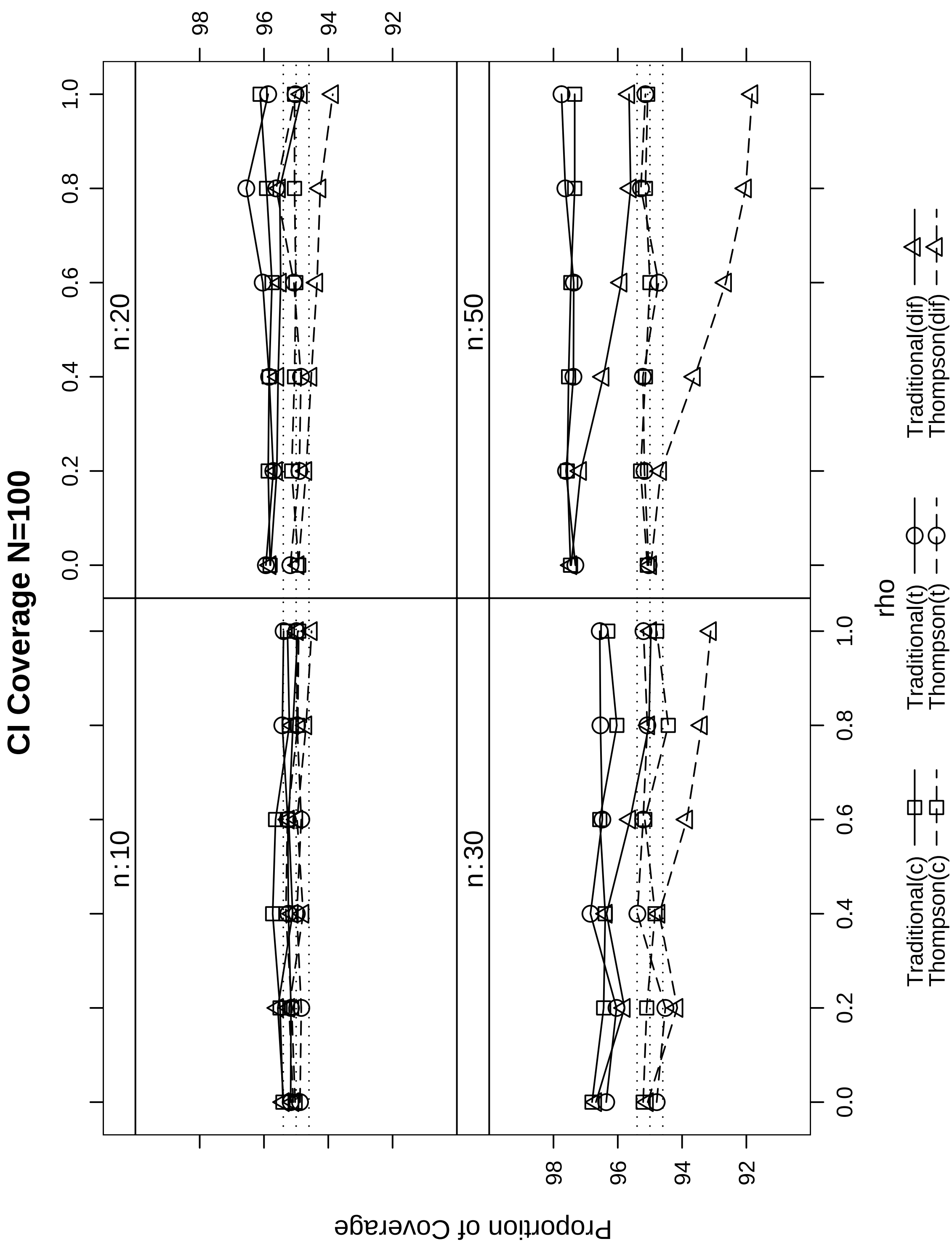


Figure 5.4: Simulation Results for Interval Width N=100

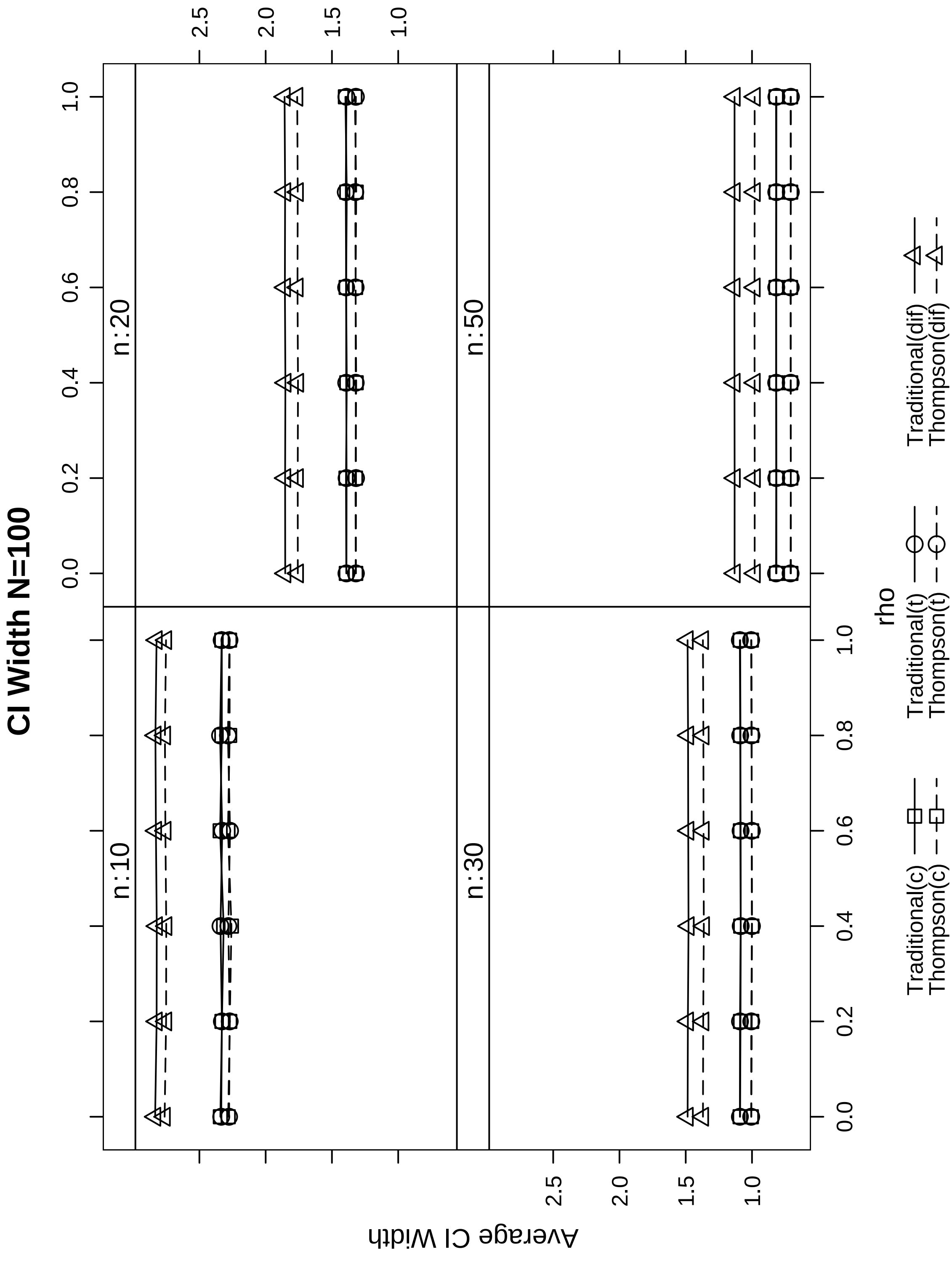


Figure 5.5: Simulation Results for Interval Coverage N=250

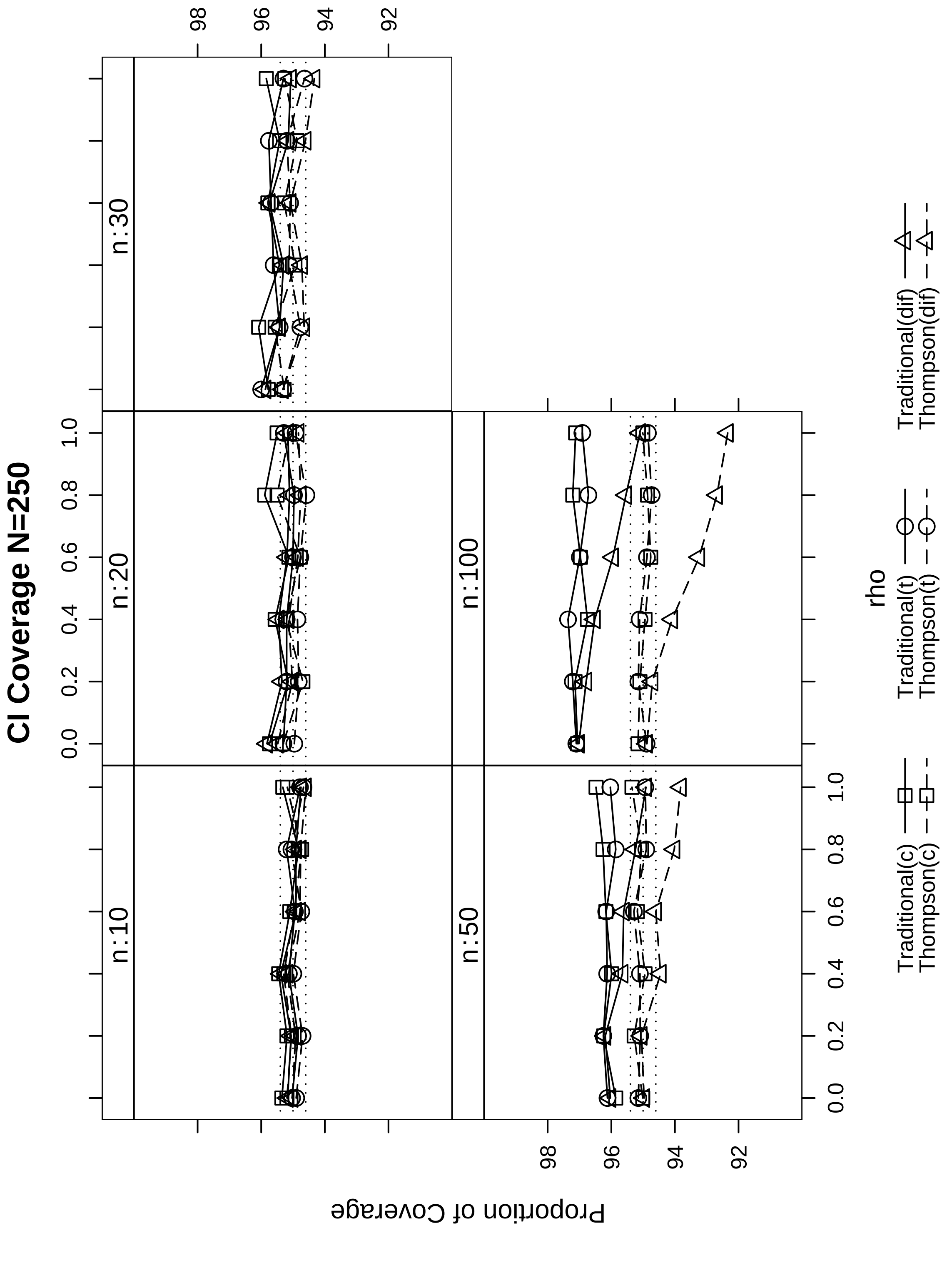


Figure 5.6: Simulation Results for Interval Width N=250

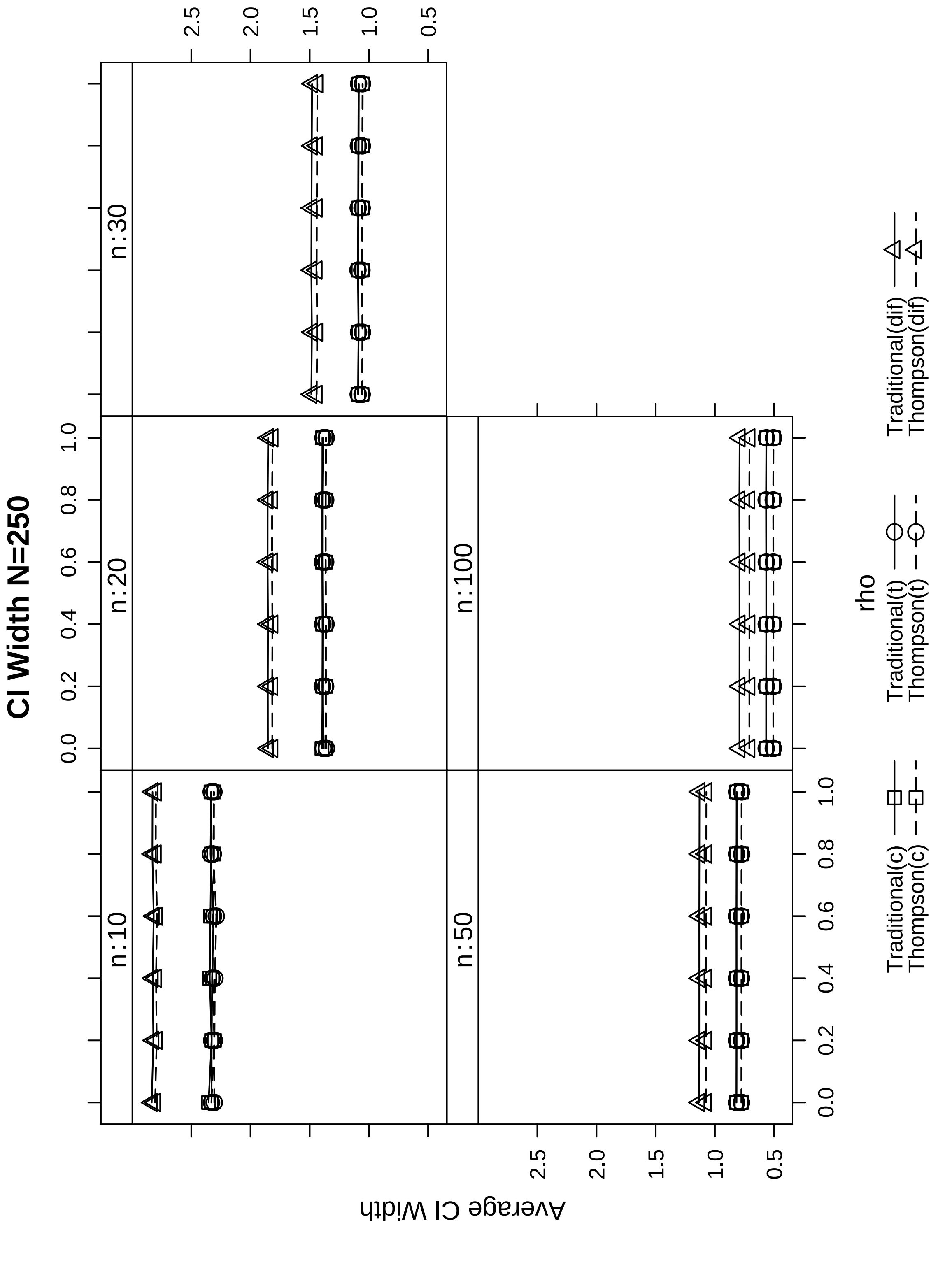


Figure 5.7: Simulation Results for Interval Coverage N=500

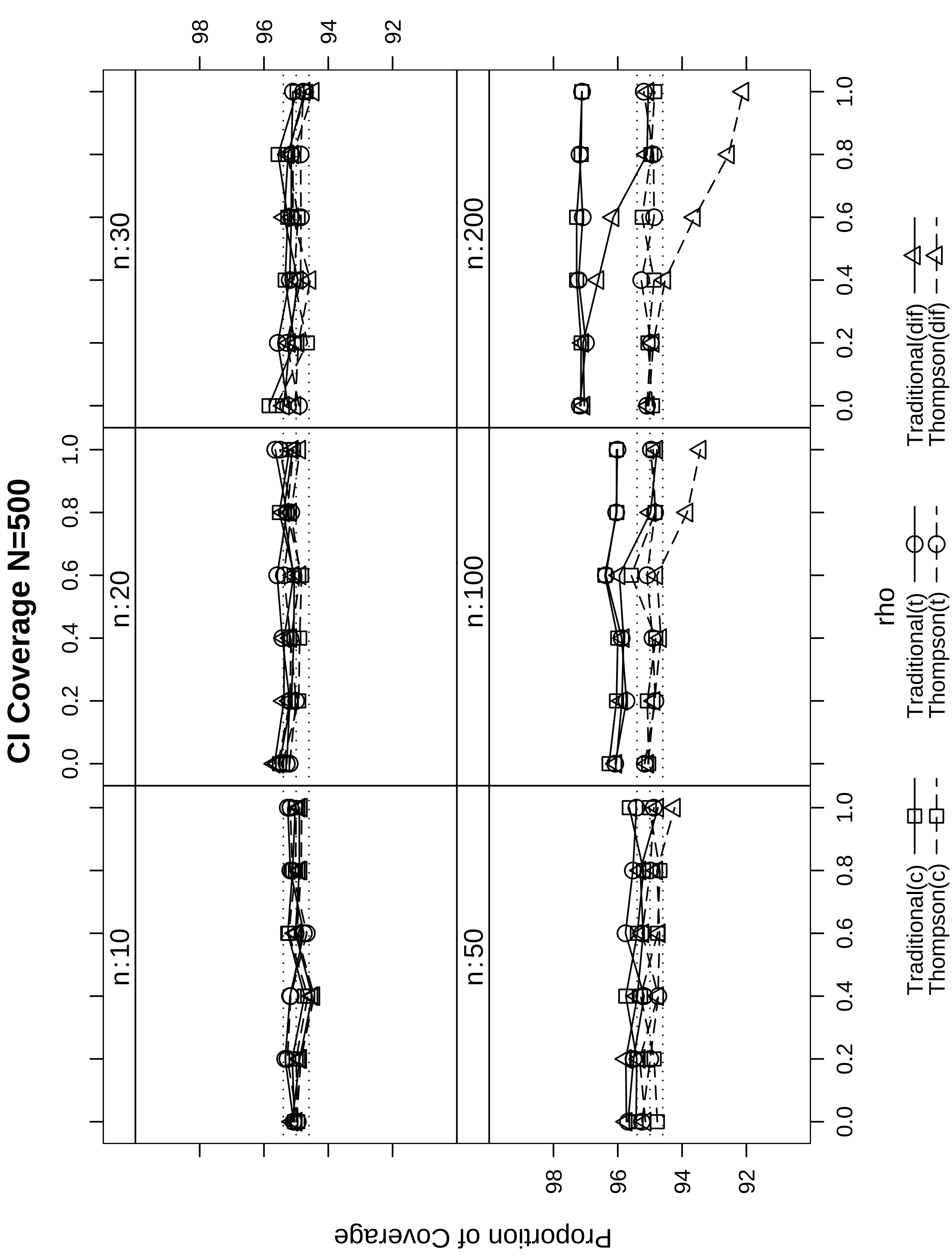


Figure 5.8: Simulation Results for Interval Width N=500

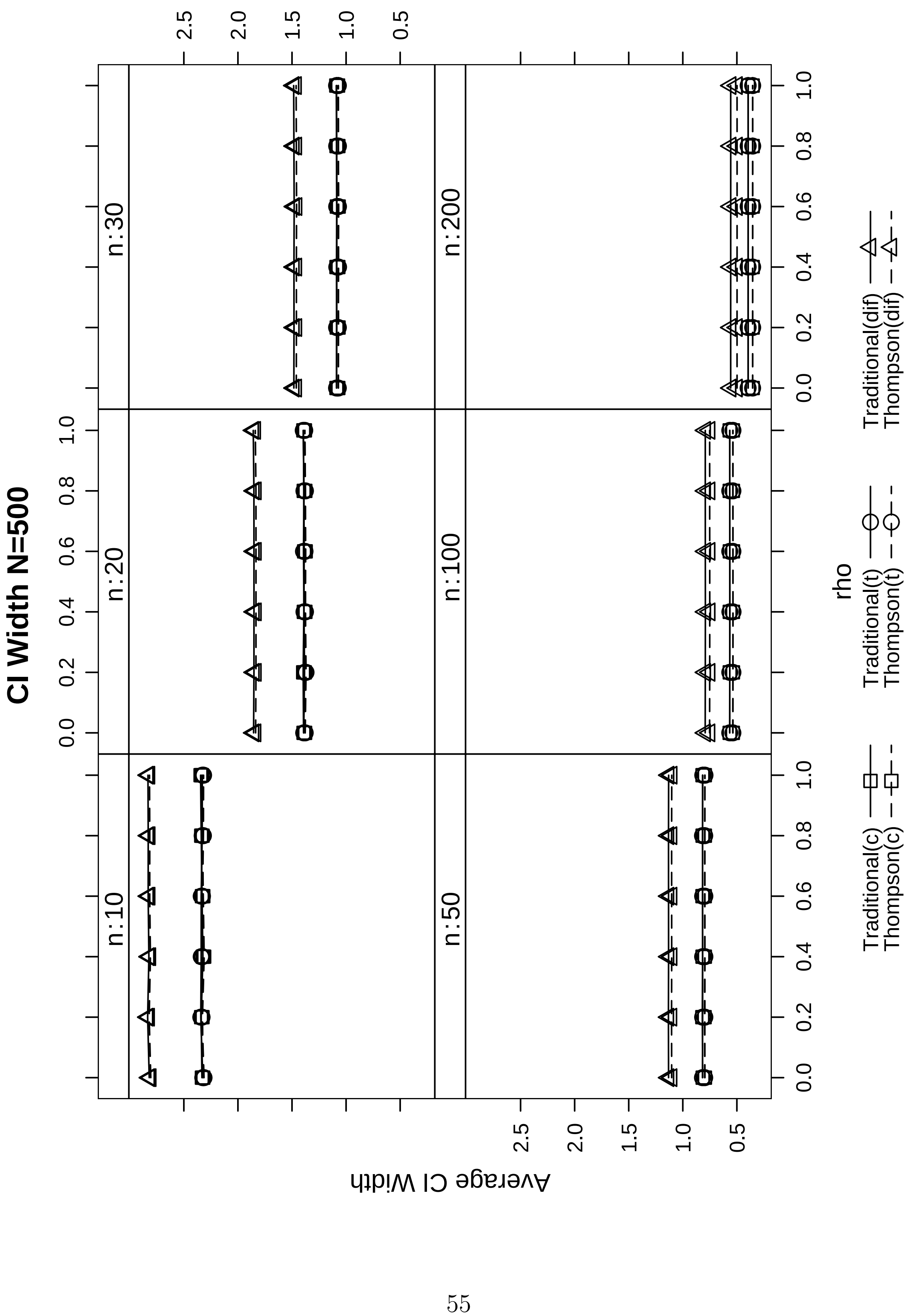


Figure 5.9: Simulation Results for Interval Coverage N=1000

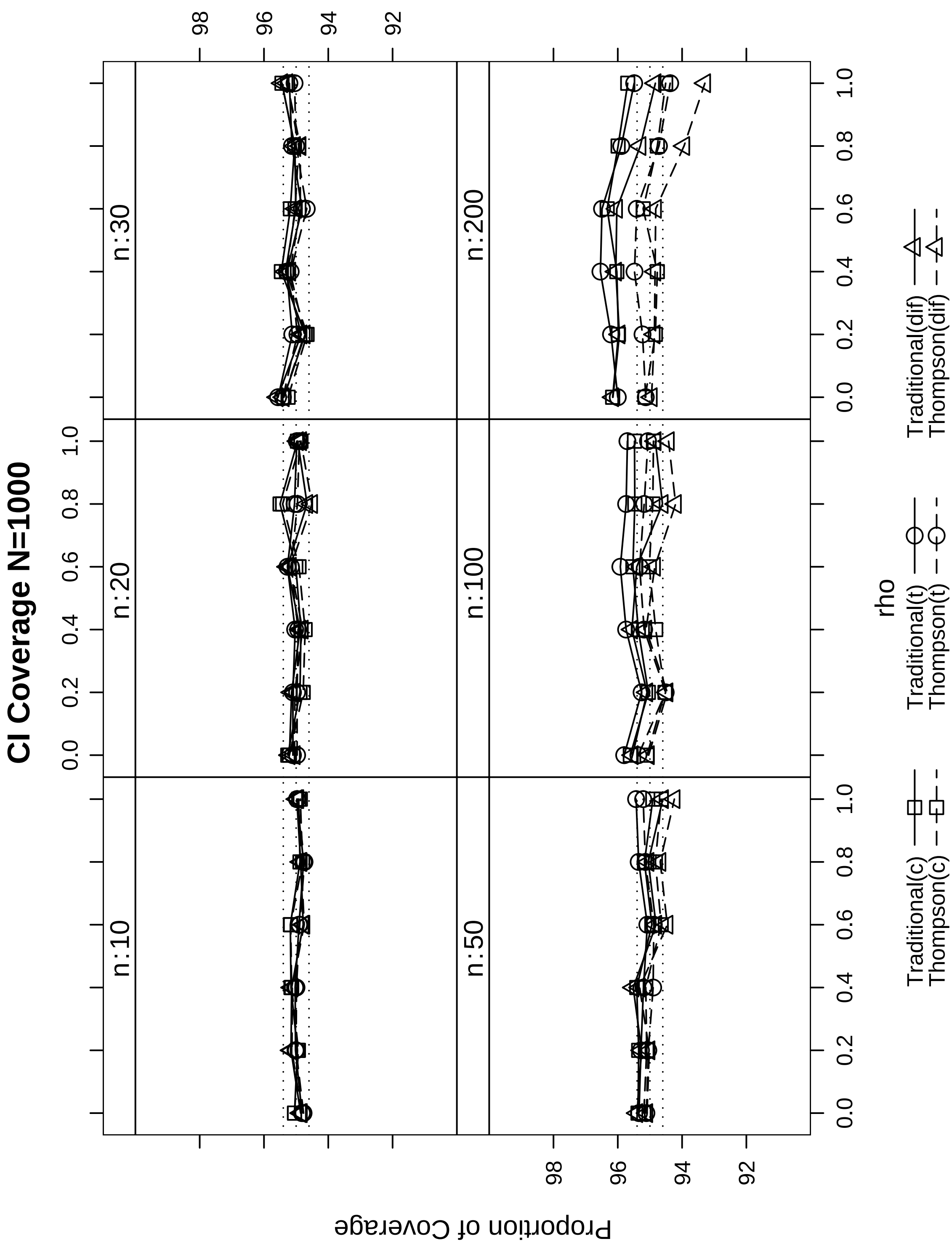


Figure 5.10: Simulation Results for Interval Width N=1000

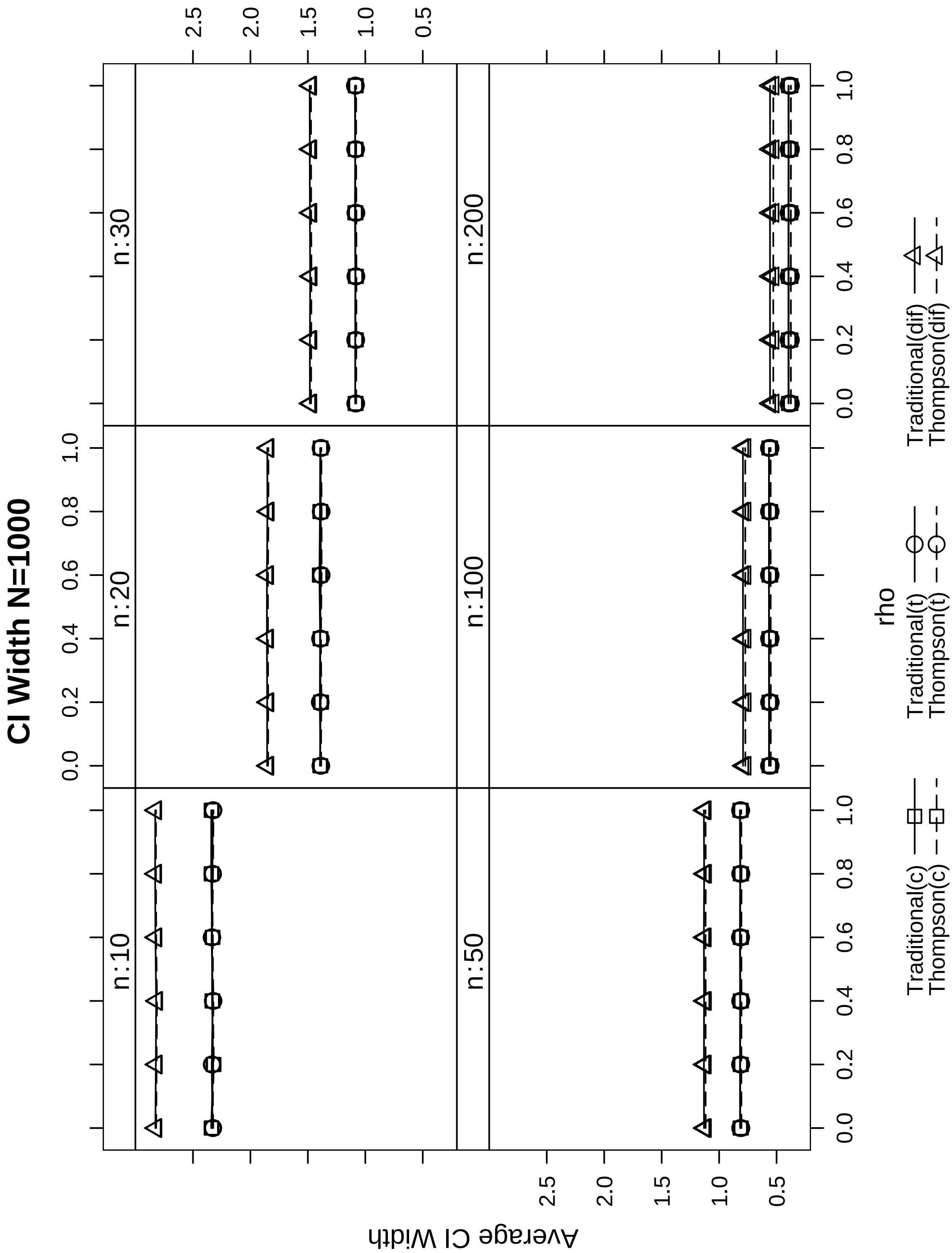


Figure 5.11: Simulation Results for Interval Coverage N=5000

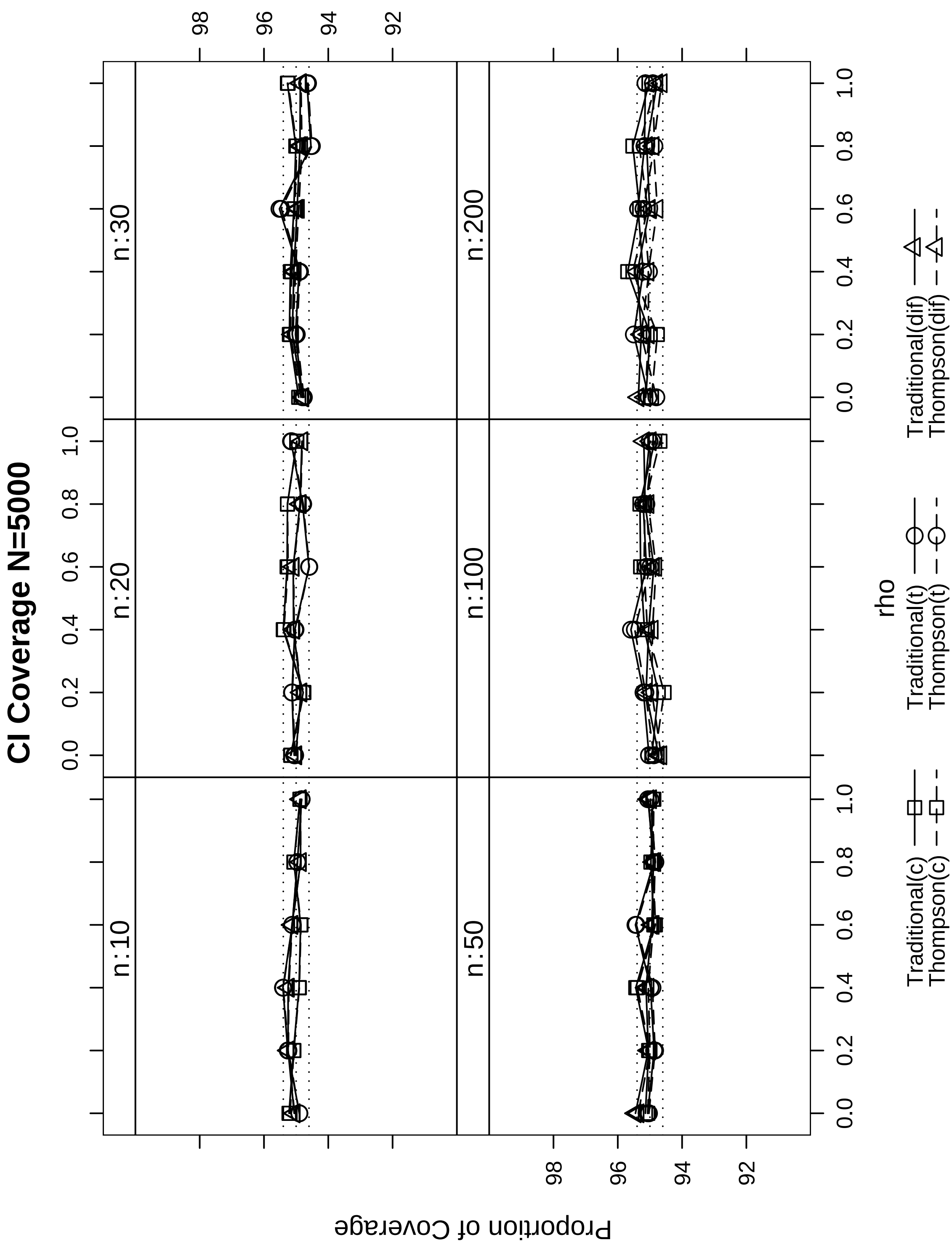
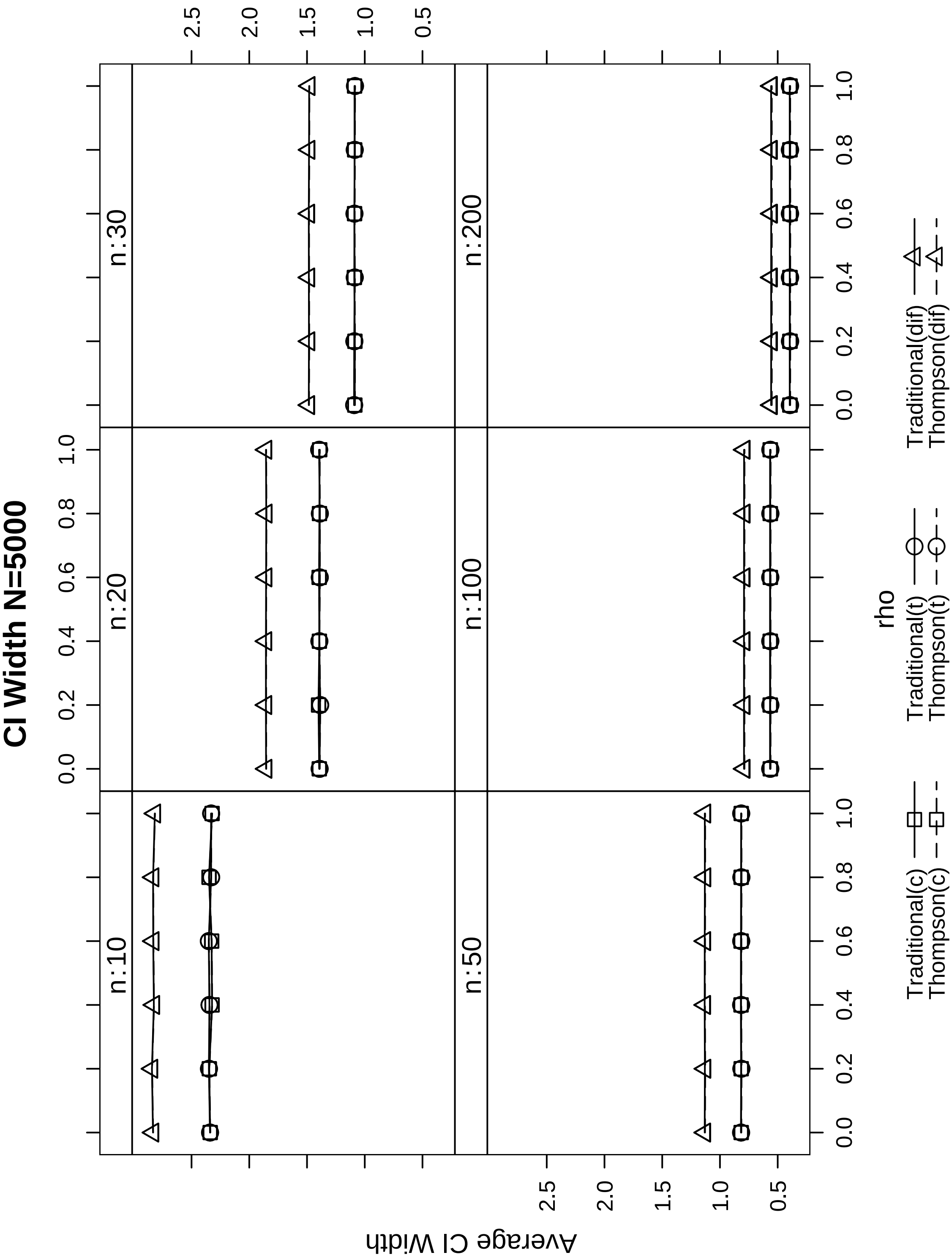


Figure 5.12: Simulation Results for Interval Width N=5000



5.4 Conclusions

Since the estimates $\hat{\mu}_k$ and $\bar{y}_k$ do have the same value under srswor and a CRD, interval differences were due to variance estimation. In general for this case, intervals based on $\widehat{var}(\hat{\mu}_k)$ and $\widehat{var}(\hat{\mu}_k - \hat{\mu}_{k'})$ had similar coverage levels as intervals based on traditional variance estimation. However, there were notable differences between the Thompson based intervals and the traditional intervals for average interval width.

Note that one can control interval coverage level but not interval width, thus intervals with smaller width are preferred over intervals with larger width. In this simulation, the intervals based on $\widehat{var}(\hat{\mu}_k)$ and $\widehat{var}(\hat{\mu}_k - \hat{\mu}_{k'})$ always had narrower width than intervals based on traditional variance estimation. This was most noticeable for small populations, in particular, when the sample size approached the population size.

Thus, the traditional intervals were never preferable to the Thompson based intervals with respect to average interval width. In fact, the interval width reduction for intervals based on $\widehat{var}(\hat{\mu}_k)$ and $\widehat{var}(\hat{\mu}_k - \hat{\mu}_{k'})$ was more than 16% for a population of size 50 and a sample of size 30 (see Table 5.3). More than 16% of the simulations had more than 10% reduction for Thompson based intervals, and more than 36% had more than a 5% reduction in width (see Table 5.3).

For all populations and sample sizes considered, the traditional intervals had higher coverage levels than the Thompson based intervals. Neither had alarmingly low coverage, but the traditional intervals did tend to have larger than expected coverage levels, especially when the ratio of sample size to population size approached one. This most likely indicates that the degrees of freedom used for the t statistics were not adequate for an interval based on traditional variance estimation under simple random sampling.

For smaller populations the traditional intervals had noticeably more intervals that covered the true population value of μ_k or $\mu_k - \mu_{k'}$. However, as the ratio of the sample size with respect to the population size decreased the two types of intervals became virtually the same. Thus, the traditional intervals were only preferable to the Thompson based intervals, with respect to interval coverage, when the ratio of the sample size to the population was large (for all population sizes). Again, this may have more to due with the degrees of freedom used than with variance estimation.

One characteristic that the traditional and Thompson based intervals had in common was that the coverage for $\mu_k - \mu_{k'}$ tended to drop as ρ increased, especially when the ratio of sample size to population size increased. As ρ increases the amount of unique information from the sample decreases, which reduces the degrees of freedom. Thus, this coverage behavior most likely indicates that the degrees of freedom used for the t statistics were not adequate when the treatment and the control responses were highly correlated on each unit. This is a potential avenue for continued study of μ_k.

These results were expected. It should be no surprise that traditional variance estimation performed better for large populations and large sample sizes, since it has good asymptotic qualities. Since the variance estimation based on Thompson's estimator accounts for the sampling design, it was expected to give more precise intervals.

The traditional and the Thompson based intervals performed nearly identically for large populations and for small sample sizes. Thus, intervals based on $\widehat{var}(\hat{\mu}_k)$ and $\widehat{var}(\hat{\mu}_k - \hat{\mu}_{k'})$ were most attractive under srswor and a CRD when the population size was small and when the sample size approached the population size.

Chapter 6

Future Research based on Thompson's Estimator

In the current project applications of Thompson's estimator were very limited. No real data were available for evaluating Thompson's estimator. Several data sets were considered as potential evaluation tools but none of them met the necessary criteria (i.e. a study that used and reported both a sampling design to collect units and an experimental designs to assign treatments). This is frequently an issue when procedure is available before application. As previously noted there are several potential applications of Thompson's estimator to real research but since the mechanism for evaluating such studies was not available researchers found other ways to design their studies in order to accommodate the available analysis tools. Thompson's estimator is a new analysis tool for meaningful research problems that has not been implemented on even a moderate scale. Hopefully this research will begin to make Thompson's estimator more accessible to researchers and useful data sets will ensue.

Since the development and application of Thompson's estimator is in it's infancy, the following sections suggest areas for further research to make Thompson's estimator more accessible for implementation.

6.1 Negative Variance Estimation

Recall that, similar to the Horvitz-Thompson estimator (HTE), estimates of the variance of Thompson's estimator can be negative. There have been suggestions for ways to deal with the negative variance estimation problem for the HTE. Some of these ideas, such as design selection, may be adapted for $\widehat{var}(\hat{\mu}_k)$. For example, a fixed-size sampling design with uniform individual and joint inclusion probabilities will satisfy conditions which guarantee $\widehat{\mathrm{V}}_{\mathrm{SYG}}(HTE) > 0$ (Hedayat and Sinha, 1991).

6.2 Demonstration through Simulation

The current project was also limited in the spectrum of simulations. Only srswor and CRD for normally distributed data were considered here. This is the simplest case and most likely to not show the true difference between $\hat{\mu}_k$ and $\bar{y}_k$. Even without real data much could be learned from additional simulations using more complex sampling and experimental designs (see the following section) or non-normal data such as Poisson or Exponential data. Also, a bootstrap simulation may shed more light on the behavior of $\hat{\mu}_k$ and it's corresponding variance estimator.

6.3 Probabilities for Complex Designs

The inclusion probabilities presented in this study are limited to just a few basic sampling designs. As the complexity of the sampling design increases the inclusion probabilities and Thompson's estimator become more complicated. Inclusion probabilities have been developed for more complicated sampling designs than the ones presented here. Thompson's estimator and it's properties should be studied for these designs as well. Some examples of more complex sampling designs include cluster, multistage, and adaptive sampling designs.

Similarly the complexity of the assignment probabilities will influence the complexity of Thompson's estimator but only the simplest experimental designs have been presented here. However, dissimilar to sampling, assignment probabilities have yet to be derived for other experimental designs. Thompson's estimator and it's properties should be studied for experimental designs such as Latin square, and split-plot designs.

6.4 Approximations for Joint Probabilities

As the noted in the previous section complicated sampling and experimental designs have complicated inclusion and assignment probabilities. The joint probabilities become extremely complicated which makes $var(\hat{\mu}_k)$, and $\widehat{var}(\hat{\mu}_k)$ quite cumbersome to calculate. An alternative is to use an approximation for the joint probabilities. This idea has been considered for the variance and its estimator for the HTE. The ideas applied to the HTE should be adaptable for joint assignment and applicable to Thompson's estimator.

For example, Stehman and Overton (1989) discussed two possible approximations for joint inclusion probabilities for use in the estimator of the variance of the HTE. One approximation is a truncation of an approximation derived by Hartley and Rao (1962),

$$\pi_{ij}^{hrt} = \frac{(n-1)\pi_i\pi_j}{n - \pi_i - \pi_j - \sum_{i=n}^{N} \pi_i^2/n} \; .$$

The other approximation, derived by Overton (1985), is given by

$$\pi_{ij}^{o} = \frac{(n-1)\pi_i\pi_j}{2n - \pi_i - \pi_j} \; .$$

The Overton approximation is preferred when the inclusion probabilities are not known for every member of the population (Stehman and Overton, 1989).

6.5 Other Parameters

Only the difference of two treatment means has been considered here. Other contrasts and parameters may be of interests to some researchers. For example, a contrast comparing means for multiple treatment means or a contrast to check for linearity may be useful applications of $\hat{\mu}_k$. Also, for example, ratio or regression estimators, may be of interest. These estimators would be Thompson type estimators in the sense that they would incorporate the sampling and experimental designs using inclusion and assignment probabilities. However, the form and properties of these estimators would likely be very different from Thompson's estimator itself.

6.6 Software

Currently there is no known software for calculating Thompson's estimator. Before Thompson's estimator can be fully functional for researchers there needs to be easy to use software for calculating $\hat{\mu}_k$, $var(\hat{\mu}_k)$, and $\widehat{var}(\hat{\mu}_k)$. One way to do this would be to create a function for use in already existing software such as R or S-Plus. Such a function should be able to accommodate several commonly used sampling and experimental designs. The programs written for conducting the simulations for this research project could potentially be a starting place for such a function.

6.7 Inferential Methodology

The interval estimation in the current study was based on the standard t-interval estimation with no adjustment for the degrees of freedom. In this case, interval coverage is equivalent to the Type I error rate of the corresponding hypothesis tests based on the t-statistic. Therefore, hypothesis testing was not even considered by the current research.

As previously noted the degrees of freedom used in this study seemed to be adequate for treatment mean estimation but not for the difference of treatment means. Alternative interval procedures based on corrected degrees of freedom or other statistics may give more precise interval estimates, especially in the non-srswor/CRD case and for non-normal populations.

Much literature can be found on the topic of hypothesis testing methodology which may shed light on appropriate ways to use Thompson's estimator to make inferences about a hypothesis. Also, to study the power of tests based on $\hat{\mu}_k$ additional simulation will be necessary. Such simulations are not done here but should be considered in future work.

6.8 Invariant Variance Estimation

A simple example can demonstrate that HTE based variance estimation is not scale invariant for unequal probability sampling. This plagues the variance estimator for the variance of Thompson's estimator as well. If one is only interested in the sampling design this invariance can be ignored, since the population responses are treated as fixed. However, this may require more consideration under combined sampling and experimental designs.

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Appendix A

Distribution of Thompson's Estimator

Figure A.1: Distribution of $\hat{\mu}_c$ for N=50

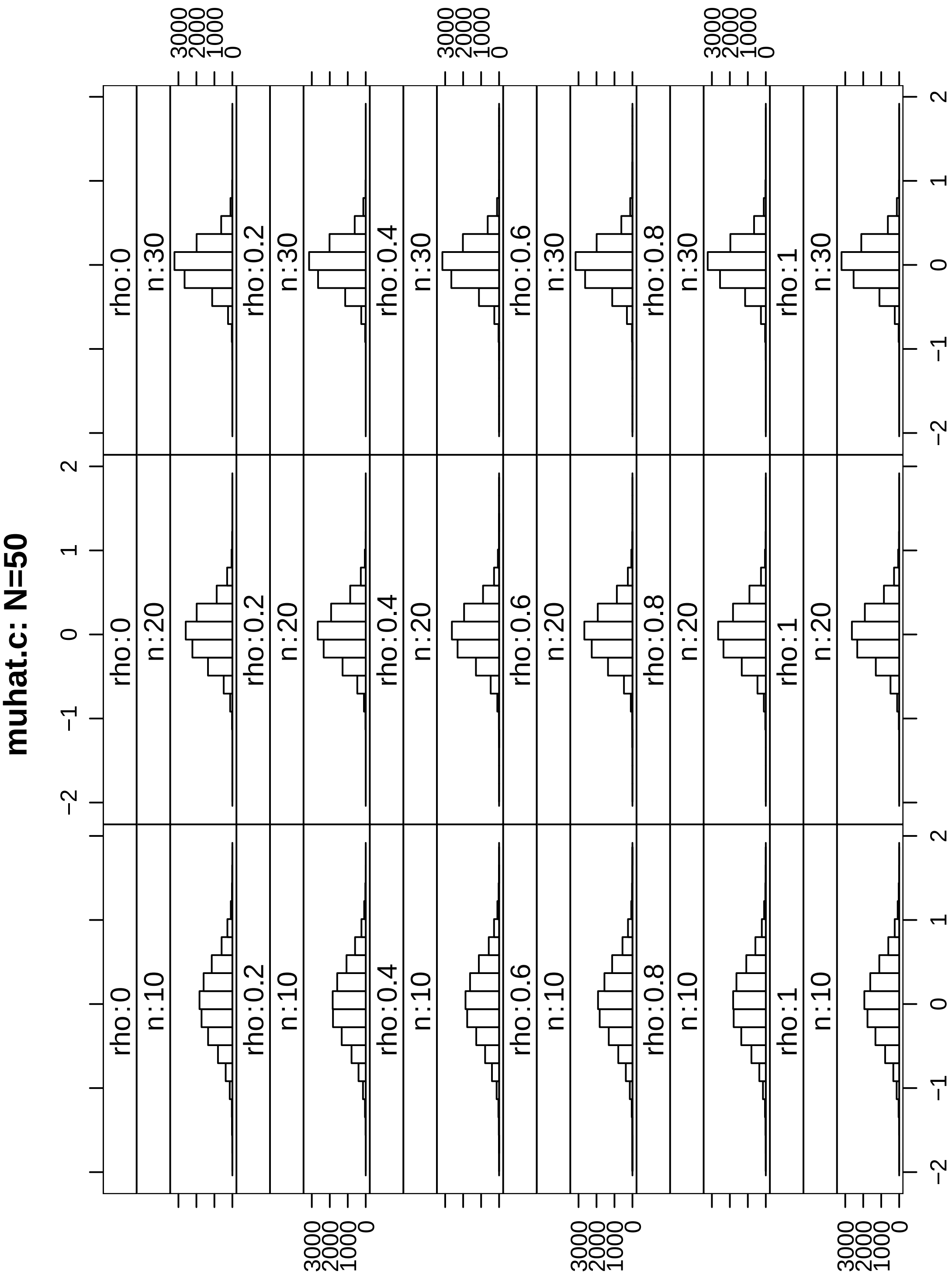


Figure A.2: Distribution of $\hat{\mu}_c - \mu_c$ for N=50

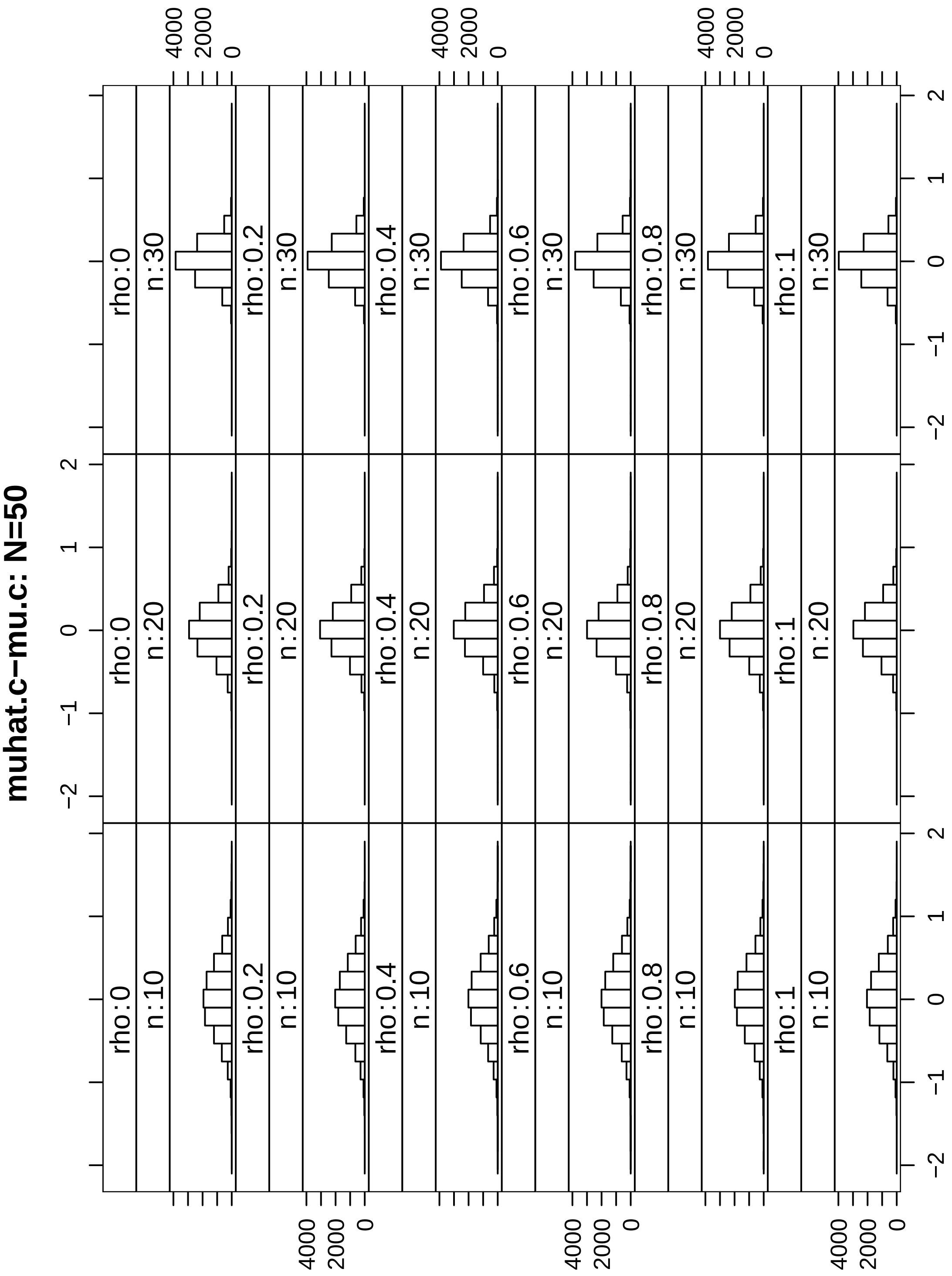


Figure A.3: Distribution of $\hat{\mu}_t$ for N=50

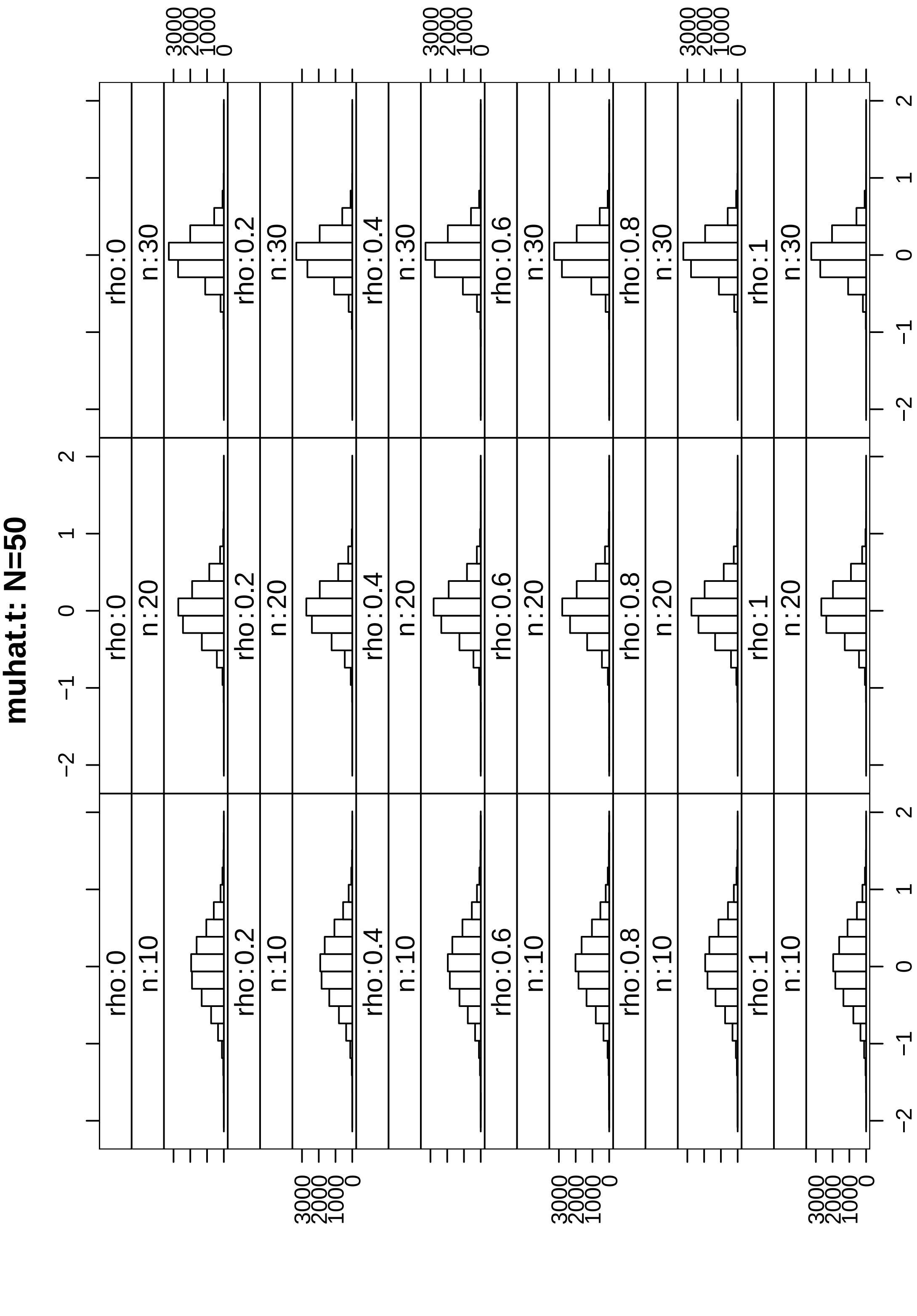


Figure A.4: Distribution of $\hat{\mu}_t - \mu_t$ for N=50

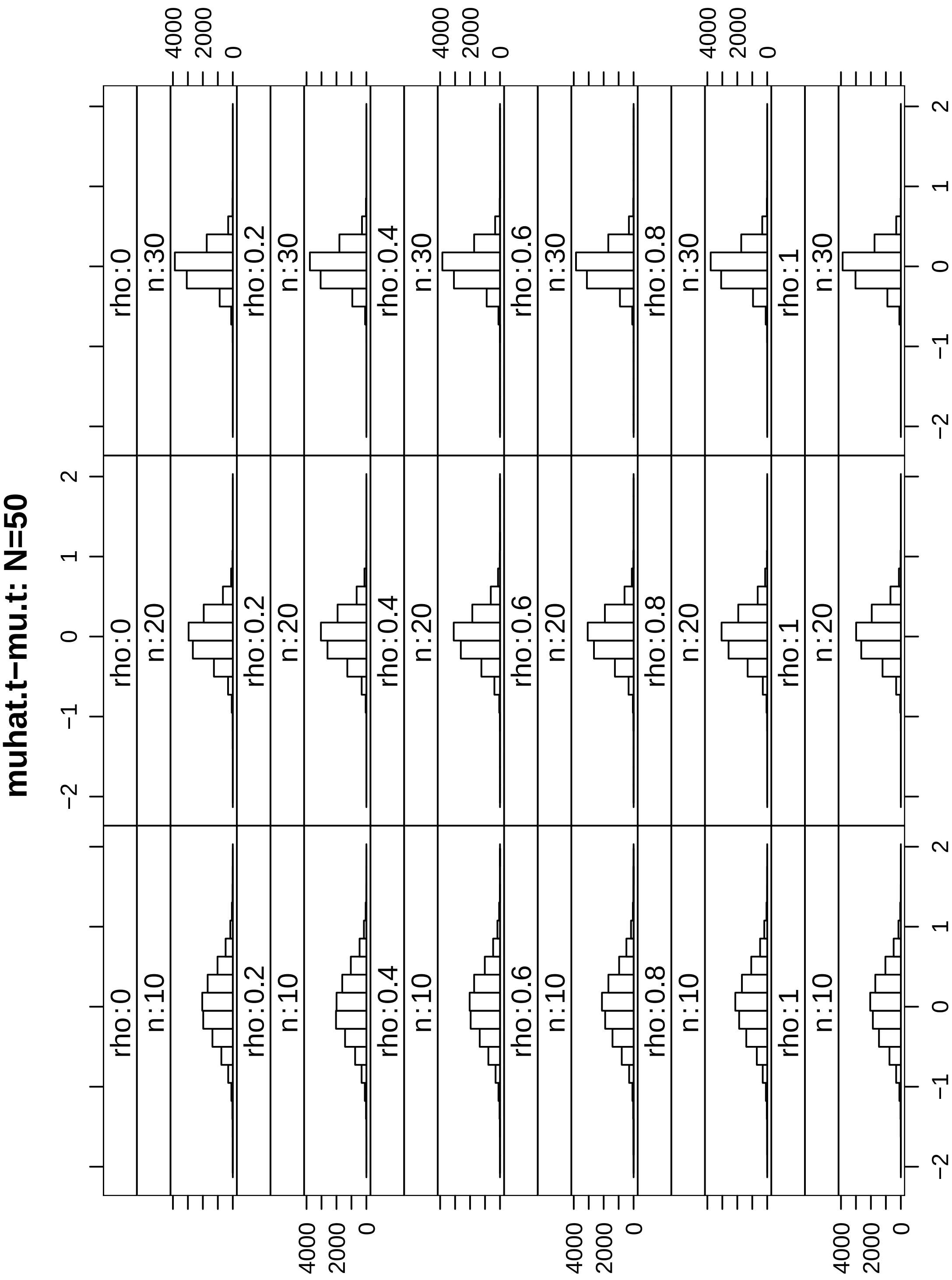


Figure A.5: Distribution of $\hat{\mu}_t - \hat{\mu}_c$ for N=50

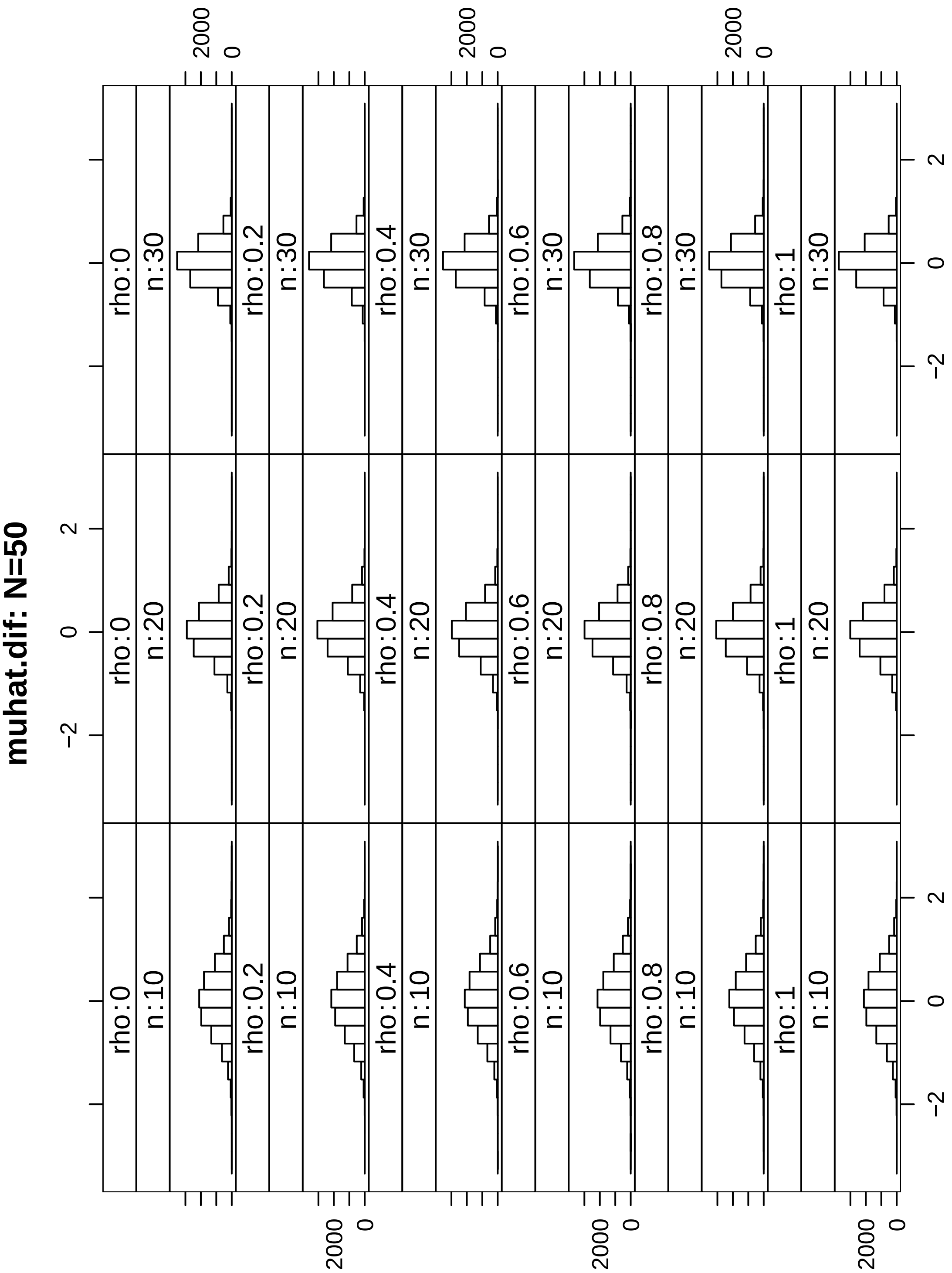


Figure A.6: Distribution of $(\hat{\mu}_t - \hat{\mu}_c) - (\mu_t - \mu_c)$ for N=50

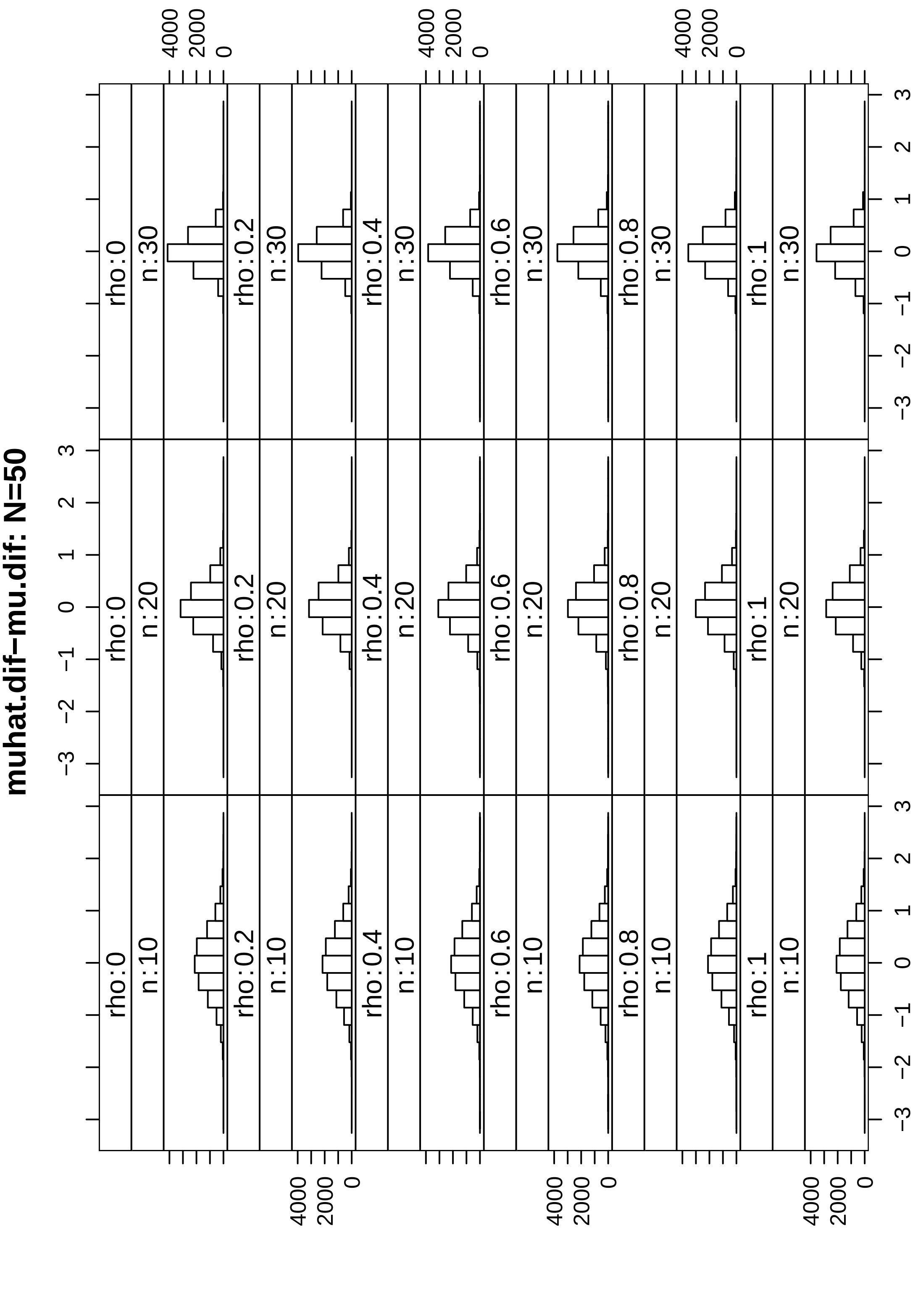


Figure A.7: Distribution of $\hat{\mu}_c$ for N=100

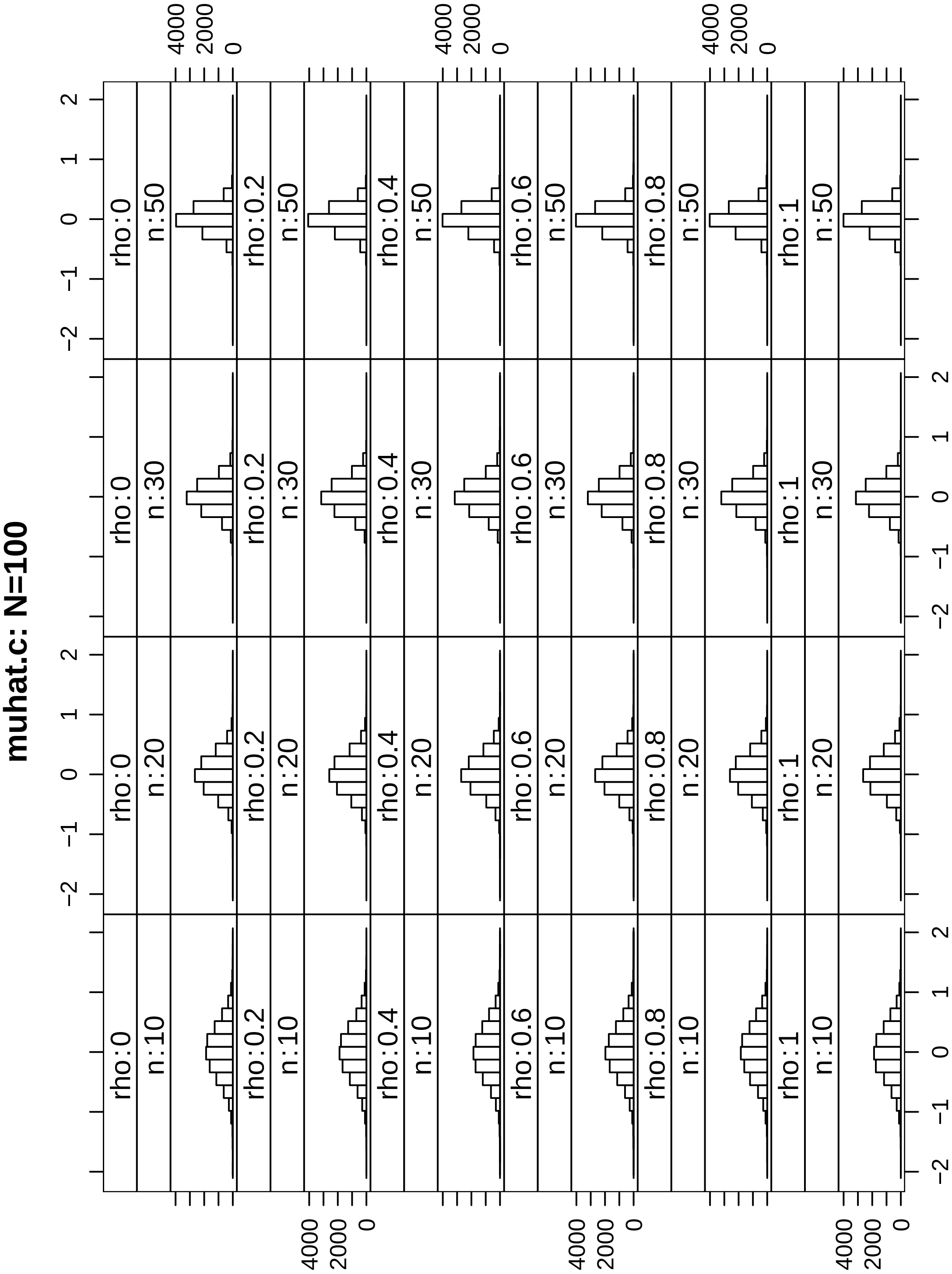


Figure A.8: Distribution of $\hat{\mu}_c - \mu_c$ for N=100

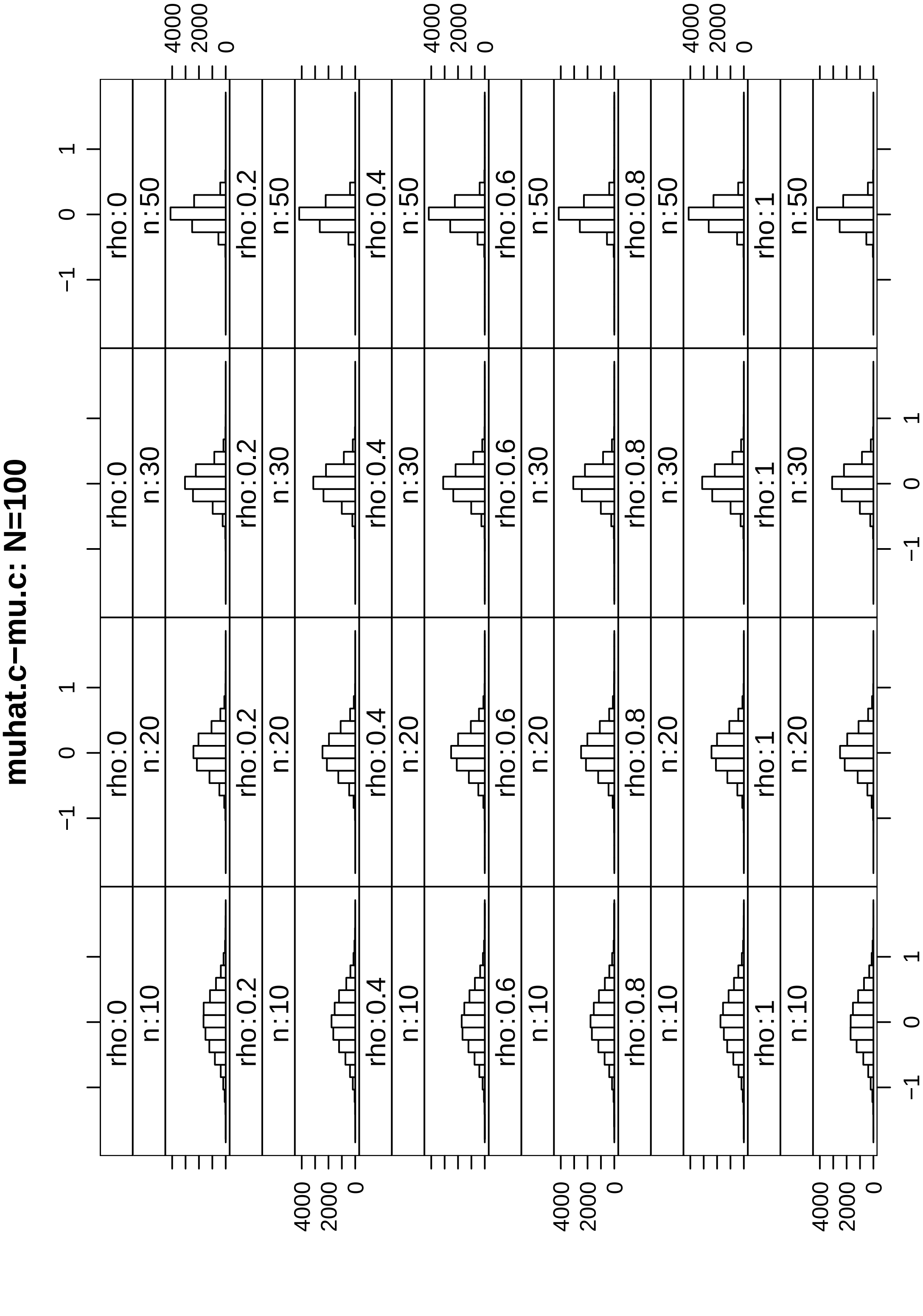


Figure A.9: Distribution of $\hat{\mu}_t$ for N=100

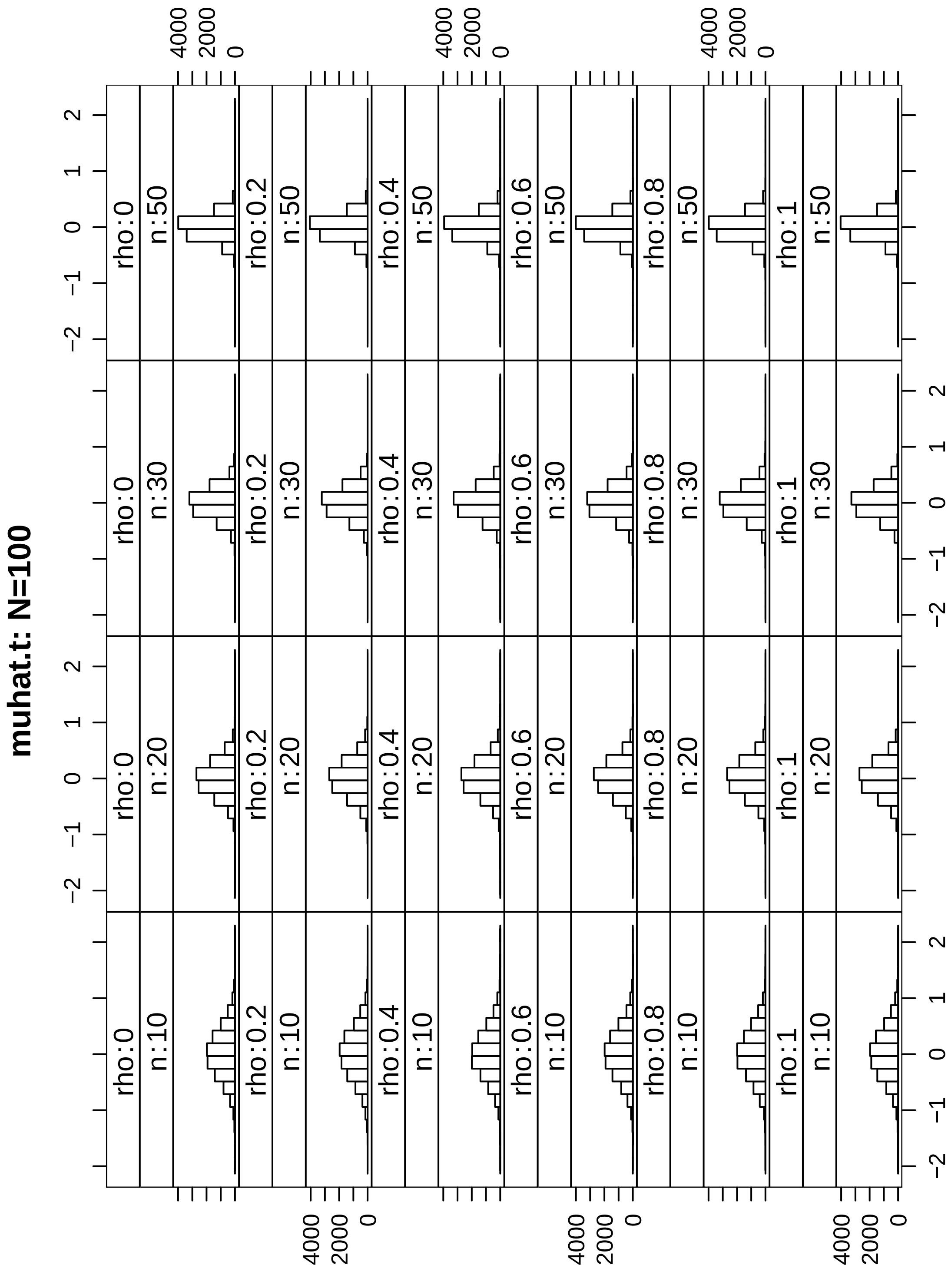


Figure A.10: Distribution of $\hat{\mu}_t - \mu_t$ for N=100

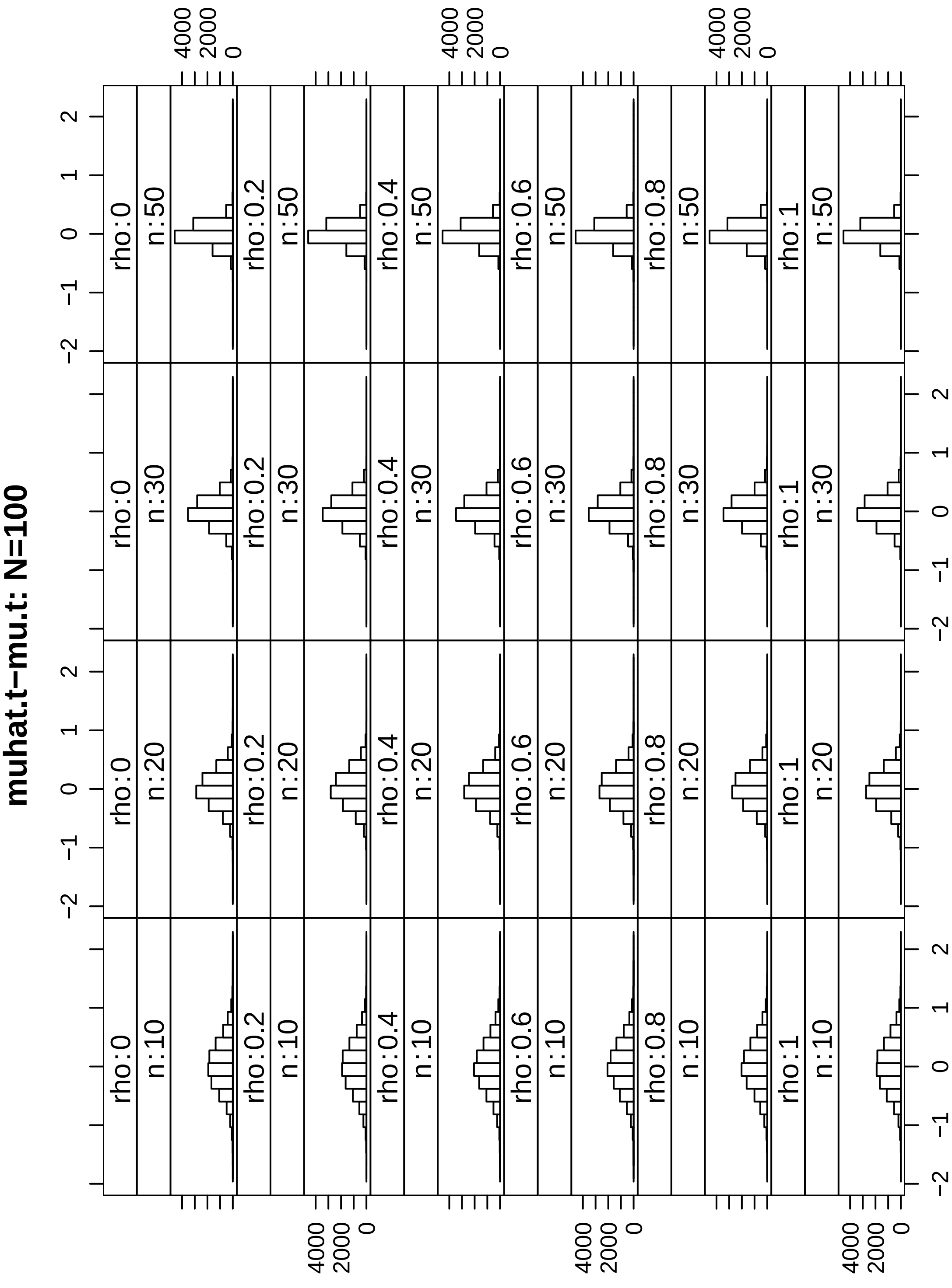


Figure A.11: Distribution of $\hat{\mu}_t - \hat{\mu}_c$ for N=100

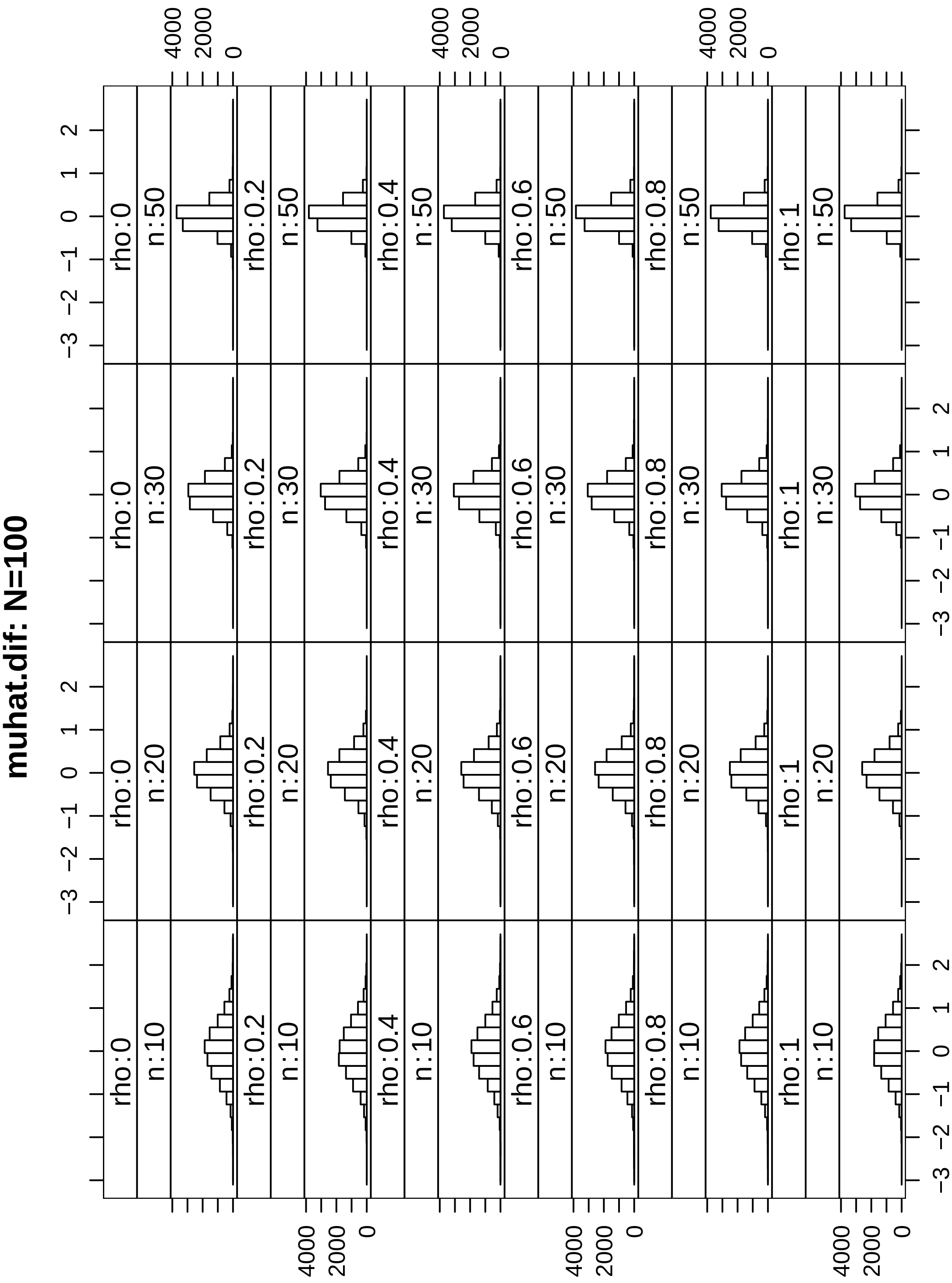


Figure A.12: Distribution of $(\hat{\mu}_t - \hat{\mu}_c) - (\mu_t - \mu_c)$ for N=100

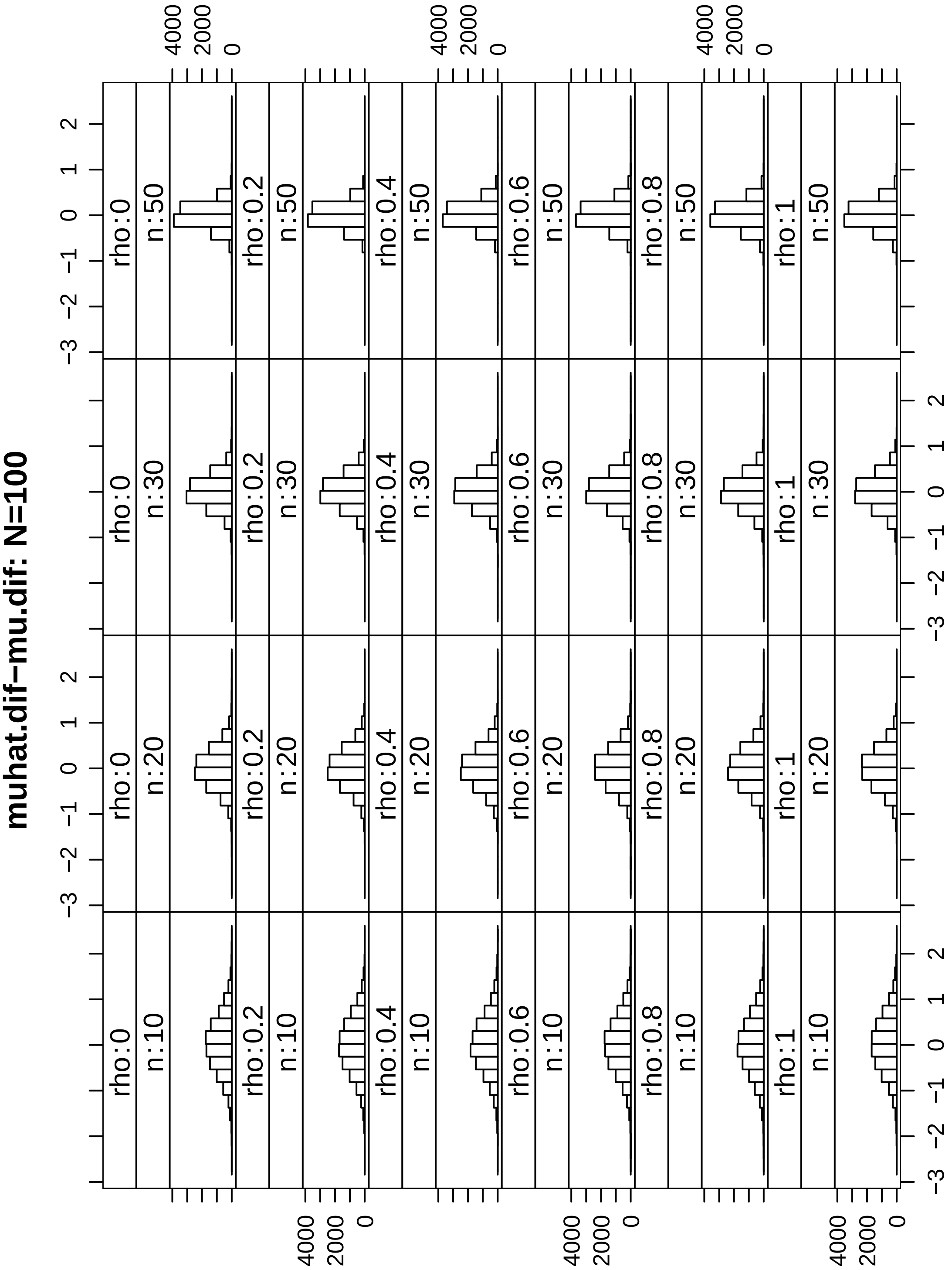


Figure A.13: Distribution of $\hat{\mu}_c$ for N=250

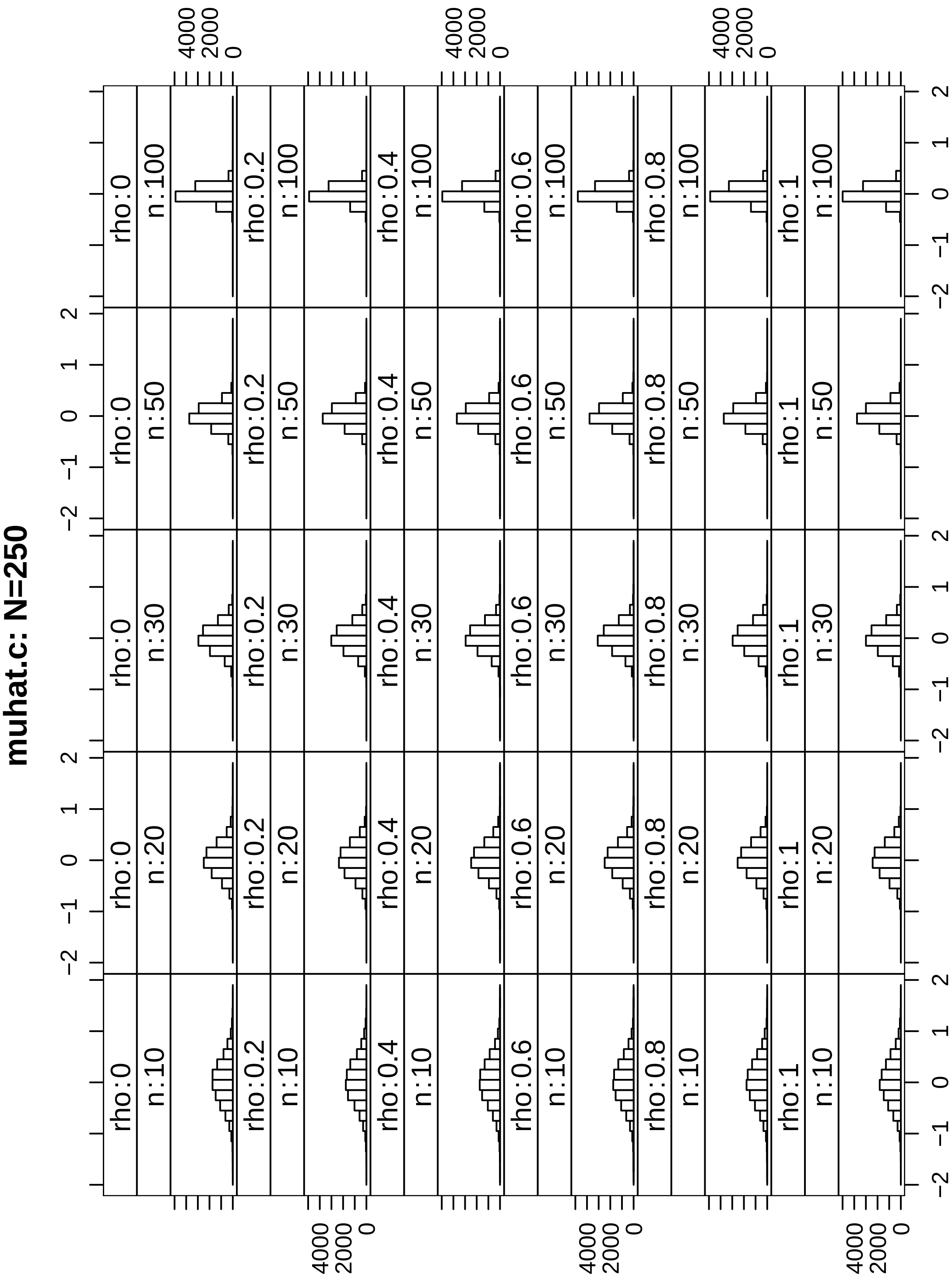


Figure A.14: Distribution of $\hat{\mu}_c - \mu_c$ for N=250

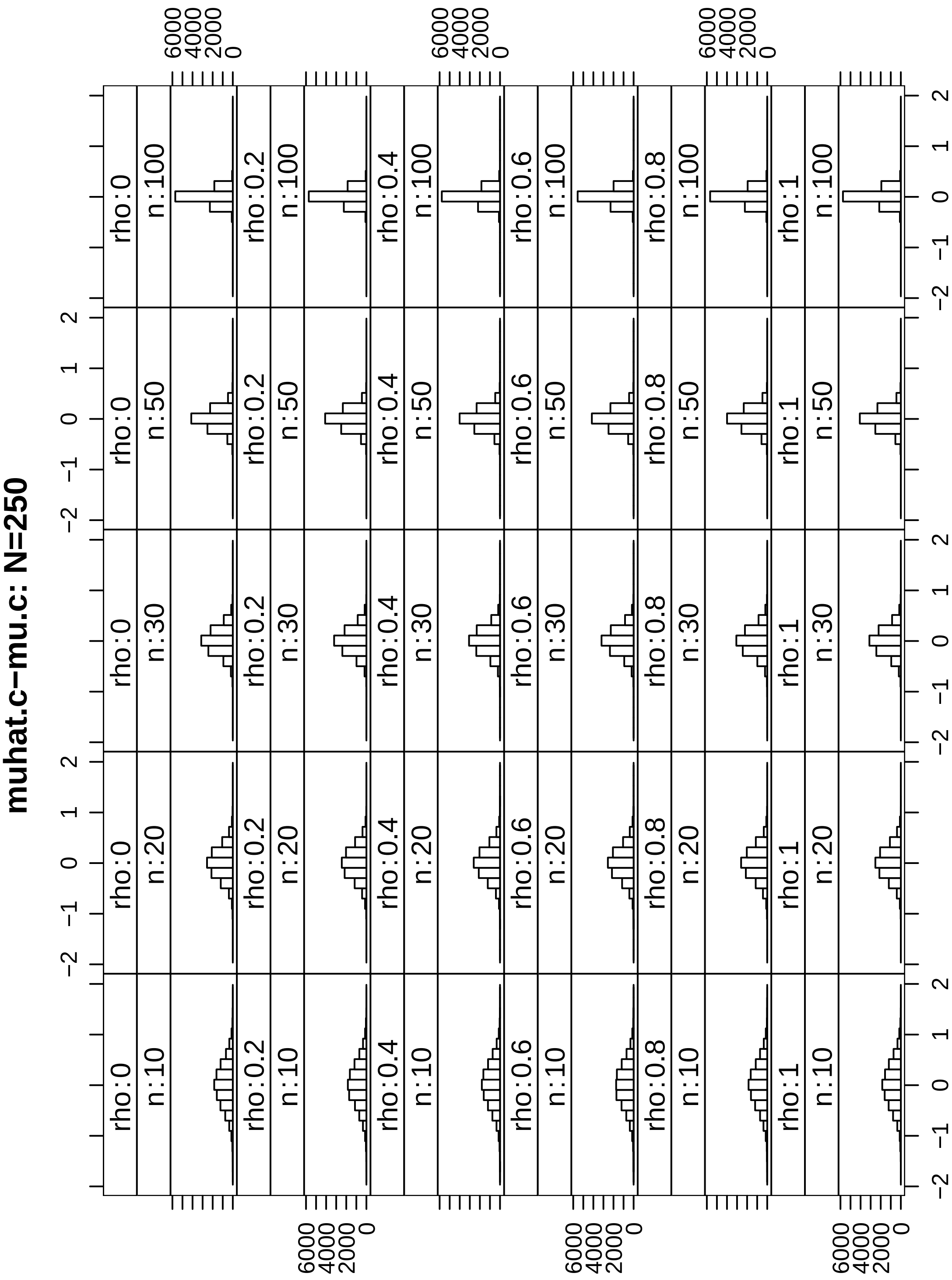


Figure A.15: Distribution of $\hat{\mu}_t$ for N=250

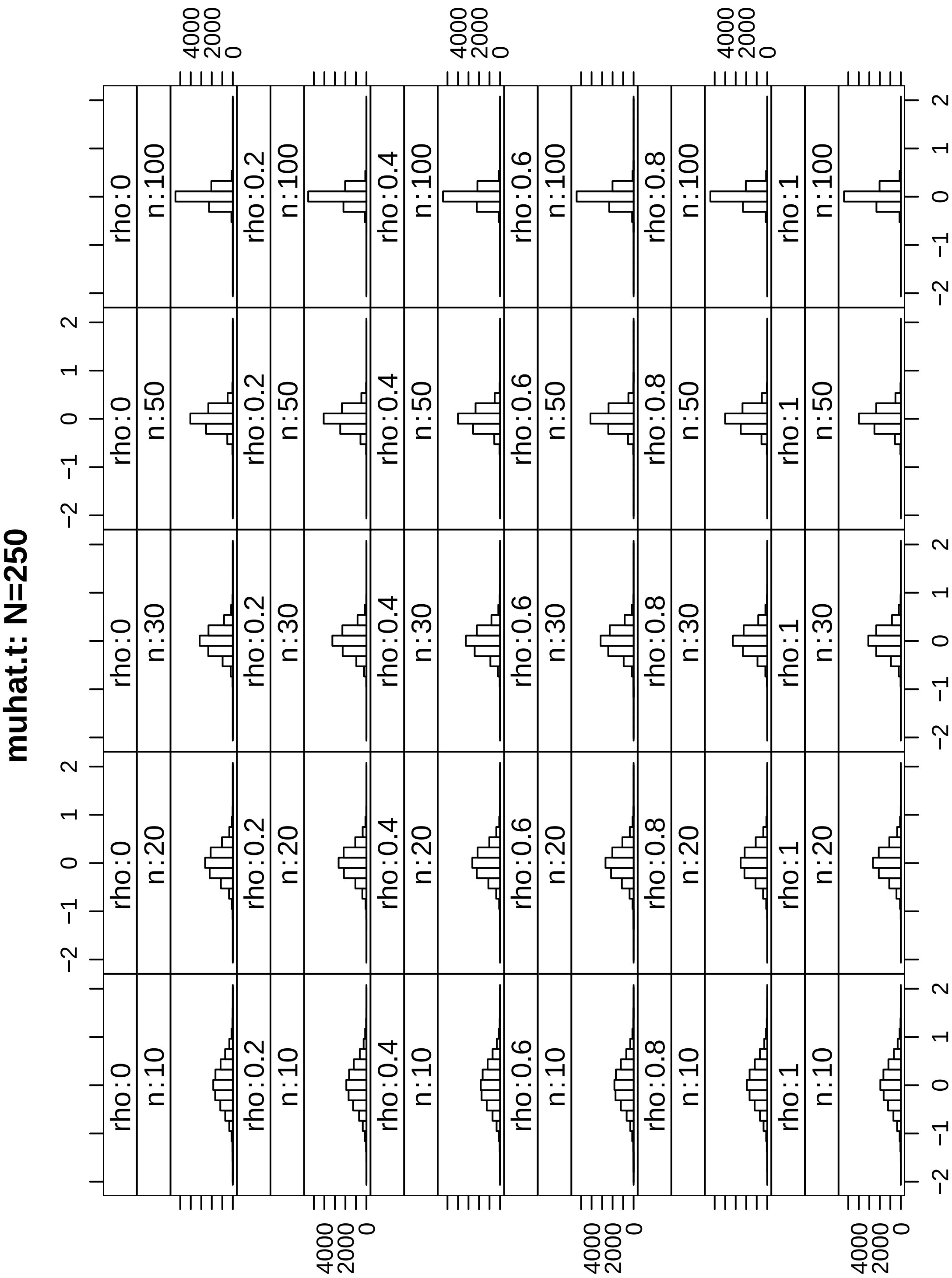


Figure A.16: Distribution of $\hat{\mu}_t - \mu_t$ for N=250

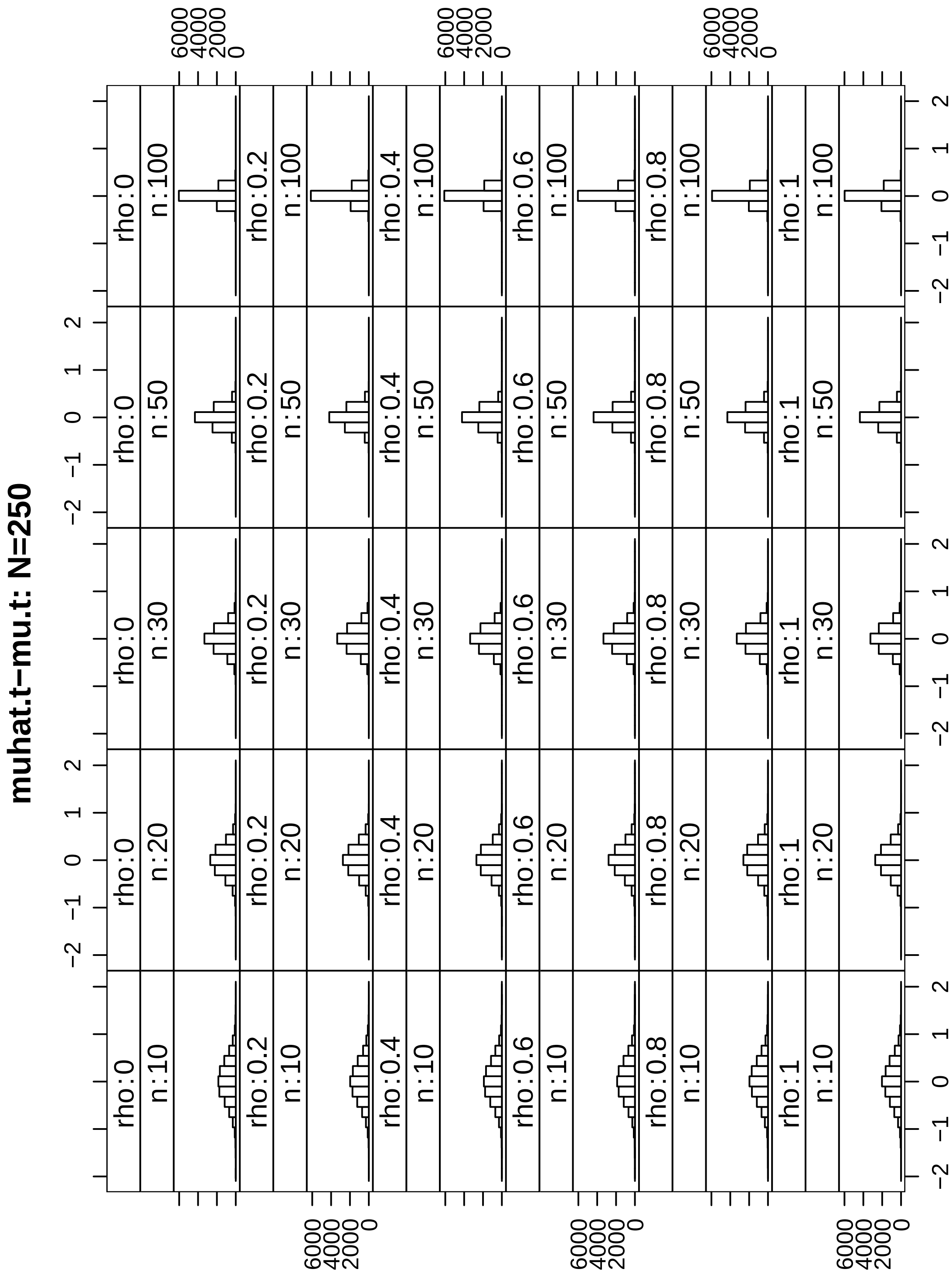


Figure A.17: Distribution of $\hat{\mu}_t - \hat{\mu}_c$ for N=250

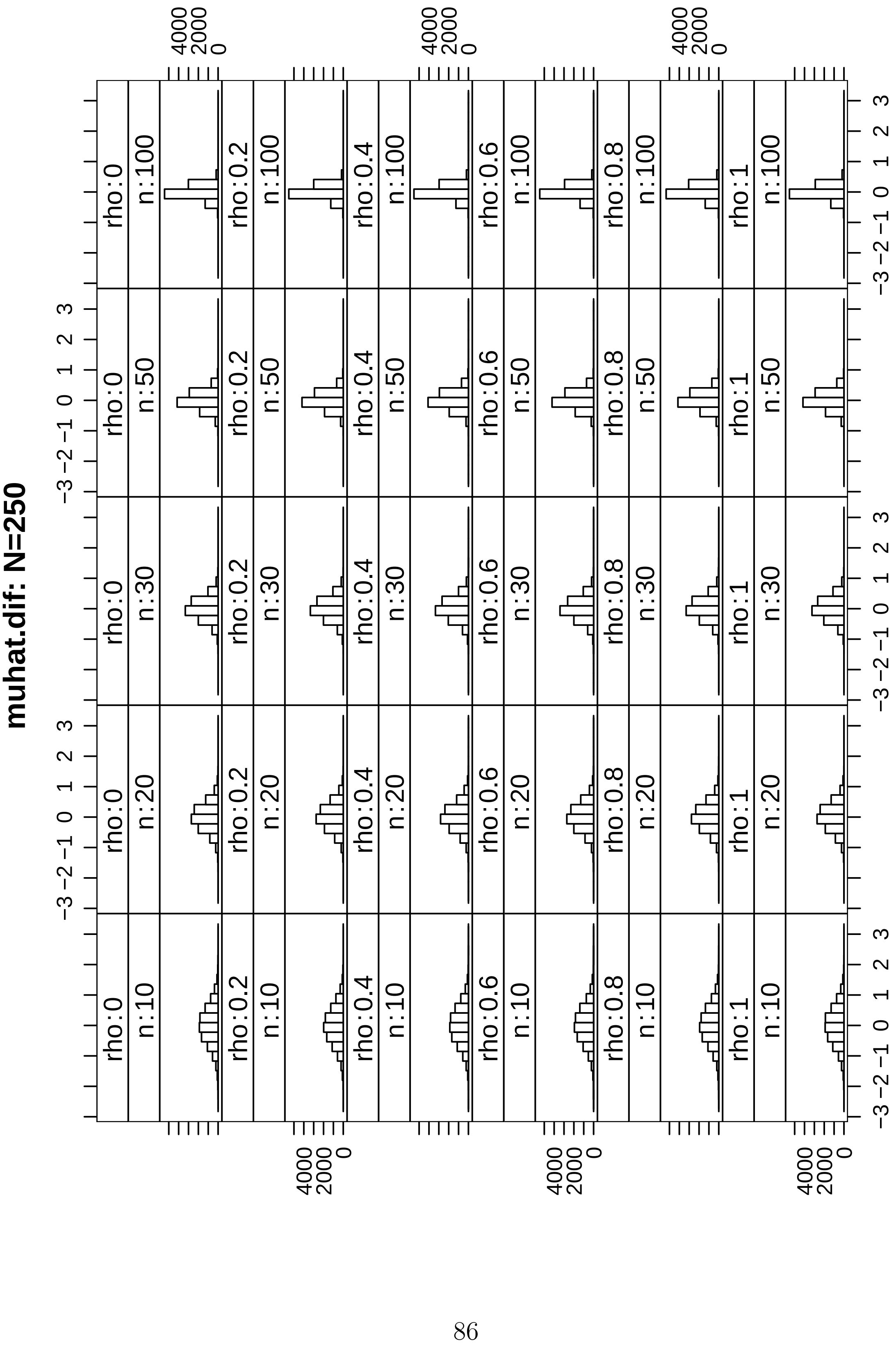


Figure A.18: Distribution of $(\hat{\mu}_t - \hat{\mu}_c) - (\mu_t - \mu_c)$ for N=250

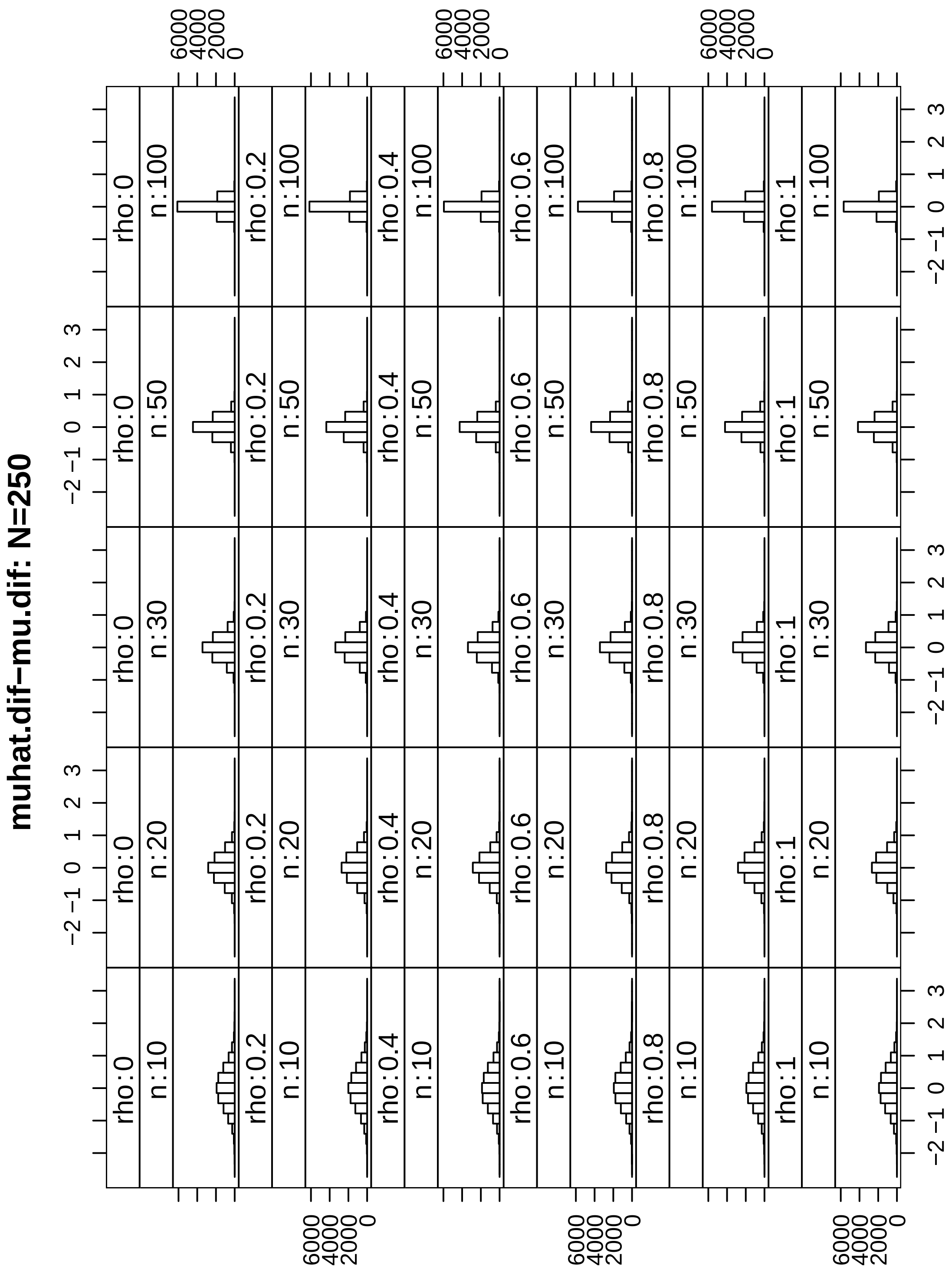


Figure A.19: Distribution of $\hat{\mu}_c$ for N=500

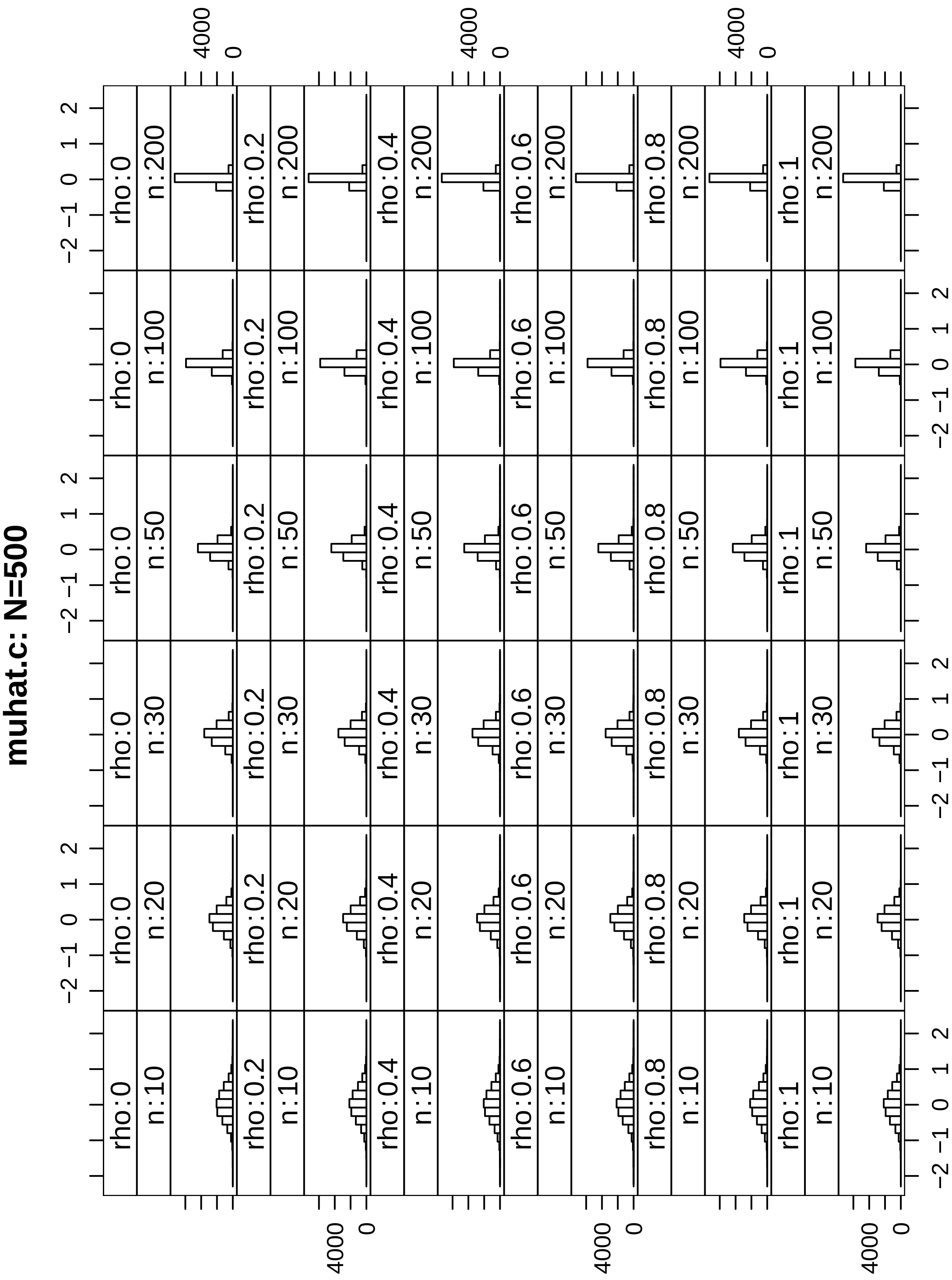


Figure A.20: Distribution of $\hat{\mu}_c - \mu_c$ for N=500

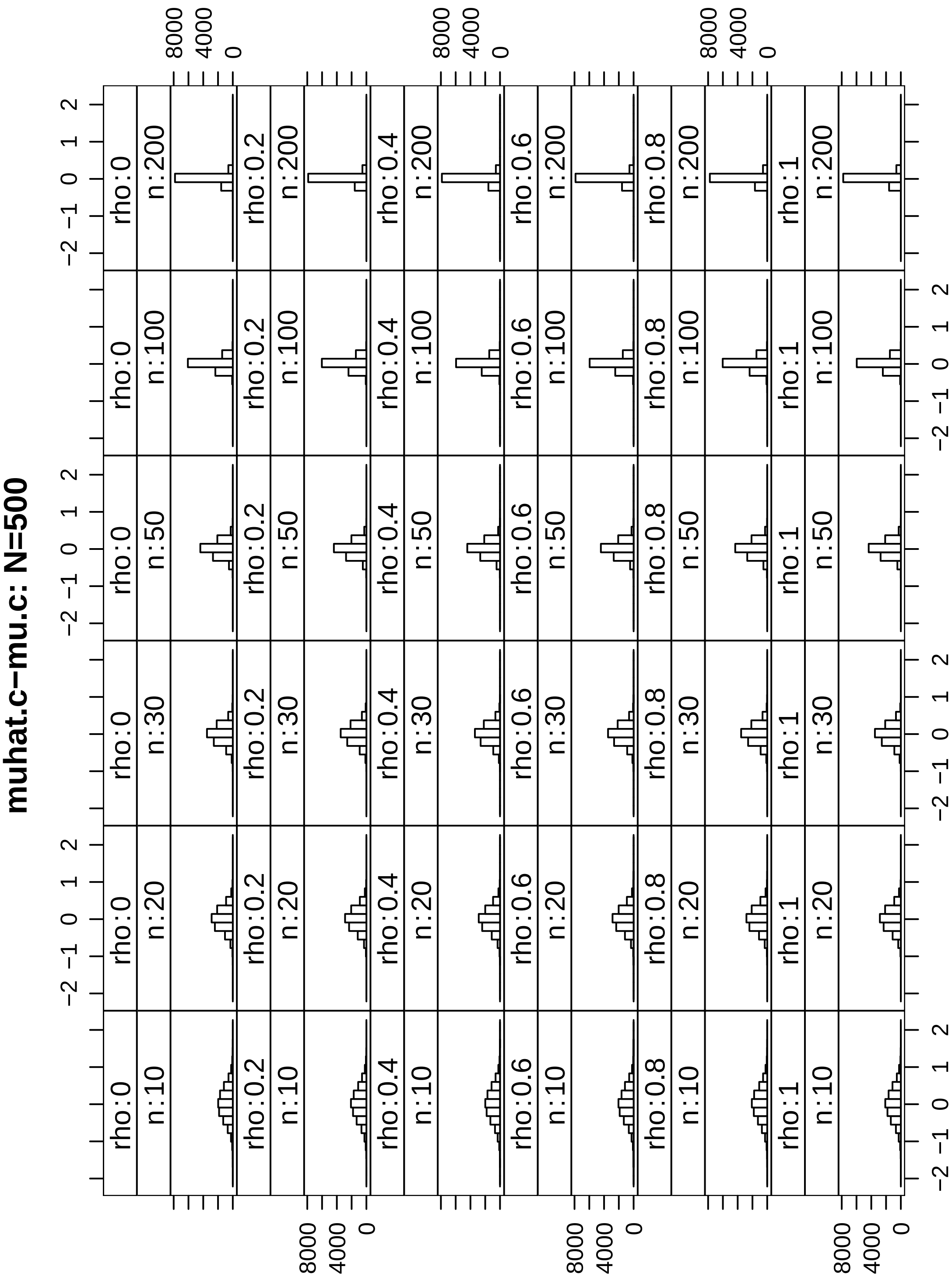


Figure A.21: Distribution of $\hat{\mu}_t$ for N=500

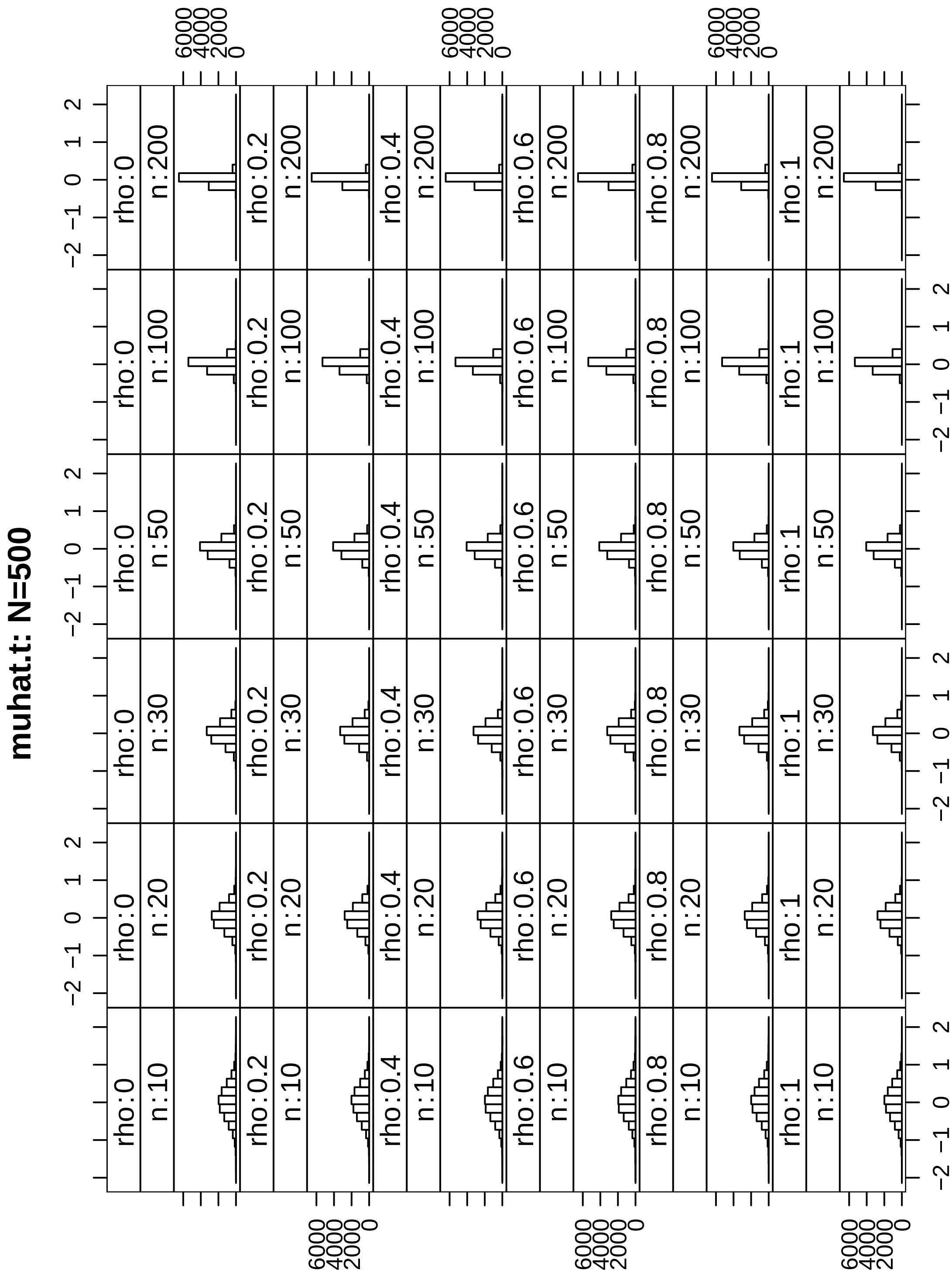


Figure A.22: Distribution of $\hat{\mu}_t - \mu_t$ for N=500

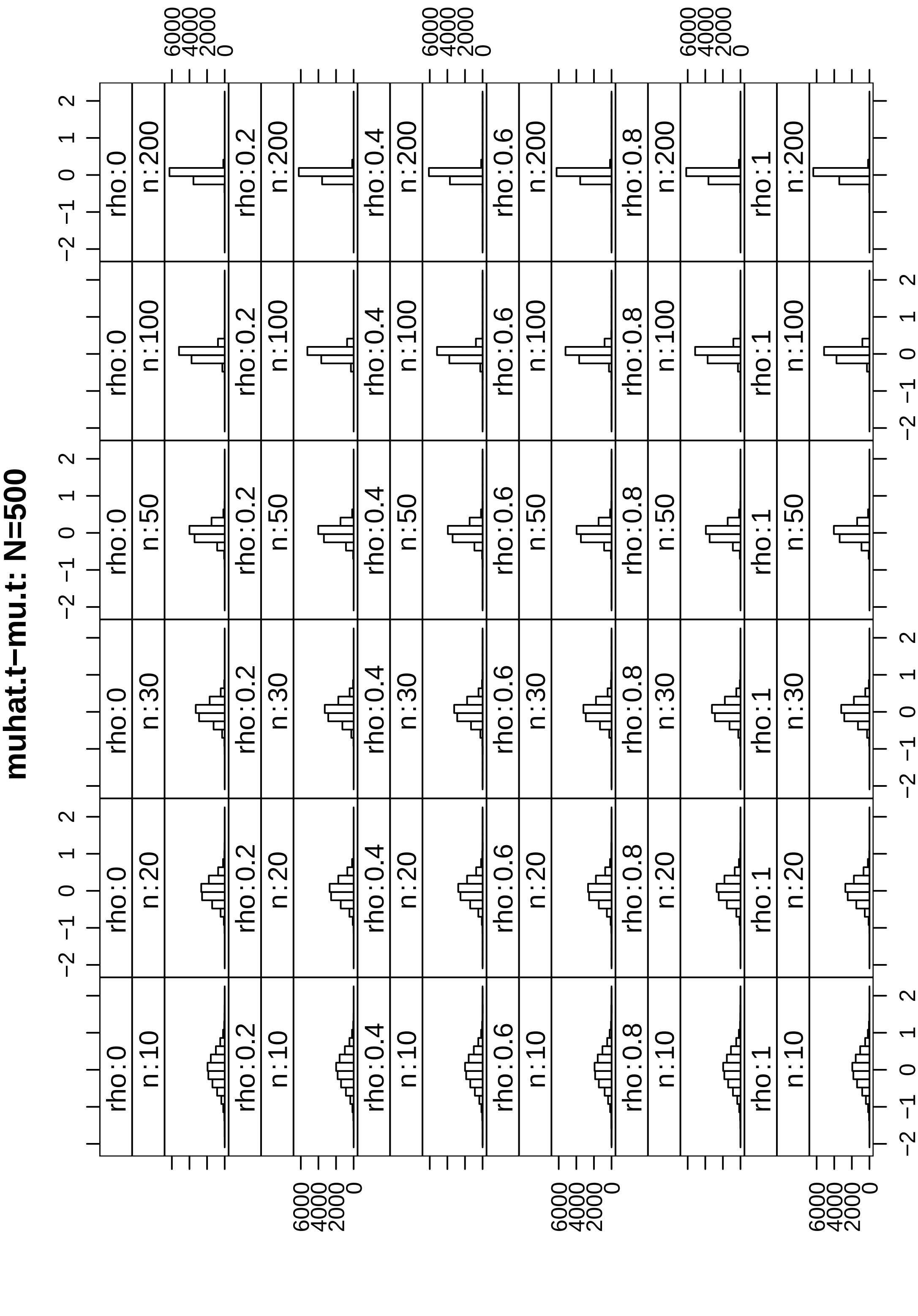


Figure A.23: Distribution of $\hat{\mu}_t - \hat{\mu}_c$ for N=500

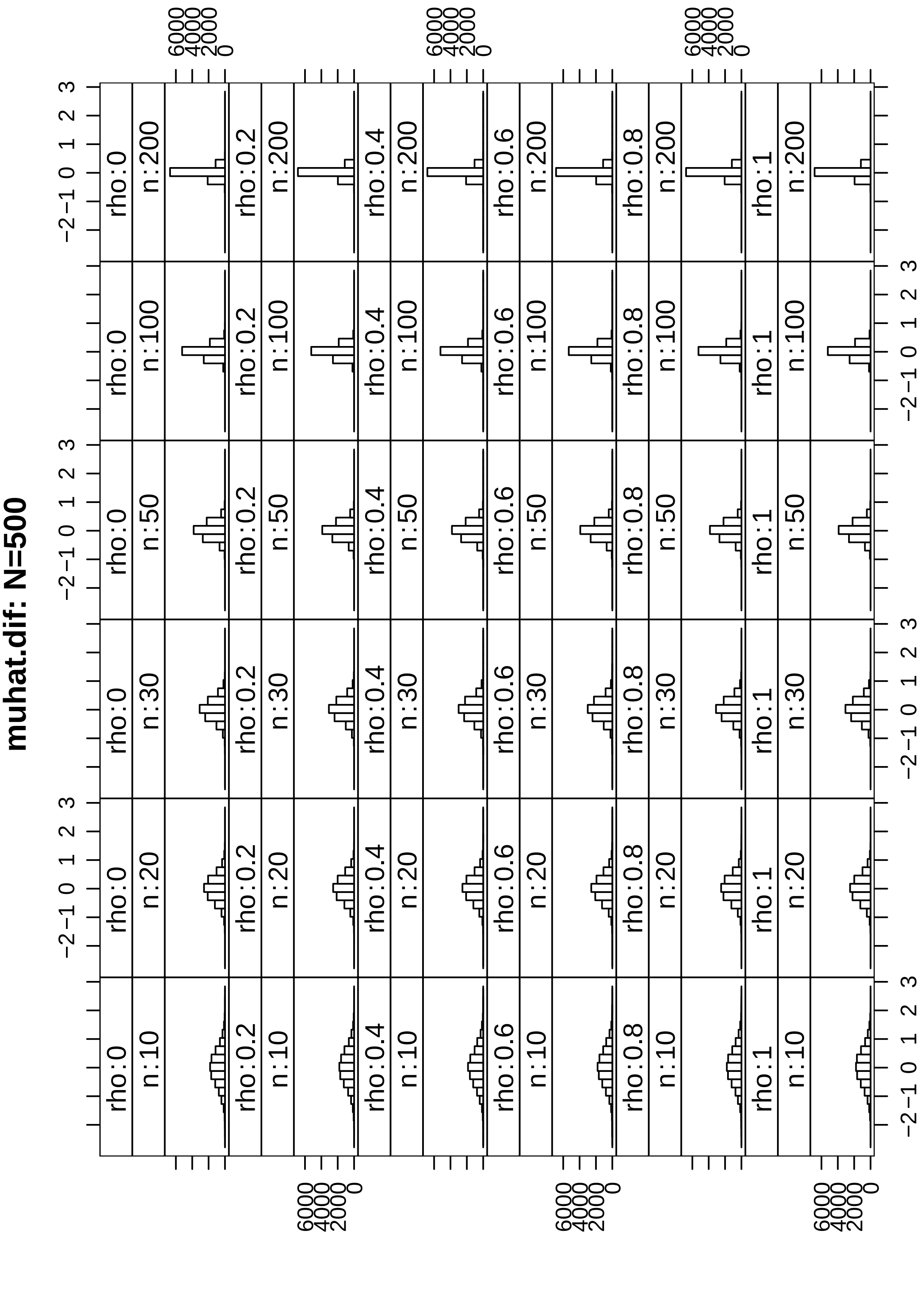


Figure A.24: Distribution of $(\hat{\mu}_t - \hat{\mu}_c) - (\mu_t - \mu_c)$ for N=500

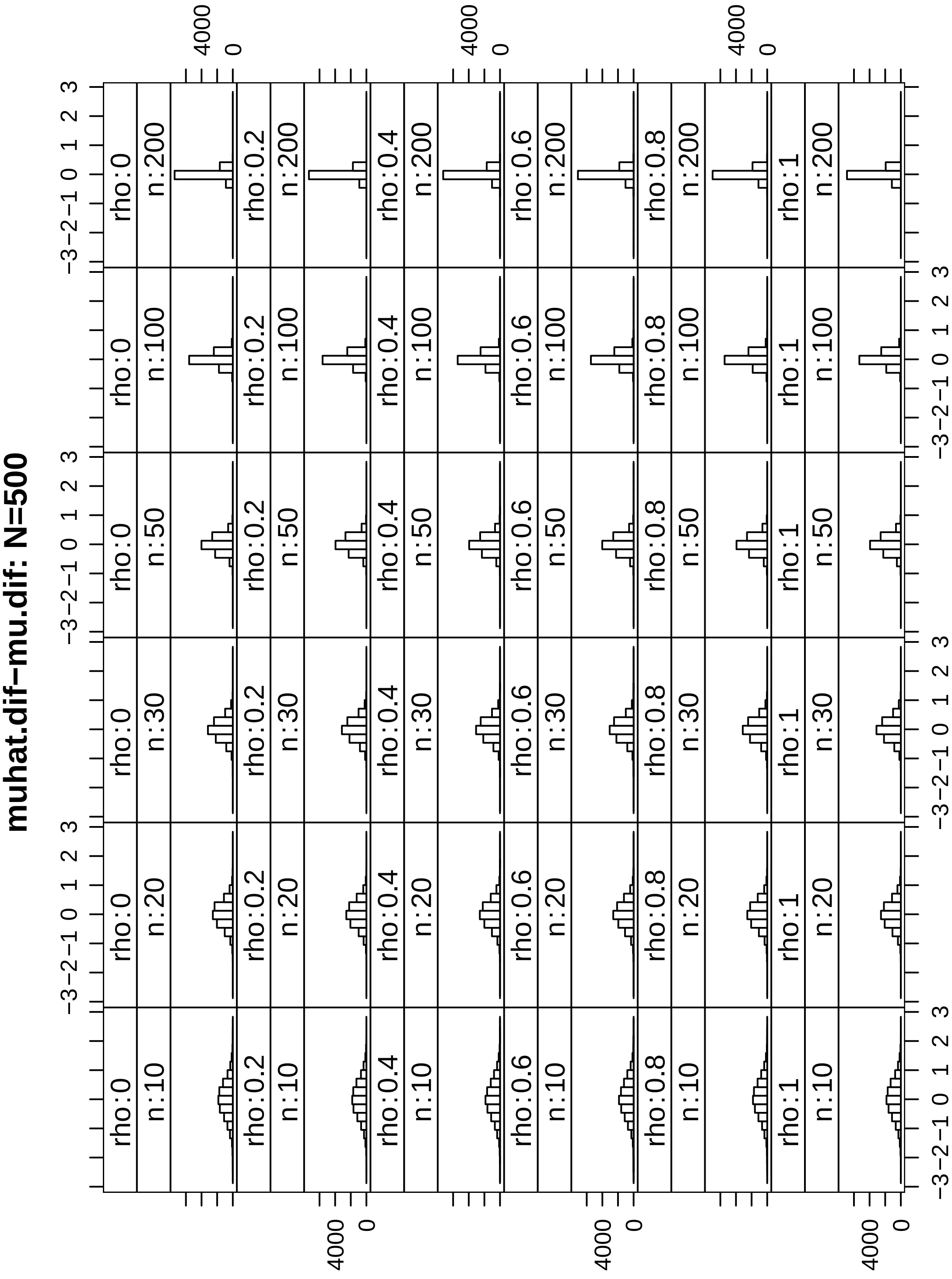


Figure A.25: Distribution of $\hat{\mu}_c$ for N=1000

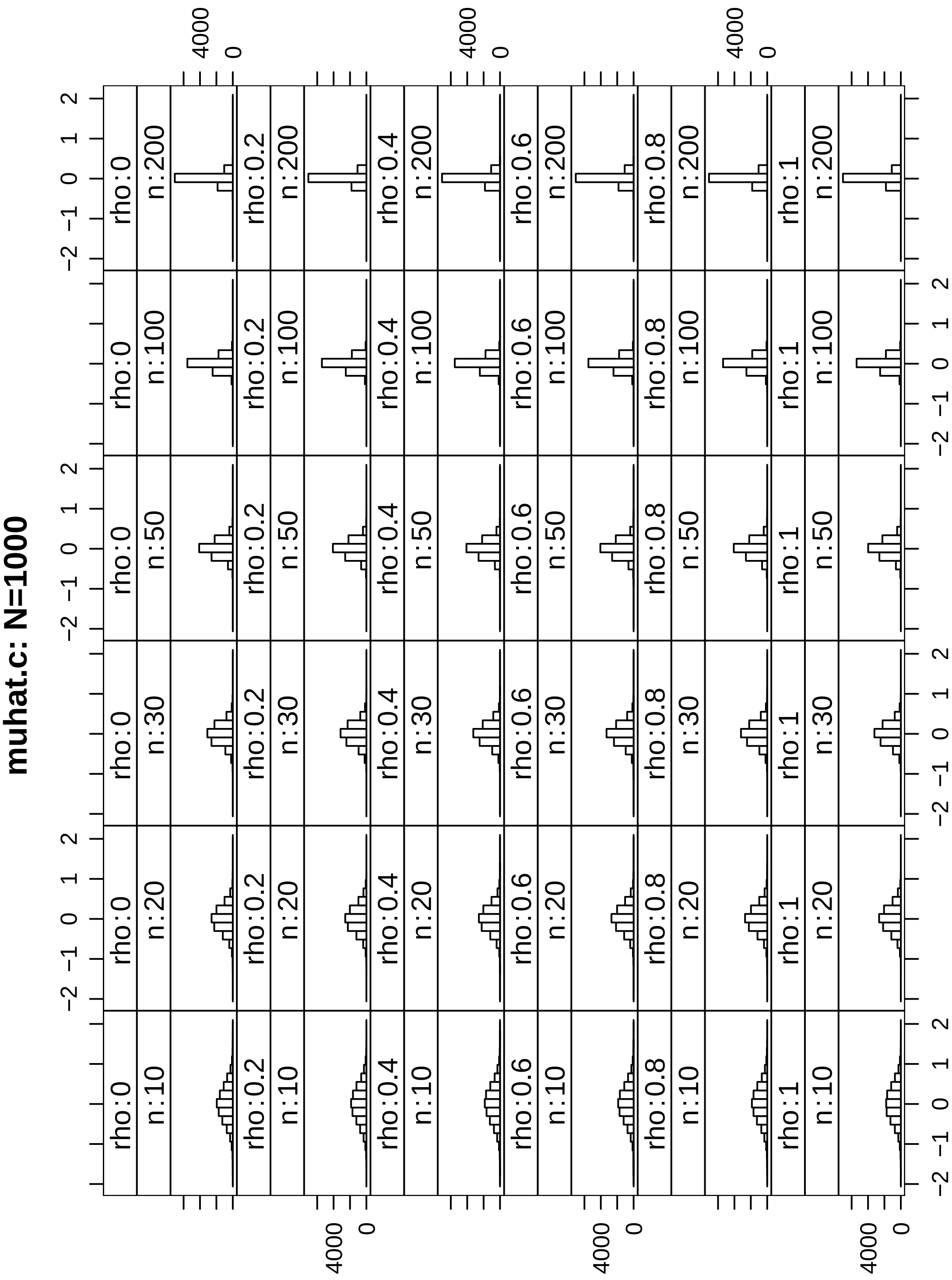


Figure A.26: Distribution of $\hat{\mu}_c - \mu_c$ for N=1000

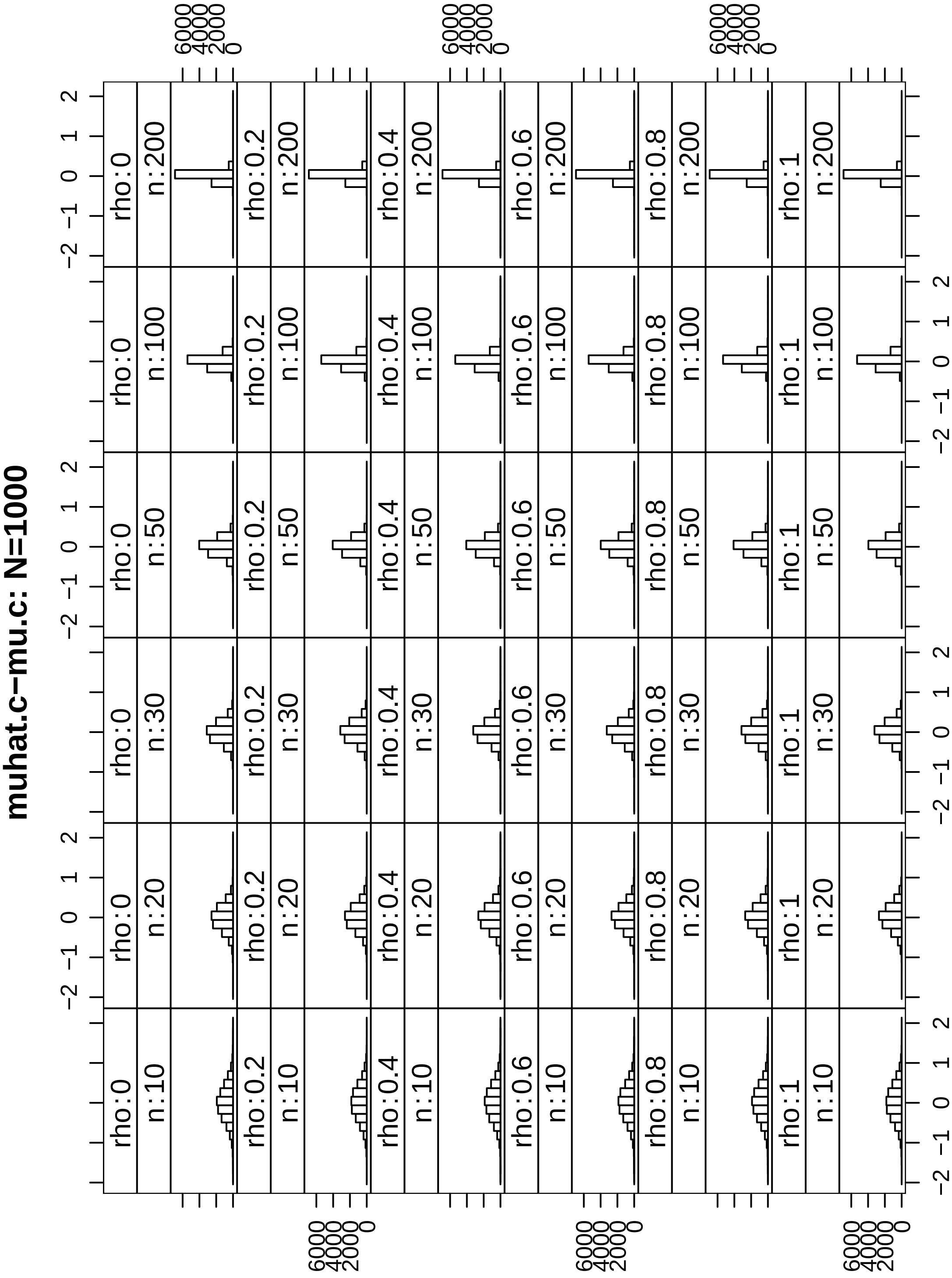


Figure A.27: Distribution of $\hat{\mu}_t$ for N=1000

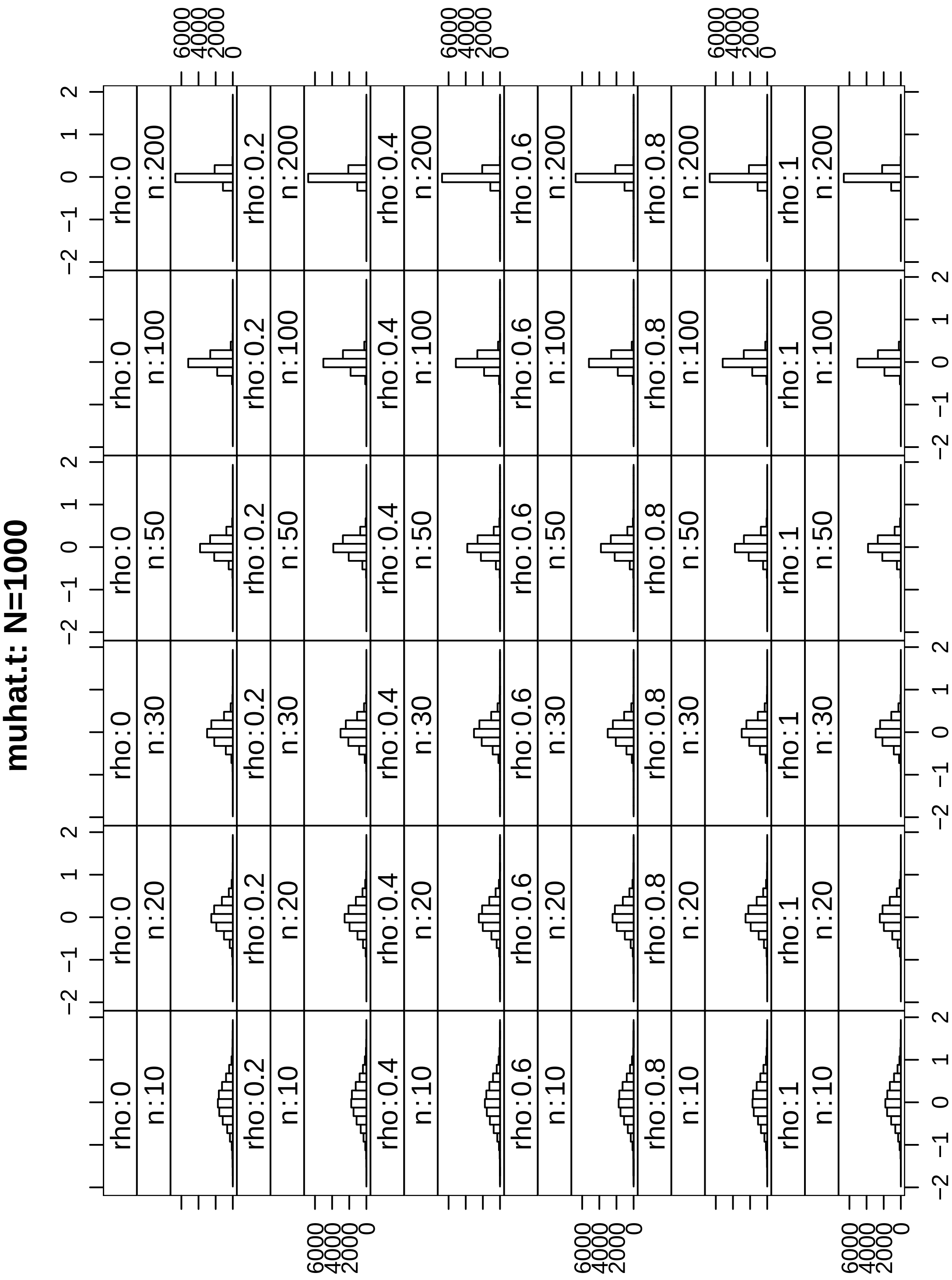


Figure A.28: Distribution of $\hat{\mu}_t - \mu_t$ for N=1000

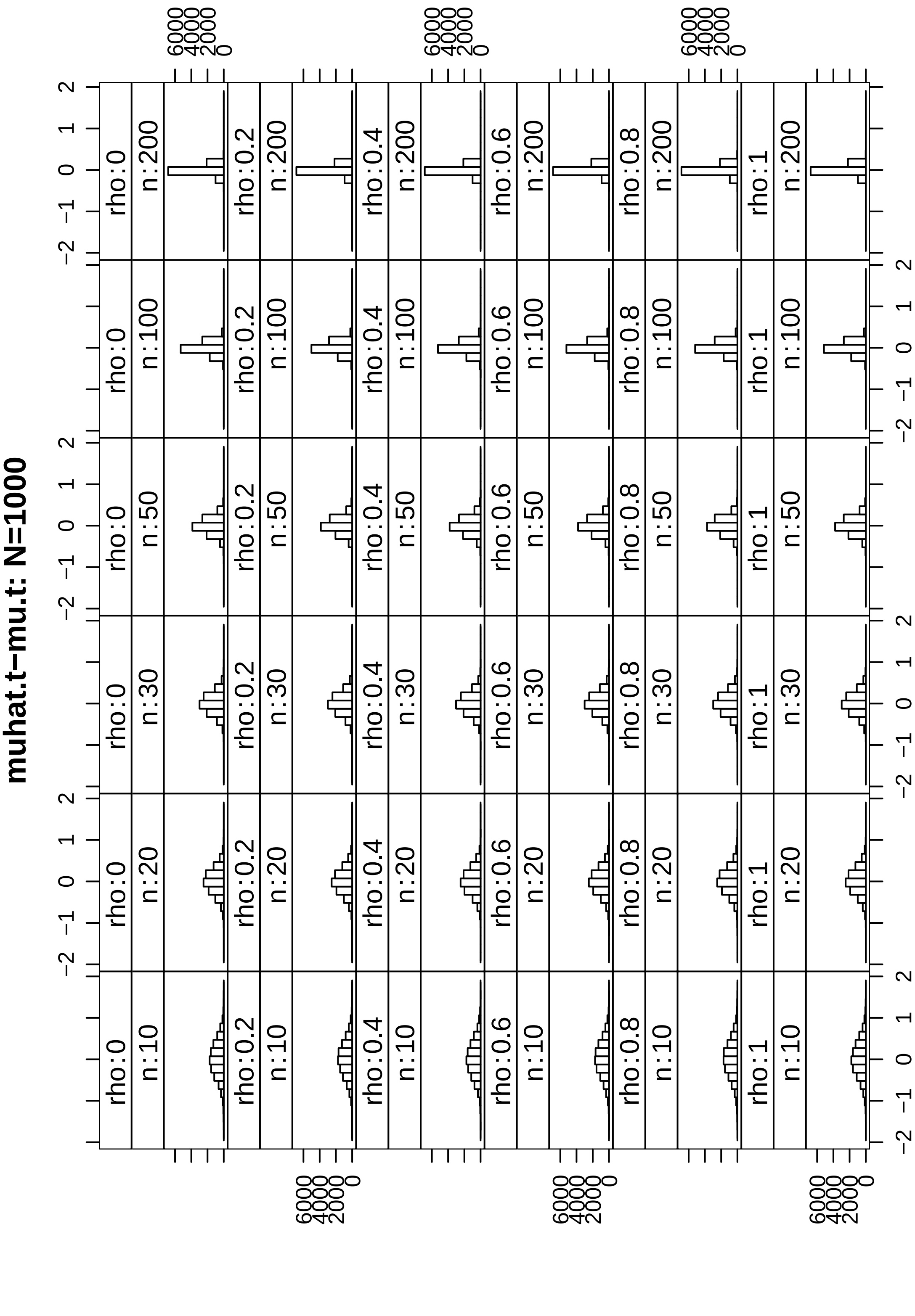


Figure A.29: Distribution of $\hat{\mu}_t - \hat{\mu}_c$ for N=1000

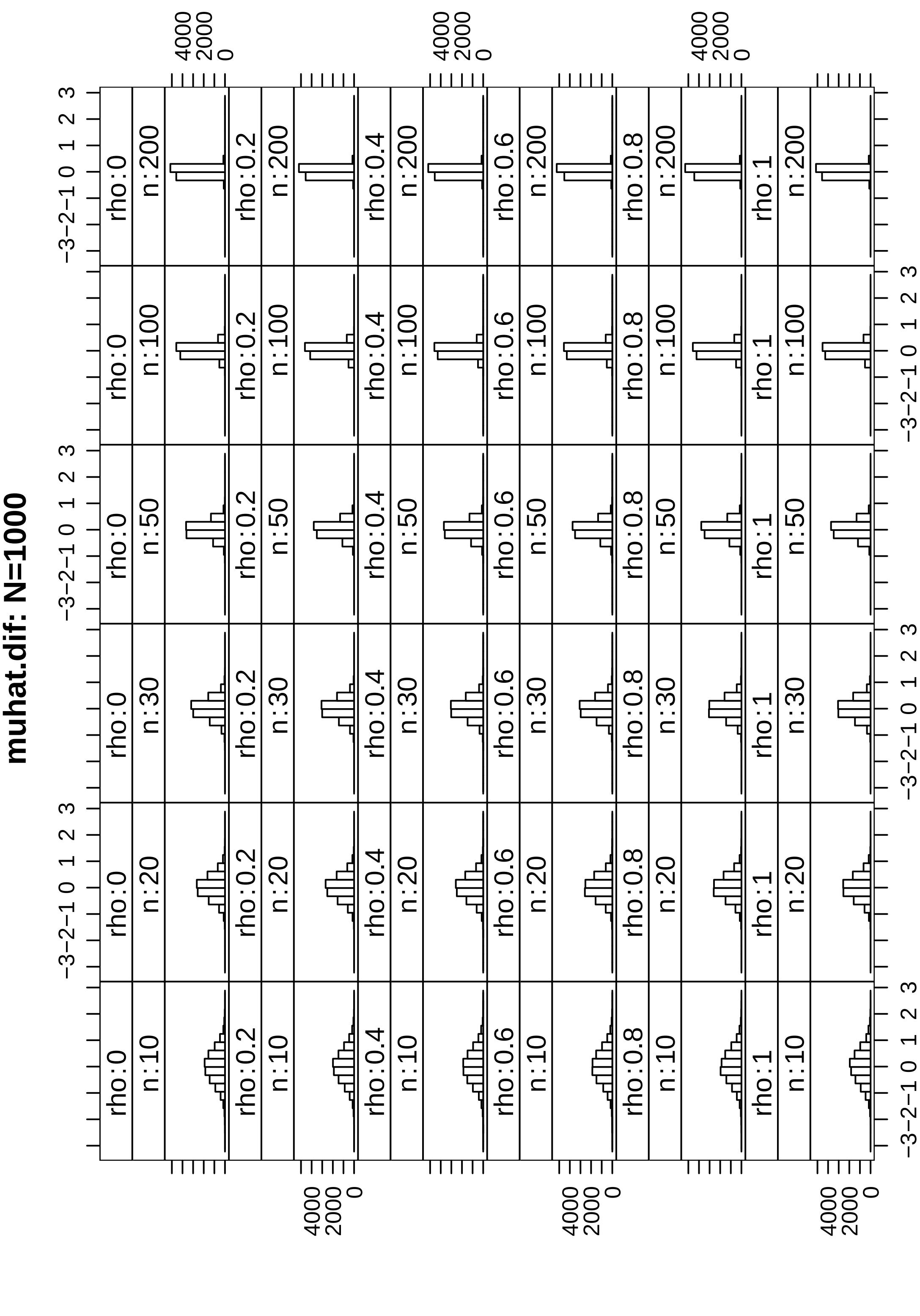


Figure A.30: Distribution of $(\hat{\mu}_t - \hat{\mu}_c) - (\mu_t - \mu_c)$ for N=1000

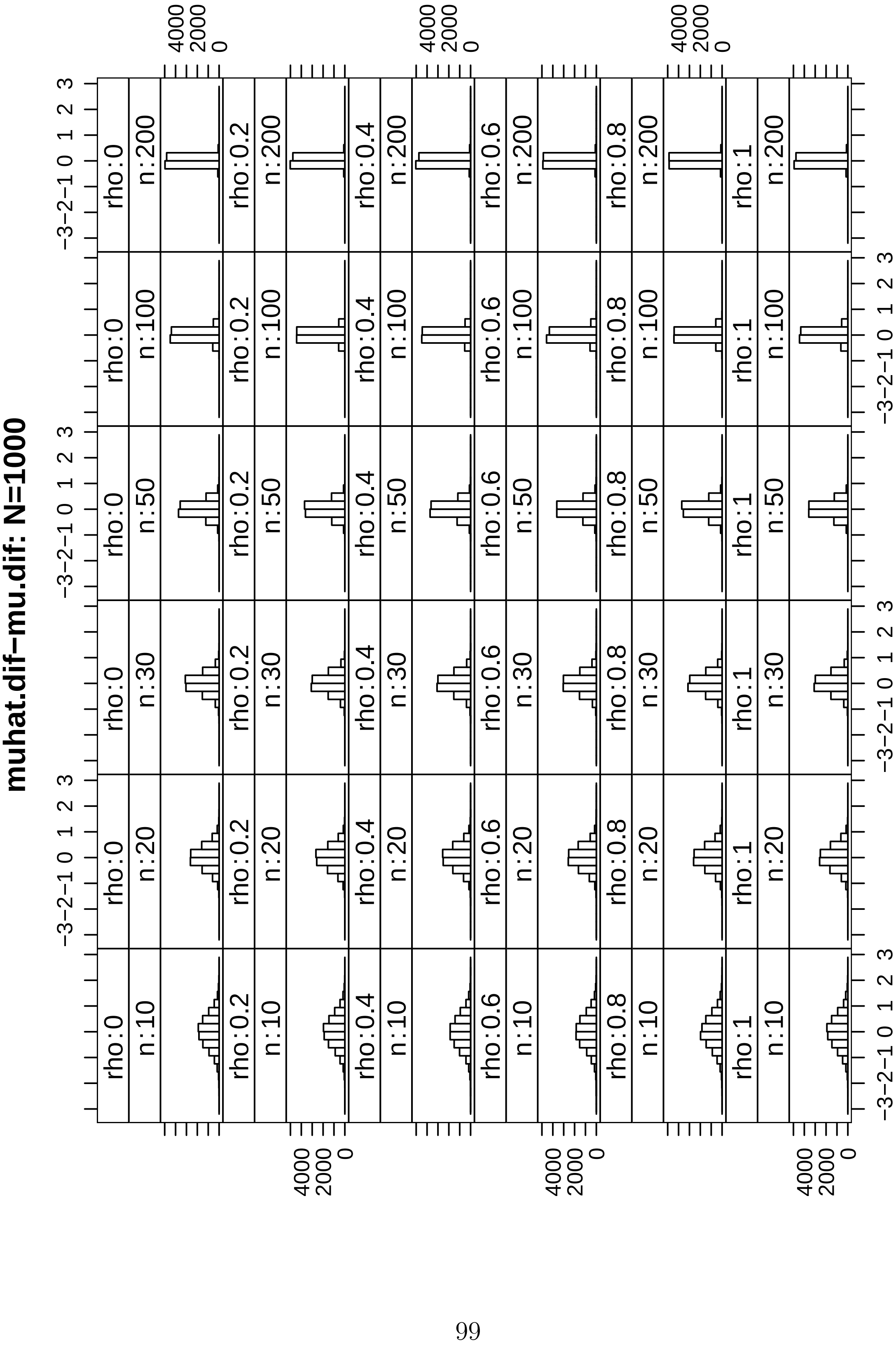


Figure A.31: Distribution of $\hat{\mu}_c$ for N=5000

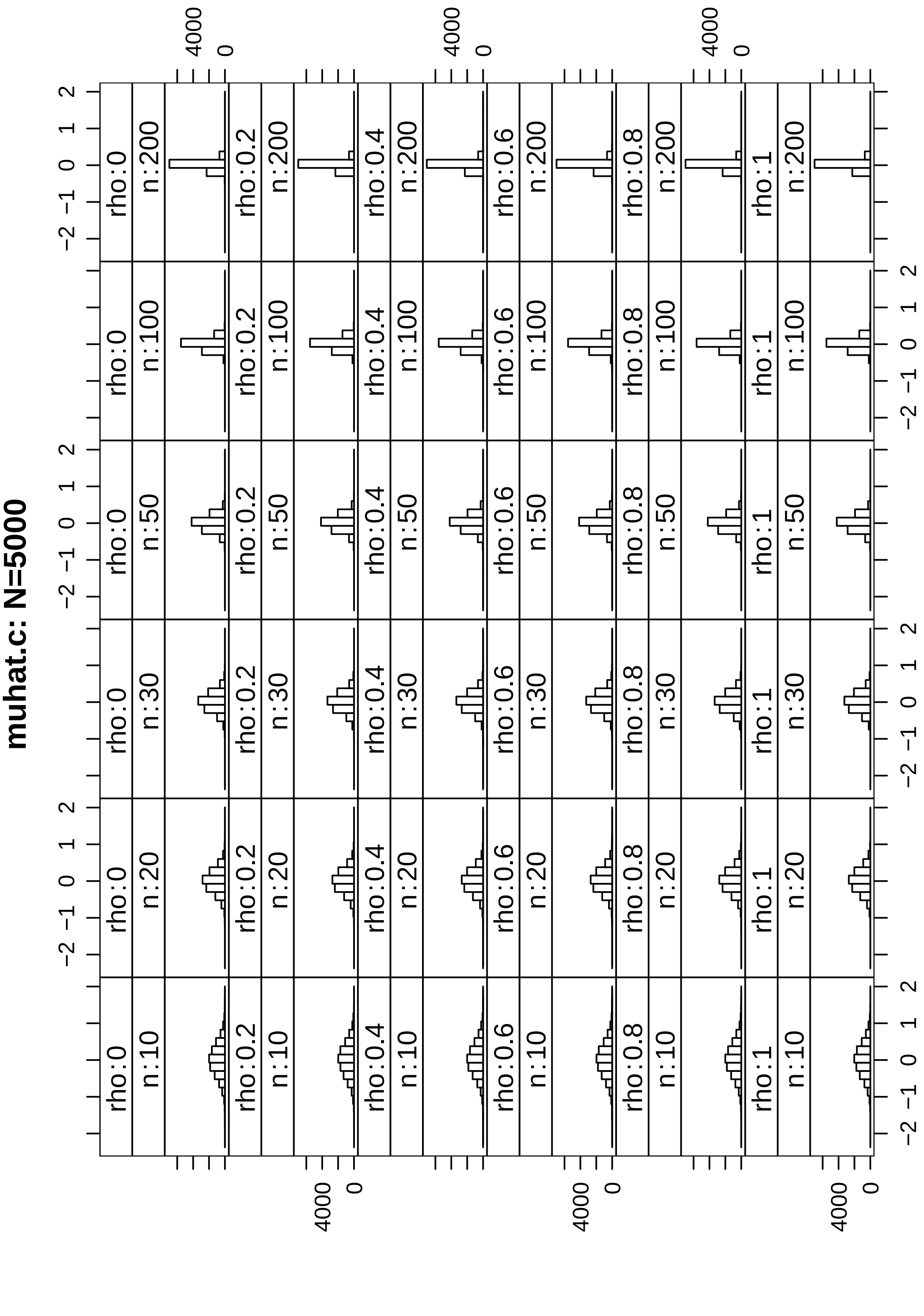


Figure A.32: Distribution of $\hat{\mu}_c - \mu_c$ for N=5000

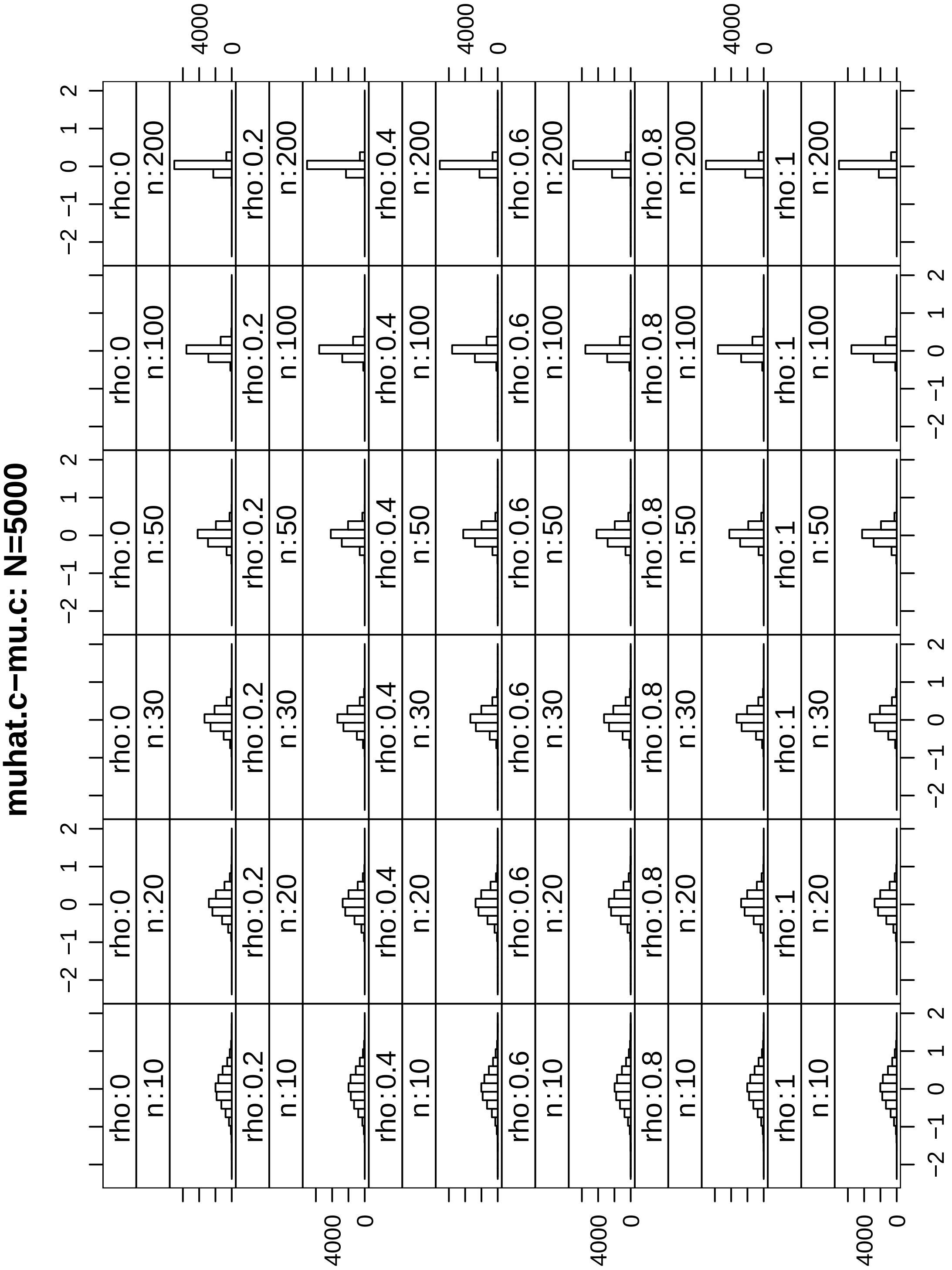


Figure A.33: Distribution of $\hat{\mu}_t$ for N=5000

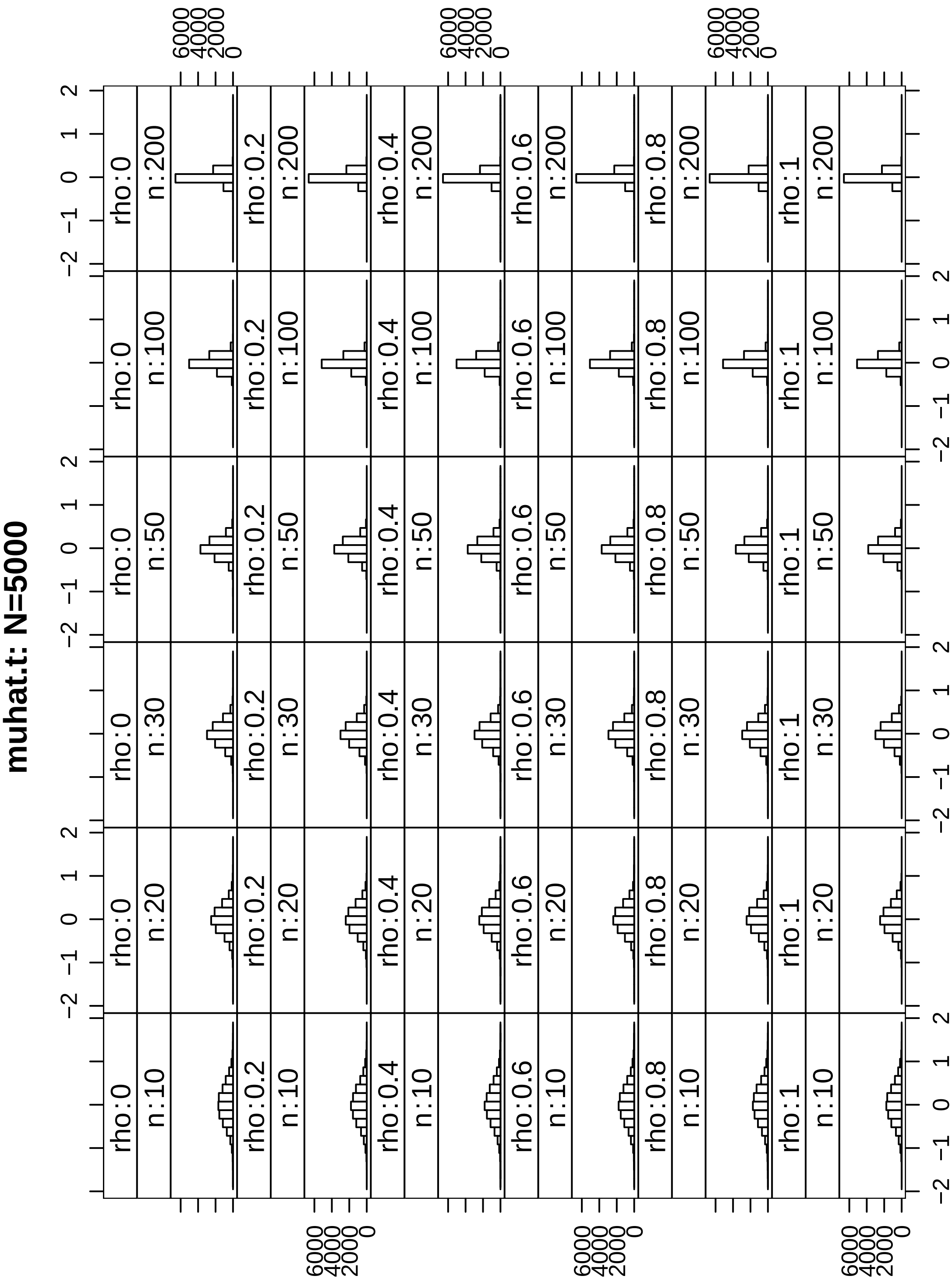


Figure A.34: Distribution of $\hat{\mu}_t - \mu_t$ for N=5000

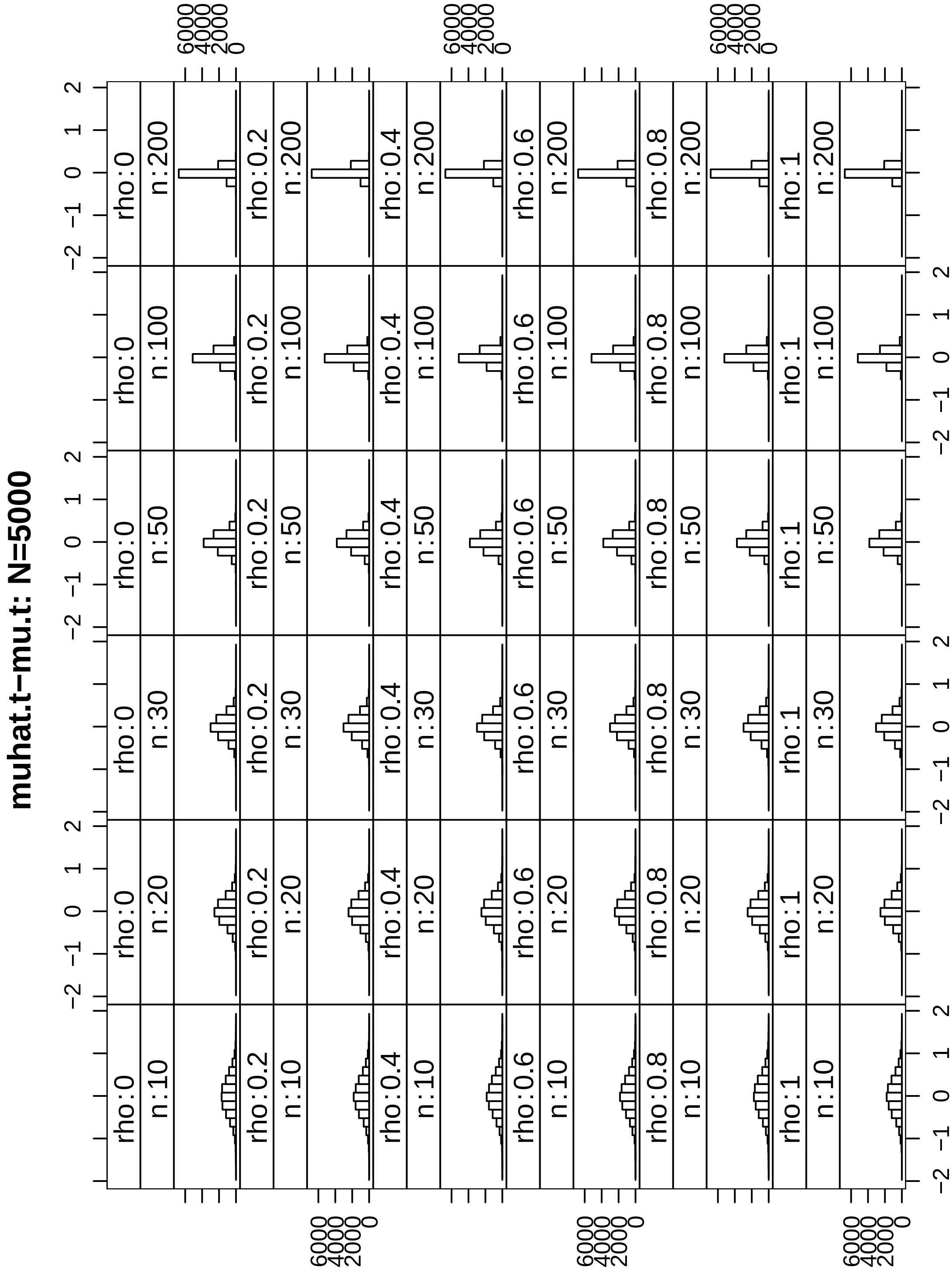


Figure A.35: Distribution of $\hat{\mu}_t - \hat{\mu}_c$ for N=5000

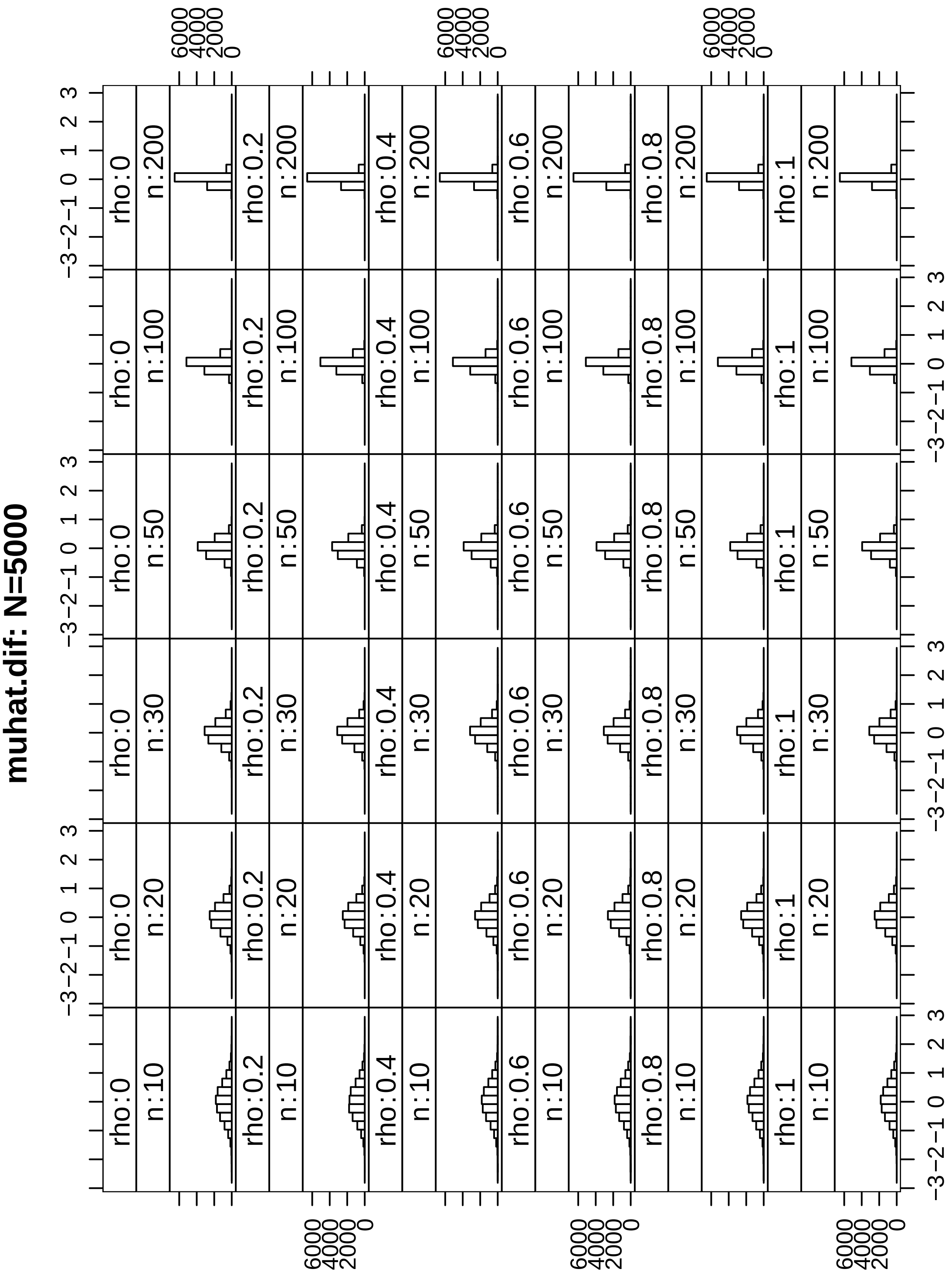
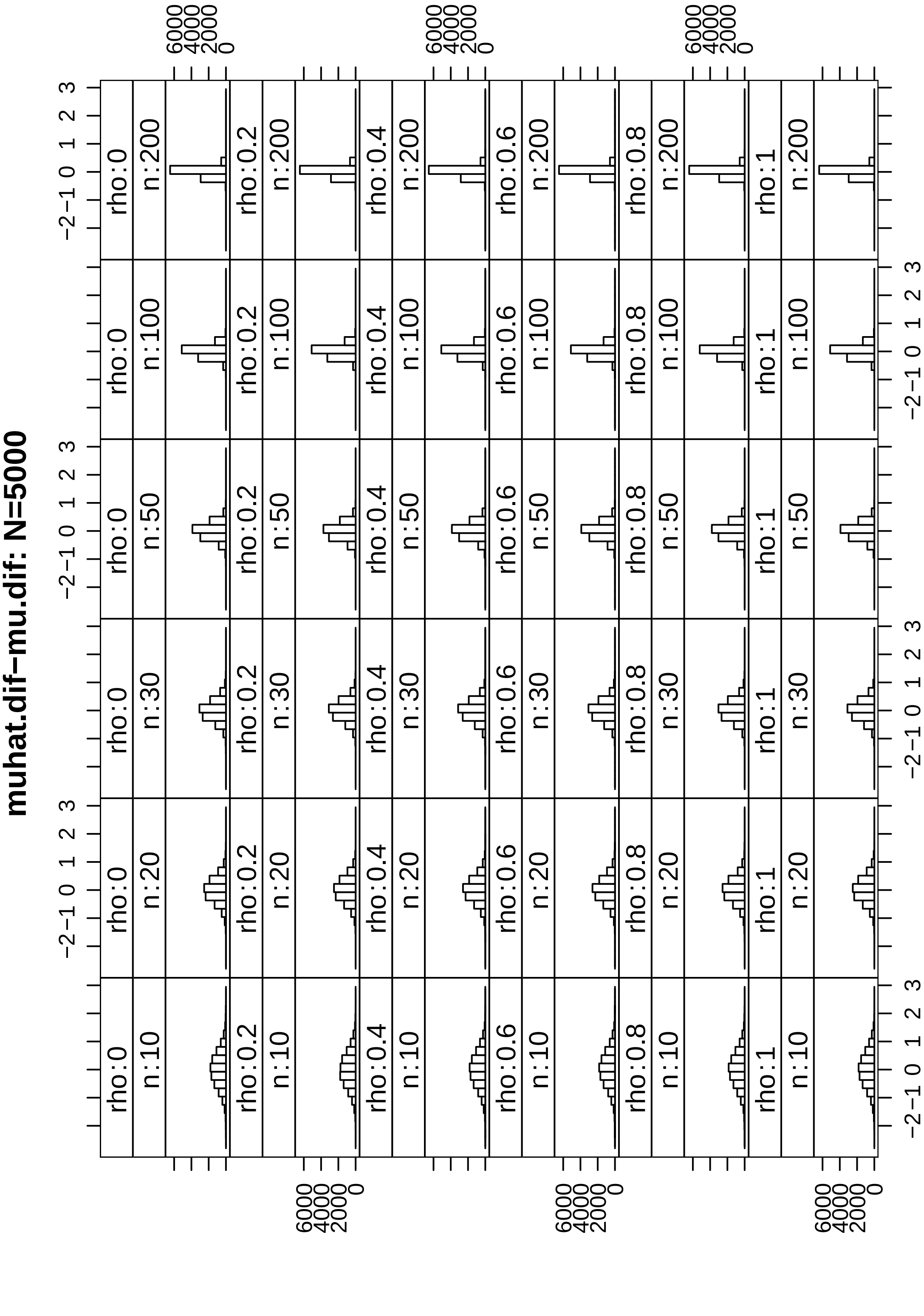


Figure A.36: Distribution of $(\hat{\mu}_t - \hat{\mu}_c) - (\mu_t - \mu_c)$ for N=5000



Appendix B

Distribution of Variance Estimators

Figure B.1: Distribution of S_c^2 for N=50

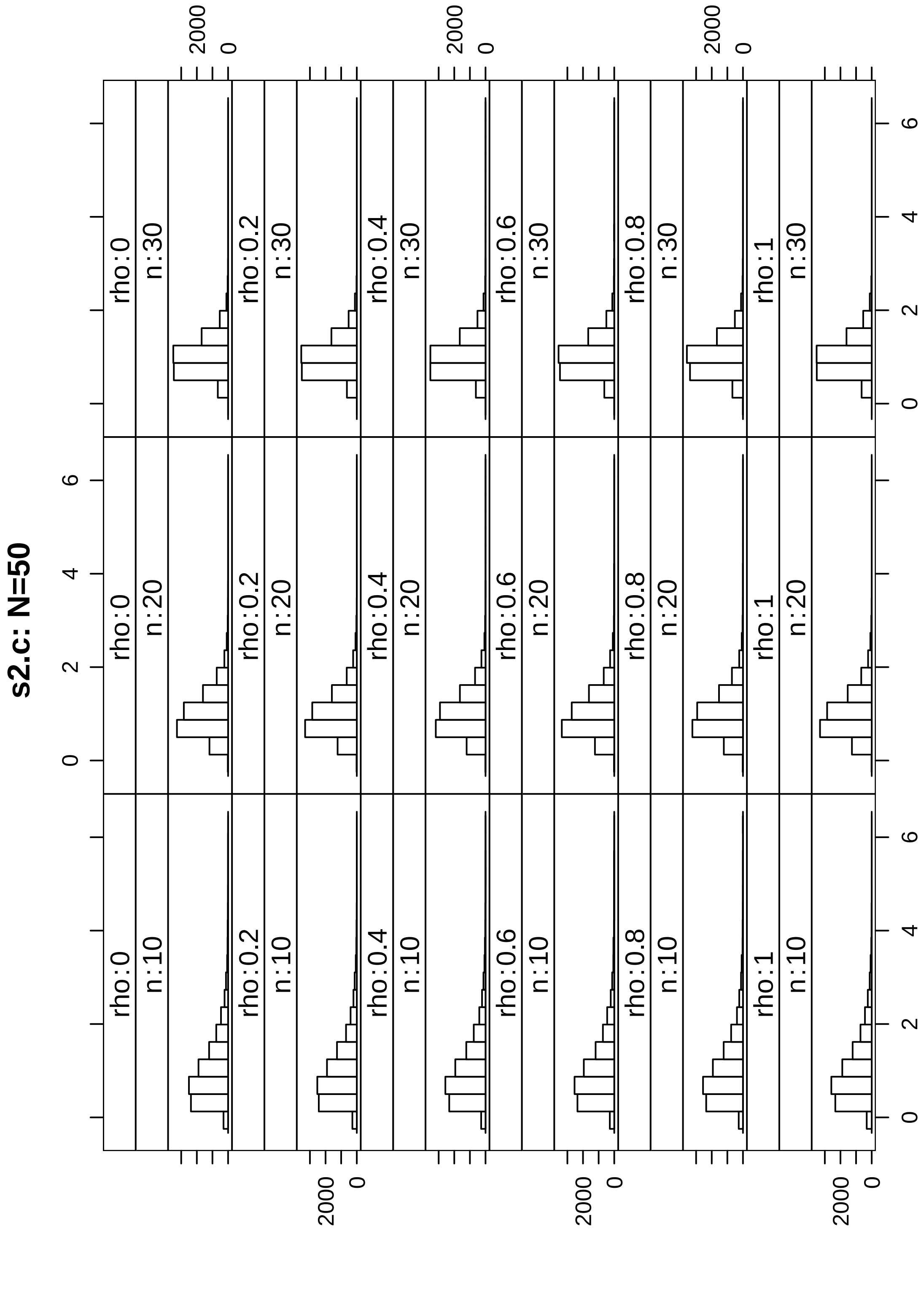


Figure B.2: Distribution of S_t^2 for N=50

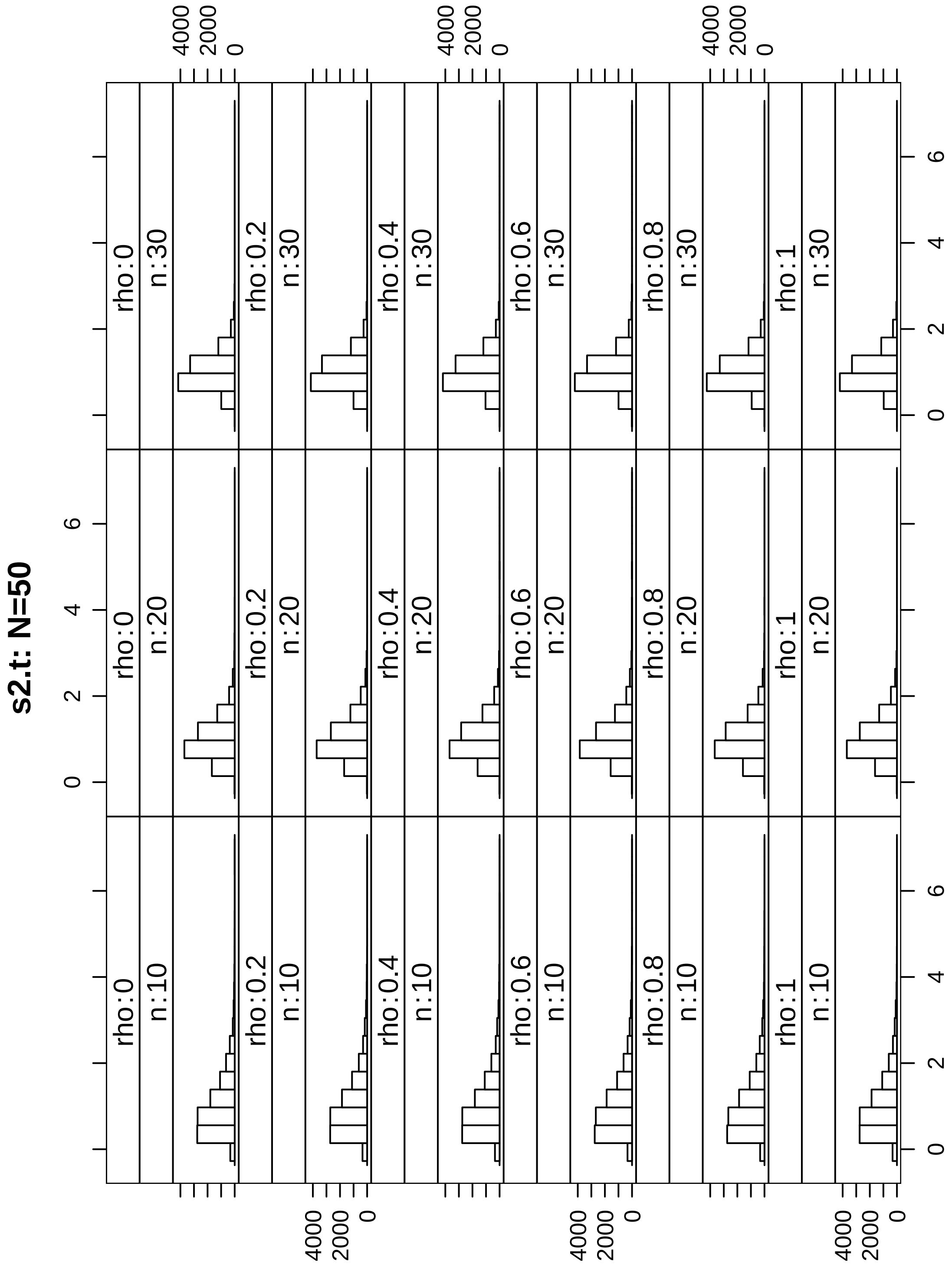


Figure B.3: Distribution of $S_t^2 + S_c^2$ for N=50

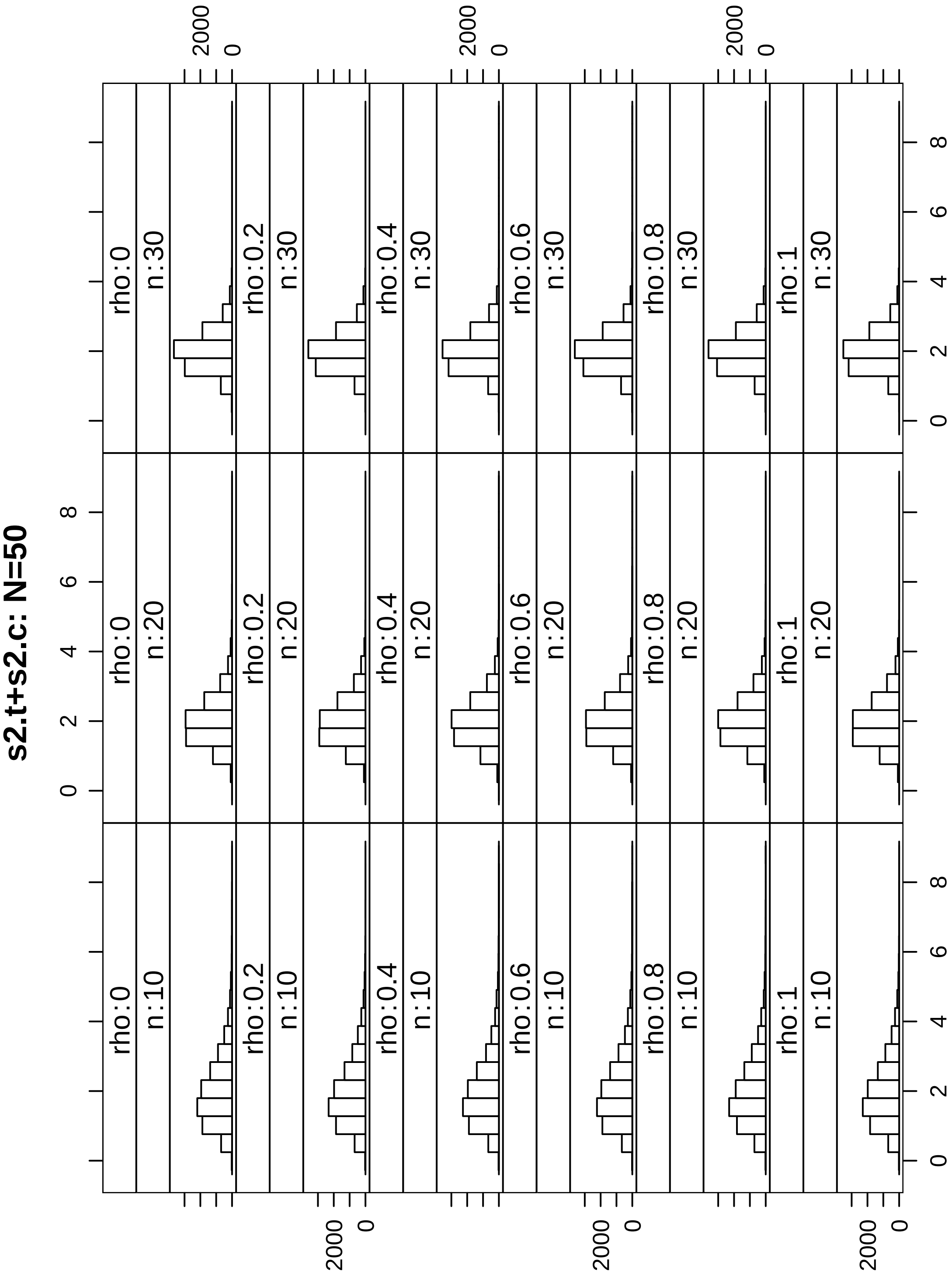


Figure B.4: Distribution of $var(\hat{\mu}_c)$ for N=50

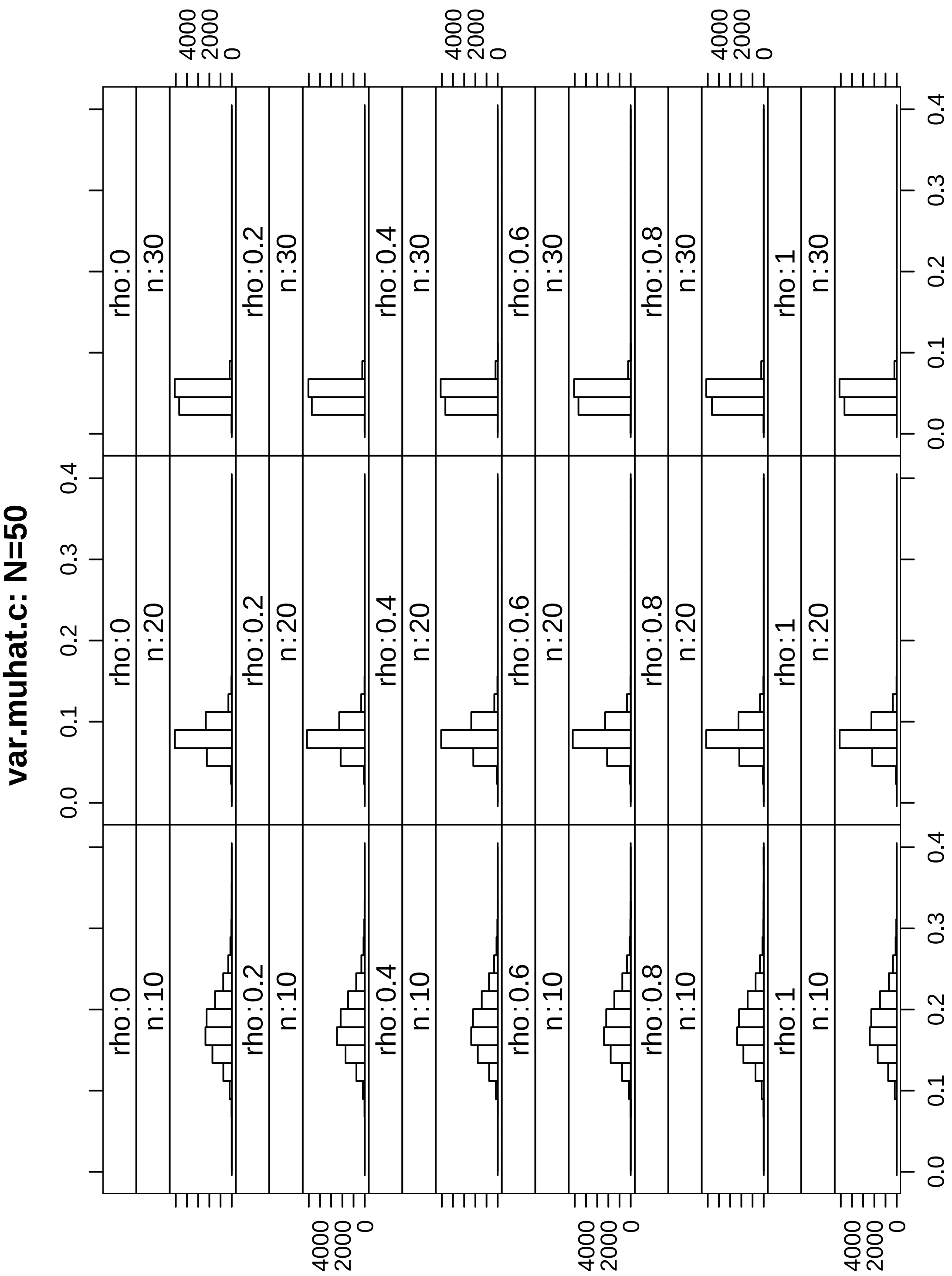


Figure B.5: Distribution of $\widehat{var}(\hat{\mu}_c)$ for N=50

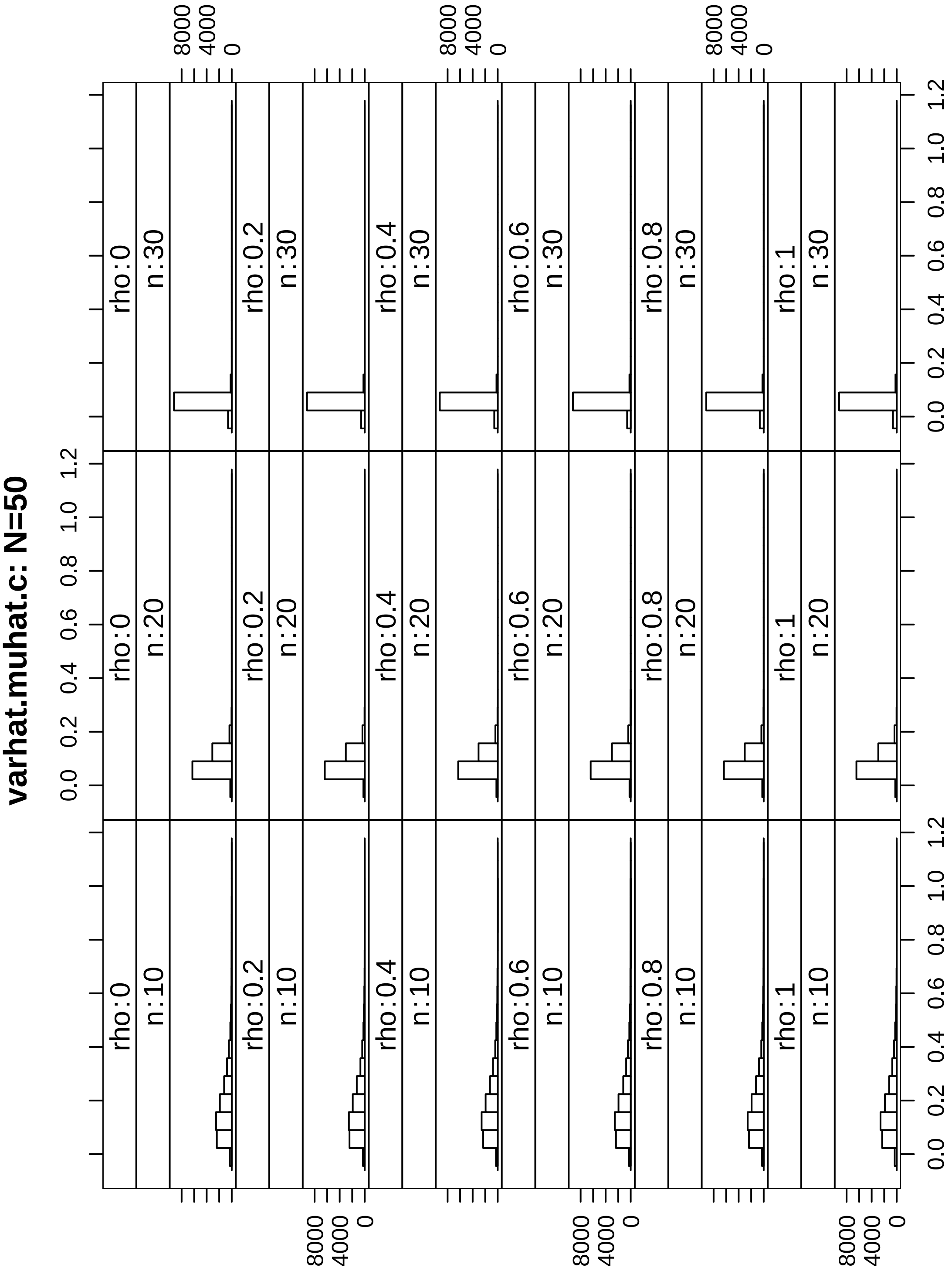


Figure B.6: Distribution of $var(\hat{\mu}_t)$ for N=50

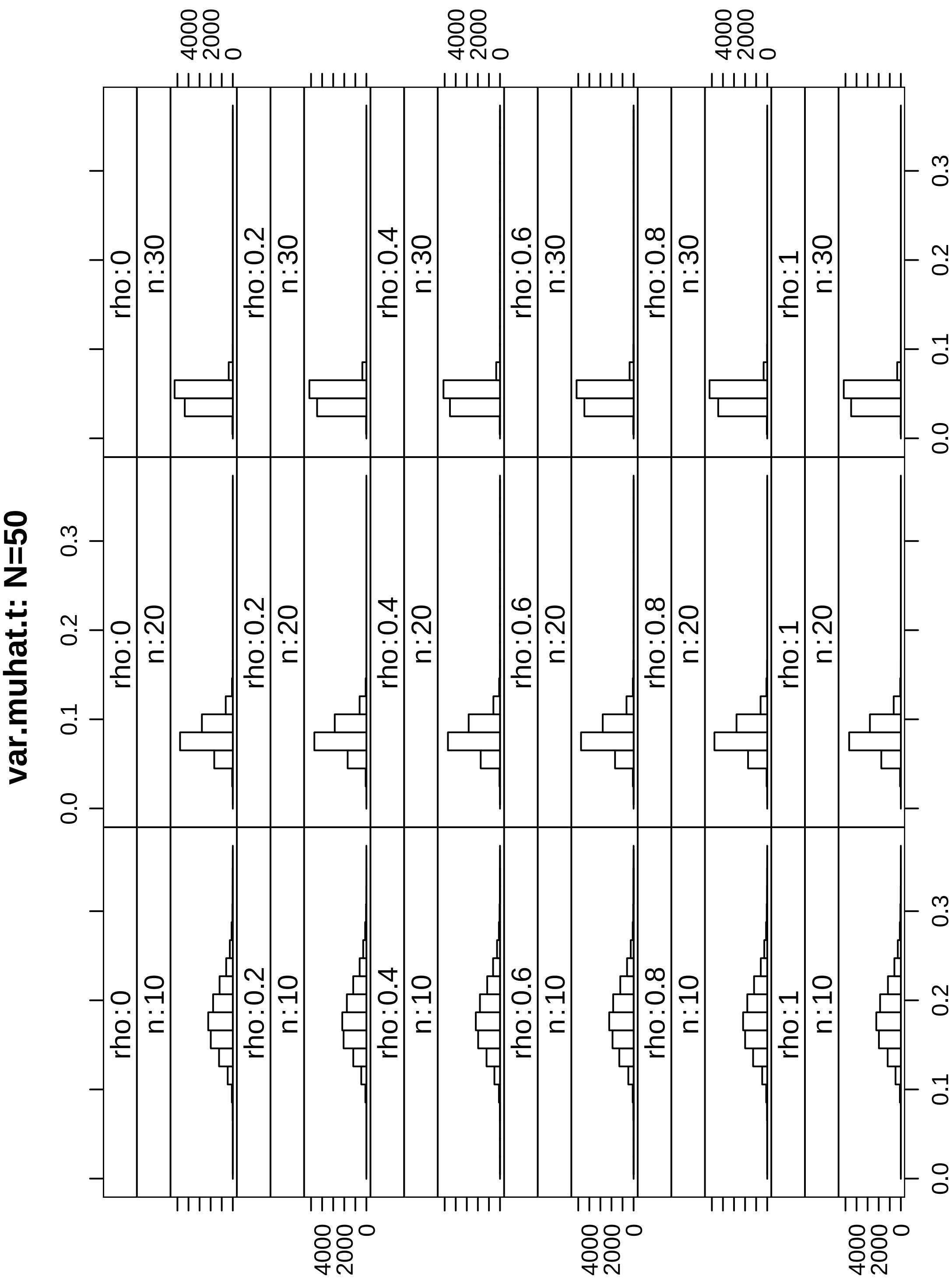


Figure B.7: Distribution of $\widehat{var}(\hat{\mu}_t)$ for N=50

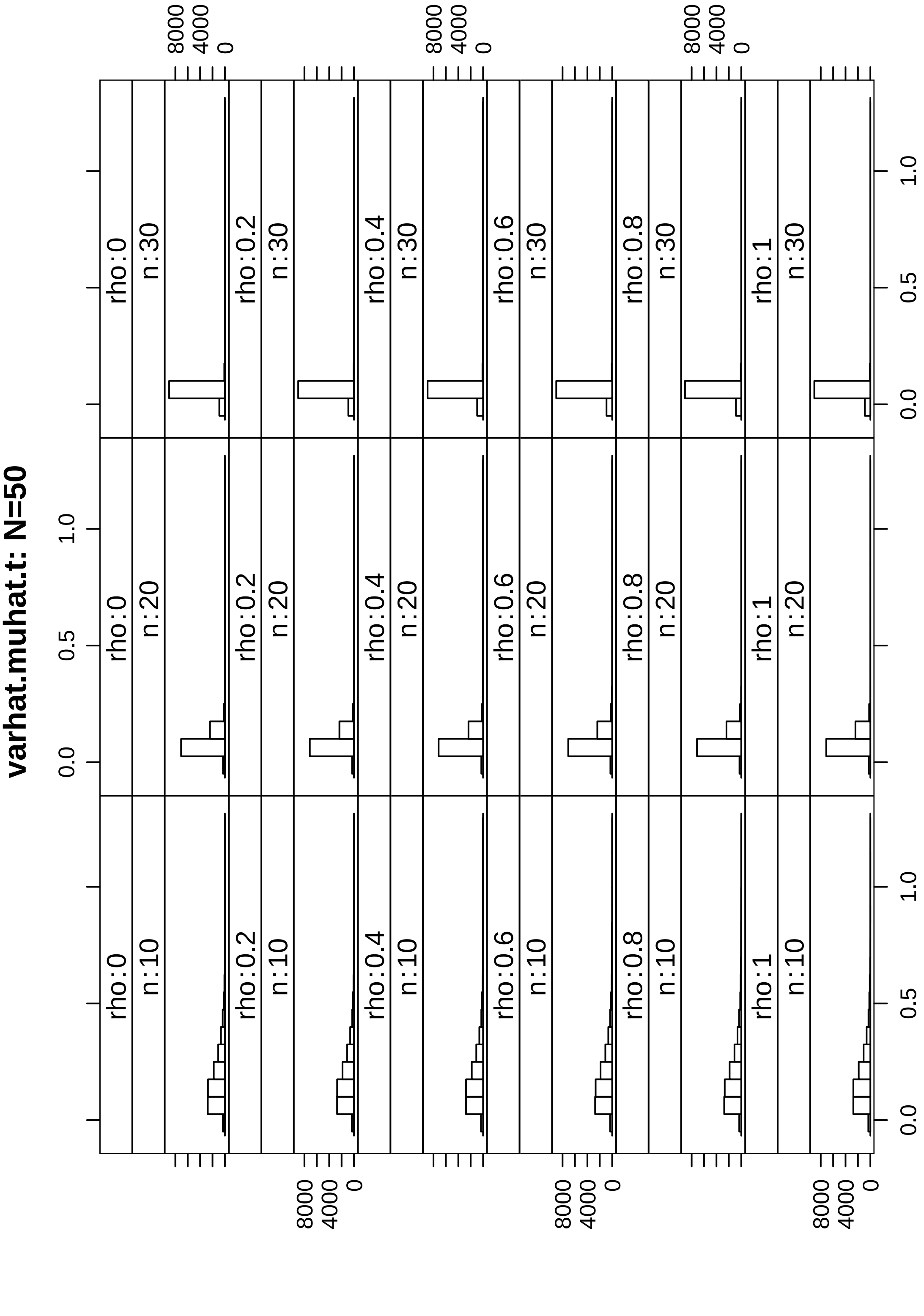


Figure B.8: Distribution of $var(\hat{\mu}_t - \hat{\mu}_c)$ for N=50

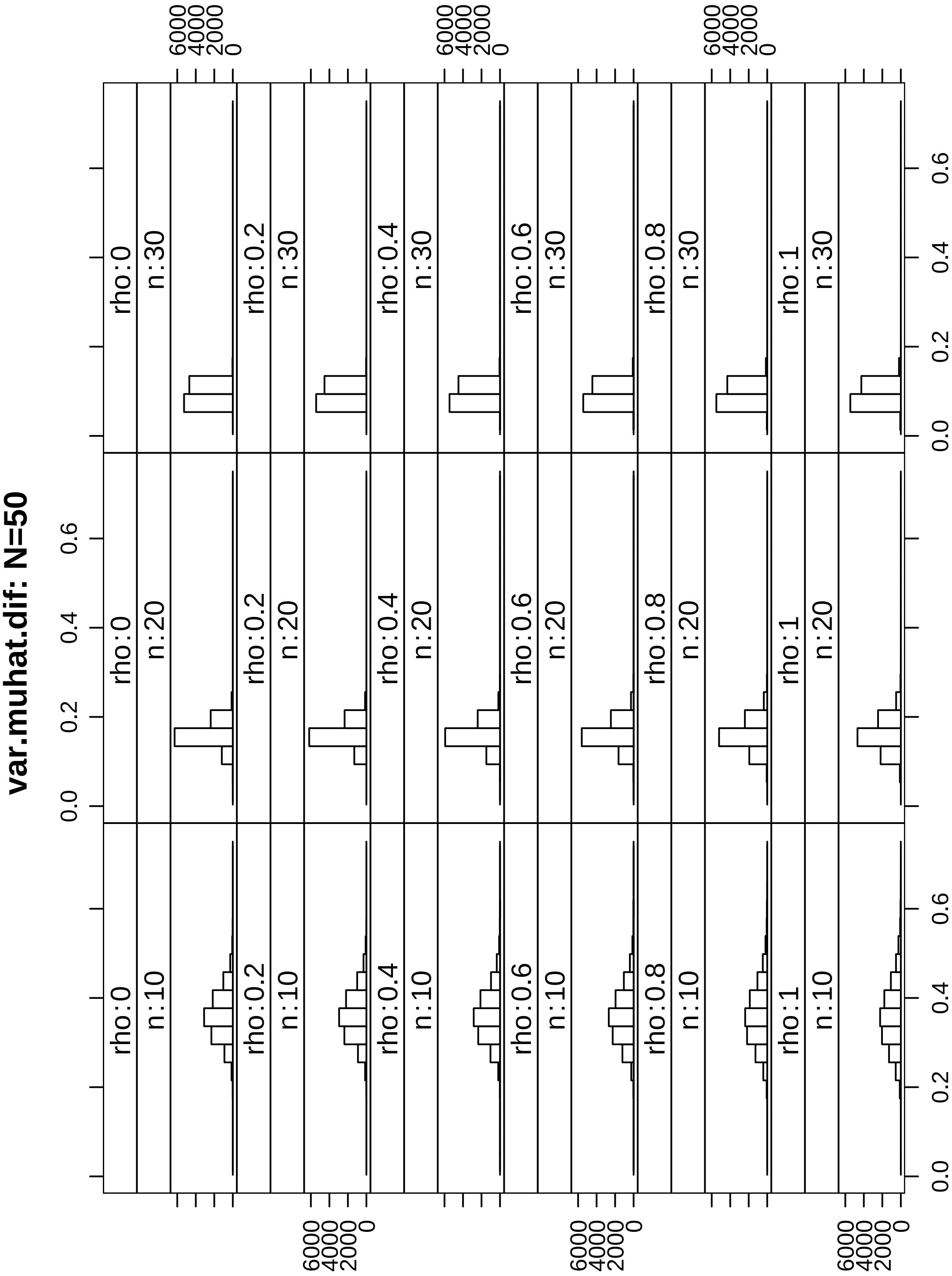


Figure B.9: Distribution of $\widehat{var}(\hat{\mu}_t - \hat{\mu}_c)$ for N=50

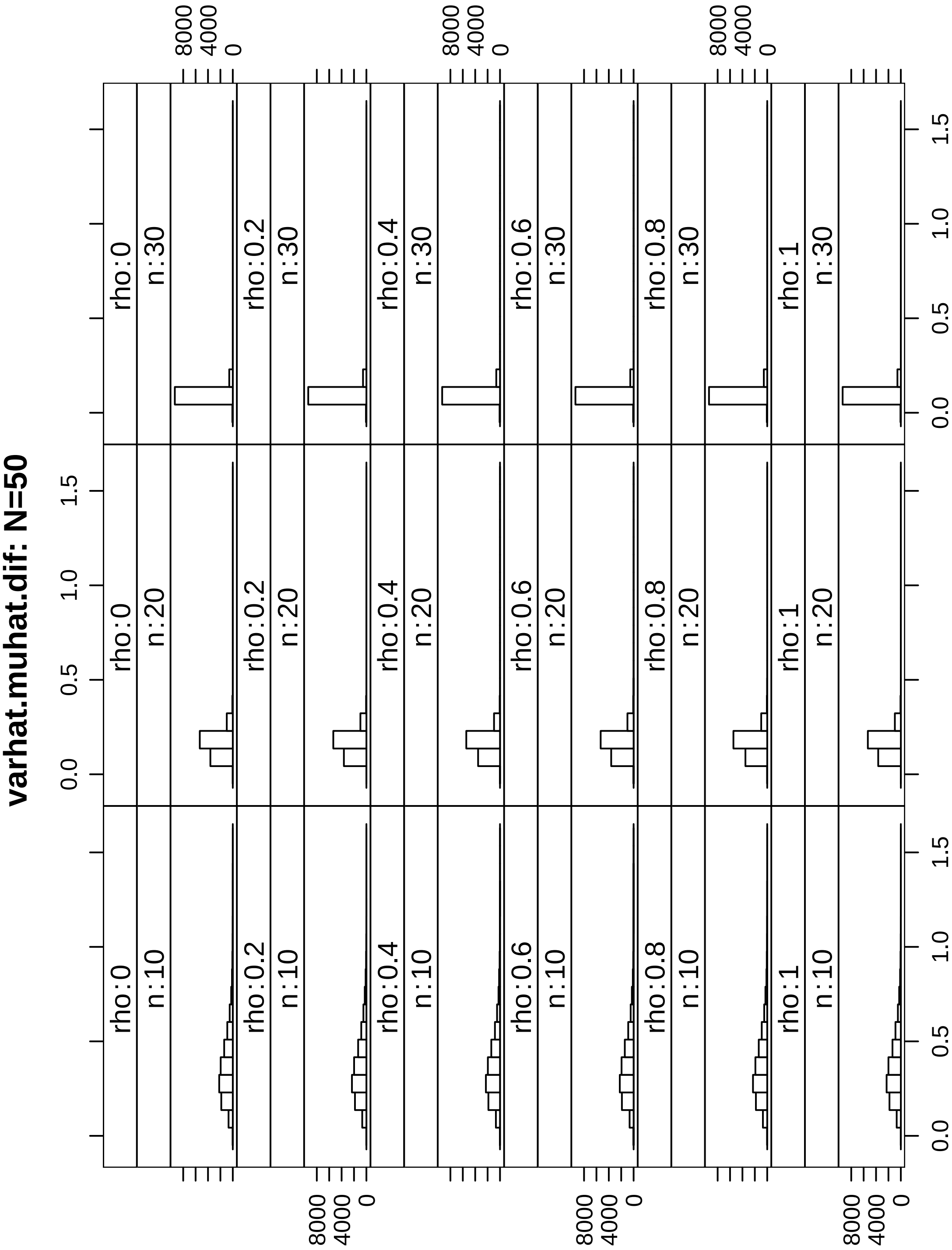


Figure B.10: Distribution of S_c^2 for N=100

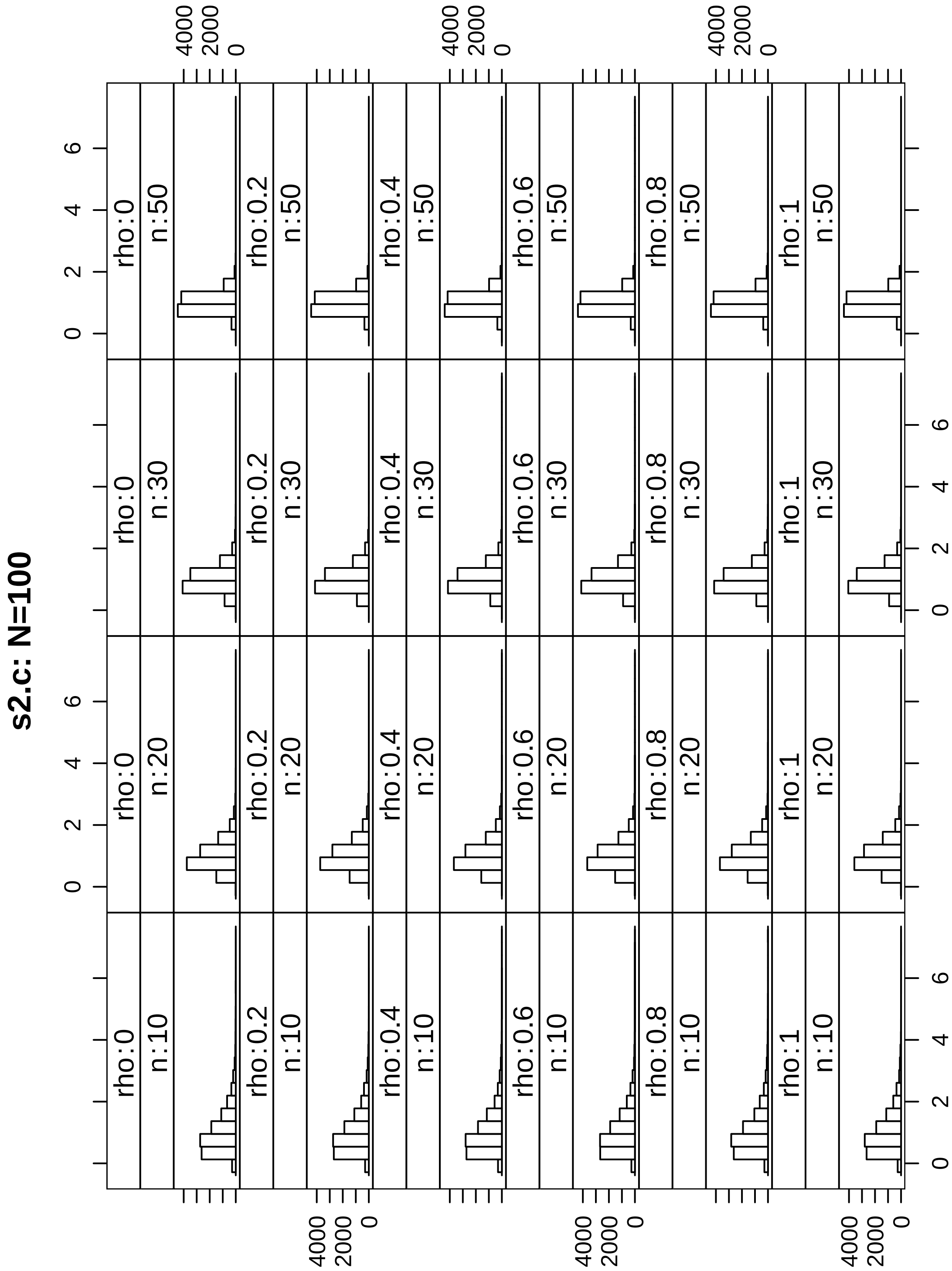


Figure B.11: Distribution of S_t^2 for N=100

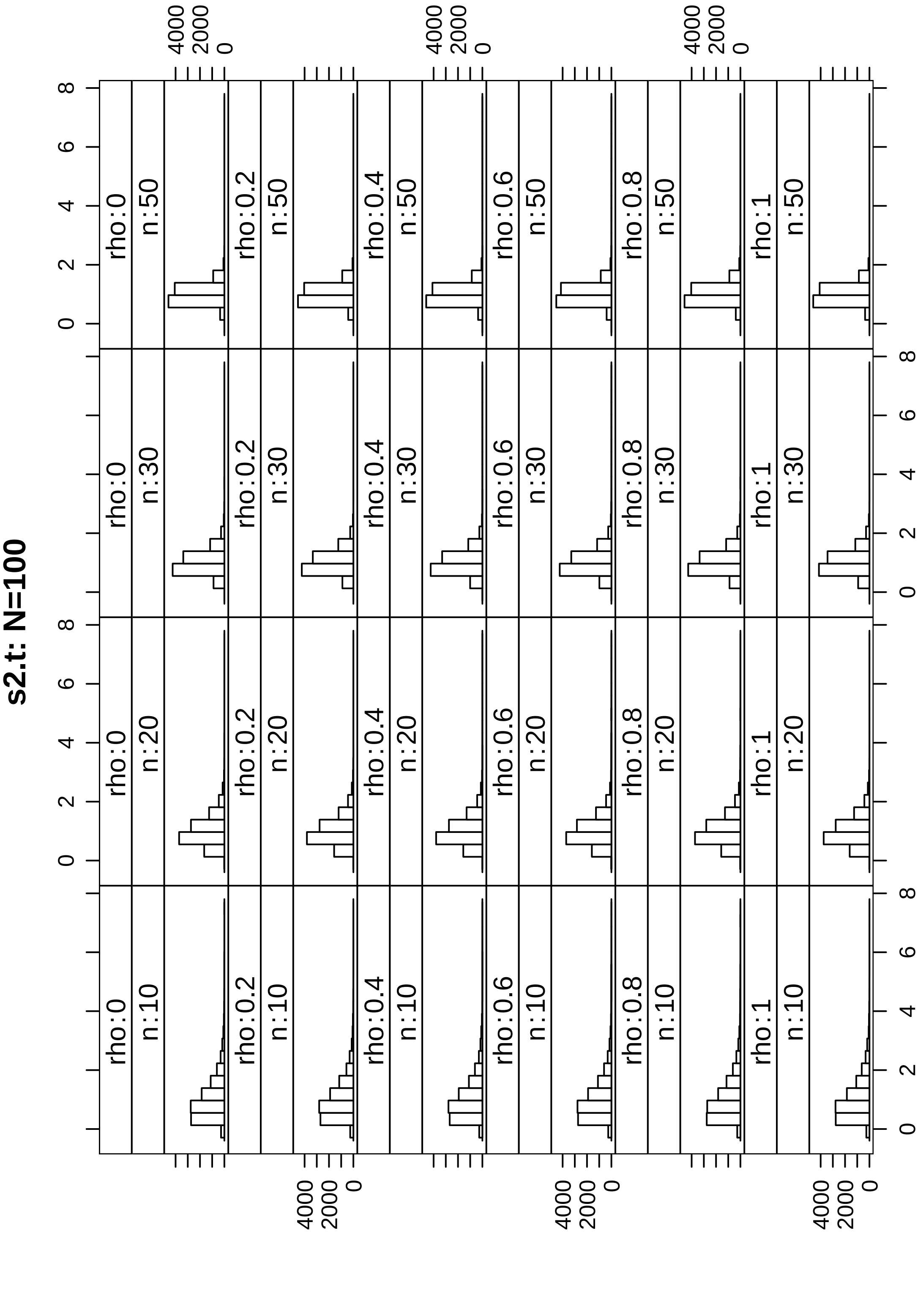


Figure B.12: Distribution of $S_t^2 + S_c^2$ for N=100

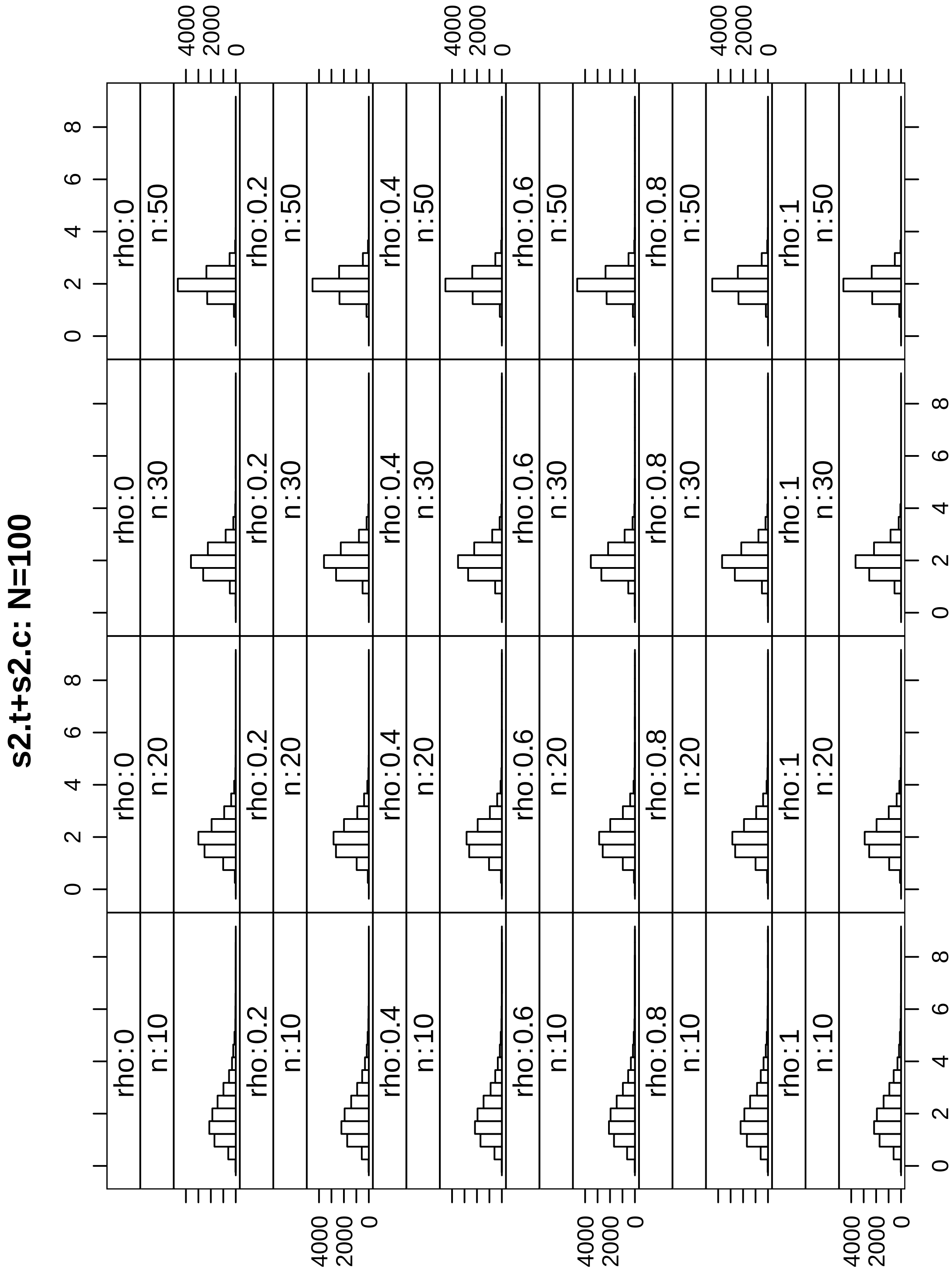


Figure B.13: Distribution of $var(\hat{\mu}_c)$ for N=100

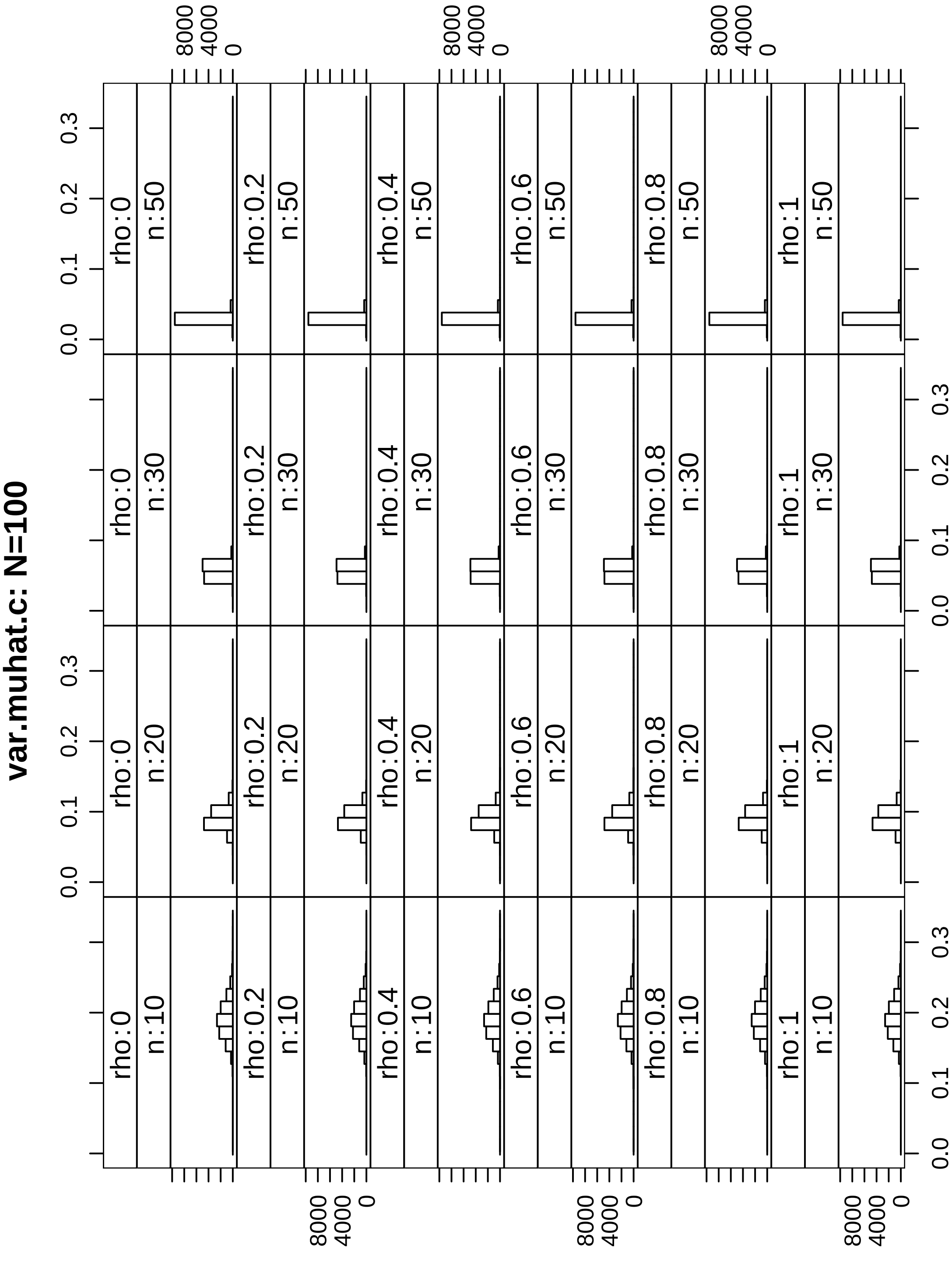


Figure B.14: Distribution of $\widehat{var}(\hat{\mu}_c)$ for N=100

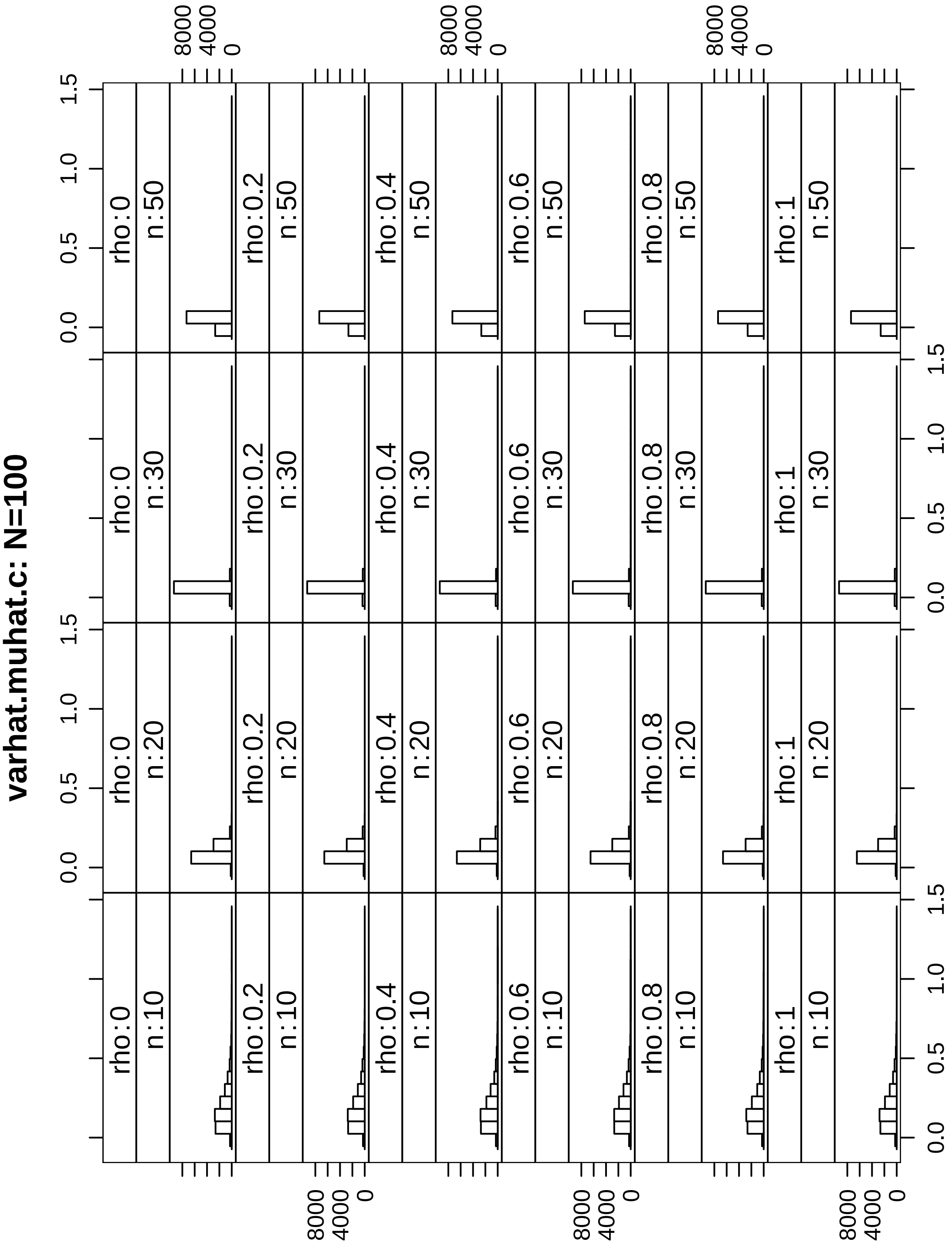


Figure B.15: Distribution of $var(\hat{\mu}_t)$ for N=100

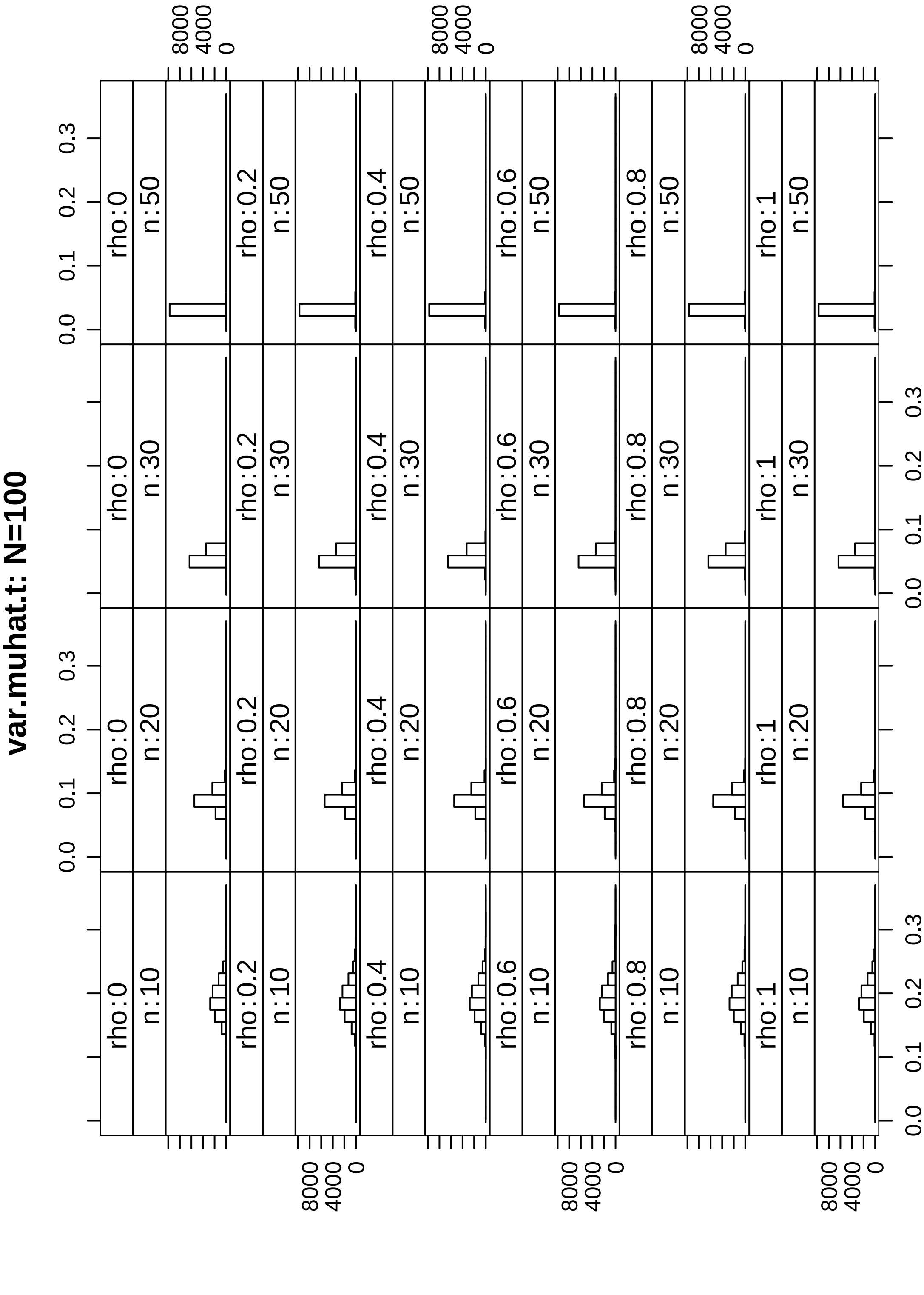


Figure B.16: Distribution of $\widehat{var}(\hat{\mu}_t)$ for N=100

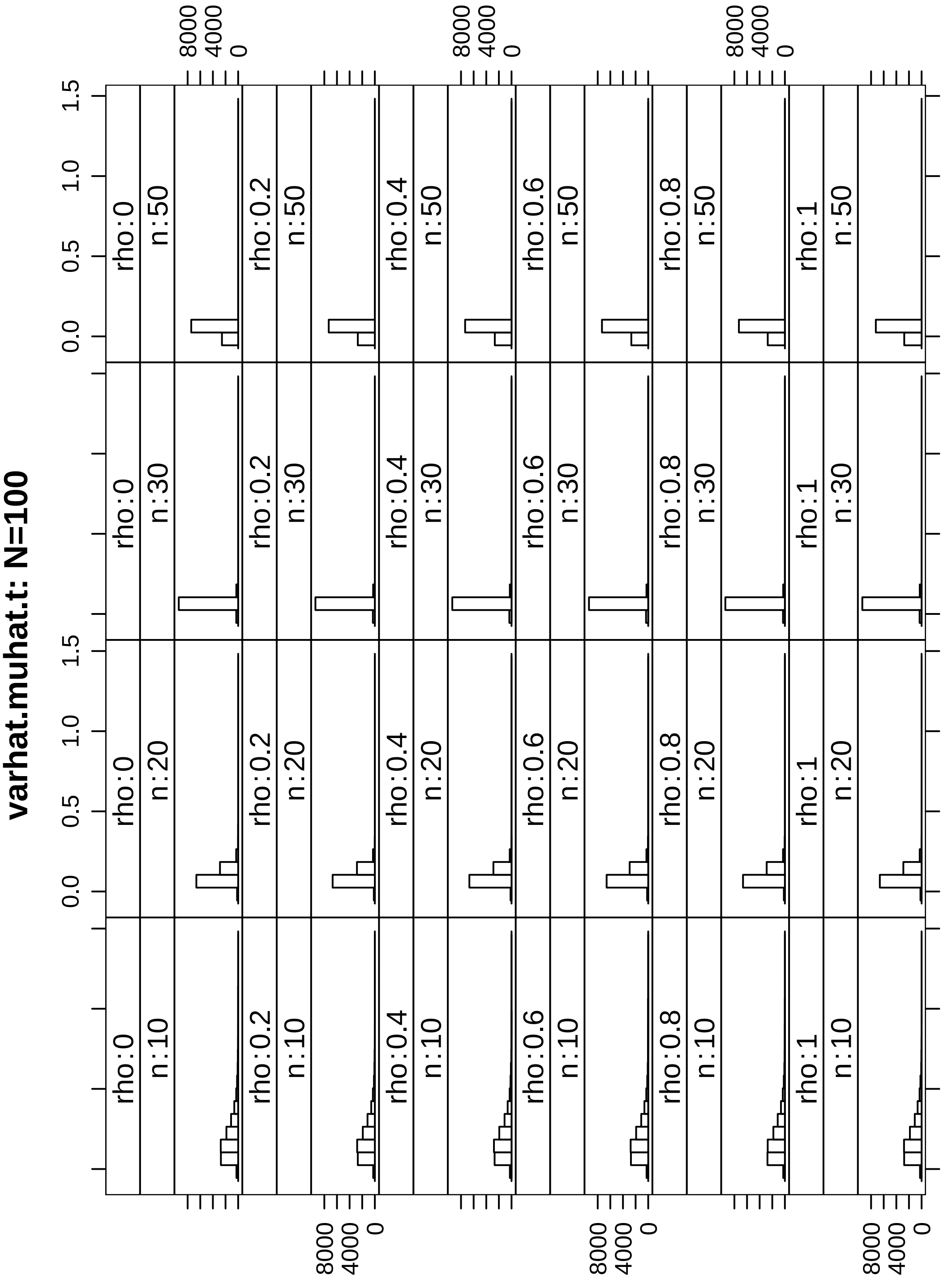


Figure B.17: Distribution of $var(\hat{\mu}_t - \hat{\mu}_c)$ for N=100

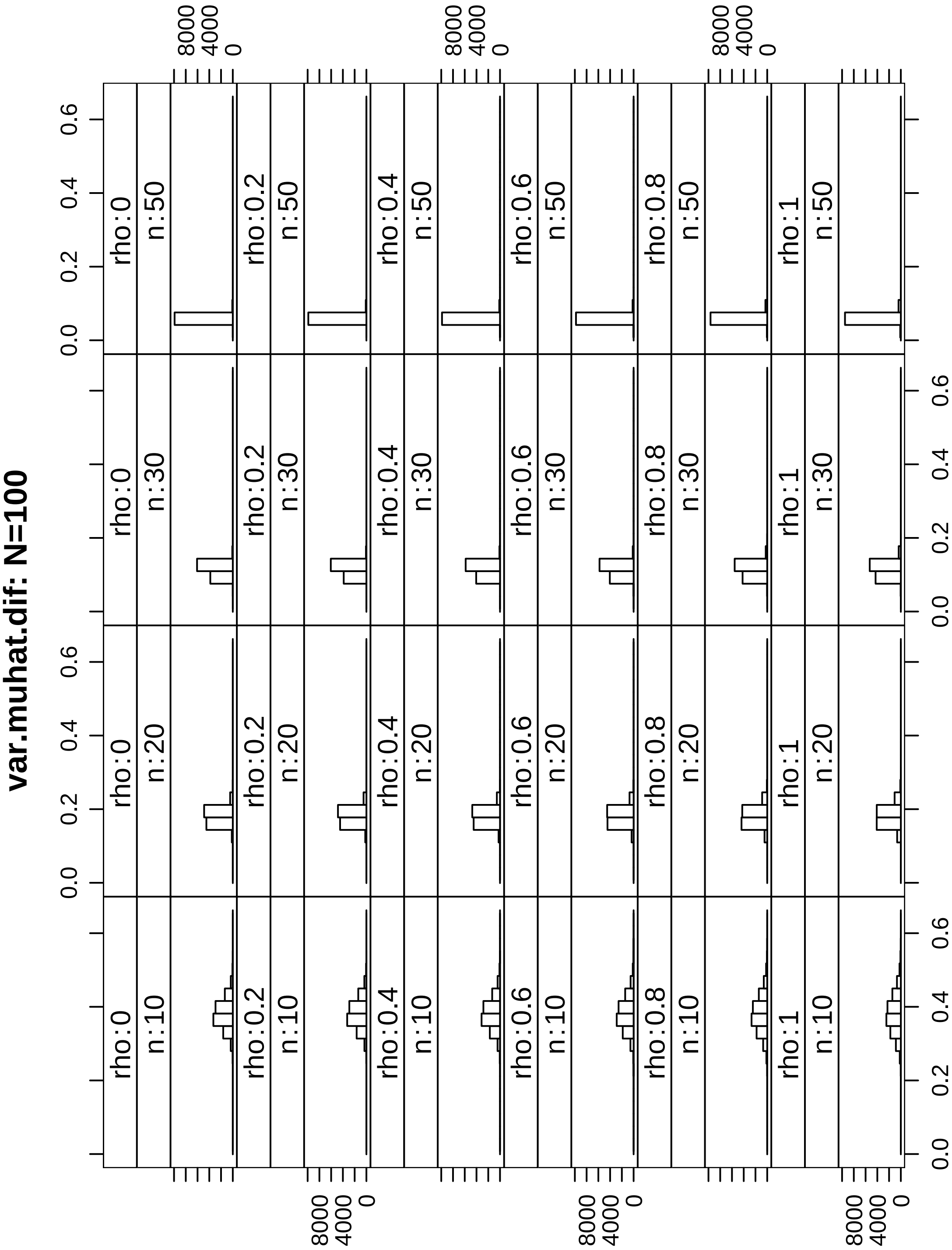


Figure B.18: Distribution of $\widehat{var}(\hat{\mu}_t - \hat{\mu}_c)$ for N=100

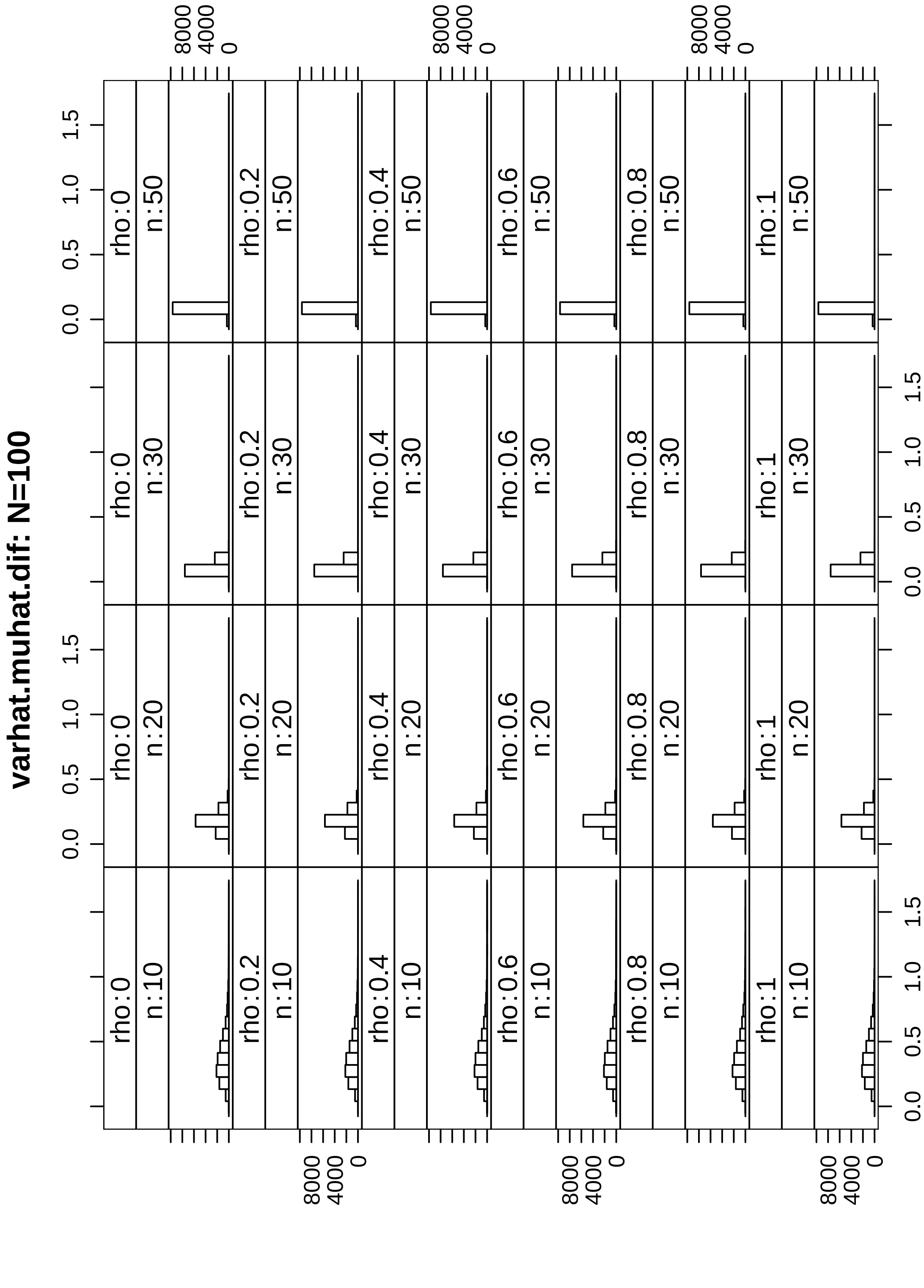


Figure B.19: Distribution of S_c^2 for N=250

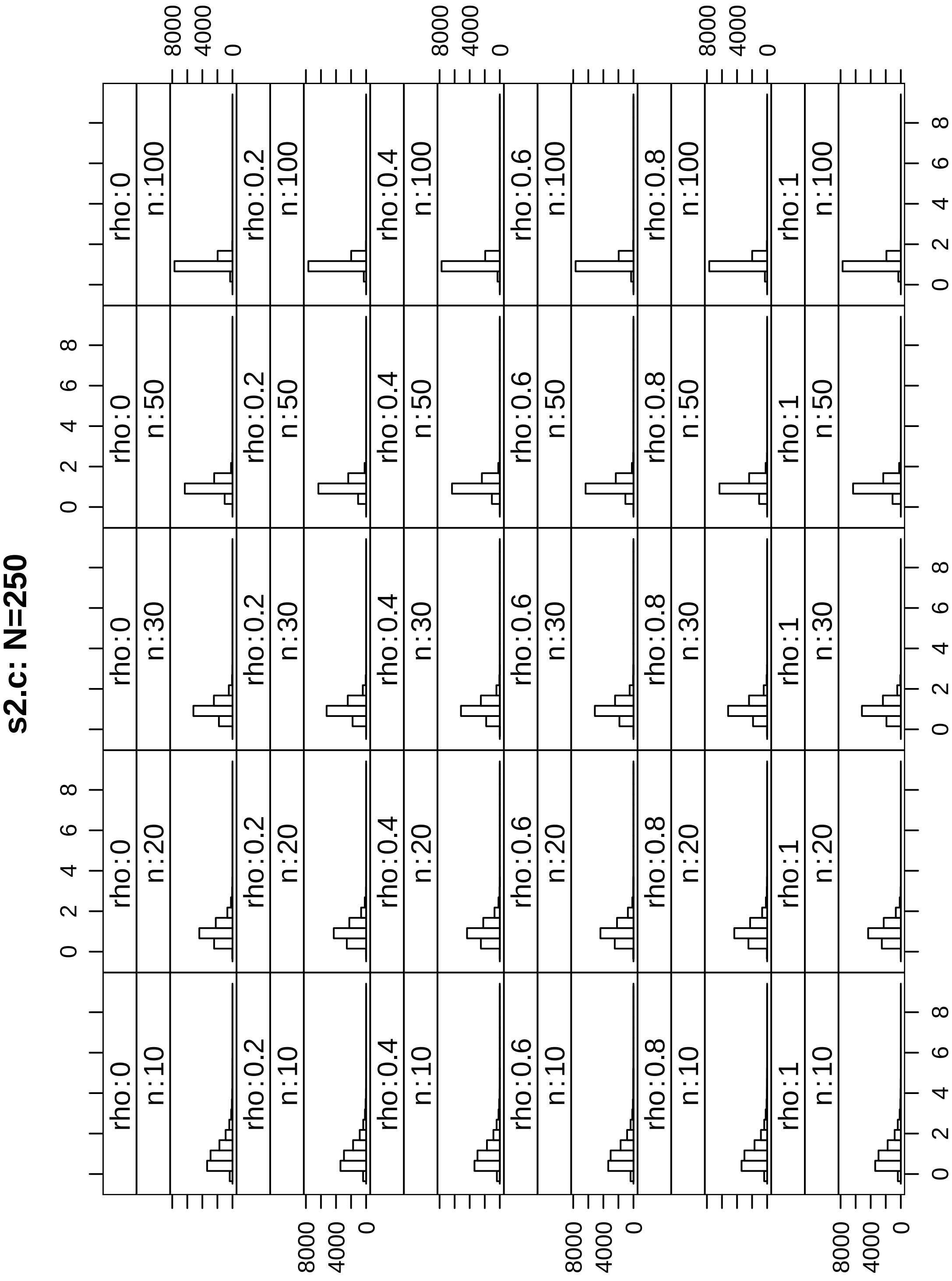


Figure B.20: Distribution of S_t^2 for N=250

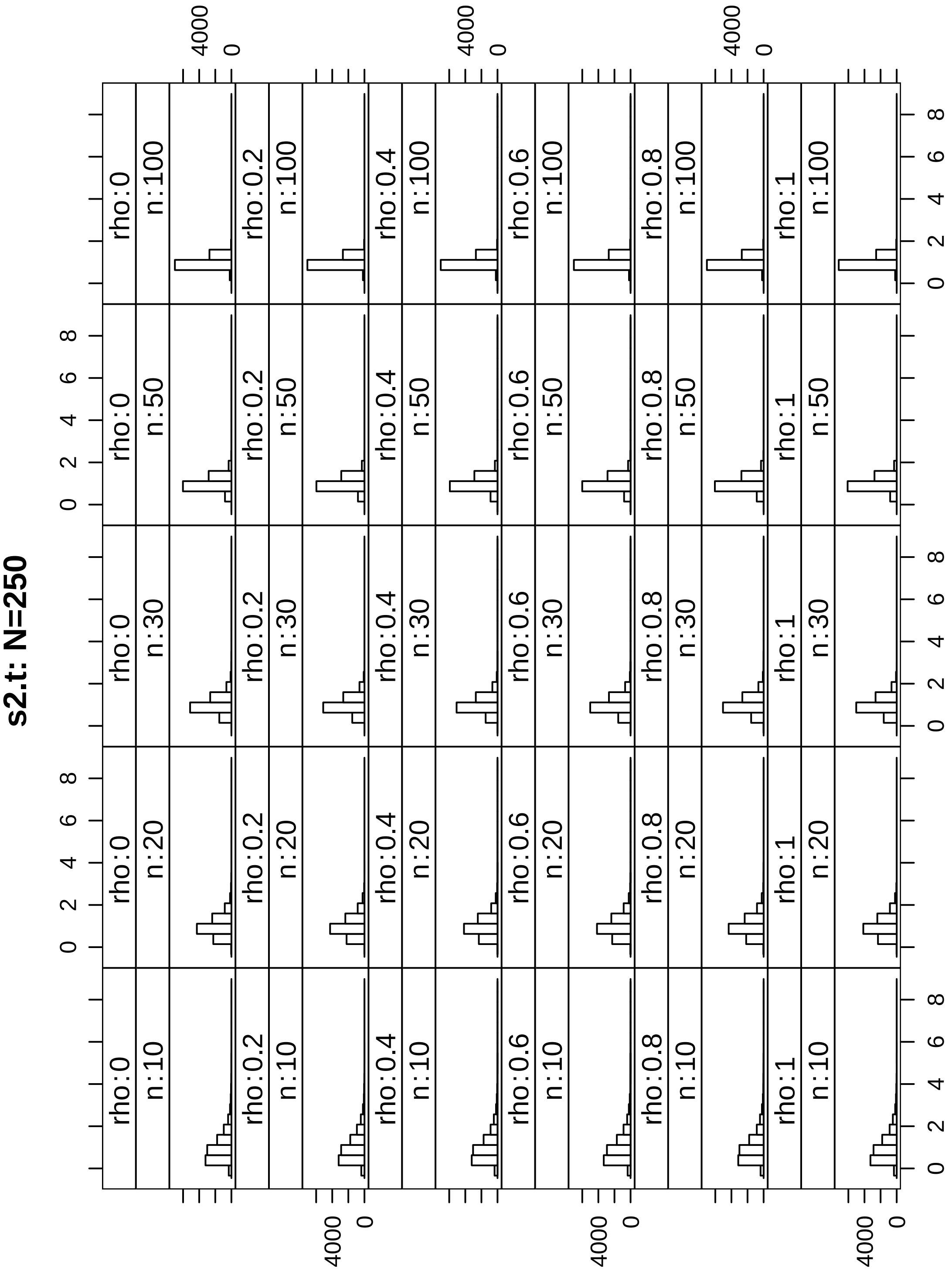


Figure B.21: Distribution of $S_t^2 + S_c^2$ for N=250

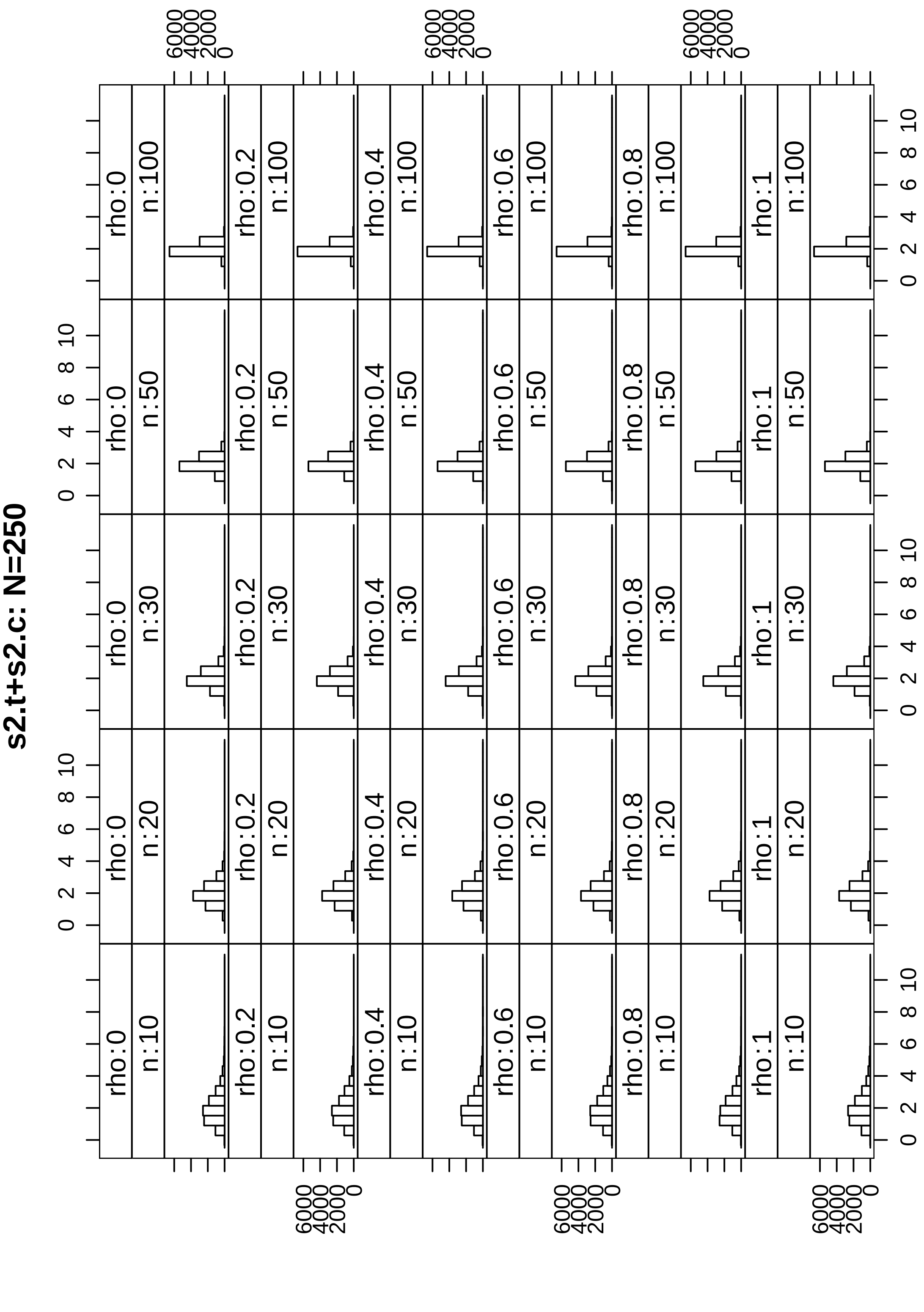


Figure B.22: Distribution of $var(\hat{\mu}_c)$ for N=250

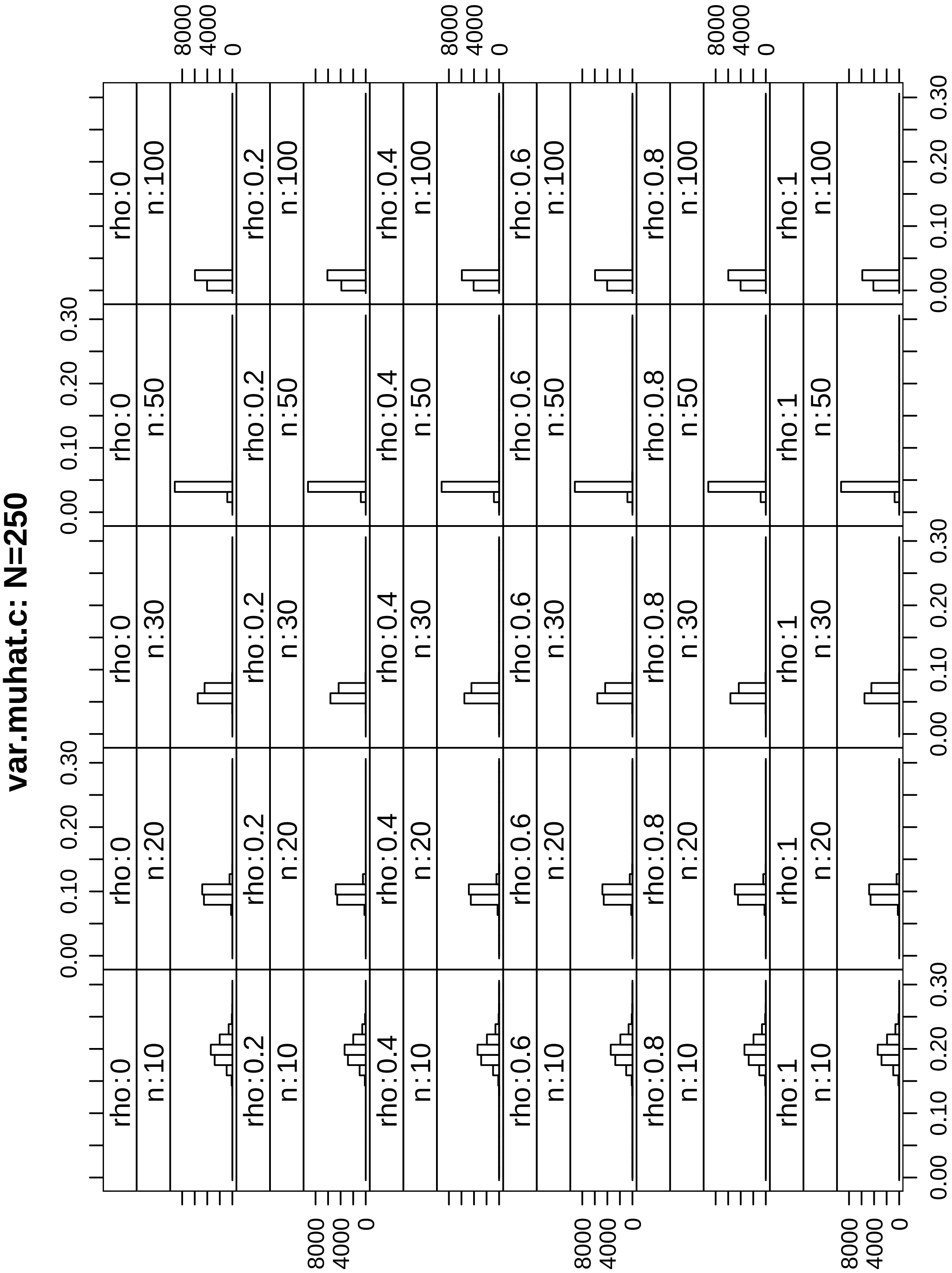


Figure B.23: Distribution of $\widehat{var}(\hat{\mu}_c)$ for N=250

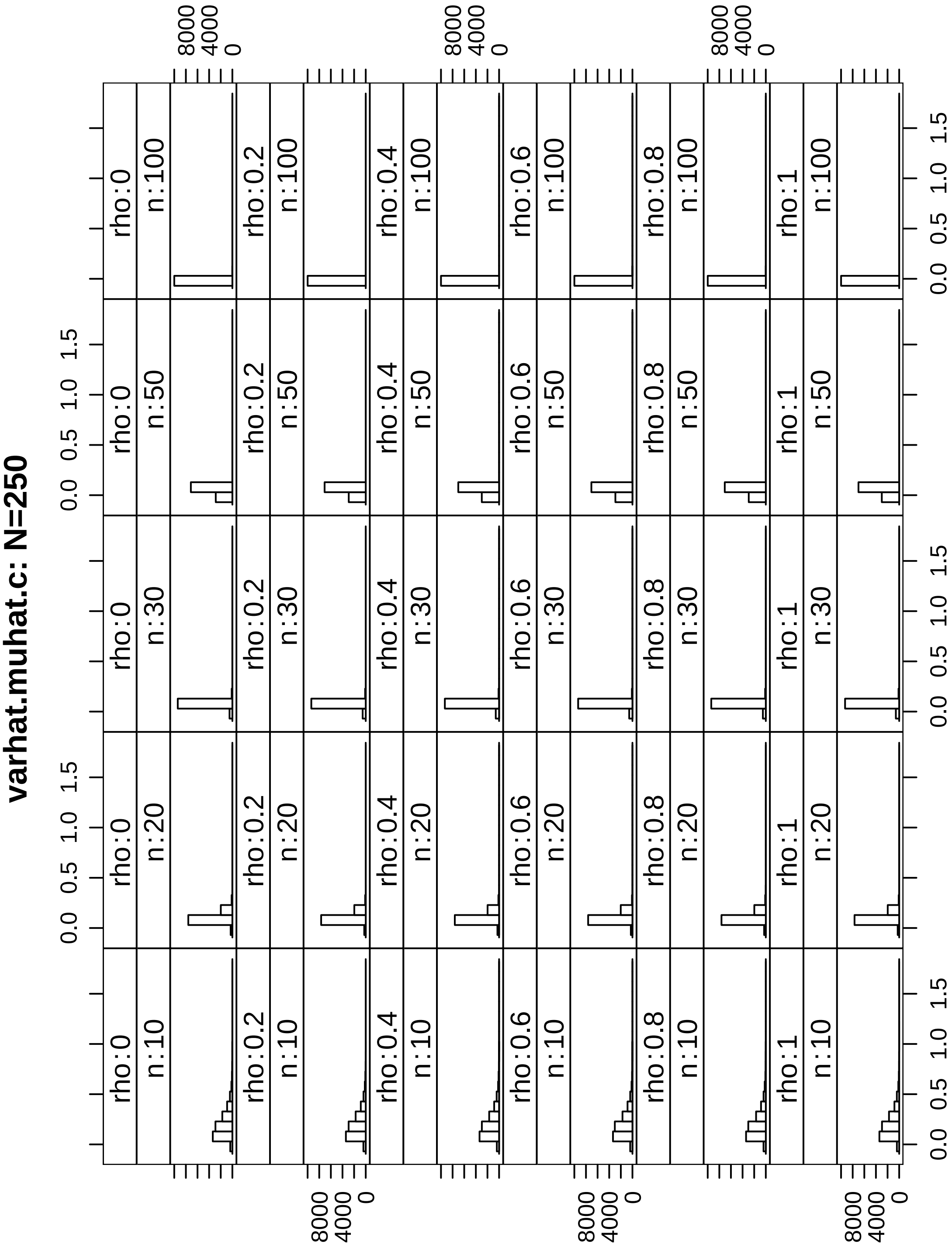


Figure B.24: Distribution of $var(\hat{\mu}_t)$ for N=250

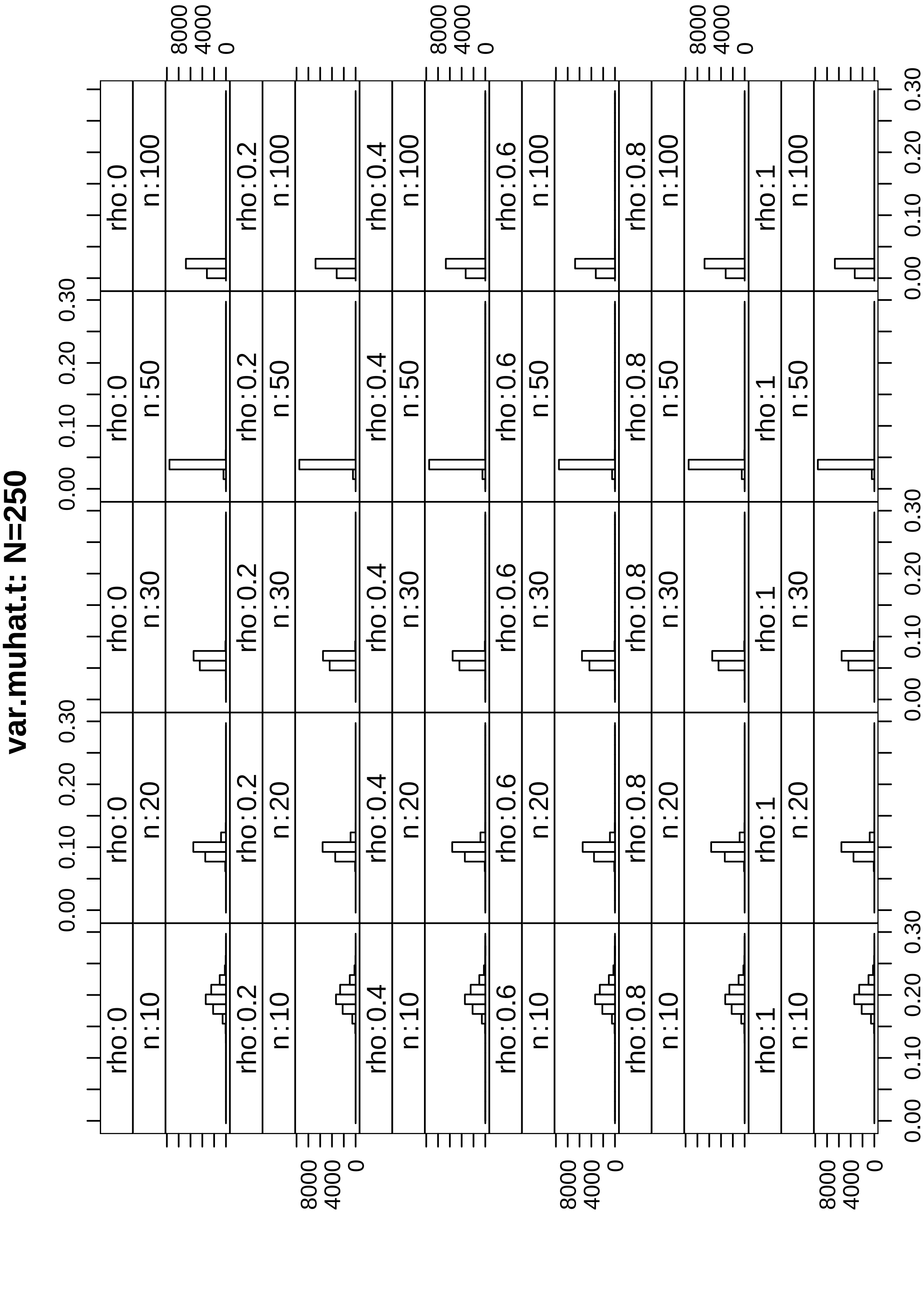


Figure B.25: Distribution of $\widehat{var}(\hat{\mu}_t)$ for N=250

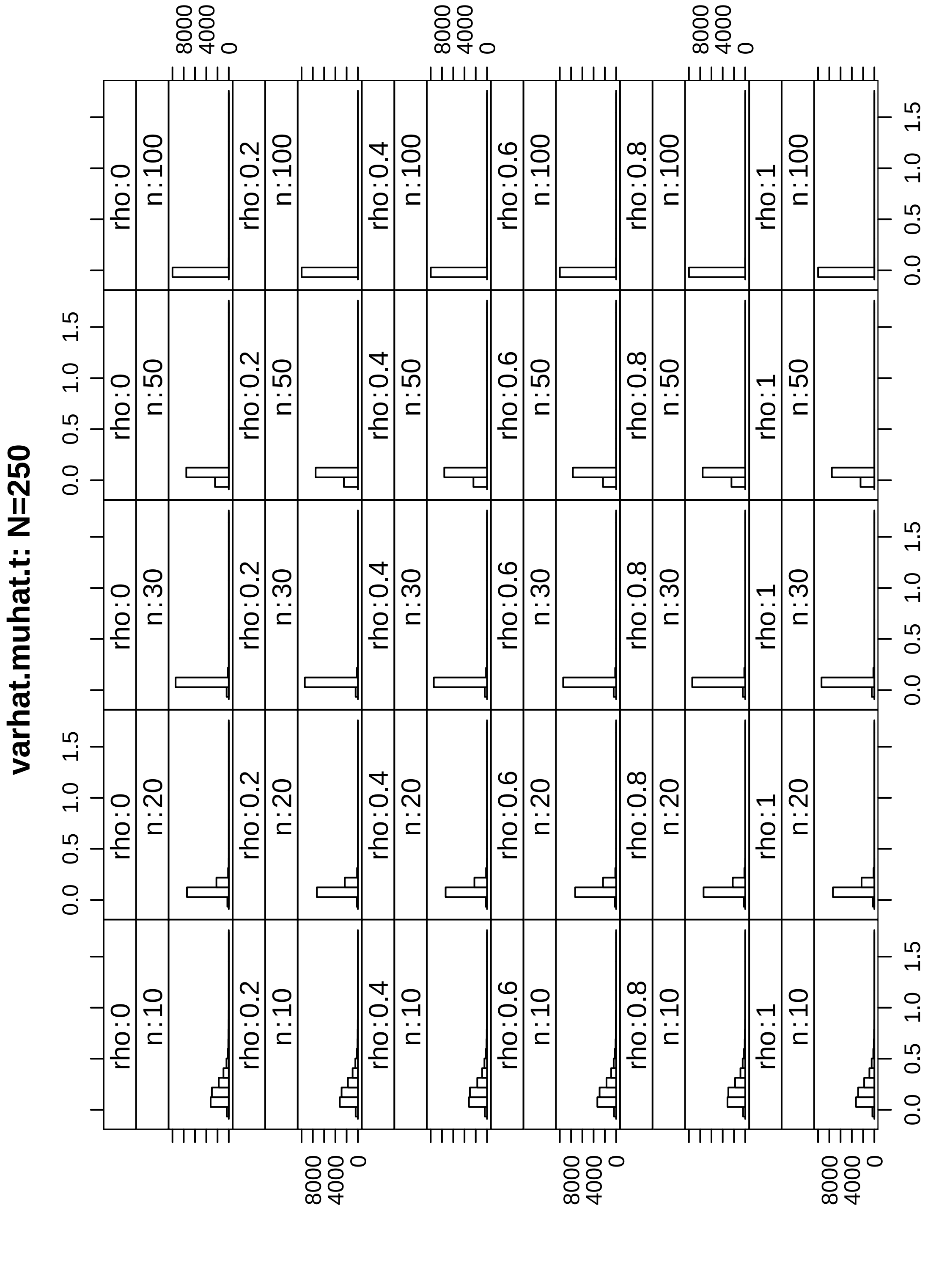


Figure B.26: Distribution of $var(\hat{\mu}_t - \hat{\mu}_c)$ for N=250

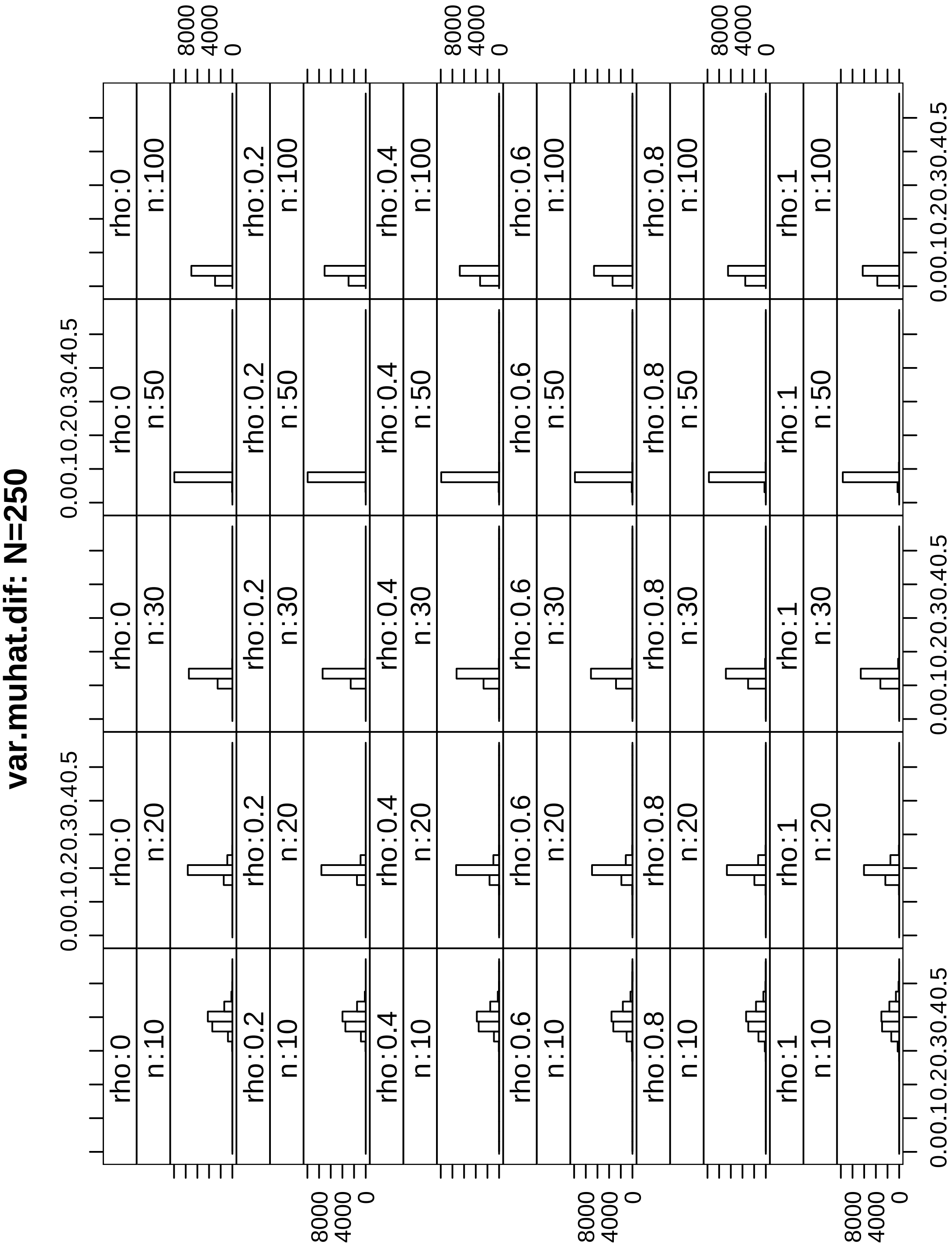


Figure B.27: Distribution of $\widehat{var}(\hat{\mu}_t - \hat{\mu}_c)$ for N=250

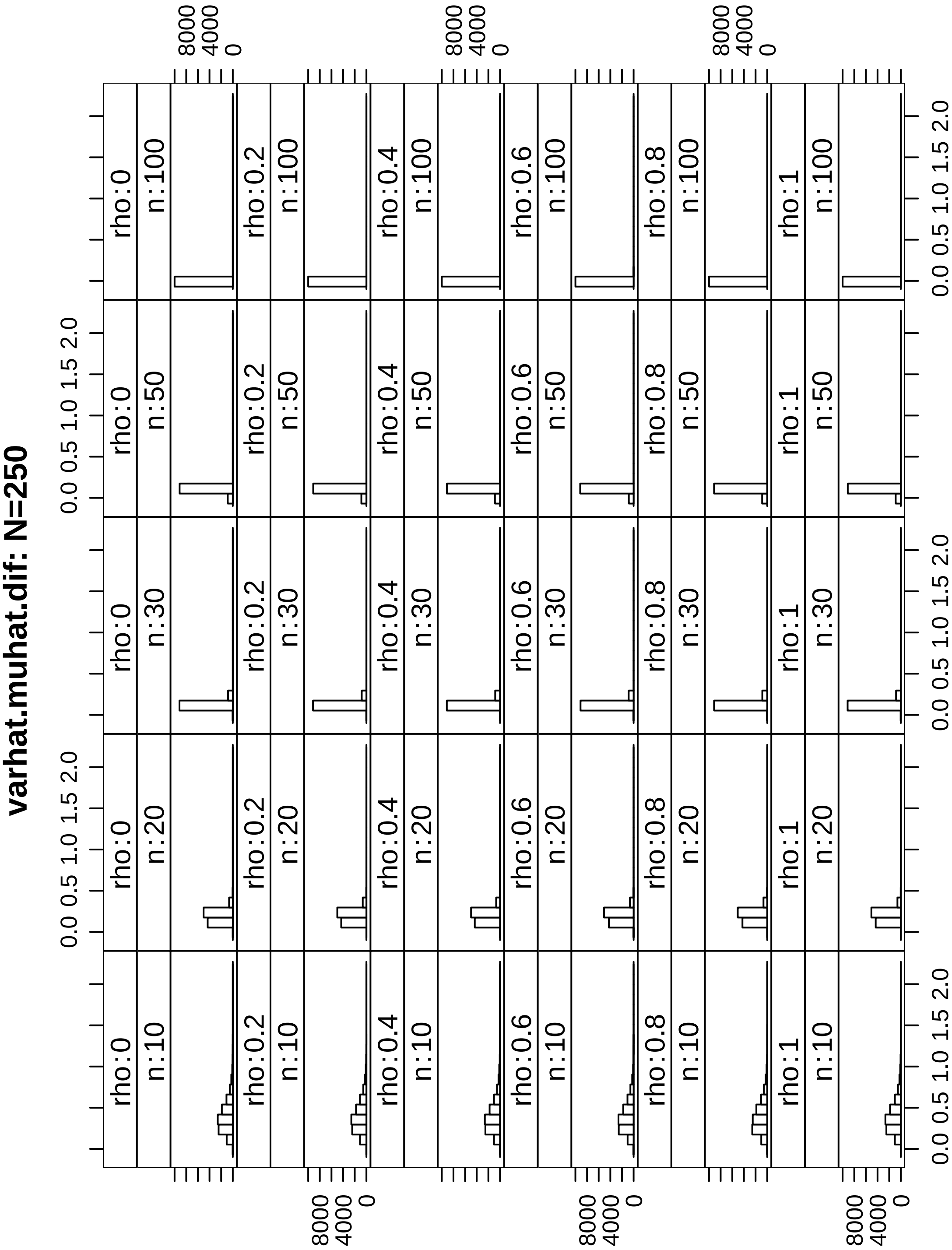


Figure B.28: Distribution of S_c^2 for N=500

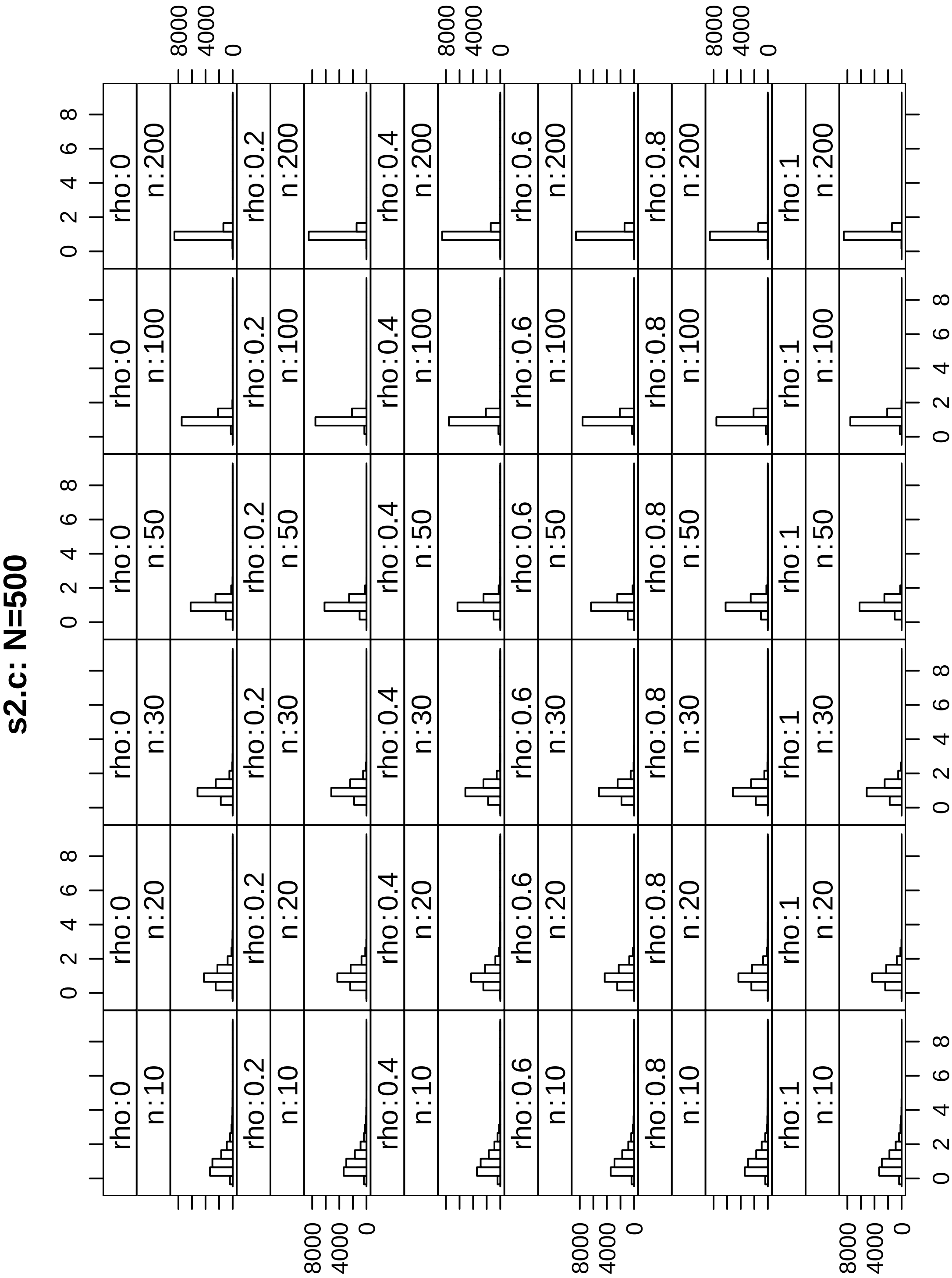


Figure B.29: Distribution of S_t^2 for N=500

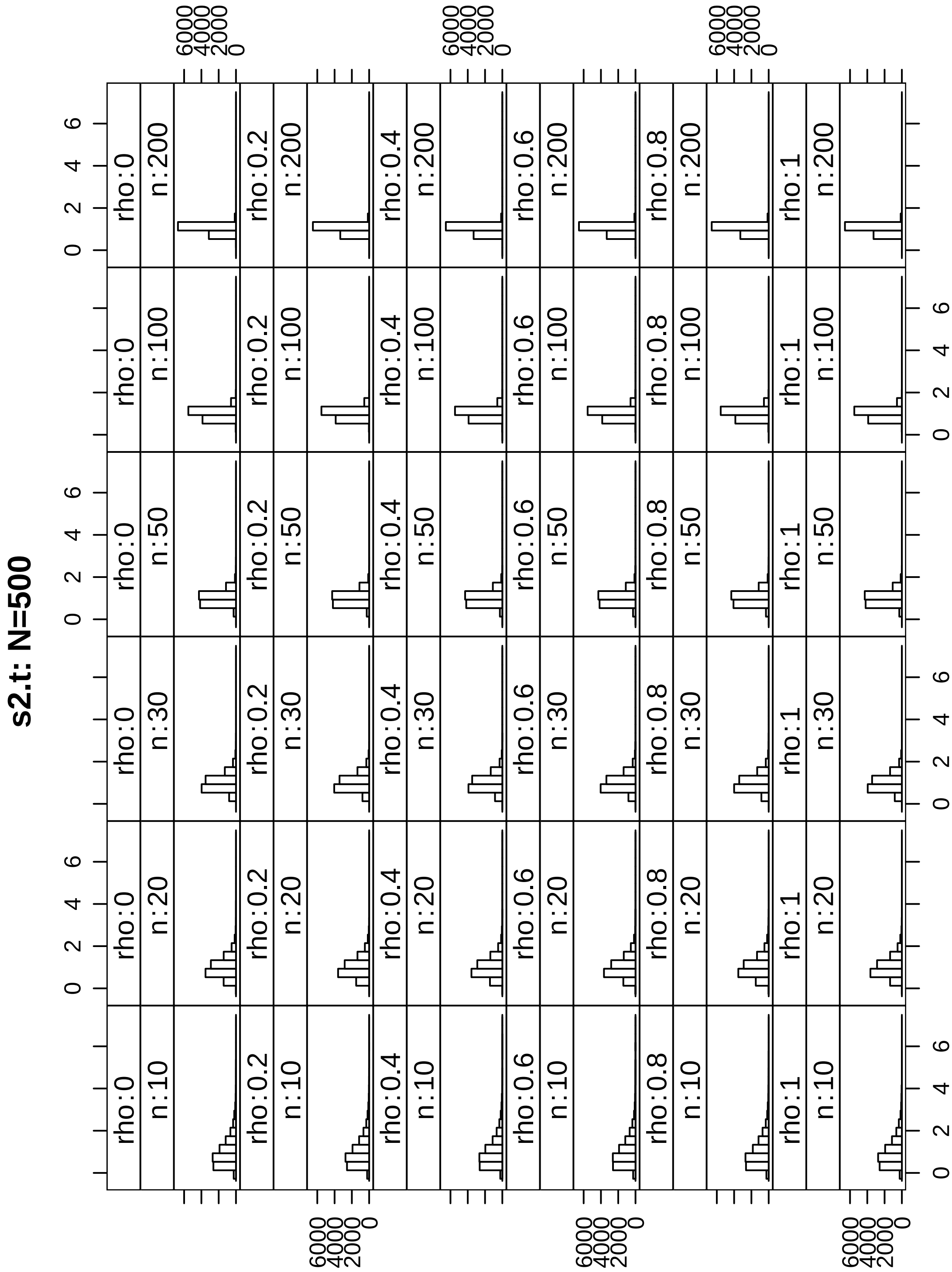


Figure B.30: Distribution of $S_t^2 + S_c^2$ for N=500

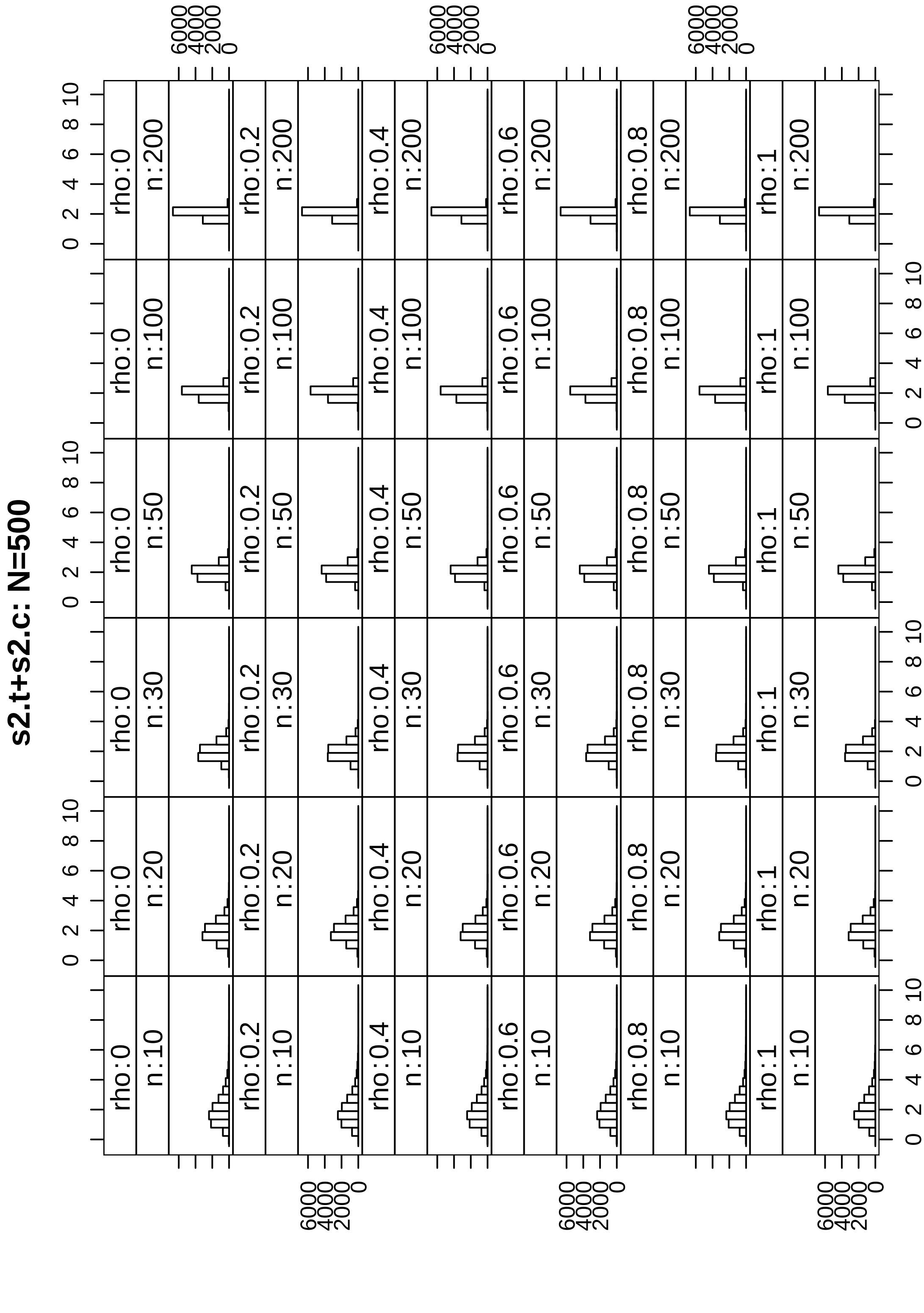


Figure B.31: Distribution of $var(\hat{\mu}_c)$ for N=500

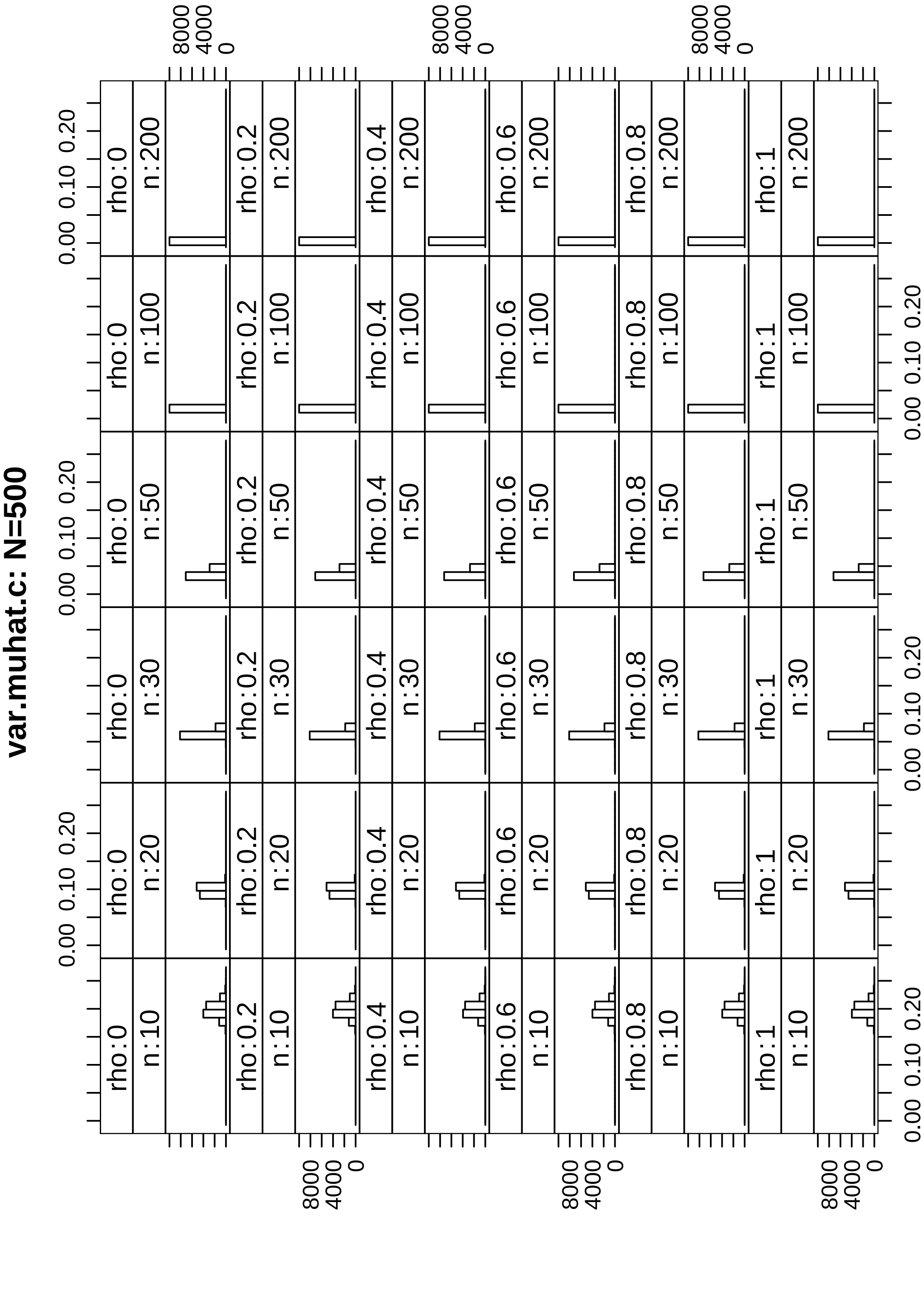


Figure B.32: Distribution of $\widehat{var}(\hat{\mu}_c)$ for N=500

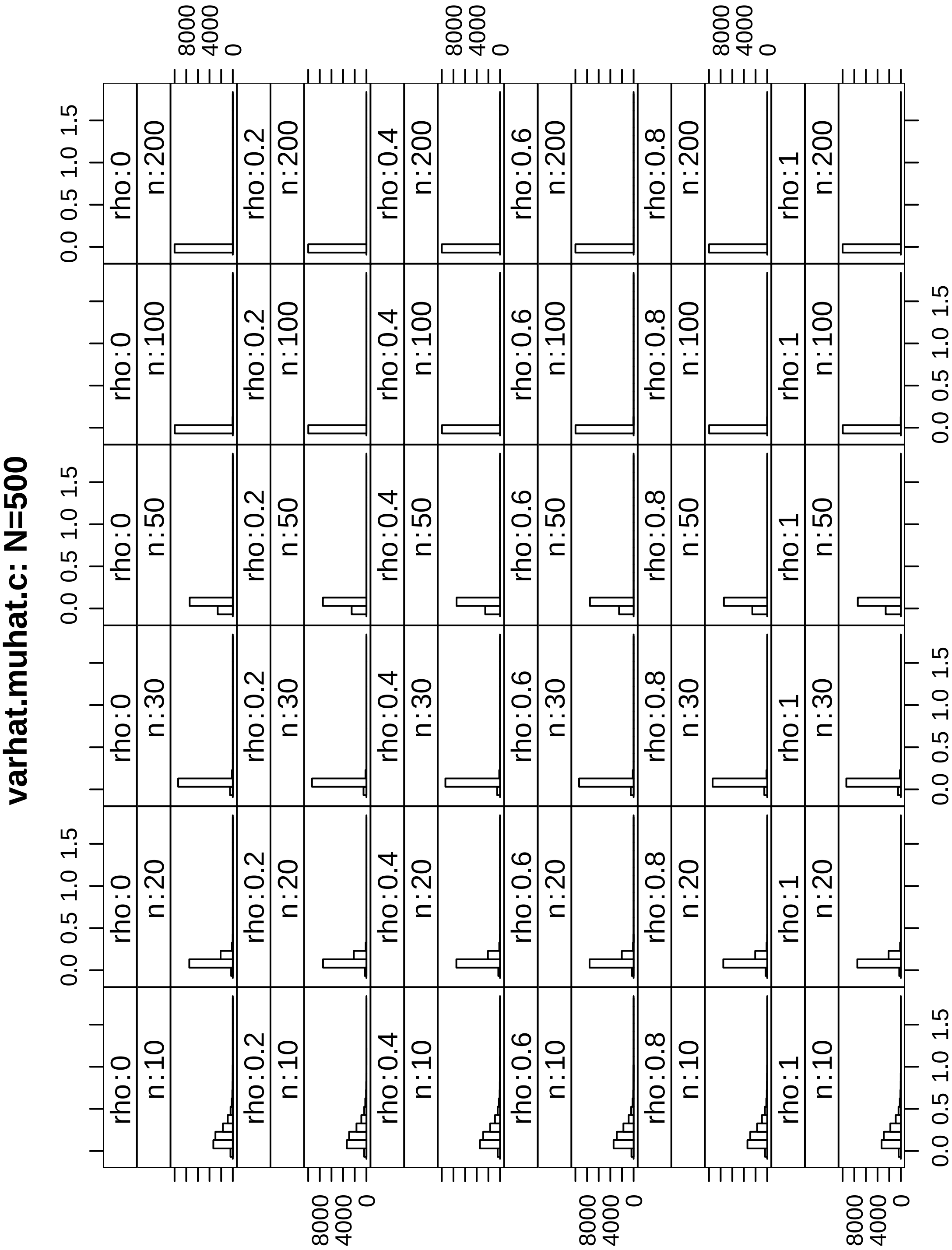


Figure B.33: Distribution of $var(\hat{\mu}_t)$ for N=500

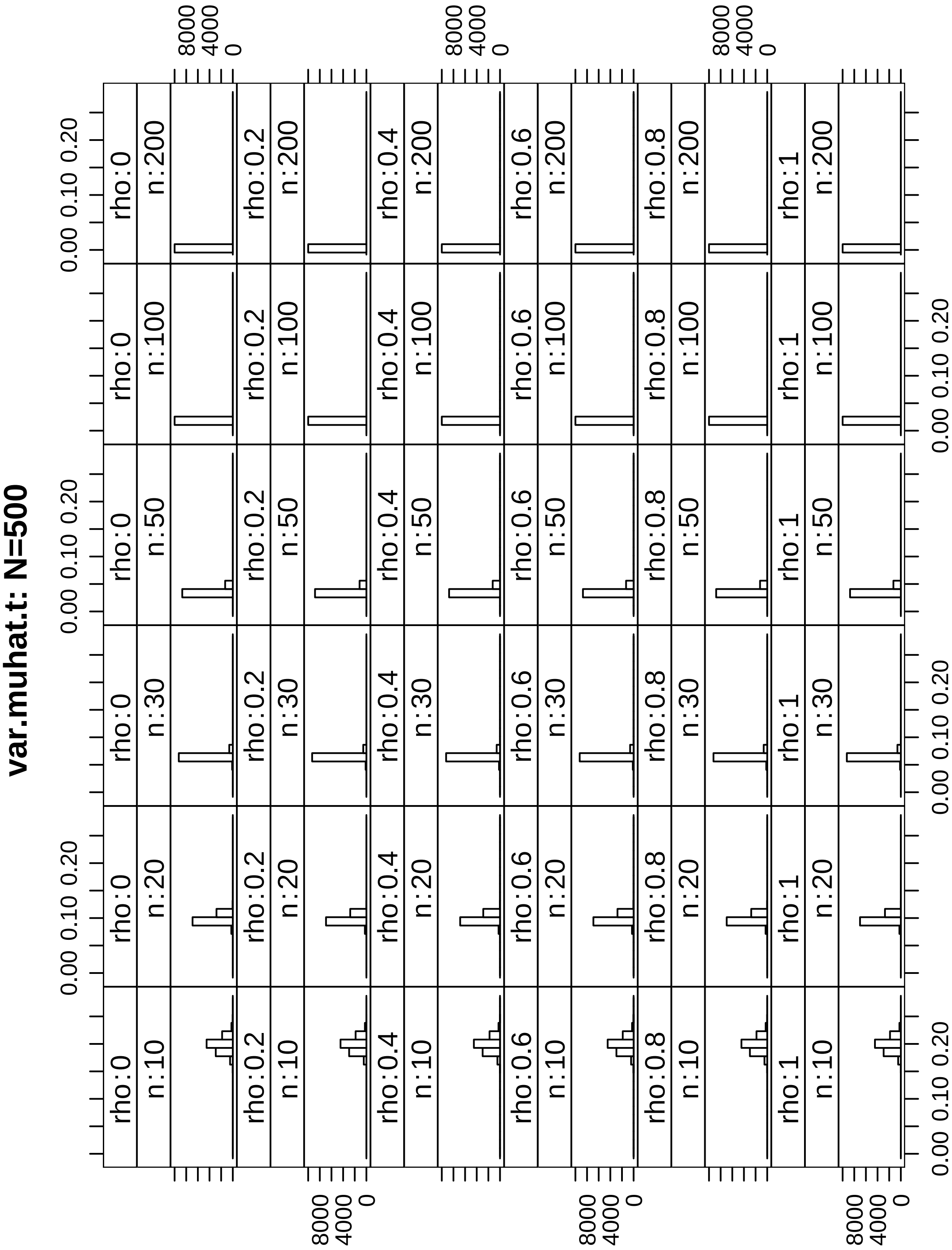


Figure B.34: Distribution of $\widehat{var}(\hat{\mu}_t)$ for N=500

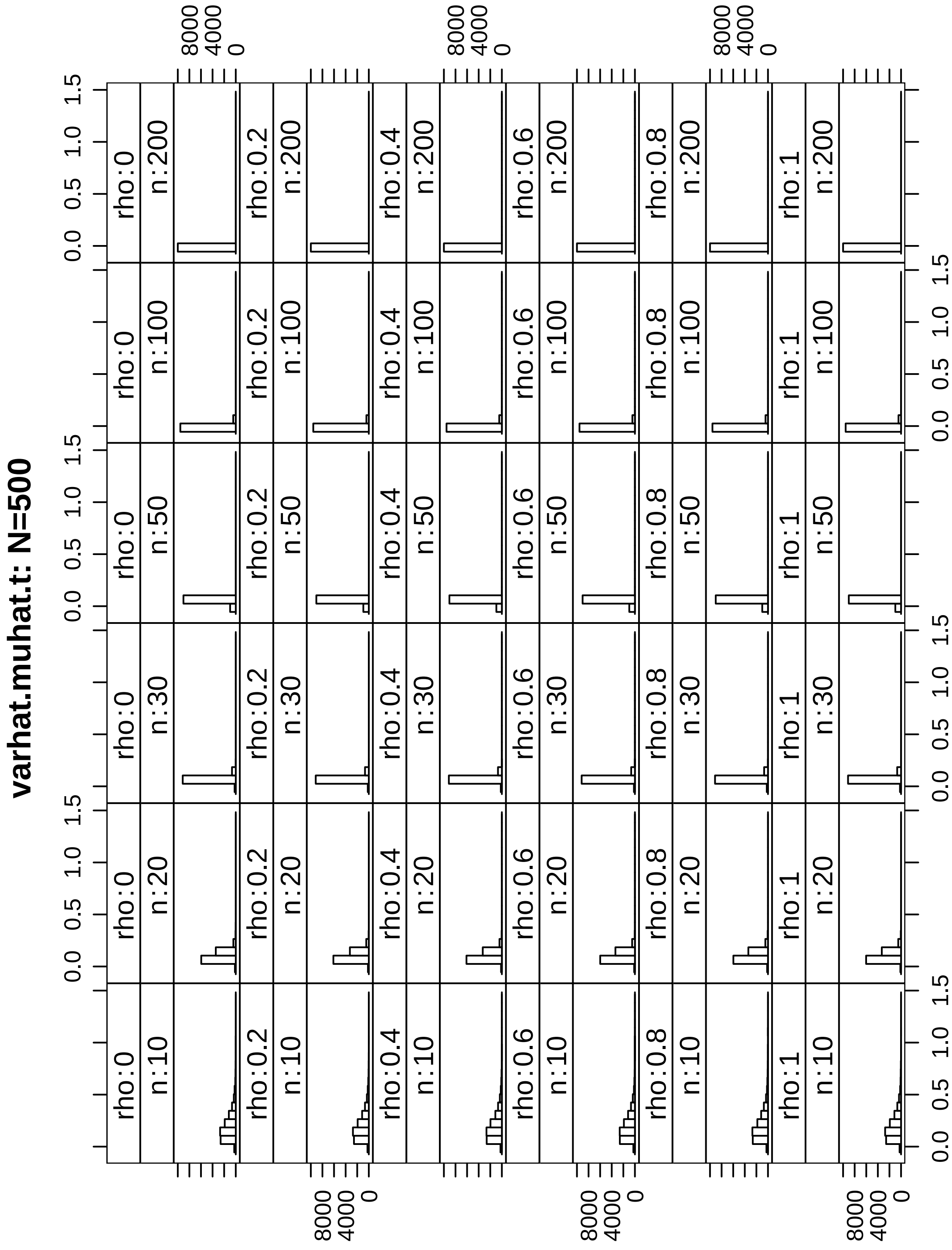


Figure B.35: Distribution of $var(\hat{\mu}_t - \hat{\mu}_c)$ for N=500

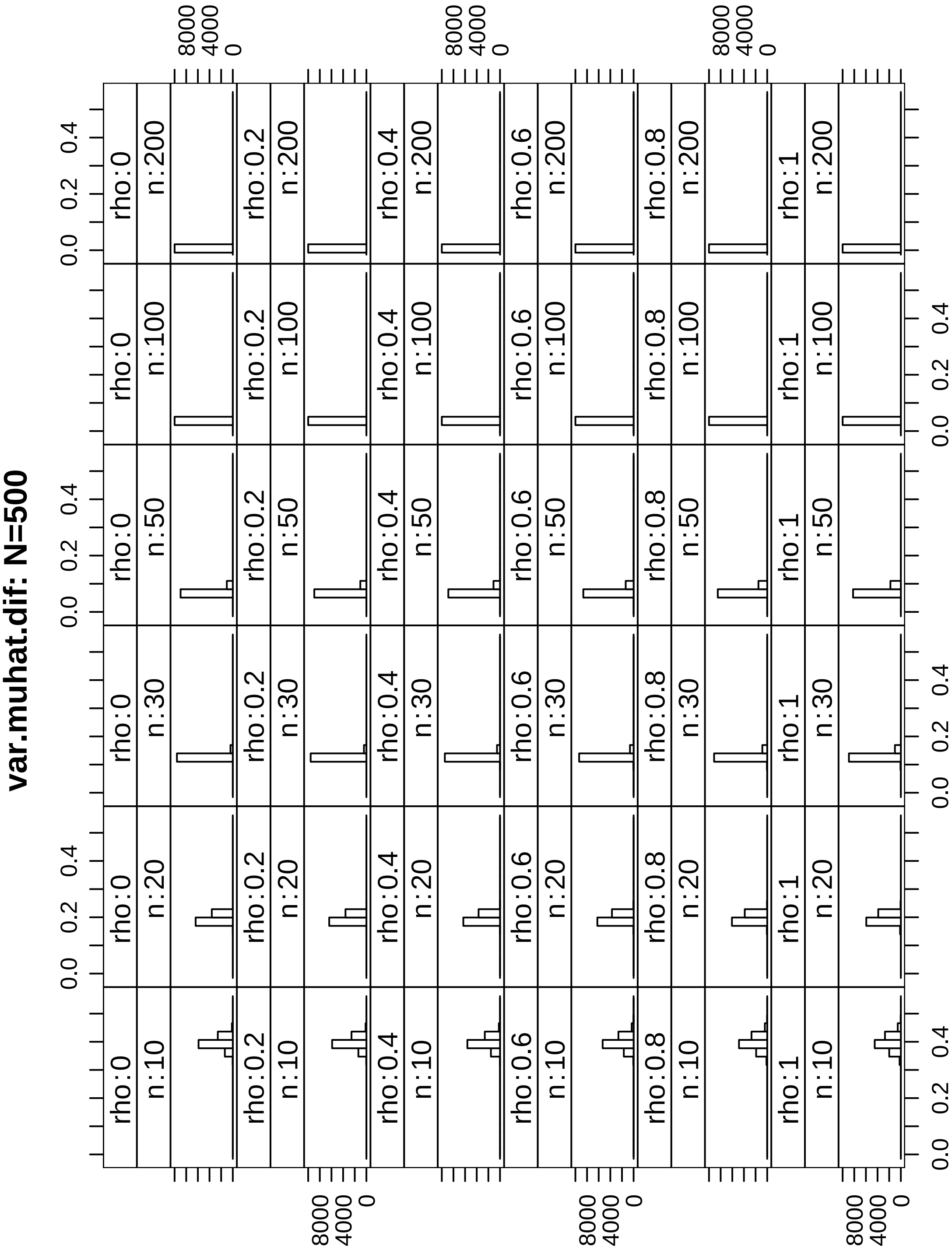


Figure B.36: Distribution of $\widehat{var}(\hat{\mu}_t - \hat{\mu}_c)$ for N=500

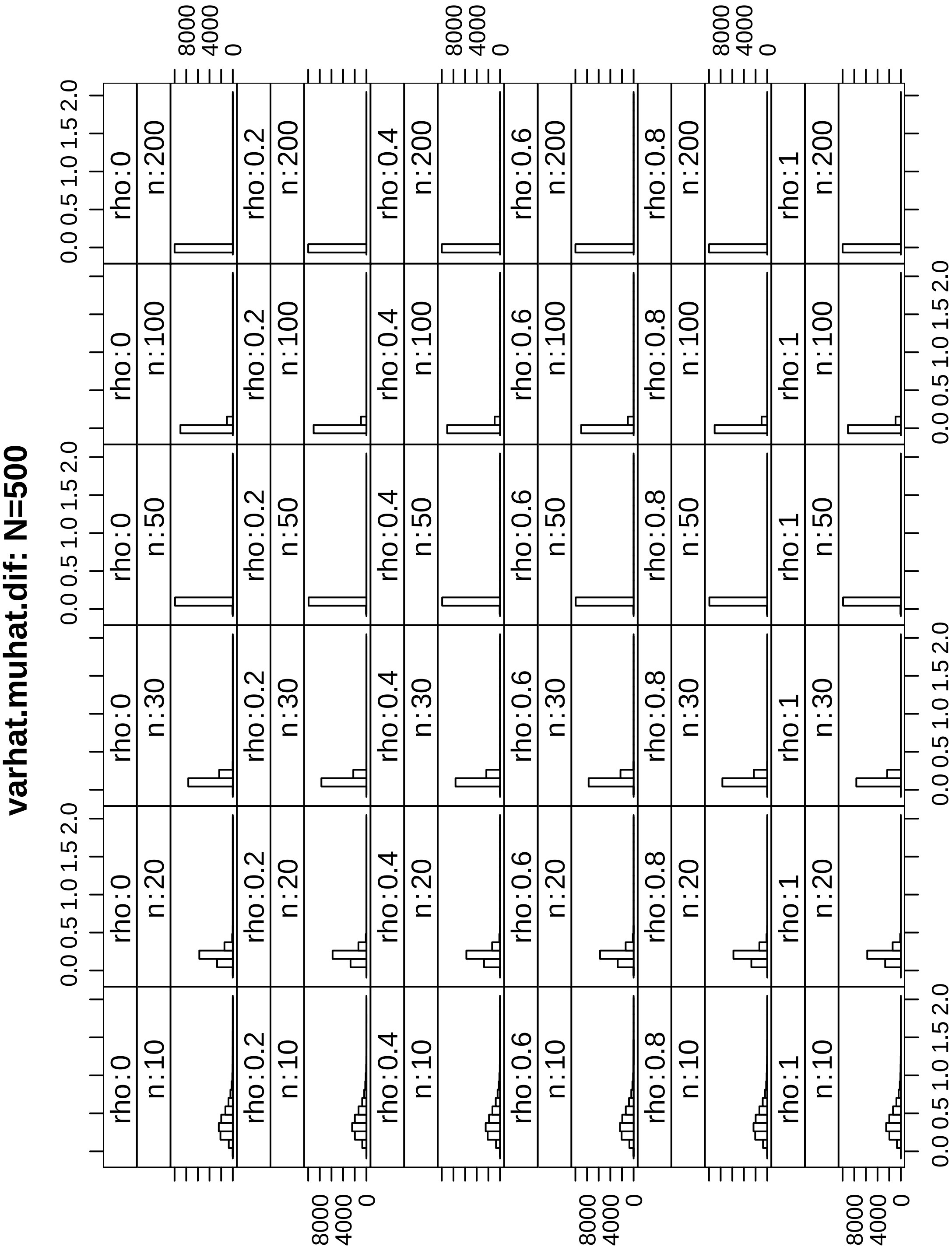


Figure B.37: Distribution of S_c^2 for N=1000

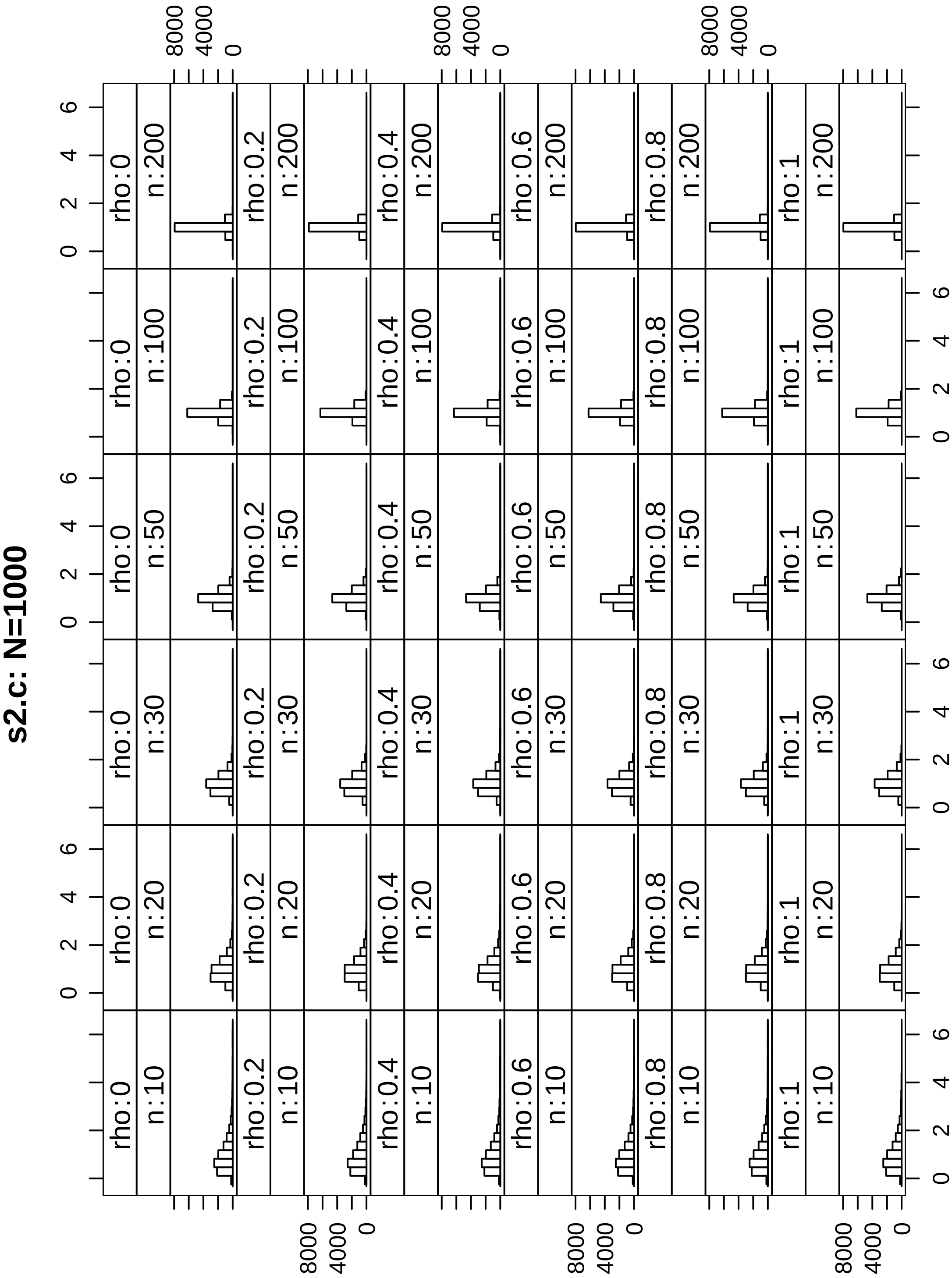


Figure B.38: Distribution of S_t^2 for N=1000

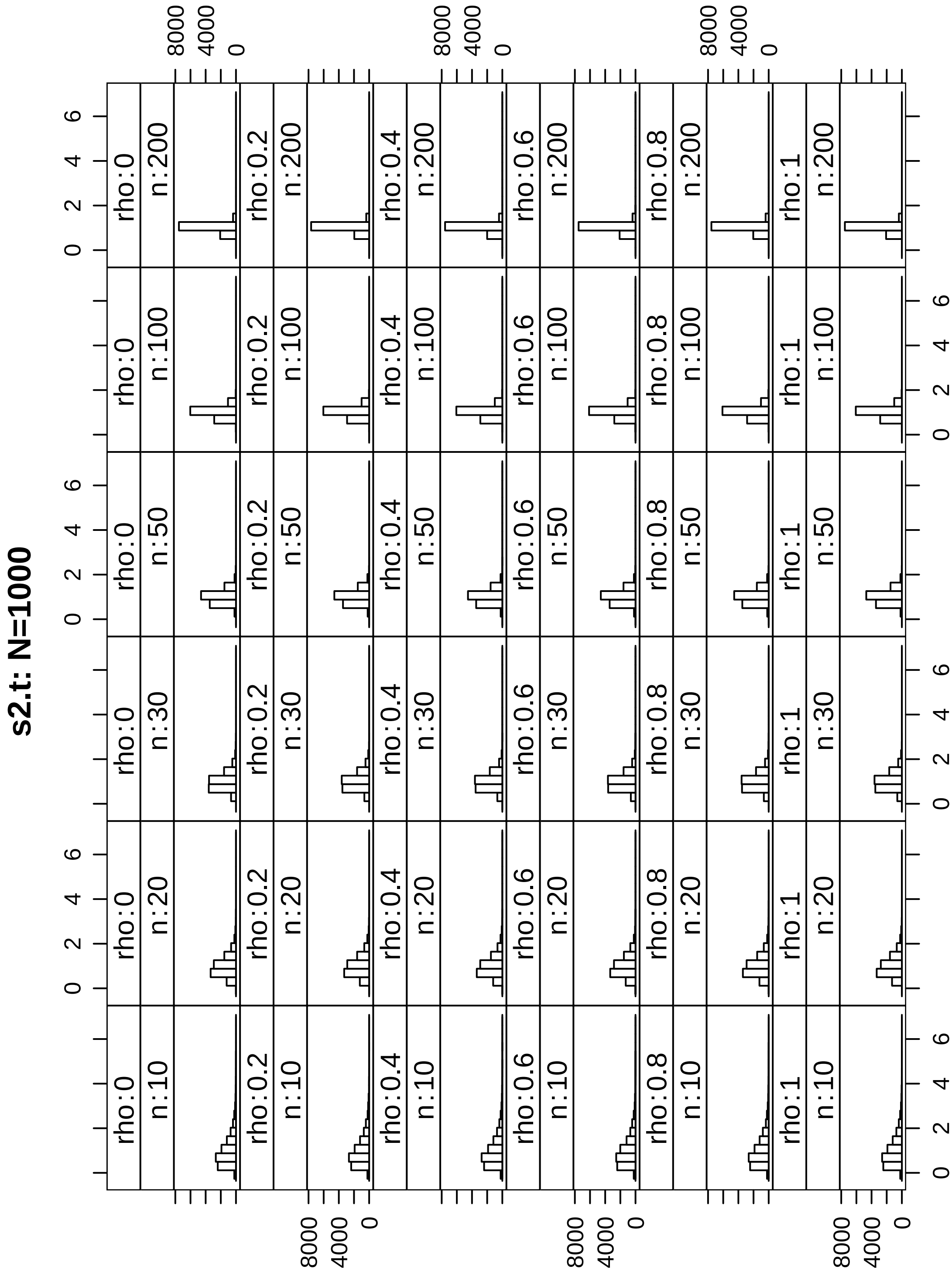


Figure B.39: Distribution of $S_t^2 + S_c^2$ for N=1000

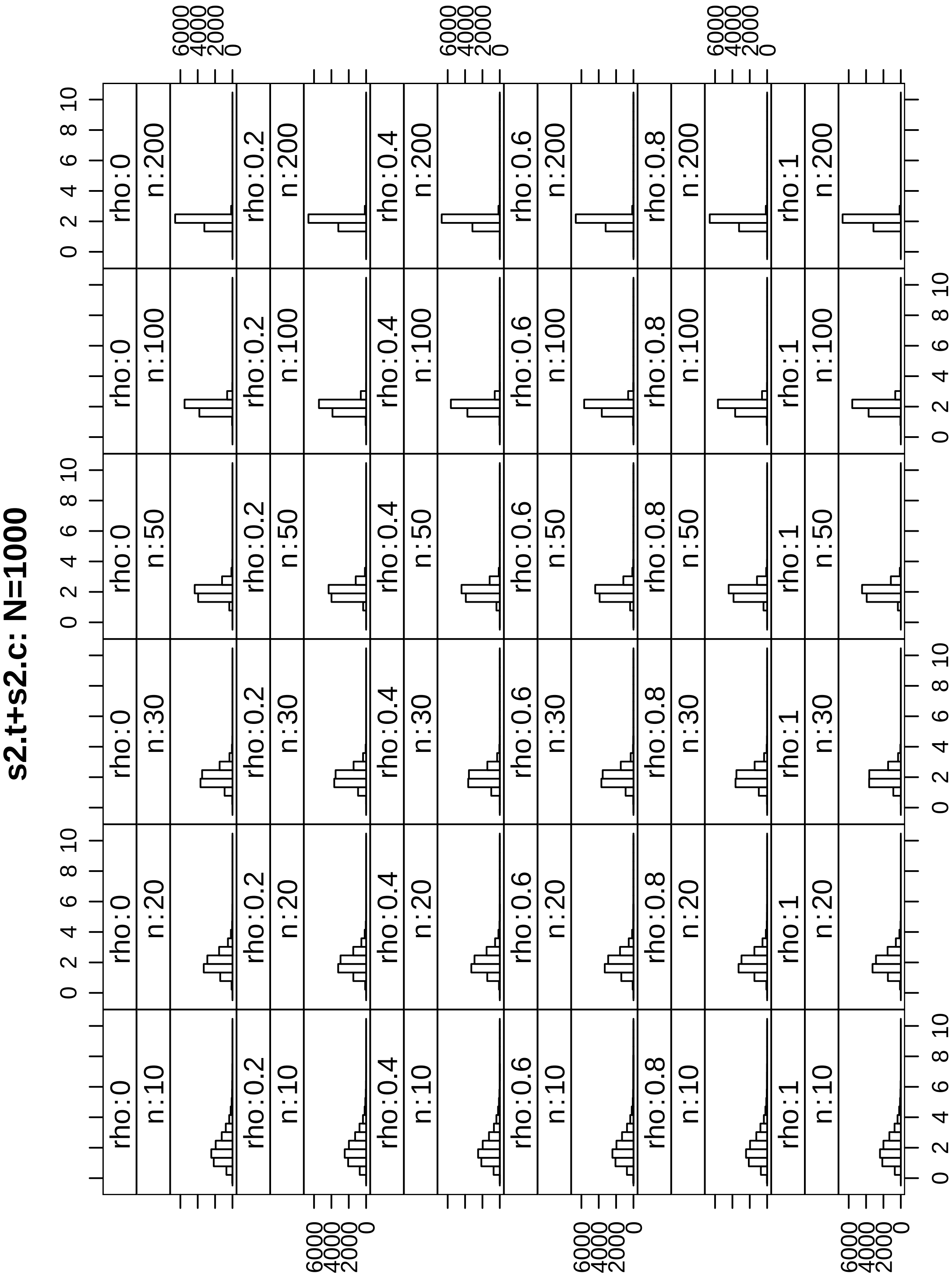


Figure B.40: Distribution of $var(\hat{\mu}_c)$ for N=1000

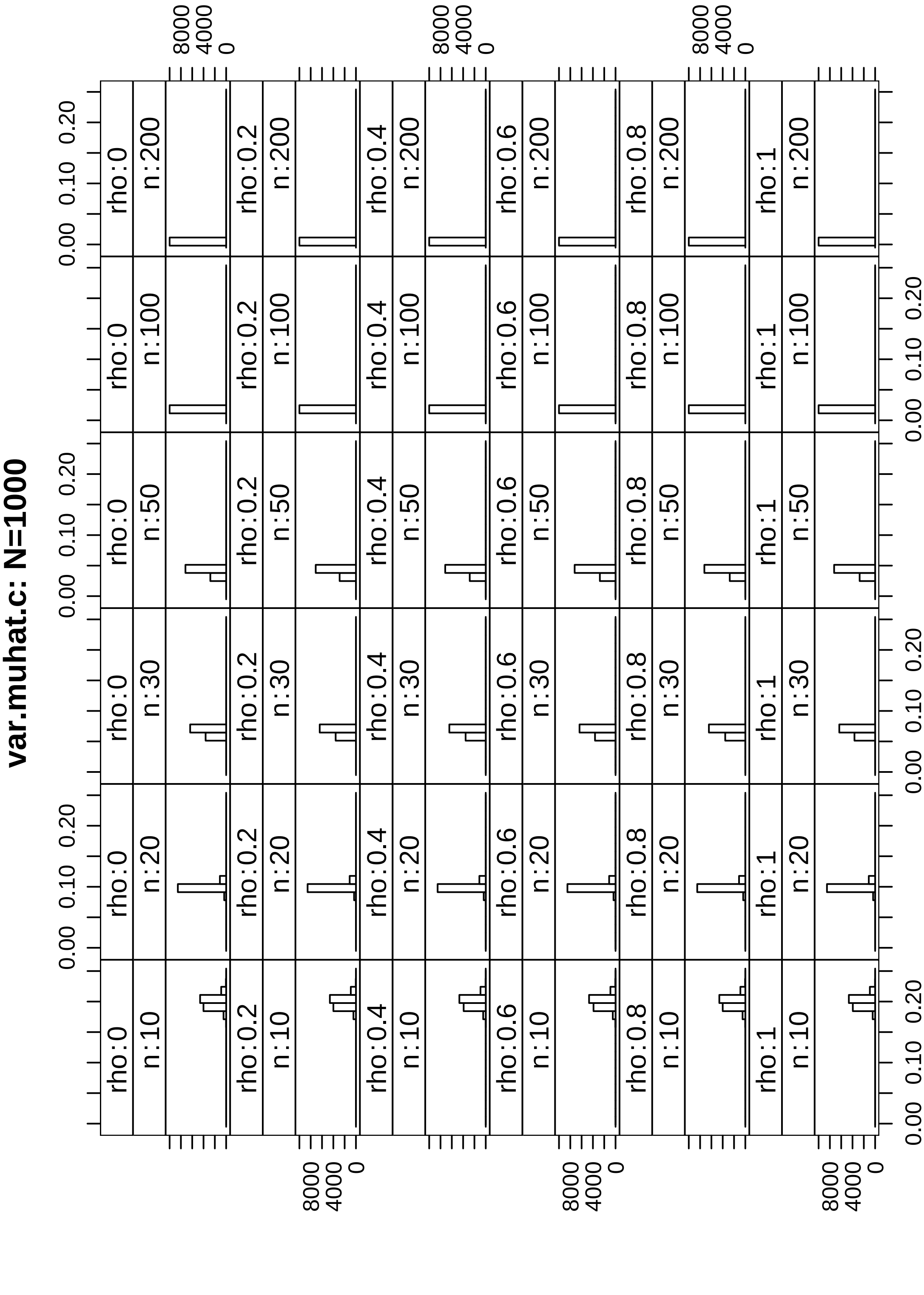


Figure B.41: Distribution of $\widehat{var}(\hat{\mu}_c)$ for N=1000

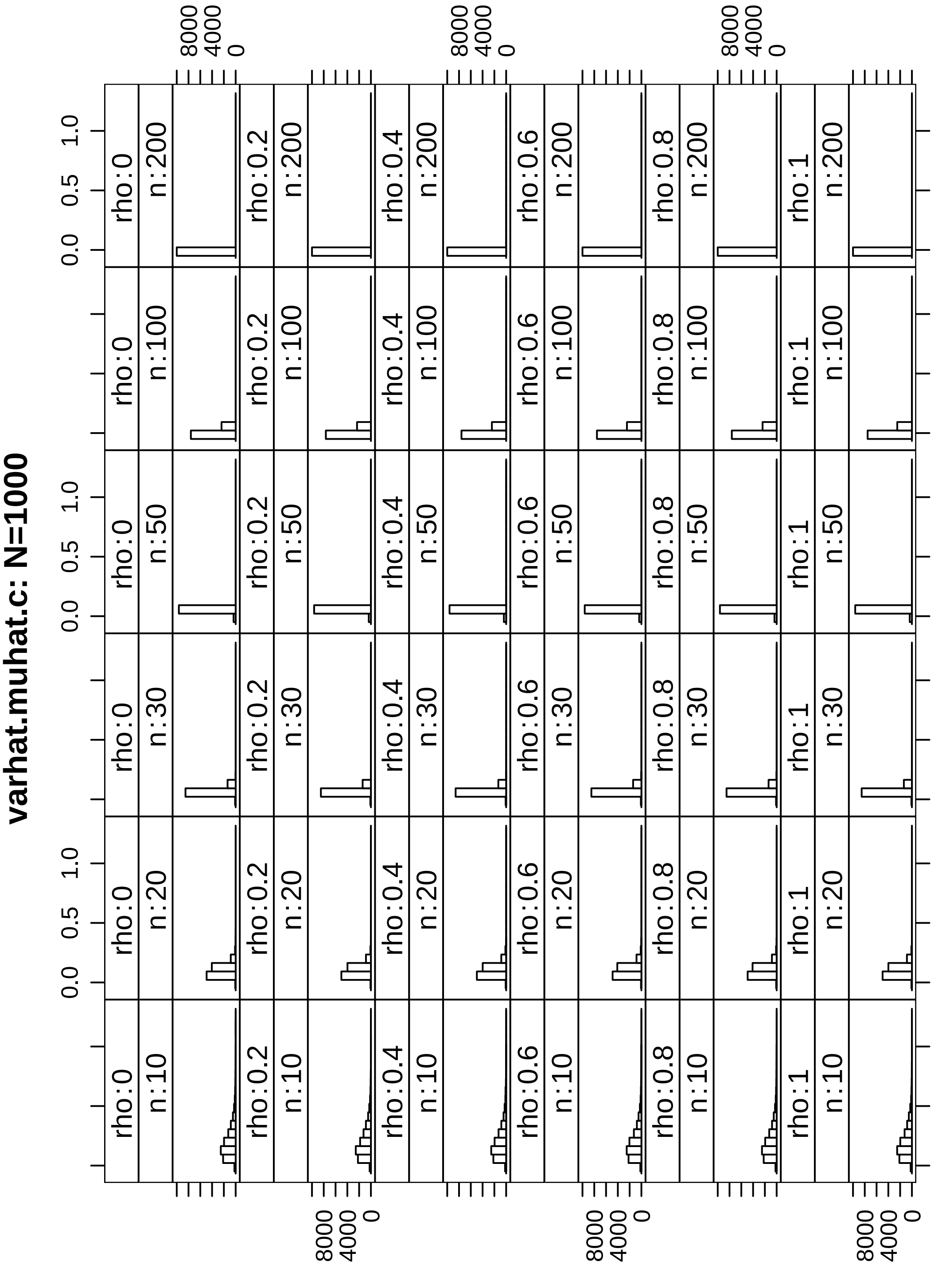


Figure B.42: Distribution of $var(\hat{\mu}_t)$ for N=1000

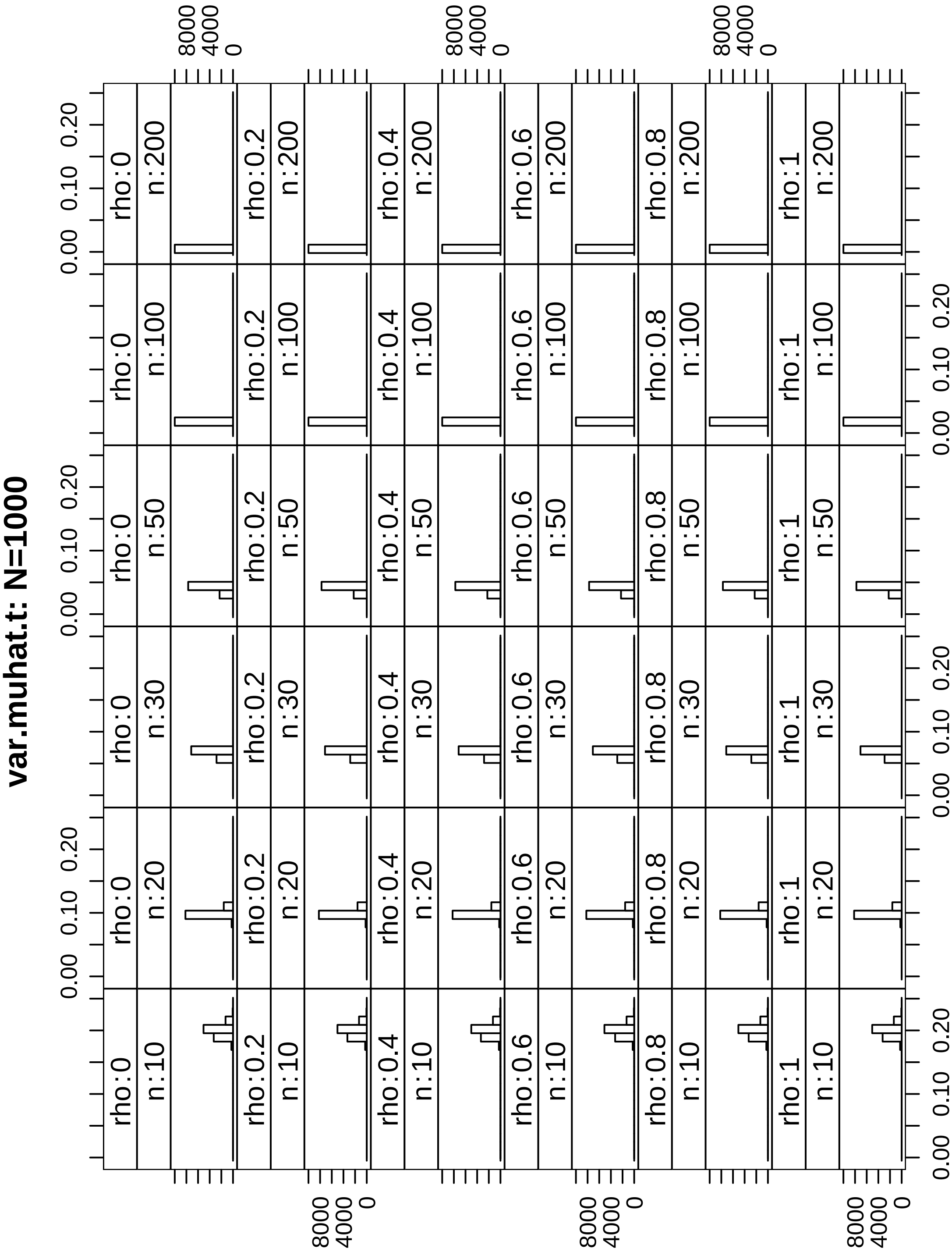


Figure B.43: Distribution of $\widehat{var}(\hat{\mu}_t)$ for N=1000

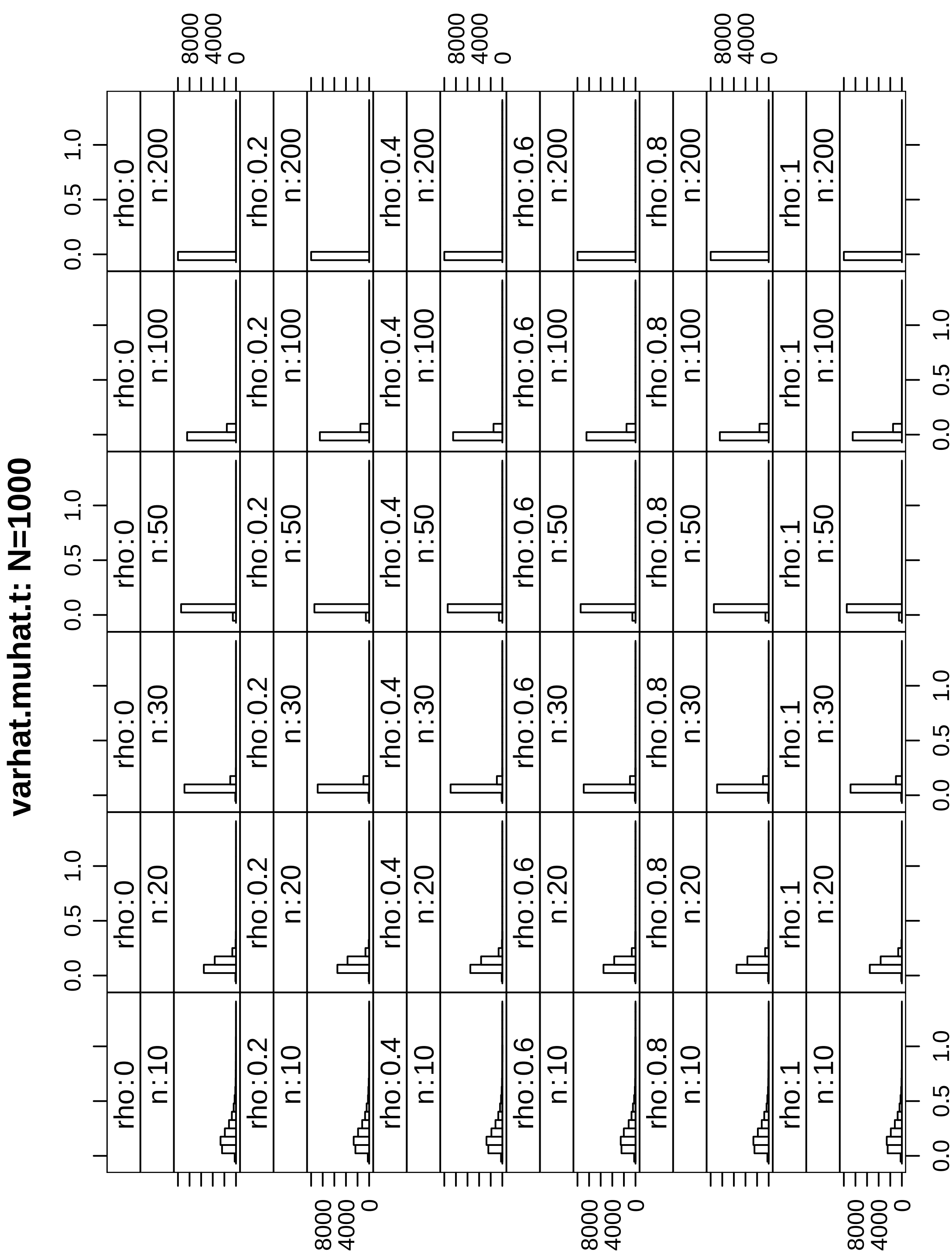


Figure B.44: Distribution of $var(\hat{\mu}_t - \hat{\mu}_c)$ for N=1000

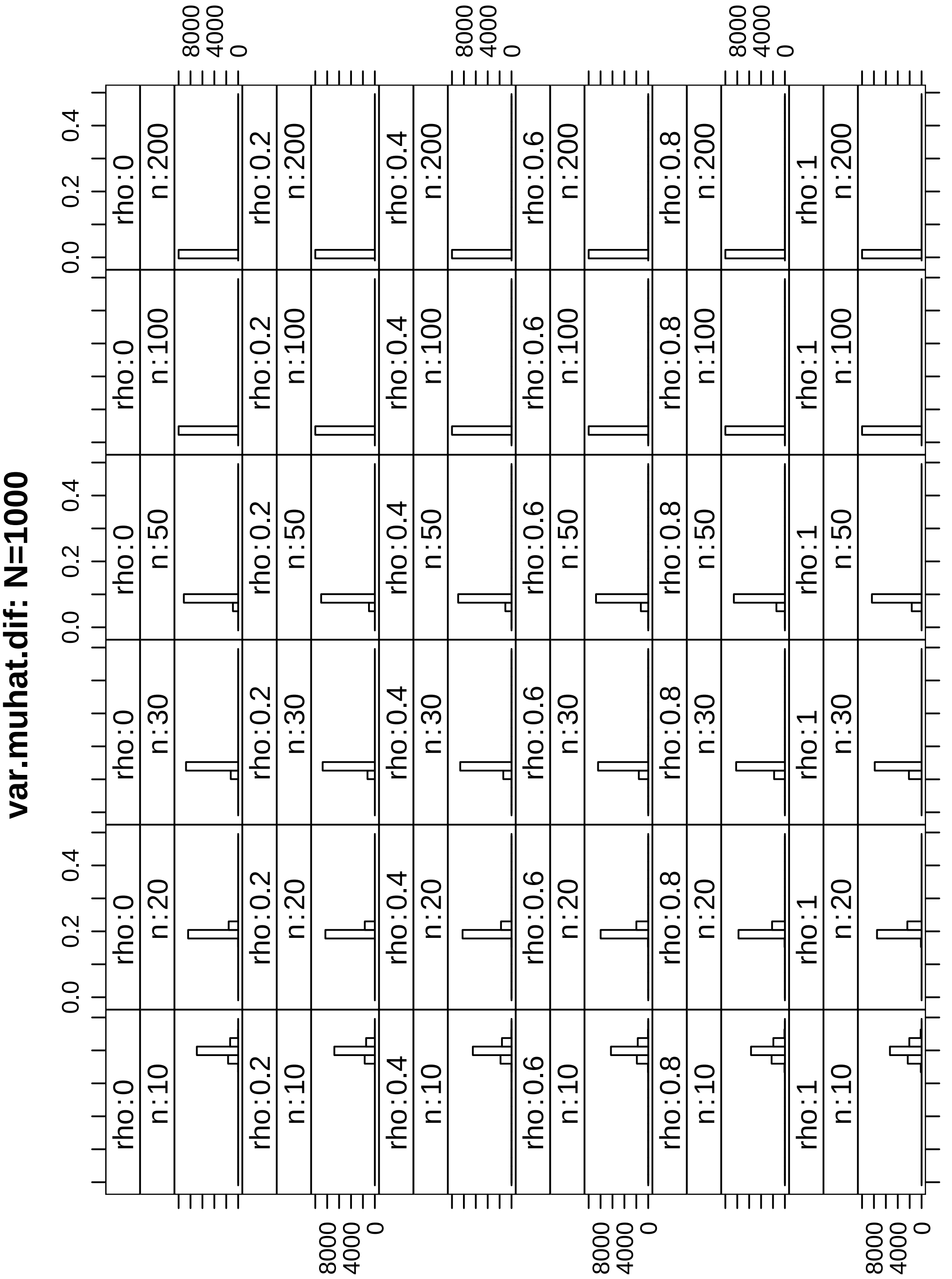


Figure B.45: Distribution of $\widehat{var}(\hat{\mu}_t - \hat{\mu}_c)$ for N=1000

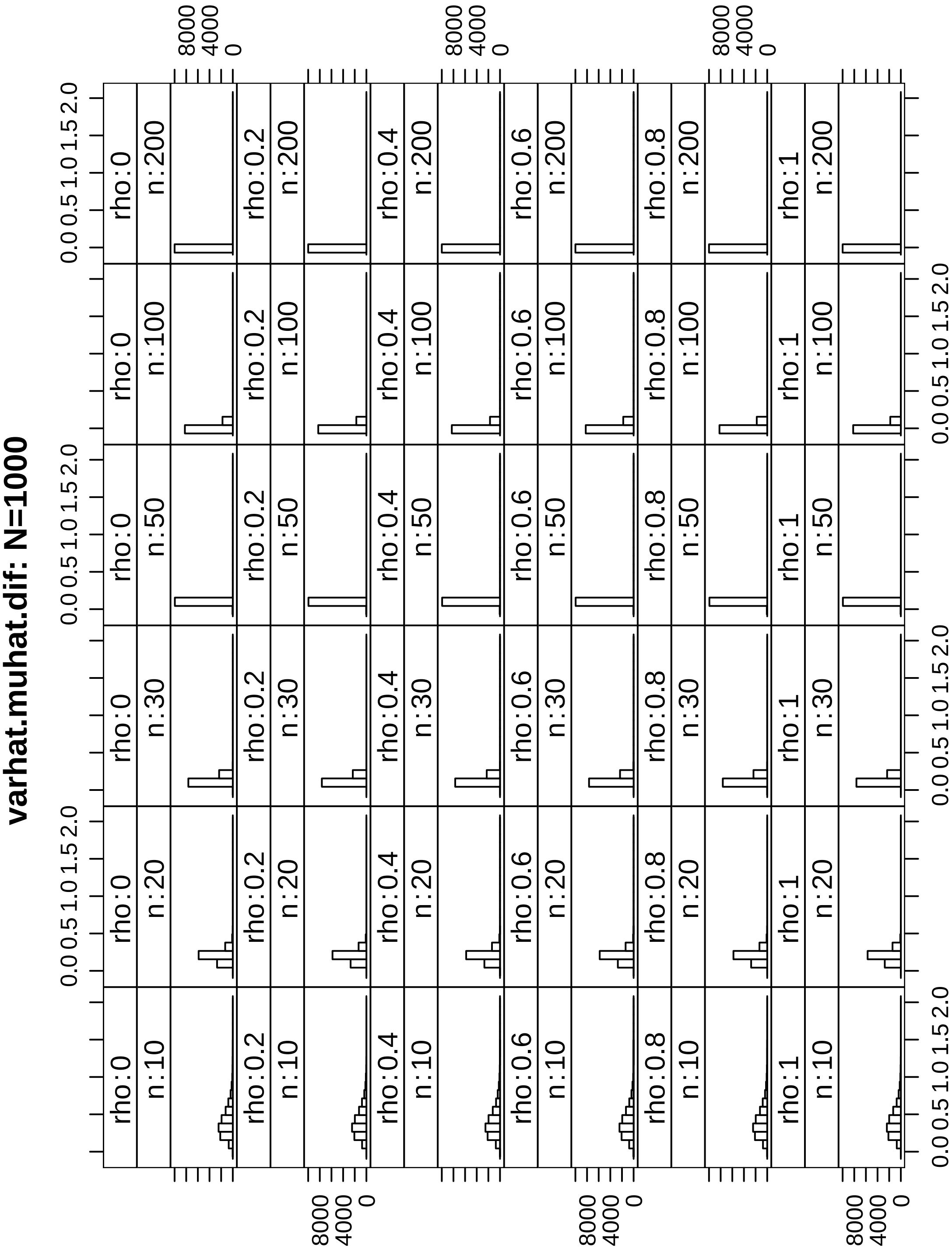


Figure B.46: Distribution of S_c^2 for N=5000

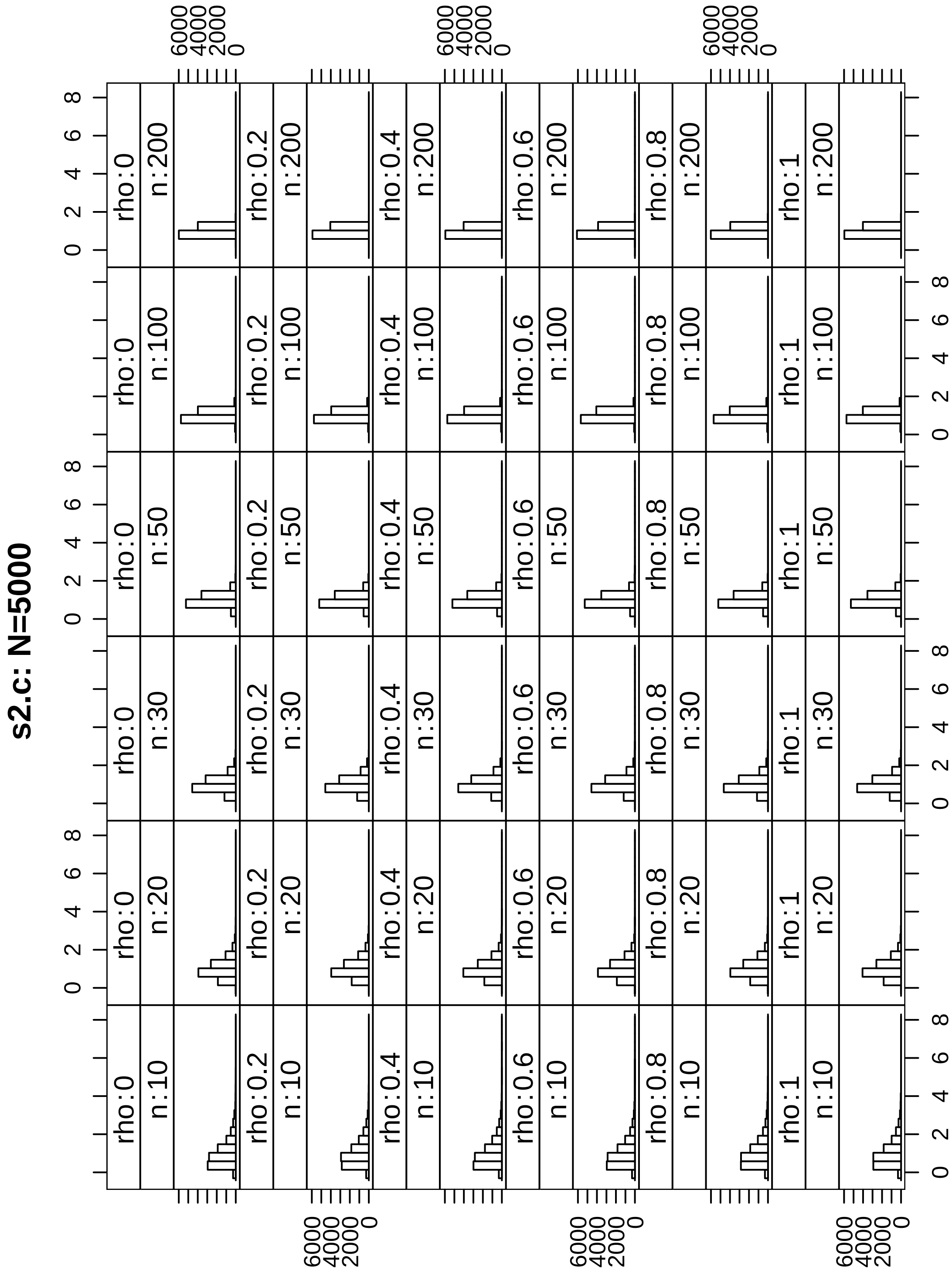


Figure B.47: Distribution of S_t^2 for N=5000

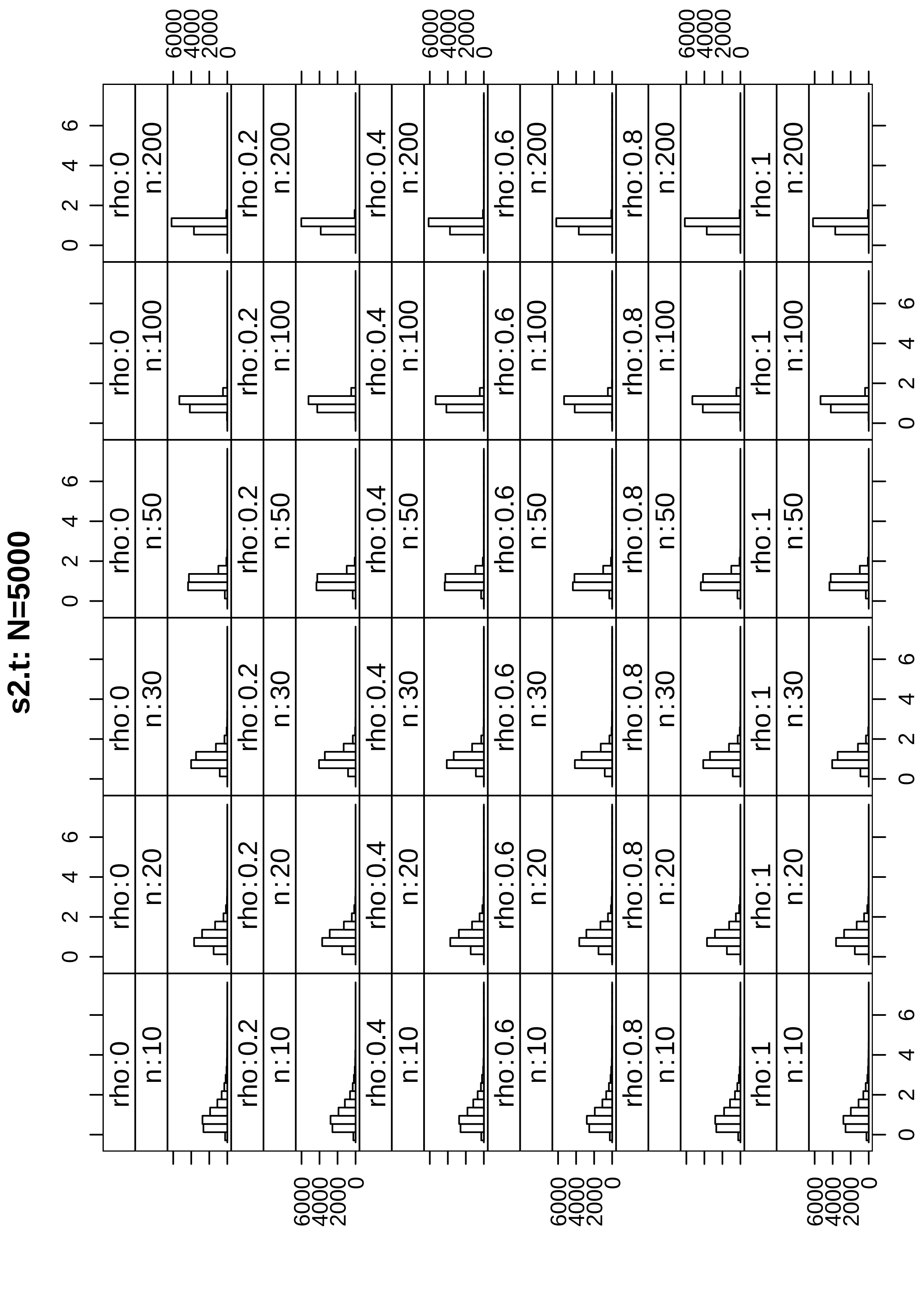


Figure B.48: Distribution of $S_t^2 + S_c^2$ for N=5000

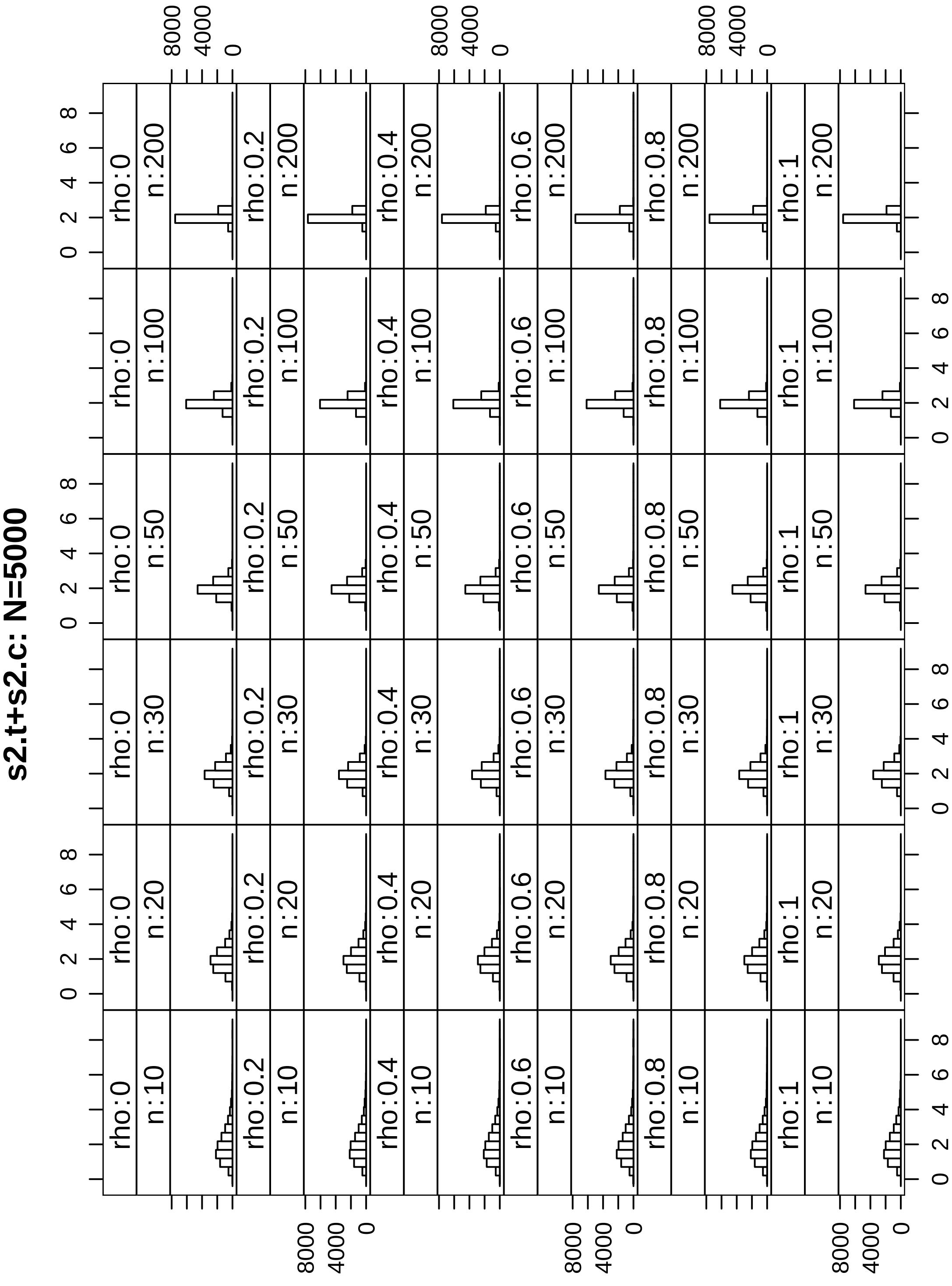


Figure B.49: Distribution of $var(\hat{\mu}_c)$ for N=5000

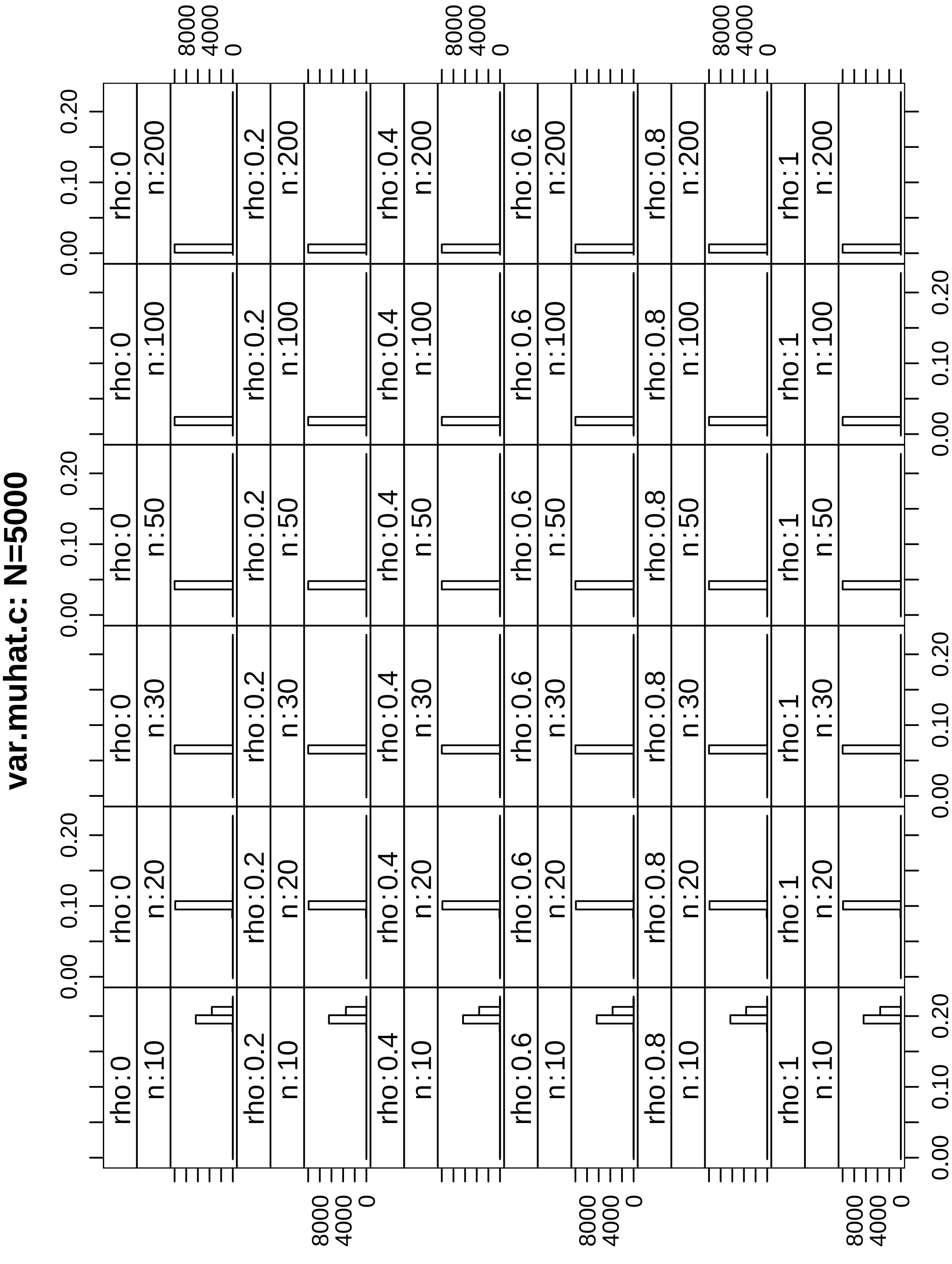


Figure B.50: Distribution of $\widehat{var}(\hat{\mu}_c)$ for N=5000

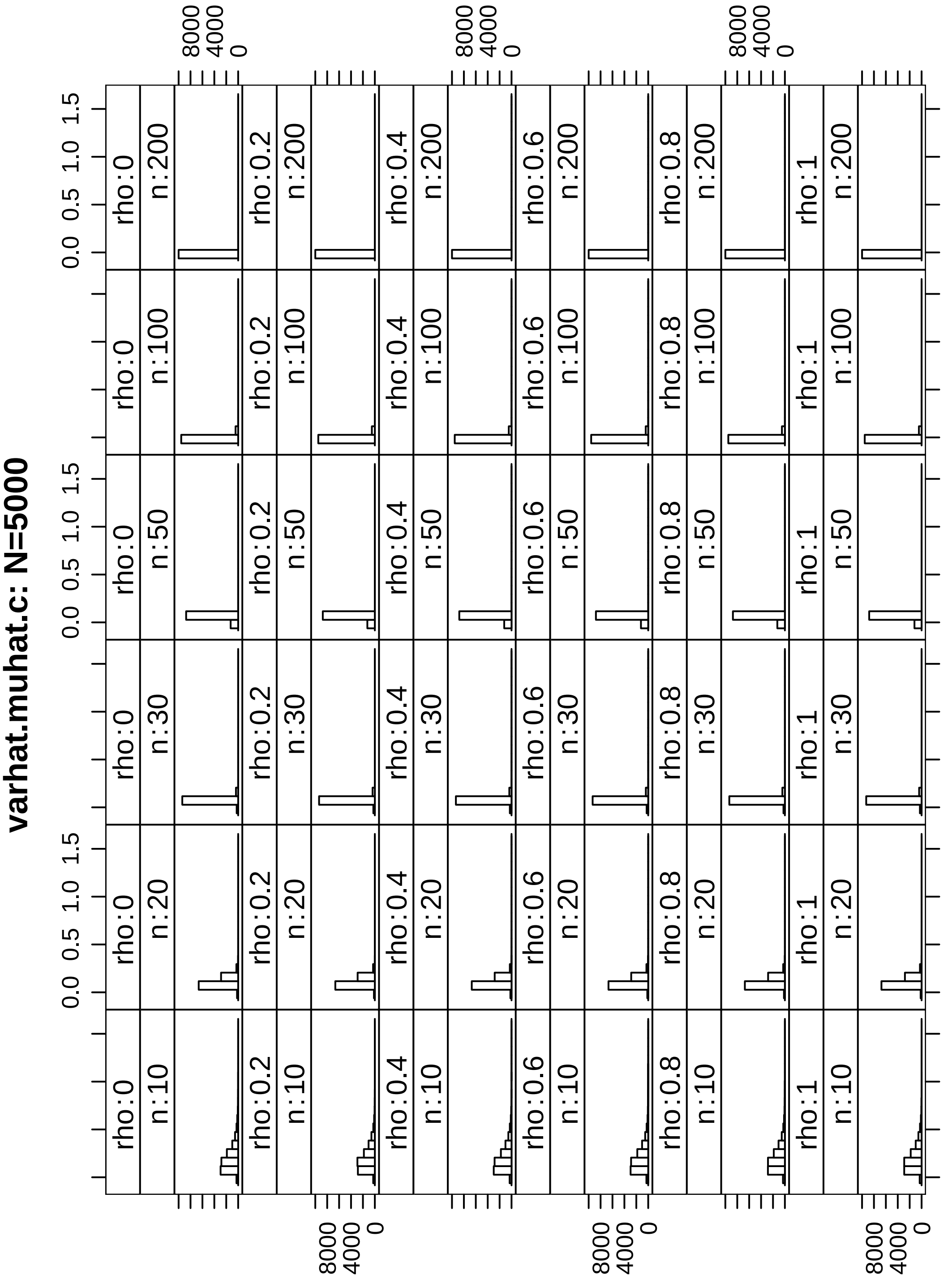


Figure B.51: Distribution of $var(\hat{\mu}_t)$ for N=5000

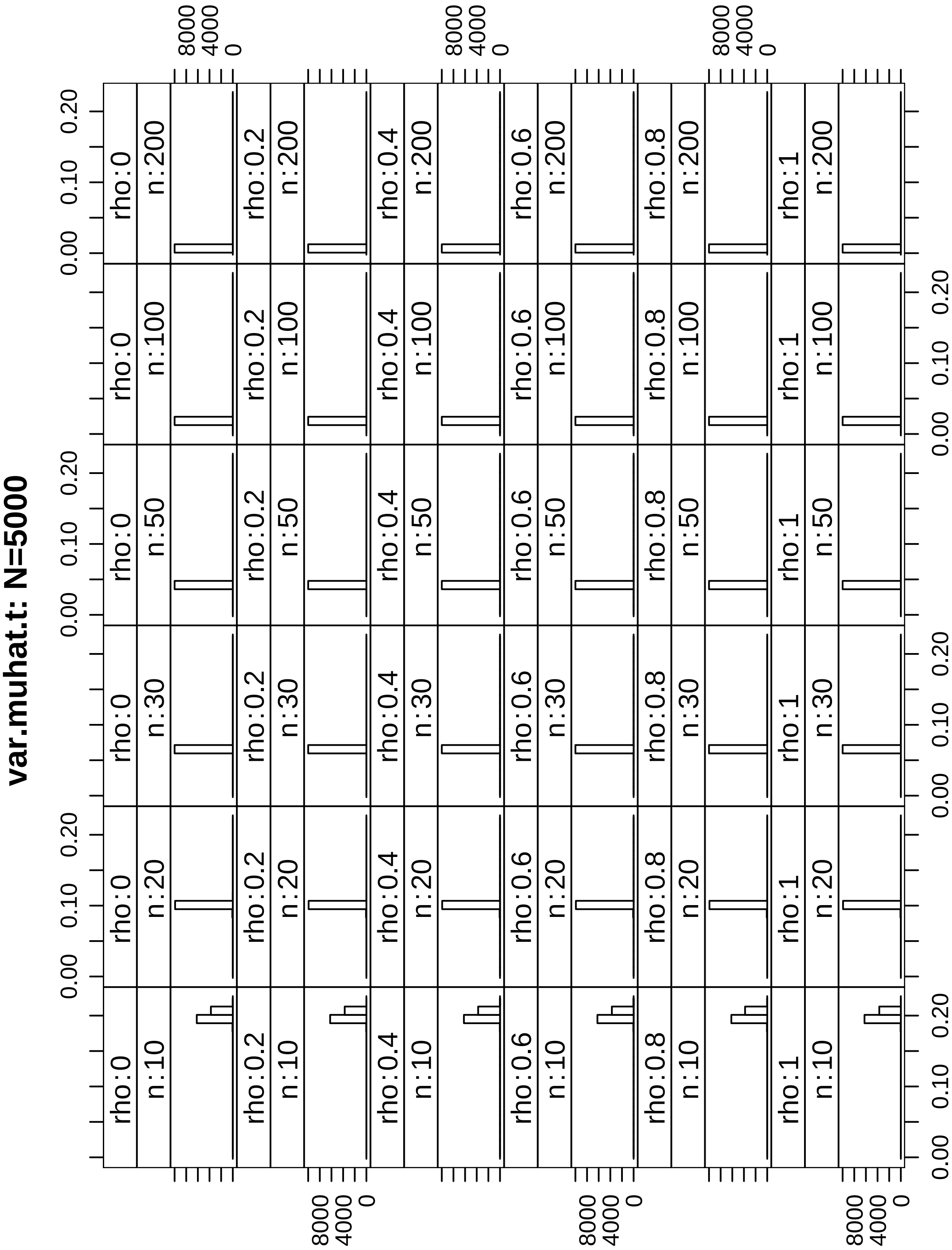


Figure B.52: Distribution of $\widehat{var}(\hat{\mu}_t)$ for N=5000

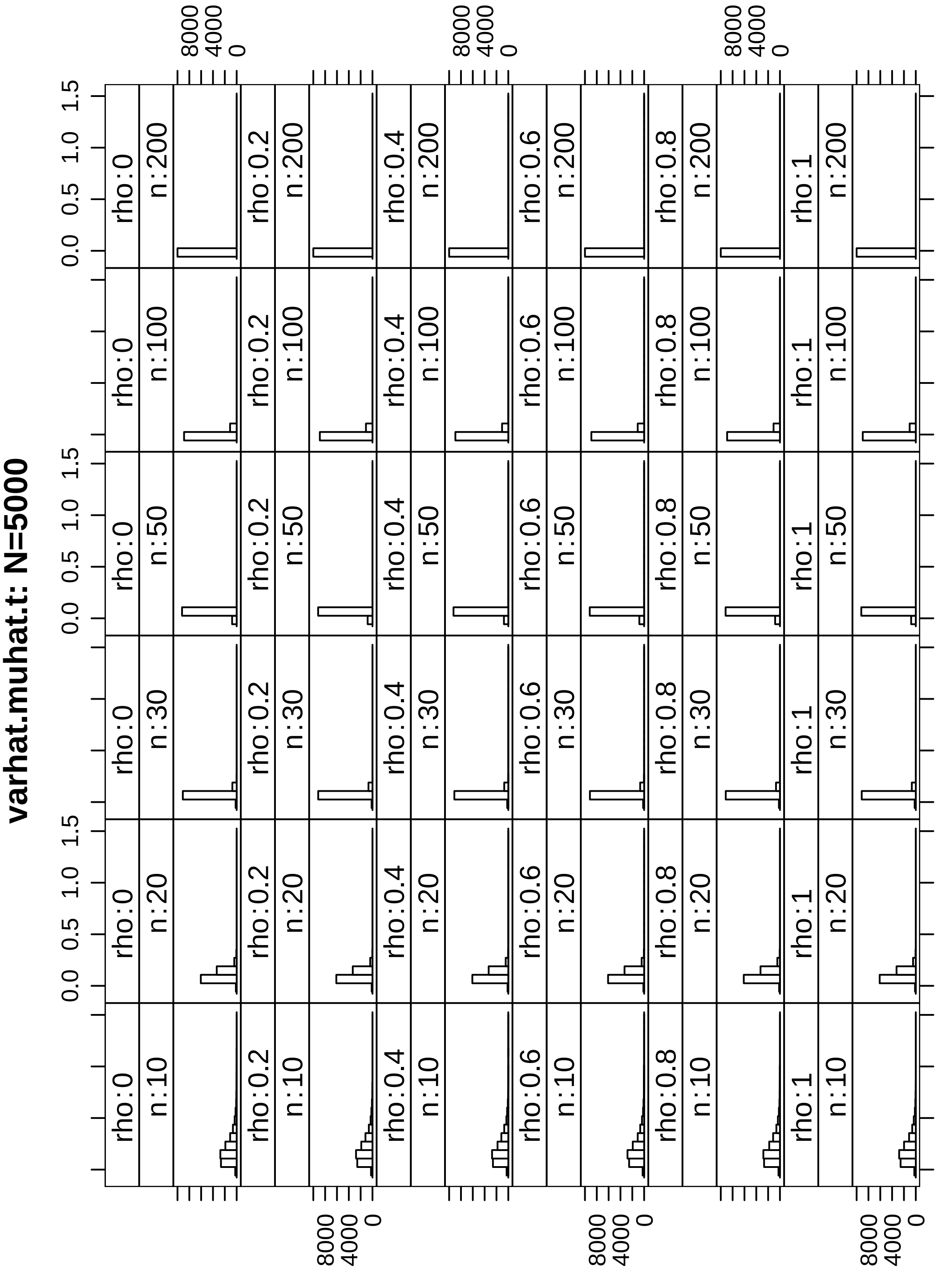


Figure B.53: Distribution of $var(\hat{\mu}_t - \hat{\mu}_c)$ for N=5000

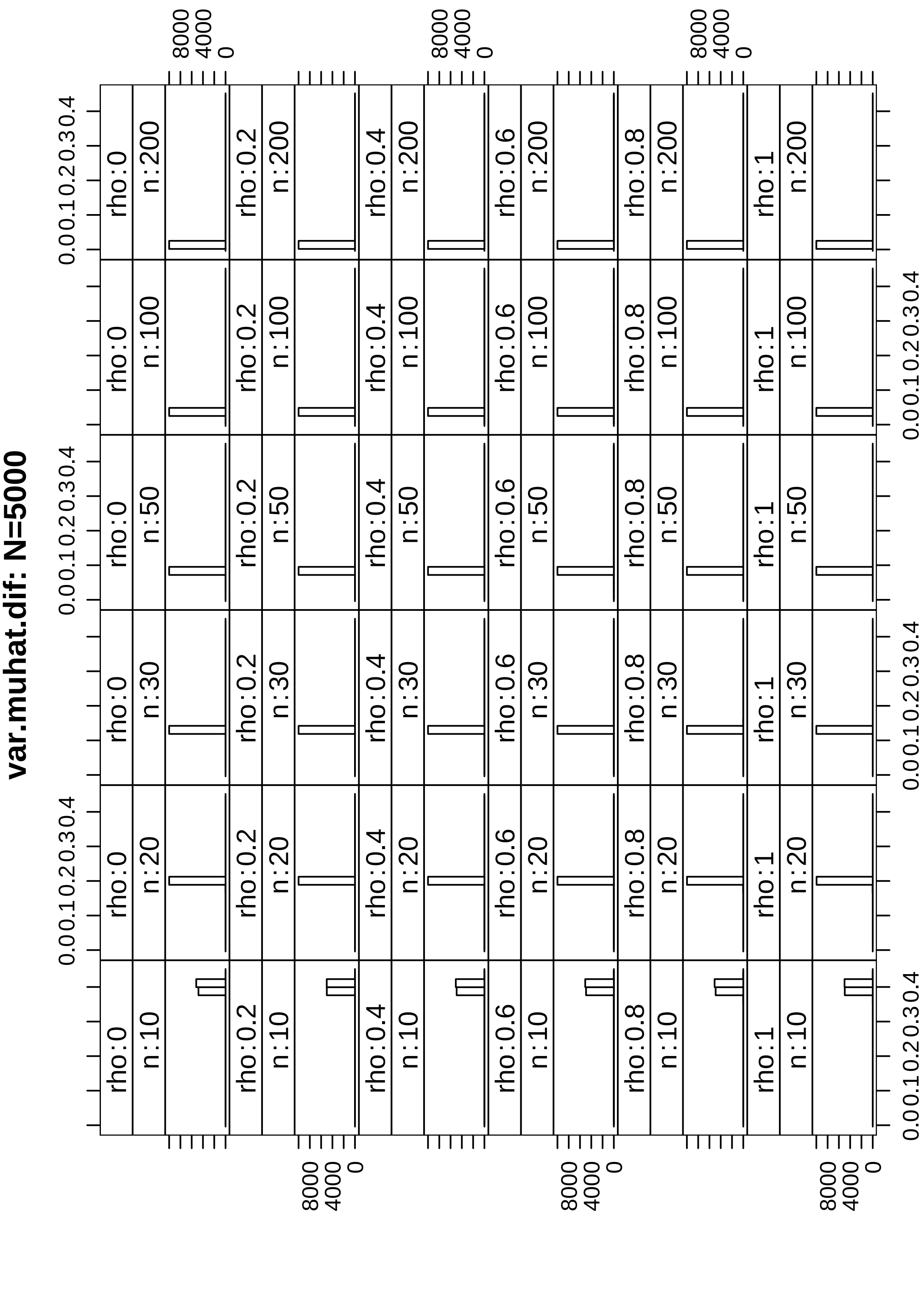
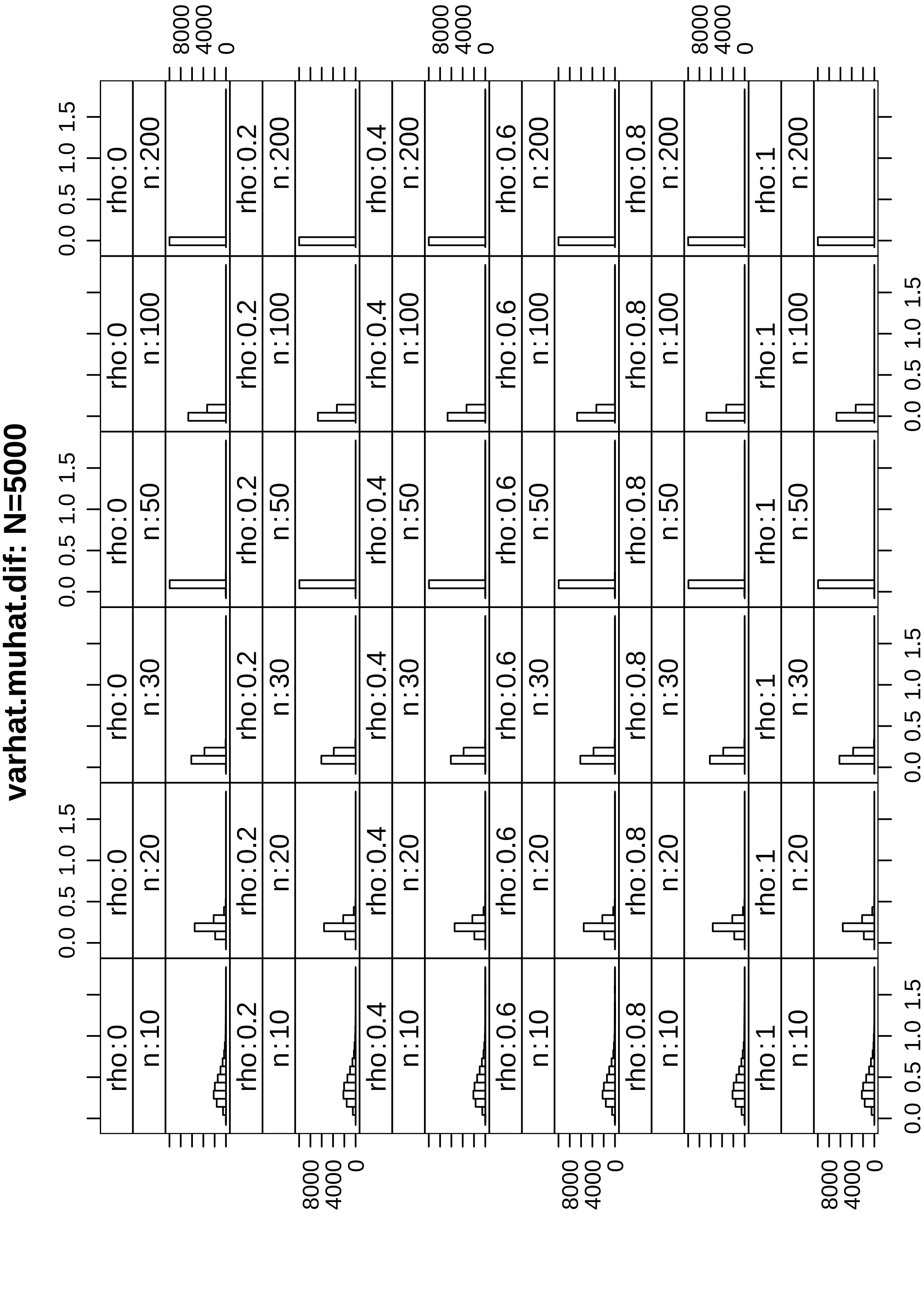


Figure B.54: Distribution of $\widehat{var}(\hat{\mu}_t - \hat{\mu}_c)$ for N=5000



Appendix C

Tables of Coverage and Width

Table C.1: Coverage, N=50

N	n	ρ	ybar.c*	muhat.c	ybar.t	muhat.t	ybar.dif	muhat.dif
50	10	0	95.55	94.96	95.63	95.03	95.82	94.92
50	10	0.2	95.86	95.25	95.37	94.63	95.76	94.94
50	10	0.4	95.77	95.15	95.54	94.81	95.76	94.96
50	10	0.6	95.89	95.23	95.74	95.05	95.6	94.77
50	10	0.8	95.59	94.91	95.91	95.17	95.36	94.61
50	10	1	95.66	95.09	95.54	94.89	95.25	94.34
50	20	0	96.53	94.81	96.6	94.77	96.74	94.89
50	20	0.2	96.71	94.98	96.7	95.08	96.78	94.82
50	20	0.4	96.91	95.18	96.63	94.6	96.14	94.1
50	20	0.6	96.88	95.13	96.98	95.21	96.17	94.21
50	20	0.8	96.85	94.99	96.81	95.22	95.6	93.11
50	20	1	96.7	94.94	96.7	94.72	94.67	92.25
50	30	0	97.79	95.34	97.91	95.25	98.06	95.37
50	30	0.2	97.8	95.07	97.65	94.88	97.42	93.94
50	30	0.4	98.04	95.22	97.8	94.84	96.88	93.24
50	30	0.6	97.65	94.8	97.83	95.07	96.22	92.41
50	30	0.8	97.79	95.04	97.56	94.57	95.44	90.99
50	30	1	97.76	95	97.68	94.97	95.02	90.49

*Note that "ybar.c", "ybar.t" and "ybar.dif" indicate results for the traditional intervals for μ_c, μ_t and $\mu_t - \mu_c$, respectively. Similarly, "muhat.c", "muhat.t" and "muhat.dif" indicate results for the Thompson based intervals for μ_c, μ_t and $\mu_t - \mu_c$, respectively. This applies to all tables in Appendix C.

Table C.2: Coverage, N=100

N	n	ρ	ybar.c	muhat.c	ybar.t	muhat.t	ybar.dif	muhat.dif
100	10	0	95.41	95.03	95.17	94.88	95.39	95.09
100	10	0.2	95.5	95.14	95.16	94.84	95.57	95.22
100	10	0.4	95.73	95.32	95.24	94.98	95.11	94.78
100	10	0.6	95.64	95.27	95.26	94.84	95.23	94.97
100	10	0.8	95.21	94.95	95.43	94.97	95.1	94.69
100	10	1	95.27	94.92	95.39	94.99	94.95	94.52
100	20	0	95.81	94.92	95.94	95.18	95.8	94.93
100	20	0.2	95.87	95.14	95.71	94.9	95.6	94.69
100	20	0.4	95.84	95.05	95.84	94.85	95.56	94.53
100	20	0.6	95.75	95.01	96.04	95.07	95.48	94.35
100	20	0.8	95.92	95.05	96.55	95.62	95.51	94.25
100	20	1	96.12	95.05	95.87	95.02	94.83	93.86
100	30	0	96.8	95.21	96.36	94.79	96.68	95.07
100	30	0.2	96.44	95.1	96.04	94.52	95.79	94.15
100	30	0.4	96.39	94.84	96.85	95.39	96.35	94.71
100	30	0.6	96.56	95.16	96.48	95.21	95.62	93.85
100	30	0.8	96.03	94.43	96.54	95.08	95.03	93.39
100	30	1	96.31	94.79	96.56	95.2	94.97	93.11
100	50	0	97.47	95.08	97.32	95.04	97.45	94.97
100	50	0.2	97.57	95.29	97.61	95.17	97.15	94.66
100	50	0.4	97.53	95.14	97.38	95.22	96.45	93.61
100	50	0.6	97.46	95	97.37	94.73	95.89	92.64
100	50	0.8	97.34	95.14	97.63	95.28	95.6	92.02
100	50	1	97.34	95.07	97.75	95.14	95.65	91.82

Table C.3: Coverage, N=250

N	n	ρ	ybar.c	muhat.c	ybar.t	muhat.t	ybar.dif	muhat.dif
250	10	0	95.35	95.19	95.03	94.9	95.17	95.02
250	10	0.2	95.19	95.05	94.84	94.7	95.06	94.92
250	10	0.4	95.45	95.29	95.13	94.99	95.38	95.18
250	10	0.6	95.1	94.96	94.94	94.74	94.91	94.78
250	10	0.8	94.83	94.73	95.2	95.05	94.91	94.76
250	10	1	95.32	95.18	94.78	94.67	94.75	94.6
250	20	0	95.74	95.31	95.3	94.96	95.82	95.48
250	20	0.2	95.15	94.69	95.2	94.82	95.35	95.02
250	20	0.4	95.56	95.22	95.2	94.86	95.45	95.14
250	20	0.6	95.13	94.77	95	94.77	95.19	94.85
250	20	0.8	95.89	95.5	94.97	94.58	95.11	94.76
250	20	1	95.5	95.11	95.29	94.92	95.16	94.83
250	30	0	95.78	95.28	96	95.31	95.87	95.32
250	30	0.2	96.08	95.56	95.42	94.77	95.42	94.65
250	30	0.4	95.43	94.94	95.61	95.12	95.3	94.72
250	30	0.6	95.79	95.27	95.69	95.09	95.74	95.08
250	30	0.8	95.42	94.87	95.76	95.19	95.15	94.61
250	30	1	95.84	95.26	95.3	94.64	95.07	94.33
250	50	0	95.86	95.01	96.12	95.15	96.03	94.97
250	50	0.2	96.24	95.28	96.25	95.09	96.19	95.05
250	50	0.4	95.99	94.94	96.13	95.1	95.65	94.45
250	50	0.6	96.17	95.18	96.16	95.29	95.61	94.6
250	50	0.8	96.26	95.05	95.86	94.9	95.24	94.02
250	50	1	96.48	95.35	96.03	94.93	94.91	93.82
250	100	0	97.06	95.16	97.1	94.89	97.02	94.88
250	100	0.2	97.14	95.1	97.21	95.15	96.79	94.71
250	100	0.4	96.75	94.94	97.36	95.11	96.52	94.09
250	100	0.6	96.97	94.76	96.99	94.88	95.95	93.24
250	100	0.8	97.21	94.86	96.72	94.73	95.54	92.68
250	100	1	97.12	95.01	96.91	94.85	95.1	92.34

Table C.4: Coverage, N=500

N	n	ρ	ybar.c	muhat.c	ybar.t	muhat.t	ybar.dif	muhat.dif
500	10	0	95.05	94.95	95.08	94.95	95.12	94.99
500	10	0.2	95.12	95	95.34	95.27	94.89	94.85
500	10	0.4	94.74	94.64	95.19	95.17	94.5	94.44
500	10	0.6	95.26	95.23	94.76	94.67	95.02	94.98
500	10	0.8	95.11	95.02	95.18	95.12	94.9	94.84
500	10	1	95.03	95	95.26	95.18	94.9	94.83
500	20	0	95.52	95.42	95.29	95.2	95.67	95.56
500	20	0.2	95.15	94.92	95.2	95	95.38	95.19
500	20	0.4	95.07	94.89	95.43	95.18	95.37	95.16
500	20	0.6	95.06	94.83	95.59	95.38	95.08	94.87
500	20	0.8	95.52	95.28	95.29	95.15	95.39	95.16
500	20	1	95.21	95.08	95.65	95.5	95.11	94.88
500	30	0	95.84	95.62	95.25	94.91	95.38	95.03
500	30	0.2	95.05	94.65	95.57	95.29	95.2	94.93
500	30	0.4	95.34	95.07	95.2	94.86	94.95	94.56
500	30	0.6	95.26	94.94	95.15	94.85	95.37	95.1
500	30	0.8	95.56	95.26	95.15	94.86	95.25	95.1
500	30	1	94.97	94.69	95.1	94.78	94.74	94.46
500	50	0	95.43	94.77	95.68	95.25	95.73	95.14
500	50	0.2	95.41	94.87	95.51	94.99	95.75	95.32
500	50	0.4	95.75	95.29	95.18	94.74	95.39	94.75
500	50	0.6	95.39	94.76	95.76	95.24	95.21	94.7
500	50	0.8	95.2	94.69	95.53	94.99	95.3	94.78
500	50	1	95.64	95.02	95.42	94.88	94.75	94.23
500	100	0	96.27	95.04	96.08	95.16	96.05	95.06
500	100	0.2	96.04	95.08	95.73	94.83	95.87	94.86
500	100	0.4	96	94.82	95.88	94.92	95.82	94.66
500	100	0.6	96.4	95.58	96.37	95.09	95.95	94.78
500	100	0.8	96.03	94.84	96.05	94.84	94.98	93.84
500	100	1	96.04	94.9	96.01	94.97	94.78	93.43
500	200	0	97.15	94.92	97.18	95.09	97.04	95.04
500	200	0.2	97.14	95.06	96.99	94.97	97.08	94.89
500	200	0.4	97.28	94.87	97.22	95.27	96.61	94.53
500	200	0.6	97.28	95.24	97.09	94.87	96.14	93.61
500	200	0.8	97.14	94.97	97.19	94.89	95.09	92.55
500	200	1	97.12	94.85	97.11	95.19	95.05	92.1

Table C.5: Coverage, N=1000

N	n	ρ	ybar.c	muhat.c	ybar.t	muhat.t	ybar.dif	muhat.dif
1000	10	0	95.05	95.05	94.82	94.77	94.86	94.83
1000	10	0.2	94.96	94.93	95.02	95	95.16	95.13
1000	10	0.4	95.16	95.14	95.03	94.99	95.14	95.12
1000	10	0.6	95.18	95.17	94.91	94.89	94.78	94.75
1000	10	0.8	94.88	94.8	94.77	94.74	94.86	94.84
1000	10	1	94.92	94.87	94.97	94.97	94.98	94.94
1000	20	0	95.26	95.21	95.1	94.97	95.21	95.05
1000	20	0.2	94.88	94.78	95.1	94.97	95.14	95.04
1000	20	0.4	94.85	94.72	95.03	94.94	94.88	94.8
1000	20	0.6	95.01	94.91	95.27	95.16	95.27	95.17
1000	20	0.8	95.5	95.42	95.05	94.96	94.66	94.51
1000	20	1	94.96	94.85	94.98	94.9	94.95	94.85
1000	30	0	95.38	95.25	95.56	95.41	95.58	95.39
1000	30	0.2	94.72	94.66	95.12	94.96	94.84	94.74
1000	30	0.4	95.46	95.3	95.29	95.17	95.33	95.19
1000	30	0.6	95.18	95.03	94.82	94.67	95.01	94.87
1000	30	0.8	95.05	94.92	95.13	95	95.04	94.87
1000	30	1	95.44	95.27	95.21	95.05	95.44	95.28
1000	50	0	95.37	95.19	95.34	95.11	95.4	95.09
1000	50	0.2	95.35	95.1	95.28	95.05	95.26	95.02
1000	50	0.4	95.41	95.13	95.2	94.9	95.52	95.32
1000	50	0.6	94.94	94.65	95.08	94.88	94.81	94.46
1000	50	0.8	95.17	94.83	95.35	95.14	95.05	94.68
1000	50	1	94.91	94.65	95.43	95.21	94.6	94.24
1000	100	0	95.61	95.09	95.8	95.36	95.54	95.04
1000	100	0.2	95.05	94.53	95.26	94.51	95.09	94.46
1000	100	0.4	95.36	94.82	95.74	95.18	95.56	95.12
1000	100	0.6	95.53	95.02	95.92	95.32	95.41	94.85
1000	100	0.8	95.47	94.91	95.74	95.18	94.61	94.2
1000	100	1	95.48	94.89	95.7	95.06	94.83	94.42
1000	200	0	96.16	95.15	96	95.13	96.15	94.95
1000	200	0.2	95.98	94.83	96.21	95.23	95.95	94.86
1000	200	0.4	96.03	94.77	96.54	95.48	96.06	94.84
1000	200	0.6	96.33	95.21	96.49	95.41	96.03	94.82
1000	200	0.8	95.99	94.77	95.89	94.72	95.3	93.94
1000	200	1	95.69	94.51	95.49	94.37	94.83	93.29

Table C.6: Coverage, N=5000

N	n	ρ	ybar.c	muhat.c	ybar.t	muhat.t	ybar.dif	muhat.dif
5000	10	0	95.22	95.19	94.9	94.9	95.06	95.05
5000	10	0.2	95.08	95.07	95.26	95.24	95.25	95.24
5000	10	0.4	94.91	94.9	95.41	95.4	95.25	95.23
5000	10	0.6	94.86	94.85	95.12	95.12	95.13	95.12
5000	10	0.8	95.07	95.06	94.96	94.96	94.86	94.86
5000	10	1	94.88	94.88	94.83	94.83	94.87	94.86
5000	20	0	95.17	95.16	95.06	95.03	95.01	95
5000	20	0.2	94.78	94.76	95.13	95.12	94.85	94.84
5000	20	0.4	95.4	95.38	95.04	95.02	95.08	95.06
5000	20	0.6	95.28	95.25	94.6	94.59	95.09	95.08
5000	20	0.8	95.27	95.27	94.8	94.78	94.88	94.86
5000	20	1	94.98	94.98	95.16	95.14	94.82	94.81
5000	30	0	94.92	94.85	94.8	94.76	94.8	94.78
5000	30	0.2	95.21	95.18	95.02	94.98	95.12	95.1
5000	30	0.4	95.18	95.14	94.91	94.88	95.07	95.02
5000	30	0.6	95.07	95.04	95.52	95.46	94.97	94.92
5000	30	0.8	95.01	94.98	94.54	94.51	94.88	94.82
5000	30	1	95.27	95.25	94.66	94.63	94.87	94.85
5000	50	0	95.14	95.07	95.07	95.03	95.45	95.36
5000	50	0.2	95.04	95	94.87	94.84	95.05	95
5000	50	0.4	95.44	95.39	94.98	94.92	95.12	95.05
5000	50	0.6	94.88	94.83	95.45	95.42	94.94	94.89
5000	50	0.8	94.97	94.9	94.88	94.83	94.89	94.84
5000	50	1	94.91	94.88	95.06	94.99	95.05	94.97
5000	100	0	94.94	94.84	95.03	94.88	94.73	94.65
5000	100	0.2	94.74	94.56	95.2	95.14	95.12	94.94
5000	100	0.4	95.17	95.08	95.59	95.47	95.04	94.92
5000	100	0.6	95.29	95.18	95.11	94.99	94.88	94.81
5000	100	0.8	95.31	95.16	95.21	95.11	95.15	95.05
5000	100	1	94.85	94.69	95.01	94.92	95.19	94.98
5000	200	0	95.13	94.95	95.04	94.8	95.36	95.15
5000	200	0.2	94.99	94.77	95.5	95.27	95.27	95.04
5000	200	0.4	95.69	95.53	95.21	95.03	95.39	95.05
5000	200	0.6	95.32	95.07	95.37	95.2	95.01	94.77
5000	200	0.8	95.53	95.33	95.17	94.86	95.1	94.89
5000	200	1	95.03	94.82	95.14	94.91	94.78	94.64

Table C.7: Width, N=50

N	n	ρ	ybar.c	muhat.c	ybar.t	muhat.t	ybar.dif	muhat.dif
50	10	0	2.3395	2.2195	2.3372	2.2173	2.8334	2.6879
50	10	0.2	2.3168	2.1979	2.3327	2.2130	2.8174	2.6727
50	10	0.4	2.3359	2.2160	2.3285	2.2090	2.8255	2.6801
50	10	0.6	2.3350	2.2152	2.3371	2.2172	2.8307	2.6854
50	10	0.8	2.3334	2.2137	2.3469	2.2265	2.8343	2.6884
50	10	1	2.3251	2.2057	2.3280	2.2085	2.8200	2.6753
50	20	0	1.3932	1.2461	1.3906	1.2438	1.8540	1.6583
50	20	0.2	1.3880	1.2415	1.3903	1.2436	1.8505	1.6552
50	20	0.4	1.3946	1.2473	1.3926	1.2455	1.8558	1.6599
50	20	0.6	1.3904	1.2436	1.3952	1.2479	1.8552	1.6593
50	20	0.8	1.3910	1.2441	1.3964	1.2490	1.8560	1.6600
50	20	1	1.3855	1.2392	1.3947	1.2474	1.8517	1.6561
50	30	0	1.0887	0.9108	1.0919	0.9136	1.4857	1.2431
50	30	0.2	1.0866	0.9091	1.0898	0.9118	1.4827	1.2406
50	30	0.4	1.0873	0.9097	1.0884	0.9106	1.4827	1.2406
50	30	0.6	1.0873	0.9097	1.0866	0.9091	1.4810	1.2391
50	30	0.8	1.0875	0.9099	1.0905	0.9123	1.4841	1.2418
50	30	1	1.0861	0.9087	1.0880	0.9103	1.4814	1.2394

Table C.8: Width, N=100

N	n	ρ	ybar.c	muhat.c	ybar.t	muhat.t	ybar.dif	muhat.dif
100	10	0	2.3419	2.2826	2.3352	2.2761	2.8339	2.7622
100	10	0.2	2.3298	2.2708	2.3304	2.2714	2.8227	2.7512
100	10	0.4	2.3170	2.2584	2.3408	2.2815	2.8215	2.7501
100	10	0.6	2.3435	2.2842	2.3278	2.2689	2.8281	2.7565
100	10	0.8	2.3306	2.2716	2.3440	2.2846	2.8318	2.7601
100	10	1	2.3312	2.2722	2.3321	2.2730	2.8232	2.7517
100	20	0	1.3918	1.3204	1.3906	1.3192	1.8524	1.7574
100	20	0.2	1.3939	1.3224	1.3896	1.3183	1.8533	1.7581
100	20	0.4	1.3880	1.3168	1.3922	1.3207	1.8512	1.7562
100	20	0.6	1.3929	1.3214	1.3927	1.3213	1.8551	1.7598
100	20	0.8	1.3879	1.3167	1.3956	1.3240	1.8533	1.7582
100	20	1	1.3985	1.3267	1.3908	1.3194	1.8574	1.7621
100	30	0	1.0898	1.0048	1.0913	1.0061	1.4859	1.3700
100	30	0.2	1.0867	1.0019	1.0905	1.0054	1.4838	1.3681
100	30	0.4	1.0853	1.0006	1.0852	1.0005	1.4790	1.3635
100	30	0.6	1.0876	1.0027	1.0860	1.0013	1.4812	1.3656
100	30	0.8	1.0869	1.0021	1.0892	1.0042	1.4825	1.3668
100	30	1	1.0894	1.0044	1.0917	1.0065	1.4863	1.3703
100	50	0	0.8156	0.7064	0.8187	0.7090	1.1316	0.9800
100	50	0.2	0.8169	0.7074	0.8168	0.7074	1.1312	0.9796
100	50	0.4	0.8176	0.7081	0.8175	0.7080	1.1322	0.9806
100	50	0.6	0.8187	0.7090	0.8169	0.7075	1.1327	0.9810
100	50	0.8	0.8176	0.7081	0.8168	0.7074	1.1318	0.9802
100	50	1	0.8186	0.7089	0.8157	0.7065	1.1317	0.9801

Table C.9: Width, N=250

N	n	ρ	ybar.c	muhat.c	ybar.t	muhat.t	ybar.dif	muhat.dif
250	10	0	2.3540	2.3303	2.3292	2.3058	2.8346	2.8061
250	10	0.2	2.3286	2.3052	2.3279	2.3045	2.8206	2.7923
250	10	0.4	2.3444	2.3209	2.3242	2.3008	2.8267	2.7983
250	10	0.6	2.3387	2.3152	2.3120	2.2887	2.8178	2.7894
250	10	0.8	2.3323	2.3088	2.3379	2.3144	2.8291	2.8007
250	10	1	2.3331	2.3096	2.3321	2.3087	2.8276	2.7992
250	20	0	1.3965	1.3683	1.3861	1.3581	1.8530	1.8155
250	20	0.2	1.3906	1.3625	1.3933	1.3652	1.8538	1.8164
250	20	0.4	1.3919	1.3638	1.3900	1.3619	1.8523	1.8148
250	20	0.6	1.3935	1.3653	1.3930	1.3648	1.8555	1.8180
250	20	0.8	1.3933	1.3651	1.3928	1.3647	1.8551	1.8177
250	20	1	1.3928	1.3646	1.3879	1.3598	1.8510	1.8136
250	30	0	1.0908	1.0575	1.0902	1.0570	1.4861	1.4408
250	30	0.2	1.0868	1.0537	1.0873	1.0542	1.4815	1.4364
250	30	0.4	1.0888	1.0557	1.0925	1.0593	1.4864	1.4410
250	30	0.6	1.0882	1.0550	1.0903	1.0571	1.4847	1.4394
250	30	0.8	1.0866	1.0535	1.0888	1.0556	1.4824	1.4372
250	30	1	1.0850	1.0520	1.0871	1.0540	1.4801	1.4350
250	50	0	0.8178	0.7758	0.8172	0.7753	1.1322	1.0741
250	50	0.2	0.8169	0.7750	0.8169	0.7749	1.1314	1.0734
250	50	0.4	0.8167	0.7748	0.8170	0.7750	1.1313	1.0733
250	50	0.6	0.8164	0.7745	0.8177	0.7758	1.1316	1.0735
250	50	0.8	0.8169	0.7749	0.8159	0.7740	1.1307	1.0727
250	50	1	0.8163	0.7744	0.8155	0.7736	1.1300	1.0720
250	100	0	0.5658	0.5060	0.5659	0.5061	0.7922	0.7086
250	100	0.2	0.5662	0.5064	0.5652	0.5055	0.7920	0.7084
250	100	0.4	0.5652	0.5055	0.5655	0.5058	0.7916	0.7080
250	100	0.6	0.5657	0.5060	0.5652	0.5055	0.7917	0.7081
250	100	0.8	0.5662	0.5065	0.5662	0.5065	0.7928	0.7091
250	100	1	0.5651	0.5054	0.5645	0.5049	0.7907	0.7072

Table C.10: Width, N=500

N	n	ρ	ybar.c	muhat.c	ybar.t	muhat.t	ybar.dif	muhat.dif
500	10	0	2.3329	2.3212	2.3252	2.3136	2.8200	2.8058
500	10	0.2	2.3398	2.3281	2.3434	2.3317	2.8348	2.8206
500	10	0.4	2.3240	2.3123	2.3396	2.3278	2.8234	2.8093
500	10	0.6	2.3324	2.3207	2.3397	2.3280	2.8299	2.8157
500	10	0.8	2.3393	2.3275	2.3310	2.3193	2.8295	2.8154
500	10	1	2.3452	2.3334	2.3293	2.3177	2.8319	2.8177
500	20	0	1.3934	1.3794	1.3921	1.3781	1.8551	1.8365
500	20	0.2	1.3967	1.3827	1.3843	1.3704	1.8519	1.8333
500	20	0.4	1.3905	1.3765	1.3895	1.3755	1.8516	1.8330
500	20	0.6	1.3875	1.3736	1.3940	1.3799	1.8517	1.8331
500	20	0.8	1.3912	1.3772	1.3899	1.3759	1.8522	1.8336
500	20	1	1.3950	1.3810	1.3963	1.3822	1.8583	1.8397
500	30	0	1.0885	1.0721	1.0884	1.0720	1.4830	1.4606
500	30	0.2	1.0865	1.0701	1.0880	1.0715	1.4816	1.4593
500	30	0.4	1.0885	1.0720	1.0862	1.0698	1.4820	1.4596
500	30	0.6	1.0851	1.0687	1.0871	1.0707	1.4801	1.4577
500	30	0.8	1.0882	1.0717	1.0872	1.0707	1.4824	1.4600
500	30	1	1.0907	1.0743	1.0881	1.0716	1.4846	1.4622
500	50	0	0.8161	0.7955	0.8181	0.7974	1.1316	1.1030
500	50	0.2	0.8193	0.7986	0.8167	0.7960	1.1329	1.1042
500	50	0.4	0.8172	0.7965	0.8167	0.7960	1.1314	1.1027
500	50	0.6	0.8179	0.7972	0.8167	0.7961	1.1318	1.1032
500	50	0.8	0.8167	0.7960	0.8180	0.7973	1.1322	1.1035
500	50	1	0.8175	0.7968	0.8155	0.7949	1.1307	1.1021
500	100	0	0.5667	0.5376	0.5659	0.5368	0.7929	0.7522
500	100	0.2	0.5656	0.5366	0.5655	0.5365	0.7919	0.7512
500	100	0.4	0.5656	0.5366	0.5646	0.5356	0.7912	0.7506
500	100	0.6	0.5650	0.5360	0.5658	0.5368	0.7917	0.7510
500	100	0.8	0.5660	0.5369	0.5657	0.5366	0.7922	0.7516
500	100	1	0.5664	0.5373	0.5654	0.5363	0.7922	0.7516
500	200	0	0.3954	0.3536	0.3964	0.3545	0.5571	0.4983
500	200	0.2	0.3961	0.3543	0.3959	0.3541	0.5573	0.4985
500	200	0.4	0.3958	0.3540	0.3960	0.3542	0.5571	0.4983
500	200	0.6	0.3956	0.3539	0.3958	0.3540	0.5569	0.4981
500	200	0.8	0.3957	0.3540	0.3956	0.3539	0.5569	0.4981
500	200	1	0.3960	0.3542	0.3958	0.3540	0.5571	0.4983

Table C.11: Width, N=1000

N	n	ρ	ybar.c	muhat.c	ybar.t	muhat.t	ybar.dif	muhat.dif
1000	10	0	2.3414	2.3356	2.3296	2.3237	2.8270	2.8200
1000	10	0.2	2.3237	2.3179	2.3366	2.3307	2.8232	2.8161
1000	10	0.4	2.3337	2.3279	2.3271	2.3213	2.8202	2.8132
1000	10	0.6	2.3325	2.3267	2.3379	2.3320	2.8267	2.8197
1000	10	0.8	2.3414	2.3355	2.3311	2.3253	2.8292	2.8221
1000	10	1	2.3409	2.3350	2.3272	2.3214	2.8286	2.8216
1000	20	0	1.3950	1.3880	1.3906	1.3836	1.8550	1.8457
1000	20	0.2	1.3888	1.3818	1.3958	1.3888	1.8543	1.8450
1000	20	0.4	1.3934	1.3864	1.3936	1.3866	1.8567	1.8474
1000	20	0.6	1.3993	1.3923	1.3873	1.3803	1.8563	1.8470
1000	20	0.8	1.3946	1.3876	1.3875	1.3806	1.8521	1.8428
1000	20	1	1.3927	1.3857	1.3896	1.3827	1.8532	1.8439
1000	30	0	1.0888	1.0806	1.0875	1.0793	1.4829	1.4717
1000	30	0.2	1.0869	1.0787	1.0880	1.0798	1.4821	1.4710
1000	30	0.4	1.0864	1.0782	1.0862	1.0780	1.4801	1.4690
1000	30	0.6	1.0902	1.0820	1.0860	1.0778	1.4825	1.4713
1000	30	0.8	1.0883	1.0801	1.0885	1.0803	1.4833	1.4721
1000	30	1	1.0875	1.0793	1.0906	1.0824	1.4841	1.4729
1000	50	0	0.8171	0.8069	0.8167	0.8064	1.1313	1.1171
1000	50	0.2	0.8173	0.8070	0.8176	0.8074	1.1321	1.1179
1000	50	0.4	0.8162	0.8059	0.8171	0.8068	1.1310	1.1168
1000	50	0.6	0.8165	0.8062	0.8190	0.8087	1.1325	1.1183
1000	50	0.8	0.8160	0.8057	0.8164	0.8061	1.1303	1.1161
1000	50	1	0.8184	0.8081	0.8179	0.8076	1.1328	1.1186
1000	100	0	0.5650	0.5507	0.5658	0.5514	0.7916	0.7716
1000	100	0.2	0.5650	0.5507	0.5651	0.5508	0.7911	0.7711
1000	100	0.4	0.5659	0.5516	0.5654	0.5511	0.7920	0.7719
1000	100	0.6	0.5659	0.5516	0.5663	0.5520	0.7926	0.7725
1000	100	0.8	0.5658	0.5515	0.5659	0.5516	0.7922	0.7722
1000	100	1	0.5661	0.5518	0.5662	0.5519	0.7927	0.7727
1000	200	0	0.3959	0.3756	0.3956	0.3753	0.5569	0.5283
1000	200	0.2	0.3962	0.3758	0.3957	0.3754	0.5572	0.5286
1000	200	0.4	0.3960	0.3757	0.3959	0.3756	0.5572	0.5286
1000	200	0.6	0.3963	0.3759	0.3955	0.3752	0.5571	0.5285
1000	200	0.8	0.3957	0.3754	0.3954	0.3751	0.5566	0.5281
1000	200	1	0.3959	0.3756	0.3956	0.3753	0.5570	0.5284

Table C.12: Width, N=5000

N	n	ρ	ybar.c	muhat.c	ybar.t	muhat.t	ybar.dif	muhat.dif
5000	10	0	2.3389	2.3377	2.3402	2.3390	2.8338	2.8324
5000	10	0.2	2.3465	2.3454	2.3497	2.3485	2.8427	2.8413
5000	10	0.4	2.3225	2.3214	2.3449	2.3437	2.8264	2.8250
5000	10	0.6	2.3263	2.3252	2.3509	2.3497	2.8317	2.8303
5000	10	0.8	2.3489	2.3478	2.3304	2.3293	2.8328	2.8314
5000	10	1	2.3225	2.3213	2.3308	2.3296	2.8180	2.8166
5000	20	0	1.3944	1.3930	1.3914	1.3900	1.8546	1.8527
5000	20	0.2	1.4006	1.3992	1.3865	1.3851	1.8558	1.8539
5000	20	0.4	1.3934	1.3920	1.3936	1.3922	1.8557	1.8539
5000	20	0.6	1.3953	1.3939	1.3900	1.3886	1.8545	1.8527
5000	20	0.8	1.3929	1.3915	1.3895	1.3881	1.8530	1.8511
5000	20	1	1.3908	1.3894	1.3955	1.3941	1.8555	1.8536
5000	30	0	1.0867	1.0850	1.0925	1.0909	1.4848	1.4826
5000	30	0.2	1.0860	1.0844	1.0908	1.0892	1.4832	1.4810
5000	30	0.4	1.0906	1.0890	1.0874	1.0858	1.4840	1.4818
5000	30	0.6	1.0891	1.0874	1.0900	1.0883	1.4848	1.4825
5000	30	0.8	1.0880	1.0864	1.0876	1.0860	1.4824	1.4801
5000	30	1	1.0890	1.0873	1.0853	1.0837	1.4815	1.4793
5000	50	0	0.8173	0.8152	0.8179	0.8159	1.1321	1.1293
5000	50	0.2	0.8171	0.8151	0.8155	0.8135	1.1306	1.1278
5000	50	0.4	0.8184	0.8164	0.8175	0.8155	1.1326	1.1298
5000	50	0.6	0.8178	0.8158	0.8168	0.8147	1.1319	1.1290
5000	50	0.8	0.8173	0.8152	0.8155	0.8135	1.1307	1.1279
5000	50	1	0.8168	0.8148	0.8169	0.8149	1.1313	1.1285
5000	100	0	0.5652	0.5624	0.5659	0.5630	0.7918	0.7879
5000	100	0.2	0.5660	0.5632	0.5651	0.5623	0.7919	0.7879
5000	100	0.4	0.5660	0.5631	0.5655	0.5626	0.7921	0.7881
5000	100	0.6	0.5653	0.5625	0.5657	0.5629	0.7918	0.7878
5000	100	0.8	0.5657	0.5629	0.5649	0.5620	0.7914	0.7875
5000	100	1	0.5658	0.5630	0.5647	0.5619	0.7914	0.7874
5000	200	0	0.3959	0.3919	0.3958	0.3919	0.5571	0.5515
5000	200	0.2	0.3960	0.3921	0.3952	0.3912	0.5568	0.5512
5000	200	0.4	0.3959	0.3920	0.3956	0.3917	0.5570	0.5514
5000	200	0.6	0.3950	0.3910	0.3957	0.3917	0.5564	0.5508
5000	200	0.8	0.3958	0.3918	0.3959	0.3919	0.5571	0.5515
5000	200	1	0.3959	0.3919	0.3958	0.3918	0.5571	0.5515

Appendix D

R Code for Simulations

The following is the R code used for generating and evaluating simulation data in this study. There are two functions. The first function, `sim.fun`, generates bivariate normal population using the R function `mvrnorm`, selects a random sample using the R function `sample`, and calculates $\bar{y}_c$, $\bar{y}_t$, S_c^2, S_t^2, $\hat{\mu}_c$, $\hat{\mu}_t$, $\widehat{var}(\hat{\mu}_c)$, $\widehat{var}(\hat{\mu}_t)$, $\bar{y}_t - \bar{y}_c$, $\hat{\mu}_t - \hat{\mu}_c$, and $\widehat{var}(\hat{\mu}_t - \hat{\mu}_c)$. The second function, `results.fun`, calculated confidence intervals for μ_c, μ_t, and $\mu_t - \mu_c$ and calculates the coverage and width of those intervals. See Chapter 5 for more details regarding the choice of intervals calculated.

```
#####################################################################
############################ SRSWOR CRD #############################
############################### Normal ##############################
#####################################################################
######################## Simulation Function ########################
#####################################################################
This function simulates two normal distributions, one for treatment
and one for control, via a bivariate normal distribution with
mean(cont)=mean(trt)=0, var(cont)=var(trt)=1, and cov(cont, trt)=rho
based on the user's input.  The function calculates sample the
population mean and variance, ybar, muhat, and varhat(muhat) for
both control and treatment, and calculates varhat(muhat.diff).  Note
that, some of the calculations for varhat are done using matrix
multiplication.  The function takes N, the desired population size,
n, the desired sample size, and rho, the desired amount of
covariance between the control and treatment responses.
#####################################################################
Note: In order to run this function, in particular mvnorm, package
MASS must be loaded.
#####################################################################

sim.fun <- function(N, n, rho) {

##generate a population of control and treatment responses via a
##multivariate normal distribution (must load pkg MASS)

sigma <- matrix(c(1,rho,rho,1), 2, 2)

pop <- mvrnorm(N, c(0,0), sigma)


##calculate population mean and variance for control and treatment

mu.c <- mean(pop[,1])
```

```
mu.t <- mean(pop[,2])

mu.dif <- mean(pop[,2]-pop[,1])

var.c <- var(pop[,1])

var.t <- var(pop[,2])

var.dif <- var(pop[,2]-pop[,1])


## collect sample from 1 to N (unit identifiers)

s <- sample(1:N, n)


##assign first half of sample to control and second half to
##treatment note that the sample is still unsorted so randomness
##is not violated

sample.c <- pop[s[1:(n/2)],1]

sample.t <- pop[s[(n/2+1):n],2]


##calculate standard statistics

ybar.c <- 2*sum(sample.c)/n

ybar.t <- 2*sum(sample.t)/n

s2.c <- (2*sum(sample.c^2) - n*ybar.c^2)/(n-2)

s2.t <- (2*sum(sample.t^2) - n*ybar.t^2)/(n-2)


##calculate Thompson's estimates

muhat.c <- 2*sum(sample.c)/n

muhat.t <- 2*sum(sample.t)/n
```

```
##calculate variance of Thompson's estimates

var.muhat.c <- sum(pop[,1]^2)*(2*N-n)/(n*N^2) +
    (sum(pop[,1])^2-sum(pop[,1]^2))*(n-2*N)/(n*N^2*(N-1))

var.muhat.t <- sum(pop[,2]^2)*(2*N-n)/(n*N^2) +
    (sum(pop[,2])^2-sum(pop[,2]^2))*(n-2*N)/(n*N^2*(N-1))

var.muhat.dif <- var.muhat.c + var.muhat.t -
    sum(pop[,1])*sum(pop[,2])*2/((N-1)*N^2)


## calculate variance estimates of Thompson's estimates

varhat.muhat.c <- sum(sample.c^2)*(4*N-2*n)/(N*n^2) +
    (sum(sample.c)^2-sum(sample.c^2))*(4*n-8*N)/(N*(n-2)*n^2)

varhat.muhat.t <- sum(sample.t^2)*(4*N-2*n)/(N*n^2) +
    (sum(sample.t)^2-sum(sample.t^2))*(4*n-8*N)/(N*(n-2)*n^2)

varhat.muhat.dif <- varhat.muhat.c + varhat.muhat.t -
    sum(sample.c)*sum(sample.t)*4/(N*n^2)


## output simulation results

sim.out <- cbind(N, n, rho, mu.c, var.c, mu.t, var.t, mu.dif,
    var.dif, ybar.c, s2.c, ybar.t, s2.t, muhat.c, var.muhat.c,
    varhat.muhat.c, muhat.t, var.muhat.t, varhat.muhat.t, var.muhat.dif,
    varhat.muhat.dif)

return(sim.out)     }


## run simulation and assign column names

sim.temp <- t(replicate(10, sim.fun(50, 10, 0)))

dimnames(sim.temp) <- list(NULL, c("N", "n", "rho", "mu.c", "var.c",
    "mu.t", "var.t", "mu.dif", "var.dif", "ybar.c", "s2.c", "ybar.t",
    "s2.t", "muhat.c", "var.muhat.c", "varhat.muhat.c", "muhat.t",
    "var.muhat.t", "varhat.muhat.t", "var.muhat.dif",
    "varhat.muhat.dif"))
```

```
####################################################################
######################## Results Function ########################
####################################################################
This function calculates the number of times varhat is negative, ana
confidence interval width and confidence level for ybar.c, ybar.t,
ybar.diff, muhat.c, muhat.t and muhat.dif.
####################################################################

results.fun <- function(sim.data) {

#input simulation results#

N <- sim.data[1,1]

n <- sim.data[1,2]

rho <- sim.data[1,3]

mu.c <- sim.data$mu.c

mu.t <- sim.data$mu.t

ybar.c <- sim.data$ybar.c

ybar.t <- sim.data$ybar.t

s2.c <- sim.data$s2.c

s2.t <- sim.data$s2.t

muhat.c <- sim.data$muhat.c

muhat.t <- sim.data$muhat.t

varhat.c <- sim.data$varhat.muhat.c

varhat.t <- sim.data$varhat.muhat.t

varhat.dif <- sim.data$varhat.muhat.dif
```

```
#lower confidence limit ybar.c#

CI.ybar.cl <- ybar.c - qt(.975,n/2-1)*sqrt(2*s2.c/n)

#upper confidence limit ybar.c#

CI.ybar.cu <- ybar.c + qt(.975,n/2-1)*sqrt(2*s2.c/n)

#lower confidence limit muhat.c#

CI.muhat.cl <- muhat.c - qt(.975,n/2-1)*sqrt(varhat.c)

#upper confidence limit muhat.c#

CI.muhat.cu <- muhat.c + qt(.975,n/2-1)*sqrt(varhat.c)

#CI width for ybar.c#

CI.width.ybar.c <- mean(CI.ybar.cu - CI.ybar.cl)

#CI coverage for ybar.c#

CI.cover.ybar.c <- sum(CI.ybar.cl < mu.c & mu.c < CI.ybar.cu)

#CI width for muhat.c#

CI.width.muhat.c <- mean(CI.muhat.cu - CI.muhat.cl)

#CI coverage for muhat.c#

CI.cover.muhat.c <- sum(CI.muhat.cl < mu.c & mu.c < CI.muhat.cu)


#lower confidence limit ybar.t#

CI.ybar.tl <- ybar.t - qt(.975,n/2-1)*sqrt(2*s2.t/n)

#upper confidence limit ybar.t#

CI.ybar.tu <- ybar.t + qt(.975,n/2-1)*sqrt(2*s2.t/n)

#lower confidence limit muhat.t#

CI.muhat.tl <- muhat.t - qt(.975,n/2-1)*sqrt(varhat.t)
```

```
#upper confidence limit muhat.t#

CI.muhat.tu <- muhat.t + qt(.975,n/2-1)*sqrt(varhat.t)

#CI width for ybar.t#

CI.width.ybar.t <- mean(CI.ybar.tu - CI.ybar.tl)

#CI coverage for ybar.t#

CI.cover.ybar.t <- sum(CI.ybar.tl < mu.t & mu.t < CI.ybar.tu)

#CI width for muhat.t#

CI.width.muhat.t <- mean(CI.muhat.tu - CI.muhat.tl)

#CI coverage for muhat.t#

CI.cover.muhat.t <- sum(CI.muhat.tl < mu.t & mu.t < CI.muhat.tu)


mu.dif <- mu.t - mu.c

ybar.dif <- ybar.t - ybar.c

muhat.dif <- muhat.t - muhat.c

#lower confidence limit ybar.dif #

CI.ybar.dif.l <- ybar.dif - qt(.975,n-2)*sqrt(2*(s2.c + s2.t)/n)

#upper confidence limit ybar.dif#

CI.ybar.dif.u <- ybar.dif + qt(.975,n-2)*sqrt(2*(s2.c + s2.t)/n)

#lower confidence limit muhat.dif#

CI.muhat.dif.l <- muhat.dif - qt(.975,n-2)*sqrt(varhat.dif)

#upper confidence limit muhat.dif#

CI.muhat.dif.u <- muhat.dif + qt(.975,n-2)*sqrt(varhat.dif)
```

```
#CI width for ybar.dif#

CI.width.ybar.dif <- mean(CI.ybar.dif.u - CI.ybar.dif.l)

#CI coverage for ybar.dif#

CI.cover.ybar.dif <- sum(CI.ybar.dif.l < mu.dif & mu.dif <
    CI.ybar.dif.u)

#CI width for muhat.dif#

CI.width.muhat.dif <- mean(CI.muhat.dif.u - CI.muhat.dif.l)

#CI coverage for muhat.dif#

CI.cover.muhat.dif <- sum(CI.muhat.dif.l < mu.dif & mu.dif <
    CI.muhat.dif.u)


#output CI coverage and width results#

CI.cover <- cbind(N, n, rho, CI.cover.ybar.c/100,
    CI.cover.muhat.c/100, CI.cover.ybar.t/100, CI.cover.muhat.t/100,
    CI.cover.ybar.dif/100, CI.cover.muhat.dif/100)

CI.width <- cbind(N, n, rho, CI.width.ybar.c, CI.width.muhat.c,
    CI.width.ybar.t, CI.width.muhat.t, CI.width.ybar.dif,
    CI.width.muhat.dif)

sim.results <- rbind(CI.cover, CI.width)

dimnames(sim.results) <- list(c("CI.cover", "CI.width"), c("N", "n",
    "rho", "ybar.c", "muhat.c", "ybar.t", "muhat.t", "ybar.dif",
    "muhat.dif"))

return(sim.results)     }
```