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INEQUALITIES

by

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INTRODUCTION

The subject of inequalities is one that has intrigued the imagination of mathematicians for generations. It is a discipline which embraces both the areas of algebra and analysis and has recently found many applications in the field of linear programming and in the computer sciences.

The primary purpose of this paper is to develop the so-called classical inequalities. These include the Holder and Minkowski inequalities, the Cauchy and triangle inequalities. The arithmetic-mean - geometric-mean inequality is considered with the classical inequalities for series but not for integrals. Some inequalities associated with matrices are also considered.

Historically one notes that the arithmetic-mean - geometric-mean inequality has perhaps captured the attention of more mathematicians than any other single inequality. There are literally dozens of proofsfor this inequality. Cauchy published the inequality bearing his name in 1821. The generalization of Cauchy's inequality is known as Holder's inequality and it appeared in 1889. A generalization of the triangle inequality was made by a famous geometer, Hermann Minkowski, (1864-1909), and is called the Minkowski inequality.

This report is in no way an exhaustive account of all of the well-known inequalities. Many lengthy volumes would be required if completeness were attempted. Inequalities dealing with prime numbers and inequalities associated with orthogonal series such as Bessel's inequality are not considered in this report. There are vast areas that deal with sets of inequalities over particular function spaces and these too lie beyond the intent of this paper.

I. ELEMENTARY ALGEBRA OF INEQUALITIES

The ideas involving inequalities are imbedded in the concept of ordering. The notion of an ordered field is essential in any discussion of inequalities. In general, in this paper, an equality is proved for an ordered field and it is noted that the inequality will hold on a subset of the ordered field if the particular quantity is an element of the subset.

In order to establish an axiomatic basis for inequalities in the real number system, the notion of a positive number is taken as an undefined term. The set P is the set of all positive numbers. The postulates of order are:

P. 1 Either $a \in P$ or $-a \in P$ or $a = 0$.

P. 2 If $a \in P$, $b \in P$ then $a+b \in P$, and $ab \in P$

P. 3 A field in which the positive elements are defined subject to postulates P_1, P_2 is ordered by the convention $a < b$ if and only if $b-a \in P$.

If the elements of a field are ordered by the above convention then the elements of that field must satisfy the following conditions:

F. 1 If $a \neq b$ then either $a < b$ or $b < a$ (Law of Trichotomy)

F. 2 If $a < b$, then $a+c < b+c$ for all c.

F. 3 If $a < b$, $c > 0$, then $a \cdot c < b \cdot c$.

F. 4 If $a < b$ and $b < c$, then $a < c$.

It is apparent that the real number system is an ordered field satisfying both $P_1 - P_3$ and $F_1 - F_4$. In showing that the above properties hold for the real number system it is important to note that the statement $a > b$ is equivalent to the statement $a - b = h$ where h is a positive real number. The properties $F_1 - F_4$ can be illustrated by rewriting the inequalities in the above form. Thus, consider F. 2 for the real number system:

If $a < b$ then $a+c < b+c$ for all c

Proof: $a < b$ implies $-a+b = h$, $h > 0$.

Therefore, $-(a+c) + (b+c) = h$ which implies $a+c < b+c$ the desired result.

Four additional elementary properties of inequalities for the real number system are $I_1 - I_4$ listed below.

I. 1 If $a > b$ and $c < 0$ then $ac < bc$.

I. 2 If $a > b$ and $ab > 0$ then $\frac{1}{a} < \frac{1}{b}$.

I. 3 If a, b, c, d are positive real numbers and if $a < b$ and $c < d$ then $ac < bd$.

I. 4 If $a > 0$, $b > 0$ and $a < b$ then $a^n < b^n$ and $a^{\frac{1}{n}} < b^{\frac{1}{n}}$ where n is any positive integer.

The inequality relation is a transitive relation by F. 4. It is a determinative relation by F. 1. However, it is not reflexive or symmetric as is easily shown.

1.) The relation $<$ is not reflexive. $2 \nless 2$

2.) The relation $<$ is not symmetric.

$2 < 3$ does not imply $3 < 2$ since $3 \nless 2$.

Hence the relation $<$ or the general inequality is not an equivalence relation. The relation $\leq$ is reflexive but not symmetric, hence it too is not an equivalence relation.

It is interesting to note that the properties of inequalities do not hold for a finite field. This is not surprising since the postulates for an ordered field upon which the basic properties of the inequality are built do not hold the a finite field. This can be demonstrated by an example of a Galois Field, which is a field modulo a prime.

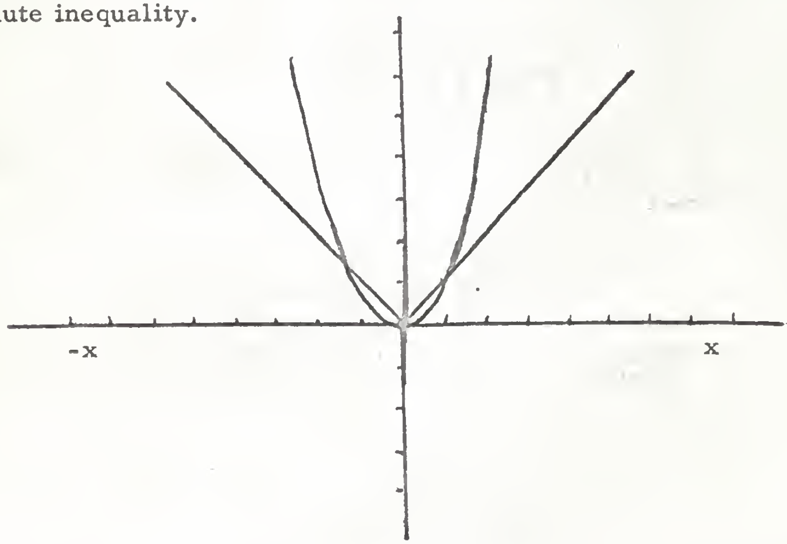
Consider the field modulo 5.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

x	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

The relationship of inequality does not satisfy property F. 1 on this finite field. Assume that $1 > 0$ and then adding 4 to both sides of the inequality gives $0 > 4$. If both sides of this inequality are multiplied by 4 one obtains $0 > 1$ which contradicts the original assumption and hence violates property F. 1.

An absolute inequality, such as $f(x) > 0$, is one which is valid for all values of the variable. A conditional inequality, such as $g(x) > 0$ is one which is valid only for certain values of the variable. In a later section the most elementary inequality, $a^2 \geq 0$, is considered. The graph of $y = x^2$ is simply a parabola as shown below. From the graph one can note that $y \geq 0$ for all values of x and hence the inequality $a^2 \geq 0$ is an absolute inequality.



The straight lines are the graph of $y = |x|$. Hence the inequality $|x| < a$, $a > 0$ is a conditional inequality since it is only true for values of x between -a, and +a.

If there is a set of inequalities then a pair of values of x and y satisfying all of the inequalities is said to be a solution of the set of inequalities. Consider the following problem involving a set of inequalities.

(1) What integral values of y satisfy the following restraints?

(2) Find the maximum y such that:

(a) $y \leq x$

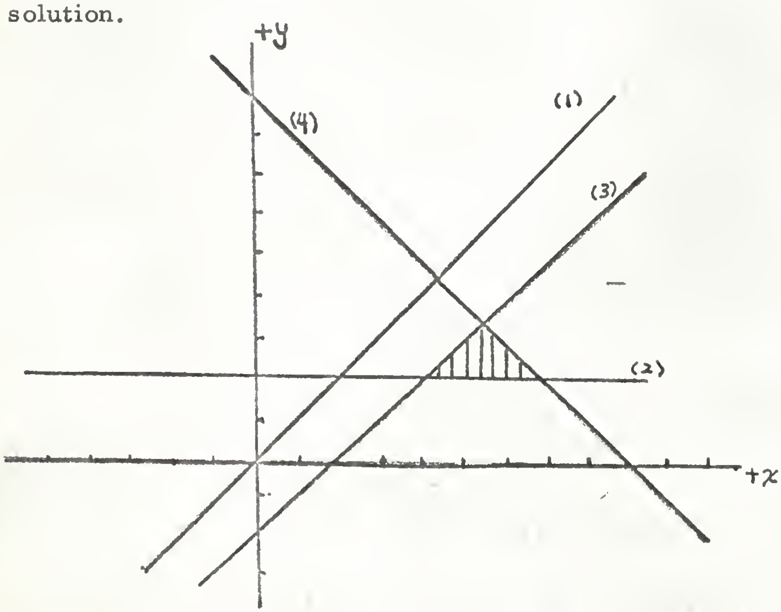
(b) $y \geq 2$

(c) $y \leq x - 2$

(d) $y \leq 9 - x$

This system is one of four linear restraints.

One might note that the values of y for points on and below the line $y = x$ satisfy inequality (a). The graphs of the corresponding equations are drawn below. The cross-hatched area represents the common solution.



(1) $y = x$

(2) $y = 2$

(3) $y = x - 2$

(4) $y = 9 - x$

The only integral values of y satisfying the restraints are $y = 2$, and $y = 3$. Note restraint (a) is less restrictive than restraint (c) and hence can be neglected from further consideration.

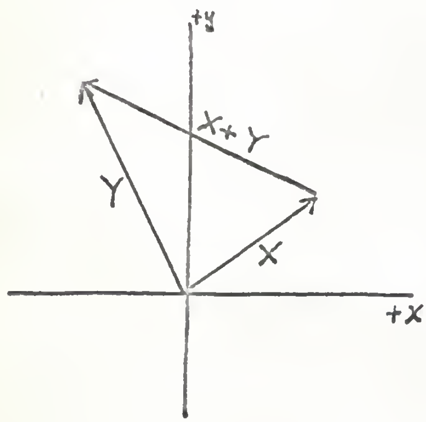
In order to find the value of x for which y is a maximum and satisfying the restraints simply solve equations (3) and (4) for x. Then

$$x - 2 = 9 - x \quad \text{and} \quad x = 5.5$$

Hence $y = 3.5$ is the maximum value of y satisfying the given restraints.

In some cases of finding the maximum value of y only integral values of y are allowed. In the above case, the maximum integral value of y is 3. The problem of satisfying a system of linear restraints is considered under the broader topic of linear programming.

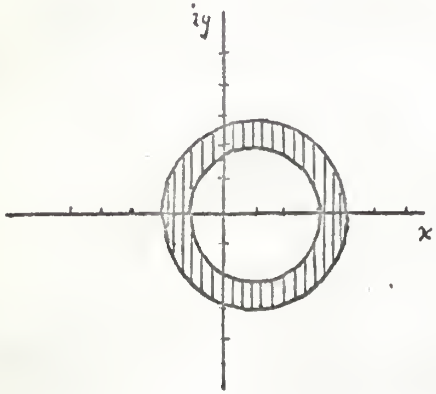
The triangle inequality $|X| + |Y| \geq |X+Y|$ which will be considered in detail later can be represented geometrically as follows. Let $X = (c, d)$ and $Y = (a, b)$ be vectors. Then $X+Y = (a+c, b+d)$. Since $|R|$ is simply the length of the vector R, the inequality implies that the sum of the lengths of two sides of a triangle is always greater than the length of the third side. Equality holds if one vector is a scalar multiple of the other, that is the two vectors are linearly dependent and hence parallel, in which case a degenerate triangle is formed.



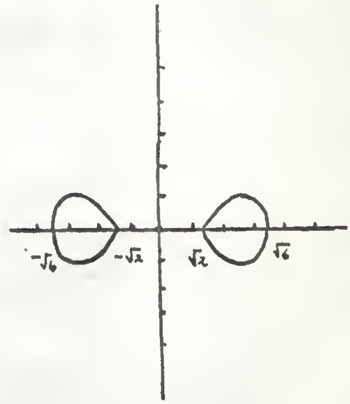
(figure 1.1)

In order to illustrate the concept of a geometrical inequality in the complex plane consider the following example:

The values of the complex valued z which satisfy both $|z-1| < 3$ and $2 < |z-1|$ are the sets of all z both outside the circle $2 = |z-1|$ and inside the circle $|z-1| = 3$. Hence it is the set of complex numbers which lie in the interior of the annulus bounded by the circles of radii 2 and 3 with the center at (1, 0). See figure 1.2.1.



(figure 1.2.1)



(figure 1.2.2)

Another example is the following. A lemniscate is the locus of points the product of whose distances to two fixed points is a constant. Consider the inequality $|z-2| \cdot |z+2| < 2$ or $|z^2 - 4| < 2$. The only values of z which satisfy the inequality are those which lie in one of the two disjoint regions bounded by the lemniscate with equation $|z^2 - 4| = 2$. See figure 1.2.2.

II. CLASSICAL INEQUALITIES FOR FINITE SEQUENCES

The aim of this section is to progress from the most elementary of all inequalities to the more subtle and interesting classical inequalities. In particular, the primary concern lies in the development of the classical inequalities for finite sequences.

It is now convenient to consider the most elementary inequality. This basic inequality is simply:

$$a^2 \geqq 0 \qquad\qquad \text{for all } a \in F$$

where F is an ordered field.

To show that this inequality is indeed valid one must first consider the postulates of order. By P.1 either $a = 0$, $a \in P$, or $-a \in P$. The inequality is valid though trivial for $a = 0$. For $a \in P$, the inequality is true by P. 2. Now consider $-a \in P$ and hence $(-a)^2 \in P$ which implies that $(-1)(-1)(a)(a) \in P$. Now assume that $(-1)^2 \in P$. Then $(-1)(+1) \in P$ but $a \in P$, $b \notin P$ implies that $ab \notin P$ and hence $(-1)(+1) \notin P$, a contradiction. Then $(-1)^2 \in P$. Hence it follows that for $a \notin P$ then $a^2 \in P$.

The following inequalities will be stated and proved for non-negative real numbers. To be valid for any real number, the inequalities would necessarily have to be stated, with absolute value signs. For the sake of simplicity of format, the inequalities will be stated without the absolute value notation, it being understood that the absolute value signs are needed when the elements can be other than positive numbers. The same restriction holds for values in the complex field. Again the inequalities hold for the modulus of a complex number but not for the complex numbers themselves since the complex field is not an ordered field.

It can be shown by altering slightly the proofs of inequalities for finite sequences that the inequalities proved are also valid for infinite sequences provided that the corresponding infinite series converges to a finite sum. The same restriction holds for infinite products.

The following is a list of the classical inequalities for finite sequences considered in this section. The a_i and b_i are sets of non-negative real numbers.

Cauchy Inequality:

$$(a_1^2 + a_2^2 + \cdots + a_n^2)^{\frac{1}{2}} (b_1^2 + b_2^2 + \cdots + b_n^2)^{\frac{1}{2}} \geqq$$

Triangle Inequality:

$$(a_1^2 + a_2^2 + \cdots + a_n^2)^{\frac{1}{2}} + (b_1^2 + b_2^2 + \cdots + b_n^2)^{\frac{1}{2}} \geqq$$

$$\left[(a_1 + b_1)^2 + (a_2 + b_2)^2 + \cdots + (a_n + b_n)^2\right]^{\frac{1}{2}}$$

Arithmetic Mean - Geometric Mean Inequality

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geqq (a_1 \cdot a_2 \cdots a_n)^{\frac{1}{n}}$$

Holder Inequality

$$(a_1^p + a_2^p + \cdots + a_n^p)^{\frac{1}{p}} (b_1^q + b_2^q + \cdots + b_n^q)^{\frac{1}{q}} \geqq$$

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

where $\frac{1}{p} + \frac{1}{q} = 1$

Minkowski Inequality:

$$(a_1^p + a_2^p + \cdots + a_n^p)^{\frac{1}{p}} + (b_1^p + b_2^p + \cdots + b_n^p)^{\frac{1}{p}}$$

$$\geqq \left[(a_1 + b_1)^p + (a_2 + b_2)^p + \cdots + (a_n + b_n)^p\right]^{\frac{1}{p}}$$

for $p > 1$

The Cauchy Inequality:

The derivation of the Cauchy Inequaity follows directly from the basic inequality, $a^2 \geq 0$. Let x and y together with two non-trivial sets $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be elements from an ordered field.

Then from the basic inequality it is evident that:

1.) $(x - y)^2 \geqq 0$

$$x^2 - 2xy + y^2 \geqq 0$$

$$x^2 + y^2 \geqq 2xy$$

$$\frac{x^2 + y^2}{2} \geqq xy$$

2.) Substitute in turn

$$x = \frac{a_1}{(a_1^2 + a_2^2 + \cdots + a_n^2)^{\frac{1}{2}}} \qquad y = \frac{b_1}{(b_1^2 + b_2^2 + \cdots + b_n^2)^{\frac{1}{2}}}$$

$$x = \frac{a_2}{(a_1^2 + a_2^2 + \cdots + a_n^2)^{\frac{1}{2}}} \qquad y = \frac{b_2}{(b_1^2 + b_2^2 + \cdots + b_n^2)^{\frac{1}{2}}}$$

$$\vdots \qquad\qquad \vdots$$

$$x = \frac{a_n}{(a_1^2 + a_2^2 + \cdots + a_n^2)^{\frac{1}{2}}} \qquad y = \frac{b_n}{(b_1^2 + b_2^2 + \cdots + b_n^2)^{\frac{1}{2}}}$$

Then if the corresponding terms of the resulting inequalities are added the following is obtained.

$$\frac{1}{2}\left[\frac{a_1^2}{(a_1^2+a_2^2+\cdots+a_n^2)}+\frac{a_2^2}{(a_1^2+a_2^2+\cdots a_n^2)}+\cdots+\frac{a_n^2}{(a_1^2+\cdots+a_n^2)}\right.$$

$$\left.+\frac{b_1^2}{(b_1^2+b_2^2+\cdots+b_n^2)}+\cdots+\frac{b_n^2}{(b_1^2+\cdots b_n^2)}\right] \geqq$$

$$\frac{a_1 b_1}{(a_1^2+a_2^2+\cdots+a_n^2)^{\frac{1}{2}}(b_1^2+b_2^2+\cdots+b_n^2)^{\frac{1}{2}}} +$$

$$\frac{a_2 b_2}{(a_1^2+a_2^2+\cdots+a_n^2)^{\frac{1}{2}}(b_1^2+b_2^2+\cdots+b_n^2)^{\frac{1}{2}}} +\cdots+$$

$$\frac{a_n b_n}{(a_1^2+a_2^2+\cdots a_n^2)^{\frac{1}{2}}(b_1^2+b_2^2+\cdots b_n^2)^{\frac{1}{2}}}$$

which is

$$1 \geqq \left(\sum_{i=1}^{n} a_i b_i\right) \Big/ \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}$$

or

$$\left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}} \geqq \sum_{i=1}^{n} a_i b_i$$

which is the Cauchy Inequality. Equality holds if and only if $a_i = kbi$ for all i.

An interesting application of the Cauchy Inequality is showing that a generalized Cos θ is always less than or equal to one in absolute value for the Euclidean n-dimensional geometries. The scalar product of two vectors over the real field is defined as

$$A \cdot B = \sum_{i=1}^{n} a_i b_i = |A| \, |B| \ \text{Cos } \theta$$

where

$$A = (A \cdot A)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}$$

$$B = (B \cdot B)^{\frac{1}{2}} = \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}$$

Hence

$$\text{Cos } \theta = \left(\sum_{i=1}^{n} a_i b_i\right) / \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}$$

and from Cauchy's Inequality it follows that

$$|\text{Cos } \theta| \leqq 1 \qquad \text{for all } \theta$$

The Cauchy Inequality will also be used to prove the triangle inequality and the arithmetic mean- geometric mean inequality.

The Triangle Inequality:

The generalized triangle inequality is

$$(x_1^2+x_2^2+\cdots+x_n^2)^{\frac{1}{2}}+(y_1^2+y_2^2+\cdots+y_n^2)^{\frac{1}{2}} \geqq \left[(x_1+y_1)^2+(x_2+y_2)^2+\cdots+(x_n+y_n)^2\right]^{\frac{1}{2}}$$

Its proof follows immediately from the Cauchy Inequality. Start with the general Cauchy Inequality:

1.) $(x_1^2+x_2^2+\cdots+x_n^2)^{\frac{1}{2}}(y_1^2+y_2^2+\cdots+y_n^2)^{\frac{1}{2}} \geqq x_1 y_1+x_2 y_2+\cdots+x_n y_n.$

Multiply both sides by two, giving

2.) $2(x_1^2+x_2^2+\cdots+x_n^2)^{\frac{1}{2}} \cdot (y_1^2+y_2^2+\cdots+y_n^2)^{\frac{1}{2}} \geqq 2x_1 y_1+2x_2 y_2+\cdots+2x_n y_n.$

Add $(x_1^2+\cdots+x_n^2)$ and $(y_1^2+y_2^2+\cdots+y_n^2)$ to both sides, so that

3.) $(x_1^2+x_2^2+\cdots+x_n^2)+2(x_1^2+x_2^2+\cdots+x_n^2)^{\frac{1}{2}} \cdot (y_1^2+y_2^2+\cdots+y_n^2)^{\frac{1}{2}} + (y_1^2+y_2^2+\cdots+y_n^2) \geqq x_1^2+2x_1y_1+y_1^2+x_2^2+2x_2y_2+y_2^2+\cdots+x_n^2+2x_ny_n+y_n^2.$

Hence

$$4.)\quad (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}} + (y_1^2 + y_2^2 + \cdots + y_n^2)^{\frac{1}{2}}\,{}^{2} \geq$$

$$(x_1 + y_1)^2 + (x_2 + y_2)^2 + \cdots + (x_n + y_n)^2 .$$

Taking the positive square root of both sides gives

$$5.)\quad (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}} + (y_1^2 + y_2^2 + \cdots + y_n^2)^{\frac{1}{2}} \geq$$

$$\left[(x_1 + y_1)^2 + (x_2 + y_2)^2 + \cdots + (x_n + y_n)^2\right]^{\frac{1}{2}}$$

and this then is the generalized triangle inequality.

In vector notation it is simply $|X| + |Y| = |X+Y|$. It is evident that equality holds if and only if x_i = kyi for all i, k R, that is, if and only if X = kY, that is, Y is scalar multiple of X and hence X and Y are parallel.

The Arithmetic Mean - Geometric Mean Inequality:

The Arithmetic Mean - Geometric Mean Inequality has intrigued the imaginations of mathematicians over the years. Literally dozens of proofs for this classic inequality have been presented. The inequality here will be proved by a method devised by Cauchy.

Consider the general Arithmetic Mean - Geometric Mean Inequality:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq (x_1\, x_2 \cdots x_n)^{\frac{1}{n}} ,$$

where the x_i are a set of non-negative elements from an ordered field and there is strict inequality unless the x_i are all equal.

The classical proof offered here is due to Cauchy. We have

$$(y_1 - y_2)^2 \geqq 0$$

hence

$$\frac{y_1^2 + y_2^2}{2} \geqq y_1 y_2$$

Now if the following transformation is made

$$x_1 = y_1^2 \qquad x_2 = y_2^2$$

and substitute into the last inequality the inequality becomes

$$\frac{x_1 + x_2}{2} \geqq \sqrt{x_1 x_2}$$

Hence the theorem is valid for any two non-negative real numbers and it is apparent that equality holds only if $x_1 = x_2$.

Now make a new transformation; $x_1 = (x_1 + x_2)/2$ $x_2 = (x_3 + x_4)/2$ and the inequality becomes

$$\frac{\frac{x_1 + x_2}{2} + \frac{x_3 + x_4}{2}}{2} \geqq \sqrt{\frac{x_1 + x_2}{2} \quad \frac{x_3 + x_4}{2}}$$

$$\frac{x_1 + x_2 + x_3 + x_4}{4} \geqq \sqrt{(x_1 x_2)^{\frac{1}{2}} (x_3 x_4)^{\frac{1}{2}}}$$

$$\frac{x_1 + x_2 + x_3 + x_4}{4} \geqq \sqrt[4]{x_1 \, x_2 \, x_3 \, x_4}$$

Proceeding in this manner, it is readily evident that the inequality can be established for n = 1, 2, 4, and in general for n a power of two. This is known as forward induction. It is now necessary to complete the proof by using backward induction.

Assume that the inequality holds for any integer n, then show that it must also be true for n-1.

Assume true for n.

$$1.) \quad \frac{x_1 + x_2 + \cdots + x_n}{n} \geqq (x_1 \, x_2 \, \cdots \, x_n)^{\frac{1}{n}}$$

assuming $n \geq 2$, replace x_n by $\dfrac{x_1 + x_2 + \cdots x_{n-1}}{n-1}$. Then the inequality becomes:

$$2.) \quad \frac{x_1 + x_2 + \cdots + x_{n-1} + \dfrac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}}{n} \geqq$$

$$\left[x_1 \cdots x_{n-1}\right]^{\frac{1}{n}} \cdot \left[\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}\right]^{\frac{1}{n}}$$

$$\frac{(n-1)\, x_1 + x_1 + (n-1)\, x_2 + x_2 + \cdots (n-1)\, x_{n-1} + x_{n-1}}{n(n-1)} \geqq$$

$$\left[x_1 \, x_2 \cdots x_{n-1}\right]^{\frac{1}{n}} \cdot \left[\frac{x_1 + x_2 + \cdots x_{n-1}}{n-1}\right]^{\frac{1}{n}}$$

$$\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1} \geqq (x_1\, x_2 \cdots x_{n-1})^{\frac{1}{n}} \left[\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}\right]^{\frac{1}{n}}$$

$$\left[\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}\right]^{n} \geqq \left[x_1\, x_2 \cdots x_{n-1}\right]^{\frac{n}{n}} \left[\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}\right]^{\frac{n}{n}}$$

$$\left[\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}\right]^{n-1} \geqq \left[x_1\, x_2 \cdots x_{n-1}\right]^{1}$$

$$\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1} \geqq \left[x_1\, x_2 \cdots x_{n-1}\right]^{\frac{1}{n-1}}$$

Hence we have the desired inequality.

That equality holds when all the x_i are equal is quite evident.

An interesting point of logic might be noticed in employing this backward induction proof. In order for a backward mathematical induction proof to be valid it is first necessary to show that the proposition holds for infinitely many k. Then it is sufficient to show that if it is true for n = k it is also true for n = k-1.

The Holder Inequality is

$$(a_1^p + a_2^p + \cdots + a_n^p)^{\frac{1}{p}}\ (b_1^q + b_2^q + \cdots + b_n^q)^{\frac{1}{q}} \geqq a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and the a_i and b_i are non-negative elements of an ordered field.

It will be proved for the case where p and q are rational, noting that by a limiting process it is possible to obtain the inequality for p and q irrational.

Since $\frac{(x_1 + x_2 + \cdots x_n)}{n} \geqq (x_1 \cdot x_2 \cdots x_n)^{\frac{1}{n}}$ is true for all non-negative x_i, it can be established that if $x = x_1 = x_2 = x_3 = \cdots = x_m$ and $x_{m+1} = x_{m+2} = \cdots = x_n = y$ it follows that

$$\frac{mx + (n-m)y}{n} \geqq (x^m y^{n-m})^{\frac{1}{n}}$$

$$\frac{m}{n}x + \frac{n-m}{n}y \geqq x^{\frac{m}{n}} y^{\frac{n-m}{n}}.$$

Note: $n > 0$ and $1 < m \leqq n$. Let $\frac{m}{n} = R$; then $Rx + (1-R)y \geqq x^R y^{(1-R)}$.

Let $R = \frac{1}{p}$ and $1-R = \frac{1}{q}$. Then $\frac{1}{p} + \frac{1}{q} = R+1-R = 1$, and $\frac{x}{p} + \frac{y}{p} \geqq x^{\frac{1}{p}} y^{\frac{1}{q}}$. *

To eliminate fractional powers, set $x = a^p$, $y = b^q$. Then $\frac{a^p}{p} + \frac{b^q}{q} \geqq ab$. Equality exists if and only if $a^p = b^q$.

---

*By a limiting process it can be shown that $Rx + (1-R)y \quad x^R y^{1-R}$ holds also for R irrational.

Substituting into $\frac{a^p}{p} + \frac{b^q}{q}$ ab , successively,

$$a = \frac{a_1}{(a_1^p + a_2^p + \cdots + a_n^p)^{\frac{1}{p}}} \qquad b = \frac{b_1}{(b_1^q + b_2^q + \cdots + b_n^q)^{\frac{1}{q}}}$$

$$a = \frac{a_2}{(a_1^p + a_2^p + \cdots + a_n^p)^{\frac{1}{p}}} \qquad b = \frac{b_2}{(b_1^q + b_2^q + \cdots + b_n^q)^{\frac{1}{q}}}$$

$$\vdots \qquad\qquad \vdots$$

$$a = \frac{a_n}{(a_1^p + a_2^p + \cdots + a_n^p)^{\frac{1}{p}}} \qquad b = \frac{b_n}{(b_1^q + b_2^q + \cdots + b_n^q)^{\frac{1}{q}}} ,$$

and adding the resulting inequalities, gives

$$\frac{1}{p}\,\frac{(a_1^p + a_2^p + \cdots + a_n^p)}{(a_1^p + a_2^p + \cdots + a_n^p)} + \frac{1}{q}\,\frac{(b_1^q + b_2^q + \cdots + b_n^q)}{(b_1^q + b_2^q + \cdots + b_n^q)} \geqq$$

$$\frac{\sum_{i=1}^{n} a_i \; b_i}{(a_1^p + a_2^p + \cdots a_n^p)^{\frac{1}{p}} \; (b_1^q + b_2^q + \cdots b_n^q)^{\frac{1}{q}}} ,$$

$$1 \geqq \frac{\sum_{i=1}^{n} a_i \; b_i}{\left[\sum_{i=1}^{n} a_i^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^{n} b_i^q\right]^{\frac{1}{q}}}$$

$$\left[\sum_{i=1}^{n} a_i^p\right]^{\frac{1}{p}} \cdot \left[\sum_{i=1}^{n} b_i^q\right]^{\frac{1}{q}} \geqq \sum_{i=1}^{n} a_i b_i ,$$

which is the Holder Inequality.

It is quite evident that the Cauchy Inequality is a special case of the Holder Inequality with $p = q = 2$. Equality holds then only when the a_i are proportional to the b_i for all i.

The Minkowski Inequality:

To complete the proofs of Classical Inequalities it will be convenient to state an elementary proof of the Minkowski Inequality. Formally, the Minkowski Inequality is:

$$\left[\sum_{i=1}^{n} (x_i + y_i)^p\right]^{\frac{1}{p}} \leqq \left[\sum_{i=1}^{n} x_i^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n} y_i^p\right]^{\frac{1}{p}}$$

for $x_i, y_i \geqq 0$, $p > 1$.

The inequality is reversed for $p < 1$ ($p \neq 0$). (For $p < 0$, it is assumed that $x_i, y_i > 0$.) In each of the cases it will be shown that equality holds if and only if the sets x_i and y_i are proportional.

Begin with the identity

$$\sum_{i=1}^{n} (x_i + y_i)^p = \sum_{i=1}^{n} x_i(x_i + y_i)^{p-1} + \sum_{i=1}^{n} y_i(x_i + y_i)^{p-1}$$

Now apply the Holder Inequality with exponents p and q to both sums on the right. Hence,

$$\left[\sum_{i=1}^{n}(x_i+y_i)^p\right] \leq \left[\sum_{i=1}^{n} x_i^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^{n}(x_i+y_i)^p\right]^{\frac{1}{q}}$$

$$+\left[\sum_{i=1}^{n} y_i^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^{n}(x_i+y_i)^p\right]^{\frac{1}{q}}$$

$$=\left[\sum_{i=1}^{n} x_i^p\right]^{\frac{1}{p}} \cdot \left[\sum_{i=1}^{n}(x_i+y_i)^p\right]^{\frac{p-1}{p}} + \left[\sum_{i=1}^{n} y_i^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^{n}(x_i+y_i)^p\right]^{\frac{p-1}{p}}$$

$$=\left[\sum_{i=1}^{n} x_i^p \left[\sum_{i=1}^{n}(x_i+y_i)^p\right]^{p-1}\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n} y_i^p \left[\sum_{i=1}^{n}(x_i+y_i)^p\right]^{p-1}\right]^{\frac{1}{p}}$$

$$=\left[\left[\sum_{i=1}^{n} x_i^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n} y_i^p\right]^{\frac{1}{p}}\right] \cdot \left[\sum_{i=1}^{n}(x_i+y_i)^p\right]^{\frac{p-1}{p}} .$$

Hence, for $p > 1$,

$$\left[\sum_{i=1}^{n}(x_i+y_i)^p\right]^{\frac{1}{p}} = \left[\sum_{i=1}^{n} x_i^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n} y_i^p\right]^{\frac{1}{p}}$$

which is the desired Minkowski Inequality.

It is also apparent that the triangle inequality is a special case of the Minkowski Inequality where $p = 2$.

Note that equality will hold only in the case where there is equality in the Holder Inequality also. Hence equality will hold only if the sets x_i and y_i are proportional.

III. CLASSICAL INEQUALITIES WITH POWER MEAN NOTATION

First consider a set a of non-negative real numbers $a_1, a_2, \cdots, a_i, \cdots, a_n$ with $a_i > 0$, and a real number r, which will be considered non-zero for the present, as an arbitrary parameter. Two sets a and b are said to be proportional if there exist two real non-zero numbers s and t such that

$$s\, a_i = t\, b_i \qquad \text{for } i = 1, 2, \ldots, n$$

The relationship of proportionality is certainly reflexive and symmetrical. It is transitive only if one does not include the null set. Hence proportionality is an equivalence relationship on the set of all non-null sets.

It is interesting to note that if a and b are proportional then $b_i = 0$ whenever $a_i = 0$ and a_i / b_i is independent of i for the remaining values of i.

In this section some of the inequalities will be stated in the stronger form of $<$ rather than $\leq$ as in the previous section. The additional restraint is that the non-negative set a is also non-trivial, that is, there exist $a_i \neq 0$.

Power mean notation is defined as follows:

If p_i 0 $(i = 1, 2, \ldots, n)$ then $M_r = M_r(a) = M_r(a, p) =$

$$\left[\frac{\sum_{i=1}^{n} p_i a_i^r}{\sum_{i=1}^{n} p_i}\right]^{\frac{1}{r}}$$

$$M_r = 0 \qquad (r < 0, \text{ and for any } a_i = 0)$$

The subscripts in general will be neglected provided no ambiguity results.

The geometric weighted mean, G, is defined as follows:

$$G = G(a) = G(a, p) = (\pi a^p)^{1/\ p}$$

The ordinary arithmetic mean with unit weights can be denoted as:

$U(a) = M_1(a, 1)$ where the weighted arithmetic mean is $U(a) = U(a, p) = M_1(a, p)$. It is sometimes necessary to use negative a_i with the arithmetic mean in which case the definition remains unchanged for U(a).

The harmonic mean is defined as:

$$H(a, p) = M_{-1}(a, p).$$

In the actual use of weighted means it is often advantageous to make the transformation $q_i = p_i / \ p_i$. Hence $\sum_{i=1}^{n} q_i = 1$. The power means then become:

$$M_r(a) = M_r(a, q) = (\Sigma q a^r)^{1/r} \qquad (\Sigma q = 1)$$

$$G(a) = G(a, q) = \pi (a)^q \qquad (\Sigma q = 1)$$

Some interesting properties of the power means are the following:

1.) $M_r(a) = \left[U(a^r)\right]^{1/r}$;

2.) $G(a) = e^{U(\log a)}$;

3.) $U(a+b) = U(a) + U(b)$;

4.) $G(ab) = G(a)\,G(b)$;

It can also be shown that $M_r(a)$ and $G(a)$ lie between $\min_i a_i$ and $\max_i a_i$. It then can be noticed that:

$\lim_{r \to 0} M_r(a) = G(a)$. If every a is positive then

$$M_r(a) = \exp\left[\frac{1}{r} \log qa^r\right] = \exp\left[\frac{1}{r} \log (1+r\ q \log a + 0(r^2)\right]$$

$$\lim_{r\ 0} M_r(a) = \exp\left[\Sigma q \log a\right] = \pi a^q = G(a).$$

If there are some zero a_i, let b_k denote a positive a_k and let s be a q corresponding to a b_k, then

$$M_r(a, q) = \left[\Sigma q\ a^r\right]^{1/r} = \left[\Sigma s\ b^r\right]^{1/r} = \left[\Sigma s\right]^{1/r} M_r(b, s')$$

Hence $\lim_{r\ +0} \left[\Sigma s\right]^{1/r} M_r(b_1 s') = 0$, since $\Sigma s < 1$.

When $r < 0$, M_r and G are both zero, so that the above result holds also for $R \to -0$.

In a similar manner it can be shown that $\lim_{r \to +\infty} M_r(a) = \mathrm{Max}(a)$ and $\lim_{r \to -\infty} M_r(a) = \mathrm{Min}(a)$. It is then convenient to write $M_0(a) = G(a)$; $M_\infty(a) = \mathrm{Max}(a)$, and $M_{-\infty}(a) = \mathrm{Min}(a)$. Then $M_{-\infty}(a) < M_r(a) < M_\infty(a)$ for all finite n unless the a are all equal, or $r \leqq 0$ and an a is zero.

An interesting alternate proof of $U(a) > G(a)$ with unit weights is the following:

Let $a_1 = \min a < \max a = a_2$. If each of a_1 and a_2 is replaced by $1/2(a_1 + a_2)$, U(a) is unaltered, but $\left[\frac{a_1 + a_2}{2}\right]^2 > a_1 a_2$ so that G(a) is increased. Suppose that the a are varied in such a manner so that U(a) is unchanged, and that further we assume the existence of a set of a' for which G(a') attains a maximum. Then the a' must be equal since if not it can be replaced by another system for which G(a) is greater. It follows then that G(a) attains a maximum only for equal a and that maximum is equal to U(a).

To prove the existence of (a'), let $f(a_1, a_2, \cdots, a_{n-1}) = a_1 a_2 \cdots a_{n-1}(nU - a_1 - \cdots - a_{n-1})$. Then f is continuous in the closed domain $a_1 \geqq 0, \cdots, a_{n-1} \geqq 0$, $a_1 + a_2 + \cdots + a_{n-1} \leqq nU$.

Therefore f attains a maximum for some system of values $a_1' \cdots a'_{n-1}$ in the domain.

It is believed that this proof originated with Maclaurin, (1698-1746). Maclaurin stated the theorem in geometrical language as follows: "If the line AB is divided into any number of parts AC, CD, DE, EB, the product of all those parts multiplied into one another will be a maximum when the parts are equal amongst themselves."*

The theorem as stated here depends on Weustrass' theorem on the maximum of a continuous function.

The Classical inequalities in Power Mean Notation.

The Cauchy Inequality in power mean notation is simply $M_r(a) < M_{2r}(a)$. Written out this becomes $\left[\sum p\, a^r\right]^2 < \sum p \, \sum pa^{2r}$.

Another proof of the Cauchy Inequality can be obtained from the following general quadratic form which is always positive:

$$\sum (xa + yb)^2 = x^2 \sum a^2 + 2xy \sum ab + y^2 \sum b^2$$

Since the quadratic form is always positive for all values of x it must then have a negative discriminant. Therefore,

$$(2y \sum ab)^2 - 4y^2 \sum a^2 \sum b^2 < 0$$

Hence,

$$(\sum ab)^2 < \sum a^2 \sum b^2 .$$

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*Hardy, Littlewood and Polya (1).

Let $a = p^{1/2}$ and $b = a^r p^{1/2}$; then $(\sum p^{1/2} a^r p^{1/2})^2 < \sum (p^{1/2})^2 \sum (a^r p^{1/2})^2$,

and $(\sum pa^r)^2 < \sum p \sum pa^{2r}$, or $M_r(a) < M_{2r}(a)$, which is the general

Cauchy Inequality with weights.

The Arithmetic Mean - Geometric Mean Inequality.

The arithmetic mean - geometric mean inequality in power mean notation is simply $G(a) < U(a)$ unless all the a are equal. Written out this becomes

$$a_1^{p_1} \cdot a_2^{p_2} \cdots a_n^{p_n} < \left[\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}\right]^{\sum_{i=1}^{n} p_i} .$$

If we let $q_i = p_i / \quad p_i$, the inequality leads to the expression:

$$a_1^{q_1} \cdot a_2^{q_2} \cdots a_n^{q_n} < q_1 a_1 + q_2 a_2 + \cdots + q_n a_n \text{ where } \sum_{i=1}^{n} q_i = 1$$

This inequality has also been proved in the previous section for unit weights. It can be noted that an extension from unit weights to rational weights can be made. On account of homogeneity the weights are integral and means can be derived with integral weights from ordinary means by replacing every number by an appropriate set of equal numbers. Means with irrational weights may be regarded as limiting cases of ordinary means. In the limiting case $U(a) \geqq G(a)$ is obtained. The following proof will demonstrate that even for irrational weights the stronger case $U(a)$ $G(a)$ holds; provided that the a_i are not all equal.

This proof of $U(a) > G(a)$ is very concise and depends upon two previous results obtained in this section. They are the Cauchy Inequality, $M_r(a) < M_{2r}(a)$, and $\lim_{r \to 0} M_r(a) = G(a)$ together with the definition $U(a) = M_1(a)$. It then follows that:

$$U(a) = M_1(a) > M_{1/2}(a) > M_{1/4}(a) > \cdots >$$

$$\cdots > \lim_{m \to \infty} M_{2^{-m}}(a) = G(a)$$

Hence $U(a) > G(a)$ if not all the a_i are equal.

In order to develop the generalized Holder Inequality it is of benefit to develop first a few theorems.

Theorem 3.1. If r, s, $\cdots$, v are positive numbers and $r+s+\cdots+v = 1$, then

$$\sum a^r b^s \cdots n^v \leqq (\sum a)^r \ (\sum b)^s \cdots \ (\sum n)^v$$

Since one must omit the null set the following relation will hold.

1.) $$\frac{\sum a^r b^s \cdots n^v}{(\sum a)^r (\sum b)^s \cdots (\sum n)^v} = \sum \left[\frac{a}{\sum a}\right]^r \left[\frac{b}{\sum b}\right]^s \cdots \left[\frac{n}{\sum n}\right]^v$$

2.) $$\leqq \sum \left[\frac{ra}{\sum a} + \frac{sb}{\sum b} + \cdots + \frac{vn}{\sum n}\right]$$

3.) $= r + s + \cdots + v = 1$

Step two is obtained from step one by applying the inequality $U(a) \geqq G(a)$. There is equality only if $\frac{a_i}{a} = \frac{b_i}{b} = \cdots = \frac{n_i}{n}$ for $i=1, 2, \ldots, n$ that is if (a), (b), $\cdots$, (n) are proportional. No limiting process is required in this proof since if r, s, $\cdots$, v are irrational it has been shown that $U(a) \geqq G(a)$ still holds.

If in the previous theorem, r, s, $\cdots$, v are replaced by $q_1, q_2, \cdots, q_n$; where q_i is a positive real number the following is obtained:

$$\Sigma a^r b^s \ldots n^v \leqq (\Sigma a)^r \cdot (\Sigma b)^s \ldots (\Sigma n)^v$$

$$\Sigma a^q b^q \ldots n^q \leqq (\Sigma a)^{q_1} \cdot (\Sigma b)^{q_2} \ldots (\Sigma n)^{q_n}$$

Consider the following rectangular array of positive real numbers.

$$\begin{matrix} a_1 & b_1 & \cdots & n_1 \\ a_2 & b_2 & \cdots & n_2 \\ \vdots & \vdots & & \vdots \end{matrix}$$

It can be shown that if every two columns are proportional then $a_i b_k - a_k b_i = 0$ and $a_i c_k - a_k c_i = 0 \cdots$ for every i and k. Then this condition is also necessary and sufficient for proportionality of all the rows. Hence if the previous theorem holds for the columns it would also hold for the rows of the rectangular array. Then $a_1^r b_1^s \cdots n_1^v + a_2^r b_2^s \cdots n_2^v + \cdots +$

$a_n^r b_n^s \cdots n_n^v \leqq (a_1 + a_2 + \cdots a_n)^r \cdots (n_1 + n_2 + \cdots n_n)^v$ becomes

$a_1^r a_2^s \cdots a_n^v + b_1^r b_2^s \cdots b_n^v + \cdots + n_1^r n_2^s \cdots n_n^v \leqq (a_1 + b_1 + \cdots + n_1)^r \cdots$

$(a_n + b_n + \cdots + n_n)^v$; $\pi a^q + \pi b^q + \cdots + \pi n^q \leqq \pi(a+b+ \cdots +n)^q$.

This result gives directly the theorem:

Theorem 3.2. $G(a) + G(b) + \cdots + G(n) \leqq G(a+b+ \cdots + n)$

It is apparent that Theorem 3.1 is equivalent to Theorem 3.2.

Theorem 3.3. If $s, t, \cdots, v$ are positive and $s+t+ \cdots + v = 1$, then $M_r(ab\ldots n) \leqq M_{r/s}(a)\, M_r(b) \cdots M_{r/v}(n)$. Equality exists for $(a^{1/s})$, $(b^{1/t})$, $\cdots$, $(n^{1/v})$ proportional. The inequality fails when one of the factors on the right-hand side is zero. If $r < 0$, the sense of the inequality is reversed.

Theorem 3.3 is proved directly from Theorem 3.1 by making the substitution $a = qa^{r/s}$, $b = qb^{r/t}$, $\cdots$, $n = qn^{r/v}$.

Two real numbers, k and k' are conjugates if $k' = \dfrac{k}{k-1}$ or written symmetrically, $(k-1)(k'-1) = 1$.

Holder's Inequality.

Theorem 3.4. Suppose that $k(k-1) \neq 0$ and that k' is conjugate to k. Then

$$\Sigma ab \leqq (\Sigma a^k)^{1/k} \cdot (\Sigma b^{k'})^{1/k'} \qquad (k > 1). \tag{3.1}$$

and

$$\Sigma ab \lesseqgtr (\Sigma a^k)^{1/k} \cdot (\Sigma b^{k'})^{1/k'} \qquad (k < 1) \tag{3.2}$$

Equality exists where (a^k) and $(b^{k'})$ are proportional. The sets a and b must be positive for inequality (3.2).

Proof of Holder Inequality:

(1) Suppose that $k > 1$. Then (3.1) is the special case of Theorem 3.1 in which there are two sets of letters and $s = 1/k$, $t = 1/k'$. This is the ordinary form of the Holder Inequality.

(2) Suppose that $0 < k < 1$. Then $k' < 0$. For inequality (3.2), every b is positive. Define j, u, v by $j = 1/k$, $u = (ab)^k$, $v = b^{-k}$ so that $j \quad 1$, $k' = -kj'$ and $ab = u^j$, $a^k = uv$, $b^{k'} = v^{j'}$. Then inequality (3.2) reduces to inequality (3.1) with u, v, j in place of a, b, k, respectively. The exceptional case is that in which (u^j) and $(v^{j'})$, that is (ab) and $(b^{k'})$ are proportional. If this is so then the sets (a) and $(b^{k'-1})$, and therefore (a^k) and $(b^{k'})$ are proportional.

(3) If $k < 0$, then $0 < k' < 1$. This case is reduced to case 2 by interchanging a and b, k and k'. Both cases 2 and 3 are included in inequality (3.2). The inequality remains true in the excluded cases $k = 0$, $k = 1$, if one adopts appropriate conventions. If $k = 0$, $k' = 0$ interpret inequality (3.2) as $\sum_{i=1}^{n} a_i b_i > n\,(a_1 \cdots a_n\; b_1 \cdots b_n)^{1/n}$. If $k = 1$ interpret k' as $+\infty$ or as

$-\infty$. In the first case one interprets inequality (3.2) as $\sum_{i=1}^{n} a_i b_i >$

Min b $\sum_{i=1}^{n} a_i$.

Inequalities (3.1) and (3.2) can be combined in the single inequality

(3.3) $$(\Sigma ab)^{kk'} \leqq (\Sigma a^k)^{k'} \quad (\Sigma b^{k'})^{k}$$

$$k(k-1) \neq 0$$

Holder's Inequality for complex a, b.

If $k > 1$ and $k' = k/(k-1)$, then $|\Sigma ab| \leq (\Sigma |a|^k)^{1/k} \cdot (\Sigma |b|^{k'})^{1/k'}$

There is equality if and only if ($a_i{}^k$) and ($b_i{}^k$) are proportional and argument of $a_i b_i$ is independent of (i).

Theorem 3.5. If r is finite and not equal to one then

(3.4) $$M_r(a) + M_r(b) + \cdots + M_r(n) \geqq M_r(a+b+ \cdots +n) \qquad (r > 1)$$

and

(3.5) $$M_r(a) + M_r(b) + \cdots + M_r(n) \leqq M_r(a+b+ \cdots +n) \qquad (r < 1)$$

Equality exists if (a), (b), $\cdots$, (n) are proportional, or if $r = 0$ and $a_k = b_k = \cdots = n_k$ for some k.

There is equality for any (a), (b), $\cdots$ when $r = 1$. Also Theorem 3.2 is a special case of Theorem 3.5 when $r = 0$. The theorem is still true as $r \to +\infty$ or $r \to -\infty$.

Proof: Let the means be weighted with q and write $a + b + \cdots + n = s$ where $M_r(s) = S$. Then

$$S^r = \Sigma qs^r = \Sigma qas^{r-1} + \Sigma qbs^{r-1} + \cdots + \Sigma qns^{r-1}$$

$$= \Sigma(q^{1/r}a)(q^{1/n}s)^{r-1} + \cdots + \Sigma(q^{1/r}n)(q^{1/r}s)^{r-1} \quad .$$

Suppose first that $r \ \ 1$. Then apply Holder Inequality, i.e., inequality (3.2) of the Theorem 3.4 to each sum on the right. These results

(3.6) $$S^r \leqq (\Sigma qa^r)^{1/r} (\Sigma qs^r)^{1/r} + \cdots = S^{r-1} \left((\Sigma qa^r)^{1/r} + \cdots \right)$$

Equality occurs only if (qa^r), (qb^r), $\cdots$ are all proportional to (qs^r); i.e., if (a), (b) are proportional. Since S is positive (except in the trivial case when every set is null) this establishes inequality (3.4).

Next consider the case where $0 < r < 1$. Except in the case where all the sets (a), (b), $\cdots$ are null, $s_k \ \ 0$ for some k. If $s_k = 0$ for some k then $a_k = b_k = \cdots = n_k = 0$ and this value of k may be omitted from the consideration. It may therefore be considered as if $s_k > 0$ for every k. In that case inequality (3.2) of Theorem 3.4 gives inequality (3.6) with the sign of inequality reversed, and the proof may be completed as before.

Now consider the case where $r < 0$. If any s_k is zero, all the means are zero. Therefore assume that $s_k > 0$ for every k. If any a_k is zero, then $M_r(a) = 0$, and the letter a may be omitted. It may therefore be argued that on the assumption that every a, b, $\cdots$ is positive, then this special case follows from inequality (3.2) of Theorem 3.4.

Theorem 3.6 If r is finite and not equal to 0 or 1, then

$$(3.7) \qquad \left(\sum(a+b+\cdots+n)^r\right)^{1/r} \leqq \left(\sum a^r\right)^{1/r} + \cdots + \left(\sum n^r\right)^{1/r} \qquad (r \quad 1)$$

and

$$(3.8) \qquad \left(\sum(a+b+\cdots+n)^r\right)^{1/r} \geqq \left(\sum a^r\right)^{1/r} + \cdots + \left(\sum n^r\right)^{1/r} \qquad (r \quad 1)$$

Equality exists if (a), (b), $\cdots$, (n) are proportional. The inequality (3.8) does not hold if r 0 and $a_k = b_k = \cdots = n_k = 0$ for some k.

This theorem follows immediately from Theorem 3.5. Minkowski's inequality is inequality (3.7). Theorem 3.5 could be deduced from Theorem 3.6 by writing $p^{1/r}a$, $p^{1/r}b$, $\cdots$, for a, b, ... Hence Theorem 3.5 only appears to be more general than Theorem 3.6.

IV. CLASSICAL INEQUALITIES WITH INTEGRALS

To begin the discussion of inequalities with integrals an elementary derivation of Young's inequality is presented. Rather than generalize the inequalities for sequences to inequalities for integrals the procedure is to obtain Holder's inequality for integrals from Young's inequality.

Minkowski's inequality is then derived making use of Holder's inequality.

The remainder of this section deals with the statement of the inequalities for the Lebesgue and Stieltjes integrals. It will be observed that the inequalities stated for the Stieltjes integral include as special cases the inequalities for finite and infinite series, and for Lebesgue and Riemann integrals.

Now consider an elementary derivation of Young's inequality. Let $f(x)$ be an increasing function of class C^1. Then for $x > 0$, it follows that $f'(x) > 0$. Then if $y = f(x)$ is solved for x the resulting equation is $x = f^{-1}(y)$ where $(f^{-1}(y))' > 0$ for $0 \leqq y$ and $0 \leqq x$. Let

$$A_1 = \int_0^a f(x)dx \qquad A_2 = \int_0^b f^{-1}(y)dy$$

where $ab = A_1 + A_2 = \int_0^a f(x)dx + \int_0^b f^{-1}(y)dy$. This is illustrated in figure 4.1.

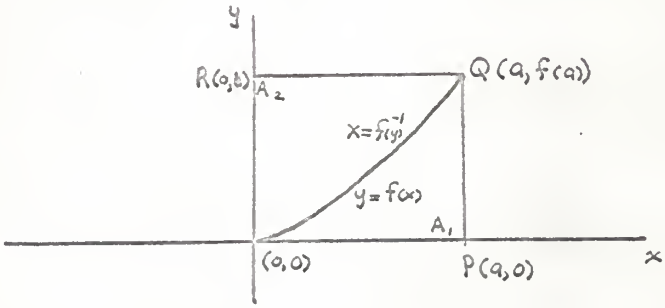

Figure 4.1

Now suppose that $f(a) \neq b$. There are two cases.

Case (a) : f (a)   b;

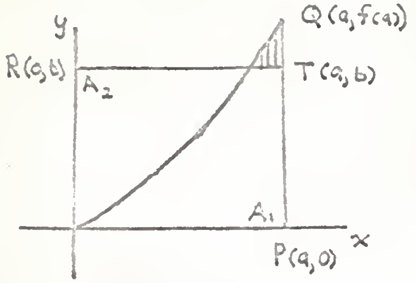

Figure 4. 2. 1

Case (b) : f(a)   b.

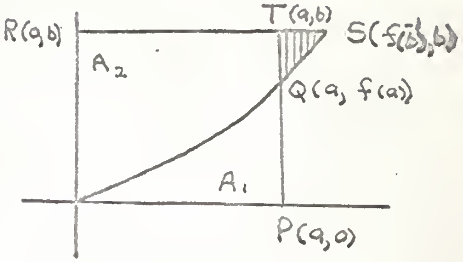

Figure 4. 2. 2

Now it is apparent that if $f(a) \neq b$ that $ab < A_1 + A_2$ and $ab = A_1 + A_2$ only when $f(a) = b$. Then in general

$$(4.1) \qquad ab \leqq A_1 + A_2 = \int_0^a f(x)dx + \int_0^b f^{-1}(y)dy$$

This then is Young's Inequality which may be stated as follows. If $f(x)$ and $f^{-1}(y)$ are two functions which vanish at the origin, are strictly increasing and inverse to each other, then for $a, b = 0$, inequality (4.1) holds. It is also clear that equality holds only for $b = f(a)$.

Classical Integral Inequalities.

It would be convenient first to state the Cauchy, Holder, and Minkowski inequalities for Riemann integrals of continuous functions. If $f(x)$ and $g(x)$ are continuous, non-negative functions on the closed interval $c = x = d$ then the following inequalities are true.

Cauchy- Schwarz Inequality.

$$\left[\int_c^d f(x)g(x)dx \right]^2 \leqq \left[\int_c^d f^2(x)dx \right] \left[\int_c^d g^2(x)dx \right]$$

where equality holds if and only if $g(x) = kf(x)$.

Holder Inequality.

$$\int_c^d f(x)g(x)dx \leqq \left[\int_c^d f^r(x)dx \right]^{\frac{1}{r}} \left[\int_c^d g^{r'}(x)dx \right]^{\frac{1}{r'}}$$

where $(1/r) + (1/r') = 1$ and r and r' are positive real numbers. It follows that they are both greater than one. Equality holds if and only if $g(x) = k\left(f(x)\right)^{r-1}$.

Minkowski Inequality

$$\left[\int_c^d f(x) + g(x)\ {}^{r}\, dx \right]^{\frac{1}{r}} \leqq \left[\int_c^d f^r(x)dx \right]^{\frac{1}{r}} + \left[\int_c^d g^r(x)dx \right]^{\frac{1}{r}}$$

where r is any real number greater than 1. It can be noted that the Cauchy-Schwarz inequality is a special case of the Holder where $r' = r = 2$.

Proof of the Holder Inequality for Riemann Integrals.

In order to prove the Holder inequality for Riemann integrals, assume first that neither $f(x)$ nor $g(x)$ is identically zero on $1 \leqq x \leqq m$. In Young's inequality $ab = \int_0^a f(x)dx + \int_0^b f^{-1}(y)dy$ let $f(x) = x^{r-1}$ where $r-1 > 0$.

Then $f^{-1}(y) = y^{\frac{1}{r-1}}$. Hence $ab \leq \int_0^a x^{r-1}dx + \int_0^b y^{\frac{1}{r-1}} dy$ and

(4.2) $$ab \leq \frac{1}{r} a^r + \frac{1}{r'} b^{r'} \quad \text{where } \frac{1}{r} + \frac{1}{r'} = 1 .$$

Now let $u = \left[\int_1^m f^r(x)dx\right]^{1/r}$ and $v = \left[\int_1^m g^{r'}(x)dx\right]^{1/r'}$. Since $u \neq 0$ and $v \neq 0$ it is convenient to let $a = f(x)/u$ and $b = g(x)/v$ in inequality (4.2). Then

(4.3) $$\frac{f(x)}{u} \cdot \frac{g(x)}{v} \leq \frac{1}{r} \frac{f^r(x)}{u^r} + \frac{1}{r'} \frac{g^{r'}(x)}{v^{r'}}$$

Since u and v are definite Riemann Integrals, they can be considered constants and thus they may be factored out from under the integral signs when we integrate both sides of (4.3). This yields

(4.4) $$\frac{1}{uv} \int_1^m f(x)g(x)dx = \frac{1}{r} \frac{\int_1^m f^r(x)dx}{u^r}$$

$$+ \frac{1}{r'} \frac{\int_1^m g^{r'}(x)dx}{v^{r'}}$$

$$= \frac{1}{r}[1] + \frac{1}{r'}[1] = 1$$

Multiplying both sides by uv and substituting back again for u and v, Holder's inequality is obtained.

A necessary and sufficient condition for equality to hold is that

$$\frac{g(x)}{v} = \left[\frac{f(x)}{u}\right]^{r-1} \quad \text{for all x in } c \leq x \leq d.$$

A proof of Minkowski's Inequality for Riemann Integrals follows:

$$\int \left(f(x) + g(x)\right)^r dx = \int \left(f(x) + g(x)\right) \cdot \left(f(x) + g(x)\right)^{r-1} dx$$

$$= \int f(x) \cdot \left(f(x) + g(x)\right)^{r-1} dx + \int g(x) \cdot \left(f(x) + g(x)\right)^{r-1} dx$$

Applying Holder's Inequality to both integrals on the right the following inequality is obtained.

$$\int \left(f(x) + g(x)\right)^r dx \leq \int f^r(x)\, dx^{\frac{1}{r}} \int \left(f(x) + g(x)\right)^{r'(r-1)} dx^{\frac{1}{r'}}$$

$$+ \int g^r(x) dx^{\frac{1}{r}} \int \left(f(x) + g(x)\right)^{r'(r-1)} dx^{\frac{1}{r'}}$$

Since for the Holder Inequality $\frac{1}{r} + \frac{1}{r'} = 1$ it follows that $(r-1)r' = r$. Factoring the right-hand side, the following is obtained.

$$\int \left(f(x) + g(x)\right)^r dx \leq \left[\int \left(f(x) + g(x)\right)^r dx\right]^{\frac{1}{r'}} \left[\left[\int f^r(x) dx\right]^{\frac{1}{r}} + \left[\int g^r(x) dx\right]^{\frac{1}{r}}\right]$$

Dividing both sides of this inequality by the first term on the right-hand side, it follows that

$$\left[\int \left(f(x) + g(x)\right)^{r} dx\right]^{\frac{1}{r}} = \left[\int f^{r}(x)dx\right]^{\frac{1}{r}} + \left[\int g^{r}(x)dx^{\frac{1}{r}}\right]$$

which is the Minkowski Inequality. In order for equality to hold, it is both necessary and sufficient that $g(x) = kf(x)$.

These inequalities may be extended to hold for Stieltjes integrals. The classical inequalities for Stieltjes integrals are as follows:

Holder Inequality for Stieltjes Integrals.

If $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, then $\int ts\, du < \left[\int t^{k} du\right]^{1/k} \left[\int s^{k'} du\right]^{1/k'}$

unless t^{k} and $s^{k'}$ are effectively proportional. The sense of the inequality is reversed when $k < 1$ and $k \neq 0$ with the exception above, or the left-hand side is zero.

Minkowski Inequality for Stieltjes Integrals.

If $k > 1$, then $\left(\int (t+s)^{k} du\right)^{1/k} \quad \left(\int t^{k} du\right)^{1/k} + \left(\int s^{k} du\right)^{1/k}$ unless the t and s are effectively proportional.

The proofs of these inequalities for the Stieltjes integral are omitted. A discussion of this topic may be found in Inequalities, by Hardy, Littlewood and Polya.

It can be shown that the Stieltjes Integral includes as a special case the Lebesgue integral. If $U(x) = x$ then the Stieltjes integral reduces to the Lebesgue integral. Stated here are the classical inequalities for the Lebesgue integral.

Holder's Inequality for the Lebesgue Integral.

If $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$ then $\int ts\,dx \quad \left(\int t^k dx\right)^{1/k} \left(\int s^{k'} dx\right)^{1/k'}$

unless t^k and $s^{k'}$ are effectively proportional. If $0 < k < 1$ or $k < 0$ then the sense of the inequality is reversed with the additional restriction that ts is null on a set of at most measure zero.

A special case of the Holder inequality is the Cauchy inequality. It is interesting, however, to note that the Cauchy inequality for finite sequences when generalized to integrals is generally called Schwarz's inequality, although it is believed that Buniakowsky was the first to state it. This inequality stated for the Lebesgue integral is as follows:

$$\left(\int ts\,dx\right)^2 < \int t^2 dx \int s^2 dx$$

unless $As = Bt$ where A and B are both constants, not both zero.

It can be shown that the sum of a series of positive terms can be expressed as a Stieltjes integral of a finite increasing step function. Then if $U(x)$ is a finite increasing step function one obtains the classical inequalities for finite series as expressed previously in this report as a special case of the classical inequalities for the Stieltjes integral.

Similarly $U(x)$ maybe a step function with infinitely many discontinuities. It can also be shown that the sum of any convergent infinite series may be expressed as a Stieltjes integral. Then

$$\sum_{k=1}^{\infty} f(x_i) = \int f(x) \, du$$

where the infinite series converges and f(x) is a step function with infinitely many discontinuities. Then it follows that special cases of the classical inequalities for the Stieltjes integral are the classical inequalities for infinite series.

V. INEQUALITIES ASSOCIATED WITH MATRICES

Let Q(x) be a quadratic form defined by Q(x) = (X, AX) = $\sum_{i=1}^{n} \sum_{i=1}^{n} x_i a_{ij} x_j$, where A' = A. Then a real symmetric matrix A is said to be positive definite if Q(x) is positive for all non-trivial sets of values of the real variable x_i, that is for some $x_i \neq 0$.

A necessary condition for A to be positive definite is for $|A| > 0$. If $|A| = 0$ then it would be possible to choose a set of non-trivial x_j so that the equation given below would be satisfied.

$$\sum_{j=1}^{n} a_{ij} x_j = 0 \quad \text{for } i = 1, 2, \ldots, n$$

Then it would follow that

$$(X, AX) = \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} a_{ij} x_j \right) = 0$$

which would contradict the fact that A is positive definite. Hence $|A| \neq 0$.

To show that $|A| > 0$ consider $|xI + (1-x)A|$ where $0 \leqq x \leqq 1$ and I is the identity matrix. Since A is positive definite it follows immediately that $(xI + (1-x)A)$ is also positive definite and that $|xI + (1-x) A|$ is non-zero. The value of $|xI + (1-x) A|$ is positive for $x = 1$ and it is a continuous function of x. Hence it follows that for $x = 0$, $|A| > 0$, by the continuity of the function.

It can also be shown that if A is positive definite then $|A_r| = |a_{ij}|$ $i, j = 1, 2, \ldots, r$; $r = 1, 2, \ldots, n$ must be positive. One might also show that $|A_r| > 0$ for $r = 1, 2, \ldots, n$ is a necessary and sufficient condition that A is positive definite. The proof of this may be found in <u>Theory of Matrices,</u> by Sam Perlis.

Theorem 5.1. If the real matrix A is positive definite then (X, A, X) $(Y, A^{-1}Y) = (X, Y)^2$ for all real X and Y. Let $A^{-1} = [s_{ij}]$. Then the theorem may be restated as

$$\left(\sum_{i=1}^{n} \sum_{j=1}^{n} x_i a_{ij} x_j \right) \left(\sum_{i=1}^{n} \sum_{j=1}^{n} y_i s_{ij} y_j \right) = (x, y)^2 .$$

Since A is positive definite there exists an orthogonal matrix T such that $T'AT = D = \mathrm{diag}\,(r_1, r_2, \cdots r_n)$. Then A may be reduced to diagonal form by means of the transformation $X = TU$ and $Y = TV$. Then if the theorem is true,

$$(TU, ATU)(TV, A^{-1}TV) \geqq (TU, TV)^2,$$

$$(U'T'ATU)(VT'A^{-1}TV) \geqq (U'T'TV)^2,$$

$$(U'DU)(VD^{-1}V) \geqq (U'V)^2,$$

$$\left(\sum_{i=1}^{n} r_i u_i^2\right)\left(\sum_{i=1}^{n} r_i^{-1} v_i^2\right) \geqq \left(\sum_{i=1}^{n} u_i v_i\right)^2 .$$

The last inequality is simply a special case of the Cauchy inequality for finite series and hence is valid. The theorem follows since the steps are reversible. The following lemma can be proved by an inductive argument from the basic definition for a quadratic form.

Lemma 5.1. Let $[A_k] = [a_{ij}]$, $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, k$ be a submatrix of A. Let $|A_k| \neq 0$ for $k = 1, 2, \ldots, n$, then

$$Q(X) = \sum_{k=1}^{n} \frac{|A_k|}{|A_{k-1}|} y_k^2 \qquad \text{where} \quad |A_0| = 1$$

and $y_k = x_k + \sum_{j=k+1}^{n} b_{kj} x_j$, $k = 1, 2, \ldots, n$ and the b_{kj} are rational functions of the a_{ij}.

From Theorem 5.1 note that $Q(A) = (Y, A^{-1}Y)^{-1} = \min_x \frac{(X, AX)}{(X, Y)^2}$.

It follows that $Q(A+B) \geqq Q(A) + Q(B)$, since

$$\min_X \frac{(X, (A+B)X)}{(X, Y)^2} \geqq \min_X \frac{(X, BX)}{(X, Y)^2} + \min_X \frac{(X, AX)}{(X, Y)^2} .$$

Then if Y is chosen to be the vector with components $y_i = 1$, $y_j = 0$ for $j \neq i$ and Lemma 5.1 is applied, the following inequality is obtained:

$$\frac{|A+B|}{|A_i+B_i|} = \frac{|A|}{|A_i|} + \frac{|B|}{|B_i|} ,$$

where $|R_i|$ denotes the determinant of the matrix R with the i^{th} row and column deleted. This inequality is valid for A and B positive definite and is known as the Bergstrom Inequality.

Before the next theorem, it is advantageous to consider the following two lemmas.

Lemma 5.2. Let A be a positive definite matrix of order n. Let

$$J_n = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(X,AX)} dx .$$

Then

$$J_n = \frac{\left[\int_{-\infty}^{\infty} e^{-x^2} dx\right]^n}{|A|^{1/2}} .$$

An outline of the proof of Lemma 5.2 follows:

Make the transformation $y_k = x_k + \sum_{j=k+1}^{n} b_{kj} x_j$ in $Q(X) = (X, AX)$.

Note that the Jacobian of the transformation is 1. Then from Lemma 5.1 and suitable transformations it follows that:

$$J_n = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\sum_{k=1}^{n} |A_k| / |A_{k-1}| \, y_k^2} \, dy$$

$$= \left(\int_{-\infty}^{\infty} e^{-y^2} dy \right)^n / \prod_{k=1}^{n} \left(|A_k| / |A_{k-1}| \right)^{\frac{1}{2}}$$

$$= \frac{\pi^{n/2}}{|A|^{1/2}}$$

Lemma 5.3. Let A be a real positive definite matrix of order n. Then $|A_{1n}| = |A_{1k}| \cdot | A_{k+1,n}|$ where the determinant $|A_{rs}|$ is defined by

$$|A_{rs}| = |a_{ij}| \quad ; \quad i, j = r, r+1, \ldots, s .$$

A special case is $|A| \leq \prod_{i=1}^{n} a_{ii}$

If in J_n of Lemma 5.2 the change of variables is made

$$x_i = - x_i \qquad i = 1, 2, \ldots, k,$$

$$x_i = x_i \qquad i = k+1, \ldots, n,$$

and the resulting integral is added to J_n the following result is obtained.

$$2J_n = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\sum_{i=1}^{k} \sum_{j=1}^{k} a_{ij}x_ix_j - \sum_{i=k+1}^{n} \sum_{j=k+1}^{n} a_{ij}x_ix_j} \cdot (R+R^{-1})dx$$

where $R = e^{-\left(\sum_{i=1}^{k} \sum_{j=k+1}^{n} a_{ij}x_ix_j + \sum_{i=k+1}^{n} \sum_{j=1}^{k} a_{ij}x_ix_j\right.}$

Since for all positive R it is apparent that $R+R^{-1} \geqq 2$ than the following inequality is obtained,

$$J_n = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij}x_ix_j} \cdot \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\sum_{i=k+1}^{n}\sum_{j=k+1}^{n} a_{ij}x_ix_k} dx .$$

Hence $\pi^{n/2}/|A| \geqq (\pi^{k/2}/|A_{1k}|) \cdot (\pi^{\frac{n-k}{2}}/|A_{k+1,n}|)$ which leads to the desired result.

Theorem 5.2. If A and B are real positive definite matrices of order n and if $|B| = 1$, then

$$\frac{\operatorname{tr}(AB)}{n} \geqq |A|^{\frac{1}{n}}$$

Note that tr(T'ATB) = tr(ATBT') and consider A in triangular form. Then $\operatorname{tr}(AB) = \sum_{i=1}^{n} r_i b_{ii}$. If the geometric mean- arithmetic mean inequality is employed, then

$$\frac{\operatorname{tr}(AB)}{n} \geqq \left[\prod_{i=1}^{n} r_i\right]^{\frac{1}{n}} \left[\prod_{i=1}^{n} b_{ii}\right]^{\frac{1}{n}} = |A|^{\frac{1}{n}} \left[\prod_{i=1}^{n} b_{ii}\right]^{\frac{1}{n}} .$$

Since $|B| = 1$, by lemma 5.3 which states that $\prod_{i=1}^{n} b_{ii} \geqq |B|$, one obtains the following:

$$\frac{\operatorname{tr}(AB)}{n} \geqq |A|^{\frac{1}{n}} |B| = |A|^{\frac{1}{n}} .$$

From Theorem 5.2 the following determinant inequality due to Minkowski may be derived.

If A and B are positive definite matrices of order n,

$$|A+B|^{\frac{1}{n}} = |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}} .$$

A discussion of the proof of this theorem is found in Inequalities by Edwin F. Beckenbach and Richard Bellman.

Hadamard's Inequality. If $|x_{ij}|$ is a real determinant of order n, and if $\|x_{ij}\|$ denotes the absolute value of $|x_{ij}|$, then

$$\|x_{ij}\| \leqq \prod_{i=1}^{n} \Big(\sum_{j=1}^{n} x_{ij}^{2} \Big)^{1/2}$$

From Lemma 5.3 one has $|A| \leqq \prod_{i=1}^{n} a_{ii}$. Let $X = [x_{ij}]$ and note that Hadamard's inequality holds if $|x_{ij}| = 0$. Assume then that $|x_{ij}| \neq 0$ and note that XX' is positive definite. Then

$$\|XX'\| = \|x_{ij}\|^{2} \leqq \prod_{i=1}^{n} \Big(\sum_{j=1}^{n} x_{ij}^{2} \Big)$$

$$\|x_{ij}\| \leqq \prod_{i=1}^{n} \Big(\sum_{j=1}^{n} x_{ij}^{2} \Big)^{1/2}$$

and hence the inequality holds.

In this section only a very few of the inequalities associated with matrices have been mentioned. One might note that there are many inequalities dealing with the eigenvalues of a matrix. For example there are inequalities which give the upper and lower bounds for a set of eigenvalues of a given matrix. Such inequalities are not considered here.

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INEQUALITIES

by

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are obtained as special cases. Care is taken to state the special cases for which the inequalities reduce to equalities.

It is necessary to develop the classical inequalities for Riemann integrals. One way to do this is, conventionally, to generalize from the classical inequalities for infinite series to the corresponding inequalities for the Riemann integral. The procedure in this report is to develop Young's inequality for Riemann integrals and then from this to prove Holder's inequality. Minkowski's inequality for Riemann integrals can then be obtained utilizing Holder's inequality. The Cauchy-Schwarz inequality and the triangle inequality follow as special cases of the Holder and Minkowski inequalities for Riemann integrals, respectively. The climactic development of the classical inequalities is the development of the classical inequalities for the Stieltjes integral. Then as special cases of the classical inequalities for the Stieltjes integral, one has the corresponding inequalities for the Lebesgue integral, the Riemann integral, and for infinite and finite series. Thus it is shown that the statement of the classical inequalities for the Stieltjes integral embodies all of the cases of the classical inequalities considered in this report.

Finally, one is concerned with the inequalities associated with matrices. A few inequalities dealing with the determinants of positive definite matrices are considered. Inequalities due to Minkowski, Bergstrom and Hadamard are considered. It is noted that there are many inequalities dealing with the bounds of a set of eigenvalues of a given matrix but the development of these inequalities lie beyond the scope of this report.

In this report, the first part is concerned with the elementary algebra of inequalities. In order to establish an axiomatic basis for inequalities the concept of a positive number is taken as an undefined term. Then the postulates of order are stated, together with the properties of an ordered field. From this the algebra of inequalities is developed, observing that the inequality relationship is not an equivalence relationship. It is illustrated that the inequality relationship does not hold on a finite field.

Next the classical inequalities for finite series are developed. The discussion is launched by considering the most elementary inequality, $a^2 \geqq 0$, for a an element of an ordered field. From this inequality the Cauchy inequality and the arithmetic-mean - geometric-mean are derived. The triangle inequality is then developed from the Cauchy inequality. Elementary derivations for the Holder and Minkowski inequalities are presented.

A representation of the classical inequalities in the power mean notation generalizes the inequalities thus far developed. The power mean notation is first defined and then some general relationships involving it are noted. The Cauchy inequality for a weighted series is developed. The arithmetic-mean - geometric-mean inequality is proved utilizing the power mean notation, thus giving a more generalized form than was developed in the previous section. Several general inequalities in power mean notation are proved. From these the Holder and Minkowski inequalities