Generalized Pak-Stanley labeling for multigraphical hyperplane arrangements for $n = 3$

by

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Abstract

Pak-Stanley labeling was originally defined by Pak and Stanley in 1998 as a bijective map from the set of regions of an extended Shi arrangement to the set of parking functions. Later this map was generalized to other hyperplane arrangements associated with graphs and directed multigraphs, but this map is not necessarily bijective in these more general cases. It was shown by in Sam Hopkins and David Perkinson in 2016 and Mikhail Mazin in 2017 that Pak-Stanley labeling is surjective to the set of G-parking functions, where G is the directed multigraph associated with the hyperplane arrangement.

This leads to the natural question of when the generalized Pak-Stanley map is bijective. We determine a necessary condition for a directed multigraph to admit a hyperplane arrangement that admits a injective Pak-Stanley labeling. For the special case $n = 3$, we present examples of directed multigraphs that satisfy our necessary condition but only admit hyperplane arrangements with a non-injective Pak-Stanley labeling, showing that the condition is not sufficient.

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Chapter 1

Introduction

Let $U \subset \mathbb{R}^n$ be given by $x_1 + x_2 + \cdots + x_n = 0$. A *linear hyperplane* is a subspace of U with codimension 1 and an *affine hyperplane* is a translate of a linear hyperplane. An affine *hyperplane arrangement* $\mathcal{A} \subset U$ is a locally finite set of affine hyperplanes in U. A *directed multigraph* is a graph G that allows vertices to have multiple directed edges but no loops.

Definition. Let $\mathcal{A}$ be a hyperplane arrangement in U consisting of finitely many hyperplanes of the form $H_{ij}^a = \{x_i - x_j = a\}$ where $i, j \in \{1, \ldots, n\}$ and $a > 0$. The directed multigraph $G_{\mathcal{A}}$ associated with $\mathcal{A}$ is defined in the following way: the vertices of $G_{\mathcal{A}}$ are $\{1, \ldots, n\}$ and edges in $G_{\mathcal{A}}$ are given by $(i \to j)$, where $i, j \in \{1, \ldots, n\}$, and each edge $(i \to j)$ has multiplicity $m_{ij} := \#\{a \in \mathbb{R}_{>0} \mid H_{i,j}^a \in \mathcal{A}\}$. We call hyperplanes arrangements of this type *multigraphical hyperplane arrangements*.

Note that in a given arrangement $\mathcal{A}$, there are a total of $m_{ij} + m_{ji}$ hyperplanes parallel to $\{x_i = x_j\}$. The multigraph $G_{\mathcal{A}}$ does not determine the combinatorial type of the the hyperplane arrangement $\mathcal{A}$ because we can shift the hyperplanes in $\mathcal{A}$ by changing the constants a_k in $\{x_i - x_j = a_k\}$ without altering the directed multigraph $G_{\mathcal{A}}$. An example is shown in Fig. 1.1.

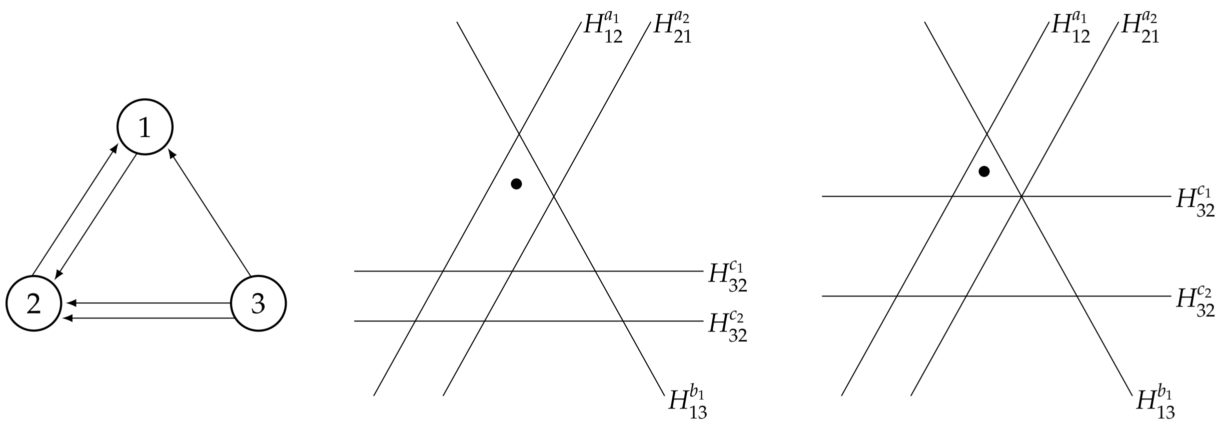


Figure 1.1: *Two hyperplane arrangements associated with the same directed multigraph*

Definition. Let $f : \{1, \ldots, n\} \to \mathbb{Z}_{\geq 0}$. We say that f is a *G-parking function* if for any non-empty $I \subset \{1, \ldots, n\}$ there exists a vertex $i \in I$ such that the number of edges $i \to j \in E_G$, where $j \notin I$, counted with multiplicity, is greater than or equal to $f(i)$.

A *region* R of a hyperplane arrangement $\mathcal{A}$ is a connected component of $U \setminus \mathcal{A}$. The *fundamental region* R_0 is the region of $\mathcal{A}$ that contains the origin.

Definition. Pak-Stanley labeling is the map from the set $\mathcal{R}$ of regions of a multigraphical arrangement $\mathcal{A}$ to the set of n-tuples of integers $\lambda : \mathcal{R} \mapsto \langle \lambda_R(1), \ldots, \lambda_R(n) \rangle$ defined in the following way: the fundamental region R_0 is labeled by $\langle 0, \ldots, 0 \rangle$. Each time we cross a hyperplane H_{ij}^a moving **away** from the origin we increase the i^{th} component by one, leaving the rest of components unchanged.

Equivalently, the Pak-Stanley labeling can be defined as follows:

Definition. [5] Let R be a region of an arrangement $\mathcal{A}$. Let $\mathcal{A}_R \subset \mathcal{A}$ be the subset of hyperplanes that separate a region R from the origin. We define the label λ_R to be the function $\lambda_R : \{1, \ldots, n\} \to \mathbb{Z}_{\geq 0}$ given by the formula

$$\lambda_R(i) := \#\{(a, j) \mid a \in \mathbb{R}_{>0}, j \in \{1, \ldots, n\}, \text{ and } H_{ij}^a \in \mathcal{A}_R\}.$$

In other words, $\lambda_R(i)$ is equal to the number of hyperplanes in the arrangement $\mathcal{A}$ of the form H_{ij}^a that separate the region R from the origin. Although i is fixed, j and a may vary.

Note that for any G-parking function λ there exists $i \in \{1, \ldots, n\}$ such that $\lambda(i) = 0$. Indeed, to see this, it suffices to take $I = \{1, \ldots, n\}$ in the definition of a G-parking function. An example of a Pak-Stanley labeling of a multigraphical arrangement is shown in Fig. 1.2.

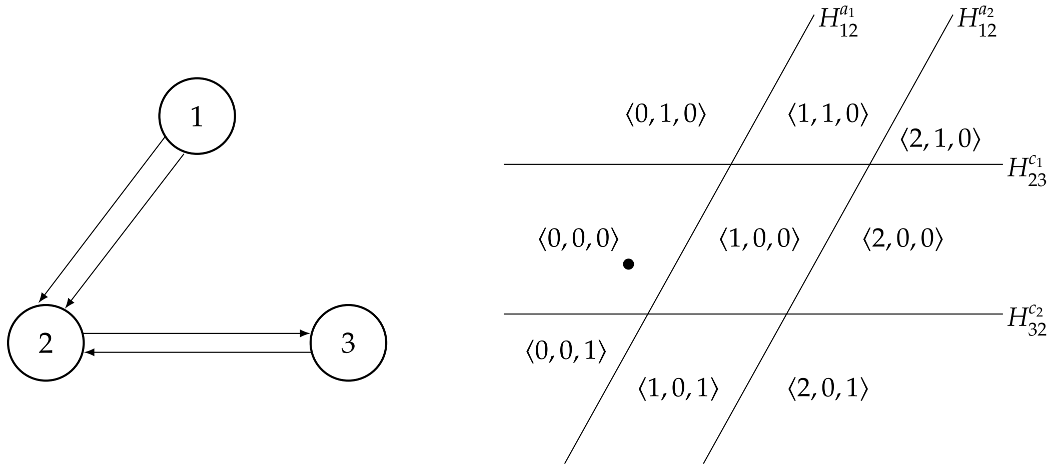


Figure 1.2: *Multigraphical arrangement with Pak-Stanley labeling*

The following results have been proved.

Theorem 1 ([2, 3])**.** *Let R be any region of a multigraphical arrangement $\mathcal{A}$. Then the corresponding label λ_R is a $G_{\mathcal{A}}$-parking function.*

Theorem 2 ([2, 3])**.** *Let $\mathcal{A}$ be a multigraphical arrangement, and let λ be any $G_{\mathcal{A}}$-parking function. Then there exists a region R of $\mathcal{A}$ such that $\lambda_R = \lambda$.*

A *central hyperplane arrangement* is a hyperplane arrangement $\mathcal{A}$ such that all hyperplanes in $\mathcal{A}$ intersect at some point $c = (c_1, c_2, \ldots, c_n)$. A *simple acyclic digraph* is directed multigraph with no cycles and no multiple edges.

Lemma 1 ([4])**.** *Let $\mathcal{A}$ be a central multigraphical arrangement. Then the corresponding directed multigraph is simple and acyclic. Vice versa, if G is a simple acyclic digraph, then there exists a central multigraphical arrangement $\mathcal{A}$ such that $G_{\mathcal{A}} = G$.*

Theorem 3 ([4]). *Let $V = \{1, \ldots, n\}$ and $G_{\mathcal{A}} = (V, E)$ be an acyclic directed graph on n vertices. Then the central multigraphical hyperplane arrangement $\mathcal{A}$ corresponding to $G_{\mathcal{A}}$ produces duplicate Pak-Stanley labels if and only if there exists $k, i, j \in V$ such that $(k \to i), (k \to j) \in E$ but $(i \to j) \notin E$ and $(j \to i) \notin E$.*

The surjectivity of generalized Pak-Stanley labeling was conjectured by Art Duval, Caroline Klivans, and Jeremy Martin in [1] and was proved for bigraphical arrangements by Sam Hopkins and David Perkinson in [2] and generalized to multigraphical arrangements by Mikhail Mazin in [3]. In the classical case of extended Shi arrangements, one can show the bijectivity of generalized Pak-Stanley labeling by using the above results and comparing the cardinalities of the two sets. In general, however, generalized Pak-Stanley labelings often fail to be injective. Based on examples studied so far, the map is not injective "globally" if it is also not injective "locally."

Conjecture ([4]). *Let $\mathcal{A}$ be a multigraphical arrangement. Then the generalized Pak-Stanley map from the regions of $\mathcal{A}$ to the set of parking functions is injective if and only if it is injective locally. More precisely, suppose that the generalized Pak-Stanley map is not injective. Then there exists a point $x \in U$ such that x belongs to the boundaries of two distinct regions with the same label.*

A natural question is to characterize the directed multigraphs for which there exist arrangements whith bijective labelings. In this thesis, we describe partial results in this direction for the special case of $n = 3$. The layout is as follows. In Chapter 2, we prove a neccessary condition for a directed multigraph to admit a corresponding hyperplane arrangement with an injective Pak-Stanley labeling. We also present a thorough example of constructing a hyperplane arrangement from a directed multigraph where an injective Pak-Stanly labeling is guaranteed. In Chapter 3, we present examples for the special case $n = 3$ in which the condition from Chapter 2 is satisfied but the directed multigraph does not admit a hyperplane arrangement with an injective Pak-Stanley labeling.

Chapter 2

A necessary condition for an injective Pak-Stanley labeling

As discussed in Chapter 1 and demonstrated in Fig. 1.1 the directed multigraph associated with a multigraphical arrangment corresponds to a family of hyperplane arrangements. We already know that the Pak-Stanley labeling is always surjective (Theorems 1 and 2). Sometimes the Pak-Stanley labeling is not bijective in an arrangement $\mathcal{A}$ associated with a directed multigraph $G_{\mathcal{A}}$ but is bijective if one or more of the hyperplanes in $\mathcal{A}$ is shifted. This shift is obtained by replacing a hyperplane $H_{ij}^{a} = \{x_i - x_j = a\}$ with the hyperplane $H_{ij}^{a'} = \{x_i - x_j = a'\}$. For an example, see Fig. 2.1.

Since the Pak-Stanley labeling is always surjective, the number of regions is bounded from below by the number of parking functions and the labeling can only be bijective if the number of regions is equal to the number of $G_{\mathcal{A}}$-parking functions. Therefore, for a given directed multigraph we always consider a hyperplane arrangement with the fewest possible regions.

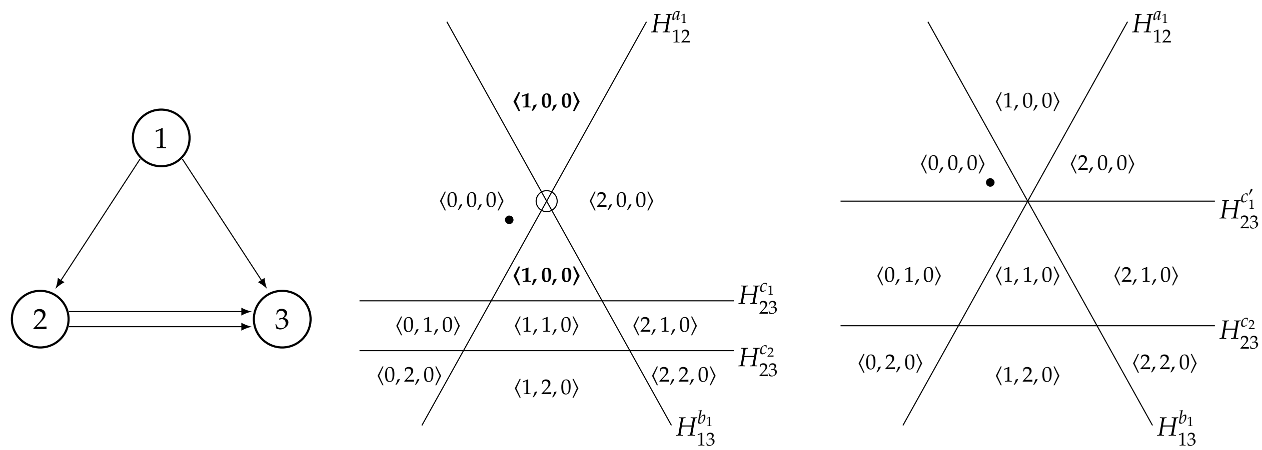


Figure 2.1: *Two hyperplanes associated with the same directed multigraph with different Pak-Stanley labelings. Both are surjective, but only the one on the right is injective.*

2.0.1 A necessary condition

Every intersection in a hyperplane arrangement $\mathcal{A}$ is locally a central multigraphical arrangement. We know from Theorem 3 that a central multigraphical arrangement $\mathcal{A}$ corresponding to $G_{\mathcal{A}}$ produces a duplicate Pak-Stanley labeling if and only if there exists $1 \leq i < j < k \leq n$ such that $(i \to j), (i \to k) \in E$ but $(k \to j) \notin E$ and $(j \to k) \notin E$. On the corresponding multigraphical arrangement $\mathcal{A}$, we will call the point of intersection of H_{ki}^a and H_{kj}^b a *bad intersection*.

A duplicate Pak-Stanley label can be eliminated if there exists a hyperplane H_{jk}^c or $H_{kj}^{c'}$ also intersecting H_{ij}^a and H_{ik}^b at the potential bad intersection. This occurs when $c = b - a$ if $b > a$ and $c' = a - b$ if $b < a$.

The following theorem provides a necessary condition for a directed multigraph to correspond to a hyperplane arrangement that admits an injective Pak-Stanley labeling.

Theorem 4. *Suppose $\mathcal{A}$ is a multigraphical arrangement with a bijective Pak-Stanley labeling and $G_{\mathcal{A}} = (V, E)$ is the corresponding directed multigraph. Fix $i, j, k \in V$ and assume $m_{ij} \neq 0$ and $m_{ik} \neq 0$. Then $m_{jk} + m_{kj} \geq m_{ij} + m_{ik} - 1$.*

Proof. Let $G_{\mathcal{A}}$ be a directed multigraph where $m_{ij} = s$ and $m_{ik} = t$. Then the arrangement

$\mathcal{A}$ contains the following hyperplanes that induce potential bad intersections:

$\{x_i - x_j = a_1\}$, $\{x_i - x_j = a_2\}$, $\ldots$, $\{x_i - x_j = a_s\}$ and $\{x_i - x_k = b_1\}$, $\{x_i - x_k = b_2\}$, $\ldots$, $\{x_i - x_k = b_t\}$.

Assume without loss of generality that $a_1 < a_2 < \ldots a_s$ and $b_1 < b_2 < \cdots < b_t$.

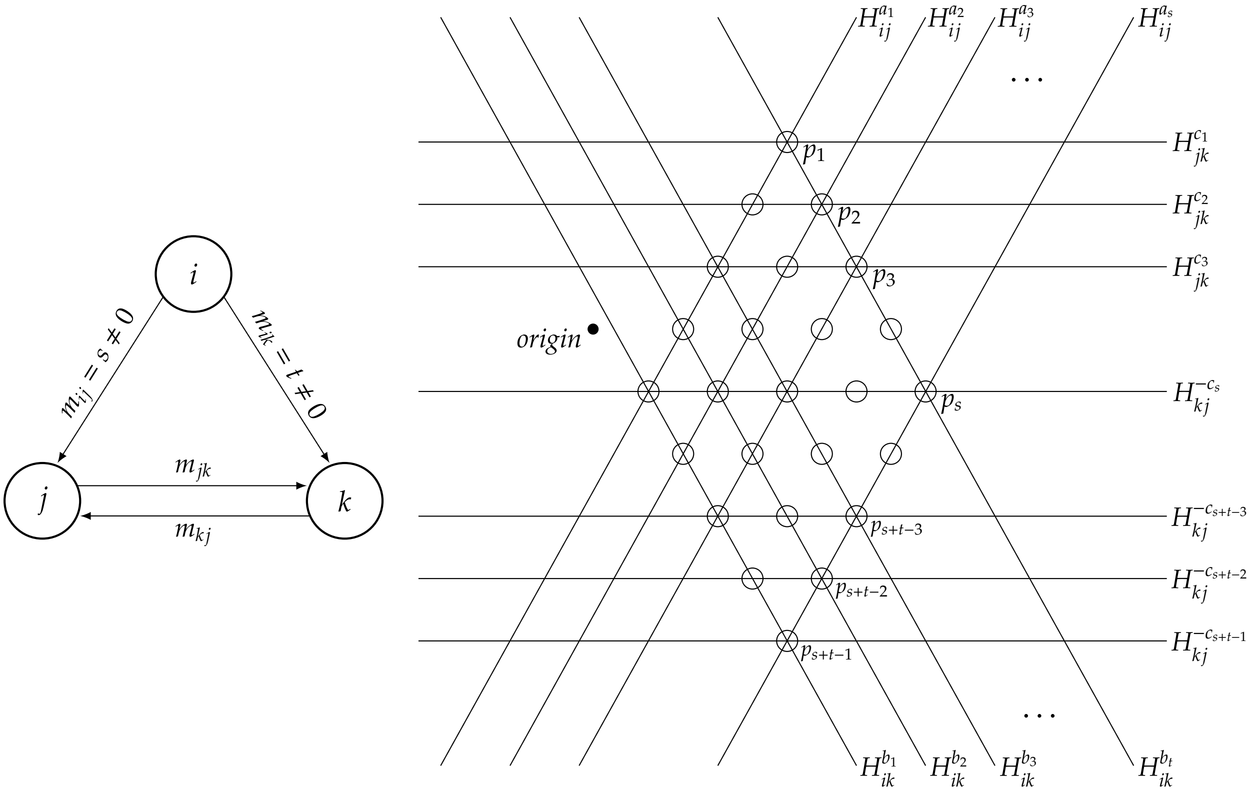


Figure 2.2: *General directed multigraph and hyperplane arrangement for Theorem 4 with potential bad intersections circled*

Define $p_1, \ldots, p_{s+t-1}$ to be the intersection points of $H_{ij}^{a_1}, \ldots, H_{ij}^{a_s}$ and $H_{ik}^{b_t}$ along with the intersection points of $H_{ik}^{b_1}, \ldots, H_{ik}^{b_t}$ and $H_{ij}^{a_s}$.

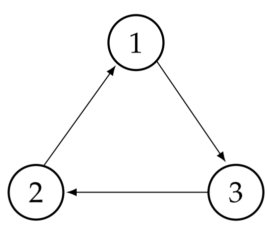
To find the hyperplane that intersects $H_{ij}^{a_1} = \{x_i - x_j = a_1\}$ and $H_{ik}^{b_t} = \{x_i - x_k = b_t\}$, we substitute $x_i = x_j + a_1$ into $H_{ik}^{b_t}$ and find that $\{x_j - x_k = b_t - a_1\}$, so $H_{jk}^{c_1 = b_t - a_1}$ intersects with $H_{ij}^{a_1}$ and $H_{ik}^{b_t}$. This eliminate their potential bad intersection. We follow a similar process to fix values for $c_1, \ldots, c_{s+t-1}$ such that each potential bad intersection has a third hyperplane to pass through it. Let $c_1 = b_t - a_1, c_2 = b_t - a_2, \ldots, c_\ell = b_t - a_\ell, c_{\ell+1} = a_s - b_{\ell+1}, \ldots, c_{s+t-2} = a_s - b_2$, and $c_{s+t-1} = a_s - b_1$. Also let ℓ be such that $c_\ell > 0 > c_{\ell+1}$.

Therefore, we have the distinct hyperplanes $H_{jk}^{c_1}, \dots, H_{jk}^{c_\ell}$ and $H_{kj}^{-c_{\ell+1}}, \dots, H_{kj}^{-c_{s+t-1}}$ to eliminate all of the potential bad intersections. So $m_{jk} \geq \ell$ and $m_{kj} \geq s+t-1-\ell$. Adding these together, we obtain $m_{jk}+m_{kj} \geq s+t-1$. Finally, since $s = m_{ij}$ and $t = m_{ik}$, $m_{jk}+m_{kj} \geq m_{ij}+m_{ik}-1$.

□

2.0.2 Multigraphical arrangements that admit an injective Pak-Stanley labeling if $m_{ij} = 0$ or $m_{ik} = 0$

Let $G_{\mathcal{A}} = (V, E)$ be a directed multigraph corresponding to a multigraphical arrangement $\mathcal{A}$. Fix $i, j, k \in V$. If either $m_{ij} = 0$ or $m_{ik} = 0$ for all $i, j, k \in \{1, \dots, n\}$, there are no bad intersections, so the directed multigraph will always admit a hyperplane arrangement with an injective Pak-Stanley labeling. Consider the multigraphical arrangement in Fig. 2.3. Note that the directed multigraph consists only of a cycle.



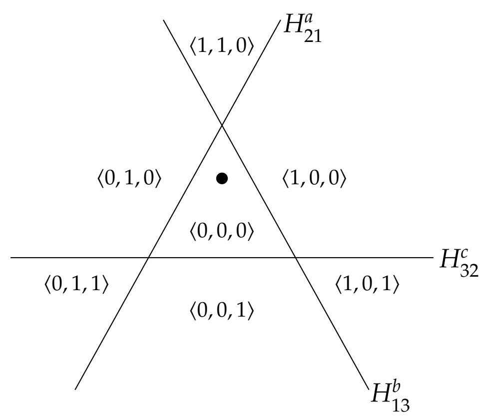


Figure 2.3: *A multigraphical arrangement with $i = 1, j = 2$, and $k = 3$. There are no bad intersections since $m_{21} = m_{13} = m_{32} = 1$ and $m_{12} = m_{23} = m_{31} = 0$.*

Let $G_{\mathcal{A}} = (V, E)$ be a directed multigraph corresponding to the the multigraphical arrangement $\mathcal{A}$ pictured in Fig. 2.4. Although the directed multigraph does not form a cycle, there are no bad intersections because $m_{12} = m_{31} = 0$ and $m_{21} = m_{13} = 0$.

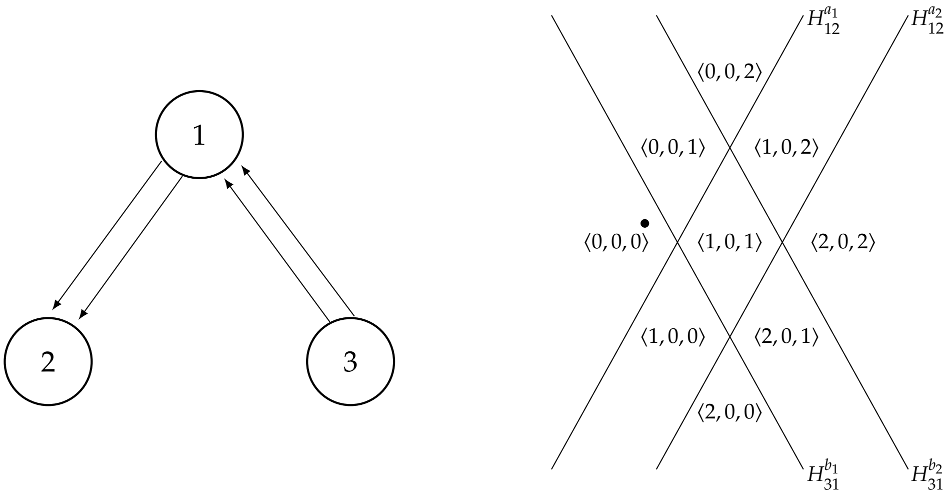


Figure 2.4: *A multigraphical arrangement with $i = 1, j = 2$, and $k = 3$. There are no bad intersections since $m_{12} = m_{31} = 0$ and $m_{21} = m_{13} = m_{23} = m_{32} = 0$.*

2.0.3 Example of multigraphical arrangement that admits an injective Pak-Stanley labeling

Now we present an example of a building a hyperplane arrangement with an injective Pak-Stanley labeling from a directed multigraph. Fix $i = 1, j = 2$, and $k = 3$. Let $G_{\mathcal{A}}$ be a directed multigraph with $m_{23} = 2, m_{32} = 1, m_{12} = 1, m_{13} = 2, m_{21} = 1$, and $m_{31} = 0$ (Fig. 2.5).

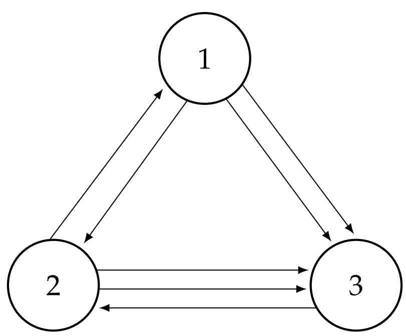


Figure 2.5: *Directed multigraph from which we will construct a hyperplane arrangement that admits an injective Pak-Stanley labeling*

We start with the hyperplanes $H_{12}^{a_1} = \{x_1 - x_2 = a_1\}, H_{21}^{a_2} = \{x_2 - x_1 = a_2\}, H_{13}^{b_1} = \{x_1 - x_3 = b_1\},$

and $H_{13}^{b_2} = \{x_1 - x_3 = b_2\}$ and fix the values a_1, a_2, b_1, and b_2. We will clarify restrictions on these values as needed.

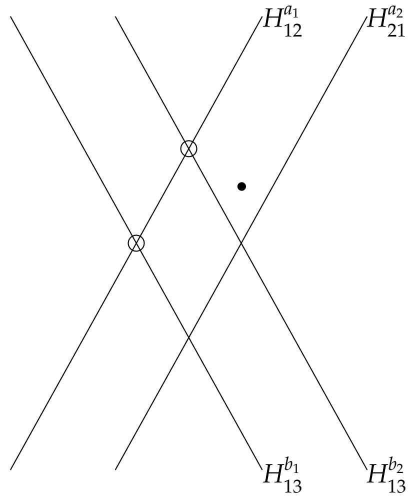


Figure 2.6: *First four hyperplanes constructed from the directed multigraph in Fig. 2.5*

Next we deal with bad intersections created by the intersection of the hyperplane $H_{12}^{a_1}$ with the hyperplanes $H_{13}^{b_1}$ and $H_{13}^{b_2}$. To find where $\{x_1 - x_2 = a_1\}$ and $\{x_1 - x_3 = b_1\}$ are equal, we add $x_1 - x_2 = a_1$ and $-(x_1 - x_3 = b_1)$ to obtain $x_3 - x_2 = a_1 - b_1$. So we need to add the hyperplane $H_{32}^{c_1} = \{x_3 - x_2 = c_1 = a_1 - b_1\}$ to the hyperplane arrangement. We know that $m_{32} = 1$. Using a similar process, the hyperplane we need to add to eliminate the bad intersection of $H_{12}^{a_1}$ with $H_{13}^{b_1}$ is $H_{23}^{c_2} = \{x_2 - x_3 = c_2 = b_2 - a_1\}$.

Now we consider potential bad intersection induced by the hyperplanes $H_{21}^{a_2}$ and $H_{23}^{c_2}$. Since $m_{31} = 0$, we need $H_{13}^{b_2} = \{x_1 - x_3 = c_2 - a_2\}$. Note that we already set $c_2 = b_2 - a_1$ and now we found that $c_2 = a_2 + b_2$. Therefore, the values a_1, a_2, b_1, and b_2 must satisfy $b_1 = a_1 + a_2 + b_2$. We still have $H_{23}^{c_3}$ left to place in the hyperplane arrangement. It must intersect with $H_{21}^{a_2}$ somewhere and can potentially cause a bad intersection. To avoid this, we need to fix $c_3 = a_2 + b_1$.

Finally, we demonstrate that there are no duplicate Pak-Stanley labels in Fig. 2.9.

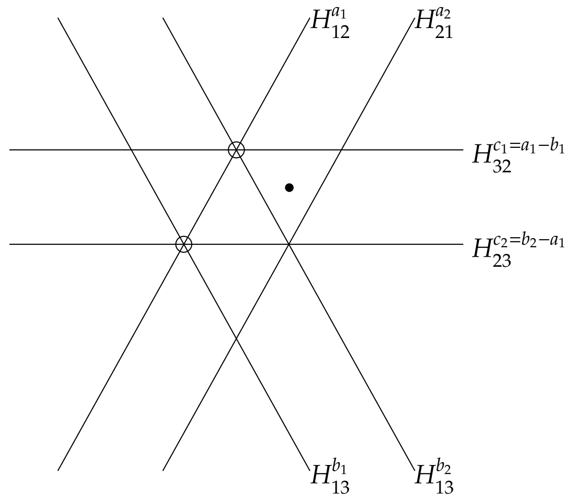


Figure 2.7: *Hyperplanes $H_{32}^{c_1}$ and $H_{23}^{c_2}$ added to the hyperplane arrangement in Fig. 2.6*

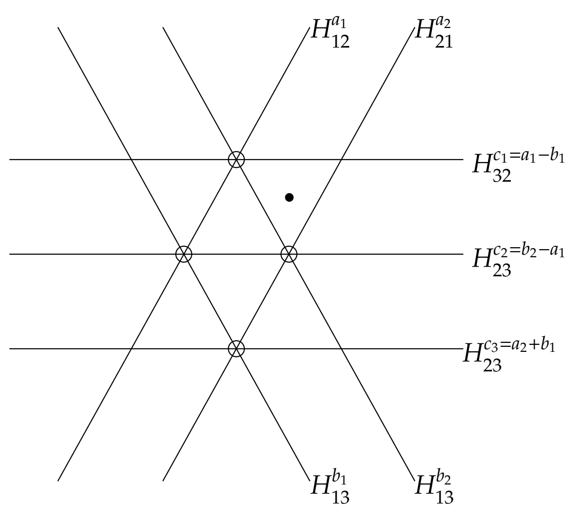


Figure 2.8: *Final hyperplane $H_{23}^{c_3}$ added to the arrangement in Fig. 2.7. There are no bad intersections, so this arrangement admits an injective Pak-Stanley labeling.*

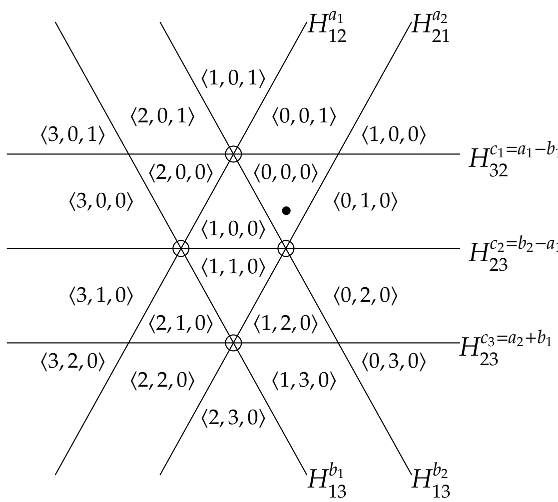


Figure 2.9: *Multigraphical arrangement that admits an injective Pak-Stanley labeling constructed from the directed multigraph in Fig. 2.5*

Chapter 3

Examples of directed multigraphs that only admit hyperplane arrangements with a non-injective Pak-Stanley labeling

This chapter contains examples of directed multigraphs that satisfy the conditions of Theorem 4 but still do not admit a hyperplane arrangement with a bijective Pak-Stanley labeling for the special case $n = 3$.

3.0.1 Directed Multigraphs with $m_{32} = m_{31} = 0$

The following lemma generalizes when multigraphical arrangements with only four different types of hyperplanes satisfies the necessary condition for an injective Pak-Stanley labeling described in Theorem 4.

Lemma 2. *Let $\mathcal{A}$ be a multigraphical arrangement such that $m_{32} = m_{31} = 0$ and all other m_{ij} be non-zero. Then $\mathcal{A}$ satisfies the conditions of Theorem 4, i. e., $m_{23} + m_{32} \geq m_{12} + m_{13} - 1$, $m_{13} + m_{31} \geq m_{21} + m_{23} - 1$, and $m_{12} + m_{21} \geq m_{31} + m_{32} - 1$, whenever $m_{12} = m_{21} = 1$ and $m_{13} = m_{23}$.*

Proof. Consider the multigraphical arrangment $\mathcal{A}$ such that $m_{32} = m_{31} = 0$ and all other m_{ij} are non-zero. From Theorem 4, we know that $m_{23} \geq m_{12} + m_{13} - 1$ and $m_{13} \geq m_{21} + m_{23} - 1$. Adding these together, we obtain that $2 \geq m_{12} + m_{21}$. Since both m_{12} and m_{21} are non-zero, $m_{12} = m_{21} = 1$. Finally, $m_{23} \geq m_{13}$ and $m_{13} \geq m_{23}$, so $m_{13} = m_{23}$. Therefore, any multigraphical arrangement with $m_{32} = m_{31} = 0, m_{12} = m_{21} = 1$, and $m_{13} = m_{23}$. □

The following examples demonstrate that the necessary condition is not sufficient for determining when a multigraphical arrangement such that $m_{32} = m_{31} = 0$ admits an injective Pak-Stanley labeling.

Example 1. Let $G_{\mathcal{A}}$ be the directed multigraph with $m_{12} = m_{21} = m_{13} = m_{23} = 1$. The corresponding hyperplanes are $H_{12}^{a_1}, H_{21}^{a_2}, H_{13}^{b}$, and H_{23}^{c} (Fig. 3.1).

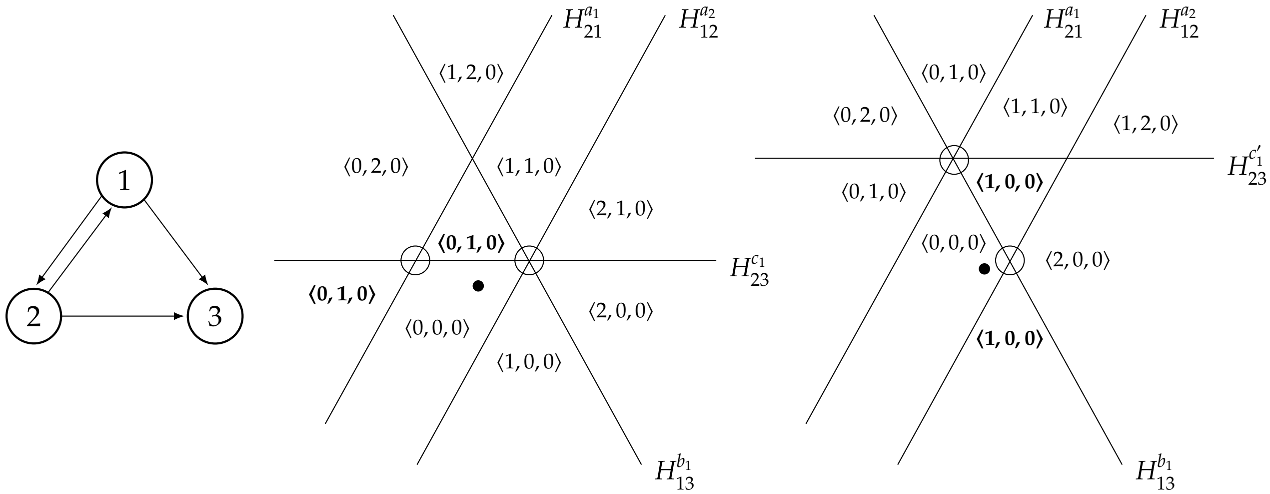


Figure 3.1: *Multigraphical arrangement with duplicate labels* $\langle 0,1,0\rangle$ *and* $\langle 1,0,0\rangle$

Potential bad intersections occur where $H_{12}^{a_1} = \{x_1 - x_2 = a_1\}$ and $H_{13}^{b} = \{x_1 - x_3 = b\}$ intersect and also where $H_{21}^{a_2} = \{x_2 - x_1 = a_2\}$ and $H_{23}^{c} = \{x_2 - x_3 = c\}$ intersect. Assume that this multigraphical arrangement admits an injective Pak-Stanley labeling. To eliminate the potential bad intersection of $H_{12}^{a_1}$ with H_{13}^{b} we need $H_{23}^{c} = \{x_2 - x_3 = c = b - a_1\}$ because $m_{32} = 0$. For the intersection of $H_{21}^{a_2}$ with H_{23}^{c} we need $H_{13}^{b} = \{x_1 - x_3 = b = a_2 - c\}$. But then since $c = b - a_1$ and $c = b + a_2$, $b = b - a_1 - a_2$, a contradiction because $m_{13} = 1$ and both a_1 and a_2 are non-zero.

Example 2. Let $G_{\mathcal{A}}$ be the directed multigraph with $m_{12} = m_{21} = 1$ and $m_{13} = m_{23} = 2$. The corresponding hyperplanes are $H_{12}^{a_1}, H_{21}^{a_2}, H_{13}^{b_1}, H_{13}^{b_2}, H_{23}^{c_1}$, and $H_{23}^{c_2}$.

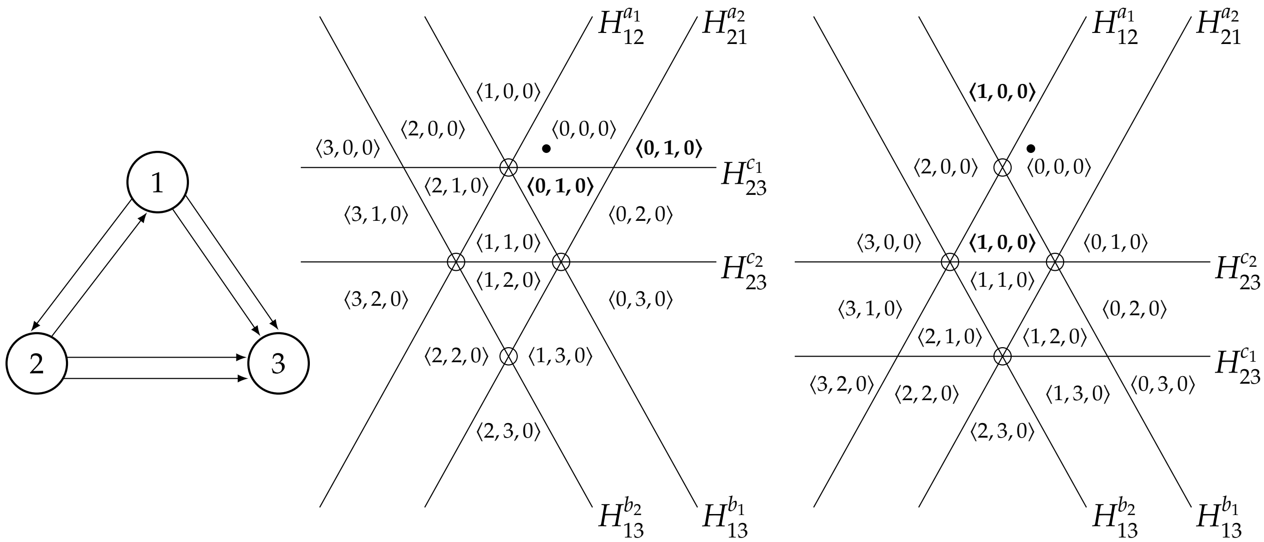


Figure 3.2: *Multigraphical arrangement with duplicate labels*

Potential bad intersections occur where the following pairs of hyperplanes intersect: $H_{12}^{a_1} = \{x_1 - x_2 = a_1\}$ and $H_{13}^{b_1} = \{x_1 - x_3 = b_1\}$, $H_{12}^{a_1} = \{x_1 - x_2 = a_1\}$ and $H_{13}^{b_2} = \{x_1 - x_3 = b_2\}$, $H_{21}^{a_2} = \{x_2 - x_1 = a_2\}$ and $H_{23}^{c_1} = \{x_2 - x_3 = c_1\}$, $H_{21}^{a_2} = \{x_2 - x_1 = a_2\}$ and $H_{23}^{c_2} = \{x_2 - x_3 = c_2\}$. For an injective Pak-Stanley labeling, we need, respectively, $c_1 = b_1 - a_1$, $c_2 = b_2 - a_1$, $b_1 = c_1 - a_1$, and $b_2 = c_2 - a_2$. Since $b_2 = c_2 + a_1$ and $b_2 = c_2 - a_2$, $c_2 = c_2 - a_1 - a_2$, a contradiction since $a_1 > 0$ and $a_2 > 0$.

The following theorem generalizes the previous two examples.

Theorem 5. *Let $\mathcal{A}$ be a multigraphical arrangement such that $m_{31} = m_{32} = 0$. If $m_{12} = m_{21} = 1$ and $m_{13} = m_{23}$, even though $m_{23} \geq m_{12} + m_{13} - 1$ and $m_{13} \geq m_{21} + m_{23} - 1$, $\mathcal{A}$ does not admit an injective Pak-Stanley labeling.*

Proof. Consider the general hyperplane arrangement represented in Fig. 3.3. Note that $b_1 > b_2 > \cdots > b_t$, and $c_1 > c_2 > \cdots > c_t$. Assume that the Pak-Stanley labeling is injective. Then every potential bad intersection has three hyperplanes that intersect it.

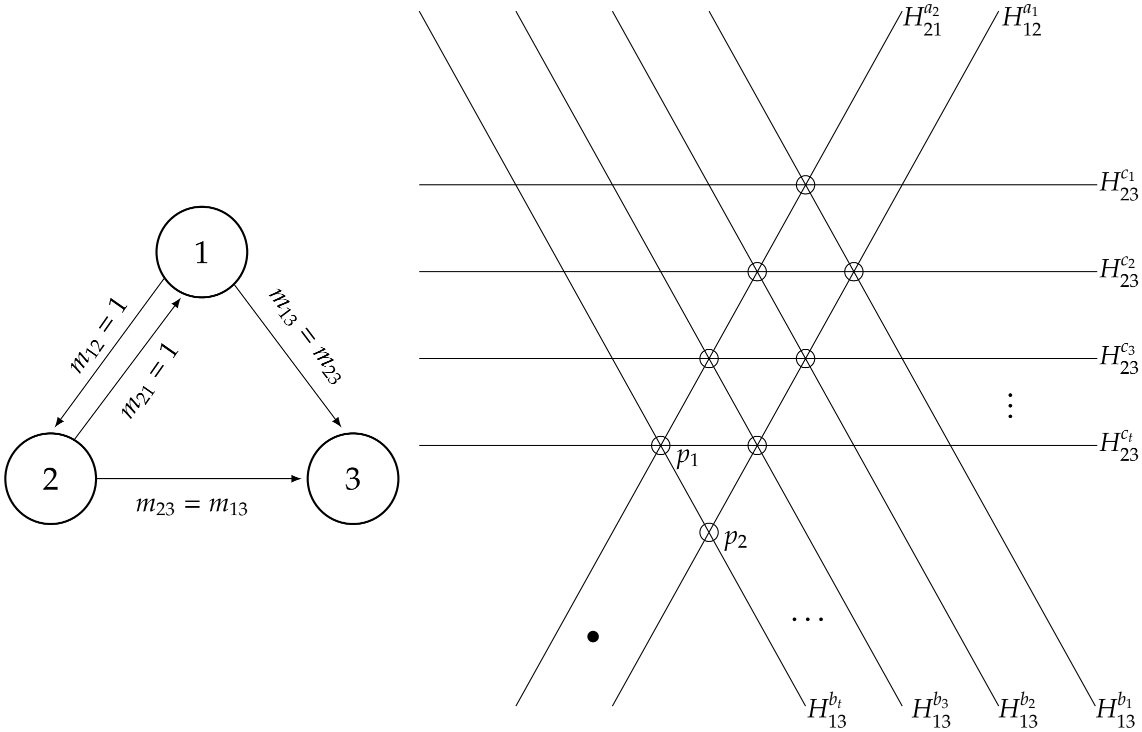


Figure 3.3: *General directed multigraph and hyperplane arrangement that satisfy the conditions of Theorem 5*

Consider the potential bad intersection p_1 of $H_{21}^{a_2} = \{x_2 - x_1 = a_2\}$ with $H_{23}^{c_k} = \{x_2 - x_3 = c_t\}$. Then there has to be a hyperplane $H_{13}^{b_j} = \{x_1 = x_3 = b_j = c_t - a_2\}$ passing through that intersection as well. Let p_2 be the potential bad intersection of $H_{12}^{a_1} = \{x_1 - x_2 = a_1\}$ with $H_{13}^{b_j} = \{x_1 - x_3 = b_j = c_t - a_2\}$. Then there has to be a hyperplane $H_{23}^{c_k} = \{x_2 - x_3 = c_k = b_j - a_1\}$ also at that intersection. So we have $c_k = c_t - a_1 - a_2$. But this is a contradiction because c_t is the smallest c_i for $1 \leq i \leq t$, so it cannot be that $c_k < c_t$. $\square$

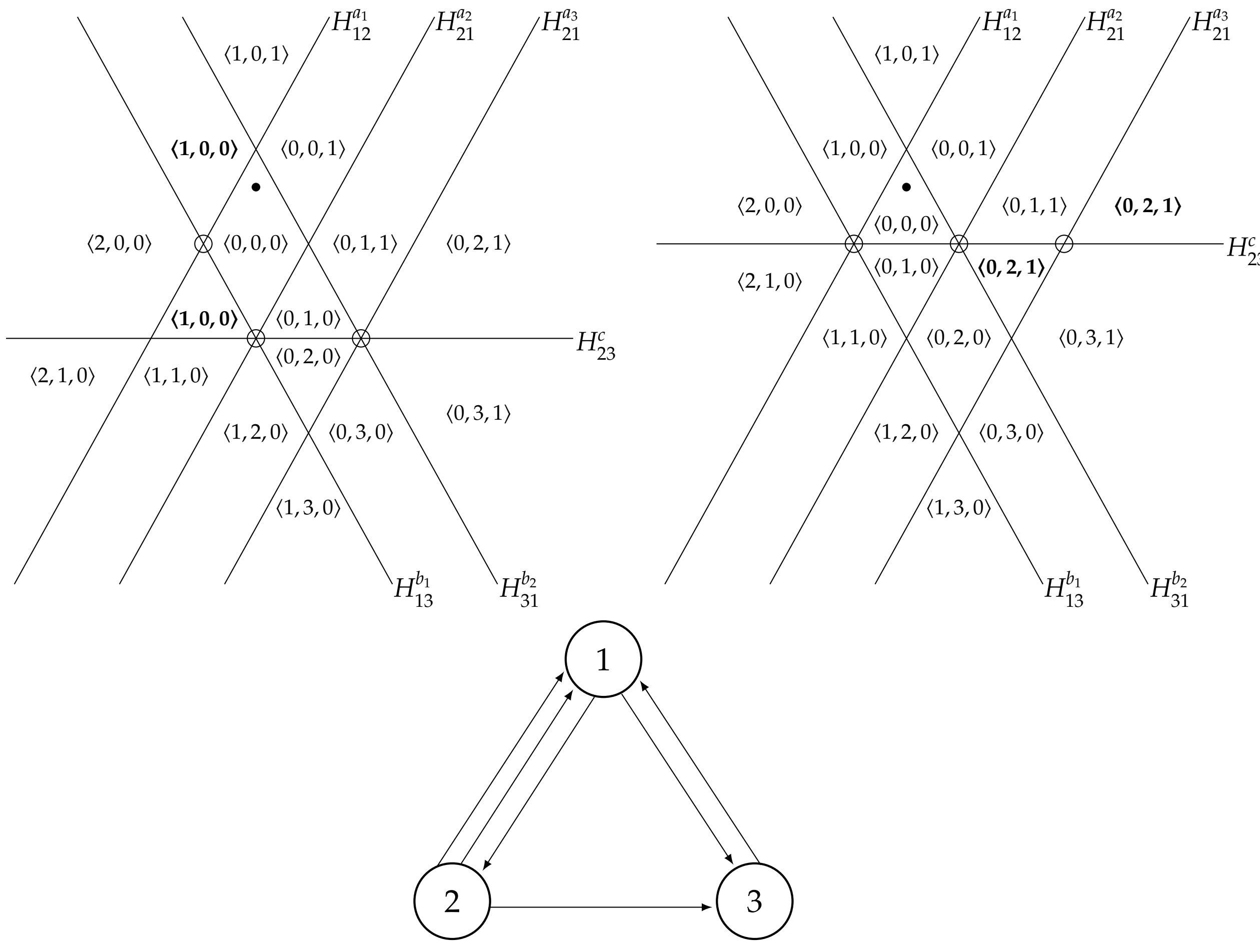


Figure 3.4: *Multigraphical arrangement with 5 types of hyperplanes that admits only a non-injective Pak-Stanley labeling despite meeting the conditions of Theorem 4*

3.0.2 Directed multigraphs with $m_{32} = 0$

Example 3. Let $G_{\mathcal{A}}$ be the directed mulitigraph with $m_{12} = 1, m_{21} = 2, m_{13} = 1, m_{31} = 1$, and $m_{23} = 1$ (Fig. 3.4). The corresponding hyperplanes are $H_{12}^{a_1}, H_{21}^{a_2}, H_{13}^{b_1}, H_{31}^{b_2}$, and H_{23}^{c}. To confirm that the conditions of Theorem 4 are satisfied,

$$m_{23} \geq m_{12} + m_{13} - 1 \qquad\qquad m_{13} + m_{31} \geq m_{21} + m_{23} - 1$$

$$1 \geq 1 + 1 - 1 \qquad\qquad 1 + 1 \geq 2 + 1 - 1$$

$$1 \geq 1 \qquad\qquad 2 \geq 2$$

Assume that this multigraphical arrangement admits an injective Pak-Stanley labeling. Then each potential bad intersection has three hyperplanes going through it. First we consider the potential bad intersections induced by $H_{21}^{a_2}$ and $H_{21}^{a_3}$ with H_{23}^{c}. To avoid duplicate Pak-Stanley labels, we will need the hyperplanes $H_{13}^{b_1}$ and $H_{13}^{b_2}$. We have $a_2 + b_1 - c = 0$ and $-a_3 + b_2 + c = 0$, so we fix the values $b_1 = c - a_2$ and $b_2 = -c + a_3$. Next we look at the potential bad intersection caused by $H_{12}^{a_2}$ and $H_{13}^{b_1}$, which occurs at $a_1 - b_1 + c = 0$. We need H_{23}^{c} to pass through the intersection of $H_{12}^{a_2}$ and $H_{13}^{b_1}$, so $c = b_1 - a_1$. Thus, $c = c - a_1 - a_2$. But this is a contradiction because $a_1 > 0$ and $a_2 > 0$. Therefore, this multigraphical arrangement does not admit an injective Pak-Stanley labeling.

3.0.3 Directed multigraphs with all m_{ij} non-zero

Example 4. Consider the multigraphical arrangement $\mathcal{A}$ with $m_{12} = m_{21} = 2$ and $m_{13} = m_{31} = m_{23} = m_{32} = 1$ (Fig. 3.5). We confirm that $\mathcal{A}$ satisfies the conditions of Theorem 4:

$$\begin{array}{ccc} m_{23} + m_{32} \geq m_{12} + m_{13} - 1 & m_{13} + m_{31} \geq m_{21} + m_{23} - 1 & m_{12} + m_{21} \geq m_{31} + m_{32} - 1 \\ 1 + 1 \geq 2 + 1 - 1 & 1 + 1 \geq 2 + 1 - 1 & 2 + 2 \geq 1 + 1 - 1 \\ 2 \geq 2 & 2 \geq 2 & 4 \geq 1 \end{array}$$

Assume that $\mathcal{A}$ admits an injective Pak-Stanley labeling. Potential bad intersections and the hyperplanes needed to avoid them are organized in Table 3.2.

Without loss of generality, fix $a_1 > a_2$. Consider the potential bad intersection of $H_{12}^{a_1}$ and $H_{13}^{b_1}$ along with that of $H_{12}^{a_2}$ and $H_{13}^{b_1}$. Note that have one hyerplane each of $H_{32}^{c_1}$ and $H_{23}^{c_2}$. Since $a_1 > a_2$, $b_1 - a_2 > b_1 - a_1$. To eliminate the potential bad intersections of $H_{12}^{a_i}$ with $H_{13}^{b_1}$ for $i = 1, 2$, it must be that $b_1 - a_2 > 0 > b_1 - a_1$. Thus we set $c_1 = b_1 - a_2$ and $c_2 = b_1 - a_1$. Note that c_1 and c_2 are both positive.

Next we consider the potential bad intersection of $H_{21}^{a_3}$ and $H_{23}^{c_2}$ along with that of $H_{21}^{a_4}$

Potential bad intersection	Possible resolving hyperplane
$H_{12}^{a_1}$ and $H_{13}^{b_1}$	$H_{23}^{c_k}$ or $H_{32}^{c_k}$ $\quad k \in \{1,2\}$
$H_{12}^{a_2}$ and $H_{13}^{b_1}$	$H_{23}^{c_k}$ or $H_{32}^{c_k}$ $\quad k \in \{1,2\}$
$H_{21}^{a_3}$ and $H_{23}^{c_2}$	$H_{13}^{b_j}$ or $H_{31}^{b_j}$ $\quad j \in \{1,2\}$
$H_{21}^{a_4}$ and $H_{23}^{b_1}$	$H_{13}^{b_j}$ or $H_{31}^{b_j}$ $\quad j \in \{1,2\}$
$H_{31}^{b_2}$ and $H_{32}^{c_1}$	$H_{12}^{a_i}$ or $H_{21}^{a_i}$ $\quad i \in \{1,2\}$

Table 3.1: *Potential bad intersections and hyperplanes that can be used to eliminate duplicate labels*

and $H_{23}^{c_2}$. We have one hyperplane on each side of the origin that can be used to eliminate either of these potential bad intersections: $H_{13}^{b_1}$ and $H_{31}^{b_2}$. Without loss of generality, assume that $a_3 < a_4$. Then $c_2 - a_3 > c_2 - a_4$ and because we have one hyperplane on each side of the origin, it must be that $c_2 - a_3 > 0 > c_2 - a_4$. Therefore, we must set $b_1 = c_2 - a_3$ and $b_2 = c_2 - a_4$, where b_1 and b_2 are both positive.

From the values we fixed to elminate the potential bad intersections of $H_{12}^{a_i}$ with $H_{13}^{b_1}$, we know that $b_1 = c_2 + a_1$. Commbining this with the values we found to eliminate the potential bad intersections of $H_{21}^{a_i}$ with $H_{23}^{c_2}$ we have $c_2 + a_1 = c_2 - a_3$. Thus $0 = -a_1 - a_3$, which is a contradiction because $a_1 > 0$ and $a_3 > 0$. Therefore, this multigraphical arrangement does not admit an injective Pak-Stanley labeling.

Example 5. Consider the multigraphical arrangement $\mathcal{A}$ (Fig. 3.6) with $m_{12} = m_{21} = 3$ and $m_{13} = m_{31} = m_{23} = m_{32} = 2$. We begin by confirming that $\mathcal{A}$ satisfies the conditions of Theorem 4:

$$m_{23} + m_{32} \geq m_{12} + m_{13} - 1 \qquad m_{13} + m_{31} \geq m_{21} + m_{23} - 1 \qquad m_{12} + m_{21} \geq m_{31} + m_{32} - 1$$
$$2 + 2 \geq 3 + 2 - 1 \qquad 2 + 2 \geq 3 + 2 - 1 \qquad 3 + 3 \geq 2 + 2 - 1$$
$$4 \geq 4 \qquad 4 \geq 4 \qquad 6 \geq 3$$

Without loss of generality, fix $a_1 > a_2 > \cdots > a_6$ and $b_1 > b_2 > \cdots > b_4$. Consider the following potential bad intersections: $H_{12}^{a_1}$ and$H_{13}^{b_2}$, $H_{12}^{a_1}$ and $H_{13}^{b_1}$, $H_{12}^{a_2}$ and $H_{13}^{b_1}$, $H_{12}^{a_3}$ and $H_{13}^{b_1}$.

Potential bad intersection	Possible resolving hyperplanes
$H_{12}^{a_1}$ and $H_{13}^{b_1}$	$H_{23}^{c_k}$ or $H_{32}^{c_k}$ $\quad k \in \{1,2,3,4\}$
$H_{12}^{a_1}$ and $H_{13}^{b_2}$	$H_{23}^{c_k}$ or $H_{32}^{c_k}$ $\quad k \in \{1,2,3,4\}$
$H_{12}^{a_2}$ and $H_{13}^{b_1}$	$H_{23}^{c_k}$ or $H_{32}^{c_k}$ $\quad k \in \{1,2,3,4\}$
$H_{12}^{a_2}$ and $H_{13}^{b_2}$	$H_{23}^{c_k}$ or $H_{32}^{c_k}$ $\quad k \in \{1,2,3,4\}$
$H_{12}^{a_3}$ and $H_{13}^{b_1}$	$H_{23}^{c_k}$ or $H_{32}^{c_k}$ $\quad k \in \{1,2,3,4\}$
$H_{12}^{a_3}$ and $H_{13}^{b_2}$	$H_{23}^{c_k}$ or $H_{32}^{c_k}$ $\quad k \in \{1,2,3,4\}$
$H_{21}^{a_4}$ and $H_{23}^{c_3}$	$H_{13}^{b_j}$ or $H_{31}^{b_j}$ $\quad j \in \{1,2,3,4\}$
$H_{21}^{a_4}$ and $H_{23}^{c_4}$	$H_{13}^{b_j}$ or $H_{31}^{b_j}$ $\quad j \in \{1,2,3,4\}$
$H_{21}^{a_5}$ and $H_{23}^{c_3}$	$H_{13}^{b_j}$ or $H_{31}^{b_j}$ $\quad j \in \{1,2,3,4\}$
$H_{21}^{a_5}$ and $H_{23}^{c_4}$	$H_{13}^{b_j}$ or $H_{31}^{b_j}$ $\quad j \in \{1,2,3,4\}$
$H_{21}^{a_6}$ and $H_{23}^{c_3}$	$H_{13}^{b_j}$ or $H_{31}^{b_j}$ $\quad j \in \{1,2,3,4\}$
$H_{21}^{a_6}$ and $H_{23}^{c_4}$	$H_{13}^{b_j}$ or $H_{31}^{b_j}$ $\quad j \in \{1,2,3,4\}$
$H_{21}^{a_4}$ and $H_{23}^{b_1}$	$H_{13}^{b_j}$ or $H_{31}^{b_j}$ $\quad j \in \{1,2,3,4\}$
$H_{31}^{b_3}$ and $H_{32}^{c_1}$	$H_{12}^{a_i}$ or $H_{21}^{a_i}$ $\quad i \in \{1,2,3,4,5,6\}$
$H_{31}^{b_3}$ and $H_{32}^{c_2}$	$H_{12}^{a_i}$ or $H_{21}^{a_i}$ $\quad i \in \{1,2,3,4,5,6\}$
$H_{31}^{b_4}$ and $H_{32}^{c_1}$	$H_{12}^{a_i}$ or $H_{21}^{a_i}$ $\quad i \in \{1,2,3,4,5,6\}$
$H_{31}^{b_4}$ and $H_{32}^{c_2}$	$H_{12}^{a_i}$ or $H_{21}^{a_i}$ $\quad i \in \{1,2,3,4,5,6\}$

Table 3.2: *Potential bad intersections and hyperplanes that can be used to eliminate duplicate labels*

To eliminate these potential bad intersections, we will need $H_{23}^{c_k=b_j-a_i}$ or $H_{32}^{c_k=a_i-b_j}$, where $i \in \{1,2,3\}$, $j \in \{1,2\}$, and $k \in \{1,2,3,4\}$.

We know that $a_1 - b_2 < a_1 - b_1$ and $b_1 - a_1 < b_1 - a_2 < b_1 - a_3$. But we cannot have both $a_1 - b_1$ and $b_1 - a_1$ because if $a_1 = b_1$, then $c_k = 0$ for some $k \in \{1,2,3,4\}$, which contradicts that $c_k > 0$. Therefore we must choose whether to position $c_k = a_1 - b_1$ above or below the origin.

Suppose $c_k = a_1 - b_1$ is below the origin. Then we have

$$a_1 - b_1 < a_2 - b_1 < a_3 - b_1 < 0 < a_1 - b_2,$$

which means we have three hyperplanes below the origin and only one above it. But this contradicts the fact that $m_{23} = m_{32} = 2$. Therefore, $c_k = b_1 - a_1$ must be above the origin.

Potential bad intersection	Resolving hyperplane
$H_{12}^{a_1}$ and $H_{13}^{b_2}$	$H_{32}^{c_1} = \{x_3 - x_2 = c_1 = a_1 - b_2\}$
$H_{12}^{a_1}$ and $H_{13}^{b_1}$	$H_{32}^{c_2} = \{x_3 - x_2 = c_2 = a_1 - b_1\}$
$H_{12}^{a_2}$ and $H_{13}^{b_1}$	$H_{23}^{c_3} = \{x_2 - x_3 = c_3 = b_1 - a_2\}$
$H_{12}^{a_3}$ and $H_{13}^{b_1}$	$H_{23}^{c_4} = \{x_2 - x_3 = c_4 = b_1 - a_3\}$

Table 3.3: *Correspondence of potential bad intersections with resolving hyperplanes*

Potential bad intersection	Resolving hyperplane
$H_{21}^{a_6}$ and $H_{23}^{c_3}$	$H_{31}^{b_4} = \{x_3 - x_1 = b_4 = a_6 - c_3\}$
$H_{21}^{a_6}$ and $H_{23}^{c_4}$	$H_{31}^{b_3} = \{x_3 - x_1 = b_3 = a_6 - c_4\}$
$H_{21}^{a_5}$ and $H_{23}^{c_2}$	$H_{13}^{b_2} = \{x_1 - x_3 = b_2 = c_4 - a_5\}$
$H_{21}^{a_4}$ and $H_{23}^{c_1}$	$H_{13}^{b_1} = \{x_1 - x_3 = b_1 = c_4 - a_4\}$

Table 3.4: *Correspondence of potential bad intersections with resolving hyperplanes*

Then $a_1 - b_1 < a_2 - b_1 < 0 < a_1 - b_2 < a_1 - b_1$, so we assign the values c_1, c_2, c_3, and c_4 according to Table 3.3.

Now we consider the following potential bad intersections: $H_{21}^{a_6}$ and $H_{23}^{c_3}$, $H_{21}^{a_6}$ and $H_{23}^{c_4}$, $H_{21}^{a_5}$ and $H_{23}^{c_4}$, $H_{21}^{a_4}$ and $H_{23}^{c_4}$. To eliminate these potential bad intersections, we will need $H_{13}^{bj=c_k-a_i}$ or $H_{31}^{b_j=a_i-c_k}$, where $i \in \{1,2,3\}$, $j \in \{1,2\}$, and $k \in \{1,2,3,4\}$.

From the values we fixed above, we know $c_4 > c_3 > 0$, so $0 < c_3 - a_6 < c_3 - a_5 < c_3 - a_4$ and $0 < a_6 - c_3 < a_6 - c_4$. But we can't have both $0 < c_3 - a_6$ and $0 < a_6 - c_3$. If $0 < a_6 - c_3$, then $c_4 - a_6 < 0 < c_3 - a_6 < c_3 - a_5 < c_3 - a_4$. Again, this leaves us with one hyperplane on one side of the origin and three on the other, contradicting the fact that $m_{13} = m_{31} = 2$.

So we must choose $0 < c_3 - a_6$ and $c_4 - a_6 < c_3 - a_6 < 0 < c_4 - a_5 < c_4 - a_4$. Thus we assign the values b_1, b_2, b_3, and b_4 according to Table 3.4.

From Table 3.3, we know that $b_1 = c_4 - a_4$ and from Table 3.4 we know that $b_1 = a_3 + c_4$. Then $c_4 - a_4 = c_4 + a_3$, so implies $0 = a_3 + a_4$, which contradicts the assumption that $a_3 > 0$ and $a_4 > 0$. Therefore, this multigraphical arrangement $\mathcal{A}$ does not admit an injective Pak-Stanley label.

The following theorem generalizes the previous two examples.

Theorem 6. *Let $\mathcal{A}$ be a multigraphical arrangement such that $m_{13} = m_{23} = m_{32} = m_{31} = t$ and $m_{12} = m_{21} = t+1$. Even though $m_{23} + m_{32} \geq m_{12} + m_{13} - 1, m_{13} + m_{31} \geq m_{21} + m_{23} - 1$, and $m_{12} + m_{21} \geq m_{31} + m_{32} - 1$ (Theorem 4), $\mathcal{A}$ does not admit an injective Pak-Stanley labeling.*

Proof. First, we confirm that $m_{23} + m_{32} \geq m_{12} + m_{13} - 1, m_{13} + m_{31} \geq m_{21} + m_{23} - 1$, and $m_{12} + m_{21} \geq m_{31} + m_{32} - 1$:

$$
\begin{array}{rrr}
m_{23} + m_{32} \geq m_{12} + m_{13} - 1 & m_{13} + m_{31} \geq m_{21} + m_{23} - 1 & m_{12} + m_{21} \geq m_{31} + m_{32} - 1 \\
t + t \geq (t+1) + t - 1 & t + t \geq (t+1) + t - 1 & (t+1) + (t+1) \geq t + t - 1 \\
2t \geq 2t & 2t \geq 2t & 2t + 2 \geq 2t - 1 \\
 & & 2t \geq 2t - 3
\end{array}
$$

Consider the multigraphical arrangement $\mathcal{A}$ in Fig. 3.7. We begin by looking at the potential bad intersections of $H_{12}^{a_1}$ with $H_{13}^{b_j}$ where $1 \leq j \leq t$ along with the potential bad intersections of $H_{13}^{b_1}$ with $H_{12}^{a_i}$ where $1 \leq i \leq t+1$. These are indicated by $p_1, p_2, \ldots, p_{2t}$ in Fig. 3.7.

To eliminate the potential bad intersection of $H_{12}^{a_i} = \{x_2 - x_1 = a_i\}$ with $H_{13}^{b_j} = \{x_1 - x_3 = b_j\}$, we will need either $H_{23}^{c_k}$ or $H_{32}^{c_k}$, where $c_k \in \{c_1, \ldots, c_{2t}\}$. Assume without loss of generality that $a_1 > a_2 > \cdots > a_{t+1}$. Then $b_1 - a_{t+1} > b_1 - a_t > \cdots > b_1 - a_1$. Similarly, assume without loss of generality that $b_1 > b_2 > \cdots > b_t$. Then $a_1 - b_t > a_1 - b_{t-1} > \cdots > a_1 - b_1$. But we cannot have both $b_1 - a_1$ and $a_1 - b_1$ because if $a_1 = b_1$, then $c_k = 0$ or $c'_k = 0$ for some k such that $1 \leq k \leq t$. Therefore we must choose whether we have $c_k = a_1 - b_1$ above the origin or $c'_k = b_1 - a_1$ below the origin.

Let $c'_k = b_1 - a_1$. Then $a_1 - b_1 < a_2 - b_1 < \cdots < a_{t+1} - b_1 < 0 < a_1 - b_2 < a_1 - b_3 < \cdots < a_1 - b_t$. So there are $t+1$ hyperplanes below the origin and $t-1$ hyperplanes above the origin. While this still adds up to the $2t$ hyperplanes we need to eliminate the potential bad

Potential bad intersection	Resolving hyperplane
p_1	$H_{32}^{c_1} = \{x_3 - x_2 = c_1 = a_1 - b_t\}$
p_2	$H_{32}^{c_2} = \{x_3 - x_2 = c_2 = a_1 - b_{t-1}\}$
$\vdots$	$\vdots$
p_t	$H_{32}^{c_t} = \{x_3 - x_2 = c_t = a_1 - b_1\}$
p_{t+1}	$H_{23}^{c'_1} = \{x_2 - x_3 = c'_1 = b_1 - a_2\}$
p_{t+2}	$H_{23}^{c'_2} = \{x_2 - x_3 = c'_2 = b_1 - a_3\}$
$\vdots$	$\vdots$
p_{2t}	$H_{23}^{c'_t} = \{x_2 - x_3 = c'_t = b_1 - a_{t+1}\}$

Table 3.5: *Correspondence of potential bad intersections with resolving hyperplanes*

intersections $p_1, \dots, p_{2t}$, this contradicts the condition that $m_{23} = m_{32} = t$. Thus we must fix $a_1 - b_1$ above the origin.

This means that we have the following string of inequalities:

$$a_1 - b_t > a_1 - b_{t-1} > \cdots > a_1 - b_1 > 0 > a_2 - b_1 > a_3 - b_1 > \cdots > a_{t+1} - b_1.$$

Now we have t hyperplanes on each side of the origin. We assign the values c_k and c'_k, where $k \in \{1, \dots, t\}$, according to Table 3.5.

Now we consider the potential bad intersections of $H_{21}^{a'_i}$ and $H_{23}^{c'_1}$ along with those of $H_{21}^{a'_{t+1}}$ and $H_{23}^{c'_k}$, indicated by $q_1, \dots, q_{2t}$ in Fig. 3.7. Assume without loss of generality that $a'_1 < a'_2 < \cdots < a'_{t+1}$. Then $c'_t - a'_1 > c'_t - a'_2 > \cdots > c'_t - a'_{t+1}$. Similarly, assume without loss of generality that $c'_1 < c'_2 < \dots c'_t$. Then $a'_{t+1} - c'_t > a'_{t+1} - c'_2 > \cdots > a'_{t+1} - c'_t$. But we cannot have both $c'_t - a'_{t+1}$ and $a'_{t+1} - c'_t$ because this implies that $b_j = 0$ or $b'_j = 0$, for some j such that $1 \leq j \leq t$. So we must decide whether $c'_t - a'_{t+1} > 0$ or $a'_{t+1} - c'_t > 0$.

Suppose $c'_t - a'_{t+1} > 0$. Then

$$c'_t - a'_1 > c'_t - a'_2 > \cdots > c'_t - a'_{t+1} > 0 > c'_{t-1} - a'_{t+1} > c'_{t-2} - a'_{t+1} > \dots c'_1 - a'_{t+1}.$$

Counting the hyperplanes on each side of the origin, there are $t + 1$ hyperplanes $H_{13}^{b_j}$ such that $b_j > 0$ and $t - 1$ hyperplanes $H_{31}^{b'_j}$ such that $b'_j > 0$. While this is enough hyperplanes to elminate the $2t$ potential bad intersections we have at $q_1, \dots, q_{2t}$, it contradicts the condition

Potential bad intersection	Resolving hyperplane
q_1	$H_{31}^{b'_t} = \{x_3 - x_1 = b'_t = a'_{t+1} - c'_1\}$
q_2	$H_{32}^{b'_{t-1}} = \{x_3 - x_1 = b'_{t-1} = a'_{t+1} - c'_2\}$
$\vdots$	$\vdots$
q_t	$H_{32}^{b'_1} = \{x_3 - x_1 = b'_1 = a'_{t+1} - c'_t\}$
q_{t+1}	$H_{13}^{b_t} = \{x_1 - x_3 = b_t = c'_t - a'_t\}$
q_{t+2}	$H_{13}^{b_{t-1}} = \{x_1 - x_3 = b_{t-1} = c'_t - a'_{t-1}\}$
$\vdots$	$\vdots$
q_{2t}	$H_{23}^{b_t} = \{x_1 - x_3 = b_1 = c'_t - a'_1\}$

Table 3.6: *Correspondence of potential bad intersections with resolving hyperplanes*

that $m_{13} = m_{31} = t$. Therefore we must fix $a'_{t+1} - c'_t > 0$.

Now we have the following string of inequalities:

$$c'_t - a'_1 > c'_t > \cdots > c'_t - a'_t > 0 > c'_t - a'_{t+1} > \cdots > c'_2 - a'_{t+1} > c'_1 - a'_{t+1}.$$

We assign the values b_j and b'_j where $1 \leq j \leq t$ according to Table 3.6.

Consider $b_1 = c'_t - a'_1$, the value fixed to eliminate the bad intersection q_{2t} along with $c'_t = b_1 - a_{t+1}$, the value fixed to eliminate the bad intersection p_{2t}. Since $b_1 = c'_t - a'_1$ and $b_1 = c'_t + a_{t+1}$, we know $c'_t - a'_1 = c'_t + a_{t+1}$, which implies $0 = a'_1 + a_{t+1}$. But this is a contradiction because $a'_1 > 0$ and $a_{t+1} > 0$. Therefore, the multigraphical arrangement does not admit an injective Pak-Stanley labeling.

□

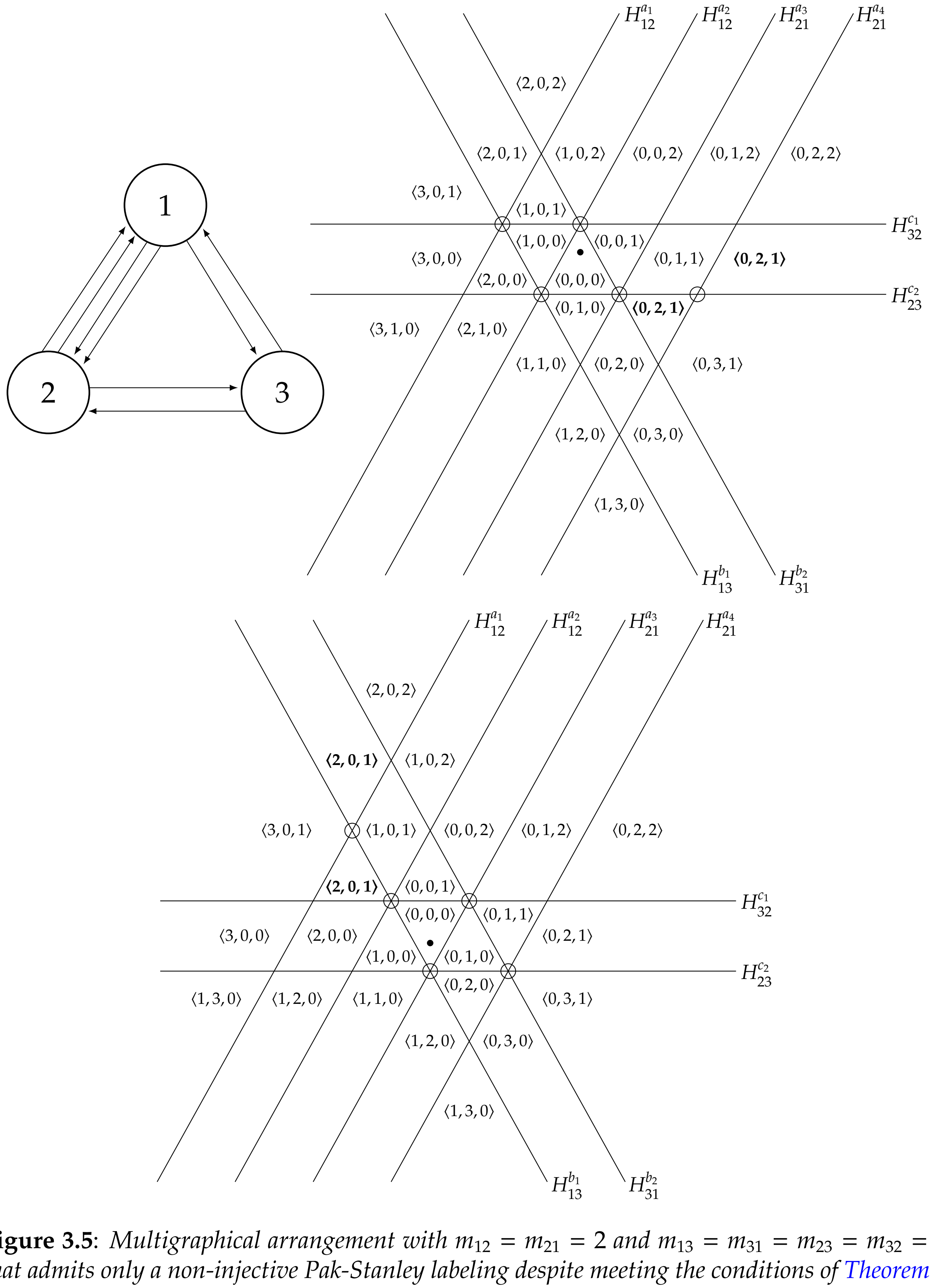


Figure 3.5: *Multigraphical arrangement with $m_{12} = m_{21} = 2$ and $m_{13} = m_{31} = m_{23} = m_{32} = 1$ that admits only a non-injective Pak-Stanley labeling despite meeting the conditions of Theorem 4*

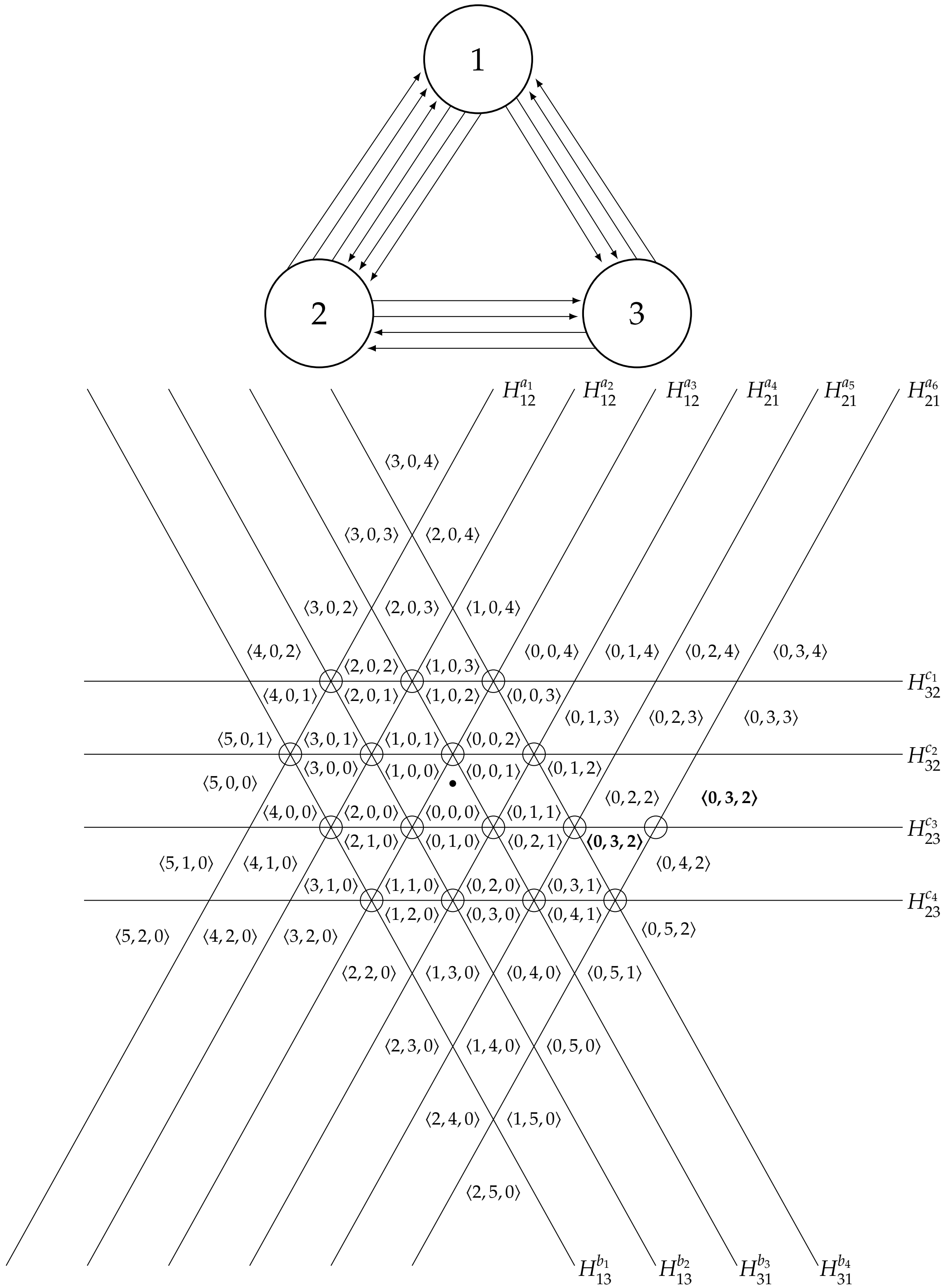


Figure 3.6: *Multigraphical arrangement with* $m_{12} = m_{21} = 3$ *and* $m_{13} = m_{31} = m_{23} = m_{32} = 2$ *that does not admit an injective Pak-Stanley labeling despite meeting the conditions of Theorem 4*

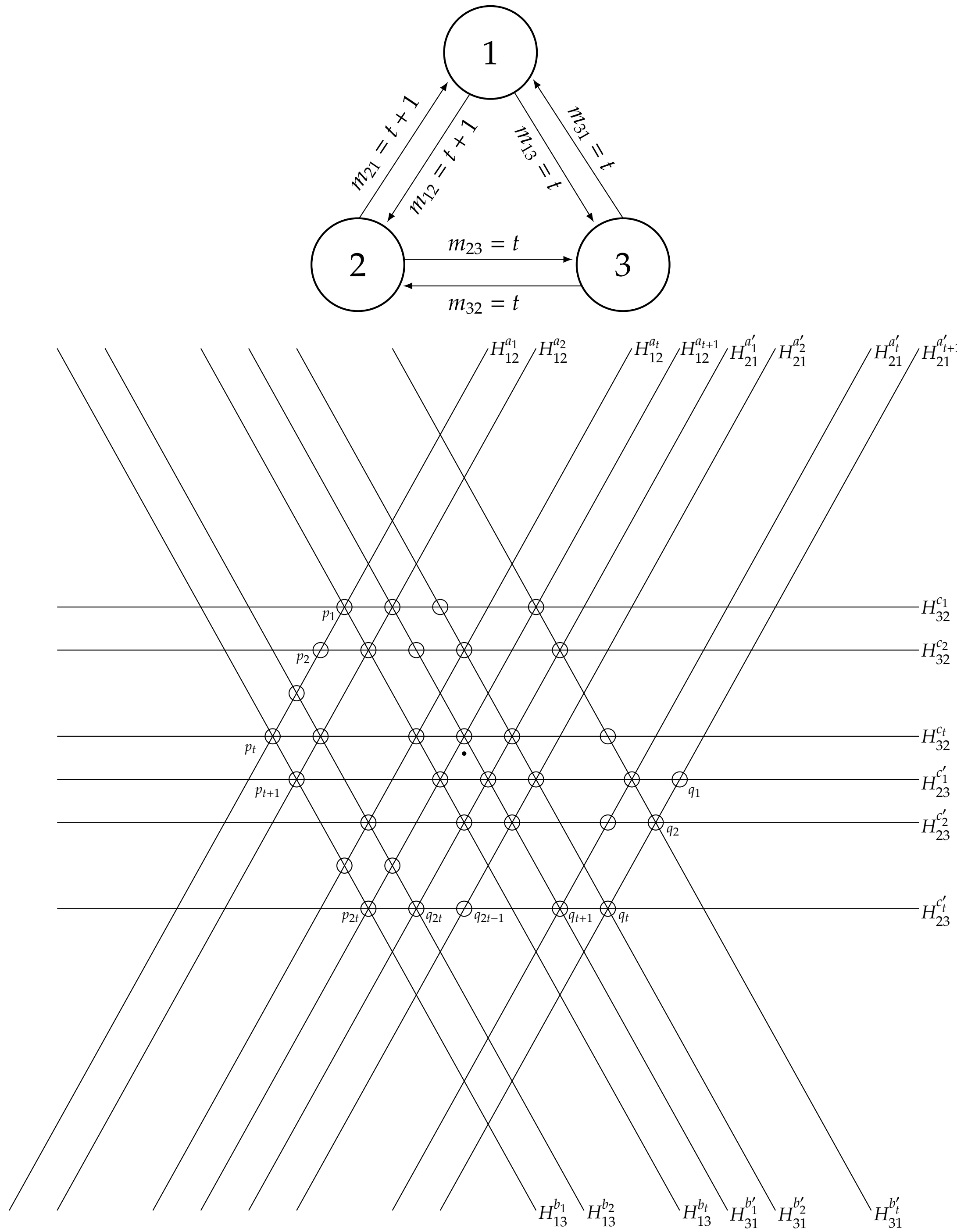


Figure 3.7: *Generalized multigraph arrangement for Theorem 6 such that* $m_{12} = m_{21} = t + 1$ *and* $m_{13} = m_{31} = m_{23} = m_{32} = t$

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