COMPUTER AIDED DESIGN OF MINIMUM SENSITIVITY FOUR-BAR LINKAGES

by

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CHAPTER I

INTRODUCTION

The digital computer has become an indispensable tool in many areas of engineering analysis and design. Computers are now being used extensively for problem solving in all disciplines of engineering and new methods for improving their effectiveness are being developed every day. Kinematic design of mechanical linkages is one of the areas that has benefited enormously from the power of the digital computer. Today, there exist hundreds of computer-oriented methods and implementations that are capable of handling several classes of mechanism design problems such as path generation, precision point synthesis, etc.

Despite this profusion of available software and methodology, there remain some critical problem areas which have not yet been satisfactorily addressed. In many cases, the problem areas were thought to be too difficult to be tackled and were therefore left untried for a number of years. The advent of the digital computer has already brought several of these previously intractable problem areas into the realm of possible solution. It is reasonable to expect that as the power of the available computer hardware increases, more research will be required to develop computer-oriented methods for solving more challenging classes of problems on state-of-the-art equipment.

The research presented in this thesis is an effort to develop a computer-based design technique for handling a very important but largely overlooked problem in mechanism design - the design of minimum sensitivity four-bar linkages. This class of problems is of interest to the design engineer as well as the manufacturing engineer. In order to manufacture the linkage, appropriate machining tolerances have to be specified on all dimensions. The tolerances on any dimension should reflect the sensitivity of the system performance to small changes or errors in that dimension. If the system performance is relatively insensitive to variations in a particular dimension, the tolerances on that dimension can be specified to be quite loose. Conversely, if the system performance is highly sensitive with respect to a particular dimension, then the tolerances on that dimension must be held very tight. Generally, it is desirable to specify tolerances to be as loose as possible because tight tolerances are associated with high manufacturing cost. Since the tolerance on any dimension is dependent on the sensitivity of the system performance to variations in that dimension, it follows that when we design a minimum sensitivity linkage, we are effectively designing a minimum cost linkage as well. Unfortunately, there has been very little work done in the area of minimum sensitivity design of four-bar linkages, although there has been some research on optimal allocation of manufacturing tolerances [1].

The approach taken in this thesis is to convert the minimum sensitivity problem into an equivalent constrained optimal design problem which can then be solved by using well-established nonlinear programming techniques. The motivation for using this approach lies in the fact that there exists a natural transformation from the minimum sensitivity problem to the constrained optimal design problem. The parameters whose values are to be determined (e.g. link lengths, coupler point location, etc.) become the design variables of the optimal design problem. The performance requirements that the design should meet become the constraint functions in the optimal design problem. Finally, the sensitivity to be minimized becomes the objective function of the optimal design problem. Once this translation is done, the methodology of optimal design gives us several systematic, semi-automated numerical schemes that will lead to the desired solution.

In order to use this approach in a computer-aided design environment, it is first necessary to develop a computer-oriented method for kinematic analysis, since the constraint functions of the optimal design problems will generally depend on the position, velocity and acceleration of the various links. Fortunately, several reliable methods for kinematic analysis are already available and so all that needs to be done is to select a method that is suitable for the present purpose. The method selected was a loop closure method [2] that is quite efficient and easy to implement in a computer code.

In addition to the kinematic analysis, a method of performing first order design sensitivity analysis is also required. This is needed for two reasons: first, the objective function is a first order sensitivity and so evaluation of the objective function requires first order sensitivity analysis; secondly, first order sensitivity analysis is needed in order to obtain the derivatives of the constraint functions so that an efficient derivative based optimization method can be used. Since methods for sensitivity analysis on four-bar linkages are not very well developed, a scheme based on the direct differentiation method [3] was derived specifically for use in the present work.

Finally, second order design sensitivity analysis must also be performed on the system. As noted earlier, it is desirable to use derivative based optimization algorithms from the point of view of efficiency. Since the objective function is itself a first order sensitivity, its derivatives can be evaluated only through second order sensitivity analysis. Methods for performing second order sensitivity analysis on four-bar linkages are practically non-existent in the literature. Consequently, a new method for computing the second order sensitivity, based on an extension of the direct differentiation technique, was developed.

Once the kinematic and design sensitivity analyses have been completed, the results must be supplied to an optimization algorithm to obtain the next updated design. As was the case with kinematic analysis, excellent optimization methods are freely available and one only needs to choose the method that is most appropriate for the purpose at hand. The method chosen was a sequential unconstrained minimization technique (SUMT) [4] using an exterior penalty function or augmented Lagrange multiplier method. The unconstrained minimization was performed using a modified steepest descent algorithm [5].

The derivation of the kinematic analysis is presented in Chapter 2. In this chapter, the loop closure equations that define the four-bar linkage are derived in order to compute the position, velocity and acceleration of the links. A detailed mobility analysis is also done to ensure that only the allowable angular regions of the crank rotation are analyzed. Chapter 3 presents the development of the first and second order design sensitivity analysis for the four-bar linkage. This chapter illustrates how the equations are derived and describes how they can be solved in a very efficient manner. The optimization methods used are explained in Chapter 4 along with the formalization of the minimum sensitivity problem as a standard nonlinear programming problem. The methods developed in Chapters 2, 3 and 4 were implemented in an interactive, user-friendly computer program that can be used for computer-aided design of minimum sensitivity four-bar linkages. The structure and capabilities of this program are presented in Chapter 5. Several numerical examples were run on this program to verify the design sensitivity analysis and to evaluate the performance of the proposed approach to minimum sensitivity design. Selected examples are described in Chapter 6. The results show the approach to be very reliable and convenient to use in addition to being computationally feasible. Finally, an assessment of the method and some recommendations for future research in this field are presented in Chapter 7.

CHAPTER II

FOUR-BAR LINKAGE ANALYSIS

In order to analyze a four-bar linkage, it is first necessary to formulate the kinematic equations that govern the behavior of the linkage. The method presented in this chapter is based on deriving position loop closure equations for the linkage from the geometry. These equations are then differentiated with respect to time to obtain the velocity and acceleration loop closure equations. This method is easy to implement in a computer program, making it possible for the linkage to be analyzed at any angular position. The only necessary input parameters required for this analysis are the link lengths and the angular position, velocity and acceleration of the input link. Knowing these inputs, the positions, velocities and accelerations of the coupler and output links can be calculated.

The following notation will be used in the derivation of the kinematic equations and in the design sensitivity analysis. Referring to Figure 2.1, the parameters are:

b_1:	Length of frame or ground link
b_2:	Length of crank or input link
b_3:	Length of coupler link
b_4:	Length of output or follower link
q_2:	Input crank angle
q_3:	Coupler link angle

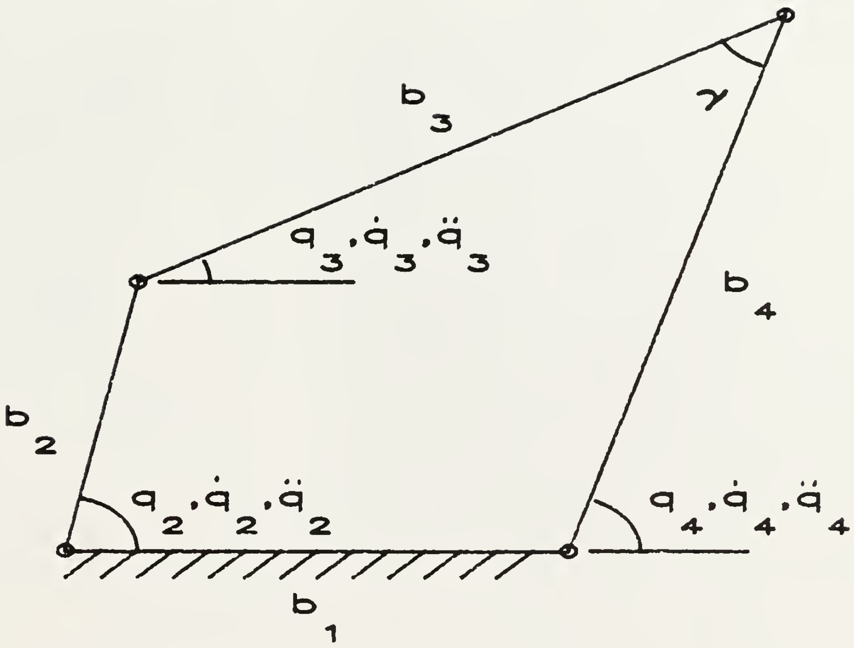


Figure 2.1 Four-bar Linkage

q_4: Output link angle

$\dot{q}_2$: Angular velocity of input link

$\dot{q}_3$: Angular velocity of coupler link

$\dot{q}_4$: Angular velocity of output link

$\ddot{q}_2$: Angular acceleration of input link

$\ddot{q}_3$: Angular acceleration of coupler link

$\ddot{q}_4$: Angular acceleration of output link

γ: Transmission angle

2.1 Position Analysis

The equations used to calculate the coupler and output link angular positions are derived using the Law of Cosines. Referring to Figure 2.2, we see that the following relationship should hold:

$$Z^2 = (b_1)^2 + (b_2)^2 - 2.0*b_1*b_2*\cos(q_2) \qquad 2.1$$

After evaluating Z from equation 2.1, we can apply the Law of Cosines to the four-bar linkage in Figure 2.2, to obtain the angles α, β and φ, as follows:

$$\alpha = \cos^{-1}((Z^2 + (b_4)^2 - (b_3)^2)/(2.0*Z*b_4)) \qquad 2.2$$

$$\beta = \cos^{-1}((Z^2 + (b_1)^2 - (b_2)^2)/(2.0*Z*b_1)) \qquad 2.3$$

$$\varphi = \cos^{-1}((Z^2 + (b_3)^2 - (b_4)^2)/(2.0*Z*b_3)) \qquad 2.4$$

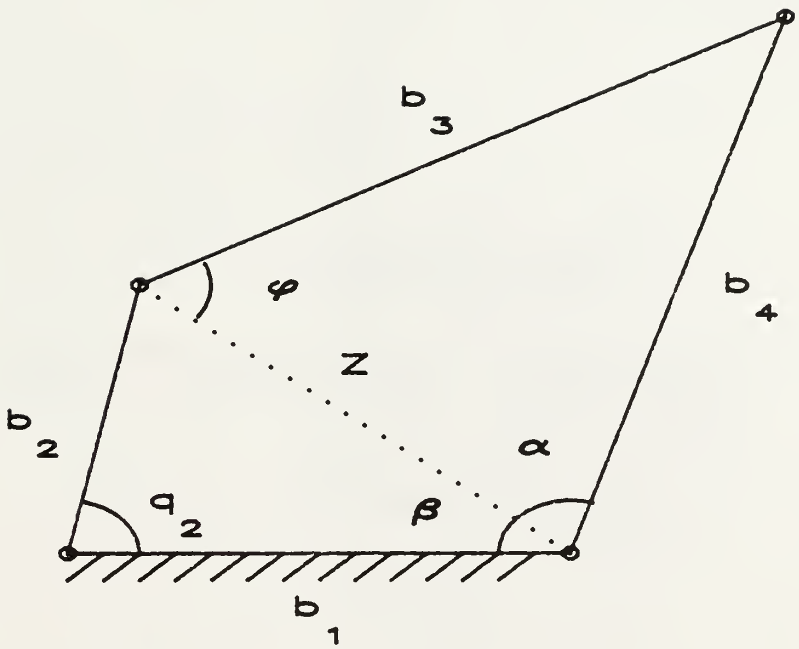


Figure 2.2 Position analysis angles

Care must be taken when evaluating the inverse cosine function on a computer since an argument value greater than +1.0 or less than -1.0 might be encountered. This problem could arise in two ways. One possibility is that it can be caused by round off error when the cosine value is near +/-1.0. A second possibility is that the linkage has not been properly defined. This situation could arise, for example, if the optimization algorithm takes too large a step.

If the absolute value of the argument does not exceed 1.0001, it is assumed that the error is due to round off. In this case, the error is ignored and the value is reset to +/-1.0. This tolerance prevents small round off errors from terminating the program prematurely.

In cases where the absolute value of the argument exceeds 1.0001 it is assumed that the linkage is improperly defined and the kinematic analysis is terminated. This problem usually occurs when the optimization algorithm takes too large a step in design space. In order to correct this problem the step size used in the optimization package should be decreased before restarting the process.

When choosing the sign of β , it must be realized that there are two possible ways to assemble the four-bar linkage. To ensure that the desired solution is computed, two conditions must be set on the angular position of the input link, q_2:

Condition 1. If($0 < q_2 < 180$)

Then($0 < \beta < 180$)

Condition 2. If($180 < q_2 < 360$)

Then($180 < \beta < 360$)

Once β has been defined in this way, α and φ will always be positive. The coupler and output link positions are calculated from the following equations:

$$q_3 = \varphi - \beta \tag{2.5}$$

$$q_4 = 180 - (\alpha + \beta) \tag{2.6}$$

The transmission angle is easily calculated at this point in the analysis once the coupler and output link positions are known. Referring to Figure 2.1 the transmission angle equation becomes:

$$\gamma = q_4 - q_3 \tag{2.7}$$

2.2 Velocity Analysis

The equations used for velocity analysis are velocity loop closure equations that are derived from the following position loop closure equations:

$$- b_3*\cos(q_3) + b_4*\cos(q_4) = - b_1 + b_2*\cos(q_2) \qquad 2.8$$

$$- b_3*\sin(q_3) + b_4*\sin(q_4) = b_2*\sin(q_2) \qquad 2.9$$

Differentiating equations 2.8 and 2.9 with respect to time, the desired velocity loop closure equations are obtained as follows:

$$b_3*\sin(q_3)*\dot{q}_3 - b_4*\sin(q_4)*\dot{q}_4 = \qquad 2.10$$
$$- b_2*\sin(q_2)*\dot{q}_2$$

$$- b_3*\cos(q_3)*\dot{q}_3 + b_4*\cos(q_4)*\dot{q}_4 = \qquad 2.11$$
$$b_2*\cos(q_2)*\dot{q}_2$$

At this point the only unknowns in equations 2.10 and 2.11 are the coupler and output link angular velocities, $\dot{q}_3$ and $\dot{q}_4$, respectively. Equations 2.10 and 2.11 can be solved simultaneously resulting in the two velocity equations:

$$\dot{q}_3 = (- b_2*\cos(q_2)*\dot{q}_2*b_4*\sin(q_4) \qquad 2.12$$
$$+ b_4*\cos(q_4)*b_2*\sin(q_2)*\dot{q}_2)/$$

$$(b_3*\cos(q_3)*b_4*\sin(q_4) - b_4*\cos(q_4)*b_3*\sin(q_3))$$

$$\dot{q}_4 = (- b_2*\sin(q_2)*\dot{q}_2*b_3*\cos(q_3) \qquad 2.13$$

$$- b_3*\sin(q_3)*b_2*\cos(q_2)*\dot{q}_2)/$$

$$(b_3*\cos(q_3)*b_4*\sin(q_4) - b_4*\cos(q_4)*b_3*\sin(q_3))$$

2.3 Acceleration Analysis

The equations used for acceleration analysis are acceleration loop closure equations that are derived from equations 2.10 and 2.11. Differentiating equations 2.10 and 2.11 with respect to time, the acceleration loop closure equations are obtained as follows:

$$- b_3*\cos(q_3)*(\dot{q}_3)^2 - b_3*\sin(q_3)*\ddot{q}_3 \qquad 2.14$$

$$+ b_4*\cos(q_4)*(\dot{q}_4)^2 + b_4*\sin(q_4)*\ddot{q}_4 =$$

$$b_2*\cos(q_2)*(\dot{q}_2)^2 + b_2*\sin(q_2)*\ddot{q}_2$$

$$- b_3*\sin(q_3)*(\dot{q}_3)^2 + b_3*\cos(q_3)*\ddot{q}_3 \qquad 2.15$$

$$+ b_4*\sin(q_4)*(\dot{q}_4)^2 - b_4*\cos(q_4)*\ddot{q}_4 =$$

$$b_2*\sin(q_2)*(\dot{q}_2)^2 - b_2*\cos(q_2)*\ddot{q}_2$$

At this point the only unknowns in equations 2.14 and 2.15 are the coupler and output link angular accelerations, $\ddot{q}_3$ and $\ddot{q}_4$, respectively. Equations 2.14 and 2.15 can be solved simultaneously resulting in the two acceleration equations:

$$\ddot{q}_3 = ((b_2*\sin(q_2)*(\dot{q}_2)^2 - b_2*\cos(q_2)*\ddot{q}_2 \qquad 2.16$$

$$+ b_3*\sin(q_3)*(\dot{q}_3)^2 - b_4*\sin(q_4)*(\dot{q}_4)^2)*b_4*\sin(q_4)$$

$$+ (b_2*\cos(q_2)*(\dot{q}_2)^2 + b_2*\sin(q_2)*\ddot{q}_2$$

$$+ b_3*\cos(q_3)*(\dot{q}_3)^2 - b_4*\cos(q_4)*(\dot{q}_4)^2)*b_4*\cos(q_4))/$$

$$(b_3*\cos(q_3)*b_4*\sin(q_4) - b_4*\cos(q_4)*b_3*\sin(q_3))$$

$$\ddot{q}_4 = ((b_2*\cos(q_2)*(\dot{q}_2)^2 + b_2*\sin(q_2)*\ddot{q}_2 \qquad 2.17$$

$$+ b_3*\cos(q_3)*(\dot{q}_3)^2 - b_4*\cos(q_4)*(\dot{q}_4)^2)*b_3*\cos(q_3)$$

$$+ (b_2*\sin(q_2)*(\dot{q}_2)^2 - b_2*\cos(q_2)*\ddot{q}_2$$

$$+ b_3*\sin(q_3)*(\dot{q}_3)^2 - b_4*\sin(q_4)*(\dot{q}_4)^2)*b_3*\sin(q_3))/$$

$$(b_3*\cos(q_3)*b_4*\sin(q_4) - b_4*\cos(q_4)*b_3*\sin(q_3))$$

The preceding scheme for kinematic analysis can be conveniently implemented in a computer program. In order to solve a given problem, it is necessary to know the initial and final crank angles and the number of grid points between these two angles at which the linkage is to be analyzed. The angular position of the input link at a particular grid point i becomes:

$$(q_2)_i = (q_2)_o + i*((q_2)_f - (q_2)_o)/n \quad , \quad i = 0,\ldots,n \qquad 2.18$$

where: $(q_2)_o$ is the initial crank angle.

$(q_2)_f$ is the final crank angle.

n is the number of grid points.

2.4 Linkage Mobility

Linkage mobility is a major concern in general purpose linkage optimization problems since a design returned from the optimization algorithm could cause the input link not to have full rotation. When the input link has full rotation (360 degrees) there is no danger of the linkage locking, and the initial and final crank angles can be set to any desired values. However, if the input link does not have full rotation, care must be taken to ensure that only allowable input crank angles are used during the analysis. This is achieved by calculating the extreme positions of crank rotation and ensuring that the initial and final crank angles lie between these extreme positions.

The extreme positions of a linkage that does not have full crank rotation must be either dead center positions or limit positions. A four-bar linkage is in its dead center position when the coupler and output link lie along a straight line with the coupler link overlapping the output link. A limit position occurs when the coupler and output link lie along a straight line with the two links being end-to-end. The positions shown in Figure 2.3 illustrate the symmetry that occurs when these extreme positions are encountered.

The allowable angular regions lie between the extreme positions and these regions are determined by the following relationships. Referring to Figure 2.3, the allowable regions are defined by:

Region 1. $$b_1 + b_2 > b_3 + b_4 \qquad 2.21$$

Region 2. $$|(b_1 - b_2)| < |(b_3 - b_4)| \qquad 2.22$$

Region 3. $$b_1 + b_2 > b_3 + b_4 \text{ and } |(b_1 - b_2)| < |(b_3 - b_4)| \qquad 2.23$$

When any of these conditions hold, the allowable angular movement of the input link must be calculated. The minimum and maximum angles of the input link in a particular region are calculated using the Law of Cosines. The minimum and maximum crank angles for region 1 are calculated from the following equations:

$$(q_2)_o = - \cos^{-1}(((b_1)^2 + (b_2)^2 - 2.0*(b_3 + b_4))/(2.0*b_1*b_2)) \qquad 2.24$$

$$(q_2)_f = \cos^{-1}(((b_1)^2 + (b_2)^2 - 2.0*(b_3 + b_4))/(2.0*b_1*b_2)) \qquad 2.25$$

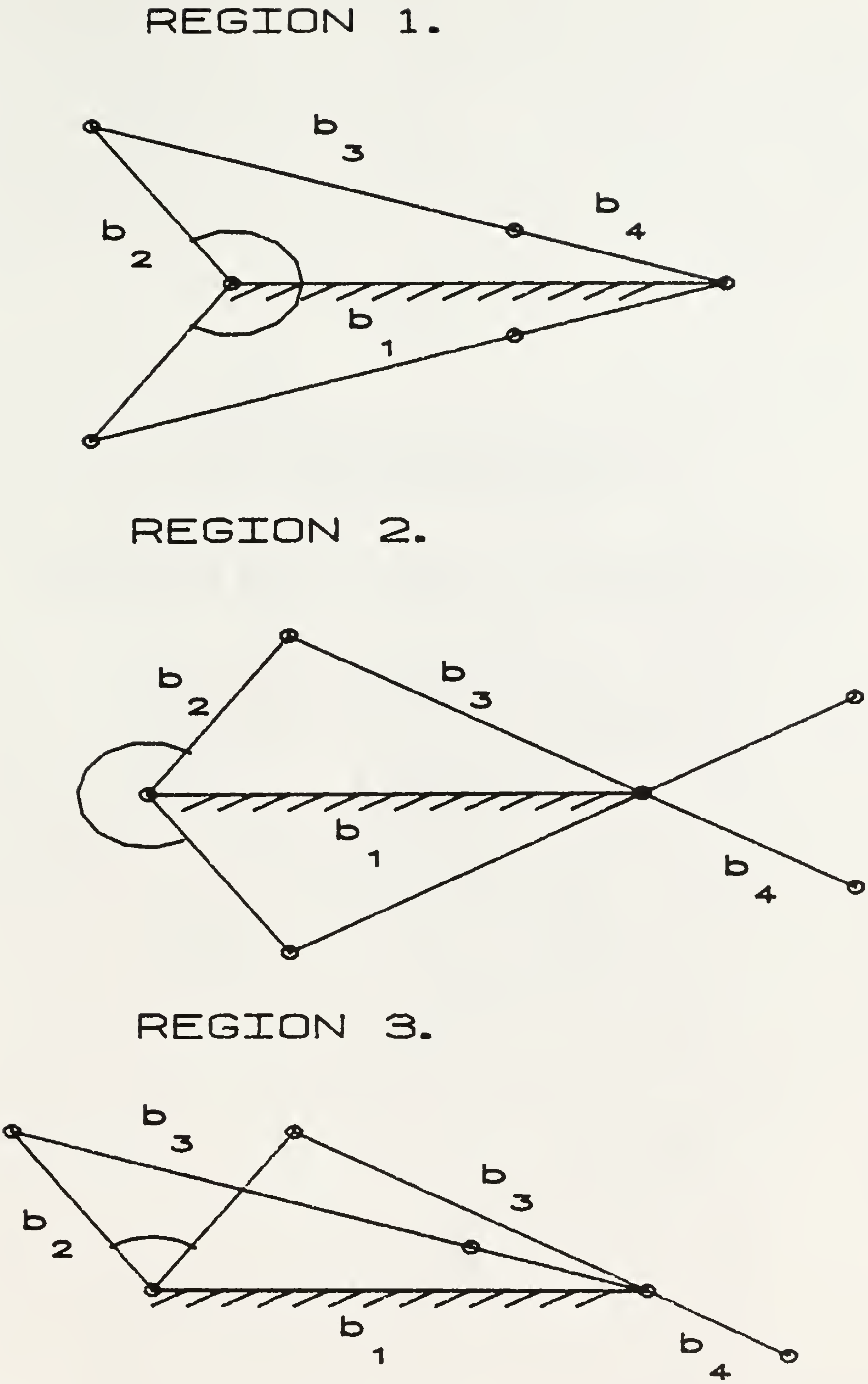


Figure 2.3 Dead center and limit positions

The minimum and maximum crank angles for region 2 are calculated from the following equations:

$$(q_2)_o = \cos^{-1}(((b_1)^2 + (b_2)^2 - 2.0*(b_3 - b_4))/(2.0*b_1*b_2)) \quad 2.26$$

$$(q_2)_f = 360 - (q_2)_o \quad 2.27$$

If both conditions of equation 2.23 are satisfied and the two regions overlap, equations 2.26 and 2.25 are used to calculate the minimum and maximum crank angles respectively. Thus, the minimum and maximum crank angles for region 3 are calculated from the following equations:

$$(q_2)_o = \cos^{-1}(((b_1)^2 + (b_2)^2 - 2.0*(b_3 - b_4))/(2.0*b_1*b_2)) \quad 2.28$$

$$(q_2)_f = \cos^{-1}(((b_1)^2 + (b_2)^2 - 2.0*(b_3 + b_4))/(2.0*b_1*b_2)) \quad 2.29$$

The input link mobility as well as the minimum and maximum crank angles can thus be determined from the link lengths.

CHAPTER III

SENSITIVITY ANALYSIS

The objective of design sensitivity analysis is the evaluation of the derivatives of relevant performance functions with respect to the design variables. First order design sensitivity analysis will yield the first partial derivatives of these functions with respect to design; similarly, second order design sensitivity analysis will yield the corresponding second partials. It is clear that the second partials can be viewed as the first partial derivatives of the first order sensitivity. Thus, if the first order sensitivity coefficients enter into the performance functions of interest, then the sensitivity calculations relating to these functions will involve second order terms. For our present purpose, we need a way to calculate the first and second order sensitivities of any performance function of the form $f = f(b,q,\dot{q},\ddot{q},x,y)$ where (x,y) are the coordinates of the coupler point. Since the state variables and coupler point position are themselves implicit functions of design, we must first devise a computational scheme for performing first and second order sensitivity analysis of these quantities. A computer-oriented method for combined first and second order design sensitivity analysis for planar four-bar linkages is presented in this chapter.

3.1 First Order Sensitivity

The kinematic equations which were derived in the preceding chapter are dependent on the link lengths of the four-bar linkage and on the position, velocity and acceleration of the links. The first order design sensitivity can be calculated by differentiating the loop closure equations with respect to the vector of desired design variables. The method used in the present work is based on the direct differentiation technique [3]. In order to apply this technique, it is necessary to first define the design vector. Referring to Figure 3.1, the components of the design vector are:

b_1: Length of frame or ground link

b_2: Length of crank or input link

b_3: Length of coupler link

b_4: Length of output or follower link

b_5: Angle of coupler point from coupler link

b_6: Distance to coupler point from reference end of coupler link

b_7: Angle of ground link

b_8: x coordinate of ground link

b_9: y coordinate of ground link

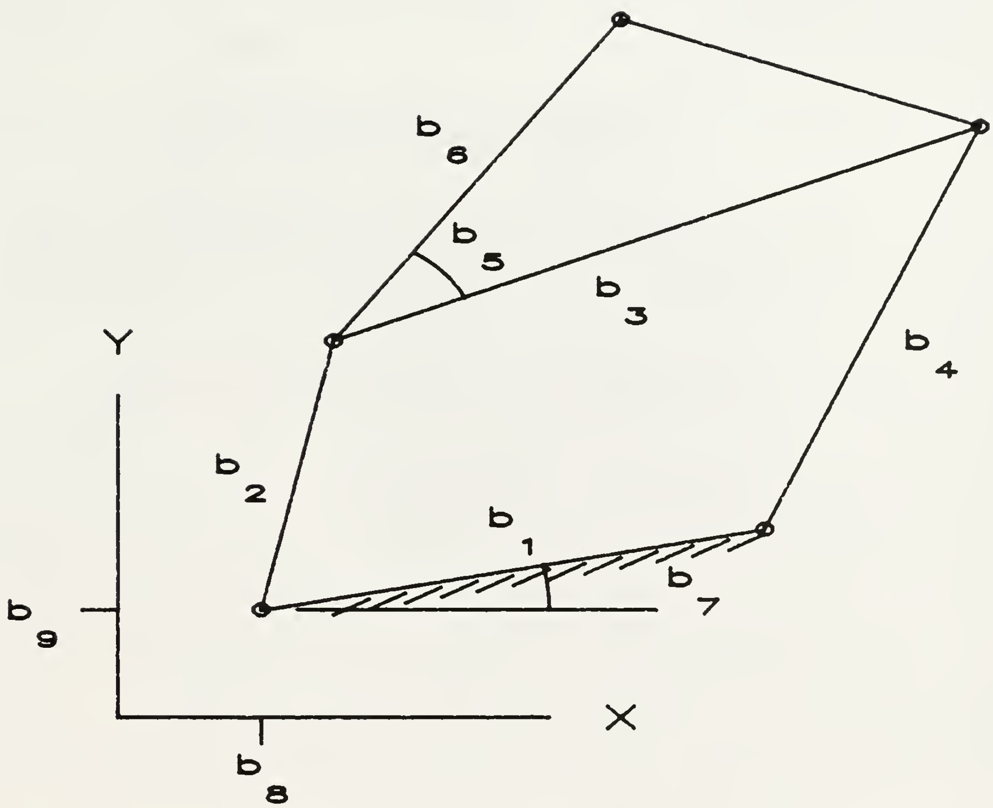


Figure 3.1 Design variables

The derivation of the partial derivatives of the loop closure equations and a technique for computing the first order design sensitivity of position, velocity, acceleration and coupler point position are discussed in the following sections.

3.1.1 Position Sensitivity

The first order position sensitivity equations are derived from the position loop closure equations 2.8 and 2.9. Differentiation of both sides of these equations with respect to the appropriate design variables produces eight equations containing twelve unknown position sensitivities. The first order position sensitivity of the input link is set equal to zero since the input link angle is independently specified and does not depend on the design variables.

The notation used for the first order position sensitivity coefficients is:

$$q_{i,j} = \frac{\partial q_i}{\partial b_j} \qquad , \; i = 2,4 \text{ and } j = 1,4 \qquad 3.1$$

Differentiating both sides of the position loop closure equations (equations 2.8 and 2.9) with respect to the link lengths b_1, b_2, b_3 and b_4 yields the following set of equations:

$$b_3*\sin(q_3)*q_{3,1} - b_4*\sin(q_4)*q_{4,1} = - 1 - b_2*\sin(q_2)*q_{2,1} \tag{3.2}$$

$$b_3*\sin(q_3)*q_{3,2} - b_4*\sin(q_4)*q_{4,2} = \cos(q_2) - b_2*\sin(q_2)*q_{2,2} \tag{3.3}$$

$$b_3*\sin(q_3)*q_{3,3} - b_4*\sin(q_4)*q_{4,3} = \cos(q_3) - b_2*\sin(q_2)*q_{2,3} \tag{3.4}$$

$$b_3*\sin(q_3)*q_{3,4} - b_4*\sin(q_4)*q_{4,4} = - \cos(q_4) - b_2*\sin(q_2)*q_{2,4} \tag{3.5}$$

$$- b_3*\cos(q_3)*q_{3,1} + b_4*\cos(q_4)*q_{4,1} = b_2*\cos(q_2)*q_{2,1} \tag{3.6}$$

$$- b_3*\cos(q_3)*q_{3,2} + b_4*\cos(q_4)*q_{4,2} = \sin(q_2) + b_2*\cos(q_2)*q_{2,2} \tag{3.7}$$

$$- b_3*\cos(q_3)*q_{3,3} + b_4*\cos(q_4)*q_{4,3} = \sin(q_3) + b_2*\cos(q_2)*q_{2,3} \tag{3.8}$$

$$- b_3*\cos(q_3)*q_{3,4} + b_4*\cos(q_4)*q_{4,4} = - \sin(q_4) + b_2*\cos(q_2)*q_{2,4} \tag{3.9}$$

The preceding eight position sensitivity equations contain eight unknown position sensitivities which occur on the left side of the equations. The right hand sides depend on position and design only.

The eight equations above can be written conveniently in the standard matrix form A*x = y where:

$$A = \begin{vmatrix} s3 & 0 & 0 & 0 & s4 & 0 & 0 & 0 \\ 0 & s3 & 0 & 0 & 0 & s4 & 0 & 0 \\ 0 & 0 & s3 & 0 & 0 & 0 & s4 & 0 \\ 0 & 0 & 0 & s3 & 0 & 0 & 0 & s4 \\ c3 & 0 & 0 & 0 & c4 & 0 & 0 & 0 \\ 0 & c3 & 0 & 0 & 0 & c4 & 0 & 0 \\ 0 & 0 & c3 & 0 & 0 & 0 & c4 & 0 \\ 0 & 0 & 0 & c3 & 0 & 0 & 0 & c4 \end{vmatrix} \qquad 3.10$$

s3, s4, c3, and c4 are defined by:

$s3 = b_3*\sin(q_3)$ $\qquad$ $c3 = -\ b_3*\cos(q_3)$

$c4 = b_4*\cos(q_4)$ $\qquad$ $s4 = -\ b_4*\sin(q_4)$

The vectors x and y are given by:

$$x = \begin{vmatrix} q_{3,1} \\ q_{3,2} \\ q_{3,3} \\ q_{3,4} \\ q_{4,1} \\ q_{4,2} \\ q_{4,3} \\ q_{4,4} \end{vmatrix} \qquad 3.11$$

$$
y = \begin{vmatrix}
-1 - b_2*\sin(q_2)*q_{2,1} \\
\cos(q_2) - b_2*\sin(q_2)*q_{2,2} \\
\cos(q_3) - b_2*\sin(q_2)*q_{2,3} \\
-\cos(q_4) - b_2*\sin(q_2)*q_{2,4} \\
b_2*\cos(q_2)*q_{2,1} \\
\sin(q_2) + b_2*\cos(q_2)*q_{2,2} \\
\sin(q_3) + b_2*\cos(q_2)*q_{2,3} \\
-\sin(q_4) + b_2*\cos(q_2)*q_{2,4}
\end{vmatrix} \qquad 3.12
$$

The system of equations above is easily solved by decoupling it into sets of two appropriate equations with each set containg the same two unknown position sensitivities. For example both the first and fifth equations contain unknown sensitivities $q_{3,1}$ and $q_{4,1}$. The two position sensitivities are computed by solving these two equations simultaneously.

3.1.2 Velocity Sensitivity

The first order velocity sensitivity equations can be derived in one of two ways. The first method is to evaluate the time derivative of the eight first order position sensitivity equations; the second option is to evaluate the time derivative of the position loop closure equations to obtain the velocity loop closure equations and then differentiate both sides of these equations with respect to design. The derivation given below is based on the second approach and it was verified by rederiving the equations through the first method and comparing the results.

The notation used for the first order velocity sensitivity coefficients is:

$$\dot{q}_{i,j} = \frac{\partial \dot{q}_j}{\partial b_j} \quad , \; i = 2,4 \text{ and } j = 1,4 \qquad 3.13$$

Differentiating both sides of the velocity loop closure equations (equations 2.10 and 2.11) with respect to the link lengths b_1, b_2, b_3 and b_4 yields the following set of equations:

$$b_3*\sin(q_3)*\dot{q}_{3,1} - b_4*\sin(q_4)*\dot{q}_{4,1} = \qquad 3.14$$
$$- b_2*(\sin(q_2)*\dot{q}_{2,1} + \cos(q_2)*\dot{q}_2*q_{2,1})$$
$$- b_3*\cos(q_3)*\dot{q}_3*q_{3,1}$$
$$+ b_4*\cos(q_4)*\dot{q}_4*q_{4,1}$$

$$b_3*\sin(q_3)*\dot{q}_{3,2} - b_4*\sin(q_4)*\dot{q}_{4,2} = \qquad 3.15$$
$$- b_2*(\sin(q_2)*\dot{q}_{2,2} + \cos(q_2)*\dot{q}_2*q_{2,2})$$
$$- b_3*\cos(q_3)*\dot{q}_3*q_{3,2}$$
$$+ b_4*\cos(q_4)*\dot{q}_4*q_{4,2} - \sin(q_2)*\dot{q}_2$$

$$b_3*\sin(q_3)*\dot{q}_{3,3} - b_4*\sin(q_4)*\dot{q}_{4,3} = \qquad 3.16$$
$$- b_2*(\sin(q_2)*\dot{q}_{2,3} + \cos(q_2)*\dot{q}_2*q_{2,3})$$
$$- b_3*\cos(q_3)*\dot{q}_3*q_{3,3}$$

$$+ b_4*\cos(q_4)*\dot{q}_4*q_{4,3} - \sin(q_3)*\dot{q}_3$$

$$b_3*\sin(q_3)*\dot{q}_{3,4} - b_4*\sin(q_4)*\dot{q}_{4,4} = \tag{3.17}$$
$$- b_2*(\sin(q_2)*\dot{q}_{2,4} + \cos(q_2)*\dot{q}_2*q_{2,4})$$
$$- b_3*\cos(q_3)*\dot{q}_3*q_{3,4}$$
$$+ b_4*\cos(q_4)*\dot{q}_4*q_{4,4} + \sin(q_4)*\dot{q}_4$$

$$- b_3*\cos(q_3)*\dot{q}_{3,1} + b_4*\cos(q_4)*\dot{q}_{4,1} = \tag{3.18}$$
$$b_2*(\cos(q_2)*\dot{q}_{2,1} - \sin(q_2)*\dot{q}_2*q_{2,1})$$
$$- b_3*\sin(q_3)*\dot{q}_3*q_{3,1}$$
$$+ b_4*\sin(q_4)*\dot{q}_4*q_{4,1}$$

$$- b_3*\cos(q_3)*\dot{q}_{3,2} + b_4*\cos(q_4)*\dot{q}_{4,2} = \tag{3.19}$$
$$b_2*(\cos(q_2)*\dot{q}_{2,2} - \sin(q_2)*\dot{q}_2*q_{2,2})$$
$$- b_3*\sin(q_3)*\dot{q}_3*q_{3,2}$$
$$+ b_4*\sin(q_4)*\dot{q}_4*q_{4,2} + \cos(q_2)*\dot{q}_2$$

$$- b_3*\cos(q_3)*\dot{q}_{3,3} + b_4*\cos(q_4)*\dot{q}_{4,3} = \tag{3.20}$$
$$b_2*(\cos(q_2)*\dot{q}_{2,3} - \sin(q_2)*\dot{q}_2*q_{2,3})$$
$$- b_3*\sin(q_3)*\dot{q}_3*q_{3,3}$$
$$+ b_4*\sin(q_4)*\dot{q}_4*q_{4,3} + \cos(q_3)*\dot{q}_3$$

$$- b_3*\cos(q_3)*\dot{q}_{3,4} + b_4*\cos(q_4)*\dot{q}_{4,4} = \tag{3.21}$$
$$b_2*(\cos(q_2)*\dot{q}_{2,4} - \sin(q_2)*\dot{q}_2*q_{2,4})$$
$$- b_3*\sin(q_3)*\dot{q}_3*q_{3,4}$$
$$+ b_4*\sin(q_4)*\dot{q}_4*q_{4,4} - \cos(q_4)*\dot{q}_4$$

The preceding eight velocity sensitivity equations contain eight unknown velocity sensitivities. The velocity sensitivity of the input link is zero since the input link velocity is independently specified. As before, the number of unknowns is equal to eight with the unknown velocity sensitivities occuring on the left side of the equations. The eight velocity sensitivity equations are solved using the same technique applied to the position sensitivity. The coefficient matrix A remains exactly the same but the vector x now contains the unknown velocity sensitivities and vector y contains the right hand sides of equations 3.14 through 3.21.

3.1.3 Acceleration Sensitivity

The first order acceleration sensitivity equations can also be derived in one of two ways. The first method is to evaluate the time derivative of the eight first order velocity sensitivity equations; the second option is to evaluate the time derivative of the velocity loop closure equations to obtain the acceleration loop closure equations and then differentiate both sides of these equations with respect to design.

The acceleration loop closure equations are derived by evaluating the time derivative of the velocity loop closure equations 2.10 and 2.11. The first order acceleration sensitivity equations are then obtained by taking the partial derivative of both sides of the acceleration loop closure equations with respect to design. Differentiation of equations 2.10 and 2.11 with respect to time and simplifying results in:

$$b_3*(\sin(q_3)*\ddot{q}_3 + \cos(q_3)*(\dot{q}_3)^2) - b_4*(\sin(q_4)*\ddot{q}_4 + \cos(q_4)*(\dot{q}_4)^2) = - b_2*(\sin(q_2)*\ddot{q}_2 + \cos(q_2)*(\dot{q}_2)^2) \qquad 3.22$$

$$- b_3*(\cos(q_3)*\ddot{q}_3 - \sin(q_3)*(\dot{q}_3)^2) + b_4*(\cos(q_4)*\ddot{q}_4 - \sin(q_4)*(\dot{q}_4)^2) = b_2*(\cos(q_2)*\ddot{q}_2 - \sin(q_2)*(\dot{q}_2)^2) \qquad 3.23$$

The notation used for the first order acceleration sensitivity coefficients is:

$$\ddot{q}_{i,j} = \frac{\partial \ddot{q}_i}{\partial b_j} \quad , \; i = 2,4 \text{ and } j = 1,4 \qquad 3.24$$

Differentiating both sides of equations 3.22 and 3.23 with respect to the link lengths b_1, b_2, b_3 and b_4 yields the following set of equations:

$$b_3*\sin(q_3)*\ddot{q}_{3,1} - b_4*\sin(q_4)*\ddot{q}_{4,1} = - b_2*(\sin(q_2)*\ddot{q}_{2,1} + \cos(q_2)*\ddot{q}_2*q_{2,1} + 2.0*\cos(q_2)*\dot{q}_2*\dot{q}_{2,1} - \sin(q_2)*(\dot{q}_2)^2*q_{2,1}) \qquad 3.25$$

$$- b_3*(\cos(q_3)*\ddot{q}_3*q_{3,1} + 2.0*\cos(q_3)*\dot{q}_3*\dot{q}_{3,1}$$
$$- \sin(q_3)*(\dot{q}_3)^2*q_{3,1}) + b_4*(\cos(q_4)*\ddot{q}_4*q_{4,1}$$
$$+ 2.0*\cos(q_4)*\dot{q}_4*\dot{q}_{4,1} - \sin(q_4)*(\dot{q}_4)^2*q_{4,1})$$

$$b_3*\sin(q_3)*\ddot{q}_{3,2} - b_4*\sin(q_4)*\ddot{q}_{4,2} = \qquad 3.26$$
$$- b_2*(\sin(q_2)*\ddot{q}_{2,2} + \cos(q_2)*\ddot{q}_2*q_{2,2}$$
$$+ 2.0*\cos(q_2)*\dot{q}_2*\dot{q}_{2,2} - \sin(q_2)*(\dot{q}_2)^2*q_{2,2})$$
$$- b_3*(\cos(q_3)*\ddot{q}_3*q_{3,2} + 2.0*\cos(q_3)*\dot{q}_3*\dot{q}_{3,2}$$
$$- \sin(q_3)*(\dot{q}_3)^2*q_{3,2}) + b_4*(\cos(q_4)*\ddot{q}_4*q_{4,2}$$
$$+ 2.0*\cos(q_4)*\dot{q}_4*\dot{q}_{4,2} - \sin(q_4)*(\dot{q}_4)^2*q_{4,2})$$
$$- (\sin(q_2)*\ddot{q}_2 + \cos(q_2)*(\dot{q}_2)^2)$$

$$b_3*\sin(q_3)*\ddot{q}_{3,3} - b_4*\sin(q_4)*\ddot{q}_{4,3} = \qquad 3.27$$
$$- b_2*(\sin(q_2)*\ddot{q}_{2,3} + \cos(q_2)*\ddot{q}_2*q_{2,3}$$
$$+ 2.0*\cos(q_2)*\dot{q}_2*\dot{q}_{2,3} - \sin(q_2)*(\dot{q}_2)^2*q_{2,3})$$
$$- b_3*(\cos(q_3)*\ddot{q}_3*q_{3,3} + 2.0*\cos(q_3)*\dot{q}_3*\dot{q}_{3,3}$$
$$- \sin(q_3)*(\dot{q}_3)^2*q_{3,3}) + b_4*(\cos(q_4)*\ddot{q}_4*q_{4,3}$$
$$+ 2.0*\cos(q_4)*\dot{q}_4*\dot{q}_{4,3} - \sin(q_4)*(\dot{q}_4)^2*q_{4,3})$$
$$- (\sin(q_3)*\ddot{q}_3 + \cos(q_3)*(\dot{q}_3)^2)$$

$$b_3*\sin(q_3)*\ddot{q}_{3,4} - b_4*\sin(q_4)*\ddot{q}_{4,4} = \qquad 3.28$$
$$- b_2*(\sin(q_2)*\ddot{q}_{2,4} + \cos(q_2)*\ddot{q}_2*q_{2,4}$$
$$+ 2.0*\cos(q_2)*\dot{q}_2*\dot{q}_{2,4} - \sin(q_2)*(\dot{q}_2)^2*q_{2,4})$$
$$- b_3*(\cos(q_3)*\ddot{q}_3*q_{3,4} + 2.0*\cos(q_3)*\dot{q}_3*\dot{q}_{3,4}$$
$$- \sin(q_3)*(\dot{q}_3)^2*q_{3,4}) + b_4*(\cos(q_4)*\ddot{q}_4*q_{4,4}$$
$$+ 2.0*\cos(q_4)*\dot{q}_4*\dot{q}_{4,4} - \sin(q_4)*(\dot{q}_4)^2*q_{4,4})$$
$$+ (\sin(q_4)*\ddot{q}_4 + \cos(q_4)*(\dot{q}_4)^2)$$

$$- b_3*\cos(q_3)*\ddot{q}_{3,1} + b_4*\cos(q_4)*\ddot{q}_{4,1} = \qquad 3.29$$

$$b_2*(\cos(q_2)*\ddot{q}_{2,1} - \sin(q_2)*\ddot{q}_2*q_{2,1}$$
$$- 2.0*\sin(q_2)*\dot{q}_2*\dot{q}_{2,1} - \cos(q_2)*(\dot{q}_2)^2*q_{2,1})$$
$$- b_3*(\sin(q_3)*\ddot{q}_3*q_{3,1} + 2.0*\sin(q_3)*\dot{q}_3*\dot{q}_{3,1}$$
$$+ \cos(q_3)*(\dot{q}_3)^2*q_{3,1}) + b_4*(\sin(q_4)*\ddot{q}_4*q_{4,1}$$
$$+ 2.0*\sin(q_4)*\dot{q}_4*\dot{q}_{4,1} + \cos(q_4)*(\dot{q}_4)^2*q_{4,1})$$

$$- b_3*\cos(q_3)*\ddot{q}_{3,2} + b_4*\cos(q_4)*\ddot{q}_{4,2} = \qquad 3.30$$

$$b_2*(\cos(q_2)*\ddot{q}_{2,2} - \sin(q_2)*\ddot{q}_2*q_{2,2}$$
$$- 2.0*\sin(q_2)*\dot{q}_2*\dot{q}_{2,2} - \cos(q_2)*(\dot{q}_2)^2*q_{2,2})$$
$$- b_3*(\sin(q_3)*\ddot{q}_3*q_{3,2} + 2.0*\sin(q_3)*\dot{q}_3*\dot{q}_{3,2}$$
$$+ \cos(q_3)*(\dot{q}_3)^2*q_{3,2}) + b_4*(\sin(q_4)*\ddot{q}_4*q_{4,2}$$
$$+ 2.0*\sin(q_4)*\dot{q}_4*\dot{q}_{4,2} + \cos(q_4)*(\dot{q}_4)^2*q_{4,2})$$
$$+ (\cos(q_2)*\ddot{q}_2 - \sin(q_2)*(\dot{q}_2)^2)$$

$$- b_3*\cos(q_3)*\ddot{q}_{3,3} + b_4*\cos(q_4)*\ddot{q}_{4,3} = \qquad 3.31$$

$$b_2*(\cos(q_2)*\ddot{q}_{2,3} - \sin(q_2)*\ddot{q}_2*q_{2,3}$$
$$- 2.0*\sin(q_2)*\dot{q}_2*\dot{q}_{2,3} - \cos(q_2)*(\dot{q}_2)^2*q_{2,3})$$
$$- b_3*(\sin(q_3)*\ddot{q}_3*q_{3,3} + 2.0*\sin(q_3)*\dot{q}_3*\dot{q}_{3,3}$$
$$+ \cos(q_3)*(\dot{q}_3)^2*q_{3,3}) + b_4*(\sin(q_4)*\ddot{q}_4*q_{4,3}$$
$$+ 2.0*\sin(q_4)*\dot{q}_4*\dot{q}_{4,3} + \cos(q_4)*(\dot{q}_4)^2*q_{4,3})$$
$$+ (\cos(q_3)*\ddot{q}_3 - \sin(q_3)*(\dot{q}_3)^2)$$

$$- b_3*\cos(q_3)*\ddot{q}_{3,4} + b_4*\cos(q_4)*\ddot{q}_{4,4} = \qquad 3.32$$

$$b_2*(\cos(q_2)*\ddot{q}_{2,4} - \sin(q_2)*\ddot{q}_2*q_{2,4}$$
$$- 2.0*\sin(q_2)*\dot{q}_2*\dot{q}_{2,4} - \cos(q_2)*(\dot{q}_2)^2*q_{2,4})$$
$$- b_3*(\sin(q_3)*\ddot{q}_3*q_{3,4} + 2.0*\sin(q_3)*\dot{q}_3*\dot{q}_{3,4}$$

$$+ \cos(q_3)*(\dot{q}_3)^2*q_{3,4}) + b_4*(\sin(q_4)*\ddot{q}_4*q_{4,4}$$

$$+ 2.0*\sin(q_4)*\dot{q}_4*\dot{q}_{4,4} + \cos(q_4)*(\dot{q}_4)^2*q_{4,4})$$

$$- (\cos(q_4)*\ddot{q}_4 - \sin(q_4)*(\dot{q}_4)^2)$$

The preceding eight acceleration sensitivity equations contain eight unknown acceleration sensitivities. The acceleration sensitivity of the input link is zero since the input link acceleration is independently specified. Once again, the equations can be arranged in matrix form with the same coefficient matrix A. The right hand side vector y now becomes the right hand sides of equations 3.25 through 3.32 and the vector x contains the unknown acceleration sensitivities. This set of equations can also be solved easily by decoupling them as described earlier.

3.1.4 Coupler Point Position Sensitivity

Another set of first order sensitivity coefficients that is of importance in designing linkages is the sensitivity of the coupler point position. The x and y position of the coupler point is defined in terms of the design variables and link angles. The two equations needed to define the x and y location of the coupler point are:

$$x = b_8 + b_2*\cos(b_7 + q_2) + b_6*\cos(b_7 + b_5 + q_3) \qquad 3.33$$

$$y = b_9 + b_2*\sin(b_7 + q_2) + b_6*\sin(b_7 + b_5 + q_3) \qquad 3.34$$

The notation used for the coupler point position sensitivity coefficients is:

$$(x)_j = \frac{\partial x}{\partial b_j} \quad , \ j = 1,9 \qquad 3.35$$

$$(y)_j = \frac{\partial y}{\partial b_j} \quad , \ j = 1,9 \qquad 3.36$$

Differentiating both sides of equations 3.33 with respect to the design variables yields the following set of equations:

$$(x)_1 = - b_6*\sin(b_7 + q_3 + b_5)*q_{3,1} \qquad 3.37$$

$$(x)_2 = - b_2*\sin(b_7 + q_2)*q_{2,2} + \cos(b_7 + q_2) - b_6*\sin(b_7 + q_3 + b_5)*q_{3,2} \qquad 3.38$$

$$(x)_3 = - b_2*\sin(b_7 + q_2)*q_{2,3} - b_6*\sin(b_7 + q_3 + b_5)*q_{3,3} \qquad 3.39$$

$$(x)_4 = - b_2*\sin(b_7 + q_2)*q_{2,4} - b_6*\sin(b_7 + q_3 + b_5)*q_{3,4} \qquad 3.40$$

$$(x)_5 = - b_6*\sin(b_7 + q_3 + b_5) \qquad 3.41$$

$$(x)_6 = \cos(b_7 + q_3 + b_5) \qquad 3.42$$

$$(x)_7 = - b_2*\sin(b_7 + q_2) - b_6*\sin(b_7 + q_3 + b_5) \qquad 3.43$$

$$(x)_8 = 1.0 \qquad 3.44$$

$$(x)_9 = 0.0 \qquad 3.45$$

Similarly, differentiation of both sides of equation 3.34 with respect to the design variables yields the following:

$$(y)_1 = b_6 * \cos(b_7 + q_3 + b_5) * q_{3,1} \tag{3.46}$$

$$(y)_2 = b_2 * \cos(b_7 + q_2) * q_{2,2} + \sin(b_7 + q_2) + b_6 * \cos(b_7 + q_3 + b_5) * q_{3,2} \tag{3.47}$$

$$(y)_3 = b_2 * \cos(b_7 + q_2) * q_{2,3} + b_6 * \cos(b_7 + q_3 + b_5) * q_{3,3} \tag{3.48}$$

$$(y)_4 = b_2 * \cos(b_7 + q_2) * q_{2,4} + b_6 * \cos(b_7 + q_3 + b_5) * q_{3,4} \tag{3.49}$$

$$(y)_5 = b_6 * \cos(b_7 + q_3 + b_5) \tag{3.50}$$

$$(y)_6 = \sin(b_7 + q_3 + b_5) \tag{3.51}$$

$$(y)_7 = b_2 * \cos(b_7 + q_2) + b_6 * \cos(b_7 + q_3 + b_5) \tag{3.52}$$

$$(y)_8 = 0.0 \tag{3.53}$$

$$(y)_9 = 1.0 \tag{3.54}$$

In general, the first order sensitivity of any function of the form $f(b,q,\dot{q},\ddot{q},x,y)$ can be computed by directly differentiating the function with respect to the design variables and using the chain rule to account for the dependency of the state variables on design as follows:

$$f_b = (f_b)_{explicit} + f_q q_b + f_{\dot{q}} \dot{q}_b + f_{\ddot{q}} \ddot{q}_b + f_x x_b + f_y y_b \tag{3.55}$$

The partial derivatives of the coupler and output link positions, velocities and accelerations are merely the corresponding first order sensitivities while the derivatives of the x and y locations of the coupler point are the coupler point position sensitivities derived in equations 3.37 through 3.54.

3.2 Second Order Sensitivity

The second order design sensitivity coefficients of a system are the partial derivatives of the first order sensitivity coefficients with respect to the design variables. We can solve for the second order position, velocity and acceleration sensitivities by finding the partial derivatives of the appropriate first order sensitivity equations with respect to the four link lengths. Since position, velocity and acceleration sensitivities of first order each have a set of eight defining equations, there are 32 available equations for the 32 corresponding second order sensitivities. However, owing to the symmetry property of the second partials (i.e. $\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1}$) only 20 of these 32 are independent.

3.2.1 Position Sensitivity

The notation used for the first order sensitivity coefficients can be extended as follows to include the second order position sensitivity coefficients also:

$$q_{i,jk} = \frac{\partial^2 q_i}{\partial b_j \partial b_k} \quad , \ i = 2,4 \ , \ j = 1,4 \text{ and } k = 1,4 \qquad 3.56$$

Evaluating the partial derivative of both sides of the eight first order position sensitivity equations (equations 3.2 through 3.9) with respect to the link lengths b_1, b_2, b_3 and b_4 and eliminating dependent equations leads to the following set of 20 equations:

$$b_3*\sin(q_3)*q_{3,11} - b_4*\sin(q_4)*q_{4,11} = \qquad 3.57$$
$$- b_2*(\sin(q_2)*q_{2,11} + \cos(q_2)*(q_{2,1})^2)$$
$$- b_3*\cos(q_3)*(q_{3,1})^2 + b_4*\cos(q_4)*(q_{4,1})^2$$

$$b_3*\sin(q_3)*q_{3,12} - b_4*\sin(q_4)*q_{4,12} = \qquad 3.58$$
$$- b_2*(\sin(q_2)*q_{2,12} + \cos(q_2)*q_{2,1}*q_{2,2})$$
$$- b_3*\cos(q_3)*q_{3,1}*q_{3,2} + b_4*\cos(q_4)*q_{4,1}*q_{4,2}$$
$$- \sin(q_2)*q_{2,1}$$

$$b_3*\sin(q_3)*q_{3,13} - b_4*\sin(q_4)*q_{4,13} = \qquad 3.59$$
$$- b_2*(\sin(q_2)*q_{2,13} + \cos(q_2)*q_{2,1}*q_{2,3})$$
$$- b_3*\cos(q_3)*q_{3,1}*q_{3,3} + b_4*\cos(q_4)*q_{4,1}*q_{4,3}$$
$$- \sin(q_3)*q_{3,1}$$

$$b_3*\sin(q_3)*q_{3,14} - b_4*\sin(q_4)*q_{4,14} = \tag{3.60}$$
$$- b_2*(\sin(q_2)*q_{2,14} + \cos(q_2)*q_{2,1}*q_{2,4})$$
$$- b_3*\cos(q_3)*q_{3,1}*q_{3,4} + b_4*\cos(q_4)*q_{4,1}*q_{4,4}$$
$$+ \sin(q_4)*q_{4,1}$$

$$b_3*\sin(q_3)*q_{3,22} - b_4*\sin(q_4)*q_{4,22} = \tag{3.61}$$
$$- b_2*(\sin(q_2)*q_{2,22} + \cos(q_2)*(q_{2,2})^2)$$
$$- b_3*\cos(q_3)*(q_{3,2})^2 + b_4*\cos(q_4)*(q_{4,2})^2$$
$$- 2.0*\sin(q_2)*q_{2,2}$$

$$b_3*\sin(q_3)*q_{3,23} - b_4*\sin(q_4)*q_{4,23} = \tag{3.62}$$
$$- b_2*(\sin(q_2)*q_{2,23} + \cos(q_2)*q_{2,2}*q_{2,3})$$
$$- b_3*\cos(q_3)*q_{3,2}*q_{3,3} + b_4*\cos(q_4)*q_{4,2}*q_{4,3}$$
$$- \sin(q_2)*q_{2,3} - \sin(q_3)*q_{3,2}$$

$$b_3*\sin(q_3)*q_{3,24} - b_4*\sin(q_4)*q_{4,24} = \tag{3.63}$$
$$- b_2*(\sin(q_2)*q_{2,24} + \cos(q_2)*q_{2,2}*q_{2,4})$$
$$- b_3*\cos(q_3)*q_{3,2}*q_{3,4} + b_4*\cos(q_4)*q_{4,2}*q_{4,4}$$
$$- \sin(q_2)*q_{2,4} + \sin(q_4)*q_{4,2}$$

$$b_3*\sin(q_3)*q_{3,33} - b_4*\sin(q_4)*q_{4,33} = \tag{3.64}$$
$$- b_2*(\sin(q_2)*q_{2,33} + \cos(q_2)*(q_{2,3})^2)$$
$$- b_3*\cos(q_3)*(q_{3,3})^2 + b_4*\cos(q_4)*(q_{4,3})^2$$
$$- 2.0*\sin(q_3)*q_{3,3}$$

$$b_3*\sin(q_3)*q_{3,34} - b_4*\sin(q_4)*q_{4,34} = \tag{3.65}$$
$$- b_2*(\sin(q_2)*q_{2,34} + \cos(q_2)*q_{2,3}*q_{2,4})$$
$$- b_3*\cos(q_3)*q_{3,3}*q_{3,4} + b_4*\cos(q_4)*q_{4,3}*q_{4,4}$$

$$- \sin(q_3)*q_{3,4} + \sin(q_4)*q_{4,3}$$

$$b_3*\sin(q_3)*q_{3,44} - b_4*\sin(q_4)*q_{4,44} = \quad 3.66$$

$$- b_2*(\sin(q_2)*q_{2,44} + \cos(q_2)*(q_{2,4})^2)$$

$$- b_3*\cos(q_3)*(q_{3,4})^2 + b_4*\cos(q_4)*(q_{4,4})^2$$

$$+ 2.0*\sin(q_4)*q_{4,4}$$

$$- b_3*\cos(q_3)*q_{3,11} + b_4*\cos(q_4)*q_{4,11} = \quad 3.67$$

$$+ b_2*(\cos(q_2)*q_{2,11} - \sin(q_2)*(q_{2,1})^2)$$

$$- b_3*\sin(q_3)*(q_{3,1})^2 + b_4*\sin(q_4)*(q_{4,1})^2$$

$$- b_3*\cos(q_3)*q_{3,12} + b_4*\cos(q_4)*q_{4,12} = \quad 3.68$$

$$+ b_2*(\cos(q_2)*q_{2,12} - \sin(q_2)*q_{2,1}*q_{2,2})$$

$$- b_3*\sin(q_3)*q_{3,1}*q_{3,2} + b_4*\sin(q_4)*q_{4,1}*q_{4,2}$$

$$+ \cos(q_2)*q_{2,1}$$

$$- b_3*\cos(q_3)*q_{3,13} + b_4*\cos(q_4)*q_{4,13} = \quad 3.69$$

$$+ b_2*(\cos(q_2)*q_{2,13} - \sin(q_2)*q_{2,1}*q_{2,3})$$

$$- b_3*\sin(q_3)*q_{3,1}*q_{3,3} + b_4*\sin(q_4)*q_{4,1}*q_{4,3}$$

$$+ \cos(q_3)*q_{3,1}$$

$$- b_3*\cos(q_3)*q_{3,14} + b_4*\cos(q_4)*q_{4,14} = \quad 3.70$$

$$+ b_2*(\cos(q_2)*q_{2,14} - \sin(q_2)*q_{2,1}*q_{2,4})$$

$$- b_3*\sin(q_3)*q_{3,1}*q_{3,4} + b_4*\sin(q_4)*q_{4,1}*q_{4,4}$$

$$- \cos(q_4)*q_{4,1}$$

$$- b_3*\cos(q_3)*q_{3,22} + b_4*\cos(q_4)*q_{4,22} = \quad 3.71$$

$$+ b_2*(\cos(q_2)*q_{2,22} - \sin(q_2)*(q_{2,2})^2)$$

$$- b_3*\sin(q_3)*(q_{3,2})^2 + b_4*\sin(q_4)*(q_{4,2})^2$$
$$+ 2.0*\cos(q_2)*q_{2,2}$$

$$- b_3*\cos(q_3)*q_{3,23} + b_4*\cos(q_4)*q_{4,23} = \tag{3.72}$$
$$+ b_2*(\cos(q_2)*q_{2,23} - \sin(q_2)*q_{2,2}*q_{2,3})$$
$$- b_3*\sin(q_3)*q_{3,2}*q_{3,3} + b_4*\sin(q_4)*q_{4,2}*q_{4,3}$$
$$+ \cos(q_2)*q_{2,3} + \cos(q_3)*q_{3,2}$$

$$- b_3*\cos(q_3)*q_{3,24} + b_4*\cos(q_4)*q_{4,24} = \tag{3.73}$$
$$+ b_2*(\cos(q_2)*q_{2,24} - \sin(q_2)*q_{2,2}*q_{2,4})$$
$$- b_3*\sin(q_3)*q_{3,2}*q_{3,4} + b_4*\sin(q_4)*q_{4,2}*q_{4,4}$$
$$+ \cos(q_2)*q_{2,4} - \cos(q_4)*q_{4,2}$$

$$- b_3*\cos(q_3)*q_{3,33} + b_4*\cos(q_4)*q_{4,33} = \tag{3.74}$$
$$+ b_2*(\cos(q_2)*q_{2,33} - \sin(q_2)*(q_{2,3})^2)$$
$$- b_3*\sin(q_3)*(q_{3,3})^2 + b_4*\sin(q_4)*(q_{4,3})^2$$
$$+ 2.0*\cos(q_3)*q_{3,3}$$

$$- b_3*\cos(q_3)*q_{3,34} + b_4*\cos(q_4)*q_{4,34} = \tag{3.75}$$
$$+ b_2*(\cos(q_2)*q_{2,34} - \sin(q_2)*q_{2,3}*q_{2,4})$$
$$- b_3*\sin(q_3)*q_{3,3}*q_{3,4} + b_4*\sin(q_4)*q_{4,3}*q_{4,4}$$
$$+ \cos(q_3)*q_{3,4} - \cos(q_4)*q_{4,3}$$

$$- b_3*\cos(q_3)*q_{3,44} + b_4*\cos(q_4)*q_{4,44} = \tag{3.76}$$
$$+ b_2*(\cos(q_2)*q_{2,44} - \sin(q_2)*(q_{2,4})^2)$$
$$- b_3*\sin(q_3)*(q_{3,4})^2 + b_4*\sin(q_4)*(q_{4,4})^2$$
$$- 2.0*\cos(q_4)*q_{4,4}$$

The preceding twenty second order position sensitivity equations contain twenty unknown second order position sensitivities (omitting the second order sensitivities of the input crank angle q_2 since they are zero). The 20 equations above can be placed in matrix form as before but the coefficient matrix B is now of dimension 20x20. The unknown vector x will contain the second order position sensitivities to be computed and the right side vector y will contain the right hand sides of equations 3.57 through 3.76. The matrix B is shown in Table 3.1 and the corresponding vector of unknown second order position sensitivities is given in Table 3.2.

This system of equations can be decoupled and solved as before to obtain the second order position sensitivity coefficients.

3.2.2 Velocity Sensitivity

The notation used for the second order velocity sensitivity coefficients is:

$$\dot{q}_{i,jk} = \frac{\partial^2 \dot{q}_i}{\partial b_j \, \partial b_k} \quad , \; i = 2,4 \; , \; j = 1,4 \text{ and } k = 1,4 \qquad 3.77$$

Evaluating the partial derivatives of both sides of the eight first order velocity sensitivity equations (equations

$$
\left|
\begin{matrix}
s3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & s3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & s3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & s3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & s3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & s3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & s3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & s3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s4 \\
c3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c4
\end{matrix}
\right|
$$

Table 3.1 Coefficient matrix for second order sensitivity

$$
\begin{vmatrix}
q_{3,11} \\
q_{3,12} \\
q_{3,13} \\
q_{3,14} \\
q_{3,22} \\
q_{3,23} \\
q_{3,24} \\
q_{3,33} \\
q_{3,34} \\
q_{3,44} \\
q_{4,11} \\
q_{4,12} \\
q_{4,13} \\
q_{4,14} \\
q_{4,22} \\
q_{4,23} \\
q_{4,24} \\
q_{4,33} \\
q_{4,34} \\
q_{4,44}
\end{vmatrix}
$$

Table 3.2 Vector of second order position sensitivities

3.14 through 3.21) with respect to the link lengths b_1, b_2, b_3 and b_4 yields the following set of equations:

$$b_3*\sin(q_3)*\dot{q}_{3,11} - b_4*\sin(q_4)*\dot{q}_{4,11} = \qquad 3.78$$

$$- b_2*(\sin(q_2)*\dot{q}_{2,11} + \cos(q_2)*\dot{q}_{2,1}*q_{2,1}$$

$$+ \cos(q_2)*(q_{2,1}*\dot{q}_{2,1} + \dot{q}_2*q_{2,11})$$

$$- \sin(q_2)*\dot{q}_2*(q_{2,1})^2)$$

$$- b_3*(\cos(q_3)*\dot{q}_{3,1}*q_{3,1} + \cos(q_3)*(q_{3,1}*\dot{q}_{3,1}$$

$$+ \dot{q}_3*q_{3,11}) - \sin(q_3)*\dot{q}_3*(q_{3,1})^2)$$

$$+ b_4*(\cos(q_4)*\dot{q}_{4,1}*q_{4,1} + \cos(q_4)*(q_{4,1}*\dot{q}_{4,1}$$

$$+ \dot{q}_4*q4,11) - \sin(q_4)*\dot{q}_4*(q_{4,1})^2)$$

$$b_3*\sin(q_3)*\dot{q}_{3,12} - b_4*\sin(q_4)*\dot{q}_{4,12} = \qquad 3.79$$

$$- b_2*(\sin(q_2)*\dot{q}2,12 + \cos(q_2)*\dot{q}_{2,1}*q_{2,2}$$

$$+ \cos(q_2)*(q_{2,1}*\dot{q}_{2,2} + \dot{q}_2*q2,12)$$

$$- \sin(q_2)*q_{2,1}*\dot{q}_2*q_{2,2})$$

$$- b_3*(\cos(q_3)*\dot{q}_{3,1}*q_{3,2} + \cos(q_3)*(q_{3,1}*\dot{q}_{3,2}$$

$$+ \dot{q}_3*q_{3,12}) - \sin(q_3)*q_{3,1}*\dot{q}_3*q_{3,2})$$

$$+ b_4*(\cos(q_4)*\dot{q}_{4,1}*q_{4,2} + \cos(q_4)*(q_{4,1}*\dot{q}_{4,2}$$

$$+ \dot{q}_4*q_{4,12}) - \sin(q_4)*q_{4,1}*\dot{q}_4*q_{4,2})$$

$$- (\sin(q_2)*\dot{q}_{2,1} + \cos(q_2)*q_{2,1}*\dot{q}_2)$$

$$b_3*\sin(q_3)*\dot{q}_{3,13} - b_4*\sin(q_4)*\dot{q}_{4,13} = \qquad 3.80$$

$$- b_2*(\sin(q_2)*\dot{q}_{2,13} + \cos(q_2)*\dot{q}_{2,1}*q_{2,3}$$

$$+ \cos(q_2)*(q_{2,1}*\dot{q}_{2,3} + \dot{q}_2*q_{2,13})$$

$$- \sin(q_2)*q_{2,1}*\dot{q}_2*q_{2,3})$$

$$- b_3*(\cos(q_3)*\dot{q}_{3,1}*q_{3,3} + \cos(q_3)*(q_{3,1}*\dot{q}_{3,3}$$

$$+ \dot{q}_3*q_{3,13}) - \sin(q_3)*q_{3,1}*\dot{q}_3*q_{3,3})$$

$$+ b_4*(\cos(q_4)*\dot{q}_{4,1}*q_{4,3} + \cos(q_4)*(q_{4,1}*\dot{q}_{4,3}$$

$$+ \dot{q}_4*q_{2,13}) - \sin(q_4)*q_{4,1}*\dot{q}_4*q_{4,3})$$

$$- (\sin(q_3)*\dot{q}_{3,1} + \cos(q_3)*q_{3,1}*\dot{q}_3)$$

$$b_3*\sin(q_3)*\dot{q}_{3,14} - b_4*\sin(q_4)*\dot{q}_{4,14} = \qquad 3.81$$

$$- b_2*(\sin(q_2)*\dot{q}_{2,14} + \cos(q_2)*\dot{q}_{2,1}*q_{2,4}$$

$$+ \cos(q_2)*(q_{2,1}*\dot{q}_{2,4} + \dot{q}_2*q2,14)$$

$$- \sin(q_2)*q_{2,1}*\dot{q}_2*q_{2,4})$$

$$- b_3*(\cos(q_3)*\dot{q}_{3,1}*q_{3,4} + \cos(q_3)*(q_{3,1}*\dot{q}_{3,4}$$

$$+ \dot{q}_3*q_{3,14}) - \sin(q_3)*q_{3,1}*\dot{q}_3*q_{3,4})$$

$$+ b_4*(\cos(q_4)*\dot{q}_{4,1}*q_{4,4} + \cos(q_4)*(q_{4,1}*\dot{q}_{4,4}$$

$$+ \dot{q}_4*q_{4,14}) - \sin(q_4)*q_{4,1}*\dot{q}_4*q_{4,4})$$

$$+ (\sin(q_4)*\dot{q}_{4,1} + \cos(q_4)*q_{4,1}*\dot{q}_4)$$

$$b_3*\sin(q_3)*\dot{q}_{3,22} - b_4*\sin(q_4)*\dot{q}_{4,22} = \qquad 3.82$$

$$- b_2*(\sin(q_2)*\dot{q}_{2,22} + \cos(q_2)*\dot{q}_{2,2}*q_{2,2}$$

$$+ \cos(q_2)*(q_{2,2}*\dot{q}_{2,2} + \dot{q}_2*q_{2,22})$$

$$- \sin(q_2)*\dot{q}_2*(q_{2,2})^2)$$

$$- b_3*(\cos(q_3)*\dot{q}_{3,2}*q_{3,2} + \cos(q_3)*(q_{3,2}*\dot{q}_{3,2}$$

$$+ \dot{q}_3*q_{3,22}) - \sin(q_3)*\dot{q}_3*(q_{3,2})^2)$$

$$+ b_4*(\cos(q_4)*\dot{q}_{4,2}*q_{4,2} + \cos(q_4)*(q_{4,2}*\dot{q}_{4,2}$$

$$+ \dot{q}_4*q_{4,22}) - \sin(q_4)*\dot{q}_4*(q_{4,2})^2)$$

$$- 2.0*(\sin(q_2)*\dot{q}_{2,2} + \cos(q_2)*\dot{q}_2*q_{2,2})$$

$$b_3*\sin(q_3)*\dot{q}_{3,23} - b_4*\sin(q_4)*\dot{q}_{4,23} = \qquad 3.83$$

$$- b_2*(\sin(q_2)*\dot{q}_{2,23} + \cos(q_2)*\dot{q}_{2,2}*q_{2,3}$$

$$+ \cos(q_2)*(q_{2,2}*\dot{q}_{2,3} + \dot{q}_2*q_{2,23})$$

$$- \sin(q_2)*q_{2,2}*\dot{q}_2*q_{2,3})$$

$$- b_3*(\cos(q_3)*\dot{q}_{3,2}*q_{3,3} + \cos(q_3)*(q_{3,2}*\dot{q}_{3,3}$$

$$+ \dot{q}_3*q_{3,23}) - \sin(q_3)*q_{3,2}*\dot{q}_3*q_{3,3})$$

$$+ b_4*(\cos(q_4)*\dot{q}_{4,2}*q_{4,3} + \cos(q_4)*(q_{4,2}*\dot{q}_{4,3}$$

$$+ \dot{q}_4*q_{4,23}) - \sin(q_4)*q_{4,2}*\dot{q}_4*q_{4,3})$$

$$- (\sin(q_2)*\dot{q}_{2,3} + \cos(q_2)*q_{2,3}*\dot{q}_2)$$

$$- (\sin(q_3)*\dot{q}_{3,2} + \cos(q_3)*q_{3,2}*\dot{q}_3)$$

$$b_3*\sin(q_3)*\dot{q}_{3,24} - b_4*\sin(q_4)*\dot{q}_{4,24} = \qquad 3.84$$

$$- b_2*(\sin(q_2)*\dot{q}_{2,24} + \cos(q_2)*\dot{q}_{2,2}*q_{2,4}$$

$$+ \cos(q_2)*(q_{2,2}*\dot{q}_{2,4} + \dot{q}_2*q_{2,24})$$

$$- \sin(q_2)*q_{2,2}*\dot{q}_2*q_{2,4})$$

$$- b_3*(\cos(q_3)*\dot{q}_{3,2}*q_{3,4} + \cos(q_3)*(q_{3,2}*\dot{q}_{3,4}$$

$$+ \dot{q}_3*q_{3,24}) - \sin(q_3)*q_{3,2}*\dot{q}_3*q_{3,4})$$

$$+ b_4*(\cos(q_4)*\dot{q}_{4,2}*q_{4,4} + \cos(q_4)*(q_{4,2}*\dot{q}_{4,4}$$

$$+ \dot{q}_4*q_{4,24}) - \sin(q_4)*q_{4,2}*\dot{q}_4*q_{4,4})$$

$$- (\sin(q_2)*\dot{q}_{2,4} + \cos(q_2)*q_{2,4}*\dot{q}_2)$$

$$+ (\sin(q_4)*\dot{q}_{4,2} + \cos(q_4)*q_{4,2}*\dot{q}_4)$$

$$b_3*\sin(q_3)*\dot{q}_{3,33} - b_4*\sin(q_4)*\dot{q}_{4,33} = \qquad 3.85$$

$$- b_2*(\sin(q_2)*\dot{q}_{2,33} + \cos(q_2)*\dot{q}_{2,3}*q_{2,3}$$

$$+ \cos(q_2)*(q_{2,3}*\dot{q}_{2,3} + \dot{q}_2*q_{2,33})$$

$$- \sin(q_2)*\dot{q}_2*(q_{2,3})^2)$$

$$- b_3*(\cos(q_3)*\dot{q}_{3,3}*q_{3,3} + \cos(q_3)*(q_{3,3}*\dot{q}_{3,3}$$
$$+ \dot{q}_3*q_{3,33}) - \sin(q_3)*\dot{q}_3*(q_{3,3})^2)$$
$$+ b_4*(\cos(q_4)*\dot{q}_{4,3}*q_{4,3} + \cos(q_4)*(q_{4,3}*\dot{q}_{4,3}$$
$$+ \dot{q}_4*q_{4,33}) - \sin(q_4)*\dot{q}_4*(q_{4,3})^2)$$
$$- 2.0*(\sin(q_3)*\dot{q}_{3,3} + \cos(q_3)*\dot{q}_3*q_{3,3})$$

$$b_3*\sin(q_3)*\dot{q}_{3,34} - b_4*\sin(q_4)*\dot{q}_{4,34} = \tag{3.86}$$
$$- b_2*(\sin(q_2)*\dot{q}_{2,34} + \cos(q_2)*\dot{q}_{2,3}*q_{2,4}$$
$$+ \cos(q_2)*(q_{2,3}*\dot{q}_{2,4} + \dot{q}_2*q_{2,34})$$
$$- \sin(q_2)*q_{2,3}*\dot{q}_2*q_{2,4})$$
$$- b_3*(\cos(q_3)*\dot{q}_{3,3}*q_{3,4} + \cos(q_3)*(q_{3,3}*\dot{q}_{3,4}$$
$$+ \dot{q}_3*q_{3,34}) - \sin(q_3)*q_{3,3}*\dot{q}_3*q_{3,4})$$
$$+ b_4*(\cos(q_4)*\dot{q}_{4,3}*q_{4,4} + \cos(q_4)*(q_{4,3}*\dot{q}_{4,4}$$
$$+ \dot{q}_4*q_{4,34}) - \sin(q_4)*q_{4,3}*\dot{q}_4*q_{4,4})$$
$$- (\sin(q_3)*\dot{q}_{3,4} + \cos(q_3)*q_{3,4}*\dot{q}_3)$$
$$+ (\sin(q_4)*\dot{q}_{4,3} + \cos(q_4)*q_{4,3}*\dot{q}_4)$$

$$b_3*\sin(q_3)*\dot{q}_{3,44} - b_4*\sin(q_4)*\dot{q}_{4,44} = \tag{3.87}$$
$$- b_2*(\sin(q_2)*\dot{q}_{2,44} + \cos(q_2)*\dot{q}_{2,4}*q_{2,4}$$
$$+ \cos(q_2)*(q_{2,4}*\dot{q}_{2,4} + \dot{q}_2*q_{2,44})$$
$$- \sin(q_2)*\dot{q}_2*(q_{2,4})^2)$$
$$- b_3*(\cos(q_3)*\dot{q}_{3,4}*q_{3,4} + \cos(q_3)*(q_{3,4}*\dot{q}_{3,4}$$
$$+ \dot{q}_3*q_{3,44}) - \sin(q_3)*\dot{q}_3*(q_{3,4})^2)$$
$$+ b_4*(\cos(q_4)*\dot{q}_{4,4}*q_{4,4} + \cos(q_4)*(q_{4,4}*\dot{q}_{4,4}$$
$$+ \dot{q}_4*q_{4,44}) - \sin(q_4)*\dot{q}_4*(q_{4,4})^2)$$
$$+ 2.0*(\sin(q_4)*\dot{q}_{4,4} + \cos(q_4)*\dot{q}_4*q_{4,4})$$

$$- b_3*\cos(q_3)*\dot{q}_{3,11} + b_4*\cos(q_4)*\dot{q}_{4,11} = \tag{3.88}$$

$$+ b_2*(\cos(q_2)*\dot{q}_{2,11} - \sin(q_2)*\dot{q}_{2,1}*q_{2,1}$$

$$- \sin(q_2)*(q_{2,1}*\dot{q}_{2,1} + \dot{q}_2*q_{2,11})$$

$$- \cos(q_2)*\dot{q}_2*(q_{2,1})^2)$$

$$- b_3*(\sin(q_3)*\dot{q}_{3,1}*q_{3,1} + \sin(q_3)*(q_{3,1}*\dot{q}_{3,1}$$

$$+ \dot{q}_3*q_{3,11}) + \cos(q_3)*\dot{q}_3*(q_{3,1})^2)$$

$$+ b_4*(\sin(q_4)*\dot{q}_{4,1}*q_{4,1} + \sin(q_4)*(q_{4,1}*\dot{q}_{4,1}$$

$$+ \dot{q}_4*q_{4,11}) + \cos(q_4)*\dot{q}_4*(q_{4,1})^2)$$

$$- b_3*\cos(q_3)*\dot{q}_{3,12} + b_4*\cos(q_4)*\dot{q}_{4,12} = \tag{3.89}$$

$$+ b_2*(\cos(q_2)*\dot{q}_{2,12} - \sin(q_2)*\dot{q}_{2,1}*q_{2,2}$$

$$- \sin(q_2)*(q_{2,1}*\dot{q}_{2,2} + \dot{q}_2*q_{2,12})$$

$$- \cos(q_2)*q_{2,1}*\dot{q}_2*q_{2,2})$$

$$- b_3*(\sin(q_3)*\dot{q}_{3,1}*q_{3,2} + \sin(q_3)*(q_{3,1}*\dot{q}_{3,2}$$

$$+ \dot{q}_3*q_{3,12}) + \cos(q_3)*q_{3,1}*\dot{q}_3*q_{3,2})$$

$$+ b_4*(\sin(q_4)*\dot{q}_{4,1}*q_{4,2} + \sin(q_4)*(q_{4,1}*\dot{q}_{4,2}$$

$$+ \dot{q}_4*q_{4,12}) + \cos(q_4)*q_{4,1}*\dot{q}_4*q_{4,2})$$

$$+ (\cos(q_2)*\dot{q}_{2,1} - \sin(q_2)*q_{2,1}*\dot{q}_2)$$

$$- b_3*\cos(q_3)*\dot{q}_{3,13} + b_4*\cos(q_4)*\dot{q}_{4,13} = \tag{3.90}$$

$$+ b_2*(\cos(q_2)*\dot{q}_{2,13} - \sin(q_2)*\dot{q}_{2,1}*q_{2,3}$$

$$- \sin(q_2)*(q_{2,1}*\dot{q}_{2,3} + \dot{q}_2*q_{2,13})$$

$$- \cos(q_2)*q_{2,1}*\dot{q}_2*q_{2,3})$$

$$- b_3*(\sin(q_3)*\dot{q}_{3,1}*q_{3,3} + \sin(q_3)*(q_{3,1}*\dot{q}_{3,3}$$

$$+ \dot{q}_3*q_{3,13}) + \cos(q_3)*q_{3,1}*\dot{q}_3*q_{3,3})$$

$$+ b_4*(\sin(q_4)*\dot{q}_{4,1}*q_{4,3} + \sin(q_4)*(q_{4,1}*\dot{q}_{4,3}$$

$$+ \dot{q}_4*q_{4,13}) + \cos(q_4)*q_{4,1}*\dot{q}_4*q_{4,3})$$

$$+ (\cos(q_3)*\dot{q}_{3,1} - \sin(q_3)*q_{3,1}*\dot{q}_3)$$

$$- b_3*\cos(q_3)*\dot{q}_{3,14} + b_4*\cos(q_4)*\dot{q}_{4,14} = \qquad 3.91$$

$$+ b_2*(\cos(q_2)*\dot{q}_{2,14} - \sin(q_2)*\dot{q}_{2,1}*q_{2,4}$$

$$- \sin(q_2)*(q_{2,1}*\dot{q}_{2,4} + \dot{q}_2*q_{2,14})$$

$$- \cos(q_2)*q_{2,1}*\dot{q}_2*q_{2,4})$$

$$- b_3*(\sin(q_3)*\dot{q}_{3,1}*q_{3,4} + \sin(q_3)*(q_{3,1}*\dot{q}_{3,4}$$

$$+ \dot{q}_3*q_{3,14}) + \cos(q_3)*q_{3,1}*\dot{q}_3*q_{3,4})$$

$$+ b_4*(\sin(q_4)*\dot{q}_{4,1}*q_{4,4} + \sin(q_4)*(q_{4,1}*\dot{q}_{4,4}$$

$$+ \dot{q}_4*q_{4,14}) + \cos(q_4)*q_{4,1}*\dot{q}_4*q_{4,4})$$

$$- (\cos(q_4)*\dot{q}_{4,1} - \sin(q_4)*q_{4,1}*\dot{q}_4)$$

$$- b_3*\cos(q_3)*\dot{q}_{3,22} + b_4*\cos(q_4)*\dot{q}_{4,22} = \qquad 3.92$$

$$+ b_2*(\cos(q_2)*\dot{q}_{2,22} - \sin(q_2)*\dot{q}_{2,2}*q_{2,2}$$

$$- \sin(q_2)*(q_{2,2}*\dot{q}_{2,2} + \dot{q}_2*q_{2,22})$$

$$- \cos(q_2)*\dot{q}_2*(q_{2,2})^2)$$

$$- b_3*(\sin(q_3)*\dot{q}_{3,2}*q_{3,2} + \sin(q_3)*(q_{3,2}*\dot{q}_{3,2}$$

$$+ \dot{q}_3*q_{3,22}) + \cos(q_3)*\dot{q}_3*(q_{3,2})^2)$$

$$+ b_4*(\sin(q_4)*\dot{q}_{4,2}*q_{4,2} + \sin(q_4)*(q_{4,2}*\dot{q}_{4,2}$$

$$+ \dot{q}_4*q_{4,22}) + \cos(q_4)*\dot{q}_4*(q_{4,2})^2)$$

$$+ 2.0*(\cos(q_2)*\dot{q}_{2,2} - \sin(q_2)*\dot{q}_2*q_{2,2})$$

$$- b_3*\cos(q_3)*\dot{q}_{3,23} + b_4*\cos(q_4)*\dot{q}_{4,23} = \qquad 3.93$$

$$+ b_2*(\cos(q_2)*\dot{q}_{2,23} - \sin(q_2)*\dot{q}_{2,2}*q_{2,3}$$

$$- \sin(q_2)*(q_{2,2}*\dot{q}_{2,3} + \dot{q}_2*q_{2,23})$$

$$- \cos(q_2)*q_{2,2}*\dot{q}_2*q_{2,3})$$

$$- b_3*(\sin(q_3)*\dot{q}_{3,2}*q_{3,3} + \sin(q_3)*(q_{3,2}*\dot{q}_{3,3}$$
$$+ \dot{q}_3*q_{3,23}) + \cos(q_3)*q_{3,2}*\dot{q}_3*q_{3,3})$$
$$+ b_4*(\sin(q_4)*\dot{q}_{4,2}*q_{4,3} + \sin(q_4)*(q_{4,2}*\dot{q}_{4,3}$$
$$+ \dot{q}_4*q_{4,23}) + \cos(q_4)*q_{4,2}*\dot{q}_4*q_{4,3})$$
$$+ (\cos(q_2)*\dot{q}_{2,3} - \sin(q_2)*q_{2,3}*\dot{q}_2)$$
$$+ (\cos(q_3)*\dot{q}_{3,2} - \sin(q_3)*q_{3,2}*\dot{q}_3)$$

$$- b_3*\cos(q_3)*\dot{q}_{3,24} + b_4*\cos(q_4)*\dot{q}_{4,24} = \qquad 3.94$$
$$+ b_2*(\cos(q_2)*\dot{q}_{2,24} - \sin(q_2)*\dot{q}_{2,2}*q_{2,4}$$
$$- \sin(q_2)*(q_{2,2}*\dot{q}_{2,4} + \dot{q}_2*q_{2,24})$$
$$- \cos(q_2)*q_{2,2}*\dot{q}_2*q_{2,4})$$
$$- b_3*(\sin(q_3)*\dot{q}_{3,2}*q_{3,4} + \sin(q_3)*(q_{3,2}*\dot{q}_{3,4}$$
$$+ \dot{q}_3*q_{3,24}) + \cos(q_3)*q_{3,2}*\dot{q}_3*q_{3,4})$$
$$+ b_4*(\sin(q_4)*\dot{q}_{4,2}*q_{4,4} + \sin(q_4)*(q_{4,2}*\dot{q}_{4,4}$$
$$+ \dot{q}_4*q_{4,24}) + \cos(q_4)*q_{4,2}*\dot{q}_4*q_{4,4})$$
$$+ (\cos(q_2)*\dot{q}_{2,4} - \sin(q_2)*q_{2,4}*\dot{q}_2)$$
$$- (\cos(q_4)*\dot{q}_{4,2} - \sin(q_4)*q_{4,2}*\dot{q}_4)$$

$$- b_3*\cos(q_3)*\dot{q}_{3,33} + b_4*\cos(q_4)*\dot{q}_{4,33} = \qquad 3.95$$
$$+ b_2*(\cos(q_2)*\dot{q}_{2,33} - \sin(q_2)*\dot{q}_{2,3}*q_{2,3}$$
$$- \sin(q_2)*(q_{2,3}*\dot{q}_{2,3} + \dot{q}_2*q_{2,33})$$
$$- \cos(q_2)*\dot{q}_2*(q_{2,3})^2)$$
$$- b_3*(\sin(q_3)*\dot{q}_{3,3}*q_{3,3} + \sin(q_3)*(q_{3,3}*\dot{q}_{3,3}$$
$$+ \dot{q}_3*q_{3,33}) + \cos(q_3)*\dot{q}_3*(q_{3,3})^2)$$
$$+ b_4*(\sin(q_4)*\dot{q}_{4,3}*q_{4,3} + \sin(q_4)*(q_{4,3}*\dot{q}_{4,3}$$
$$+ \dot{q}_4*q_{4,33}) + \cos(q_4)*\dot{q}_4*(q_{4,3})^2)$$

$$+ 2.0*(\cos(q_3)*\dot{q}_{3,3} - \sin(q_3)*\dot{q}_3*q_{3,3})$$

$$- b_3*\cos(q_3)*\dot{q}_{3,34} + b_4*\cos(q_4)*\dot{q}_{4,34} = \quad 3.96$$

$$+ b_2*(\cos(q_2)*\dot{q}_{2,34} - \sin(q_2)*\dot{q}_{2,3}*q_{2,4}$$

$$- \sin(q_2)*(q_{2,3}*\dot{q}_{2,4} + \dot{q}_2*q_{2,34})$$

$$- \cos(q_2)*q_{2,3}*\dot{q}_2*q_{2,4})$$

$$- b_3*(\sin(q_3)*\dot{q}_{3,3}*q_{3,4} + \sin(q_3)*(q_{3,3}*\dot{q}_{3,4}$$

$$+ \dot{q}_3*q_{3,34}) + \cos(q_3)*q_{3,3}*\dot{q}_3*q_{3,4})$$

$$+ b_4*(\sin(q_4)*\dot{q}_{4,3}*q_{4,4} + \sin(q_4)*(q_{4,3}*\dot{q}_{4,4}$$

$$+ \dot{q}_4*q_{4,34}) + \cos(q_4)*q_{4,3}*\dot{q}_4*q_{4,4})$$

$$+ (\cos(q_3)*\dot{q}_{3,4} - \sin(q_3)*q_{3,4}*\dot{q}_3)$$

$$- (\cos(q_4)*\dot{q}_{4,3} - \sin(q_4)*q_{4,3}*\dot{q}_4)$$

$$- b_3*\cos(q_3)*\dot{q}_{3,44} + b_4*\cos(q_4)*\dot{q}_{4,44} = \quad 3.97$$

$$+ b_2*(\cos(q_2)*\dot{q}_{2,44} - \sin(q_2)*\dot{q}_{2,4}*q_{2,4}$$

$$- \sin(q_2)*(q_{2,4}*\dot{q}_{2,4} + \dot{q}_2*q_{2,44})$$

$$- \cos(q_2)*\dot{q}_2*(q_{2,4})^2)$$

$$- b_3*(\sin(q_3)*\dot{q}_{3,4}*q_{3,4} + \sin(q_3)*(q_{3,4}*\dot{q}_{3,4}$$

$$+ \dot{q}_3*q_{3,44}) + \cos(q_3)*\dot{q}_3*(q_{3,4})^2)$$

$$+ b_4*(\sin(q_4)*\dot{q}_{4,4}*q_{4,4} + \sin(q_4)*(q_{4,4}*\dot{q}_{4,4}$$

$$+ \dot{q}_4*q_{4,44}) + \cos(q_4)*\dot{q}_4*(q_{4,4})^2)$$

$$- 2.0*(\cos(q_4)*\dot{q}_{4,4} - \sin(q_4)*\dot{q}_4*q_{4,4})$$

The preceding twenty velocity sensitivity equations contain twenty unknown second order velocity sensitivities (omitting the second order velocity sensitivities of the input crank $\dot{q}_2$ since they are zero). The 20 equations above can be written in matrix form with the right side vector containing only known values. The coefficient matrix that results is identical to that shown in Table 3.1 and the same solution procedure can be applied.

3.2.3 Acceleration Sensitivity

The notation used for the second order acceleration sensitivity coefficients is:

$$\ddot{q}_{i,jk} = \frac{\partial^2 \ddot{q}_i}{\partial b_j \partial b_k} \quad , \ i = 2,4 \ , \ j = 1,4 \text{ and } k = 1,4 \qquad 3.98$$

Evaluating the partial derivative of both sides of the eight first order acceleration sensitivity equations (equations 3.25 through 3.32) with respect to the link lengths b_1, b_2, b_3 and b_4 yields the following set of 20 equations:

$$b_3*\sin(q_3)*\ddot{q}_{3,11} - b_4*\sin(q_4)*\ddot{q}_{4,11} = \qquad 3.99$$

$$- b_2*(\sin(q_2)*\ddot{q}_{2,11} + \cos(q_2)*\ddot{q}_{2,1}*q_{2,1}$$

$$+ \cos(q_2)*(q_{2,1}*\ddot{q}_{2,1} + \ddot{q}_2*q_{2,11})$$

$$- \sin(q_2)*\ddot{q}_2*(q_{2,1})^2$$

$$+ 2.0*(\cos(q_2)*(\dot{q}_2*\dot{q}_{2,11} + (\dot{q}_{2,1})^2)$$

$$- \sin(q_2)*\dot{q}_{2,1}*\dot{q}_2*q_{2,1}) - \sin(q_2)*(2.0*\dot{q}_2*q_{2,1}*\dot{q}_{2,1}$$

$$+ (\dot{q}_2)^2*q_{2,11}) - \cos(q_2)*(\dot{q}_2)^2*(q_{2,1})^2)$$

$$- b_3*(\cos(q_3)*\ddot{q}_{3,1}*q_{3,1} + \cos(q_3)*(q_{3,1}*\ddot{q}_{3,1}$$

$$+ \ddot{q}_3*q_{3,11}) - \sin(q_3)*\ddot{q}_3*(q_{3,1})^2$$
$$+ 2.0*(\cos(q_3)*(\dot{q}_3*\dot{q}_{3,11} + (\dot{q}_{3,1})^2)$$
$$- \sin(q_3)*\dot{q}_{3,1}*\dot{q}_3*q_{3,1}) - \sin(q_3)*(2.0*\dot{q}_3*q_{3,1}*\dot{q}_{3,1}$$
$$+ (\dot{q}_3)^2*q_{3,11}) - \cos(q_3)*(\dot{q}_3)^2*(q_{3,1})^2)$$
$$+ b_4*(\cos(q_4)*\ddot{q}_{4,1}*q_{4,1} + \cos(q_4)*(q_{4,1}*\ddot{q}_{4,1}$$
$$+ \ddot{q}_4*q_{4,11}) - \sin(q_4)*\ddot{q}_4*(q_{4,1})^2$$
$$+ 2.0*(\cos(q_4)*(\dot{q}_4*\dot{q}_{4,11} + (\dot{q}_{4,1})^2)$$
$$- \sin(q_4)*\dot{q}_{4,1}*\dot{q}_4*q_{4,1}) - \sin(q_4)*(2.0*\dot{q}_4*q_{4,1}*\dot{q}_{4,1}$$
$$+ (\dot{q}_4)^2*q_{4,11}) - \cos(q_4)*(\dot{q}_4)^2*(q_{4,1})^2)$$

$$b_3*\sin(q_3)*\ddot{q}_{3,12} - b_4\sin(q_4)*\ddot{q}_{4,12} = \qquad 3.100$$
$$- b_2*(\sin(q_2)*\ddot{q}_{2,12} + \cos(q_2)*\ddot{q}_{2,1}*q_{2,2}$$
$$+ \cos(q_2)*(q_{2,1}*\ddot{q}_{2,2} + \ddot{q}_2*q_{2,12})$$
$$- \sin(q_2)*q_{2,1}*\ddot{q}_2*q_{2,2}$$
$$+ 2.0*(\cos(q_2)*(\dot{q}_2*\dot{q}_{2,12} + \dot{q}_{2,1}*\dot{q}_{2,2})$$
$$- \sin(q_2)*\dot{q}_{2,1}*\dot{q}_2*q_{2,2}) - \sin(q_2)*(2.0*\dot{q}_2*q_{2,1}*\dot{q}_{2,2}$$
$$+ (\dot{q}_2)^2*q_{2,12}) - \cos(q_2)*q_{2,1}*(\dot{q}_2)^2*q_{2,2})$$
$$- b_3*(\cos(q_3)*\ddot{q}_{3,1}*q_{3,2} + \cos(q_3)*(q_{3,1}*\ddot{q}_{3,2}$$
$$+ \ddot{q}_3*q_{3,12}) - \sin(q_3)*q_{3,1}*\ddot{q}_3*q_{3,2}$$
$$+ 2.0*(\cos(q_3)*(\dot{q}_3*\dot{q}_{3,12} + \dot{q}_{3,1}*\dot{q}_{3,2})$$
$$- \sin(q_3)*\dot{q}_{3,1}*\dot{q}_3*q_{3,2}) - \sin(q_3)*(2.0*\dot{q}_3*q_{3,1}*\dot{q}_{3,2}$$
$$+ (\dot{q}_3)^2*q_{3,12}) - \cos(q_3)*q_{3,1}*(\dot{q}_3)^2*q_{3,2})$$
$$+ b_4*(\cos(q_4)*\ddot{q}_{4,1}*q_{4,2} + \cos(q_4)*(q_{4,1}*\ddot{q}_{4,2}$$
$$+ \ddot{q}_4*q_{4,12}) - \sin(q_4)*q_{4,1}*\ddot{q}_4*q_{4,2}$$
$$+ 2.0*(\cos(q_4)*(\dot{q}_4*\dot{q}_{4,12} + \dot{q}_{4,1}*\dot{q}_{4,2})$$

$- \sin(q_4)*\dot{q}_{4,1}*\dot{q}_4*q_{4,2}) - \sin(q_4)*(2.0*\dot{q}_4*q_{4,1}*\dot{q}_{4,2}$

$+ (\dot{q}_4)^2*q_{4,12}) - \cos(q_4)*q_{4,1}*(\dot{q}_4)^2*q_{4,2})$

$- (\sin(q_2)*\ddot{q}_{2,1} + \cos(q_2)*q_{2,1}*\ddot{q}_2$

$+ 2.0*\cos(q_2)*\dot{q}_2*\dot{q}_{2,1} - \sin(q_2)*q_{2,1}*(\dot{q}_2)^2)$

$b_3*\sin(q_3)*\ddot{q}_{3,13} - b_4*\sin(q_4)*\ddot{q}_{4,13} =$ 3.101

$- b_2*(\sin(q_2)*\ddot{q}_{2,13} + \cos(q_2)*\ddot{q}_{2,1}*q_{2,3}$

$+ \cos(q_2)*(q_{2,1}*\ddot{q}_{2,3} + \ddot{q}_2*q_{2,13})$

$- \sin(q_2)*q_{2,1}*\ddot{q}_2*q_{2,3}$

$+ 2.0*(\cos(q_2)*(\dot{q}_2*\dot{q}_{2,13} + \dot{q}_{2,1}*\dot{q}_{2,3})$

$- \sin(q_2)*\dot{q}_{2,1}*\dot{q}_2*q_{2,3}) - \sin(q_2)*(2.0*\dot{q}_2*q_{2,1}*\dot{q}_{2,3}$

$+ (\dot{q}_2)^2*q_{2,13}) - \cos(q_2)*q_{2,1}*(\dot{q}_2)^2*q_{2,3})$

$- b_3*(\cos(q_3)*\ddot{q}_{3,1}*q_{3,3} + \cos(q_3)*(q_{3,1}*\ddot{q}_{3,3}$

$+ \ddot{q}_3*q_{3,13}) - \sin(q_3)*q_{3,1}*\ddot{q}_3*q_{3,3}$

$+ 2.0*(\cos(q_3)*(\dot{q}_3*\dot{q}_{3,13} + \dot{q}_{3,1}*\dot{q}_{3,3})$

$- \sin(q_3)*\dot{q}_{3,1}*\dot{q}_3*q_{3,3}) - \sin(q_3)*(2.0*\dot{q}_3*q_{3,1}*\dot{q}_{3,3}$

$+ (\dot{q}_3)^2*q_{3,13}) - \cos(q_3)*q_{3,1}*(\dot{q}_3)^2*q_{3,3})$

$+ b_4*(\cos(q_4)*\ddot{q}_{4,1}*q_{4,3} + \cos(q_4)*(q_{4,1}*\ddot{q}_{4,3}$

$+ \ddot{q}_4*q_{4,13}) - \sin(q_4)*q_{4,1}*\ddot{q}_4*q_{4,3}$

$+ 2.0*(\cos(q_4)*(\dot{q}_4*\dot{q}_{4,13} + \dot{q}_{4,1}*\dot{q}_{4,3})$

$- \sin(q_4)*\dot{q}_{4,1}*\dot{q}_4*q_{4,3}) - \sin(q_4)*(2.0*\dot{q}_4*q_{4,1}*\dot{q}_{4,3}$

$+ (\dot{q}_4)^2*q_{4,13}) - \cos(q_4)*q_{4,1}*(\dot{q}_4)^2*q_{4,3})$

$- (\sin(q_3)*\ddot{q}_{3,1} + \cos(q_3)*q_{3,1}*\ddot{q}_3$

$+ 2.0*\cos(q_3)*\dot{q}_3*\dot{q}_{3,1} - \sin(q_3)*q_{3,1}*(\dot{q}_3)^2)$

$$b_3*\sin(q_3)*\ddot{q}_{3,14} - b_4*\sin(q_4)*\ddot{q}_{4,14} = \tag{3.102}$$

$$- b_2*(\sin(q_2)*\ddot{q}_{2,14} + \cos(q_2)*\ddot{q}_{2,1}*q_{2,4}$$

$$+ \cos(q_2)*(q_{2,1}*\ddot{q}_{2,4} + \ddot{q}_2*q_{2,14})$$

$$- \sin(q_2)*q_{2,1}*\ddot{q}_2*q_{2,4}$$

$$+ 2.0*(\cos(q_2)*(\dot{q}_2*\dot{q}_{2,14} + \dot{q}_{2,1}*\dot{q}_{2,4})$$

$$- \sin(q_2)*\dot{q}_{2,1}*\dot{q}_2*q_{2,4}) - \sin(q_2)*(2.0*\dot{q}_2*q_{2,1}*\dot{q}_{2,4}$$

$$+ (\dot{q}_2)^2*q_{2,14}) - \cos(q_2)*q_{2,1}*(\dot{q}_2)^2*q_{2,4})$$

$$- b_3*(\cos(q_3)*\ddot{q}_{3,1}*q_{3,4} + \cos(q_3)*(q_{3,1}*\ddot{q}_{3,4}$$

$$+ \ddot{q}_3*q_{3,14}) - \sin(q_3)*q_{3,1}*\ddot{q}_3*q_{3,4}$$

$$+ 2.0*(\cos(q_3)*(\dot{q}_3*\dot{q}_{3,14} + \dot{q}_{3,1}*\dot{q}_{3,4})$$

$$- \sin(q_3)*\dot{q}_{3,1}*\dot{q}_3*q_{3,4}) - \sin(q_3)*(2.0*\dot{q}_3*q_{3,1}*\dot{q}_{3,4}$$

$$+ (\dot{q}_3)^2*q_{3,14}) - \cos(q_3)*q_{3,1}*(\dot{q}_3)^2*q_{3,4})$$

$$+ b_4*(\cos(q_4)*\ddot{q}_{4,1}*q_{4,4} + \cos(q_4)*(q_{4,1}*\ddot{q}_{4,4}$$

$$+ \ddot{q}_4*q_{4,14}) - \sin(q_4)*q_{4,1}*\ddot{q}_4*q_{4,4}$$

$$+ 2.0*(\cos(q_4)*(\dot{q}_4*\dot{q}_{4,14} + \dot{q}_{4,1}*\dot{q}_{4,4})$$

$$- \sin(q_4)*\dot{q}_{4,1}*\dot{q}_4*q_{4,4}) - \sin(q_4)*(2.0*\dot{q}_4*q_{4,1}*\dot{q}_{4,4}$$

$$+ (\dot{q}_4)^2*q_{4,14}) - \cos(q_4)*q_{4,1}*(\dot{q}_4)^2*q_{4,4})$$

$$- (\sin(q_4)*\ddot{q}_{4,1} + \cos(q_4)*q_{4,1}*\ddot{q}_4$$

$$+ 2.0*\cos(q_4)*\dot{q}_4*\dot{q}_{4,1} - \sin(q_4)*q_{4,1}*(\dot{q}_4)^2)$$

$$b_3*\sin(q_3)*\ddot{q}_{3,22} - b_4*\sin(q_4)*\ddot{q}_{4,22} = \tag{3.103}$$

$$- b_2*(\sin(q_2)*\ddot{q}_{2,22} + \cos(q_2)*\ddot{q}_{2,2}*q_{2,2}$$

$$+ \cos(q_2)*(q_{2,2}*\ddot{q}_{2,2} + \ddot{q}_2*q_{2,22})$$

$$- \sin(q_2)*\ddot{q}_2*(q_{2,2})^2$$

$$+ 2.0*(\cos(q_2)*(\dot{q}_2*\dot{q}_{2,22} + (\dot{q}_{2,2})^2)$$

$$- \sin(q_2)*\dot{q}_{2,2}*\dot{q}_2*q_{2,2}) - \sin(q_2)*(2.0*\dot{q}_2*q_{2,2}*\dot{q}_{2,2}$$

$$+ (\dot{q}_2)^2*q_{2,22}) - \cos(q_2)*(\dot{q}_2)^2*(q_{2,2})^2)$$

$$- b_3*(\cos(q_3)*\ddot{q}_{3,2}*q_{3,2} + \cos(q_3)*(q_{3,2}*\ddot{q}_{3,2}$$

$$+ \ddot{q}_3*q_{3,22}) - \sin(q_3)*\ddot{q}_3*(q_{3,2})^2$$

$$+ 2.0*(\cos(q_3)*(\dot{q}_3*\dot{q}_{3,22} + (\dot{q}_{3,2})^2)$$

$$- \sin(q_3)*\dot{q}_{3,2}*\dot{q}_3*q_{3,2}) - \sin(q_3)*(2.0*\dot{q}_3*q_{3,2}*\dot{q}_{3,2}$$

$$+ (\dot{q}_3)^2*q_{3,22}) - \cos(q_3)*(\dot{q}_3)^2*(q_{3,2})^2)$$

$$+ b_4*(\cos(q_4)*\ddot{q}_{4,2}*q_{4,2} + \cos(q_4)*(q_{4,2}*\ddot{q}_{4,2}$$

$$+ \ddot{q}_4*q_{4,22}) - \sin(q_4)*\ddot{q}_4*(q_{4,2})^2$$

$$+ 2.0*(\cos(q_4)*(\dot{q}_4*\dot{q}_{4,22} + (\dot{q}_{4,2})^2)$$

$$- \sin(q_4)*\dot{q}_{4,2}*\dot{q}_4*q_{4,2}) - \sin(q_4)*(2.0*\dot{q}_4*q_{4,2}*\dot{q}_{4,2}$$

$$+ (\dot{q}_4)^2*q_{4,22}) - \cos(q_4)*(\dot{q}_4)^2*(q_{4,2})^2)$$

$$- 2.0*(\sin(q_2)*\ddot{q}_{2,1} + \cos(q_2)*q_{2,1}*\ddot{q}_2$$

$$+ 2.0*\cos(q_2)*\dot{q}_2*\dot{q}_{2,1} - \sin(q_2)*q_{2,1}*(\dot{q}_2)^2)$$

$$b_3*\sin(q_3)*\ddot{q}_{3,23} - b_4*\sin(q_4)*\ddot{q}_{4,23} = \qquad 3.104$$

$$- b_2*(\sin(q_2)*\ddot{q}_{2,23} + \cos(q_2)*\ddot{q}_{2,2}*q_{2,3}$$

$$+ \cos(q_2)*(q_{2,2}*\ddot{q}_{2,3} + \ddot{q}_2*q_{2,23})$$

$$- \sin(q_2)*q_{2,2}*\ddot{q}_2*q_{2,3}$$

$$+ 2.0*(\cos(q_2)*(\dot{q}_2*\dot{q}_{2,23} + \dot{q}_{2,2}*\dot{q}_{2,3})$$

$$- \sin(q_2)*\dot{q}_{2,2}*\dot{q}_2*q_{2,3}) - \sin(q_2)*(2.0*\dot{q}_2*q_{2,2}*\dot{q}_{2,3}$$

$$+ (\dot{q}_2)^2*q_{2,23}) - \cos(q_2)*q_{2,2}*(\dot{q}_2)^2*q_{2,3})$$

$$- b_3*(\cos(q_3)*\ddot{q}_{3,2}*q_{3,3} + \cos(q_3)*(q_{3,2}*\ddot{q}_{3,3}$$

$$+ \ddot{q}_3*q_{3,23}) - \sin(q_3)*q_{3,2}*\ddot{q}_3*q_{3,3}$$

$$+ 2.0*(\cos(q_3)*(\dot{q}_3*\dot{q}_{3,23} + \dot{q}_{3,2}*\dot{q}_{3,3})$$

$- \sin(q_3)*\dot{q}_{3,2}*\dot{q}_3*q_{3,3}) - \sin(q_3)*(2.0*\dot{q}_3*q_{3,2}*\dot{q}_{3,3}$

$+ (\dot{q}_3)^2*q_{3,23}) - \cos(q_3)*q_{3,2}*(\dot{q}_3)^2*q_{3,3})$

$+ b_4*(\cos(q_4)*\ddot{q}_{4,2}*q_{4,3} + \cos(q_4)*(q_{4,2}*\ddot{q}_{4,3}$

$+ \ddot{q}_4*q_{4,23}) - \sin(q_4)*q_{4,2}*\ddot{q}_4*q_{4,3}$

$+ 2.0*(\cos(q_4)*(\dot{q}_4*\dot{q}_{4,23} + \dot{q}_{4,2}*\dot{q}_{4,3})$

$- \sin(q_4)*\dot{q}_{4,2}*\dot{q}_4*q_{4,3}) - \sin(q_4)*(2.0*\dot{q}_4*q_{4,2}*\dot{q}_{4,3}$

$+ (\dot{q}_4)^2*q_{4,23}) - \cos(q_4)*q_{4,2}*(\dot{q}_4)^2*q_{4,3})$

$- (\sin(q_2)*\ddot{q}_{2,3} + \cos(q_2)*q_{2,3}*\ddot{q}_2$

$+ 2.0*\cos(q_2)*\dot{q}_2*q_{2,3} - \sin(q_2)*q_{2,3}*(\dot{q}_2)^2)$

$- (\sin(q_3)*\ddot{q}_{3,2} + \cos(q_3)*q_{3,2}*\ddot{q}_3$

$+ 2.0*\cos(q_3)*\dot{q}_3*q_{3,2} - \sin(q_3)*q_{3,2}*(\dot{q}_3)^2)$

$b_3*\sin(q_3)*\ddot{q}_{3,24} - b_4*\sin(q_4)*\ddot{q}_{4,24} =$ 3.105

$- b_2*(\sin(q_2)*\ddot{q}_{2,24} + \cos(q_2)*\ddot{q}_{2,2}*q_{2,4}$

$+ \cos(q_2)*(q_{2,2}*\ddot{q}_{2,4} + \ddot{q}_2*q_{2,24})$

$- \sin(q_2)*q_{2,2}*\ddot{q}_2*q_{2,4}$

$+ 2.0*(\cos(q_2)*(\dot{q}_2*\dot{q}_{2,24} + \dot{q}_{2,2}*\dot{q}_{2,4})$

$- \sin(q_2)*\dot{q}_{2,2}*\dot{q}_2*q_{2,4}) - \sin(q_2)*(2.0*\dot{q}_2*q_{2,2}*\dot{q}_{2,4}$

$+ (\dot{q}_2)^2*q_{2,24}) - \cos(q_2)*q_{2,2}*(\dot{q}_2)^2*q_{2,4})$

$- b_3*(\cos(q_3)*\ddot{q}_{3,2}*q_{3,4} + \cos(q_3)*(q_{3,2}*\ddot{q}_{3,4}$

$+ \ddot{q}_3*q_{3,24}) - \sin(q_3)*q_{3,2}*\ddot{q}_3*q_{3,4}$

$+ 2.0*(\cos(q_3)*(\dot{q}_3*\dot{q}_{3,24} + \dot{q}_{3,2}*\dot{q}_{3,4})$

$- \sin(q_3)*\dot{q}_{3,2}*\dot{q}_3*q_{3,4}) - \sin(q_3)*(2.0*\dot{q}_3*q_{3,2}*\dot{q}_{3,4}$

$+ (\dot{q}_3)^2*q_{3,24}) - \cos(q_3)*q_{3,2}*(\dot{q}_3)^2*q_{3,4})$

$+ b_4*(\cos(q_4)*\ddot{q}_{4,2}*q_{4,4} + \cos(q_4)*(q_{4,2}*\ddot{q}_{4,4}$

$$+ \ddot{q}_4*q_{4,24}) - \sin(q_4)*q_{4,2}*\ddot{q}_4*q_{4,4}$$

$$+ 2.0*(\cos(q_4)*(\dot{q}_4*\dot{q}_{4,24} + \dot{q}_{4,2}*\dot{q}_{4,4})$$

$$- \sin(q_4)*\dot{q}_{4,2}*\dot{q}_4*q_{4,4}) - \sin(q_4)*(2.0*\dot{q}_4*q_{4,2}*\dot{q}_{4,4}$$

$$+ (\dot{q}_4)^2*q_{4,24}) - \cos(q_4)*q_{4,2}*(\dot{q}_4)^2*q_{4,4})$$

$$+ (\sin(q_4)*\dot{q}_{4,2} + \cos(q_4)*q_{4,2}*\ddot{q}_4$$

$$+ 2.0*\cos(q_4)*\dot{q}_4*\dot{q}_{4,2} - \sin(q_4)*q_{4,2}*(\dot{q}_4)^2)$$

$$b_3*\sin(q_3)*\ddot{q}_{3,33} - b_4*\sin(q_4)*\ddot{q}_{4,33} = \qquad 3.106$$

$$- b_2*(\sin(q_2)*\ddot{q}_{2,33} + \cos(q_2)*\ddot{q}_{2,3}*q_{2,3}$$

$$+ \cos(q_2)*(q_{2,3}*\ddot{q}_{2,3} + \ddot{q}_2*q_{2,33})$$

$$- \sin(q_2)*\ddot{q}_2*(q_{2,3})^2$$

$$+ 2.0*(\cos(q_2)*(\dot{q}_2*\dot{q}_{2,33} + (\dot{q}_{2,3})^2)$$

$$- \sin(q_2)*\dot{q}_{2,3}*\dot{q}_2*q_{2,3}) - \sin(q_2)*(2.0*\dot{q}_2*q_{2,3}*\dot{q}_{2,3}$$

$$+ (\dot{q}_2)^2*q_{2,33}) - \cos(q_2)*(\dot{q}_2)^2*(q_{2,3})^2)$$

$$- b_3*(\cos(q_3)*\ddot{q}_{3,3}*q_{3,3} + \cos(q_3)*(q_{3,3}*\ddot{q}_{3,3}$$

$$+ \ddot{q}_3*q_{3,33}) - \sin(q_3)*\ddot{q}_3*(q_{3,3})^2$$

$$+ 2.0*(\cos(q_3)*(\dot{q}_3*\dot{q}_{3,33} + (\dot{q}_{3,3})^2)$$

$$- \sin(q_3)*\dot{q}_{3,3}*\dot{q}_3*q_{3,3}) - \sin(q_3)*(2.0*\dot{q}_3*q_{3,3}*\dot{q}_{3,3}$$

$$+ (\dot{q}_3)^2*q_{3,33}) - \cos(q_3)*(\dot{q}_3)^2*(q_{3,3})^2)$$

$$+ b_4*(\cos(q_4)*\ddot{q}_{4,3}*q_{4,3} + \cos(q_4)*(q_{4,3}*\ddot{q}_{4,3}$$

$$+ \ddot{q}_4*q_{4,33}) - \sin(q_4)*\ddot{q}_4*(q_{4,3})^2$$

$$+ 2.0*(\cos(q_4)*(\dot{q}_4*\dot{q}_{4,33} + (\dot{q}_{4,3})^2)$$

$$- \sin(q_4)*\dot{q}_{4,3}*\dot{q}_4*q_{4,3}) - \sin(q_4)*(2.0*\dot{q}_4*q_{4,3}*\dot{q}_{4,3}$$

$$+ (\dot{q}_4)^2*q_{4,33}) - \cos(q_4)*(\dot{q}_4)^2*(q_{4,3})^2)$$

$$- 2.0*(\sin(q_3)*\ddot{q}_{3,3} + \cos(q_3)*q_{3,3}*\ddot{q}_3$$

$$+\ 2.0*\cos(q_3)*\dot{q}_3*\dot{q}_{3,3} - \sin(q_3)*q_{3,3}*(\dot{q}_3)^2)$$

$$b_3*\sin(q_3)*\ddot{q}_{3,34} - b_4*\sin(q_4)*\ddot{q}_{4,34} = \qquad 3.107$$

$$-\ b_2*(\sin(q_2)*\ddot{q}_{2,34} + \cos(q_2)*\ddot{q}_{2,3}*q_{2,4}$$

$$+\ \cos(q_2)*(q_{2,3}*\ddot{q}_{2,4} + \ddot{q}_2*q_{2,34})$$

$$-\ \sin(q_2)*q_{2,3}*\ddot{q}_2*q_{2,4}$$

$$+\ 2.0*(\cos(q_2)*(\dot{q}_2*\dot{q}_{2,34} + \dot{q}_{2,3}*\dot{q}_{2,4})$$

$$-\ \sin(q_2)*\dot{q}_{2,3}*\dot{q}_2*q_{2,4}) - \sin(q_2)*(2.0*\dot{q}_2*q_{2,3}*\dot{q}_{2,4}$$

$$+\ (\dot{q}_2)^2*q_{2,34}) - \cos(q_2)*q_{2,3}*(\dot{q}_2)^2*q_{2,4})$$

$$-\ b_3*(\cos(q_3)*\ddot{q}_{3,3}*q_{3,4} + \cos(q_3)*(q_{3,3}*\ddot{q}_{3,4}$$

$$+\ \ddot{q}_3*q_{3,34}) - \sin(q_3)*q_{3,3}*\ddot{q}_3*q_{3,4}$$

$$+\ 2.0*(\cos(q_3)*(\dot{q}_3*\dot{q}_{3,34} + \dot{q}_{3,3}*\dot{q}_{3,4})$$

$$-\ \sin(q_3)*\dot{q}_{3,3}*\dot{q}_3*q_{3,4}) - \sin(q_3)*(2.0*\dot{q}_3*q_{3,3}*\dot{q}_{3,4}$$

$$+\ (\dot{q}_3)^2*q_{3,34}) - \cos(q_3)*q_{3,3}*(\dot{q}_3)^2*q_{3,4})$$

$$+\ b_4*(\cos(q_4)*\ddot{q}_{4,3}*q_{4,4} + \cos(q_4)*(q_{4,3}*\ddot{q}_{4,4}$$

$$+\ \ddot{q}_4*q_{4,34}) - \sin(q_4)*q_{4,3}*\ddot{q}_4*q_{4,4}$$

$$+\ 2.0*(\cos(q_4)*(\dot{q}_4*\dot{q}_{4,34} + \dot{q}_{4,3}*\dot{q}_{4,4})$$

$$-\ \sin(q_4)*\dot{q}_{4,3}*\dot{q}_4*q_{4,4}) - \sin(q_4)*(2.0*\dot{q}_4*q_{4,3}*\dot{q}_{4,4}$$

$$+\ (\dot{q}_4)^2*q_{4,34}) - \cos(q_4)*q_{4,3}*(\dot{q}_4)^2*q_{4,4})$$

$$+\ (\sin(q_4)*\ddot{q}_{4,3} + \cos(q_4)*q_{4,3}*\ddot{q}_4$$

$$+\ 2.0*\cos(q_4)*\dot{q}_4*\dot{q}_{4,3} - \sin(q_4)*q_{4,3}*(\dot{q}_4)^2)$$

$$-\ (\sin(q_3)*\ddot{q}_{3,4} + \cos(q_3)*q_{3,4}*\ddot{q}_3$$

$$+\ 2.0*\cos(q_3)*\dot{q}_3*\dot{q}_{3,4} - \sin(q_3)*q_{3,4}*(\dot{q}_3)^2)$$

$$b_3*\sin(q_3)*\ddot{q}_{3,44} - b_4*\sin(q_4)*\ddot{q}_{4,44} = \tag{3.108}$$

$$- b_2*(\sin(q_2)*\ddot{q}_{2,44} + \cos(q_2)*\ddot{q}_{2,4}*q_{2,4}$$

$$+ \cos(q_2)*(q_{2,4}*\ddot{q}_{2,4} + \ddot{q}_2*q_{2,44})$$

$$- \sin(q_2)*\ddot{q}_2*(q_{2,4})^2$$

$$+ 2.0*(\cos(q_2)*(\dot{q}_2*\dot{q}_{2,44} + (\dot{q}_{2,4})^2)$$

$$- \sin(q_2)*\dot{q}_{2,4}*\dot{q}_2*q_{2,4}) - \sin(q_2)*(2.0*\dot{q}_2*q_{2,4}*\dot{q}_{2,4}$$

$$+ (\dot{q}_2)^2*q_{2,44}) - \cos(q_2)*(\dot{q}_2)^2*(q_{2,4})^2)$$

$$- b_3*(\cos(q_3)*\ddot{q}_{3,4}*q_{3,4} + \cos(q_3)*(q_{3,4}*\ddot{q}_{3,4}$$

$$+ \ddot{q}_3*q_{3,44}) - \sin(q_3)*\ddot{q}_3*(q_{3,4})^2$$

$$+ 2.0*(\cos(q_3)*(\dot{q}_3*\dot{q}_{3,44} + (\dot{q}_{3,4})^2)$$

$$- \sin(q_3)*\dot{q}_{3,4}*\dot{q}_3*q_{3,4}) - \sin(q_3)*(2.0*\dot{q}_3*q_{3,4}*\dot{q}_{3,4}$$

$$+ (\dot{q}_3)^2*q_{3,44}) - \cos(q_3)*(\dot{q}_3)^2*(q_{3,4})^2)$$

$$+ b_4*(\cos(q_4)*\ddot{q}_{4,4}*q_{4,4} + \cos(q_4)*(q_{4,4}*\ddot{q}_{4,4}$$

$$+ \ddot{q}_4*q_{4,44}) - \sin(q_4)*\ddot{q}_4*(q_{4,4})^2$$

$$+ 2.0*(\cos(q_4)*(\dot{q}_4*\dot{q}_{4,44} + (\dot{q}_{4,4})^2)$$

$$- \sin(q_4)*\dot{q}_{4,4}*\dot{q}_4*q_{4,4}) - \sin(q_4)*(2.0*\dot{q}_4*q_{4,4}*\dot{q}_{4,4}$$

$$+ (\dot{q}_4)^2*q_{4,44}) - \cos(q_4)*(\dot{q}_4)^2*(q_{4,4})^2)$$

$$+ 2.0*(\sin(q_4)*\ddot{q}_{4,4} + \cos(q_4)*q_{4,4}*\ddot{q}_4$$

$$+ 2.0*\cos(q_4)*\dot{q}_4*\dot{q}_{4,4} - \sin(q_4)*q_{4,4}*(\dot{q}_4)^2)$$

$$- b_3*\cos(q_3)*\ddot{q}_{3,11} + b_4*\cos(q_4)*\ddot{q}_{4,11} = \tag{3.109}$$

$$+ b_2*(\cos(q_2)*\ddot{q}_{2,11} - \sin(q_2)*\ddot{q}_{2,1}*q_{2,1}$$

$$- \sin(q_2)*(q_{2,1}*\ddot{q}_{2,1} + \ddot{q}_2*q_{2,11})$$

$$- \cos(q_2)*\ddot{q}_2*(q_{2,1})^2$$

$$- 2.0*(\sin(q_2)*(\dot{q}_2*\dot{q}_{2,11} + (\dot{q}_{2,1})^2)$$

$$- \cos(q_2)*\dot{q}_{2,1}*\dot{q}_2*q_{2,1}) - \cos(q_2)*(2.0*\dot{q}_2*q_{2,1}*\dot{q}_{2,1}$$

$$+ (\dot{q}_2)^2*q_{2,11}) + \sin(q_2)*(\dot{q}_2)^2*(q_{2,1})^2)$$

$$- b_3*(\sin(q_3)*\ddot{q}_{3,1}*q_{3,1} + \sin(q_3)*(q_{3,1}*\ddot{q}_{3,1}$$

$$+ \ddot{q}_3*q_{3,11}) + \cos(q_3)*\ddot{q}_3*(q_{3,1})^2$$

$$+ 2.0*(\sin(q_3)*(\dot{q}_3*\dot{q}_{3,11} + (\dot{q}_{3,1})^2)$$

$$+ \cos(q_3)*\dot{q}_{3,1}*\dot{q}_3*q_{3,1}) + \cos(q_3)*(2.0*\dot{q}_3*q_{3,1}*\dot{q}_{3,1}$$

$$+ (\dot{q}_3)^2*q_{3,11}) - \sin(q_3)*(\dot{q}_3)^2*(q_{3,1})^2)$$

$$+ b_4*(\sin(q_4)*\ddot{q}_{4,1}*q_{4,1} + \sin(q_4)*(q_{4,1}*\ddot{q}_{4,1}$$

$$+ \ddot{q}_4*q_{4,11}) + \cos(q_4)*\ddot{q}_4*(q_{4,1})^2$$

$$+ 2.0*(\sin(q_4)*(\dot{q}_4*\dot{q}_{4,11} + (\dot{q}_{4,1})^2)$$

$$+ \cos(q_4)*\dot{q}_{4,1}*\dot{q}_4*q_{4,1}) + \cos(q_4)*(2.0*\dot{q}_4*q_{4,1}*\dot{q}_{4,1}$$

$$+ (\dot{q}_4)^2*q_{4,11}) - \sin(q_4)*(\dot{q}_4)^2*(q_{4,1})^2)$$

$$- b_3*\cos(q_3)*\ddot{q}_{3,12} + b_4*\cos(q_4)*\ddot{q}_{4,12} = \tag{3.110}$$

$$+ b_2*(\cos(q_2)*\ddot{q}_{2,12} - \sin(q_2)*\ddot{q}_{2,1}*q_{2,2}$$

$$- \sin(q_2)*(q_{2,1}*\ddot{q}_{2,2} + \ddot{q}_2*q_{2,12})$$

$$- \cos(q_2)*q_{2,1}*\ddot{q}_2*q_{2,2}$$

$$- 2.0*(\sin(q_2)*(\dot{q}_2*\dot{q}_{2,12} + \dot{q}_{2,1}*\dot{q}_{2,2})$$

$$- \cos(q_2)*\dot{q}_{2,1}*\dot{q}_2*q_{2,2}) - \cos(q_2)*(2.0*\dot{q}_2*q_{2,1}*\dot{q}_{2,2}$$

$$+ (\dot{q}_2)^2*q_{2,12}) + \sin(q_2)*q_{2,1}*(\dot{q}_2)^2*q_{2,2})$$

$$- b_3*(\sin(q_3)*\ddot{q}_{3,1}*q_{3,2} + \sin(q_3)*(q_{3,1}*\ddot{q}_{3,2}$$

$$+ \ddot{q}_3*q_{3,12}) + \cos(q_3)*q_{3,1}*\ddot{q}_3*q_{3,2}$$

$$+ 2.0*(\sin(q_3)*(\dot{q}_3*\dot{q}_{3,12} + \dot{q}_{3,1}*\dot{q}_{3,2})$$

$$+ \cos(q_3)*\dot{q}_{3,1}*\dot{q}_3*q_{3,2}) + \cos(q_3)*(2.0*\dot{q}_3*q_{3,1}*\dot{q}_{3,2}$$

$$+ (\dot{q}_3)^2*q_{3,12}) - \sin(q_3)*q_{3,1}*(\dot{q}_3)^2*q_{3,2})$$

$$+ b_4*(\sin(q_4)*\ddot{q}_{4,1}*q_{4,2} + \sin(q_4)*(q_{4,1}*\ddot{q}_{4,2}$$
$$+ \ddot{q}_4*q_{4,12}) + \cos(q_4)*q_{4,1}*\ddot{q}_4*q_{4,2}$$
$$+ 2.0*(\sin(q_4)*(\dot{q}_4*\dot{q}_{4,12} + \dot{q}_{4,1}*\dot{q}_{4,2})$$
$$+ \cos(q_4)*\dot{q}_{4,1}*\dot{q}_4*q_{4,2}) + \cos(q_4)*(2.0*\dot{q}_4*q_{4,1}*\dot{q}_{4,2}$$
$$+ (\dot{q}_4)^2*q_{4,12}) - \sin(q_4)*q_{4,1}*(\dot{q}_4)^2*q_{4,2})$$
$$+ (\cos(q_2)*\ddot{q}_{2,1} + \sin(q_2)*q_{2,1}*\ddot{q}_2$$
$$+ 2.0*\sin(q_2)*\dot{q}_2*\dot{q}_{2,1} - \cos(q_2)*q_{2,1}*(\dot{q}_2)^2)$$

$$- b_3*\cos(q_3)*\ddot{q}_{3,13} + b_4*\cos(q_4)*\ddot{q}_{4,13} = \qquad 3.111$$
$$+ b_2*(\cos(q_2)*\ddot{q}_{2,13} - \sin(q_2)*\ddot{q}_{2,1}*q_{2,3}$$
$$- \sin(q_2)*(q_{2,1}*\ddot{q}_{2,3} + \ddot{q}_2*q_{2,13})$$
$$- \cos(q_2)*q_{2,1}*\ddot{q}_2*q_{2,3}$$
$$- 2.0*(\sin(q_2)*(\dot{q}_2*\dot{q}_{2,13} + \dot{q}_{2,1}*\dot{q}_{2,3})$$
$$- \cos(q_2)*\dot{q}_{2,1}*\dot{q}_2*q_{2,3}) - \cos(q_2)*(2.0*\dot{q}_2*q_{2,1}*\dot{q}_{2,3}$$
$$+ (\dot{q}_2)^2*q_{2,13}) + \sin(q_2)*q_{2,1}*(\dot{q}_2)^2*q_{2,3})$$
$$- b_3*(\sin(q_3)*\ddot{q}_{3,1}*q_{3,3} + \sin(q_3)*(q_{3,1}*\ddot{q}_{3,3}$$
$$+ \ddot{q}_3*q_{3,13}) + \cos(q_3)*q_{3,1}*\ddot{q}_3*q_{3,3}$$
$$+ 2.0*(\sin(q_3)*(\dot{q}_3*\dot{q}_{3,13} + \dot{q}_{3,1}*\dot{q}_{3,3})$$
$$+ \cos(q_3)*\dot{q}_{3,1}*\dot{q}_3*q_{3,3}) + \cos(q_3)*(2.0*\dot{q}_3*q_{3,1}*\dot{q}_{3,3}$$
$$+ (\dot{q}_3)^2*q_{3,13}) + \sin(q_3)*q_{3,1}*(\dot{q}_3)^2*q_{3,3})$$
$$+ b_4*(\sin(q_4)*\ddot{q}_{4,1}*q_{4,3} + \sin(q_4)*(q_{4,1}*\ddot{q}_{4,3}$$
$$+ \ddot{q}_4*q_{4,13}) + \cos(q_4)*q_{4,1}*\ddot{q}_4*q_{4,3}$$
$$+ 2.0*(\sin(q_4)*(\dot{q}_4*\dot{q}_{4,13} + \dot{q}_{4,1}*\dot{q}_{4,3})$$
$$+ \cos(q_4)*\dot{q}_{4,1}*\dot{q}_4*q_{4,3}) + \cos(q_4)*(2.0*\dot{q}_4*q_{4,1}*\dot{q}_{4,3}$$
$$+ (\dot{q}_4)^2*q_{4,13}) - \sin(q_4)*q_{4,1}*(\dot{q}_4)^2*q_{4,3})$$

$$+ (\cos(q_3)*\ddot{q}_{3,1} + \sin(q_3)*q_{3,1}*\ddot{q}_3$$

$$+ 2.0*\sin(q_3)*\dot{q}_3*\dot{q}_{3,1} - \cos(q_3)*q_{3,1}*(\dot{q}_3)^2)$$

$$- b_3*\cos(q_3)*\ddot{q}_{3,14} + b_4*\cos(q_4)*\ddot{q}_{4,14} = \qquad 3.112$$

$$+ b_2*(\cos(q_2)*\ddot{q}_{2,14} - \sin(q_2)*\ddot{q}_{2,1}*q_{2,4}$$

$$- \sin(q_2)*(q_{2,1}*\ddot{q}_{2,4} + \ddot{q}_2*q_{2,14})$$

$$- \cos(q_2)*q_{2,1}*\ddot{q}_2*q_{2,4}$$

$$- 2.0*(\sin(q_2)*(\dot{q}_2*\dot{q}_{2,14} + \dot{q}_{2,1}*\dot{q}_{2,4})$$

$$- \cos(q_2)*\dot{q}_{2,1}*\dot{q}_2*q_{2,4}) - \cos(q_2)*(2.0*\dot{q}_2*q_{2,1}*\dot{q}_{2,4}$$

$$+ (\dot{q}_2)^2*q_{2,14}) + \sin(q_2)*q_{2,1}*(\dot{q}_2)^2*q_{2,4})$$

$$- b_3*(\sin(q_3)*\ddot{q}_{3,1}*q_{3,4} + \sin(q_3)*(q_{3,1}*\ddot{q}_{3,4}$$

$$+ \ddot{q}_3*q_{3,14}) + \cos(q_3)*q_{3,1}*\ddot{q}_3*q_{3,4}$$

$$+ 2.0*(\sin(q_3)*(\dot{q}_3*\dot{q}_{3,14} + \dot{q}_{3,1}*\dot{q}_{3,4})$$

$$+ \cos(q_3)*\dot{q}_{3,1}*\dot{q}_3*q_{3,4}) + \cos(q_3)*(2.0*\dot{q}_3*q_{3,1}*\dot{q}_{3,4}$$

$$+ (\dot{q}_3)^2*q_{3,14}) - \sin(q_3)*q_{3,1}*(\dot{q}_3)^2*q_{3,4})$$

$$+ b_4*(\sin(q_4)*\ddot{q}_{4,1}*q_{4,4} + \sin(q_4)*(q_{4,1}*\ddot{q}_{4,4}$$

$$+ \ddot{q}_4*q_{4,14}) + \cos(q_4)*q_{4,1}*\ddot{q}_4*q_{4,4}$$

$$+ 2.0*(\sin(q_4)*(\dot{q}_4*\dot{q}_{4,14} + \dot{q}_{4,1}*\dot{q}_{4,4})$$

$$+ \cos(q_4)*\dot{q}_{4,1}*\dot{q}_4*q_{4,4}) + \cos(q_4)*(2.0*\dot{q}_4*q_{4,1}*\dot{q}_{4,4}$$

$$+ (\dot{q}_4)^2*q_{4,14}) - \sin(q_4)*q_{4,1}*(\dot{q}_4)^2*q_{4,4})$$

$$- (\cos(q_4)*\ddot{q}_{4,1} - \sin(q_4)*q_{4,1}*\ddot{q}_4$$

$$- 2.0*\sin(q_4)*\dot{q}_4*\dot{q}_{4,1} - \cos(q_4)*q_{4,1}*(\dot{q}_4)^2)$$

$$- b_3*\cos(q_3)*\ddot{q}_{3,22} + b_4*\cos(q_4)*\ddot{q}_{4,22} = \qquad 3.113$$

$$+ b_2*(\cos(q_2)*\ddot{q}_{2,22} - \sin(q_2)*\ddot{q}_{2,2}*q_{2,2}$$

$- \sin(q_2)*(q_{2,2}*\ddot{q}_{2,2} + \ddot{q}_2*q_{2,22})$

$- \cos(q_2)*\ddot{q}_2*(q_{2,2})^2$

$- 2.0*(\sin(q_2)*(\dot{q}_2*\dot{q}_{2,22} + (\dot{q}_{2,2})^2)$

$- \cos(q_2)*\dot{q}_{2,2}*\dot{q}_2*q_{2,2}) - \cos(q_2)*(2.0*\dot{q}_2*q_{2,2}*\dot{q}_{2,2}$

$+ (\dot{q}_2)^2*q_{2,22}) + \sin(q_2)*(\dot{q}_2)^2*(q_{2,2})^2)$

$- b_3*(\sin(q_3)*\ddot{q}_{3,2}*q_{3,2} + \sin(q_3)*(q_{3,2}*\ddot{q}_{3,2}$

$+ \ddot{q}_3*q_{3,22}) + \cos(q_3)*\ddot{q}_3*(q_{3,2})^2$

$+ 2.0*(\sin(q_3)*(\dot{q}_3*\dot{q}_{3,22} + (\dot{q}_{3,2})^2)$

$+ \cos(q_3)*\dot{q}_{3,2}*\dot{q}_3*q_{3,2}) + \cos(q_3)*(2.0*\dot{q}_3*q_{3,2}*\dot{q}_{3,2}$

$+ (\dot{q}_3)^2*q_{3,22}) - \sin(q_3)*(\dot{q}_3)^2*(q_{3,2})^2)$

$+ b_4*(\sin(q_4)*\ddot{q}_{4,2}*q_{4,2} + \sin(q_4)*(q_{4,2}*\ddot{q}_{4,2}$

$+ \ddot{q}_4*q_{4,22}) + \cos(q_4)*\ddot{q}_4*(q_{4,2})^2$

$+ 2.0*(\sin(q_4)*(\dot{q}_4*\dot{q}_{4,22} + \dot{q}_{4,2}*\dot{q}_{4,2})$

$+ \cos(q_4)*\dot{q}_{4,2}*\dot{q}_4*q_{4,2}) + \cos(q_4)*(2.0*\dot{q}_4*q_{4,2}*\dot{q}_{4,2}$

$+ (\dot{q}_4)^2*q_{4,22}) - \sin(q_4)*(\dot{q}_4)^2*(q_{4,2})^2)$

$+ 2.0*(\cos(q_2)*\ddot{q}_{2,2} + \sin(q_2)*q_{2,2}*\ddot{q}_2$

$+ 2.0*\sin(q_2)*\dot{q}_2*\dot{q}_{2,2} - \cos(q_2)*q_{2,2}*(\dot{q}_2)^2)$

$- b_3*\cos(q_3)*\ddot{q}_{3,23} + b_4*\cos(q_4)*\ddot{q}_{4,23} =$ 3.114

$+ b_2*(\cos(q_2)*\ddot{q}_{2,23} - \sin(q_2)*\ddot{q}_{2,2}*q_{2,3}$

$- \sin(q_2)*(q_{2,2}*\ddot{q}_{2,3} + \ddot{q}_2*q_{2,23})$

$- \cos(q_2)*q_{2,2}*\ddot{q}_2*q_{2,3}$

$- 2.0*(\sin(q_2)*(\dot{q}_2*\dot{q}_{2,23} + \dot{q}_{2,2}*\dot{q}_{2,3})$

$- \cos(q_2)*\dot{q}_{2,2}*\dot{q}_2*q_{2,3}) - \cos(q_2)*(2.0*\dot{q}_2*q_{2,2}*\dot{q}_{2,3}$

$+ (\dot{q}_2)^2*q_{2,23}) + \sin(q_2)*q_{2,2}*(\dot{q}_2)^2*q_{2,3})$

$$- b_3*(\sin(q_3)*\ddot{q}_{3,2}*q_{3,3} + \sin(q_3)*(q_{3,2}*\ddot{q}_{3,3}$$

$$+ \ddot{q}_3*q_{3,23}) + \cos(q_3)*q_{3,2}*\ddot{q}_3*q_{3,3}$$

$$+ 2.0*(\sin(q_3)*(\dot{q}_3*\dot{q}_{3,23} + \dot{q}_{3,2}*\dot{q}_{3,3})$$

$$+ \cos(q_3)*\dot{q}_{3,2}*\dot{q}_3*q_{3,3}) + \cos(q_3)*(2.0*\dot{q}_3*q_{3,2}*\dot{q}_{3,3}$$

$$+ (\dot{q}_3)^2*q_{3,23}) - \sin(q_3)*q_{3,2}*(\dot{q}_3)^2*q_{3,3})$$

$$+ b_4*(\sin(q_4)*\ddot{q}_{4,2}*q_{4,3} + \sin(q_4)*(q_{4,2}*\ddot{q}_{4,3}$$

$$+ \ddot{q}_4*q_{4,23}) + \cos(q_4)*q_{4,2}*\ddot{q}_4*q_{4,3}$$

$$+ 2.0*(\sin(q_4)*(\dot{q}_4*\dot{q}_{4,23} + \dot{q}_{4,2}*\dot{q}_{4,3})$$

$$+ \cos(q_4)*\dot{q}_{4,2}*\dot{q}_4*q_{4,3}) + \cos(q_4)*(2.0*\dot{q}_4*q_{4,2}*\dot{q}_{4,3}$$

$$+ (\dot{q}_4)^2*q_{4,23}) - \sin(q_4)*q_{4,2}*(\dot{q}_4)^2*q_{4,3})$$

$$+ (\cos(q_2)*\ddot{q}_{2,3} - \sin(q_2)*q_{2,3}*\ddot{q}_2$$

$$- 2.0*\sin(q_2)*\dot{q}_2*\dot{q}_{2,3} - \cos(q_2)*q_{2,3}*(\dot{q}_2)^2)$$

$$+ (\cos(q_3)*\ddot{q}_{3,2} - \sin(q_3)*q_{3,2}*\ddot{q}_3$$

$$- 2.0*\sin(q_3)*\dot{q}_3*\dot{q}_{3,2} - \cos(q_3)*q_{3,2}*(\dot{q}_3)^2)$$

$$- b_3*\cos(q_3)*\ddot{q}_{3,24} + b_4*\cos(q_4)*\ddot{q}_{4,24} = \tag{3.115}$$

$$+ b_2*(\cos(q_2)*\ddot{q}_{2,24} - \sin(q_2)*\ddot{q}_{2,2}*q_{2,4}$$

$$- \sin(q_2)*(q_{2,2}*\ddot{q}_{2,4} + \ddot{q}_2*q_{2,24})$$

$$- \cos(q_2)*q_{2,2}*\ddot{q}_2*q_{2,4}$$

$$- 2.0*(\sin(q_2)*(\dot{q}_2*\dot{q}_{2,24} + \dot{q}_{2,2}*\dot{q}_{2,4})$$

$$- \cos(q_2)*\dot{q}_{2,2}*\dot{q}_2*q_{2,4}) - \cos(q_2)*(2.0*\dot{q}_2*q_{2,2}*\dot{q}_{2,4}$$

$$+ (\dot{q}_2)^2*q_{2,24}) + \sin(q_2)*q_{2,2}*(\dot{q}_2)^2*q_{2,4})$$

$$- b_3*(\sin(q_3)*\ddot{q}_{3,2}*q_{3,4} + \sin(q_3)*(q_{3,2}*\ddot{q}_{3,4}$$

$$+ \ddot{q}_3*q_{3,24}) + \cos(q_3)*q_{3,2}*\ddot{q}_3*q_{3,4}$$

$$+ 2.0*(\sin(q_3)*(\dot{q}_3*\dot{q}_{3,24} + \dot{q}_{3,2}*\dot{q}_{3,4})$$

$$+ \cos(q_3)*\dot{q}_{3,2}*\dot{q}_3*q_{3,4}) + \cos(q_3)*(2.0*\dot{q}_3*q_{3,2}*\dot{q}_{3,4}$$

$$+ (\dot{q}_3)^2*q_{3,24}) - \sin(q_3)*q_{3,2}*(\dot{q}_3)^2*q_{3,4})$$

$$+ b_4*(\sin(q_4)*\ddot{q}_{4,2}*q_{4,4} + \sin(q_4)*(q_{4,2}*\ddot{q}_{4,4}$$

$$+ \ddot{q}_4*q_{4,24}) + \cos(q_4)*q_{4,2}*\ddot{q}_4*q_{4,4}$$

$$+ 2.0*(\sin(q_4)*(\dot{q}_4*\dot{q}_{4,24} + \dot{q}_{4,2}*\dot{q}_{4,4})$$

$$+ \cos(q_4)*\dot{q}_{4,2}*\dot{q}_4*q_{4,4}) + \cos(q_4)*(2.0*\dot{q}_4*q_{4,2}*\dot{q}_{4,4}$$

$$+ (\dot{q}_4)^2*q_{4,24}) - \sin(q_4)*q_{4,2}*(\dot{q}_4)^2*q_{4,4})$$

$$- (\cos(q_4)*\ddot{q}_{4,2} - \sin(q_4)*q_{4,2}*\ddot{q}_4$$

$$- 2.0*\sin(q_4)*\dot{q}_4*\dot{q}_{4,2} - \cos(q_4)*q_{4,2}*(\dot{q}_4)^2)$$

$$- b_3*\cos(q_3)*\ddot{q}_{3,33} + b_4*\cos(q_4)*\ddot{q}_{4,33} = \qquad 3.116$$

$$+ b_2*(\cos(q_2)*\ddot{q}_{2,33} - \sin(q_2)*\ddot{q}_{2,3}*q_{2,3}$$

$$- \sin(q_2)*(q_{2,3}*\ddot{q}_{2,3} + \ddot{q}_2*q_{2,33})$$

$$- \cos(q_2)*\ddot{q}_2*(q_{2,3})^2$$

$$- 2.0*(\sin(q_2)*(\dot{q}_2*\dot{q}_{2,33} + (\dot{q}_{2,3})^2)$$

$$- \cos(q_2)*\dot{q}_{2,3}*\dot{q}_2*q_{2,3}) - \cos(q_2)*(2.0*\dot{q}_2*q_{2,3}*\dot{q}_{2,3}$$

$$+ (\dot{q}_2)^2*q_{2,33}) + \sin(q_2)*(\dot{q}_2)^2*(q_{2,3})^2)$$

$$- b_3*(\sin(q_3)*\ddot{q}_{3,3}*q_{3,3} + \sin(q_3)*(q_{3,3}*\ddot{q}_{3,3}$$

$$+ \ddot{q}_3*q_{3,33}) + \cos(q_3)*\ddot{q}_3*(q_{3,3})^2$$

$$+ 2.0*(\sin(q_3)*(\dot{q}_3*\dot{q}_{3,33} + (\dot{q}_{3,3})^2)$$

$$+ \cos(q_3)*\dot{q}_{3,3}*\dot{q}_3*q_{3,3}) + \cos(q_3)*(2.0*\dot{q}_3*q_{3,3}*\dot{q}_{3,3}$$

$$+ (\dot{q}_3)^2*q_{3,33}) - \sin(q_3)*(\dot{q}_3)^2*(q_{3,3})^2)$$

$$+ b_4*(\sin(q_4)*\ddot{q}_{4,3}*q_{4,3} + \sin(q_4)*(q_{4,3}*\ddot{q}_{4,3}$$

$$+ \ddot{q}_4*q_{4,33}) + \cos(q_4)*\ddot{q}_4*(q_{4,3})^2$$

$$+ 2.0*(\sin(q_4)*(\dot{q}_4*\dot{q}_{4,33} + (\dot{q}_{4,3})^2)$$

$$+ \cos(q_4)*\dot{q}_{4,3}*\dot{q}_4*q_{4,3}) + \cos(q_4)*(2.0*\dot{q}_4*q_{4,3}*\dot{q}_{4,3}$$

$$+ (\dot{q}_4)^2*q_{4,33}) - \sin(q_4)*(\dot{q}_4)^2*(q_{4,3})^2)$$

$$+ 2.0*(\cos(q_3)*\ddot{q}_{3,3} - \sin(q_3)*q_{3,3}*\ddot{q}_3$$

$$- 2.0*\sin(q_3)*\dot{q}_3*\dot{q}_{3,3} - \cos(q_3)*q_{3,3}*(\dot{q}_3)^2)$$

$$- b_3*\cos(q_3)*\ddot{q}_{3,34} + b_4*\cos(q_4)*\ddot{q}_{4,34} = \qquad 3.117$$

$$+ b_2*(\cos(q_2)*\ddot{q}_{2,34} - \sin(q_2)*\ddot{q}_{2,3}*q_{2,4}$$

$$- \sin(q_2)*(q_{2,3}*\ddot{q}_{2,4} + \ddot{q}_2*q_{2,34})$$

$$- \cos(q_2)*q_{2,3}*\ddot{q}_2*q_{2,4}$$

$$- 2.0*(\sin(q_2)*(\dot{q}_2*\dot{q}_{2,34} + \dot{q}_{2,3}*\dot{q}_{2,4})$$

$$- \cos(q_2)*\dot{q}_{2,3}*\dot{q}_2*q_{2,4}) - \cos(q_2)*(2.0*\dot{q}_2*q_{2,3}*\dot{q}_{2,4}$$

$$+ (\dot{q}_2)^2*q_{2,34}) + \sin(q_2)*q_{2,3}*(\dot{q}_2)^2*q_{2,4})$$

$$- b_3*(\sin(q_3)*\ddot{q}_{3,3}*q_{3,4} + \sin(q_3)*(q_{3,3}*\ddot{q}_{3,4}$$

$$+ \ddot{q}_3*q_{3,34}) + \cos(q_3)*q_{3,3}*\ddot{q}_3*q_{3,4}$$

$$+ 2.0*(\sin(q_3)*(\dot{q}_3*\dot{q}_{3,34} + \dot{q}_{3,3}*\dot{q}_{3,4})$$

$$+ \cos(q_3)*\dot{q}_{3,3}*\dot{q}_3*q_{3,4}) + \cos(q_3)*(2.0*\dot{q}_3*q_{3,3}*\dot{q}_{3,4}$$

$$+ (\dot{q}_3)^2*q_{3,34}) - \sin(q_3)*q_{3,3}*(\dot{q}_3)^2*q_{3,4})$$

$$+ b_4*(\sin(q_4)*\ddot{q}_{4,3}*q_{4,4} + \sin(q_4)*(q_{4,3}*\ddot{q}_{4,4}$$

$$+ \ddot{q}_4*q_{4,34}) + \cos(q_4)*q_{4,3}*\ddot{q}_4*q_{4,4}$$

$$+ 2.0*(\sin(q_4)*(\dot{q}_4*\dot{q}_{4,34} + \dot{q}_{4,3}*\dot{q}_{4,4})$$

$$+ \cos(q_4)*\dot{q}_{4,3}*\dot{q}_4*q_{4,4}) + \cos(q_4)*(2.0*\dot{q}_4*q_{4,3}*\dot{q}_{4,4}$$

$$+ (\dot{q}_4)^2*q_{4,34}) - \sin(q_4)*q_{4,3}*(\dot{q}_4)^2*q_{4,4})$$

$$- (\cos(q_4)*\ddot{q}_{4,3} - \sin(q_4)*q_{4,3}*\ddot{q}_4$$

$$- 2.0*\sin(q_4)*\dot{q}_4*\dot{q}_{4,3} - \cos(q_4)*q_{4,3}*(\dot{q}_4)^2)$$

$$+ (\cos(q_3)*\ddot{q}_{3,4} - \sin(q_3)*q_{3,4}*\ddot{q}_3$$

$$- 2.0*\sin(q_3)*\dot{q}_3*\dot{q}_{3,4} - \cos(q_3)*q_{3,4}*(\dot{q}_3)^2)$$

$$- b_3*\cos(q_3)*\ddot{q}_{3,44} + b_4*\cos(q_4)*\ddot{q}_{4,44} = \qquad 3.118$$

$$+ b_2*(\cos(q_2)*\ddot{q}_{2,44} - \sin(q_2)*q_{2,4}*\ddot{q}_{2,4}$$
$$- \sin(q_2)*(q_{2,4}*\ddot{q}_{2,4} + \ddot{q}_2*q_{2,44})$$
$$- \cos(q_2)*\ddot{q}_2*(q_{2,4})^2$$
$$- 2.0*(\sin(q_2)*(\dot{q}_2*\dot{q}_{2,44} + (\dot{q}_{2,4})^2)$$
$$- \cos(q_2)*\dot{q}_{2,4}*\dot{q}_2*q_{2,4}) - \cos(q_2)*(2.0*\dot{q}_2*q_{2,4}*\dot{q}_{2,4}$$
$$+ (\dot{q}_2)^2*q_{2,44}) + \sin(q_2)*(\dot{q}_2)^2*(q_{2,4})^2)$$
$$- b_3*(\sin(q_3)*\ddot{q}_{3,4}*q_{3,4} + \sin(q_3)*(q_{3,4}*\ddot{q}_{3,4}$$
$$+ \ddot{q}_3*q_{3,44}) + \cos(q_3)*\ddot{q}_3*(q_{3,4})^2$$
$$+ 2.0*(\sin(q_3)*(\dot{q}_3*\dot{q}_{3,44} + (\dot{q}_{3,4})^2)$$
$$+ \cos(q_3)*\dot{q}_{3,4}*\dot{q}_3*q_{3,4}) + \cos(q_3)*(2.0*\dot{q}_3*q_{3,4}*\dot{q}_{3,4}$$
$$+ (\dot{q}_3)^2*q_{3,44}) - \sin(q_3)*(\dot{q}_3)^2*(q_{3,4})^2)$$
$$+ b_4*(\sin(q_4)*\ddot{q}_{4,4}*q_{4,4} + \sin(q_4)*(q_{4,4}*\ddot{q}_{4,4}$$
$$+ \ddot{q}_4*q_{4,44}) + \cos(q_4)*\ddot{q}_4*(q_{4,4})^2$$
$$+ 2.0*(\sin(q_4)*(\dot{q}_4*\dot{q}_{4,44} + (\dot{q}_{4,4})^2)$$
$$+ \cos(q_4)*\dot{q}_{4,4}*\dot{q}_4*q_{4,4}) + \cos(q_4)*(2.0*\dot{q}_4*q_{4,4}*\dot{q}_{4,4}$$
$$+ (\dot{q}_4)^2*q_{4,44}) - \sin(q_4)*(\dot{q}_4)^2*(q_{4,4})^2)$$
$$- 2.0*(\cos(q_4)*\ddot{q}_{4,4} - \sin(q_4)*q_{4,4}*\ddot{q}_4$$
$$- 2.0*\sin(q_4)*\dot{q}_4*\dot{q}_{4,4} - \cos(q_4)*q_{4,4}*(\dot{q}_4)^2)$$

The preceding twenty acceleration sensitivity equations contain twenty second order acceleration sensitivities (omitting the second order acceleration sensitivities of the input crank $\ddot{q}_2$ since they are zero). The 20 equations above can be written in matrix form with the same coefficient matrix as before. Again, the right side vector contains only known values. This system of equations can obviously be solved by the same decoupling technique that was used for the position and velocity sensitivity equations .

CHAPTER IV

THE MINIMUM SENSITIVITY DESIGN PROBLEM

Design optimization theory has been successfully applied to a large number of problems in the engineering field. Optimization methods are usually iterative numerical procedures that typically require a large amount of computing time for the solution process. The design optimization approach provides a semi-automatic tool for making design decisions which must otherwise be based on the designer's intuition and experience. The computer can be used as a resource for performing the repetitive calculations required at each iteration.

There are many well developed optimization packages available to date that require only the initial design, cost/constraint functions and their gradients as input. In the present work the optimization and kinematic/sensitivity analysis segments were kept independent. This allows some flexibility in choosing an optimization package. The numerical examples presented in Chapter 6 were obtained by sequential unconstrained minimization using a modified steepest descent algorithm for the required first order unconstrained nonlinear optimization. The subroutine used for this was the routine VA06A from the Harwell Subroutine Library [5].

The aim of the optimization process is to find the design that minimizes a suitable objective function subject to specified constraints. The standard nonlinear constrained optimization problem is generally defined as follows:

Minimize: $F(b)$ (objective function) 4.1

Subject to:

$g_j(b) <= 0.0 \quad j = 1,m$ (inequality constraints) 4.2

$h_i(b) = 0.0 \quad i = m+1,m+k$ (equality constraints) 4.3

where: b is the vector of design variables.

In order to solve any design problem using optimization techniques it is necessary to first convert the design problem into a standard nonlinear programming problem in the above format.

4.1 Formulation of the Minimum Sensitivity Problem

The first order sensitivity coefficients derived in Chapter 3 are the derivatives of the change in position, velocity, acceleration and coupler point position with respect to the design variables. In general, the first order sensitivity coefficients of any function of design and state may be viewed as measures of the change in the value of the function for a small change in design. Manufacturing errors can be viewed as being small changes in design. Thus, the problem of minimizing the sensitivity of the system performance with respect to manufacturing error is equivalent to minimizing the first order sensitivity coefficients of a suitable function of the form $f = f(b,q,\dot{q},\ddot{q},x,y)$.

The minimization of the first order sensitivity coefficients can be achieved through the use of nonlinear programming methods. To do this, however, we must restate the problem in the form of a standard nonlinear programming problem as described in the preceding section. First of all, in order to minimize the maximum first order sensitivity requires the introduction of an artificial design variable that will represent the maximum sensitivity at the optimum. Accordingly, an artificial design variable b_{10} is introduced in addition to the design variables b_1 - b_9 that are used to define the four-bar linkage. The objective function is then chosen to be the artificial design variable while added constraints are set to ensure that the magnitude of the appropriate first order sensitivity coefficient is less than the artificial design variable. Upper and lower bound constraints are also set for the artificial design variable. After taking these steps, the original minimum sensitivity problem can be converted from a minmax problem into a standard nonlinear problem as given below:

Minimize: $F(b) = b_{10}$ 4.4

Subject to: $g_j(b) <= 0.0 \quad j=1,m$ 4.5

$$h_i(b) = 0.0 \quad i=m+1,m+k \tag{4.6}$$

The inequality constraints of equation 4.5 include those that specify that the magnitude of all sensitivity coefficients of interest are less than b_{10}. These constraints can be written as:

$$(S_k)^2 - (b_{10})^2 <= 0.0 \quad k = 1,2,.. \tag{4.7}$$

where: S_k are the first order sensitivity coefficients of interest

Implementing this into a general nonlinear constrained optimization algorithm will require the second order sensitivities since the gradient of the first order sensitivity constraint of equation 4.7 will be second order sensitivities. The gradient of the objective function with respect to b_{10} is 1.0 and with respect to all other design variables it is zero. The gradient of the sensitivity constraint of equation 4.7 with respect to the design variables depends on the choice of sensitivity coefficients to be considered. For example, if we wish to minimize the maximum position sensitivity, then the constraint equation becomes:

$$(q_{i,j})^2 - (b_{10})^2 <= 0.0 \qquad i = 2,4 \qquad j = 1,4 \qquad 4.8$$

The corresponding gradients, G are given by:

$$G = 2.0*q_{i,jk}*q_{i,j} \qquad i = 2,4 \qquad j = 1,4 \qquad k = 1,4 \qquad 4.9$$

The gradient of the bound constraints on each design variable with respect to itself is 1.0 for the upper bound constraint and -1.0 for the lower bound constraint; the gradient with respect to all other design variables is zero.

The number of sensitivity constraints required depends on the number of grid points to be considered since the sensitivity is calculated at each grid point. However, for any number of grid points the general statement of the problem still conforms to the format of the standard nonlinear problem and can therefore be solved using suitable optimization techniques. In the present work, sequential unconstrained minimization techniques (SUMT) were used for this purpose [4]. These techniques are described briefly in the next section.

4.2 Sequential Unconstrained Minimization Techniques

To solve the constrained optimization problem through a sequence of unconstrained minimizations, the objective function must be modified to reflect the influence of the constraints. This is done by creating a pseudo-objective function that is formed from the true objective function by the addition of a penalty term as follows:

$$F_p(b, \lambda, r_p) = F(b) + r_p * P(b) \qquad 4.10$$

Here, F(b) is the original objective function defined by equation 4.1, P(b) is a measure of the constraint violation, and r_p is a multiplier used to control the magnitude of the penalty term. The multiplier r_p is increased slowly from one unconstrained minimization to the next in order to avoid the problem of ill-conditioning. An ill-conditioned problem occurs when the pseudo-objective function or its derivatives become discontinuous or ill-behaved at the constraint boundaries.

The penalty function method adds a penalty to the pseudo-objective function depending on the violations in the constraints. The first method discussed in the next section is the exterior penalty function method; it was the easiest to incorporate but it has some disadvantages. The second method used was the augmented Lagrangian multiplier method which is more complex but less sensitive to numerical ill-conditioning.

4.2.1 Exterior Penalty Function Method

The exterior penalty function method is the easiest to incorporate into an unconstrained optimization algorithm. No penalties are imposed if all the constraints are satisfied: however, if one inequality or equality constraint is violated the penalty imposed is of the form:

$$P(b) = \sum_{j=1}^{m} (\max(0.0, g_j(b)))^2 + \sum_{i=m+1}^{m+k} (h_i(b))^2 \qquad 4.11$$

Squaring the terms in equation 4.11 ensures a slope of zero for the penalty function at the constraint boundary. This, in turn, ensures a continuous slope for the first derivative of the pseudo-objective function at the constraint boundary.

The multiplier r_p is a very critical parameter and is increased from iteration to iteration by multiplying the current value by a fixed scalar γ. For the first unconstrained minimization, r_p is kept small (r_p = 2.0) and the pseudo-objective function is minimized. However, the solution that is found might have large constraint violations. The multiplier r_p is then increased by a factor of γ, which is usually in the range of 2.0 to 5.0. After r_p is updated, the next unconstrained minimization is performed using the latest estimate for the design variables. If the design ever goes into the infeasible region, the design approaches the true constrained optimum from the infeasible region as r_p is increased and becomes feasible only in the limit as r_p approaches infinity. This is one major disadvantage of the exterior penalty function method because if the minimization is stopped before the optimum is reached the design will be in the infeasible region and therefore will not be acceptable.

4.2.2 Augmented Lagrange Multiplier Method

The augmented Lagrange multiplier method (ALM) is a better penalty function method since it reduces the probability of numerical ill-conditioning. The augmented Lagrange multiplier method helps reduce the dependency of the algorithm on the choice of penalty parameters and the way in which they are updated. The general augmented Lagrange psuedo-objective function becomes:

$$A(b,\lambda,r_p) = F(b) + \sum_{j=1}^{m}(\lambda_j*\psi_j + r_p*(\psi_j)^2) + \sum_{i=m+1}^{m+k}(\lambda_i*h_i(b) + r_p*(h_i(b))^2) \qquad 4.12$$

where: $\psi_j = \max(g_j(b),- \lambda_j/2.0*r_p)$ 4.13

The major difference between the exterior penalty function method and the ALM method is the presence of the multiplier λ. If the λ in equation 4.12 were equal to zero, the penalty function for the ALM method would reduce to the penalty function for the exterior penalty function method in equation 4.11. The update formulas for the Lagrange multipliers are:

$$(\lambda_j)^{p+1} = (\lambda_j)^p + 2.0*r_p(\max(g_j(b),(-\lambda_j)^p/2.0*r_p)) \qquad 4.14$$

$$(\lambda_i)^{p+1} = (\lambda_i)^p + 2.0*r_p*h_i(b) \qquad 4.15$$

This method is insensitive to the value of r_p and there is no need to increase r_p to infinity in order to reach the optimum. The factor r_p is multiplied at each iteration by γ, but only up to a preset maximum value; after that, it is held constant throughout the remainder of the minimization process. Some advantages of the ALM method are:

1. The starting point may be either feasible or infeasible.
2. Acceleration to the optimal solution is achieved by updating the Lagrange multipliers.
3. Precise $g_j(b) = 0.0$ and $h_i(b) = 0.0$ is possible.
4. At the optimum, the value of $(\lambda_j)^* \neq 0.0$ will automatically identify the active constraint set.

There are many other penalty function methods available that could be used for this type of problems. The interior penalty function and extended penalty function methods both offer attractive features. Furthermore, SUMT is not the only gradient based method available. Other methods such as gradient projection techniques and the generalized reduced gradient method (GRG) can also be applied effectively.

CHAPTER V

IMPLEMENTATION

The methods derived in the preceding chapters were implemented in an interactive, menu driven program which was used to solve the numerical examples presented later in this thesis. The program consists of four modules, each of which has a well-defined function. These modules are: input, analysis, optimization and output. The main program serves as the driver from which any one of the four options can be interactively selected. When the user selects an option, the program enters that particular module and may be returned to the main driver by selecting the return option within the module. Each of the modules is described in detail in the following sections.

5.1 Input Module

The parameters that must be read in by the input module are the design variables b_1-b_9, the initial conditions for the input link (i.e., the initial angular velocity and the angular acceleration) and the number of grid points. The input can be read from one of two files (named D.INP1 and D.INP2) which must be generated prior to execution of the program. The input can also be provided interactively from the keyboard. If desired, the design variables can be input interactively from the screen using the tablet to draw each link's end points. Before the screen input, a grid is displayed to represent units of length and an option is provided to change the grid size. Upon completing the screen input for the design variables, the velocity, acceleration and number of grid points are read from a file generated beforehand. After all the input has been given to the program, the user can return to the main driver and choose to analyze or optimize the design linkage.

5.2 Analysis Module

The analysis module does not support any subcommands and control of the program is automatically returned to the main driver upon completion of the analysis of the linkage. The input link's mobility is first calculated depending on the link lengths, as explained in Chapter 2. Once the minimum and maximum crank angles are defined, the kinematic and design sensitivity analyses are simultaneously performed. The analysis is done at each grid point. The kinematic and design sensitivity analysis are done in separate subroutines (VELAC and SENS respectively).

5.3 Optimization Module

The optimization module may be called from the main program at any time after the first call to the input module. Since the analysis module is called from within the optimization module, it is not necessary to perform an analysis before the first call to the optimization module. The user is allowed to select one of the two penalty function methods discussed in Chapter 4 to perform the optimization. Each penalty function method requires additional parameter values to be input. The input parameters required for the exterior penalty function method are:

NITER	Number of r_p updates
STEP	Initial design change
MAXFUN	Number of function evaluations within an update
r_p	Multiplier for penalty term
γ	Scalar for the multiplier r_p

The input parameters for the ALM method are the same as the exterior penalty function method with the addition of the following:

$(r_p)_{max}$	The limit for the multiplier r_p
λ	The initial values for the Lagrangian multipliers

The program automatically reads the appropriate penalty function method's input file, which must be generated prior to execution. Once the optimization method has been selected, a flag is set within the program to store this information.

After the optimization method has been selected and the appropriate input parameters are read, the program flow within the optimization module enters a loop. From within this loop, it calls an unconstrained optimization subroutine (VA06A from the Harwell subroutine library) to obtain the design updates. The number of cycles within the loop is determined by the parameter NITER which also controls the number of updates for the multiplier r_p. Within subroutine VA06A, a routine CALCFG is called to perform function evaluations for the pseudo-objective function and its gradients. The constraints and gradients of the constraint functions are provided through a subroutine (called SETUP) before the pseudo-objective function and its gradients are calculated. The subroutine SETUP is provided by the user prior to execution and contains the equations for the constraint functions and their gradients for the particular problem being solved.

The output from the optimization module is written to two separate files whose file names can be specified by the user. The final constraint violations and minimized pseudo-objective function are printed to the screen and to a file specified by the user for storing the kinematic/design sensitivity analysis output. The optimization output for each function evaluation is written to a different file selected by the user prior to exiting the optimization module.

5.4 Output Module

The output module has a local driver that allows the user to select different types of output to display the final results. The user can select from one of four options: file, screen, plots or pictorial representations. If the user chooses to display analysis results to the screen or to a file they may select from various types of output. Once this selection is made and an optimization is performed these results will be printed to the specified file or screen. The type of output can be chosen from the following: kinematic, first order design sensitivity, second order design sensitivity or all of the kinematic/design sensitivity analysis. The kinematic/design sensitivity results can also be plotted against the crank angle. The user can interactively select the predetermined y-axis variables (maximum of two per plot) and select between two choices of x-axis variable (crank angle or grid point number). The pictorial representations consist of a graphical display of the four-bar linkage. The user can choose from one of the following three types of pictorial representations: superposition, single position and animation.

The selections made from the menus were done by using a tablet and very little keyboard interaction was required from the user. The program ran on a Harris 800 supermini computer with DI-3000 graphics.

CHAPTER VI

NUMERICAL EXAMPLES

The techniques developed in Chapters 2, 3 and 4 were implemented in the computer program described in Chapter 5 and tested on several numerical examples.

6.1 Sensitivity Analysis Verification

This section discusses the results obtained for the first and second order sensitivity analysis of selected linkages. In order to verify the sensitivity analysis using a finite difference technique, the linkage was analyzed for a given set of design variables, b. One design variable was then given a small perturbation Δb_j, so that the new value of this design variable became:

$$(b_j)^* = b_j + \Delta b_j \tag{6.1}$$

The four-bar linkage was then analyzed at the new design. The first order position sensitivity value at a particular grid point should be approximated by:

$$q_{i,j} \cong (q_i(b_j)^* - q_i(b_j))/((b_j)^* - b_j) \tag{6.2}$$

$$i = 2,4$$

$$j = 1,4$$

Similarly the second order position sensitivity value at a particular grid point can be approximated by:

$$q_{i,jk} \cong (q_{i,j}(b_k)^* - q_{i,j}(b_k))/((b_k)^* - b_k) \qquad 6.3$$

$i = 2,4$

$j = 1,4$

$k = 1,4$

The preceding method can be used to check the first and second order velocity and acceleration sensitivities as well.

The following example illustrates the use of a small perturbation in design variable b_1 in checking the first and second order position sensitivity for q_3. The initial values of the design variables corresponding to the link lengths are:

$$b_1 = 7.0$$

$$b_2 = 3.0$$

$$b_3 = 8.0$$

$$b_4 = 6.0$$

Using a perturbation of 0.001 in design variable b_1, the following data was obtained:

$$q_{3,1} = 0.06455$$

$$q_3((b_1)^*) = 0.81282$$

$$q_3(b_1) = 0.81276$$

$$(b_1)^* = 7.001$$

$$b_1 = 7.000$$

Using equation 6.2 to check the first order position sensitivity of q_3 with respect to b_1, we see that we require:

$$0.06455 \cong (0.81282 - 0.81276)/(7.001 - 7.000)$$

i.e. $0.06455 \cong 0.0646$

Thus, the first order position sensitivity calculated matches up to the third significant figure when compared to the finite difference approximation of the first order position sensitivity.

The following calculation was used to check the second order position sensitivity of q_3 using the same perturbation in link length b_1:

$$q_{3,11} = -\ 0.07918$$

$$q_{3,1}((b_1)^*) = 0.06447$$

$$q_{3,1}(b_1) = 0.06455$$

Using equation 6.3 to check the second order position sensitivity coefficient $q_{3,11}$, we see that we should have:

$$-\ 0.07918 \cong (0.06447 - 0.06455)/(7.001 - 7.000)$$

i.e. $-\ 0.07918 \cong -\ 0.07920$

The second order position sensitivity is accurate to the third significant figure when compared to the finite difference approximation the second order position sensitivity calculated from the perturbation analysis. Similar calculations were done for first and second order velocity and acceleration sensitivities for several cases. The agreement with the finite difference perdiction was uniformly good (within 1%) and indicates that the proposed technique for sensitivity analysis works with a very high degree of accuracy.

6.2 Minimum Sensitivity Results

The second order sensitivity analysis was incorporated into an optimization scheme for semi-automated design of minimum sensitivity four-bar linkages. Some examples of minimum sensitivity design using this method are presented in this section. The objective in all the examples was to minimize the maximum first order position sensitivity of the coupler link with respect to the link lengths. Each example was run for one full rotation of the crank with 16 grid points. Since there are four position sensitivity constraints for each grid point, 64 inequality constraints are required to enforce the condition specified in equation 4.8. In addition to these constraints there are upper and lower bound constraints for all the design variables, including the artificial design variable b_{10}. Additional performance constraints may also be required, depending on the problem to be solved.

In all the examples presented in this section, the crank is driven at an angular velocity of 1.0 with a constant angular acceleration of 0.0. In addition, all the examples used the following initial estimate for the design vector:

$$\begin{aligned}
b_1 &= 7.0 \\
b_2 &= 3.0 \\
b_3 &= 8.0 \\
b_4 &= 6.0 \\
b_5 &= 1.0 \\
b_6 &= 6.0 \\
b_7 &= 0.0 \\
b_8 &= 0.0 \\
b_9 &= 0.0
\end{aligned}$$

The parameters required for the optimization algorithm also remained the same for all the examples. The values chosen for the exterior penalty function and ALM method were:

NITER = 8	Number of multiplier updates
r_p = 2.0	Multiplier r_p, initial value
γ = 5.0	Multiplying factor for updating r_p

The additional parameters required for the ALM method were:

$(r_p)_{max}$ = 500.0   Maximum value for r_p

λ = 1.0   The Lagrangian multiplier

The multiplier r_p for each example was increased up to a final value of 156250.0 to insure a reasonably effective correction of constraint violations.

<u>Example 1: Straight Line Generator</u>

A straight line generator should satisfy the requirement that the coupler point trace an approximate straight line during a portion of the complete rotation of the input link. One linkage that can be used for this purpose is the Chebyshev linkage, which is defined by the relative proportions of the link lengths. The equality constraints needed to ensure that these proportions hold in the final design are as follows:

$$b_2 - 2.0*b_1 = 0.0 \tag{6.8}$$

$$b_3 - b_4 = 0.0 \tag{6.9}$$

$$b_3 - 2.5*b_2 = 0.0 \tag{6.10}$$

$$b_6 - 2.0*b_3 = 0.0 \tag{6.11}$$

$$b_5 = 0.0 \tag{6.12}$$

Initially, the values for the violated equality constraints in equations 6.8 through 6.12 were relatively large but after optimization they were very close to zero. The values of the constraint functions of equations 6.8 through 6.12 before and after optimization were:

Before:	After:
- 0.94388	- 0.22E-04
0.92722	- 0.41E-04
3.11640	- 0.33E-04
2.15500	0.33E-04
- 0.23E-10	0.13E-21

The final values of the design variables after optimization were:

$$b_1 = 3.7924$$

$$b_2 = 1.8962$$

$$b_3 = 4.7404$$

$$b_4 = 4.7403$$

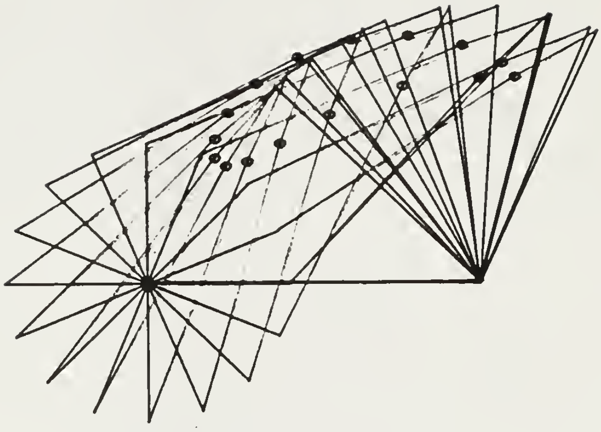
$$b_5 = -0.23E-23$$

$$b_6 = 9.4808$$

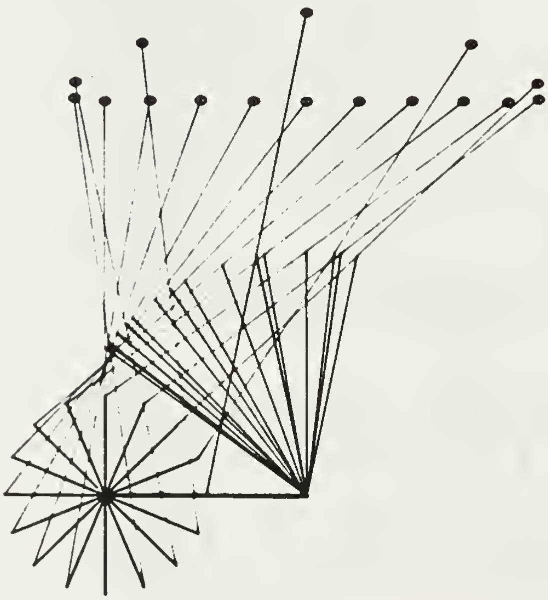
$$b_7 = 0.0$$

$$b_8 = 0.0$$

$$b_9 = 0.0$$



a. Straight line generator at initial design



b. Straight line generator at final design

Figure 6.1 Example 1: Straight line generator

In addition to correcting the performance constraints as described above, the cost function, i.e. the maximum position sensitivity of the coupler link showed an increase from 0.2582 at the initial design to 0.538 at the final design. This increase in cost was due to the strict requirements placed by the equality constraints in equations 6.8 through 6.12. The exact proportions of the design variables were analyzed and compared with the final design. In this case, the final design cost function was reduced by nearly 50%.

Example 2: Perpendicular Line Generator

The perpendicular line generator is to be designed so that the coupler point traces two straight line segments that are approximately perpendicular to each other during a portion of the rotation of the input link. This can be ensured by maintaining certain proportions between the lengths of the links. The equality constraints required for this are as follows:

$$b_1 - 2.83*b_2 = 0.0 \qquad 6.13$$

$$b_3 - b_2*2.17 = 0.0 \qquad 6.14$$

$$b_3 - b_4 = 0.0 \qquad 6.15$$

$$b_6 - 2.0*b_3 = 0.0 \qquad 6.16$$

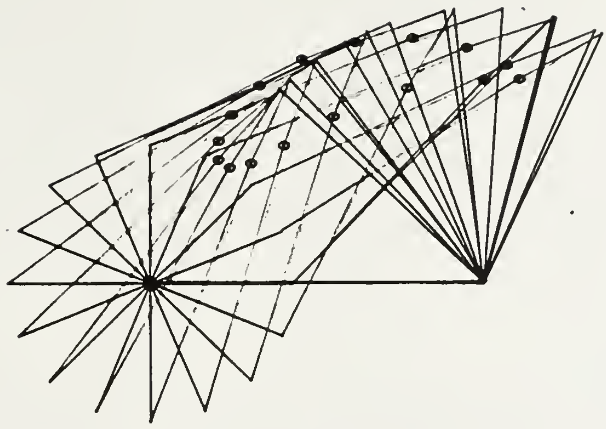
$$b_5 = 0.0 \qquad 6.17$$

Initially, the values of the violations in the constraints of equations 6.13 through 6.17 were relatively large but after optimization they were almost exactly satisfied. The values of the constraint functions of equations 6.13 through 6.17 before and after optimization were:

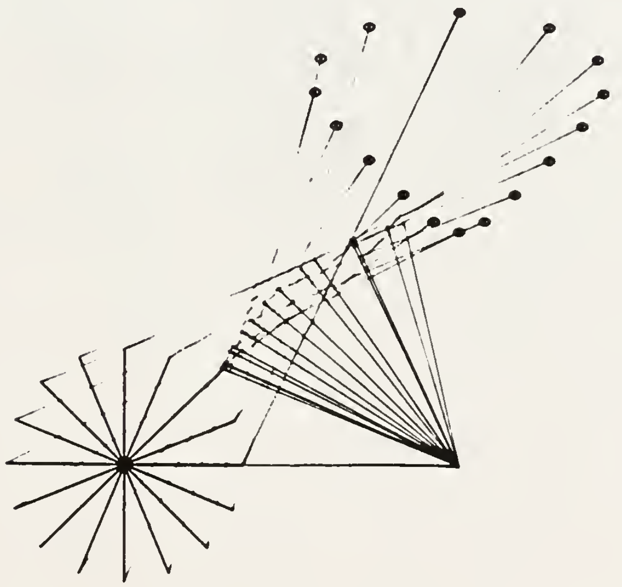
Before:	After:
0.94857	0.267E-05
3.14760	0.349E-05
0.56386	0.892E-05
0.32159	0.358E-05
- 0.24E-10	- 0.347E-15

The final values of the design variables after optimization were:

$$b_1 = 5.9585$$
$$b_2 = 2.1055$$
$$b_3 = 4.5689$$
$$b_4 = 4.5689$$
$$b_5 = 0.61E-17$$
$$b_6 = 9.1378$$
$$b_7 = 0.0$$
$$b_8 = 0.0$$
$$b_9 = 0.0$$



a. Perpendicular line generator at initial design



b. Perpendicular line generator at final design

Figure 6.2 Example 2: Perpendicular line generator

Once again the cost function in this example increased from 0.2582 at the initial design to 0.286 at the final design due to the requirements for the constraints. However, when the cost functions from the exact proportions of the design variables were compared to the final design and there was a reduction.

Example 3: Circle Generator

The circle generator is required to satisfy the condition that the coupler point trace an approximate circle during one complete rotation of the input link. The equality constraints needed to maintain the correct proportions between the link lengths are:

$$b_1 - 1.41*b_3 = 0.0 \tag{6.18}$$

$$b_2 - 0.136*b_3 = 0.0 \tag{6.19}$$

$$b_3 - b_4 = 0.0 \tag{6.20}$$

$$b_6 - 2.0*b_3 = 0.0 \tag{6.21}$$

$$b_5 = 0.0 \tag{6.22}$$

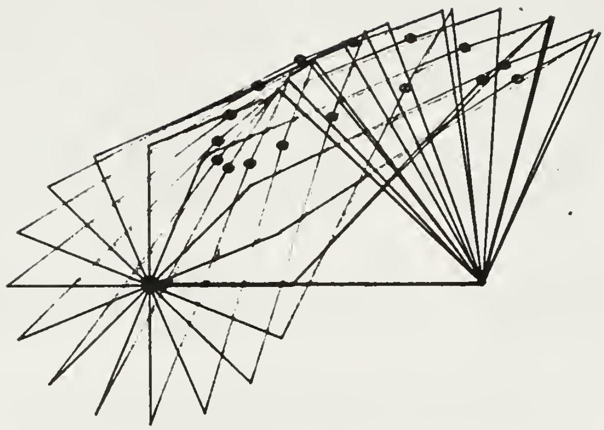
At the initial design, the values of the violated constraints of equations 6.18 through 6.22 were relatively large but after optimization they were almost exactly satisfied. The values of the constraint functions of equations 6.18 through 6.22 before and after optimization were:

Before:	After:
0.13215	- 0.8158E-02
2.01370	0.16299
- 0.98492	0.5099E-02
- 0.243E-10	- 0.1227E-28
- 3.19420	- 0.1024E-01

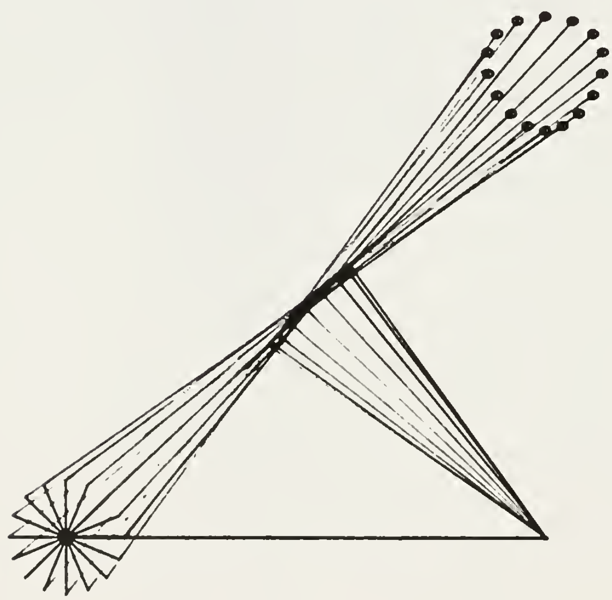
The final values of the design variables were:

$$b_1 = 6.9728$$
$$b_2 = 0.83634$$
$$b_3 = 4.9511$$
$$b_4 = 4.9460$$
$$b_5 = 0.214E-30$$
$$b_6 = 9.8919$$
$$b_7 = 0.0$$
$$b_8 = 0.0$$
$$b_9 = 0.0$$

The observed cost reduction in this example was from 0.2582 at the initial design to 0.208 at the final design.



a. Circle generator at initial design



b. Circle generator at final design

Figure 6.3 Example 3: Circle generator

Example 4: Four-bar Linkage Design with Transmission Angle Limits

In this example it is required that the transmission angle remain between 80 and 100 degrees throughout the rotation of the input link. The constraints needed to enforce these limits on the transmission angle at each grid point are:

$$\gamma - 100.0 \le 0.0 \qquad (6.23)$$

$$-\gamma + 80.0 \le 0.0 \qquad (6.24)$$

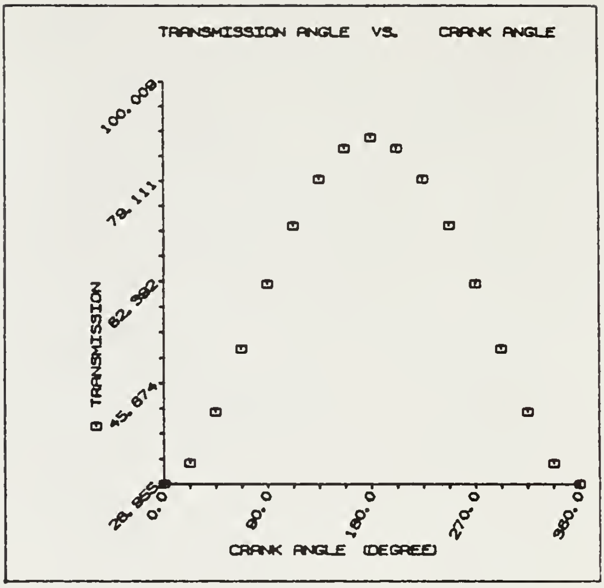
Initially the violated constraints were relatively large but after optimization they were almost fully corrected. The three highest constraint violations before and after optimization were:

Before:	After:
13.3	0.109
12.0	0.221
8.5	0.144

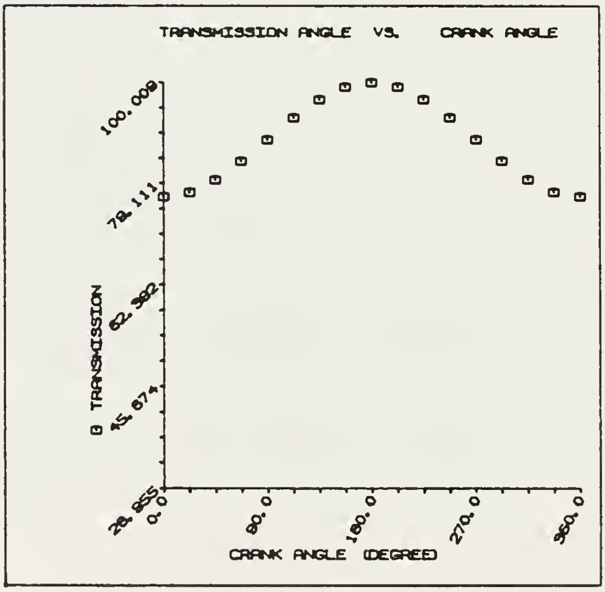
The maximum constraint violation was reduced from 13.3 to 0.221. The final values of the design variables were:

$$b_1 = 8.99$$

$$b_2 = 7.73$$



a. Design with transmission angle limits at initial design



b. Design with transmission angle limits at final design

Figure 6.4 Example 4: Design with transmission angle limits

$b_3 = 7.02$

$b_4 = 5.68$

$b_5 = 1.0$

$b_6 = 6.0$

$b_7 = 0.0$

$b_8 = 0.0$

$b_9 = 0.0$

The violated constraints at the final design were caused by the transmission angle falling below 80 degrees to a value of 79.9 and going above 100 degrees to a value of 100.008. The maximum position sensitivity i.e. cost function was minimized from 0.2582 at the initial design to a value of 0.145 at the final design.

Example 5: Design for Coupler Link Angular Velocity

In this example, it is required that the angular velocity of the coupler link remain between 0.1 and -0.1 (radians/sec) throughout the rotation of the input link. The constraints needed to impose this requirement are:

$$\dot{q}_3 - 0.1 <= 0.0 \qquad 6.25$$

$$-\dot{q}_3 - 0.1 <= 0.0 \qquad 6.26$$

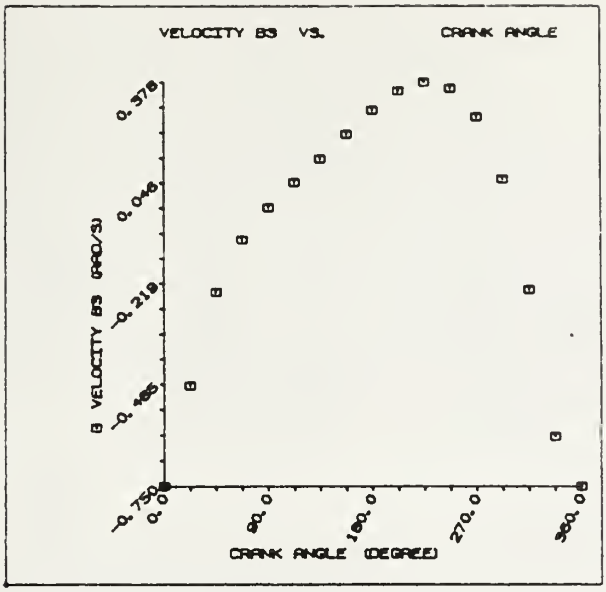
Initially, the constraint violations were relatively large but after optimization the constraints were almost fully corrected. The three highest constraint violations before and after optimization were:

Before:	After:
0.65	0.022
0.37	0.018
0.11	0.012

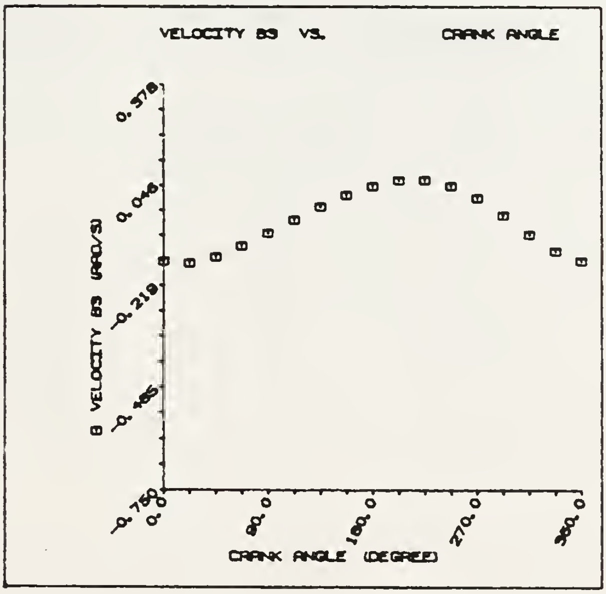
The maximum constraint violation was reduced from 0.65 to 0.022. The final values of the design variables were:

$$b_1 = 9.4241$$
$$b_2 = 0.99466$$
$$b_3 = 8.7684$$
$$b_4 = 5.9697$$
$$b_5 = 1.0$$
$$b_6 = 6.0$$
$$b_7 = 0.0$$
$$b_8 = 0.0$$
$$b_9 = 0.0$$

The angular velocity condition was not exactly satisfied but the largest negative and positive velocities were equal to -0.122 and 0.112, respectively. The maximum position



a. Design for coupler link angular velocity at initial design



b. Design for coupler link angular velocity at final design

Figure 6.5 Example 5: Design for coupler link angular velocity sensitivity was minimized from 0.2582 at the initial design to a value of 0.124 at the final design.

Example 6: Rigid Body Guidance

In this rigid body guidance problem, it is required that the coupler link remain at a 45 degree angle throughout the rotation of the input link. The constraint needed to enforce this requirement is:

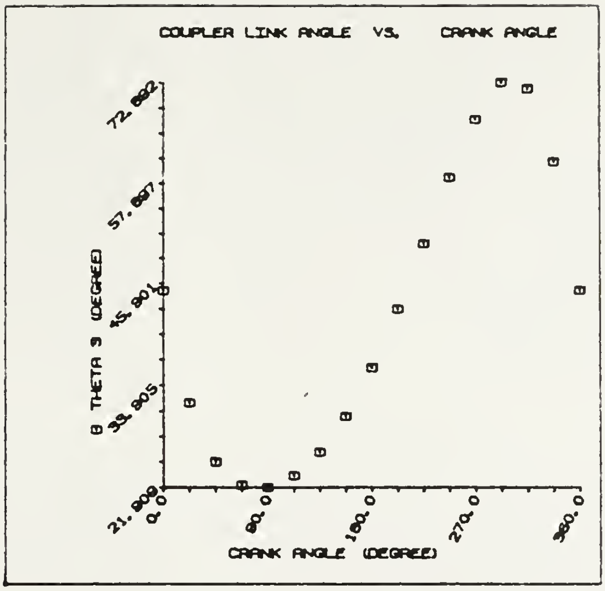
$$q_3 - 45 = 0.0 \tag{6.27}$$

The performance constraint of equation 6.27 must be converted to radians before verifying these results. The three highest constraint violations before and after optimization were:

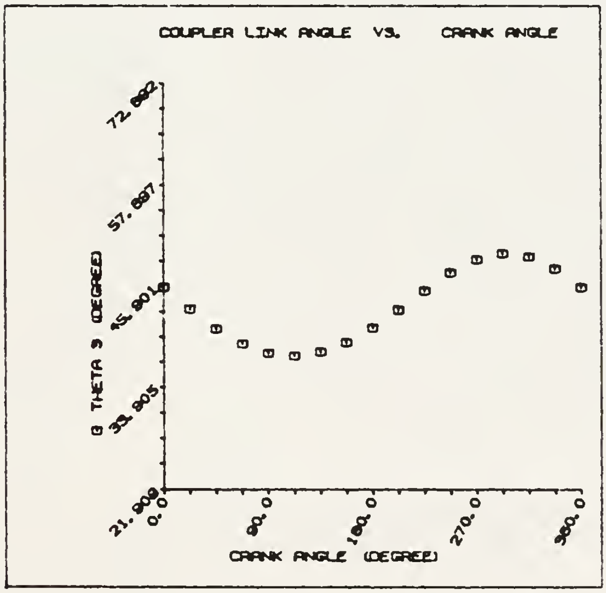
Before:	After:
- 0.403	- 0.105
0.487	0.114
0.407	0.101

The maximum constraint violation was reduced from 0.487 to 0.114. The final values of the design variables were:

$$b_1 = 8.3873$$
$$b_2 = 0.87126$$



a. Rigid body guidance at initial design



b. Rigid body guidance at final design

Figure 6.6 Example 6: Rigid body guidance

$$b_3 = 8.3150$$

$$b_4 = 6.3946$$

$$b_5 = 1.0$$

$$b_6 = 6.0$$

$$b_7 = 0.0$$

$$b_8 = 0.0$$

$$b_9 = 0.0$$

The largest and smallest angles for the coupler link were equal to 38.6 and 51.5 degrees, respectively. The maximum sensitivity was minimized from 0.2582 at the initial design to a value of 0.139 at the final design.

Example 7: Coupler Curve Synthesis

In this example, it is required that the coupler point trace a straight line at 45 degrees to the horizontal throughout the rotation of the input link. The slope and y-intercept for the coupler curve were required for the specification of this example problem. In order to achieve this desired path a slope of 1.0 and y-intercept of 0.0 were used. The constraint needed to enforce this requirement is:

$$\Delta^2 = 0.0 \tag{6.28}$$

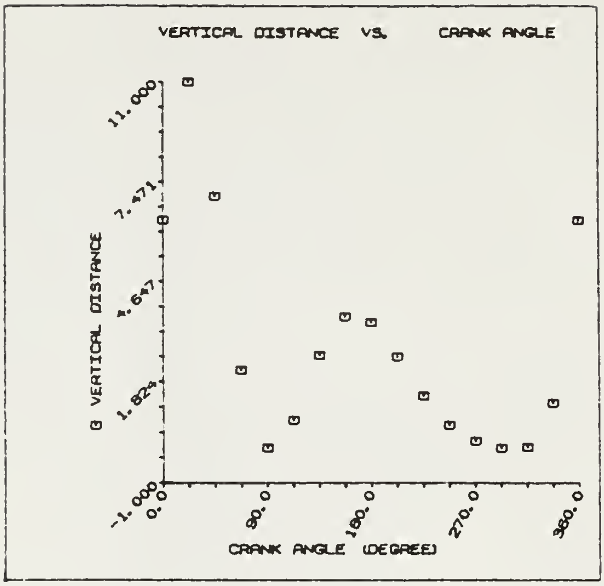
The three highest constraint violations before and after optimization were:

Before:	After:
2.210	0.931E-01
3.737	0.1313
2.862	0.1141

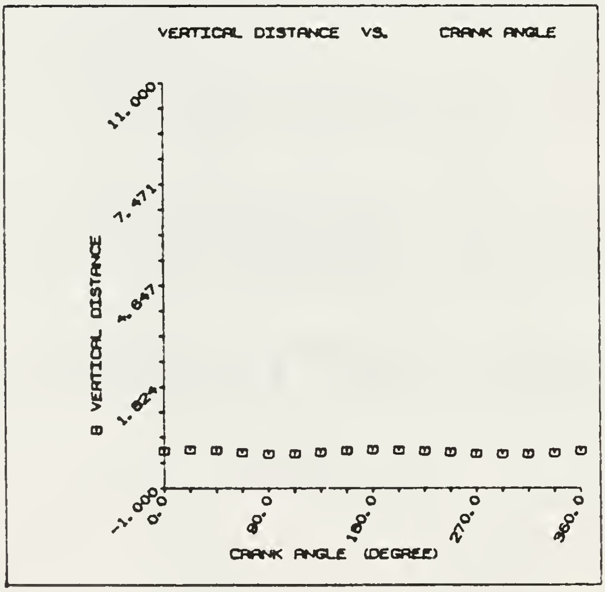
The maximum constraint violation was reduced from 3.737 to 0.1313. The final values of the design variables were:

$$b_1 = 8.13$$
$$b_2 = 0.86$$
$$b_3 = 7.78$$
$$b_4 = 6.06$$
$$b_5 = 0.00$$
$$b_6 = 6.52$$
$$b_7 = 0.0$$
$$b_8 = 0.0$$
$$b_9 = 0.0$$

The violated constraints were due to the fact that the coupler point did not trace an exact 45 degree line but had a maximum vertical deviation of 0.131. This deviation seems large, but compared to the maximum vertical deviation of 11.0 before the optimization, it is seen to be considerably smaller. The maximum sensitivity was minimized from 0.2582



a. Coupler curve synthesis at initial design



b. Coupler curve synthesis at final design

Figure 6.7 Example 7: Coupler curve synthesis at the initial design to a value of 0.149 at the final design.

CHAPTER VVI

CONCLUSIONS

The research presented in this thesis was to develop a computer-based design technique for the design of minimum sensitivity four-bar linkages. In order to manufacture the linkage, appropriate tolerances have to be specified on the link lengths. The tolerances on any dimension reflect the sensitivity of the system performance to small variations in that dimension. If the system performance is relatively insensitive to variations in a particular dimension, the tolerances on that dimension can be specified to be quite loose. Since the tolerance on any dimension is dependent on the sensitivity of the system performance to variations in that dimension, it follows that in designing a minimum sensitivity linkage, we are effectively designing a minimum cost linkage as well.

The primary objective of the research described in this thesis was the development of a general method for the design of minimum sensitivity four-bar linkages using a nonlinear programming approach. The underlying idea was to convert the minimum sensitivity design problem into a nonlinear optimal design problem which would then be solved through the use of a gradient-based optimization technique. It was realized that this would require not only kinematic analysis of the linkage but first and second order design sensitivity analysis as well. The method that was adopted for the kinematic analysis was a well-known loop closure formulation. Since suitable methods for first order design sensitivity analysis for the four-bar linkages were not readily available in the literature, a set of first order sensitivity equations was derived from the kinematic equations by a direct differentiation approach. This direct differentiation method was applied again to the first order sensitivity equations to obtain a set of equations for the second order design sensitivity analysis. The results of the kinematic and design sensitivity analyses were supplied to an optimization algorithm to obtain the next improved design. The optimization method used was a sequential unconstrained minimization technique that could make use of an exterior penalty function or an augmented Lagrangian function.

A second goal of the present work was the implementation of the above solution method in an interactive computer-aided design program that could be used for efficient design of minimum sensitivity four-bar linkages. This goal was also accomplished successfully. The program developed offers several attractive features and is highly interactive and user-friendly. The kinematic/design sensitivity analysis and optimization sections are completely independent, allowing the optimization package to be interchanged quite easily. The program does not require much user involvement other than the input of an initial design, specification of cost/constraint functions and their gradients and selection of a penalty function method. The program also offers a variety of graphical displays for inputting the problem description and for interpreting the output.

The program described in the preceding paragraph was used to run several examples in order to verify the sensitivity analysis schemes that were developed and to evaluate the performance of the proposed scheme for the design of minimum sensitivity four-bar linkages. The results indicate that the sensitivity analysis is very accurate (within 1% when checked by perturbation analysis) and the optimization scheme works very effectively and reliably in reducing the sensitivity of the system and in satisfying specified performance requirements.

The work that has been presented in this thesis offers many possilities for future development in several areas. The loop closure and direct diffentiation techniques can be extended to cover a wide range of dynamic systems. The Harwell subroutine could be replaced with other routines to improve the efficiency of the optimization algorithm. Second order optimization techniques should also be tried to improve efficiency. The program could be made more user-friendly to give the user greater control over the design process. Other uses of the second order sensitivity information should also be investigated. Possible uses for this information include second order optimization, reliabilty design and approximation of system behavior in the neighborhood of a design point.

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COMPUTER AIDED DESIGN OF MINIMUM SENSITIVITY FOUR-BAR LINKAGES

by

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ABSTRACT

The objective of this research endeavor was the development of a general scheme for the minimum sensitivity design of four-bar linkages using mathematical programming techniques. An algorithm that utilizes gradient-based optimization was derived for this purpose. This algorithm required not only the kinematic analysis of a four-bar linkage but the first and second order design sensitivity analyses as well. The kinematic analysis of the four-bar linkage was performed using a loop closure technique. The first order sensitivity analysis was obtained by direct differentiation of the loop closure equations with respect to the appropriate design variables. The second order sensitivity analysis was obtained by direct differentiation of the first order sensitivity equations with respect to the appropriate design variables. The constrained minimum sensitivity problem was solved using exterior penalty and augmented Lagrangian methods. An interactive, user-friendly computer program was developed for computer-aided design of minimum sensitivity four-bar linkages based on this algorithm. Finally, several numerical examples were solved in order to evaluate the performance and reliability of the proposed solution technique.