DISTRIBUTION-FREE LIMITS INVOLVING RANDOM VARIABLES

by

TZE-U CHO

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Approved by:

Major Professor

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INTRODUCTION

Many common statistical methods are concerned with estimating the parameters of a distribution function of known or assumed form. Thus the population may be assumed to be normal with parameters mean and variance, and these may be estimated by the sample mean and sample variance; or we may use a t-test or a F-test to make some test about them. Such statistical methods, since they deal with population parameters, are called parametric.

There are, however, other methods which do not require any assumptions about the nature of the population from which the sample is drawn. These are called distribution-free, since they are free of specific assumptions about the distribution in the parent population. These method are valid for any parent population, hence they also could be validly applied to samples from normal distributions. It is true that these methods may be less efficient, in some sense, than the parametric methods when the population is normal. But for some kinds of data, particularly in behavioral science and enginneering experiments, it would be inappropriate to assume normal distributions. In such cases, distribution-free methods should be used. Median, quartiles, etc., may be far more appropriate to describe the location and dispersion of the population than the mean and/or variance. Besides, distribution-free methods are generally simple to apply, not involving much computation.

As is well-known, it is convenient to use the t and χ^2 distributions to find confidence limits on the parameters of a normal distribution. In this paper we will discuss some methods of finding the confidence limits when no assumption is suitable to be made about the parent population.

CONFIDENCE INTERVAL FOR QUARTILES

There are three quartiles, usually designated Q_1, Q_2 or median, Q_3, which divide a given series of measurements into four equal parts. The first quartile, Q_1, is the value at or below which one-fourth of all items in the series fall; the second quartile, Q_2, is the value such that half of the items are smaller and half of the items exceed it aside from ties; and the third quartile, Q_3, is the value at or below which three-fourths of the items lie. The simplest way to find confidence intervals for quartiles is to use the concept of the binomial distribution. Let X_1, X_2, ... X_n present a ramdom sample from a population, and arrange the n observations in increasing order: $X_{(1)} \le X_{(2)} \le \ldots\ldots\ldots \le X_{(r)} \le \ldots\ldots\ldots X_{(s)} \le \ldots\ldots\ldots \le X_{(n)}$, where $1 \le r \le s \le n$. Suppose the distribution is continous and 1- is the desired confidence coefficient. To find the confidence interval for an unknown quartile is to find statistics r and s such that

(1) $$P(X_{(r)} \le Q_i \le X_{(s)}) = 1 - \alpha$$

Because of continuity, (1) also can be written as

(2) $$\begin{aligned} P(X_{(r)} \le Q_i \le X_{(s)}) &= P(Q_i \le X_{(s)}) - P(Q_i < X_{(r)}) \\ &= P(Q_i < X_{(s)}) - P(Q_i < X_{(r)}) \end{aligned}$$

Consider the statement $(Q_i < X_{(1)})$ where $X_{(1)}$ is the smallest value in the sample. This statement is true only if all n observations in the sample are greater than Q_i.

Next consider $(Q_i < X_{(2)})$. The statement is true either when n-1 observations are greater than Q_i, in which case $X_{(1)} \leq Q_i < X_{(2)}$; or all observations are greater than Q_i, in which case $Q_i < X_{(1)} < X_{(2)}$. A picture can make the situation clear.

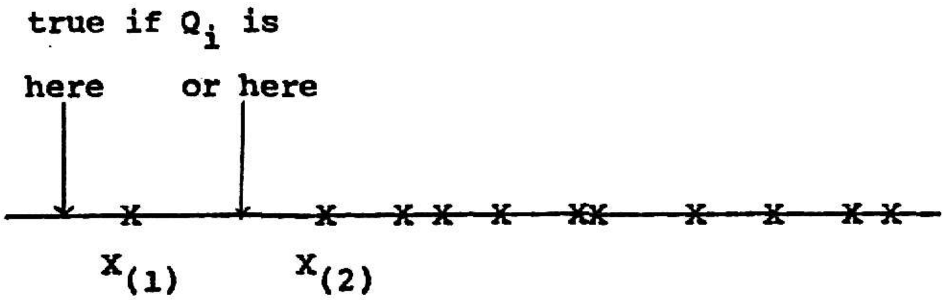

Figure 1

Each random X_i has probability p_1=.25 of being less than or equal to Q_1; has probability p_2=.50 of being less or equal to Q_2; and has probability p_3=.75 of being less or equal to Q_3. Therefore, with the aid of the binomial distribution function we add the appropriate probabilities and obtain

$$
\begin{aligned}
(3) \qquad P(Q_i < X_{(1)}) &= P(\text{all } X_j \text{ exceed } Q_i) \\
&= P(\text{none of } X_j \text{ below } Q_i) \\
&= (1 - p_i)^n \qquad\qquad i=1,\ 2,\ 3.
\end{aligned}
$$

$$
\begin{aligned}
(4) \qquad P(Q_i < X_{(2)}) &= P(Q_i < X_{(1)}) + P(X_{(1)} \leq Q_i < X_{(2)}) \\
&= P(\text{at least n-1 } X_j \text{ exceed } Q_i) \\
&= P(\text{1 or none of the } X_j \text{ below } Q_i) \\
&= \sum_{a=0}^{1} \binom{n}{a} p_i^{\,a} (1-p_i)^{n-a} \qquad\qquad i=1,\ 2,\ 3.
\end{aligned}
$$

This result can be extended as follows:

(5) $$P(Q_i < X_{(r)}) = P(\text{at least } n-r+1 \text{ of } X_j \text{ exceed } Q_i)$$
$$= P(r-1 \text{ or less of } X_j \text{ below } Q_i)$$
$$= \sum_{a=0}^{r-1} \binom{n}{a} p_i^{\,a}(1-p_i)^{n-a} \qquad i=1,\ 2,\ 3.$$

(6) $$P(Q_i < X_{(s)}) = P(\text{at least } n-s+1 \text{ of } X_j \text{ exceed } Q_i)$$
$$= P(s-1 \text{ or less of } X_j \text{ below } Q_i)$$
$$= \sum_{a=0}^{s-1} \binom{n}{a} p_i^{\,a}(1-p_i)^{n-a} \qquad i=1,\ 2,\ 3.$$

Tables IA, IB, IC are the binomial distribution tables with p=.25, .50, .75. We can use these tables to find the confidence intervals for the three quartiles.

For the sample size ≤ 20, Table I is used to find the value of r and s. From previous discuss we know that

(7) $$1 - \alpha = P(X_{(r)} \leq Q_i \leq X_{(s)})$$
$$= P(Q_i < X_{(s)}) - P(Q_i < X_{(r)})$$

In order to obtain a two-sided confidence interval, set

(8) $$P(Q_i < X_{(s)}) = 1 - \alpha/2$$

(9) $$P(Q_i < X_{(r)}) = \alpha/2$$

Enter Table I with proper probability p and sample size n. Read across the row until reaching a value close to $\alpha/2$, the corresponding value of m is r-1. Add 1 to get r. Then continue reading across the same row until reaching a value close to $1-\alpha/2$. and add one to the corresponding m to get s.

When the sample size exceed 20, use the normal distribution to approximate the binomial distribution with mean=np and standard deviation $\sqrt{np(1-p)}$. The formulas are

(10) $$r^* = np + Z_{\alpha/2}\ \sqrt{np(1-p)}$$

and

(11) $$s^* = np + Z_{\alpha/2}\ \sqrt{np(1-p)}$$

Where the Z is obtained from the cumulative standard normal distribution table (Table II). The values r^*, s^* may not always be integers so *round* them upward to the next higher ingeger to get r and s.

A one-sided confidence interval with confidence cofficient 1- can be found by using the above method to find either r or s, so that

(12) $$P(Q_i < X_{(s)}) = 1 - \alpha$$

or

(13) $$P(Q_i < X_{(r)}) = \alpha$$

The method described above applies only for a continous variate. In practice the distribution function may be discrete. Scheffé and Tukey (1945) have investigated confidence intervals for quantiles and have noted that if in the discrete case the confidence interval bounded by the appropriate ordered statistic has confidence coefficient exactly equal $1-\alpha$, the corresponding closed confidence interval (with endpoints) has confidence coefficient at least $1-\alpha$, whereas the open interval has confidence coefficient of $1-\alpha$ at most. Therefore we shall write the confidence interval for quartiles as

(14) $$P(X_{(r)} \leq Q_i \leq X_{(s)}) \geq 1 - \alpha$$

Example 1: An experiment is made by the fibers division of a chemical company to determine the breaking strength of corn yarn, represented by pounds per square inch required to break a skein of yarn. A random sample of size 20 is taken, and the data obtained are:

98, 112, 108, 86, 124, 92, 102, 91, 95, 104, 89, 129, 83, 98, 92, 99, 113, 116, 122 and 85.

We wish to find the 95% confidence interval on Q_3.

First arrange the observed data in an ordered array:

83, 85, 86, 89, 91, 92, 92, 93, 95, 98, 98, 99, 102, 104, 108, 112, 113, 116, 122, 129.

Enter Table IA with n=20, reading across the row, the probability closest to .025 is .014. The m value corresponding to .014 is 10. Therefore s equals 11. In the same row the probability is chosen as being close to .975 is .976. The corresponding m value is 18, therefore s=19. The confidence interval

$$P(X_{11)} \leq Q_i \leq X_{(19)}) = .976 - .014 = .962$$
$$\geq 95\%$$

It is reasonable to consider this case as a discrete distribution, becaise $X_{(11)}$ equals 98, $X_{(19)}$ equals 122, we may say "the interval from 98 to 122 pounds is at least a 95% confidence interval for the third quartile."

Example 2: Find a 95% confidence interval for the Q_2 of the population of blood lactates of patients with an anxiety neurosis, given the following random sample of size 27 from the population:

32, 36, 42, 33, 49, 98, 51, 46, 24, 51, 56, 45, 47, 51, 56, 24, 95, 22, 31, 34, 38, 44, 49, 52, 54, 42, 57 milligrams per 100 millilitres.

Arrange them in order, thus:

22, 24, 24, 31, 32, 33, 34, 36, 38, 42, 42, 44, 45, 46, 47, 49, 49, 51, 51, 51, 52, 54, 56, 56, 57, 95, 98.

Because the sample is greater than 20, use Equation (10) and (11).

$$\begin{aligned} r^* &= (27)(.50) + (-1.96)(27)(.50)(.50) \\ &= 13.5 - 5.09 \\ &= 8.41 \\ s^* &= 13.5 + 5.09 \\ &= 18.59 \end{aligned}$$

Therefore r equals 9, s equals 19. Hence the 95% confidence interval on the median is: $38 \leq Q_2 \leq 51$ mg/100ml.

CONFIDENCE INTERVAL FOR MEDIAN

Median is identical with Q_2. In a list of values arranged in increasing magnitude, half of them at or above median, half at or below it. We will discuss three ways of finding the confidence interval for the median in this section.

The first method is based on binomial distribution. This method resembles the method shown in the previous section of finding a confidence interval for Q_2. The only difference is that we now take the same number of items on both tails. The merit of this method is ease of caculation. We can find out the confidence interval easily even without the aid of tables.

Obtain a random sample X_1, X_2, ... X_n from a population with unknown median, arrange them in order in two ways, denoted as follow:

(1) $$X_{(1)} \leq X_{(2)} \leq \dots\dots\dots\dots \leq X_{(n-1)} \leq X_{(n)}$$

(2) $$X^{(n)} \leq X^{(n-1)} \leq \dots\dots\dots\dots \leq X^{(2)} \leq X^{(1)}$$

In Equation (1) we express the smallest value as $X_{(1)}$, second smallest as $X_{(2)}$ .... etc. In Equation (2) we express the largest values as $X^{(1)}$, second largest as $X^{(2)}$ .... etc.

From definition of the median we know that the probability that a random X_i will be larger than (or smaller than) the median is ½. The probability that the median is between $X_{(c)}$ and $X^{(c)}$ for a given c can be calculated.

The statement ($X_{(c)} \leq md \leq X^{(c)}$) (denoting the true population median by md) is false if either less than c observations are above median, in which case $md > X^{(c)}$; or less than c observations are below median, in which case $md < X_{(c)}$. Hence, by adding appropriate binomial probabilities we can obtain the error probability and confidence probability.

(3) $$\text{error probability} = P(md > X^{(c)}) + P(md < X_{(c)})$$
$$= 2\sum_{a=0}^{c-1}\binom{n}{a}(\tfrac{1}{2})^n$$
$$= \sum_{a=0}^{c-1}\binom{n}{a}(\tfrac{1}{2})^{n-1}$$

(4) $$\text{confidence probability} = P(X_{(c)} \leq md \leq X^{(c)})$$
$$= 1 - \sum_{a=0}^{c-1}\binom{n}{a}(\tfrac{1}{2})^{n-1}$$

For example, to find the confidence level of the interval ($X_{(3)}$, $X^{(3)}$)

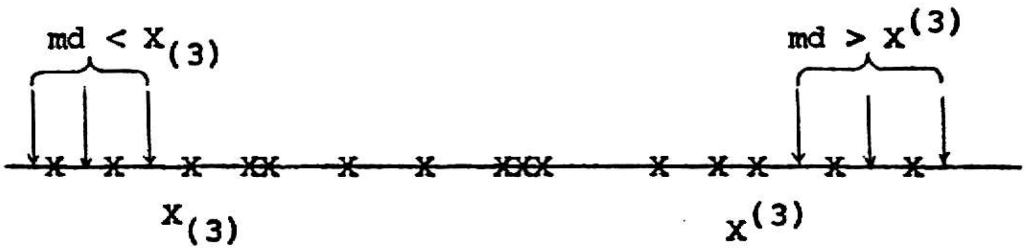

Figure 2

$$\text{error probability} = \sum_{a=0}^{2}\binom{n}{a}(\tfrac{1}{2})^{n-1}$$

$$\text{confidence probability} = 1 - \sum_{a=0}^{2}\binom{n}{a}(\tfrac{1}{2})^{n-1}$$
$$= 1 - (n^2+n+2)(\tfrac{1}{2})^n$$

The confidence coefficient for a continous distribution is exactly 1-α, but the confidence coefficient for a discrete distribution is at least 1-α. Therefore, the general formulas for confidence probabilities for intervals: $X_{(c)} \leq \text{md} \leq X^{(c)}$ are:

(5) c=1 confidence probability $\geq 1 - (\frac{1}{2})^{n-1}$

c=2 confidence probability $\geq 1 - (n+1)(\frac{1}{2})^{n-1}$

c=3 confidence probability $\geq 1 - (n^2+n+2)(\frac{1}{2})^{n}$

c=4 confidence probability $\geq 1 - (n^2+5n+6)/3(\frac{1}{2})^{n}$

$\vdots$

Table III shows the confidence level for sample sizes up to 40, and all possible intervals. If 1-α is the chosen confidence coefficient, enter the table with sample size n, find a probability closest to but not smaller than 1-α, and read the corresponding c from the top of the table.

Example: Twenty electron tubes were tested and the following life limes (hours) were reported:

7.2, 37.7, 49.6, 21.4, 67.2, 41.1, 3.8, 8.1, 23.2, 72.2,

11.4, 17.5, 29.8, 57.8, 84.6, 12.8, 2.9, 42.7, 7.4, 33.4.

Find the 95% confidence interval for the population median.

List data in increasing order:

2.9, 3.8, 7.2, 7.4, 8.1, 11.4, 12.8, 17.5, 21.4, 23.2,

29.8, 33.4, 37.8, 41.1, 42.7, 49.6, 57.8, 67.2, 72.1, 84.6.

Enter Table III with sample size n=20, the probability .96 expressed as a percentage is found under c=6. We find that $X_{(6)}$=11.4, $X^{(6)}$=42.7; therefore, the interval from 11.4 to

42.7 hours is a 96% confidence interval for the median life of electron tubes. Or, we could write:

$$CI_{96} : 11.4 \leq md \leq 42.7 \text{ hours}$$

where md is true average life of the population of electron tubes sampled.

The second method is based on the Wilcoxon signed rank test. The important assumptions of the Wilcoxon test are that the random variable X is continous and its distribution is symmetric. Thus, the left half of the graph of the probability function is the same as the right half. So the median is also the mean, because both are located exactly in the middle of the distribution at the line of symmetry. Because of these assumptions, especially the symmetry assumption, a relatively short confidence interval can be found.

The observations $X_1, X_2, \ldots\ldots X_n$ are a random sample from a population. To obtain the $1-\alpha$ confidence interval, find the critical value $W_{\alpha/2}$ from Table IV, compute all $n(n+1)/2$ possible averages of $(X_i+X_j)/2$ for all i and j, including i=j. A confidence interval is bounded by the $W_{\alpha/2}$th largest and $W_{\alpha/2}$th smallest of these averages. When the sample size is large, it is not necessary to count all the averages, compute only the averages near the largest and smallest.

This method is equvalent to a graphic procedure. Mark the heavy dots representing the sample data on the vertical axis as in Figure 3. $X_{(n)}$ is the top point denoted by A, $X_{(1)}$ is the bottom point denoted by B. A point half-way between A and B is found and marked C. On the horizontal line segment through C mark point D at some convenient distance. The line segments AD, AB, and BD form an isosceles triangle. Through each dot on the vertical draw two lines parallel to the sides of the triangle and extending to them. Each intersection is marked with a heavy dot. The number of such dots is n(n+1)/2. Triangle ABD is an isosceles triangle, and each smaller triangle on the graph is an isosceles triangle similiar to triangle ABD. Then the ordinate of each point of intersection is the average of pairs of original data points. So the n-th largest average is represented by the n-th dot from the top.

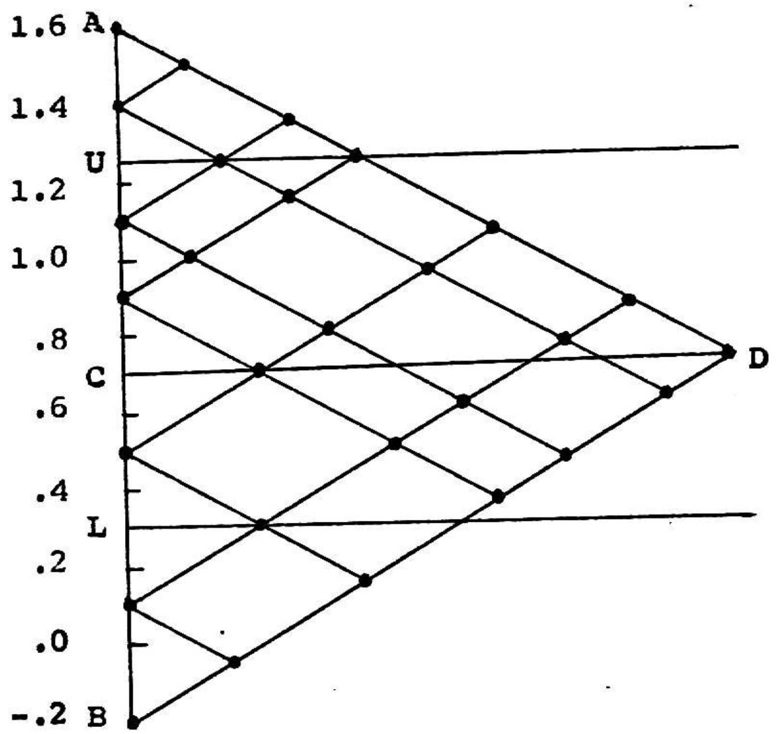

Figure 3

A confidence interval will be an interval on the vertical scale. To find it, read from Table IV the critical value $W_{\alpha/2}$ for given n. As usual, $1-\alpha$ is the confidence coefficient. Count down from the top dot A to the $W_{\alpha/2}$th dot, considering all dots in the figure. Draw a horizontal line through that dot until it intersects the vertical axis, then indicate this intersection as U (see Figure 3 where $W_{\alpha/2}=5$). Now find the $W_{\alpha/2}$th of all dots from the bottom of the figure. Draw a horizontal line through it, and indicate the intersection of this line with the vertical axis as L. The points U and L are the endpoints of the confidence interval with coefficient $1-\alpha$. If two or more observtions are equal, graph with a tiny distance between them. Say if r values are equal to the same value d, each dot on the line segment extend from d is counted r times, and dot d is counted $r(r+1)/2$ times.

Example: Suppose that 10 samples of a particular type of wire are obtained and resistances in ohms are measured, and the following values obtained:

9.0, 15.0, 13.5, 7.5, 10.5, 9.5, 12.5, 10.5, 17.5, 11.5.

Find a 90% confidence interval for the median resistance.

For a 90% confidence interval, from Table IV, $W_{\alpha/2}$ for n=10 is found to equal 11. The algebraic method of finding the confidence interval is to find the eleventh largest and the eleventh smallest averages. The largest averages are:

17.50, 16.25, 15.50, 15.00, 15.00, 14.50, 14.25, 14.00, 14.00, 13.75, <u>13.50</u>, 13.50, 13.25, ...;

and the smallest averages are:

7.50, 8.25, 8.25, 9.00, 9.00, 9.00, 9.25, 9.50, 9.50,

9.75, 9.75, 10.00,.........

Hence, the 90% confidence interval on the median resistance is from 13.5 to 9.75 ohms.

To use the graphic method, first mark heavy dots on the vertical axis representing the data, as shown in Figure 4. The mid-line is drawn through 12.5, an arbitrary point D is selected, and the line segments are drawn. Heavy dots are placed at the intersections. Circles are placed around those dots that are to be counted twice because of the two readings of 10.5, and a square is placed around the dot that is to be counted three times.

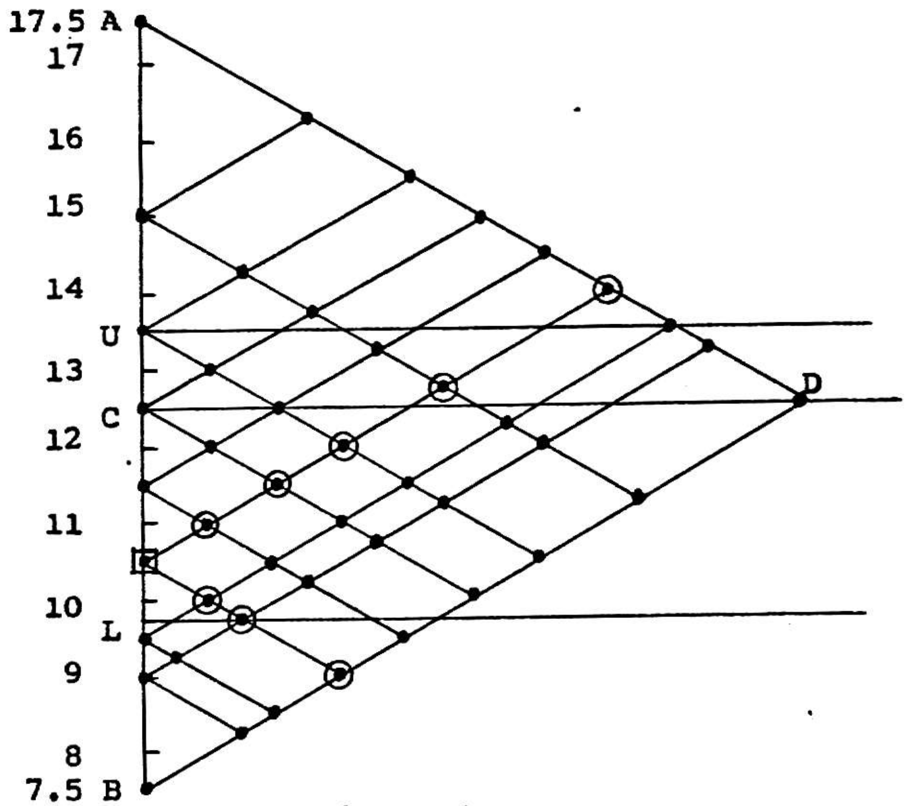

Figure 4

The eleventh dot from the top is at 13.5, and the eleventh dot from the bottom is at 9.75. Therefore, the 90% confidence interval on the median is from 9.75 to 13.50, the same answer as before.

Another method for obtaining a confidence interval on the median is to use the binomial distribution. Table III is entered with n=10. The probability 0.89 is selected as the nearest to 0.90 in Table III. The value of c associated with it is c=3. $X_{(3)}$ is equal to 9.5, $X^{(3)}$ is equal to 13.5. Therefore, The confidence interval found by this method is from 9.5 to 13.5, which agrees quite well with Wilcoxon method.

The third method is provided by John E. Walsh (1949). He proposed a significance test for the population median, which can also be used to find the confidence interval on the median. This method is efficient for small sample size, and can be applied with little computation. If the n sample observations are arrenged in increasing order, the two-sided confidence interval has the form

(6) $$P(\min(X_h,\tfrac{1}{2}(X_m+X_n)) \leq md \leq \max(X_r,\tfrac{1}{2}(X_s+X_t))) \geq 1-\alpha$$

The one-sided confidence intervals are of the forms

(7) $$P(md \geq \min(X_h,\tfrac{1}{2}(X_m+X_n))) \geq 1-\alpha$$

and

(8) $$P(md \leq \max(X_r,\tfrac{1}{2}(X_s+X_t))) \geq 1-\alpha$$

where h, m, n, r, s, t are the values used in Table V with proper sample size and confidence level.

Example: An experiment is made to study the effect of certain hormones on the growth of cucumber seeds. Following are the lengths in certimeters of the roots developed from the seeds grown in the presence of a certain small concentration of the hormone, in increasing order of size:

7.2, 8.0, 8.1, 8.7, 11.5, 12.1, 12.8, 13.2, 15.2.

Find the 95% confidence interval for the median length of those roots.

Table V is entered with n=9, in the symmetrical column we find 4.3% is closest to 5%. Reading from that row we see that the median is expected to be between $\max(X_7, \frac{1}{2}(X_5+X_9))$ and $\min(X_3, \frac{1}{2}(X_1+X_5))$. It is found that:

$$\min(X_3, \tfrac{1}{2}(X_1+X_5)) = \min(8.1,\ 9.35) = 8.1$$

$$\max(X_7, \tfrac{1}{2}(X_5+X_9)) = \max(12.8,\ 13.35) = 13.35$$

Therefore the approximate 95% confidence interval on the true median length of roots is between 8.1 and 13.35 cms.

CONFIDENCE BAND FOR POPULATION DISTRIBUTION FUNCTION

Let X_1, X_2, ... X_n be mutually independent random variables with the common but unknown distribution function F(x). Here the interest is to obtain a confidence band on F(x). Let S(x) be the empirical cumulative probability distribution function for the n observations. Thus S(x) is a step function of x which shows the proportion of X_i's whose values are less than or equal to x for arbitrary x. If k of the X_i's are equal to or less than x then S(x)=k/n. Clearly, for any value of x from $-\infty$ to $+\infty$, S(x) must be an integer multiplied by 1/n, and ranges from 0 to 1.

A confidence band can be based on the largest distance between S(x) and F(x) measured in a vertical direction. The statistic is in the form

(1) $$d_\alpha = \max|S(x) - F(x)|$$

This is the statistic suggest by Kolmogorov (1933). The distance of d_α (also the distribution of $\max|S(x)-F(x)|$) is tabled as Table VI.

To form a confidence band graphically for F(x) with confidence coefficient $1-\alpha$, first draw a praph of the empirical distribution function S(x) based on the random sample. Find the critical value d of the Kolmogorov test statistic from Table VI for the two-sided test and for the appropriate sample size n. Then $P(\max|S(x)-F(x)| \geq d_\alpha)=\alpha$ or contraiwise, $P(\max|S(x)-F(x)| < d_\alpha)=1-\alpha$. This means that we can state with $1-\alpha$ confidence that F(x) will be within $+d_\alpha$ of $S(x)$. Thus we may plot two step functions $S(x)+d_\alpha$, called $U(x)$, and $S(x)-d_\alpha$, called $L(x)$; and expect that the unknown function $F(x)$ lies entirely within the enclosing area with confidence level $1-\alpha$. The two step functions $U(x)$ and $L(x)$ together constitute a $1-\alpha$ confidence band on $F(x)$. Because $F(x)$ and $S(x)$ both are cumulative functions, $U(x)$ and $L(x)$ must be bounded by 1 and 0 so the band obtained is limited to that range. The formal mathematical definitions of $U(x)$ and $L(x)$ are as follows:

(2) $$U(x) = S(x) + d_\alpha \qquad \text{if } S(x) + d_\alpha \leq 1$$
$$U(x) = 1 \qquad \text{if } S(x) + d_\alpha > 1$$

(3) $$L(x) = S(x) - d_\alpha \qquad \text{if } S(x) - d_\alpha \geq 0$$
$$L(x) = 0 \qquad \text{if } S(x) - d_\alpha < 0$$

The Kolmogorov statistic for the discrete data was investigated by John Walsh (1963) who showed that the confidence coefficient for a band is exactly $1-\alpha$ if $F(x)$ is continous, otherwise the confidence coefficient is at least as large as $1-\alpha$.

Let d_α^+ be the critical value for a one-sided test corresponding exactly to the significande level α. Then by the same reasoning as above, we may state with confidence level $1-\alpha$ that $F(x)$ lies entirely below $S(x)+d_\alpha^+$.

Example: A random sample of size 10 is drawn and arranged in increasing order of magnitude as follows:

9.1, 10.3, 11.6, 12.4, 12.5, 13.0, 13.6, 14.2, 16.1 and 18.7.

We wish to find a 90% confidence band for $F(x)$.

The empirical cumulative probability function $S(x)$ is shown by the solid line in Figure 5. Entering Table VI with $n=10$, $\alpha=.10$, we find that $d_{.10}=.369$. Then one dotted line is drawn parallel to the sample line in Figure 5 and 0.369 above it. A second dotted line is drawn parallel to the sample line and 0.369 below it. We may now assert with confidence measured by .90 that the cumulative distribution from which our sample was drawn lies inside the band bounded by these dotted lines.

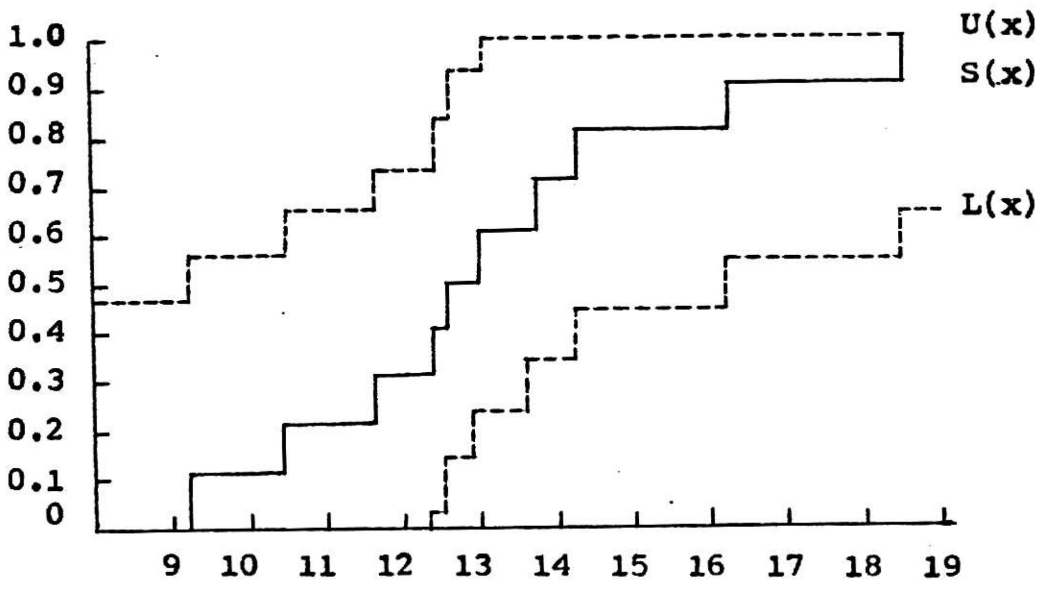

Figure 5

TOLERANCE LIMITS

Suppose a random sample of size n is drawn from a population having a continous cumulative distribution. Let the sample values arranged in order of increasing magnitude be $X_{(1)}, X_{(2)}, \ldots\ldots X_{(n)}$. Let the proportion of the population which is included between $X_{(r)}$ (the r-th smallest value in the sample) and $X_{(n-s+1)}$ (the s-th largest value) be greater or equal to p. Also, let $1-\alpha$ be the probability that the interval $(X_{(r)}, X_{(n-s+1)})$ will contain the proportion p or more of the population. Then the random interval $(X_{(r)}, X_{(n-s+1)})$ has probability $1-\alpha$ of containing at least 100p% of the population and is called the $1-\alpha$ tolerance interval for 100p per cent of the population. $X_{(r)}$ amd $X_{(n-s+1)}$ are called the $1-\alpha$ tolerance limits for 100p per cent of the population. The quantity p is between 0 and 1 and has been called the population coverage. The restriction to the continous distribution function has been removed by Scheffe and Tukey (1942); hence if the distribution is not continous the statement of the tolerance limit must be "the probability is at least $1-\alpha$ that at least the proportion p of the population is between $X_{(r)}$ and $X_{(n-s+1)}$".

The one-sided tolerance limit can be obtained by using the same theory as in finding the confidence interval of quartiles, already shown in previous sections. The one-sided tolerance limits are of the form:

(1) $P(\text{at least } p \text{ of the population} \leq X_{(n-s+1)}) \geq 1 - \alpha$

or

(2) $P(X_{(r)} \leq \text{at least } p \text{ of the population}) \geq 1 - \alpha$

The statement "at least p of the population $\geq X_{(n-s+1)}$" is saying each X has the probability p of being smaller than or equal to $X_{(n-s+1)}$. Therefore, we can write:

(3) $P(\text{at least } p \text{ of the population} \leq X_{(n-s+1)})$

$$= \sum_{a=0}^{n-s} \binom{n}{a} p^a (1-p)^{n-a}$$

Rewrite the right side of (3) as

(4) $$\sum_{a=0}^{n-s} \binom{n}{a} p^a (1-p)^{n-a} = 1 - \sum_{a=n-s+1}^{n} \binom{n}{a} p^a (1-p)^{n-a}$$

because the sum of the binomial probabilities equals unity. Again, rewrite the right side of equation (4) as

(5) $$1 - \sum_{a=n-s+1}^{n} \binom{n}{a} p^a (1-p)^{n-a} = 1 - \sum_{a=0}^{s-1} \binom{n}{a} (1-p)^a p^{n-a}$$

because the probability of n-s+1 or more successes equals the probability of s-1 or fewer failures. Combine equations (3), (4) and (5) we get

(6) $$1 - \sum_{a=0}^{s-1} \binom{n}{a} (1-p)^a p^{n-a} \geq 1 - \alpha$$

and also can be written as

(7) $$\sum_{a=0}^{s-1} \binom{n}{a} (1-p)^a p^{n-a} \leq \alpha$$

The other one-sided tolerance limit is shown in Equation (2), and equals

(8) $P(X_{(r)} \leq$ at least p of the population)

$= 1 - P($at least 1-p of the population $< X_{(r)})$

Combine Equations (7) and (8),

(9) $P($at least 1-p of the population $< X_{(r)}) \leq \alpha$

In this case each X has probability 1-p of being smaller than $X_{(r)}$. Therefore

(10) $P($at least 1-p of the population $< X_{(r)})$

$$= \sum_{a=0}^{r-1} \binom{n}{a} (1-p)^a p^{n-a}$$

so that

(11) $$\sum_{a=0}^{r-1} \binom{n}{a} (1-p)^a p^{n-a} \leq \alpha$$

With the aid of calculus, we can obtain the following formula to express the relations between r, s, n and p for two-sided tolerance limit and for both types of one-sided tolerance limits.

(12) $$\sum_{a=0}^{r+s-1} \binom{n}{a} (1-p)^a p^{n-a} \leq \alpha$$

This formula still holds if either r=0 or s=0, provided we interpret $X_{(0)}$ to be lower bound on X, or to be $-\infty$ if no lower bound exists. When s=0, we interpret $X_{(n+1)}$ to be the upper bound on X, or to be $+\infty$ if no upper bound exists. There will be one-sided tolerance limits when either r=0 or s=0. In other words, we can obtain an upper tolerance limit when r=0 and a lower tolerance limit when s=0.

It is obvious that Equation (12) depends on the sum r+s. The most popular values of r+s are 1 and 2.

R. B. Murphy (1948) presented a graph of p as a function of n for r+s from 1 to 100 and for confidence coefficients $1-\alpha=.90$, .95, .99. There are three tables: Table VIIIA for $1-\alpha=.99$, Table VIIIB for $1-\alpha=.95$ and Table VIIIC for $1-\alpha=.90$. The most common use of the graphs is in the prediction of p in sampling from a distribution. If our interest is in how large should be the sample size so that we can be ninety per cent sure that p (already a known value) proportion of the population is between $X_{(r)}$ and $X_{(n-s+1)}$, enter Table VIIIC with m=r+s and appropriate p and read off the n.

The graphs can be used to find either the value of p or the sample size, while the above method is only good to find the sample size. Here we recommend use of Murphy's graph to solve the tolerance limits problems.

Example: Suppose in the mass production of a certain type of screw one is interested in the least proportion of all screws manufactured that have lengths between the smallest and largest lengths appearing in a random sample of 100 screws. At the level of .99 we obtain from Table VIIIA p equals 93.5%. If we draw a random sample of 100 screws and find the smallest and largest screw lengths to be 1.40 and 1.60 inches respectively, we may say with at least 99% confidence that at least 93.5% of the all screws from the population sampled have lengths between 1.40 and 1.60 inches.

Now assume one wants to find the proper length of box for shipping those manufactured screws. One desires to obtain an

When $r+s=1$, Equation (12) is reduced to $p^n \le \alpha$ or $n \le \ln\alpha/\ln p$. This corresponds to the case where a one-sided tolerance limit only is desired and the largest (or smallest) sample value is taken as that limit. Table VIIA gives values of n which solve this equation for given values of p and $1-\alpha$.

When $r+s=2$, Equation (12) reduces to $np^{n-1}-(n-1)p^n \le \alpha$. This corresponds to the case where the largest and the smallest sample values are taken as the tolerance limits, i.e., we can say that at least 100p% of the population is between the largest and smallest sample values or equal to one of them. Table VIIB gives values of n for given values of p and $1-\alpha$ when $r+s=2$.

If $r+s=1$ or 2 we can read n directly from Table VIIA or Table VIIB. If those tables are not appropriate, use the following approximation:

$$(13) \qquad n = \tfrac{1}{4}\chi^2_{1-\alpha}\,\frac{1+p}{1-p} + \tfrac{1}{2}(r+s-1)$$

where $\chi^2_{1-\alpha}$ is the $1-\alpha$ critical value of a chi-square random variable with $2(r+s)$ degrees of freedom.

Another method of finding tolerance limits is obtained by using K. Pearson's incomplete Beta-distribution. Wilks (1942) noted that the probability of p proportion of a population being in the interval $(X_{(r)}, X_{(n-s+1)})$ is

$$(14) \qquad 1-\alpha = 1 - I_p(n-r-s+1, r+s)$$

where

$$I_x(k,m) = \int_0^x t^{k-1}(1-t)^{m-1}dt \Big/ \int_0^1 t^{k-1}(1-t)^{m-1}dt$$

is the incomplete Beta-distribution tabulated by K. Pearson.

upper and lower limit, with which one is ninety per cent certain, that the box is adequate for 90% of the manufactured screws. What must n be so that $X_{(1)}$ and $X_{(n)}$ furnish the upper and lower limits? Examining the intersection of the curves in Table VIIIC with the line p=.90 and m=r+s=2, we get n=38. Alternatively Table VIIB can be entered with 1-α=.90, p=.90 to obtain the value n=38 again.

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APPENDIX

Table IA Binomial Distribution, p = .75

n \ m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	300	*																			
2	063	436	*																		
3	016	156	578	*																	
4	004	051	262	687	*																
5	001	016	104	367	763	*															
6		005	038	169	466	822	*														
7		001	013	071	244	555	867	*													
8			004	027	114	321	633	900	*												
9			001	010	049	166	399	700	925	*											
10				004	020	078	224	474	756	944	*										
11				001	008	034	115	287	545	803	958	*									
12					003	014	054	158	351	609	842	968	*								
13					001	006	024	080	206	416	667	873	976	*							
14						002	010	038	113	258	479	719	899	982	*						
15						001	004	017	057	148	314	539	764	920	987	*					
16							002	007	027	080	190	370	595	803	937	990	*				
17							001	003	012	040	107	235	426	647	836	950	993	*			
18								001	005	019	057	139	283	481	694	865	961	995	*		
19									002	009	029	077	175	332	535	737	889	969	996	*	
20									001	004	014	041	102	214	383	585	775	909	976	997	*

Derived from Table 3 by W. J. Conover, "Practical nonparametric statistics", 1971.

Table IB Binomial Distribution, p = .50

n \ m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	500	*																			
2	250	750	*																		
3	125	500	875	*																	
4	063	313	688	938	*																
5	031	188	500	812	969	*															
6	016	109	344	656	891	984	*														
7	008	062	227	500	773	938	992	*													
8	004	035	145	363	637	855	965	996	*												
9	002	020	090	254	500	746	910	980	998	*											
10	001	011	055	172	377	623	828	945	989	999	*										
11		006	033	113	274	500	726	887	967	994	*										
12		003	019	073	194	387	613	806	927	981	997	*									
13		002	011	046	133	291	500	709	867	954	989	998	*								
14		001	006	029	090	212	395	605	788	910	971	994	999	*							
15			004	018	059	151	304	500	696	849	941	982	996	*							
16			002	011	038	105	227	402	598	773	895	962	989	998	*						
17			001	006	025	072	166	315	500	685	834	928	975	994	999	*					
18			001	004	015	048	119	240	407	593	760	881	952	985	996	999	*				
19				002	010	032	084	180	324	500	676	820	916	968	990	998	*				
20				001	006	021	058	132	253	412	588	748	868	942	979	994	999	*			

Derived from Table 3 by W. J. Conover, "Practical nonparametric statistics", 1971.

Table IC Binomial Distribution, p = .25

n \ m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	750	*														
2	563	938	*													
3	422	844	984	*												
4	316	738	949	996	*											
5	237	633	896	984	999	*										
6	178	534	831	962	995	*										
7	133	445	756	929	987	999	*									
8	100	367	679	886	973	996	*									
9	075	300	601	834	951	990	999	*								
10	056	244	526	776	922	980	996	*								
11	042	197	455	713	885	966	992	999	*							
12	032	158	391	649	842	946	986	997	*							
13	024	127	333	584	794	920	976	994	999	*						
14	018	101	281	521	742	888	962	990	998	*						
15	013	080	236	461	686	852	943	983	996	999	*					
16	010	063	197	405	630	810	920	973	993	998	*					
17	008	050	164	353	574	765	893	960	988	997	999	*				
18	006	039	135	306	519	717	861	943	981	995	999	*				
19	004	031	111	263	465	668	825	923	971	991	998	*				
20	003	024	091	225	415	617	786	898	959	986	996	999	*			

Derived from Table 3 by W. J. Conover, "Practical nonparametric Statistics", 1971.

Table II Normal Distribution

z_p	p	z_p	p	z_p	p
-3.719	.0001	-.468	.32	.524	.70
-3.291	.0005	-.440	.33	.554	.71
-3.090	.001	-.413	.34	.583	.72
-2.576	.005	-.385	.35	.613	.73
-2.326	.01	-.359	.36	.643	.74
-2.170	.015	-.332	.37	.675	.75
-2.054	.02	-.306	.38	.706	.76
-1.960	.025	-.279	.39	.739	.77
-.1881	.03	-.253	.40	.772	.78
-1.751	.04	-.228	.41	.806	.79
-1.645	.05	-.202	.42	.842	.80
-1.555	.06	-.176	.43	.878	.81
-1.476	.07	-.151	.44	.915	.82
-1.440	.075	-.126	.45	.954	.83
-1.405	.08	-.100	.46	.994	.84
-1.341	.09	-.075	.47	1.036	.85
-1.282	.10	-.050	.48	1.080	.86
-1.227	.11	-.025	.49	1.126	.87
-1.175	.12	.000	.50	1.175	.88
-1.126	.13	.025	.51	1.127	.89
-1.080	.14	.050	.52	1.282	.90
-1.036	.15	.075	.53	1.341	.91
-.995	.16	.100	.54	1.405	.92
-.954	.17	.125	.55	1.440	.925
-.915	.18	.151	.56	1.476	.93
-.878	.19	.176	.57	1.555	.94
-.842	.20	.202	.58	1.645	.95
-.806	.21	.228	.59	1.751	.96
-.772	.22	.253	.60	1.881	.97
-.739	.23	.279	.61	1.960	.975
-.706	.24	.306	.62	2.054	.98
-.675	.25	.332	.63	2.170	.985
-.643	.26	.359	.64	2.326	.99
-.613	.27	.385	.65	2.576	.995
-.583	.28	.413	.66	3.090	.999
-.553	.29	.440	.67	3.291	.9995
-.524	.30	.468	.68	3.719	.9999
-.496	.31	.496	.69		

From W. J. Conover, "Practical nonparametric statistics", 1971.

Table III

Confidence Coefficients (in percentages) for Confidence Limits on the Median Using Extremes of Samples of Size n

{Sample: $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(6)} \leq \cdots \leq X^{(6)} \cdots \leq X^{(2)} \leq X^{(1)}$}.

$c (\leq n/2)$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	75																			
4	88	38																		
5	94	62																		
6	97	78	31																	
7	98	88	55																	
8	99	93	71	27																
9		96	82	49																
10		98	89	66	25															
11		99	94	77	45															
12		99	96	85	61	23														
13			98	91	73	42														
14			99	94	82	58	21													
15			99	96	88	70	39													
16				98	92	79	55	20												
17				99	95	86	67	37												
18				99	97	90	76	52	18											
19					98	94	83	64	35											
20					99	96	88	74	50	18										
21					99	97	92	81	62	34										
22						98	95	87	71	48	17									
23						99	96	91	79	60	32									
24						99	98	94	85	69	46	16								
25							99	96	89	77	58	31								
26							99	97	92	83	67	44	16							
27							99	98	95	88	75	56	30							
28								99	96	91	82	66	43	15						
29								99	98	94	86	74	54	29						
30									98	96	90	80	64	42	14					
31									99	97	93	85	72	53	28					
32									99	98	95	89	78	62	40	14				
33										99	96	92	84	70	51	27				
34										99	98	94	88	77	61	39	14			
35										99	98	96	91	82	69	50	26			
36											99	97	94	87	76	60	38	13		
37											99	98	95	90	81	68	49	26		
38												99	97	93	86	74	58	37	13	
39												99	98	95	89	80	66	48	25	
40												99	98	96	92	85	73	57	36	17

Computed by G. Milliken through APL.

Table IV Critical Values of The Wilcoxon Signed Ranks Test Statistic

	$W_{.005}$	$W_{.01}$	$W_{.025}$	$W_{.05}$	$W_{.10}$	$W_{.20}$	$W_{.30}$	$W_{.40}$	$W_{.50}$	$n(n+1)/2$
n=4	0	0	0	0	1	3	3	4	5	10
5	0	0	0	1	3	4	5	6	7.5	15
6	0	0	1	3	4	6	8	9	10.5	21
7	0	1	3	4	6	9	11	12	14	28
8	1	2	4	6	9	12	14	16	18	36
9	2	4	6	9	11	15	18	20	22.5	45
10	4	6	9	11	15	19	22	25	27.5	55
11	6	8	11	14	18	23	27	30	33	66
12	8	10	14	18	22	28	32	36	39	78
13	10	13	18	22	27	33	38	42	45.5	91
14	13	16	22	26	32	39	44	48	52.5	105
15	16	20	26	31	37	45	51	55	60	120
16	20	24	30	36	43	51	58	63	68	136
17	24	28	35	42	49	58	65	71	76.5	153
18	28	33	41	48	56	66	73	80	85.5	171
19	33	38	47	54	63	74	82	89	95	190
20	38	44	53	61	70	82	91	98	105	210

From McCornack, R. L., "Extended tables of the Wilcoxon matched pairs signed rank statistics". Journal of the American Statistical Association, 60, 1965.

Table V

Some One-sided and Symmetrical Significance Test for $n \leq 15$

n	Significance Level of Tests: Open Sided	Significance Level of Tests: Symmetrical	Tests (Symmetrical: Accept $\phi \neq \phi_0$ if either) One-sided: Accept $\phi < \phi_0$ if	One-sided: Accept $\phi > \phi_0$ if:	Approx. Efficiency for normality
4	6.2%	12.5%	$x_4 < \phi_0$	$x_1 > \phi_0$	95%
5	6.2%	12.5%	$\frac{1}{2}(x_4+x_5) < \phi_0$	$\frac{1}{2}(x_1+x_1) > \phi_0$	98%
	3.1%	6.2%	$x_4 < \phi_0$	$x_1 > \phi_0$	96%
6	4.7%	9.4%	$\max[x_4, \frac{1}{2}(x_4+x_6)] < \phi_0$	$\min[x_2, \frac{1}{2}(x_1+x_3)] > \phi_0$	97%
	3.1%	6.2%	$\frac{1}{2}(x_5+x_6) < \phi_0$	$\frac{1}{2}(x_1+x_2) > \phi_0$	98%
	1.6%	3.1%	$x_6 < \phi_0$	$x > \phi_0$	95%
7	5.5%	10.9%	$\max[x_5, \frac{1}{2}(x_5+x_7)] < \phi_0$	$\min[x_3, \frac{1}{2}(x_1+x_4)] > \phi_0$	95%
	2.3%	4.7%	$\max[x_6, \frac{1}{2}(x_5+x_7)] < \phi_0$	$\min[x_2, \frac{1}{2}(x_1+x_3)] > \phi_0$	98%
	1.6%	3.1%	$\frac{1}{2}(x_6+x_7) < \phi_0$	$\frac{1}{2}(x_1+x_2) > \phi_0$	98%
	0.8%	1.6%	$x_7 < \phi_0$	$x_1 > \phi_0$	95%
8	4.3%	8.6%	$\max[x_6, \frac{1}{2}(x_4+x_8)] < \phi_0$	$\min[x_3, \frac{1}{2}(x_1+x_5)] > \phi_0$	94.5%
	2.7%	5.5%	$\max[x_6, \frac{1}{2}(x_6+x_8)] < \phi_0$	$\min[x_3, \frac{1}{2}(x_1+x_4)] > \phi_0$	96%
	1.2%	2.3%	$\max[x_7, \frac{1}{2}(x_6+x_8)] < \phi_0$	$\min[x_3, \frac{1}{2}(x_1+x_3)] > \phi_0$	98%
	0.8%	1.6%	$\frac{1}{2}(x_7+x_8) < \phi_0$	$\frac{1}{2}(x_1+x_2) > \phi_0$	98%
	0.4%	0.8%	$x_8 < \phi_0$	$x_1 > \phi_0$	95%
9	5.1%	10.2%	$\max[x_6, \frac{1}{2}(x_4+x_9)] < \phi_0$	$\min[x_4, \frac{1}{2}(x_1+x_6)] > \phi_0$	91%
	2.2%	4.3%	$\max[x_7, \frac{1}{2}(x_5+x_9)] < \phi_0$	$\min[x_3, \frac{1}{2}(x_1+x_5)] > \phi_0$	96%
	1.0%	2.0%	$\max[x_8, \frac{1}{2}(x_5+x_9)] < \phi_0$	$\min[x_2, \frac{1}{2}(x_1+x_5)] > \phi_0$	95.5%
	0.6%	1.2%	$\max[x_8, \frac{1}{2}(x_7+x_9)] < \phi_0$	$\min[x_2, \frac{1}{2}(x_1+x_3)] > \phi_0$	99%
	0.4%	0.8%	$\frac{1}{2}(x_8+x_9) < \phi_0$	$\frac{1}{2}(x_1+x_2) > \phi_0$	98%
10	5.6%	11.1%	$\max[x_6, \frac{1}{2}(x_4+x_{10})] < \phi_0$	$\min[x_5, \frac{1}{2}(x_1+x_7)] > \phi_0$	87.5%
	2.5%	5.1%	$\max[x_7, \frac{1}{2}(x_5+x_{10})] < \phi_0$	$\min[x_4, \frac{1}{2}(x_1+x_6)] > \phi_0$	93%
	1.1%	2.1%	$\max[x_8, \frac{1}{2}(x_6+x_{10})] < \phi_0$	$\min[x_3, \frac{1}{2}(x_1+x_5)] > \phi_0$	96.5%
	0.5%	1.0%	$\max[x_9, \frac{1}{2}(x_6+x_{10})] < \phi_0$	$\min[x_2, \frac{1}{2}(x_1+x_5)] > \phi_0$	96.5%
11	4.8%	9.7%	$\max[x_7, \frac{1}{2}(x_4+x_{11})] < \phi_0$	$\min[x_5, \frac{1}{2}(x_1+x_8)] > \phi_0$	
	2.8%	5.6%	$\max[x_7, \frac{1}{2}(x_5+x_{11})] < \phi_0$	$\min[x_5, \frac{1}{2}(x_1+x_7)] > \phi_0$	89%
	1.1%	2.1%	$\max[\frac{1}{2}(x_6+x_{11}), \frac{1}{2}(x_8+x_9)] < \phi_0$	$\min[\frac{1}{2}(x_1+x_6), \frac{1}{2}(x_3+x_4)] > \phi_0$	
	0.5%	1.1%	$\max[x_9, \frac{1}{2}(x_7+x_{11})] < \phi_0$	$\min[x_3, \frac{1}{2}(x_1+x_5)] > \phi_0$	97%
12	4.7%	9.4%	$\max[\frac{1}{2}(x_4+x_{12}), \frac{1}{2}(x_5+x_{11})] < \phi_0$	$\min[\frac{1}{2}(x_1+x_9), \frac{1}{2}(x_2+x_8)] > \phi_0$	
	2.4%	4.8%	$\max[x_8, \frac{1}{2}(x_5+x_{12})] < \phi_0$	$\min[x_5, \frac{1}{2}(x_1+x_8)] > \phi_0$	
	1.0%	2.0%	$\max[x_9, \frac{1}{2}(x_6+x_{12})] < \phi_0$	$\min[x_4, \frac{1}{2}(x_1+x_7)] > \phi_0$	93.5%
	0.5%	1.1%	$\max[\frac{1}{2}(x_7+x_{12}), \frac{1}{2}(x_9+x_{10})] < \phi_0$	$\min[\frac{1}{2}(x_1+x_6), \frac{1}{2}(x_3+x_4)] > \phi_0$	
13	4.7%	9.4%	$\max[\frac{1}{2}(x_4+x_{13}), \frac{1}{2}(x_5+x_{12})] < \phi_0$	$\min[\frac{1}{2}(x_1+x_{10}), \frac{1}{2}(x_2+x_9)] > \phi_0$	
	2.3%	4.7%	$\max[\frac{1}{2}(x_5+x_{13}), \frac{1}{2}(x_6+x_{12})] < \phi_0$	$\min[\frac{1}{2}(x_1+x_9), \frac{1}{2}(x_2+x_8)] > \phi_0$	
	1.0%	2.0%	$\max[\frac{1}{2}(x_6+x_{13}), \frac{1}{2}(x_9+x_{10})] < \phi_0$	$\min[\frac{1}{2}(x_1+x_8), \frac{1}{2}(x_4+x_5)] > \phi_0$	
	0.5%	1.0%	$\max[x_{10}, \frac{1}{2}(x_7+x_{13})] < \phi_0$	$\min[x_4, \frac{1}{2}(x_1+x_7)] > \phi_0$	94.5%
14	4.7%	9.4%	$\max[\frac{1}{2}(x_4+x_{14}), \frac{1}{2}(x_5+x_{13})] < \phi_0$	$\min[\frac{1}{2}(x_1+x_{11}), \frac{1}{2}(x_2+x_{10})] > \phi_0$	
	2.3%	4.7%	$\max[\frac{1}{2}(x_5+x_{14}), \frac{1}{2}(x_6+x_{13})] < \phi_0$	$\min[\frac{1}{2}(x_1+x_{10}), \frac{1}{2}(x_2+x_9)] > \phi_0$	
	1.0%	2.0%	$\max[x_{10}, \frac{1}{2}(x_5+x_{14})] < \phi_0$	$\min[x_5, \frac{1}{2}(x_1+x_9)] > \phi_0$	90.5%
	0.5%	1.0%	$\max[\frac{1}{2}(x_7+x_{14}), \frac{1}{2}(x_{10}+x_{11})] < \phi_0$	$\min[\frac{1}{2}(x_1+x_8), \frac{1}{2}(x_4+x_5)] > \phi_0$	
15	4.7%	9.4%	$\max[\frac{1}{2}(x_4+x_{15}), \frac{1}{2}(x_5+x_{14})] < \phi_0$	$\min[\frac{1}{2}(x_1+x_{12}), \frac{1}{2}(x_2+x_{11})] > \phi_0$	
	2.3%	4.7%	$\max[\frac{1}{2}(x_5+x_{15}), \frac{1}{2}(x_6+x_{14})] < \phi_0$	$\min[\frac{1}{2}(x_1+x_{11}), \frac{1}{2}(x_2+x_{10})] > \phi_0$	
	1.0%	2.0%	$\max[\frac{1}{2}(x_6+x_{15}), \frac{1}{2}(x_{10}+x_{11})] < \phi_0$	$\min[\frac{1}{2}(x_1+x_{10}), \frac{1}{2}(x_5+x_6)] > \phi_0$	
	0.5%	1.0%	$\max[x_{11}, \frac{1}{2}(x_7+x_{15})] < \phi_0$	$\min[x_5, \frac{1}{2}(x_1+x_9)] > \phi_0$	92%

From J. E. Walsh, "Applications of some significande tests for the median which are valid under very general conditions". Journal of the American Statistical Association, 44, 1949.

Table VI Critical Values of the Kolmogorov Test Statistic

One-sided Test										
α = .10	.05	.025	.01	.005		.10	.05	.025	.01	.005
Two-sided Test										
α = .20	.10	.05	.02	.01		.20	.10	.05	.02	.01
n = 1	.900	.950	.975	.990	.995	n = 21	.226	.259	.287	.321	.344
2	.684	.776	.842	.900	.929	22	.221	.253	.281	.314	.337
3	.565	.636	.708	.785	.829	23	.216	.247	.275	.307	.330
4	.493	.565	.624	.689	.734	24	.212	.242	.269	.301	.323
5	.447	.509	.563	.627	.669	25	.208	.238	.264	.295	.317
6	.410	.468	.519	.577	.617	26	.204	.233	.259	.290	.311
7	.381	.436	.483	.538	.576	27	.200	.229	.254	.284	.305
8	.358	.410	.454	.507	.542	28	.197	.225	.250	.279	.300
9	.339	.387	.430	.480	.513	29	.193	.221	.246	.275	.295
10	.323	.369	.409	.457	.489	30	.190	.218	.242	.270	.290
11	.308	.352	.391	.437	.468	31	.187	.214	.238	.266	.285
12	.296	.338	.375	.419	.449	32	.184	.211	.234	.262	.281
13	.285	.325	.361	.404	.432	33	.182	.208	.231	.258	.277
14	.275	.314	.349	.390	.418	34	.179	.205	.227	.254	.273
15	.266	.304	.338	.377	.404	35	.177	.202	.224	.251	.269
16	.285	.295	.327	.366	.392	36	.174	.199	.221	.247	.265
17	.250	.286	.318	.355	.381	37	.172	.196	.218	.244	.262
18	.244	.279	.309	.346	.371	38	.170	.194	.215	.241	.258
19	.237	.271	.301	.337	.361	39	.168	.191	.213	.238	.255
20	.232	.265	.294	.329	.352	40	.165	.189	.210	.235	.252

From H. L. Miller, "Table of percentage points of Kolmogorov statistics". Journal of the American Statistical Association, 51, 1956.

Table VIIA Sample Sizes for One-sided Tolerance Limits

1-α	p=.500	.700	.750	.800	.850	.900	.950	.975	.980	.990
.500	1	2	3	4	5	7	14	28	35	69
.700	2	4	5	6	8	12	24	48	60	120
.750	2	4	5	7	9	14	28	55	69	138
.800	3	5	6	8	10	16	32	64	80	161
.850	3	6	7	9	12	19	37	75	94	189
.900	4	7	9	11	15	22	45	91	144	230
.950	5	9	11	14	19	29	59	119	149	299
.975	6	11	13	17	23	36	72	146	183	368
.980	6	11	14	18	25	38	77	155	194	390
.990	7	13	17	21	29	44	90	182	228	459
.995	8	15	19	24	33	51	104	210	263	528
.999	10	20	25	31	43	66	135	273	342	688

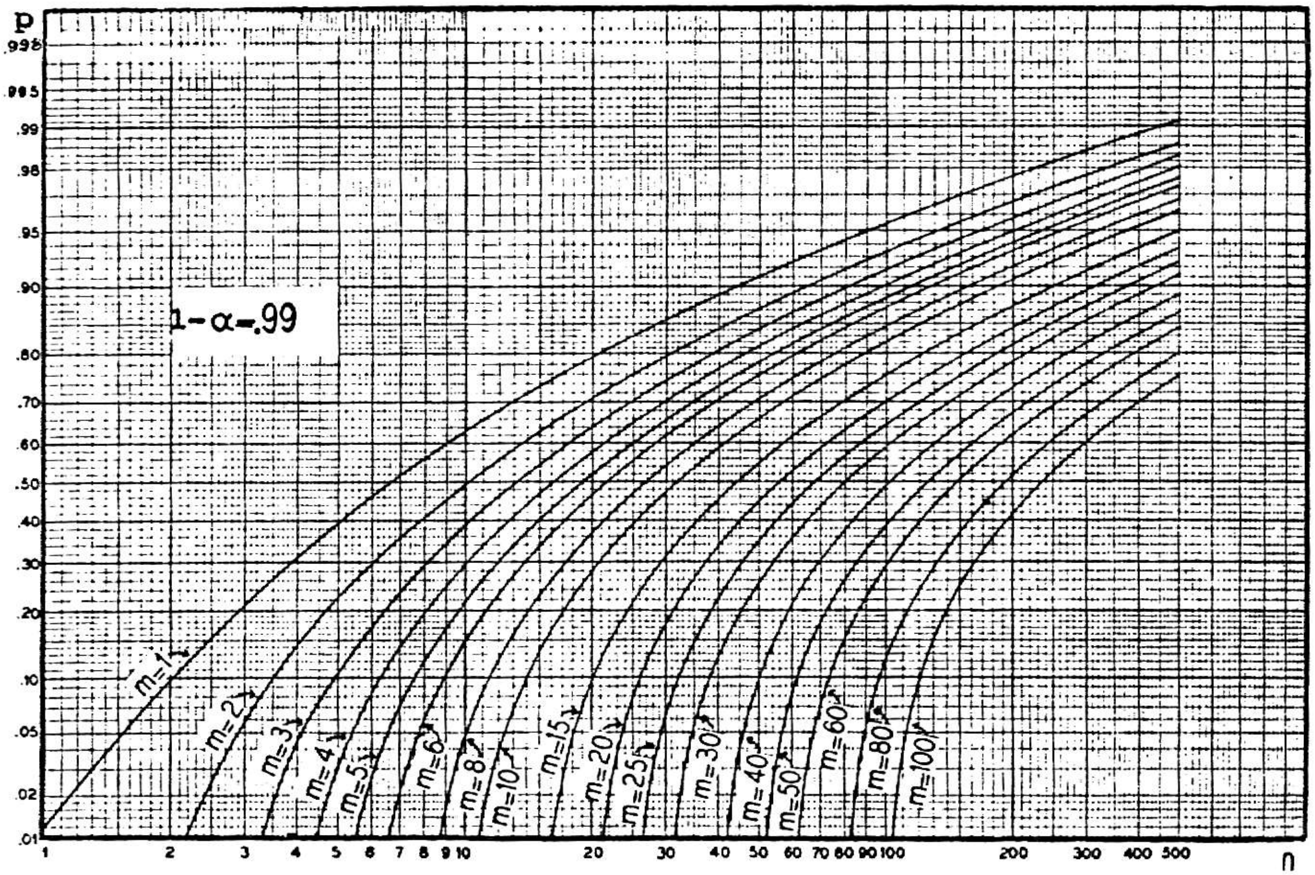
From W. J. Conover, "Practical nonparametic statistics", 1971.

Table VIIB Sample Sizes for Two-sided Tolerance Limits

1-α	p=.500	.700	.750	.800	.850	.900	.950	.975	.980	.990
.500	3	6	7	9	11	17	34	67	84	168
.700	5	8	10	12	16	24	49	97	122	244
.750	5	9	10	13	18	27	53	107	134	269
.800	5	9	11	14	19	29	59	119	149	299
.850	6	10	13	16	22	33	67	134	168	337
.900	7	12	15	18	25	38	77	155	194	388
.950	8	14	18	22	30	46	93	188	236	473
.975	9	17	20	26	35	54	110	221	277	555
.980	9	17	21	27	37	56	115	231	290	581
.990	11	20	24	31	42	64	130	263	330	662
.995	12	22	27	34	47	72	146	294	369	740
.999	14	27	33	42	58	89	181	366	458	920

From W. J. Conover, "Practical nonparametric statistics", 1971.

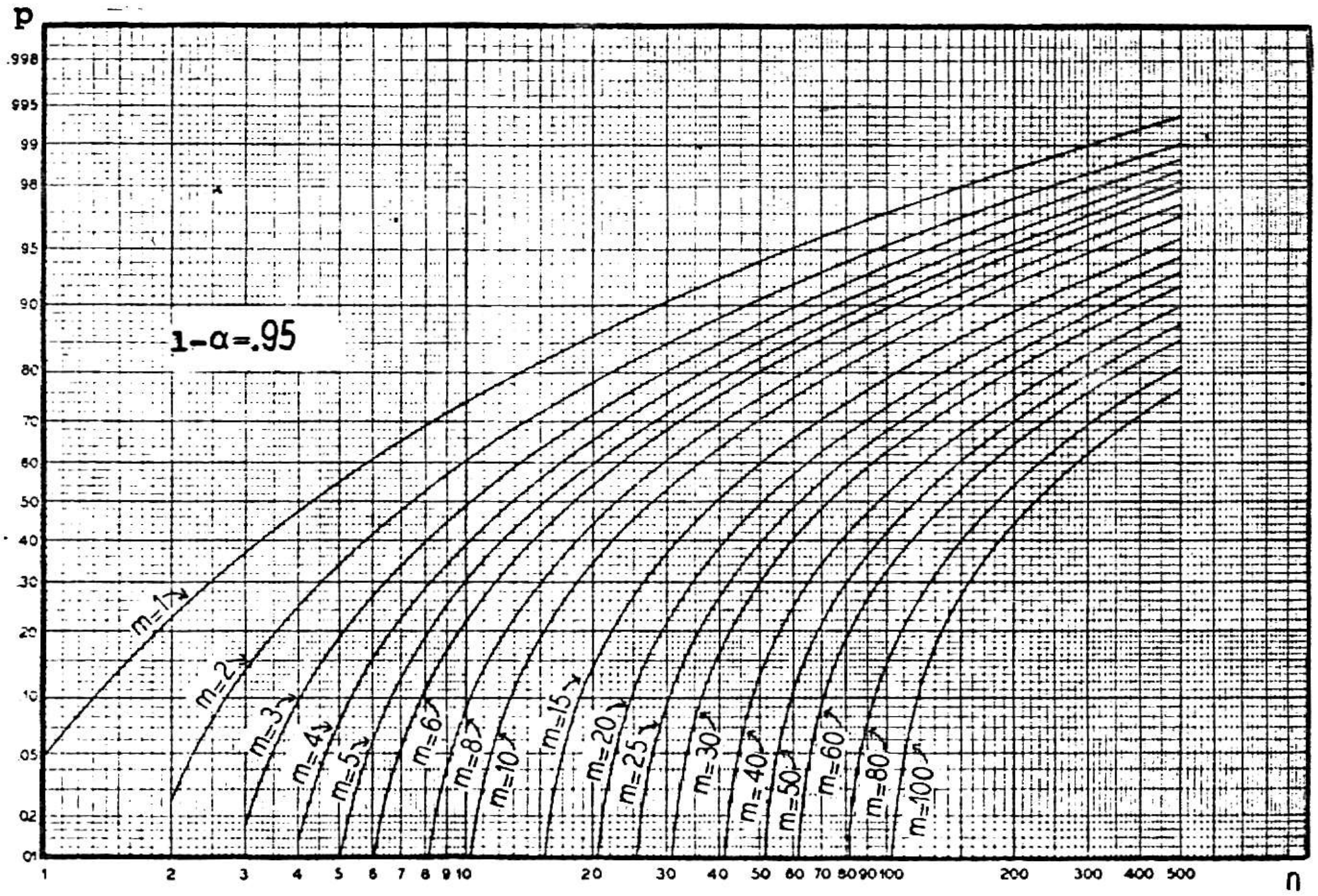
Table VIIIA Graphs of Population Coverage for Tolerance Levels .99



m = r+s

From R. B. Murphy, "Nonparametric tolerance limits". The Annals of Mathematical Statistics, 19, 1948.

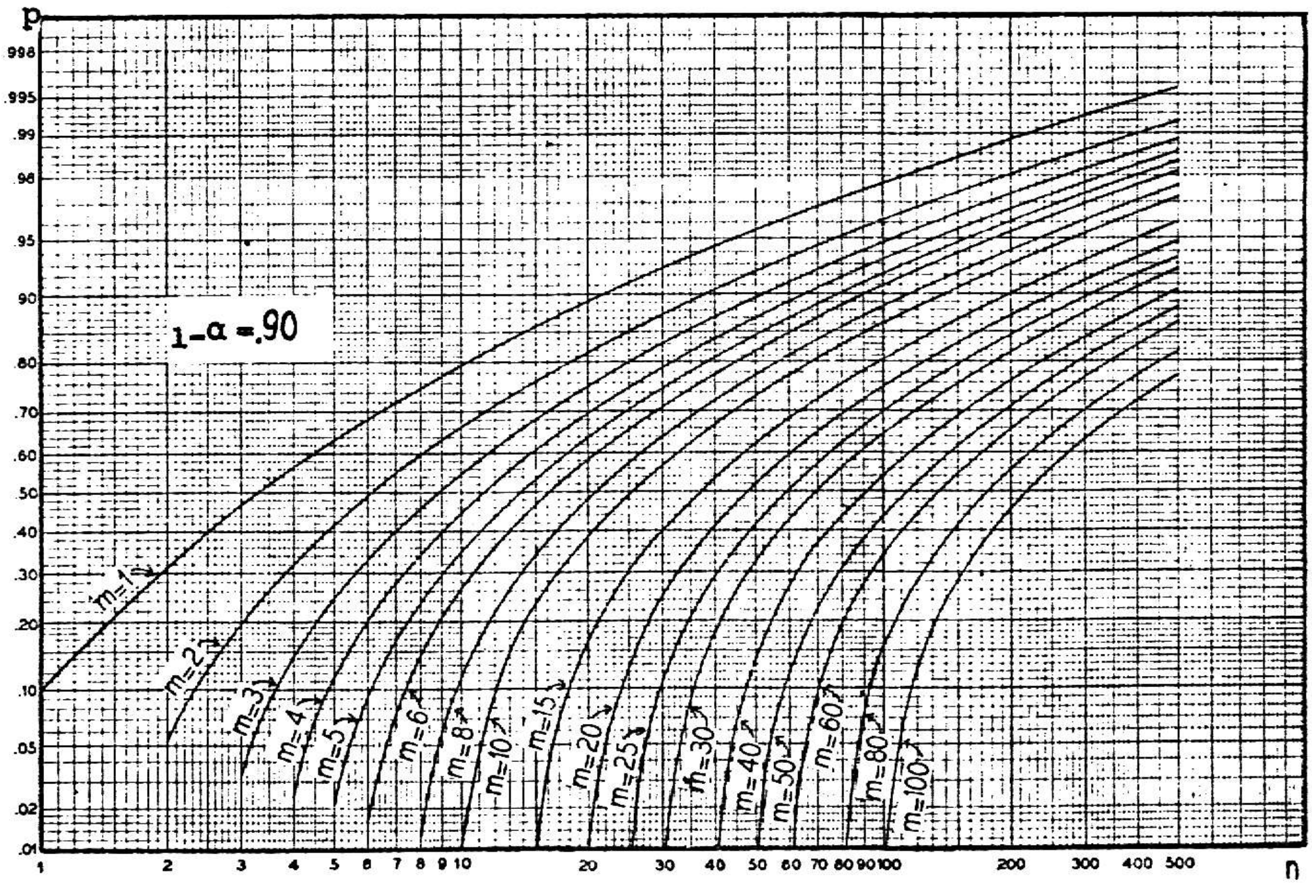
Table VIIIB Graphs of Population Coverage for Tolerance Level .95



m = r+s

From R. B. Murphy, "Nonparametric tolerance limits". The Annals of Mathematical Statistics, 19, 1948.

Table VIIIC Graphs of Population Coverage for Tolerance Level .90



m = r+s

From R. B. Murphy, "Nonparametric tolerance limits". The Annals of Mathematical Statistics, 19, 1948.

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DISTRIBUTION-FREE LIMITS INVOLVING RANDOM VARIABLES

by

TZE-U CHO

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AN ABSTRACT OF A MASTER'S REPORT

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ABSTRACT

Most of the theory of statistics inference has been developed for samples known or assumed to have originated from populations having normal, multinormal and other specified forms of distribution. However, there are many situations encountered where it is not possible to specify the functional form of the populations distribution. And it is in these situations that the distribution-free methods are particularly useful.

The main effort of this paper is directed toward an orderly codification and relevant discussions of these distribution-free methods which are adopted mainly for the purpose of finding out the confidence limits of certain population to which no suitable assumption can be made. In general, there are three types of such confidence limits. Specifically, they are:

1. Confidence interval for an unknown quartile (especially the median).
2. Confidence band for an unknown cumulative distribution function.
3. Population tolerance limits.

Each of these methods is explaned and evaluated with respect to its problem-solving potential. Formulas and tables for getting the pertinent answer are thus compiled.