# Riesz representation theorem

by

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# Abstract

This master's report presents a proof of the Riesz Representation Theorem for Hilbert spaces. The proof is developed from the basic geometry of inner product spaces, beginning with the Cauchy-Bunyakowski-Schwartz Inequality, the norm induced by an inner product, and the Parallelogram Law, and then using the Closest Point Theorem and the Orthogonal Projection Theorem. The report also discusses two applications of the Riesz Representation Theorem: the existence and basic properties of adjoint operators, and a special case of the Radon-Nikodym Theorem for finite measure spaces in which one measure is bounded above by a constant multiple of another.

# Table of Contents

# 1 Inner Product Spaces

Let $V$ be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. An **inner product** on $V$ is a map

$$\langle \cdot \mid \cdot \rangle : V \times V \to \mathbb{K}, \qquad (v, w) \mapsto \langle v \mid w \rangle,$$

satisfying the following properties:

(i) Positive Definiteness:

$\forall v \in V, \quad \langle v \mid v \rangle \geq 0, \quad$ and $\quad \langle v \mid v \rangle = 0 \Longleftrightarrow v = 0;$

(ii) Linearity (Additivity and Conjugate Homogeneity):

$\forall v \in V, \quad$ the map $V \ni w \mapsto \langle v \mid w \rangle \in \mathbb{K}$ is $\mathbb{K}$-linear;

(iii) Conjugate Symmetry:

$\forall v, w \in V, \quad \langle v \mid w \rangle = \overline{\langle w \mid v \rangle}.$

Whenever $\langle \cdot \mid \cdot \rangle$ is an inner product on $V$, we call the pair $(V, \langle \cdot \mid \cdot \rangle)$ an **inner product space**.

Here, for $\mathbb{K} = \mathbb{C}$, we use the convention that the inner product is linear in the second argument and conjugate-linear in the first argument (that is, it is sesquilinear). For $\mathbb{K} = \mathbb{R}$, the inner product is bilinear (linear in both arguments) and conjugate symmetry is ordinary symmetry.

The first fundamental result in an inner product space is the **Cauchy-Bunyakowski-Schwartz Inequality**. Its statement and proof are as follows.

**Theorem 1** (Cauchy-Bunyakowski-Schwartz Inequality)**.** *If $\langle \cdot \mid \cdot \rangle$ is an inner product on a vector space $V$ (over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ), then* $\quad |\langle v \mid w\rangle|^2 \leqslant \langle v \mid v\rangle \cdot \langle w \mid w\rangle, \quad \forall v, w \in V.$

*Proof.* Consider the function $f : \mathbb{R} \to \mathbb{C}$, defined by $f(t) = \langle tv + w \mid tv + w\rangle$.

On the one hand, we have $f(t) \geq 0 \quad \forall t \in \mathbb{R}$ by Property (i).

On the other hand, we have

$$\begin{aligned}
f(t) &= \langle tv + w \mid tv + w\rangle \\
&= \langle tv \mid tv + w\rangle + \langle w \mid tv + w\rangle \\
&= \langle tv \mid tv\rangle + \langle tv \mid w\rangle + \langle w \mid tv\rangle + \langle w \mid w\rangle \\
&= t^2\langle v \mid v\rangle + t\langle v \mid w\rangle + t\langle w \mid v\rangle + \langle w \mid w\rangle \\
&\qquad \text{by conjugate linearity and the fact that } t \in \mathbb{R} \implies \bar{t} = t. \\
&= t^2\langle v \mid v\rangle + t\langle v \mid w\rangle + t\overline{\langle v \mid w\rangle} + \langle w \mid w\rangle \\
&\qquad \text{since } \langle w \mid v\rangle = \overline{\langle v \mid w\rangle} \text{ by conjugate symmetry.} \\
&= \underbrace{\langle v \mid v\rangle}_{a} t^2 + \underbrace{\left(\langle v \mid w\rangle + \overline{\langle v \mid w\rangle}\right)}_{b} t + \underbrace{\langle w \mid w\rangle}_{c} \\
&= at^2 + bt + c,
\end{aligned}$$

where $a = \langle v \mid v\rangle \geq 0$ and $b = \langle v \mid w\rangle + \overline{\langle v \mid w\rangle} = 2\,\mathrm{Re}\,(\langle v \mid w\rangle) \in \mathbb{R}$. Hence, $f(t) = at^2 + bt + c$ with real coefficients.

Since $f(t) \geq 0$ for all $t \in \mathbb{R}$, one of the following must happen

(a) $a = 0$, in which case $f$ is linear and non-negative, which forces $b = 0$; or

(b) $a > 0$, in which case $f$ is quadratic and non-negative, which forces

$$b^2 - 4ac \leq 0. \tag{1.1}$$

Note that (1.1) also holds true in case (a).

Substituting $a, b$, and $c$, we get $(2\,\mathrm{Re}\,(\langle v \mid w\rangle))^2 \leq 4\langle v \mid v\rangle \cdot \langle w \mid w\rangle$, which implies

$$(\mathrm{Re}\,(\langle v \mid w\rangle))^2 \leq \langle v \mid v\rangle \cdot \langle w \mid w\rangle. \tag{1.2}$$

Now,

- If $\mathbb{K} = \mathbb{R}$, then $\langle v \mid w\rangle \in \mathbb{R}$, and so $\mathrm{Re}\,(\langle v \mid w\rangle) = \langle v \mid w\rangle$. Hence $(\mathrm{Re}\,(\langle v \mid w\rangle))^2 = \langle v \mid w\rangle^2 = |\langle v \mid w\rangle|^2$.

  Therefore, (1.2) becomes

  $$|\langle v \mid w\rangle|^2 \leq \langle v \mid v\rangle \cdot \langle w \mid w\rangle.$$

- If $\mathbb{K} = \mathbb{C}$, since (1.2) holds for all vectors, we have

  $$(\mathrm{Re}\,(\langle v \mid \lambda w\rangle))^2 \leq \langle v \mid v\rangle\cdot\langle \lambda w \mid \lambda w\rangle = |\lambda|^2\langle v \mid v\rangle\cdot\langle w \mid w\rangle, \quad \forall v, w \in V, \quad \lambda \in \mathbb{C}. \tag{1.3}$$

  If we fix for the moment $v, w \in V$, using trigonometric form, we can find $\theta \in \mathbb{R}$ such that $\langle v \mid w\rangle = |\langle v \mid w\rangle| e^{i\theta}$, so if we let $\lambda = e^{-i\theta}$ in (1.3), the left-hand side of (1.3) is $\left(\mathrm{Re}\,\left(\langle v \mid e^{-i\theta} w\rangle\right)\right)^2 = \left(\mathrm{Re}\,\left(e^{-i\theta}\langle v \mid w\rangle\right)\right)^2 = (\mathrm{Re}\,(|\langle v \mid w\rangle|))^2 = |\langle v \mid w\rangle|^2$, while the right-hand side is $\langle v \mid v\rangle \cdot \langle w \mid w\rangle$. $\square$

The following result shows that the Cauchy-Bunyakowski-Schwartz Inequality allows us to endow an inner product space with a normed vector space structure. Recall that a **norm** on a vector space $V$ over $\mathbb{K}$ is a map $\|\cdot\| : V \to [0, \infty), \quad v \mapsto \|v\|$, satisfying the following properties:

(i) Positive Definiteness:

$\forall v \in V, \quad \|v\| \geq 0, \quad$ and $\|v\| = 0 \Longleftrightarrow v = 0$;

(ii) Absolute Homogeneity:

$\forall \alpha \in \mathbb{K},\ v \in V, \quad \|\alpha v\| = |\alpha| . \|v\|$;

(iii) Triangle Inequality:

$$\forall v, w \in V, \quad \|v + w\| \leq \|v\| + \|w\|.$$

A vector space equipped with a norm is called a **normed vector space**.

**Theorem 2** (Norm Induced by an Inner Product)**.** *If $\langle \cdot \mid \cdot \rangle$ is an inner product on a vector space $V$ (over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ), then the function $\|\cdot\| : V \to [0, \infty)$ given by*

$$\|v\| = \sqrt{\langle v \mid v \rangle}$$

*defines a norm on $V$.*

*Proof.* To prove the positive definiteness property, we notice that since $\|v\| = \sqrt{\langle v \mid v \rangle}$ and $\langle v \mid v \rangle \geq 0$ with $\langle v \mid v \rangle = 0 \iff v = 0$, it follows that $\|v\| \geq 0$ and $\|v\| = 0 \iff v = 0$.

To prove the absolute homogeneity property, let $\alpha \in \mathbb{K}$. Then we have

$$\|\alpha v\|^2 = \langle \alpha v \mid \alpha v \rangle = \overline{\alpha} \alpha \langle v \mid v \rangle = |\alpha|^2 \langle v \mid v \rangle = |\alpha|^2 \cdot \|v\|^2.$$

Thus (by taking square roots),

$$\|\alpha v\| = |\alpha| \cdot \|v\|.$$

Finally, to prove the triangle inequality property, we first note that for $\|v\| = \sqrt{\langle v \mid v \rangle}$ and $\|w\| = \sqrt{\langle w \mid w \rangle}$, the Cauchy-Bunyakowski-Schwartz Inequality becomes $|\langle v \mid w \rangle| \leq \|v\| \cdot \|w\|$. We have

$$\begin{aligned}
\|v + w\|^2 &= \langle v + w \mid v + w \rangle \\
&= \langle v \mid v + w \rangle + \langle w \mid v + w \rangle \\
&= \langle v \mid v \rangle + \langle v \mid w \rangle + \langle w \mid v \rangle + \langle w \mid w \rangle \\
&= \|v\|^2 + \|w\|^2 + \langle v \mid w \rangle + \overline{\langle v \mid w \rangle} \\
&= \|v\|^2 + \|w\|^2 + 2\,\mathrm{Re}(\langle v \mid w \rangle).
\end{aligned}$$

Now, $\mathrm{Re}(\langle v \mid w\rangle) \le |\langle v \mid w\rangle|$ since $\mathrm{Re}(z) \le |z| \quad \forall z \in \mathbb{C}$.

Hence, we get $\mathrm{Re}(\langle v \mid w\rangle) \le |\langle v \mid w\rangle| \le \|v\| \cdot \|w\|$.

Therefore,

$$\begin{aligned} & \|v+w\|^2 \le \|v\|^2 + \|w\|^2 + 2\|v\| \cdot \|w\| = (\|v\| + \|w\|)^2 \\ \implies \quad & \|v+w\| \le \|v\| + \|w\|. \end{aligned} \qquad \square$$

The norm induced by an inner product satisfies the following property.

**Theorem 3** (Parallelogram Law). *If $(V, \langle \cdot \mid \cdot \rangle)$ is an inner product space, then the induced norm satisfies the equality:*

$$\|v+w\|^2 + \|v-w\|^2 = 2\big(\|v\|^2 + \|w\|^2\big), \quad \forall v, w \in V.$$

*Proof.* Using the definition of the norm, all we have to do is add the identities:

$$\begin{aligned} \|v+w\|^2 &= \|v\|^2 + \|w\|^2 + \langle v \mid w\rangle + \langle w \mid v\rangle \qquad \text{and} \\ \|v-w\|^2 &= \|v\|^2 + \|w\|^2 - \langle v \mid w\rangle - \langle w \mid v\rangle. \end{aligned} \qquad \square$$

# 2 Hilbert Spaces and the Closest Point Theorem

A normed vector space $(V, \|\cdot\|)$ carries a natural distance function

$$dist(v, w) = ||v - w||.$$

More generally, for a subset $S \subset V$, and $v \in V$, one defines

$$dist(v, S) = \inf\{||v - w|| : w \in S\}.$$

As usual (when working in a **metric space**), a sequence $(a_n)_n$ is declared to be **convergent to $\boldsymbol{a \in V}$** if, for all $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$n \geq N \Longrightarrow \|a_n - a\| < \varepsilon.$$

With this notation in mind, the **norm topology** on $(V, \|\cdot\|)$ is built by declaring $S \subset V$ **closed** if whenever $(a_n) \subset S$ converges to $a$, it follows that $a \in S$. Equivalently, whenever $dist(a, S) = 0$, it follows that $a \in S$.

As in the case of general metric spaces, given normed vector spaces $(V, \|\cdot\|)$ and $(W, \|\cdot\|)$, a map $F : (V, \|\cdot\|) \to (W, \|\cdot\|)$ is **continuous at $\boldsymbol{a \in V}$** if whenever $a_n \to a$, it follows that $f(a_n) \to f(a)$.

A **Banach space** is a normed vector space $(V, \|\cdot\|)$ in which every **Cauchy sequence** is convergent. Recall that $(a_n)_n$ is a Cauchy sequence in $(V, \|\cdot\|)$ if, for all $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$m, n \geq N \Longrightarrow \|a_n - a_m\| < \varepsilon.$$

A **Hilbert space** is an inner product vector space $(V, \langle \cdot \mid \cdot \rangle)$ over $\mathbb{R}$ or $\mathbb{C}$, which is a Banach space when equipped with the norm induced by $\langle \cdot \mid \cdot \rangle$.

A subset $C \subset V$ is called **convex** if, for all $v, w \in C$ and all $t \in [0, 1]$, we have

$$tv + (1-t)w \in C.$$

That is, the straight line segment connecting any two points in $C$ stays entirely inside $C$.

**Theorem 4** (Closest Point Theorem)**.** *Suppose $V$ is a Hilbert space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\emptyset \neq C \subset V$ is a closed convex set. Then for every $v \in V$, there exists a unique $v_0 \in C$ such that*

$$\|v - w\| \geq \|v - v_0\|, \qquad \forall w \in C. \tag{2.1}$$

*Proof.* Existence:

Fix $v \in V$. If $v \in C$, we let $v_0 = v$ and so for all $w \in C$, $\|v - w\| \geq 0 = \|v - v_0\|$. If $v \notin C$, let $\delta = \mathrm{dist}(v, C) = \inf\{\|v - w\| : w \in C\} > 0$. Then, by the definition of infimum, there exists a sequence $(w_n)_{n=1}^{\infty} \subset C$ such that $\|v - w_n\| \longrightarrow \delta$. We claim that $(w_n)_{n=1}^{\infty}$ is a Cauchy sequence. Indeed, since $C$ is convex, we have

$$\frac{w_m + w_n}{2} \in C, \qquad \forall m, n \in \mathbb{N}.$$

Hence, $\forall m, n \in \mathbb{N}$,

$$\begin{aligned}
&\left\| v - \frac{w_m + w_n}{2} \right\| \geq \delta \\
\implies &\left\| \frac{1}{2}(2v - (w_m + w_n)) \right\| \geq \delta \\
\implies &\|2v - (w_m + w_n)\| \geq 2\delta \\
\implies &\|(v - w_m) + (v - w_n)\| \geq 2\delta \\
\implies &\|(v - w_m) + (v - w_n)\|^2 \geq 4\delta^2. \qquad (2.2)
\end{aligned}$$

Now, by the Parallelogram Law,

$$\begin{aligned}
&\|(v - w_m) + (v - w_n)\|^2 + \|(v - w_m) - (v - w_n)\|^2 = 2\left(\|v - w_m\|^2 + \|v - w_n\|^2\right) \\
\implies &\|(v - w_m) + (v - w_n)\|^2 + \|w_n - w_m\|^2 = 2\left(\|v - w_m\|^2 + \|v - w_n\|^2\right) \\
\implies &\|w_n - w_m\|^2 = 2\left(\|v - w_m\|^2 + \|v - w_n\|^2\right) - \|(v - w_m) + (v - w_n)\|^2.
\end{aligned}$$

Thus, by (2.2),

$$\|w_n - w_m\|^2 \leq 2\left(\|v - w_m\|^2 + \|v - w_n\|^2\right) - 4\delta^2.$$

As $m, n \to \infty$, $\|v - w_m\|^2 \to \delta^2$ and $\|v - w_n\|^2 \to \delta^2$. Hence, $\|w_n - w_m\|^2 \to 0$. Therefore, $\|w_n - w_m\| \to 0$ as $m, n \to \infty$. Thus, $(w_n)_{n=1}^{\infty}$ is Cauchy.

Since $V$ is a Hilbert space, it is complete. Thus, every Cauchy sequence converges, and so $\|w_n - v_0\| \to 0$ for some $v_0 \in V$.

Since $C$ is closed and $w_n \in C$ for every $n$, we get that $v_0 \in C$.

Since $w_n \to v_0$, we have $v - w_n \to v - v_0$. Hence, by continuity of the norm, $\|v - v_0\| = \lim_{n\to\infty} \|v - w_n\| = \delta$.

Therefore, $\|v - w\| \geq \delta = \|v - v_0\|$. Thus, proving existence.

Uniqueness:

Let $u_0 \in C$ be another point satisfying (2.1). Then $\|v - u_0\| = \delta$.

Since $C$ is convex, $\frac{u_0+v_0}{2} \in C$. Therefore,

$$\begin{aligned}
&\left\| v - \frac{u_0+v_0}{2} \right\| \geq \delta \\
\implies &\|2v - (u_0+v_0)\| \geq 2\delta \\
\implies &\|(v-u_0) + (v-v_0)\| \geq 2\delta \\
\implies &\|(v-u_0) + (v-v_0)\|^2 \geq 4\delta^2. \qquad (2.3)
\end{aligned}$$

Now, by the Parallelogram Law,

$$\begin{aligned}
&\|(v-u_0) + (v-v_0)\|^2 + \|(v-u_0) - (v-v_0)\|^2 = 2\left(\|v-u_0\|^2 + \|v-v_0\|^2\right) \\
\implies &\|(v-u_0) + (v-v_0)\|^2 + \|v_0-u_0\|^2 = 2\left(\|v-u_0\|^2 + \|v-v_0\|^2\right) \\
\implies &\|v_0-u_0\|^2 = 2\left(\|v-u_0\|^2 + \|v-v_0\|^2\right) - \|(v-u_0) + (v-v_0)\|^2.
\end{aligned}$$

Hence, by (2.3),

$$\|v_0-u_0\|^2 \leq 2\left(\|v-u_0\|^2 + \|v-v_0\|^2\right) - 4\delta^2 = 2\left(\delta^2+\delta^2\right) - 4\delta^2 = 0.$$

Since $\|v_0-u_0\|^2 \geq 0$, the reverse inequality above forces $\|v_0-u_0\|^2 = 0$, which by the definition of the norm yields $v_0 = u_0$. $\square$

As an application of the Closest Point Theorem, we obtain the Orthogonal Projection Theorem. Recall that if $X \subset V$ is a linear subspace, then a vector $y \in V$ is said to be **orthogonal** to $X$, written $y \perp X$, if $\langle y \mid x \rangle = 0, \quad \forall x \in X$.

**Theorem 5** (Orthogonal Projection Theorem)**.** *Suppose $V$ is a Hilbert space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $X \subset V$ is a closed linear subspace. Then for every $v \in V$, there exists a unique $v_0 \in X$ such that*

$$v - v_0 \perp X.$$

*Proof.* Existence:

Since $X$ is a linear subspace, it is convex.

Since $X$ is closed and convex, we can apply the Closest Point Theorem with $C = X$. Hence for all $v \in V$, there exists a unique $v_0 \in X$ such that

$$\|v - w\| \geq \|v - v_0\|, \qquad \forall w \in X. \tag{2.4}$$

We claim that $v - v_0 \perp X$.

Indeed, for fixed $0 \neq x \in X$ and for all $t \in \mathbb{R}$, consider the function defined by

$$g : \mathbb{R} \to \mathbb{R}, \qquad g(t) = \|v - (v_0 - tx)\|^2 = \|v - v_0 + tx\|^2.$$

On the one hand, (2.4) implies that $\forall t \in \mathbb{R}$,

$$\begin{aligned} &\|v - (v_0 - tx)\| \geq \|v - v_0\|, \\ \implies &\|v - (v_0 - tx)\|^2 \geq \|v - v_0\|^2. \end{aligned}$$

Hence, $g(t) \geq g(0), \quad \forall t \in \mathbb{R}$. That is, $g(t)$ attains its global minimum value at $t = 0$.

On the other hand, exactly as in the proof of the Cauchy-Bunyakowski-Schwartz Inequality, we have

$$g(t) = \|v - v_0 + tx\|^2 = \langle v - v_0 + tx \mid v - v_0 + tx \rangle = at^2 + bt + c,$$

where $a = \|x\|^2$, $b = 2\,\mathrm{Re}\,(\langle v - v_0 \mid x \rangle) \in \mathbb{R}$, and $c = \|v - v_0\|^2$. Hence, $g(t) = at^2 + bt + c$ with real coefficients.

Since $g(t)$ attains its global minimum at $t = 0$, it follows that $g'(0) = 0$. That is, $2\,\mathrm{Re}\,(\langle v - v_0 \mid x\rangle) = 0$, which implies that

$$\mathrm{Re}\,(\langle v - v_0 \mid x\rangle) = 0. \tag{2.5}$$

Now,

- If $\mathbb{K} = \mathbb{R}$, then $\mathrm{Re}\,(\langle v - v_0 \mid x\rangle) = \langle v - v_0 \mid x\rangle$. Hence, $\langle v - v_0 \mid x\rangle = 0, \quad \forall x \in X$.
- If $\mathbb{K} = \mathbb{C}$, since (2.5) holds for all $x \in X$, it holds for $ix \in X$. Hence, $\mathrm{Re}\,(\langle v - v_0 \mid -ix\rangle) = 0, \quad \forall x \in X$. Thus, $0 = \mathrm{Re}\,(\langle v - v_0 \mid -ix\rangle) = \mathrm{Re}\,(-i\langle v - v_0 \mid x\rangle) = \mathrm{Im}\,(\langle v - v_0 \mid x\rangle)$. Therefore, we have $\mathrm{Im}\,(\langle v - v_0 \mid x\rangle) = 0$ and $\mathrm{Re}\,(\langle v - v_0 \mid x\rangle) = 0$ for all $x \in X$. Hence, $\langle v - v_0 \mid x\rangle = 0, \quad \forall x \in X$.

Uniqueness:

Let $u_0 \in X$ be another element that satisfies $\langle v - u_0 \mid x\rangle = 0, \quad \forall x \in X$. Since $X$ is a linear subspace, $v_0 - u_0 \in X$. Now, for all $x \in X$,

$$\langle v_0 - u_0 \mid x\rangle = \langle (v - u_0) - (v - v_0) \mid x\rangle = \langle v - u_0 \mid x\rangle - \langle v - v_0 \mid x\rangle = 0 - 0 = 0.$$

Using $x = v_0 - u_0 \in X$, we get $\|v_0 - u_0\| = \langle v_0 - u_0 \mid v_0 - u_0\rangle = 0$. Therefore, $v_0 - u_0 = 0$ by positive definiteness of the norm. Hence, $v_0 = u_0$. $\square$

# 3 The Riesz Representation Theorem

Let $V$ and $W$ be vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A map $A : V \to W$ is called a **linear operator** if

$$A(\alpha v + \beta w) = \alpha Av + \beta Aw, \qquad \forall \alpha, \beta \in \mathbb{K},\ \forall v, w \in V.$$

**Proposition 1.** *Given normed vector spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, a linear operator $A : V \to W$ is **continuous** on $V$ if and only if there exists a constant $M \geq 0$ such that*

$$\|Av\|_W \leq M\|v\|_V, \qquad \forall v \in V. \tag{3.1}$$

*Proof.* For the "if" implication, we observe that if $M \geq 0$ satisfies (3.1) and $(v_n)_n$ converges to $v \in V$, then

$$\|Av_n - Av\|_W = \|A(v_n - v)\|_W \leq M\|v_n - v\|_V \longrightarrow 0.$$

Thus, $Av_n \to Av$, and so $A$ is continuous at $v$.

For the "only if" implication, we argue that the continuity of $A$ implies that $A$ is **bounded**, that is

$$\sup\{\|Av\|_W : v \in V,\ \|v\|_V \leq 1\} < \infty. \tag{3.2}$$

Indeed, if the above supremum were infinite, we could find a sequence $(v_n)_n$ in $V$ such that $\|v_n\|_V \leq 1$ while $\|Av_n\|_W \to \infty$, in which case the vectors $u_n = \frac{v_n}{\|Av_n\|_W}$ will form a sequence convergent to 0 $\left(\text{since} \quad \|u_n\|_V = \frac{\|v_n\|_V}{\|Av_n\|_W} \leq \frac{1}{\|Av_n\|_W} \to 0\right)$ while also satisfying $\|Au_n\|_W = 1$, which will make it impossible for $A$ to be continuous at 0.

Now, we are done by letting $M$ be the left-hand side of (3.2), because every $v \in V$ can be presented as $v = \|v\|_V v_0$ with $\|v_0\|_V \leq 1$. So

$$\|Av\|_W = \|v\|_V \cdot \|Av_0\|_W \leq M\|v\|_V. \qquad \square$$

For the normed vector spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$, denote the **space of all continuous linear operators** from $V$ to $W$ by

$$\mathcal{L}(V, W) = \{A : V \to W : A \text{ is linear and continuous}\}.$$

Then, for $A \in \mathcal{L}(V, W)$, the **operator norm** is defined by

$$\|A\| = \sup\{\|Av\|_W : v \in V,\ \|v\|_V \leq 1\}.$$

It is easy to see that $\|A\|$ is the smallest constant $M$ that satisfies (3.1).

A particularly important case of linear operators occurs when $W = \mathbb{K}$. A linear map $\varphi : V \to \mathbb{K}$ is called a **linear functional**. If $\varphi$ is continuous on $V$, then $\varphi \in \mathcal{L}(V, \mathbb{K})$, and the space $V^* = \mathcal{L}(V, \mathbb{K})$ is called the **continuous dual space** of $V$. The norm of $\varphi$ is given by

$$\|\varphi\| = \sup\{|\varphi(v)| : v \in V,\ \|v\|_V \leq 1\}.$$

Thus, for every $v \in V$,

$$|\varphi(v)| \leq \|\varphi\| \cdot \|v\|_V.$$

We now state and prove the main theorem of this report.

**Theorem 6** (Riesz Representation Theorem)**.** *Suppose $V$ is a Hilbert space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and*

$$\varphi : V \to \mathbb{K}$$

*is linear and continuous. Then there exists a unique vector $v \in V$ such that*

$$\varphi(w) = \langle v \mid w \rangle, \qquad \forall w \in V. \tag{3.3}$$

*Furthermore, $\|\varphi\| = \|v\|$.*

*Proof.* Existence:

If $\varphi = 0$, then take $v = 0$.

Now assume that $\varphi \neq 0$. Then there exists $v_1 \in V$ such that $\varphi(v_1) = 1$. Let

$$X = \ker \varphi = \{w \in V : \varphi(w) = 0\} = \varphi^{-1}(\{0\}).$$

Since $\varphi$ is linear, $X$ is a linear subspace of $V$. Since $\varphi$ is continuous, $X$ is closed, being the pre-image of the closed subset $\{0\} \subset \mathbb{K}$.

Since $X$ is a closed linear subspace of the Hilbert space $V$, the Orthogonal Projection Theorem applies. Hence, there exists a unique $v_0 \in X$ such that $v_1 - v_0 \perp X$. Set $z = v_1 - v_0$ so by linearity we have $\varphi(z) = \varphi(v_1) - \varphi(v_0) = \varphi(v_1) = 1$. In particular, $z \neq 0$.

We now show that the vector $v = \frac{z}{\|z\|^2}$ does satisfy (3.3).
Fix $w \in V$ and observe that the vector $\widetilde{w} = w - \varphi(w)z$ satisfies $\varphi(\widetilde{w}) = 0$. So $\widetilde{w}$ belongs to $X$.
Since $z \perp X$, it follows that

$$0 = \langle z \mid \widetilde{w} \rangle = \langle z \mid w \rangle - \varphi(w)\langle z \mid z \rangle = \langle z \mid w \rangle - \|z\|^2 \varphi(w).$$

Dividing everything by $\|z\|^2$ yields

$$\varphi(w) = \frac{1}{\|z\|^2}\langle z \mid w\rangle = \left\langle \frac{z}{\|z\|^2} \mid w \right\rangle = \langle v \mid w\rangle,$$

and we are done.

Uniqueness:

Suppose $u \in V$ is another element that satisfies (3.3). Then

$$\langle v - u \mid w\rangle = 0, \qquad \forall w \in V.$$

Taking $w = v - u$, we get $\|v - u\|^2 = \langle v - u \mid v - u\rangle = 0$. Hence, by positive definiteness, $v - u = 0$. That is, $v = u$.

We now check that $\|\varphi\| = \|v\|$. Indeed, the Cauchy-Bunyakowski-Schwartz Inequality gives

$$|\varphi(w)| = |\langle v \mid w\rangle| \leq \|v\| \cdot \|w\|, \quad \forall w \in V.$$

Hence, taking the supremum over all $w$ with $\|w\| \leq 1$, we obtain $\|\varphi\| \leq \|v\|$.
If $v \neq 0$, take $w = \frac{v}{\|v\|}$. Then $\|w\| = 1$, and

$$|\varphi(w)| = \left|\left\langle v \mid \frac{v}{\|v\|} \right\rangle\right| = \frac{1}{\|v\|}\langle v \mid v\rangle = \|v\|.$$

Thus, $\|\varphi\| \geq \|v\|$, and therefore, $\|\varphi\| = \|v\|$. If $v = 0$, then $\varphi = 0$, and the equality is immediate. $\square$

# 4 Applications of the Riesz Representation Theorem

The Riesz Representation Theorem is of fundamental importance because it completely characterizes the continuous dual of a Hilbert space in terms of the space itself. More specifically, it associates each continuous linear functional with a unique vector while preserving norms. This correspondence transforms questions about functionals into questions about vectors and inner products, making the theorem a powerful tool in Hilbert space theory. We now illustrate its usefulness through adjoint operators and a special case of the Radon-Nikodym Theorem.

**Theorem 7** (Existence and Properties of the Adjoint Operator)**.** *Let $H_1$ and $H_2$ be Hilbert spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $A \in \mathcal{L}(H_1, H_2)$. Then there exists a unique linear operator $A^* \in \mathcal{L}(H_2, H_1)$ such that*

$$\langle A^* y \mid x \rangle_{H_1} = \langle y \mid Ax \rangle_{H_2}, \qquad \forall x \in H_1,\ \forall y \in H_2. \tag{4.1}$$

*The operator $A^*$ is called the adjoint operator of $A$. Moreover, $\|A^*\| = \|A\|$ and $(A^*)^* = A$.*

*Proof.* Fix for the moment $y \in H_2$. Define $\varphi_y : H_1 \to \mathbb{K}$ by

$$\varphi_y(x) = \langle y \mid Ax \rangle_{H_2}.$$

Since $A$ is linear and the inner product is linear in the second argument, $\varphi_y$ is linear. Also, by the Cauchy-Bunyakowski-Schwartz Inequality,

$$|\varphi_y(x)| = |\langle y \mid Ax\rangle_{H_2}| \leq \|y\|_{H_2} \cdot \|Ax\|_{H_2}.$$

Since $A$ is bounded,

$$\|Ax\|_{H_2} \leq \|A\| \cdot \|x\|_{H_1}.$$

Therefore,

$$|\varphi_y(x)| \leq \|y\|_{H_2}\|A\| \cdot \|x\|_{H_1}.$$

Thus, $\varphi_y$ is a continuous linear functional on $H_1$.

By the Riesz Representation Theorem, there exists a unique vector $z_y \in H_1$ such that

$$\varphi_y(x) = \langle z_y \mid x\rangle_{H_1}, \qquad \forall x \in H_1.$$

Doing this for every $y \in H_2$, we can define the map $A^* : H_2 \ni y \longmapsto z_y \in H_1$, which satisfies (4.1).

Also, by the Riesz Representation Theorem, $A^*$ is the unique map satisfying (4.1).

We now show that $A^*$ is linear. Fix $y_1, y_2 \in H_2$ and $\alpha, \beta \in \mathbb{K}$. For every $x \in H_1$,

$$\begin{aligned}\langle A^*(\alpha y_1 + \beta y_2) \mid x\rangle_{H_1} &= \langle \alpha y_1 + \beta y_2 \mid Ax\rangle_{H_2} \\ &= \overline{\alpha}\langle y_1 \mid Ax\rangle_{H_2} + \overline{\beta}\langle y_2 \mid Ax\rangle_{H_2} \\ &= \overline{\alpha}\langle A^* y_1 \mid x\rangle_{H_1} + \overline{\beta}\langle A^* y_2 \mid x\rangle_{H_1}.\end{aligned}$$

Since the inner product is conjugate-linear in the first argument,

$$\overline{\alpha}\langle A^* y_1 \mid x\rangle_{H_1} + \overline{\beta}\langle A^* y_2 \mid x\rangle_{H_1} = \langle \alpha A^* y_1 + \beta A^* y_2 \mid x\rangle_{H_1}.$$

Hence,

$$\langle A^*(\alpha y_1 + \beta y_2) \mid x\rangle_{H_1} = \langle \alpha A^* y_1 + \beta A^* y_2 \mid x\rangle_{H_1}, \qquad \forall x \in H_1.$$

Using the uniqueness part in the Riesz Representation Theorem, applied to the continuous linear functional $\psi(x) = \langle A^*(\alpha y_1 + \beta y_2) \mid x\rangle_{H_1} = \langle \alpha A^* y_1 + \beta A^* y_2 \mid x\rangle_{H_1}, \quad \forall x \in H_1$, we obtain

$$A^*(\alpha y_1 + \beta y_2) = \alpha A^* y_1 + \beta A^* y_2,$$

thus proving the linearity of $A^*$.

Next, we prove that $A^*$ is bounded. By the norm identity in the Riesz Representation Theorem, $\|A^* y\|_{H_1} = \|\varphi_y\|$. But for every $x \in H_1$, $y \in H_2$, $|\varphi_y(x)| \leq \|A\| \cdot \|y\|_{H_2} \cdot \|x\|_{H_1}$, and therefore, $\|\varphi_y\| \leq \|A\| \cdot \|y\|_{H_2}$, which yields,

$$\|A^* y\|_{H_1} \leq \|A\| \cdot \|y\|_{H_2}, \qquad \forall y \in H_2.$$

This proves that $A^*$ is bounded, and $\|A^*\| \leq \|A\|$.

It remains to show that $(A^*)^* = A$ and $\|A^*\| = \|A\|$. By definition of the adjoint, for all $x \in H_1$ and $y \in H_2$, $\langle A^* y \mid x\rangle_{H_1} = \langle y \mid Ax\rangle_{H_2}$. By conjugate symmetry, $\langle x \mid A^* y\rangle_{H_1} = \langle Ax \mid y\rangle_{H_2}$. Now, the adjoint of $A^*$ is the unique operator $(A^*)^* \in \mathcal{L}(H_1, H_2)$ such that

$$\langle (A^*)^* x \mid y\rangle_{H_2} = \langle x \mid A^* y\rangle_{H_1}, \qquad \forall x \in H_1,\ \forall y \in H_2.$$

Combining the previous identities gives

$$\langle (A^*)^* x \mid y\rangle_{H_2} = \langle Ax \mid y\rangle_{H_2} = \overline{\varphi_y(x)}, \qquad \forall x \in H_1,\ \forall y \in H_2.$$

If we fix $x \in H_1$, then using the uniqueness part in the Riesz Representation Theorem applied to the functional $y \longmapsto \overline{\varphi_y(x)}$, we get $(A^*)^* x = Ax$. Since $x \in H_1$ was arbitrary, $(A^*)^* = A$.

Finally, since $\|A^*\| \leq \|A\|$, applying the same inequality to $A^*$ gives

$$\|A\| = \|(A^*)^*\| \leq \|A^*\|,$$

which together with $\|A^*\| \leq \|A\|$, yields $\|A^*\| = \|A\|$. $\square$

As a second application of the Riesz Representation Theorem, we prove a special case of the Radon-Nikodym Theorem in which the measure $\nu$ is dominated by the measure $\mu$.

**Theorem 8** ("Easy" Radon-Nikodym Theorem)**.** *Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Assume $M > 0$ is some constant and $\nu$ is another measure on $\mathcal{A}$ such that*

$$\nu(A) \leq M\mu(A), \quad \forall A \in \mathcal{A}.$$

*Then there exists a measurable function $h : X \to [0, \infty)$ such that*

$$\nu(A) = \int_A h d\mu, \quad \forall A \in \mathcal{A}.$$

*Moreover, $h$ is unique $\mu$-a.e. and satisfies $0 \leq h \leq M$ $\mu$-a.e.*

*Proof.* Note that by the finiteness of $\mu$ and the Cauchy-Bunyakowski-Schwartz Inequality, we have

$$\int_X |f| d\mu = \langle 1 \mid |f| \rangle_{L^2(X,\mathcal{A},\mu)} \leq \|1\|_{L^2(X,\mathcal{A},\mu)} \cdot \|f\|_{L^2(X,\mathcal{A},\mu)} < \infty, \qquad \forall f \in L^2(X, \mathcal{A}, \mu),$$

which gives the inclusion

$$L^2(X, \mathcal{A}, \mu) \subset L^1(X, \mathcal{A}, \mu).$$

We now justify that the given assumption also yields the inclusion

$$L^1(X, \mathcal{A}, \mu) \subset L^1(X, \mathcal{A}, \nu),$$

for which it suffices to show that, for every non-negative measurable function $g$, we have the inequality

$$\int_X g d\nu \leq M \int_X g d\mu. \tag{4.2}$$

If

$$s = \sum_{j=1}^{N} a_j \varkappa_{E_j}, \qquad a_j \geq 0,\ E_j \in \mathcal{A},$$

is a non-negative simple function, then by the assumption $\nu(A) \leq M\mu(A)$, for all $A \in \mathcal{A}$, we have

$$\int_X s d\nu = \sum_{j=1}^{N} a_j \nu(E_j) \leq M \sum_{j=1}^{N} a_j \mu(E_j) = M \int_X s d\mu.$$

Now let $g : X \to [0, \infty]$ be measurable. Choose a sequence $(s_n)_{n=1}^{\infty}$ of non-negative simple functions such that

$$0 \leq s_1 \leq s_2 \leq \cdots \leq g \quad \text{and} \quad s_n(x) \to g(x), \quad \forall x \in X.$$

By the Monotone Convergence Theorem,

$$\int_X g d\nu = \lim_{n\to\infty} \int_X s_n d\nu \leq M \lim_{n\to\infty} \int_X s_n d\mu = M \int_X g d\mu.$$

Using the inclusion $L^2(X, \mathcal{A}, \mu) \subset L^1(X, \mathcal{A}, \mu) \subset L^1(X, \mathcal{A}, \nu)$, we can now define the linear map $\varphi : L^2(X, \mathcal{A}, \mu) \to \mathbb{R}$ by $\varphi(f) = \int f d\nu$.

In fact, by (4.2), we have

$$\begin{aligned} |\varphi(f)| = \left| \int_X f d\nu \right| &\leq \int |f| d\nu \leq M \int |f| d\mu \\ &= M \cdot \langle 1 \mid |f| \rangle_{L^2(X,\mathcal{A},\mu)} \leq M \cdot \|1\|_{L^2(X,\mathcal{A},\mu)} \cdot \|f\|_{L^2(X,\mathcal{A},\mu)}. \end{aligned}$$

Therefore, $\varphi$ is continuous. The linearity of $\varphi$ follows from the linearity of the integral. So $\varphi$ is a continuous linear functional on $L^2(X, \mathcal{A}, \mu)$.

Now, by the Riesz Representation Theorem, $\exists! h \in L^2(X, \mathcal{A}, \mu)$ such that

$$\varphi(f) = \langle h \mid f \rangle_{L^2(X,\mathcal{A},\mu)} = \int h f d\mu, \quad \forall f \in L^2(X, \mathcal{A}, \mu).$$

Choose a measurable representative of $h$, still denoted by $h$.

Specializing to $f = \varkappa_A$, which belongs to $L^2(X, \mathcal{A}, \mu)$ since $\mu$ is finite, yields

$$\nu(A) = \int_X \varkappa_A d\nu = \varphi(\varkappa_A) = \int_X h\varkappa_A d\mu = \int_A hd\mu,$$

and hence

$$\nu(A) = \int_A hd\mu, \quad \forall A \in \mathcal{A}.$$

To show that $0 \leq h \leq M$ $\mu$-a.e., consider the sets $A_n = h^{-1}\left(\left[-\infty, -\frac{1}{n}\right]\right)$ and $B_n = h^{-1}\left(\left[M + \frac{1}{n}, \infty\right]\right)$, which satisfy $\bigcup_{n=1}^{\infty} A_n = h^{-1}([-\infty, 0))$ and $\bigcup_{n=1}^{\infty} B_n = h^{-1}((M, \infty])$, and note that

$$0 \leq \nu\left(A_n\right) \leq -\frac{1}{n}\mu\left(A_n\right).$$

(Indeed, since $h \leq -\frac{1}{n}$ on $A_n$, we have $\nu(A_n) = \int_{A_n} hd\mu \leq -\frac{1}{n}\mu(A_n)$. And because $\nu$ is a measure, $\nu(A_n) \geq 0$.)

Then

$$M\mu\left(B_n\right) \geq \nu\left(B_n\right) \geq \left(M + \frac{1}{n}\right)\mu\left(B_n\right)$$

because the first inequality follows from the assumption $\nu(A) \leq M\mu(A)$ and the second from

$$\nu(B_n) = \int_{B_n} hd\mu \geq \left(M + \frac{1}{n}\right)\mu(B_n),$$

as $h \geq M + \frac{1}{n}$ on $B_n$. These inequalities force

$$\mu\left(A_n\right) = \mu\left(B_n\right) = 0.$$

So by $\sigma$-subadditivity, we obtain

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = 0,$$

meaning precisely that

$$h \geq 0 \quad \mu - a.e. \quad \text{and}$$

$$h \leq M \quad \mu - a.e.$$

We now prove uniqueness. Suppose $p : X \to [0, \infty)$ is another measurable function such that

$$\nu(A) = \int_A p d\mu, \qquad \forall A \in \mathcal{A}, \tag{4.3}$$

and consider the partition $X = D \cup E \cup N$, where

$$D = \{x \in X : h(x) > p(x)\},$$

$$E = \{x \in X : h(x) < p(x)\}, \quad \text{and}$$

$$N = \{x \in X : h(x) = p(x)\}.$$

Applying (4.3) to $A = D$ and $A = E$, we have

$$\int_D (h - p) d\mu = \nu(D) - \nu(D) = 0,$$

$$\int_E (h - p) d\mu = \nu(E) - \nu(E) = 0,$$

and therefore,

$$\int_X |h - p| d\mu = \int_D (h - p) d\mu - \int_E (h - p) d\mu + \int_N (h - p) d\mu = 0,$$

which forces the non-negative function $|h - p|$ to vanish $\mu$-a.e. □

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