

OPTIMIZATION OF MANAGEMENT SYSTEMS  
USING THE TWO-PHASE METHOD OF  
QUASILINEARIZATION

by 4589

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A MASTER'S THESIS

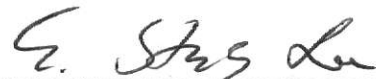
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## CHAPTER 1

### INTRODUCTION

This thesis is concerned with the optimization of deterministic, non-linear management models. The basic approach to solve such models is to determine the decision variables so that the performance criteria is optimized and the performance equations of the system are also satisfied. Quasilinearization technique has been used to solve these management models.

In the past, much work has been done to apply quasilinearization combined with the calculus of variation to trajectory optimization problems. The biggest drawback in the whole procedure is to guess the unknown profiles to start the process. If the initial guess is far from the exact profile, the procedure might not produce any solution. This difficulty occurs frequently in very unstable problems. Recent studies on this procedure are concentrated on increasing the range in which successful initial guess can be made. This work is devoted to enhance the chances of getting optimal solution to the complex industrial management problems even in the case of bad initial guess.

A brief review of the various methods is given in Chapter 2. Emphasis is placed on the modifications in and applications of quasilinearization. The procedure generally used to solve the non-linear boundary-value problems is described in Chapter 3. Chapters 4 and 5 deal with two different industrial management systems. The first management system is a four dimensional inventory and advertising scheduling problem. The second

management system is a twelve dimensional advertising and production scheduling problem. The effectiveness of the approach is summarized in the concluding chapter of this work.

## CHAPTER 2

### LITERATURE REVIEW

#### 2.1 Introduction

The theory of ordinary differential equations of the initial-value type is an extensively developed branch of mathematical analysis. However, for two-point boundary-value problems, analytical and computational difficulties are present. There are two general numerical approaches for the optimal control problem: direct methods and indirect methods. The direct methods proceeded by solving a sequence of non-optimal problems with the property that each successive set of solution functions yields an improved value for the functional being optimized. An example of such a procedure is the method of gradients which has been applied with considerable success to a variety of problems. The direct numerical optimization method was first suggested in 1960 by Kelley<sup>12</sup>. The direct methods have the inherent disadvantage of slow convergence in the neighborhood of the optimal solution. In this chapter, only indirect methods will be reviewed since this work is primarily concerned with one of such methods.

#### 2.2 Indirect Methods

The indirect methods are concerned with finding, by numerical means, a set of functions which satisfy the necessary conditions for an extremal. Tapley et al.<sup>31</sup> mentioned that indirect methods were suggested as early as 1949 by Hestenes<sup>10</sup> who applied a calculus of variations approach to the study of time-optimal solutions to a fixed end point problem.



The indirect numerical methods can be divided into two groups. The main difference is in the way the reference trajectory is generated. In the perturbation methods, the reference trajectory is generated by integrating the non-linear differential equations. In the quasilinearization methods the same is achieved by integration of the linearized differential equations.

The perturbation methods are divided into two classes: the methods that use perturbation equations and the methods that use adjoint equations. Breakwell et al.<sup>5</sup> described a scheme using perturbation equations for solving the non-linear two-point boundary-value problems. The scheme is based on a linear perturbation from a nominal optimum path and involves the second variation of the calculus of variations. Tapley et al.<sup>31</sup> mentioned that Goodman and Lance<sup>7</sup> used the method of adjoint systems for solving a non-linear, two-point boundary-value problem with fixed terminal time, and Jurovics and McIntyre<sup>11</sup> extended the method to handle variable terminal time problems. Roberts and Shipman<sup>25</sup> described a continuation procedure which, when applied to the method of adjoints can solve problems normally beyond the capacity of the method of adjoints.

The quasilinearization methods are reviewed in the next section.

### 2.3 Quasilinearization Method

The Newton-Raphson Method as an algorithm for solving boundary-value problems was first suggested by Hestenes<sup>10</sup>, who called it "differential variations." It was developed further by Bellman and Kalaba<sup>3</sup>, who refer to the technique as "quasilinearization." One of the problems discussed by Bellman et al.<sup>3</sup> is the non-linear second-order differential equation with two-point boundary-conditions:

$$\ddot{x} = g(x, t) \quad (0 \leq t \leq b) \quad (2.1)$$

$$x(0) = c \text{ and } \dot{x}(b) = 0$$

The quasilinearization scheme consists in producing a sequence of approximating functions  $x_n$ ,  $n = 0, 1, 2, \dots$ , which converge rapidly to the solution of the original problem. Assuming that the  $n$ th approximation is known, we find the  $(n+1)^{\text{th}}$  approximation as the solution of the linear two-point boundary-value problem

$$\ddot{x}_{n+1} = g(x_n) + (x_{n+1} - x_n) \dot{g}(x_n)$$

$$x_{n+1}(0) = c \text{ and } \dot{x}_{n+1}(b) = 0$$

where the right hand side of the equation is obtained by taking the first two terms in the Taylor series expansion of the function  $g(x_{n+1})$  about the point  $x_n$ .

The following generalized Newton-Raphson Method has been extensively applied to various optimization problems<sup>3,23</sup>. The system of non-linear differential equation is

$$\dot{X} - F(X, t) = 0 \quad (t_0 \leq t \leq t_f) \quad (2.2)$$

where  $X$  and  $F$  are  $N$  dimensional vectors.

The following sequence of linear problems is obtained

$$\dot{X}_{n+1} = J(X_n, t) (X_{n+1} - X_n) + F(X_n, t) \quad (n = 1, 2, \dots)$$

where  $J(X, t)$  is the Jacobian matrix of partial derivatives of  $F$  with respect to  $X$ .

Thus the method is different from the usual indirect procedure of successively integrating the non-linear equations and adjusting arbitrary initial conditions until the remaining boundary conditions are satisfied. McGill and Kenneth<sup>23</sup> suggested to refer Breakwell et al.<sup>5</sup>, Scharmack<sup>27</sup>, and Kelley et al.<sup>13</sup> for comparison with other methods for solving the

non-linear boundary-value problem. Kopp et al.<sup>15</sup> made a direct comparison of gradient, second variation, and generalized Newton-Raphson techniques. Tabak<sup>30</sup> made a comparative study of various mathematical programming techniques and found the quasilinearization technique to be most efficient in terms of the number of iterations and the computing time required.

In 1965, Long<sup>21</sup> used a simpler procedure to solve problems having undetermined end point. He replaced the independent variable  $t$  with a new independent variable  $S = t/a$ , where the constant ' $a$ ' is adjusted so that  $0 \leq S \leq 1$ .

Now one can write

$$\dot{X} = F(X, t) \quad (2.3)$$

as

$$\dot{X} = aF(X, aS) \quad (2.4)$$

The constant ' $a$ ' is considered as a new state variable. The value of ' $a$ ' is assumed initially and corrected on each iteration. The value of  $t_f$  is given by  $t_f = a$ .

In 1966, Kenneth and McGill<sup>14</sup> dealt with the convergence theorems and applied the generalized Newton-Raphson Method to

- i) the boundary-value problems resulting from the application of the Pontryagin maximum principle.
- ii) the problems having inequality constraints on the state space.
- iii) the problems with bounded control variables.

In 1967, Lewallen<sup>20</sup> extended the method to handle terminal conditions which are general functions of the problem variables rather than just specific values of the variables, and it also had variable final time.

The choice of initial profile for the quasilinearization method is

often difficult. In many cases, bad guess poses convergence difficulties. Recent studies are oriented towards extending the range of convergence in quasilinearization method. The range of convergence is the interval within which the initial guess should be in order for the solution to converge. In 1968, Leondes and Paine<sup>19</sup> gave a method which increases the range of convergence but the rate of convergence is reduced. In this method

$$\begin{aligned} x^{k+1} &= \epsilon \bar{x}^{k+1} + (1-\epsilon) x^k & 0 \leq t \leq 1 & \quad k=1, \dots, P \\ x^{k+1} &= x^{k+1}, & k &= P+1, \dots, N \end{aligned} \quad (2.5)$$

Where  $x^k$  is the  $k^{\text{th}}$  profile,  $\bar{x}^{k+1}$  is the  $(k+1)^{\text{th}}$  profile as computed in the usual quasilinearization manner.  $x^{k+1}$  is the  $(k+1)^{\text{th}}$  profile stored and used for computing the  $(k+2)^{\text{th}}$  profile.  $P$  is chosen on discretion and it shows the stage up to which quadratic convergence is lost. Obviously it is kept as small as possible. For the theoretical proof that this method extends the range of convergence, the reader should refer to Leondes et al.<sup>19</sup>.

During the same time, Roberts et al.<sup>26</sup> described a continuation method for quasilinearization to solve unstable two-point boundary-value problems which cannot be solved directly over the specified interval  $(t_0, t_f)$ , because of instability. He observed that the two-point boundary-value problem over the smaller interval  $(t_0, t_1)$ , where the boundary conditions at  $t_f$  have been assigned to  $t_1$ , is easier to solve. A brief description of this procedure is given below

1. Use the solution of the non-linear, two-point boundary-value problem over  $(t_0, t_1)$  as the nominal profile for an application of quasilinearization over  $(t_0, t_1)$ . The solution is obtained by the perturbation method.

2. Solve the resulting linear two-point boundary-value problem over the interval  $(t_0, t_1)$ .

3. Apply the continuation idea as follows: To form the nominal profiles over the extended interval  $(t_0, t_2)$ , where  $t_2 = t_1 + \Delta t_1$ , use the solution of the linearized problem over  $(t_0, t_1)$ . The value of the variables at  $t_1$  are used for the interval  $(t_1, t_2)$ .

4. Repeat step (2) for the extended interval  $(t_0, t_2)$ .

5. Repeat steps (2) through (4) until  $t_i = t_f$ , at which time the original problem is solved.

In the same year, Heidman<sup>9</sup> combined two mathematical techniques, quasilinearization and the method of particular solutions to solve the non-linear, two-point boundary-value problems. Specifically, the non-linearity of problem is removed by quasilinearization, and the resulting linear, two-point boundary-value problem is solved by the method of particular solutions. Then, the procedure is applied iteratively until convergence to the desired solution is obtained.

Lee<sup>17</sup> has given a detailed description of the application of quasilinearization technique to the non-linear boundary-value problems under various types of constraints. He has also explored the possibility of combining maximum principle with quasilinearization. Much work has been done in the application of quasilinearization to the problems of chemical engineering and control engineering. He has combined the concept of invariant imbedding with quasilinearization to form some useful predictor-corrector formulas. Various efficient algorithms are produced which are suited for different types of boundary-value problems.

Quasilinearization technique has been successfully used in the parameter estimation problems. Those interested in such application can

refer to Lee<sup>17</sup>, Schulz<sup>28</sup>, Lee<sup>18</sup>, and Ramaker et al.<sup>24</sup>.

Shah<sup>29</sup>, apparently for the first time applied quasilinearization to the industrial management systems. A diffusion model for the advertisement problem developed by Teichroew<sup>32</sup>, and a twelve dimensional model of advertisement and production have been successfully solved by the quasilinearization technique.

Reviewing the literature it is observed that the difficult part in solving an unstable problem is to choose the initial profile to start the quasilinearization process. In this work, a recent version of the quasilinearization technique is applied to industrial management systems.

## CHAPTER 3

### QUASILINEARIZATION

#### 3.1 Introduction

Engineering problems which ultimately lead to boundary-value problems are numerous. In optimal-control studies, one invariably faces two-point boundary-value problem, whether one attacks the problem from the point of view of the classical variational calculus or by the application of the Pontryagin's maximum principle. Since these problems are, in general, non-linear, one faces analytical and numerical difficulties. Analytically, the proof for the existence and uniqueness of the solution is missing. Numerically, since all the conditions are not given at one point, one has to apply a trial and error procedure. This is done by a systematic variation of arbitrarily chosen initial conditions until the remaining boundary conditions are met. This technique has proved unsuccessful on high-speed digital computers. In many problems if the guessed value is not close enough to the exact solution, the procedure does not converge. To overcome these difficulties an algorithm has been developed which proceeds by converting a non-linear boundary-value problem into a sequence of linear boundary-value problems. These linear boundary-value problems are solved by Runge-Kutta integration scheme with the help of the superposition principle.

This chapter is devoted to the whole procedure of solving a two-point non-linear boundary-value problem. First, the initial-value problem and its solution will be described. Then the superposition principle

will be explained which deals with the linear two-point boundary-value problems. In the end quasilinearization method will be explained.

### 3.2 Numerical Solution of Initial-Value Problems

In general, the methods can be grouped into two: the single-step methods and the multiple-step methods.

The single-step methods need only one known point to start the process. Runge-Kutta method is perhaps the best known and most frequently used. Antosiewicz et al.<sup>1</sup> deal in detail with the Runge-Kutta formulas of different orders. Another popular single-step method is the Euler's formula. McCormick et al.<sup>22</sup> deal with the theory involved and the variations developed in the Euler's formula. Babuska et al.<sup>2</sup>, Antosiewicz et al.<sup>1</sup> deal with the error analysis, convergence and stability of the single-step methods.

The multiple-step methods need more than one known point to start the process. Milne's predictor-corrector formula is the one most popularly used. Weeg et al.<sup>34</sup>, Tompkins et al.<sup>33</sup>, Babuska et al.<sup>2</sup> have discussed the predictor-corrector formulas of Milne and of Adam-Bashforth. McCormick et al.<sup>22</sup> has given a computer program for a first-order ordinary differential equation. Stability of such methods is discussed by Antosiewicz et al.<sup>1</sup>, and Hamming<sup>8</sup>.

In this work the fourth-order Runge-Kutta method is used. It gives fairly accurate results. Moreover, in using the multiple-step methods the starting values must be calculated by a single-step method, but no such calculations are necessary in the fourth-order Runge-Kutta method.

### 3.3 Numerical Solution of Linear Boundary-Value Problems

The methods described in the last section cannot be applied directly



to boundary-value problems. One approach is to find the particular solution and all the homogeneous solutions of the systems. Then, by virtue of the linearity of the differential equation, the superposition principle can be used. Lee<sup>17</sup> has discussed this procedure in detail. The general solution for a system of  $n$  simultaneous first order linear differential equations with  $m$  initial conditions, where  $m < n$ , can be represented by

$$x_i(t) = x_{i,p}(t) + \sum_{k=1}^{n-m} A_k x_{i,kh}(t) \quad i = 1, 2, \dots, n \quad (3.1)$$

### 3.4 Numerical Solution of Non-linear Two-Point Boundary-Value Problem by Quasilinearization

In the last section, a brief description is given of the procedure to numerically solve the linear two-point boundary-value problem. In order to use the superposition principle the non-linear differential equations must be linearized. Quasilinearization is the procedure which serves the purpose of linearizing. Besides this, the quasilinearization technique provides a sequence of functions which in general converges quadratically to the solution of the original non-linear equations.

Consider a set of non-linear differential equations

$$\frac{dx}{dy} = f(x, y) \quad (3.2)$$

$$\frac{dy}{dt} = g(x, y) \quad t_i \leq t \leq t_f \quad (3.3)$$

with boundary conditions

$$x(t_0) = x^0 \quad \text{and} \quad y(t_f) = y^1 \quad (3.4)$$

Here  $x$  and  $y$  are functions of  $t$ .

The first two terms of the Taylor series expansion of functions

$f(x,y)$  and  $g(x,y)$  about  $x = a$  and  $y = b$  lead to

$$\frac{dx}{dt} = f(a,b) + (x-a) f_x(a,t) + (y-b) f_y(b,t) \quad (3.5)$$

$$\frac{dy}{dt} = g(a,b) + (x-a) g_x(a,t) + (y-b) g_y(b,t) \quad (3.6)$$

The symbol  $f_x(a,t)$  means the partial derivative of  $f$  with respect to  $x$  at  $x=a$ . In Equations (3.5) and (3.6), since  $a$  and  $b$  are known profiles, all the terms except  $x$  and  $y$  on the right hand side are known. Equations (3.5) and (3.6) are two linear differential equations with boundary conditions given by Equation (3.4). Equations (3.5) and (3.6) can be written in the following recurrence form.

$$\frac{dx_{n+1}}{dt} = f(x_n, y_n) + (x_{n+1} - x_n) f_x(x_n, y_n) + (y_{n+1} - y_n) f_y(x_n, y_n) \quad (3.7)$$

$$\frac{dy_{n+1}}{dt} = g(x_n, y_n) + (x_{n+1} - x_n) g_x(x_n, y_n) + (y_{n+1} - y_n) g_y(x_n, y_n) \quad (3.8)$$

where  $n = 1, 2, \dots$

In spite of the advantages already mentioned, the whole procedure involves a few drawbacks. First is the ill conditioning phenomena which may take place in solving a set of linear algebraic equations to get the integration constants. Second is the problem of convergence. If the initial profile is too approximate the solution may not converge at all.

The whole period of time will be divided into a number of phases for the sole purpose of reducing the occurrence of convergence problem. For illustration let the whole period  $t_0$  to  $t_f$  be divided into two parts,  $t_0$  to  $t_1$  and  $t_1$  to  $t_f$ . Phase 1 consists of solving the same problem in the period  $t_0$  to  $t_1$  with the condition at  $t_f$  to be taken as the condition at  $t_1$ . Some initial profile is chosen for this phase. Now quasilinearization is applied. The problem is solved and an optimum profile is

obtained for all the variables. This optimum profile is taken as the initial profile for phase 2 for the interval  $(t_0, t_f)$ .

## CHAPTER 4

### APPLICATION TO AN INVENTORY AND ADVERTISING SCHEDULING PROBLEM

#### 4.1 Introduction

In this chapter, the quasilinearization technique is applied to an inventory and advertising scheduling problem. The optimal advertising rate is obtained for the well known diffusion model<sup>32</sup>. In Section 2 of this chapter, the diffusion model is developed. Sections 3 and 4 deal with the definition and the formulation of the problem. Section 5 illustrates the quasilinearization procedure for this problem. Numerical and computational aspects of this problem are mentioned in Section 6. Sections 7 and 8 deal with results and discussion, respectively.

#### 4.2 Development of the Model

The diffusion model developed by Teichroew<sup>32</sup> is used for this problem. Consider a group of people having a constant number of persons of which only certain members possess a piece of information about a firm's product. Suppose that the diffusion of information occurs only through personal contact. In a contact, the contactee receives information only if he does not have it; if he already has it, the contact is wasted, this means that the number of informed people remains the same. The number of contacts made by an average informed person in an arbitrary unit of time is given by a contact coefficient. Let  $N$  represent the total number of persons in the group and  $c_c$  represent the contact coefficient, which is the number of contacts made by one informed person per unit time. Let

$K(t)$  represent the number of informed person at time  $t$ . Then

$$\frac{K(t)}{N} = \text{proportion of the informed persons at time } t.$$

$$\frac{N-K(t)}{N} = \text{proportion of the uninformed persons at time } t.$$

$$c_c K(t) \Delta t = \text{contacts made during the time interval } \Delta t.$$

The increase in total number of informed persons during time  $\Delta t$  is given as  $K(t) = K(t + \Delta t) - K(t) =$  the contacts made with the uninformed persons.

So,

$$\Delta K(t) = (c_c K(t) \Delta t) \left( \frac{N-K(t)}{N} \right) \quad (4.1)$$

For a very short interval of time  $d(t)$ , Equation (4.1) can be written in differential form

$$\frac{dK(t)}{dt} = c_c K(t) \left( 1 - \frac{K(t)}{N} \right) \quad (4.2)$$

Suppose next that the firm can increase the contact coefficient by spending money on advertising. This will result in an increase in the number of contacts made with the uninformed people. Let the increase in the contact coefficient be represented by  $A(t)$ . Thus,

$$\frac{dK(t)}{dt} = (c_c + A(t)) K(t) \left( 1 - \frac{K(t)}{N} \right) \quad (4.3)$$

If each informed person buys one unit of the firm's product per unit time, the sales rate  $S(t)$  can be represented by the following equation.

$$S(t) = K(t) \quad (4.4)$$

On substituting  $S(t)$  for  $K(t)$  in Equation (4.3), the following differential equation is obtained.

$$\frac{dS(t)}{dt} = S(t) (c_c + A(t)) \left( 1 - \frac{S(t)}{N} \right) \quad (4.5)$$

The rate of change of the inventory level  $I(t)$  can be represented by

$$\frac{dI(t)}{dt} = P(t) - S(t) \quad (4.6)$$

where  $P(t)$  is the production rate at time  $t$ . For simplicity the production rate is assumed to be a linear function given by

$$P(t) = a + bt \quad (4.7)$$

where  $a$  and  $b$  are known constants.

For such a system, the management has to select an optimal policy so that the profit function is maximized.

$$J = \int_{t_0}^{t_f} (c_S S(t) - c_I (I_m - I(t))^2 - c_A S(t) A^2(t)) dt \quad (4.8)$$

where  $J$  is the net profit,  $c_S$  is the revenue from the sale of one unit of the product,  $c_I$  is the inventory carrying cost,  $I_m$  is the capacity for the storage of inventory and  $c_A$  is the cost of advertising.

#### 4.3 Definition of the Problem

The problem is to find the advertising rate  $A(t)$  such that the following net profit is maximized

$$J(A) = \int_{t_0}^{t_f} (c_S S(t) - c_I (I_m - I(t))^2 - c_A S(t) A^2(t)) dt \quad (4.9)$$

subject to

$$P(t) = a + bt \quad (4.10)$$

$$\frac{dI(t)}{dt} = P(t) - S(t) \quad (4.11)$$

$$\frac{dS(t)}{dt} = S(t) (c_c + A(t)) \left(1 - \frac{S(t)}{N}\right) \quad (4.12)$$

with boundary conditions

$$I(t_0) = I^0 \quad \text{and} \quad S(t_0) = S^0 \quad (4.13)$$

The system is represented by two differential equations with two initial conditions.

#### 4.4 Formulation of the Problem

When calculus of variations is used, the above optimization problem is converted into a two-point boundary-value problem. For detailed discussion of the calculus of variations, the reader is referred to Bliss<sup>4</sup> and Elsgolc<sup>6</sup>. The resulting non-linear equations are solved with the help of the quasilinearization technique.

Equations (4.10) through (4.12) can be rewritten as

$$\frac{dI(t)}{dt} - (a+bt) + S(t) = 0 \quad (4.14)$$

$$\frac{dS(t)}{dt} - S(t) (c_c + A(t)) \left(1 - \frac{S(t)}{N}\right) = 0 \quad (4.15)$$

The above problem has two state variables,  $S$  and  $I$ , and one control variable  $A$ . To solve this problem, introduce the Lagrange multipliers  $\lambda_1(t)$  and  $\lambda_2(t)$  and the constant multipliers  $\theta_1(t)$  and  $\theta_2(t)$ . Then define the following functions.

$$F = \lambda_1(\dot{I} - (a+bt) + S) + \lambda_2\left(\dot{S} - S\left(c_c + A - \frac{Sc}{N} - \frac{SA}{N}\right) + c_S S - c_I(I_m - I)^2 - c_A S A^2\right) \quad (4.16)$$

and

$$G = \theta_1(I(0) - I^0) + \theta_2(S(0) - S^0) \quad (4.17)$$

where the notation  $\dot{I}$  represents the first derivative with respect to time,  $t$ ,  $\frac{dI}{dt}$ .

Now consider maximization of the following function

$$J(A) = \int_0^{t_f} (F(I(t), S(t), A(t), \lambda_1(t), \lambda_2(t), t)) dt \quad (4.18)$$

The Euler-Lagrange equations can be written as

$$\frac{d}{dt} \frac{\partial F}{\partial \dot{x}_i} - \frac{\partial F}{\partial x_i} = 0, \quad \frac{\partial F}{\partial A_j} = 0 \quad (4.19)$$

where  $x_{i=1}$  and  $x_{i=2}$  correspond to  $I$  and  $S$  in Equation (4.18). Since there is only one control variable, the subscript  $j = 1$ .

To obtain the relations for Lagrange multipliers  $\lambda_1$  and  $\lambda_2$ , Equation (4.19) is applied to Equation (4.16)

$$\frac{d\lambda_1}{dt} = 2c_I (I_m - I) \quad (4.20)$$

$$\begin{aligned} \frac{d\lambda_2}{dt} = & \lambda_1 + c_S - c_A A^2 - c_c \lambda_2 - \lambda_2 A + \frac{2c_c S \lambda_2}{N} \\ & + \frac{2AS \lambda_2}{N} \end{aligned} \quad (4.21)$$

$$A = \frac{\lambda_2}{2c_A} \left( \frac{S}{N} - 1 \right) \quad (4.22)$$

Since A is an explicit function of S, the control variable is eliminated from Equations (4.15), (4.16), and (4.21) by using Equation (4.22). The four performance equations can be written as

$$\frac{dI}{dt} = a + bt - S \quad (4.23)$$

$$\frac{dS}{dt} = c_c S - \frac{c_c S^2}{N} + \frac{S^2 \lambda_2}{c_A N} - \frac{S \lambda_2}{2c_A} - \frac{S^3 \lambda_2}{2c_A N^2} \quad (4.24)$$

$$\frac{d\lambda_1}{dt} = 2c_I (I_m - I) \quad (4.25)$$

$$\frac{d\lambda_2}{dt} = \lambda_1 + c_S - c_c \lambda_2 + \frac{3S^2 \lambda_2^2}{4c_A N^2} - \frac{\lambda_2^2 S}{c_A N} + \frac{\lambda_2^2}{4c_A} + \frac{2c_c S \lambda_2}{N} \quad (4.26)$$

There are four equations but only two boundary conditions given by Equation (4.13). The other two boundary conditions are obtained by applying the transversality conditions.

$$\left. \frac{\partial G}{\partial x_i} \right|_{t_i} - \left. \frac{\partial F}{\partial \dot{x}_i} \right|_{t_i} = 0$$

$$\text{or} \quad \left. \frac{\partial G}{\partial x_i} \right|_{t_f} - \left. \frac{\partial F}{\partial \dot{x}_i} \right|_{t_f} = 0 \quad (4.27)$$

Applying this condition to Equations (4.16) and (4.17), we obtain

$$\lambda_1(t_f) = 0; \quad \lambda_2(t_f) = 0 \quad (4.28)$$



Now we have four different equations (4.23) through (4.26), with two initial conditions, Equation (4.13), and two final conditions, Equation (4.28). The solution of these differential equations will give the stationary point of the problem. Whether this stationary point is a maximum or not can be determined from physical considerations.

#### 4.5 Quasilinearization

As discussed in Chapter 3, by the use of quasilinearization the non-linear differential equations can be represented by a set of linear differential equations. Once the system is represented by a set of linear differential equations, it can be solved by the combined use of the single-step method and the superposition principle. Equations (4.24) and (4.26) can be linearized by the use of Taylor series expansion. The linearized equations can be developed according to Equations (3.7) and (3.8)

$$\frac{dI_{n+1}}{dt} = a+bt - S_{n+1} \quad (4.29)$$

$$\begin{aligned} \frac{dS_{n+1}}{dt} = & (c_c S_n - \frac{c_c S_n^2}{N} + \frac{S_n^2 \lambda_{2,n}}{c_A N} - \frac{S_n^2 \lambda_{2,n}}{2c_A} - \frac{S_n^3 \lambda_{2,n}}{2c_A N^2}) \\ & + (S_{n+1} - S_n) (c_c - \frac{2c_c S_n}{N} + \frac{2S_n \lambda_{2,n}}{c_A N} - \frac{\lambda_{2,n}}{2c_A} - \frac{3S_n^2 \lambda_{2,n}}{2c_A N^2}) \\ & + (\lambda_{2,n+1} - \lambda_{2,n}) (\frac{S_n^2}{c_A N} - \frac{S_n}{2c_A} - \frac{S_n^3}{2c_A N^2}) \end{aligned} \quad (4.30)$$

$$\frac{d\lambda_{1,n+1}}{dt} = 2c_I I_m - 2c_I I_{n+1} \quad (4.31)$$

$$\begin{aligned}
\frac{d\lambda_{2,n+1}}{dt} = & (\lambda_{1,n+1} + c_S - c_c \lambda_{2,n} + \frac{3S_n^2 \lambda_{2,n}}{4c_A N^2} + \frac{\lambda_{2,n}^2}{4c_A} - \frac{S_n \lambda_{2,n}^2}{c_A N} \\
& + \frac{2c_c S_n \lambda_{2,n}}{N}) + (S_{n+1} - S_n) (\frac{3S_n \lambda_{2,n}^2}{2c_A N^2} - \frac{\lambda_{2,n}^2}{c_A N} + \frac{2c_c \lambda_{2,n}}{N}) \\
& + (\lambda_{2,n+1} - \lambda_{2,n}) (\frac{\lambda_{2,n}}{2c_A} - c_c + \frac{3S_n^2 \lambda_{2,n}}{2c_A N^2} - \frac{2S_n \lambda_{2,n}}{c_A N} + \frac{2c_c S_n}{N})
\end{aligned} \tag{4.32}$$

The boundary conditions are

$$I(t_0) = I^0; \quad S(t_0) = S^0; \quad \lambda_1(t_f) = 0; \quad \lambda_2(t_f) = 0 \tag{4.33}$$

Equations (4.29) through (4.32) are ordinary first order linear differential equations. The boundary conditions are given by Equation (4.33). As discussed in Chapter 3, this problem will be solved by using the superposition principle. The general solution can be written as

$$\begin{aligned}
I(t) &= I_p(t) + A_1 I_{1h}(t) + A_2 I_{2h}(t) \\
S(t) &= S_p(t) + A_1 S_{1h}(t) + A_2 S_{2h}(t) \\
\lambda_1(t) &= \lambda_{1,p}(t) + A_1 \lambda_{1,1h}(t) + A_2 \lambda_{1,2h}(t) \\
\lambda_2(t) &= \lambda_{2,p}(t) + A_1 \lambda_{2,1h}(t) + A_2 \lambda_{2,2h}(t)
\end{aligned} \tag{4.34}$$

These equations are derived according to Equation (3.1). The initial values chosen for the particular and the homogeneous solutions are explained in the next section. Runge-Kutta method of integration is used to get the solution for each set of initial conditions.

For each iteration, the integration constants are determined by applying the final conditions of Equation (4.33) to the last two equations of Equation (4.34). The profit and the advertisement is calculated using Equations (4.18) and (4.22), respectively.

#### 4.6 Numerical and Computational Aspects

The above problem was solved with the following numerical constants.

$$\begin{array}{lll}
a = 70 & c_A = 1.5 & I_{n+1}(o) = I^o = 20 \\
b = 100 & c_I = 0.15 & S_{n+1}(o) = S^o = 20 \\
c_c = 2 & N = 150 & t_i = 0 \\
c_S = 10 & I_m = 50 & \Delta t = 0.01
\end{array} \quad (4.35)$$

This problem is solved in two phases. The duration of time in phase 1 is 0 to 0.1 and in phase 2 is 0 to 1.0.  $\Delta t$  is the step size for integration. To obtain the particular solution, the following initial values were used, for all the iterations.

$$\begin{array}{ll}
I_p(o) = 20 & \lambda_{1,p}(o) = 0 \\
S_p(o) = 20 & \lambda_{2,p}(o) = 0
\end{array} \quad (4.36)$$

For the homogeneous solutions, the following initial values were used for all the iterations.

$$\begin{array}{llll}
I_{1h}(o) = 0 & , & S_{1h}(o) = 0 & , & \lambda_{1,1h}(o) = 1 & , & \lambda_{2,1h}(o) = 0; \\
I_{2h}(o) = 0 & , & S_{2h}(o) = 0 & , & \lambda_{1,2h}(o) = 0 & , & \lambda_{2,2h}(o) = 1.
\end{array} \quad (4.37)$$

This selection of initial values for the particular and homogeneous solutions keeps the computation of the number of homogeneous solutions needed to the minimum. In this case it is two.

In order to start the iteration initial profiles for  $S$  and  $\lambda_2$  must be assumed. The various initial profiles used for  $S$  and  $\lambda_2$  are listed in Table 4.1. These are used only for phase 1 calculation.

This problem was solved on an IBM 360/50 computer. To obtain the homogeneous solutions the constant terms in Equations (4.29) through (4.32) were set to zero and the resulting homogeneous equations were solved. The computer program is given in Appendix (1).

The control variable  $A$  and the objective function,  $J$ , were determined using Equations (4.22) and (4.18) respectively. The following

approximation was used to calculate the total profit on the digital computer.

$$J = \sum_{t_i}^{t_f} (c_S S(t) - c_I (I_m - I(t))^2 - (c_A S(t) A^2(t)) \Delta t$$

In phase 1, only one iteration was allowed and the problem was solved for the period 0 to 0.1. The optimal profile obtained for this short period is used as the initial profile for phase 2. The initial profile for phase 2 for the period 0.1 to 1.0 was assumed to be a constant and is equal to the value of the variable at  $t = 0.1$ .

$$\begin{aligned} [S_i(t)]_2 &= [S_o(t)]_1 & \text{for } 0 \leq t \leq .1 \\ [\lambda_{2,i}(t)]_2 &= [\lambda_{2,o}(t)]_1 & \text{for } 0 \leq t \leq .1 \\ [S_i(t)]_2 &= [S_o(.1)]_1 & \text{for } .1 \leq t \leq 1 \\ [\lambda_{2,i}(t)]_2 &= [\lambda_{2,o}(.1)]_1 & \text{for } .1 \leq t \leq 1 \end{aligned}$$

The subscript outside the bracket represents the phase, 'i' represents the initial profile and 'o' represents the optimal profile. Using this initial profile the problem was solved over the whole period in phase 2. Nine iterations were allowed in this phase. It is expected that convergence should be obtained within 9 iterations.

#### 4.7 Results

The results are shown in Tables 4.2 through 4.11 and Figs. 4.1 through 4.16. The iteration which gives the optimal profile is the one at which it becomes difficult to differentiate between this profile and that of the next iteration.

No convergence was obtained when Set 1 of Table 4.1 is used as the initial profile and the following results were obtained in the first iteration of phase 1.

$$I(o.1) = 1.47 \times 10^5$$

$$S(o.1) = -3.36 \times 10^7$$

$$A(o.1) = -9.9 \times 10^{11}$$

$$J(o.1) = 4.9 \times 10^{29}$$

The computer experienced overflow while calculating the particular solution in the second iteration of phase 1.

Set 2 did not produce convergence when only one iteration was allowed in phase 1, and overflow was experienced in iteration seven of phase 2. However, convergence was obtained when two iterations were allowed in phase 1.

The remaining sets converged to the optimal solution with one iteration in phase 1. In general, five digit accuracy was obtained within six iterations in phase 2. The optimal results are given below.

$$I(o) = 20.0$$

$$S(o) = 20.0$$

$$A(o) = 3.98$$

$$I(1) = 66.15$$

$$S(1) = 115.69$$

$$A(1) = 0.00$$

$$\text{Profit} = 587.80$$

The convergence rates of sales, inventory, and the advertisement are shown in Figs. 4.1 through 4.9 for the sets 3, 8, and 11. The convergence rate of profit function for these sets is shown in Figs. 10, 11, and 12. The convergence rates of all the three variables is shown in Table 4.2 for the set 8. The convergence rates of the final values of the state variables in phase 2 are drawn in Tables 4.3 and 4.4 for sets 2 through 11.

To study the effect of the number of iterations in phase 1, all the sets were again solved with nine iterations in phase 1. All the variables converged in five iterations to five digit accuracy in phase 1. In general, five digit accuracy is obtained with five iterations in phase 2. Table 4.7 through 4.10 summarize the convergence rates of the variables with nine iterations in phase 1. Table 4.11 shows the convergence rate of the

variables and the objective function in phase 2. The same values are obtained for all the sets because the initial profile for phase 2 is the same for all the sets. Figs. 4.13 through 4.16 show the profiles of the variables and the profit function for Set 11.

This problem took 2.1 minutes in Fortran 4, G level for one iteration in phase 1 and nine iterations in phase 2.

#### 4.8 Discussion

This problem converged with ten sets of different and far from optimal initial profiles. The same problem was solved by Shah<sup>29</sup> using the one-phase quasilinearization technique. On comparing the results, it is observed that the range of convergence is doubled by using the two-phase method. Shah found that the problem converged for the set with  $S_0(t) = 250$  and  $\lambda_{2,o}(t) = 10$  but it failed to converge for the set with  $S_0(t) = 275$  and  $\lambda_{2,o}(t) = 10$ . With the two-phase method, it is found that the problem converged for the set with  $S_0(t) = 500$  and  $\lambda_{2,o}(t) = 20$  but it failed to converge for the set with  $S_0(t) = 550$  and  $\lambda_{2,o}(t) = 30$ .

The data given in Tables 4.3 and 4.4 show that only one iteration is sufficient in phase 1, for most of the problems. Five digit accuracy was obtained with six iterations in phase 2 for all the sets. It is found that the convergence rate is nearly independent of the initial approximation.

Table 4.5 illustrates the effect of the number of iterations in phase 1 on the convergence rate in phase 2. It is found that if the initial guess is not too far from the correct value, only one iteration in phase 1 is needed.

To study the influence of the length of the period on convergence, some computation was carried out over longer periods in phase 1. Table 4.6 shows that shorter periods in phase 1 are very helpful for cases where the initial guess is very far from the correct value. It was found that with fairly good initial guess such as Sets 7 through 11, convergence was obtained with all the time periods listed in Table 4.6.

**THIS BOOK  
CONTAINS  
NUMEROUS PAGES  
WITH DIAGRAMS  
THAT ARE CROOKED  
COMPARED TO THE  
REST OF THE  
INFORMATION ON  
THE PAGE.**

**THIS IS AS  
RECEIVED FROM  
CUSTOMER.**



TABLE 4.1. Sets of Initial Profile for Phase 1,  
in Inventory and Advertising Scheduling  
Problem.

| Set | $S_o(t)$ | $\lambda_{2,o}(t)$ |
|-----|----------|--------------------|
| 1   | 550      | 30                 |
| 2   | 500      | 20                 |
| 3   | 450      | 20                 |
| 4   | 400      | 20                 |
| 5   | 350      | 15                 |
| 6   | 350      | 10                 |
| 7   | 150      | 5                  |
| 8   | 20       | 0                  |
| 9   | 0        | -5                 |
| 10  | 0        | -15                |
| 11  | 0        | -25                |



TABLE 4.3. Convergence Rate of S(1.0) in Phase 2.  
(1 Iteration in Phase 1)

| Set       | 2*     | 3      | 4      | 5      | 6      | 7      | 8      | 9      | 10     | 11     |
|-----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| Iteration |        |        |        |        |        |        |        |        |        |        |
| 1         | 236.68 | 144.00 | 226.00 | 211.16 | 204.20 | 167.55 | 208.80 | 198.29 | 173.44 | 145.28 |
| 2         | 166.55 | 171.00 | 169.67 | 154.09 | 148.58 | 120.85 | 127.81 | 125.99 | 121.75 | 117.69 |
| 3         | 123.85 | 135.53 | 128.83 | 120.27 | 118.57 | 115.73 | 115.86 | 115.82 | 115.74 | 115.69 |
| 4         | 115.91 | 116.73 | 116.06 | 115.79 | 115.73 | 115.69 | 115.69 | 115.69 | 115.69 | 115.69 |
| 5         | 115.69 | 115.71 | 115.69 | 115.69 | 115.69 | 115.69 | 115.69 | 115.69 | 115.69 | 115.69 |
| 6         | 115.69 | 115.69 | 115.69 | 115.69 | 115.69 | 115.69 | 115.69 | 115.69 | 115.69 | 115.69 |
| 7         |        | 115.69 |        |        |        |        |        |        |        |        |

\* Two iterations were needed in Phase 1.

TABLE 4.4. Convergence Rate of I(1.0) in Phase 2.  
(1 Iteration in Phase 1)

| Set       | 2*      | 3        | 4       | 5       | 6       | 7      | 8      | 9      | 10     | 11     |
|-----------|---------|----------|---------|---------|---------|--------|--------|--------|--------|--------|
| Iteration |         |          |         |         |         |        |        |        |        |        |
| 1         | -57.636 | -72.000  | -71.125 | -40.757 | -27.687 | 48.520 | 38.024 | 40.325 | 47.027 | 56.030 |
| 2         | 29.062  | - 9.8669 | 8.2083  | 38.742  | 47.579  | 64.665 | 63.207 | 63.539 | 64.468 | 65.543 |
| 3         | 65.74   | 52.527   | 61.414  | 66.382  | 66.452  | 66.147 | 66.131 | 66.136 | 66.146 | 66.153 |
| 4         | 66.017  | 66.618   | 66.199  | 66.080  | 66.123  | 66.154 | 66.154 | 66.154 | 66.154 | 66.154 |
| 5         | 66.154  | 66.136   | 66.153  | 66.154  | 66.165  | 66.154 | 66.154 | 66.154 | 66.154 | 66.154 |
| 6         | 66.154  | 66.154   | 66.154  | 66.154  | 66.154  |        |        |        |        |        |
| 7         |         | 66.154   | 66.154  |         |         |        |        |        |        |        |

\* Two iterations were needed in Phase 1.

TABLE 4.5. The Effect of the Number of Iterations in Phase 1 on the Convergence Rate in Phase 2.

Set                      Number of Iterations required in Phase 2  
for 5 Digit Accuracy.

| Set | 1 Iteration in<br>Phase 1 | 9 Iterations in<br>Phase 1 |
|-----|---------------------------|----------------------------|
|     |                           |                            |
| 2   | 6*                        | 5                          |
| 3   | 6                         | 5                          |
| 4   | 6                         | 5                          |
| 5   | 5                         | 5                          |
| 6   | 5                         | 5                          |
| 7   | 4                         | 5                          |
| 8   | 4                         | 5                          |
| 9   | 4                         | 5                          |
| 10  | 4                         | 5                          |
| 11  | 4                         | 5                          |

\* Two iterations were allowed in Phase 1 for Set 2.

TABLE 4.6. Convergence for Different Time  
Periods in Phase 1.

| $S_o(t)$ | $\lambda_{2,o}(t)$ | $t_f$        |              |              |           |
|----------|--------------------|--------------|--------------|--------------|-----------|
|          |                    | 1.0**        | 0.5          | 0.2          | 0.1       |
| 275      | 10                 | not converge | converged    | converged    | converged |
| 300      | 10                 | not converge | converged    | converged    | converged |
| 300      | 12.5               | not converge | converged    | converged    | converged |
| 350      | 15.0               | not converge | not converge | converged    | converged |
| 450      | 20.0               | not converge | not converge | not converge | converged |

\*\* This column is obtained from Shah<sup>29</sup>.



TABLE 4.8. Convergence Rate of I(.1) in Phase 1.

[illegible]



TABLE 4.9. Convergence Rate of  $A(o)$  in Phase 1.

| $S_o(t)$           | 500     | 450     | 400     | 350     | 300     | 150     | 20     | 0       | -5      | -15      | -25 |
|--------------------|---------|---------|---------|---------|---------|---------|--------|---------|---------|----------|-----|
| $\lambda_{2,o}(t)$ | 20      | 20      | 20      | 15      | 10      | 5       | 0      | 0       | -5      | -15      | -25 |
| Iteration          |         |         |         |         |         |         |        |         |         |          |     |
| 1                  | -7.1451 | -7.2067 | -7.2875 | -5.3962 | -3.3890 | -.33382 | .29909 | -.01788 | -2.9306 | -114.26  |     |
| 2                  | -5.8615 | -5.8679 | -5.6593 | -2.6961 | -.94069 | .29069  | .29937 | .29794  | .19601  | -0.66021 |     |
| 3                  | -2.4238 | -1.4950 | -.56701 | .15392  | 0.26948 | .29938  | .29937 | .29937  | .29936  | .29541   |     |
| 4                  | .16305  | .24401  | .28992  | .29931  | 0.29939 | .29937  | .29937 | .29937  | .29937  | .29937   |     |
| 5                  | .29932  | .29939  | .29938  | .29937  | 0.29937 | .29937  |        |         | .29937  | .29937   |     |
| 6                  | .29937  | .29937  | .29937  | .29937  | 0.29937 |         |        |         |         |          |     |
| 7                  | .29937  | .29937  | .29937  |         |         |         |        |         |         |          |     |

TABLE 4.10. Convergence Rate of  $J(1)$  in Phase 1.

| $S_o(t)$           | 500     | 450     | 400     | 350    | 350    | 10     | 5      | 0      | -5     | -15     | -25 |
|--------------------|---------|---------|---------|--------|--------|--------|--------|--------|--------|---------|-----|
| $\lambda_{2,o}(t)$ | 20      | 20      | 20      | 15     | 15     | 10     | 5      | 0      | -5     | -15     | -25 |
| Iteration          |         |         |         |        |        |        |        |        |        |         |     |
| 1                  | -2350.5 | -1297.2 | -526.65 | 106.25 | 174.55 | 22.467 | 11.669 | 14.113 | 9.482  | -119.03 |     |
| 2                  | 68.612  | 108.67  | 90.215  | 50.709 | 37.478 | 11.758 | 11.673 | 11.679 | 11.812 | 11.966  |     |
| 3                  | 51.264  | 31.132  | 18.055  | 12.413 | 11.929 | 11.673 | 11.673 | 11.673 | 11.673 | 11.671  |     |
| 4                  | 12.383  | 11.826  | 11.680  | 11.673 | 11.673 | 11.673 | 11.673 | 11.673 | 11.673 | 11.673  |     |
| 5                  | 11.673  | 11.673  | 11.673  | 11.673 | 11.673 |        |        |        |        | 11.673  |     |
| 6                  | 11.673  | 11.673  | 11.673  |        |        |        |        |        |        |         |     |

TABLE 4.11. Convergence Rate in Phase 2 with 9 iterations in Phase 1.

| Iteration | S(1.0) | I(1.0) | A(0.0) | J      |
|-----------|--------|--------|--------|--------|
| 0         | 24.021 |        |        |        |
| 1         | 208.79 | 38.025 | 5.4996 | 720.80 |
| 2         | 127.81 | 63.208 | 4.0913 | 611.40 |
| 3         | 115.86 | 66.131 | 3.9804 | 588.04 |
| 4         | 115.69 | 66.154 | 3.9816 | 587.80 |
| 5         | 115.69 | 66.154 | 3.9816 | 587.80 |

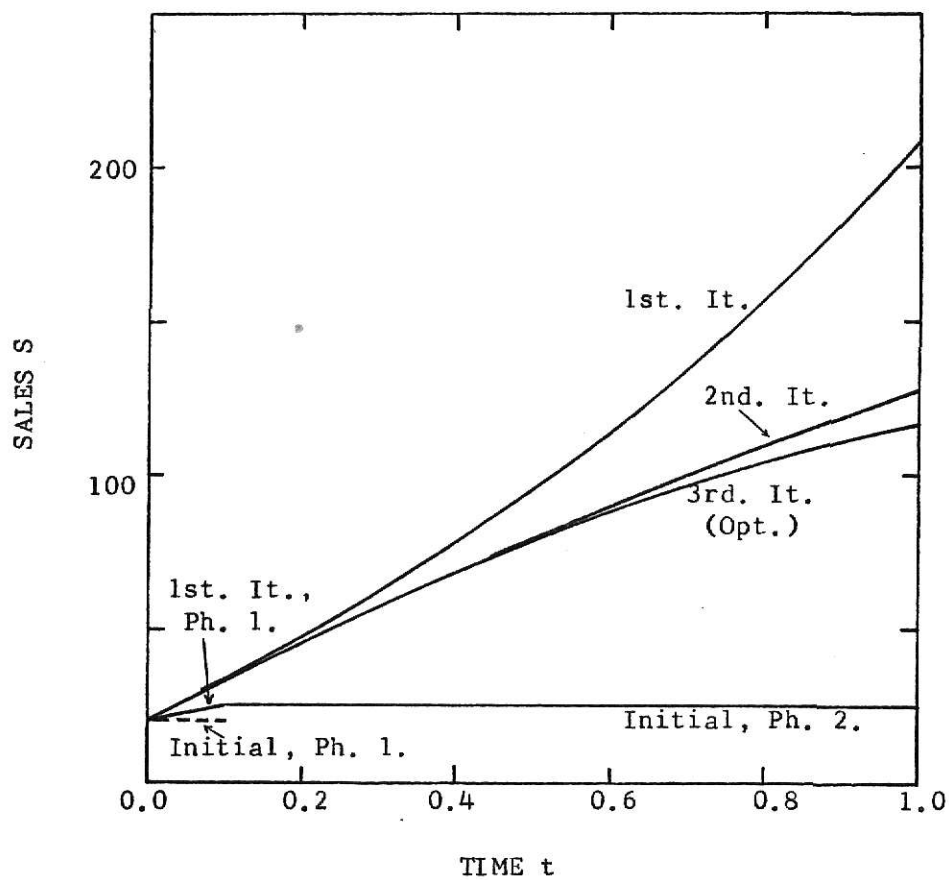


Fig. 4.1. Convergence Rate of Sales. Set 8.  
1 iteration in phase 1.

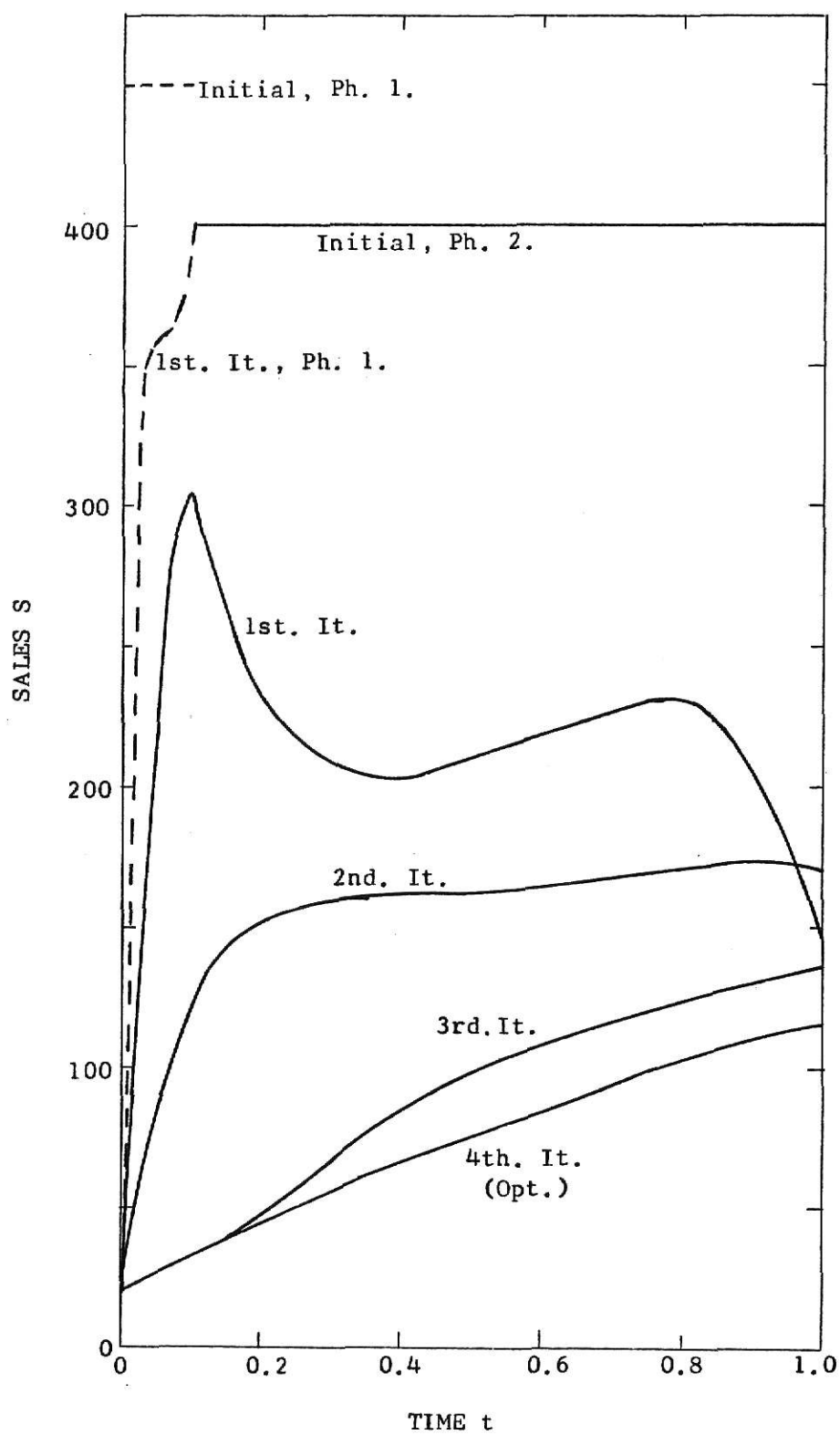


Fig. 4.2. Convergence Rate of Sales. Set 3.  
1 iteration in phase 1.

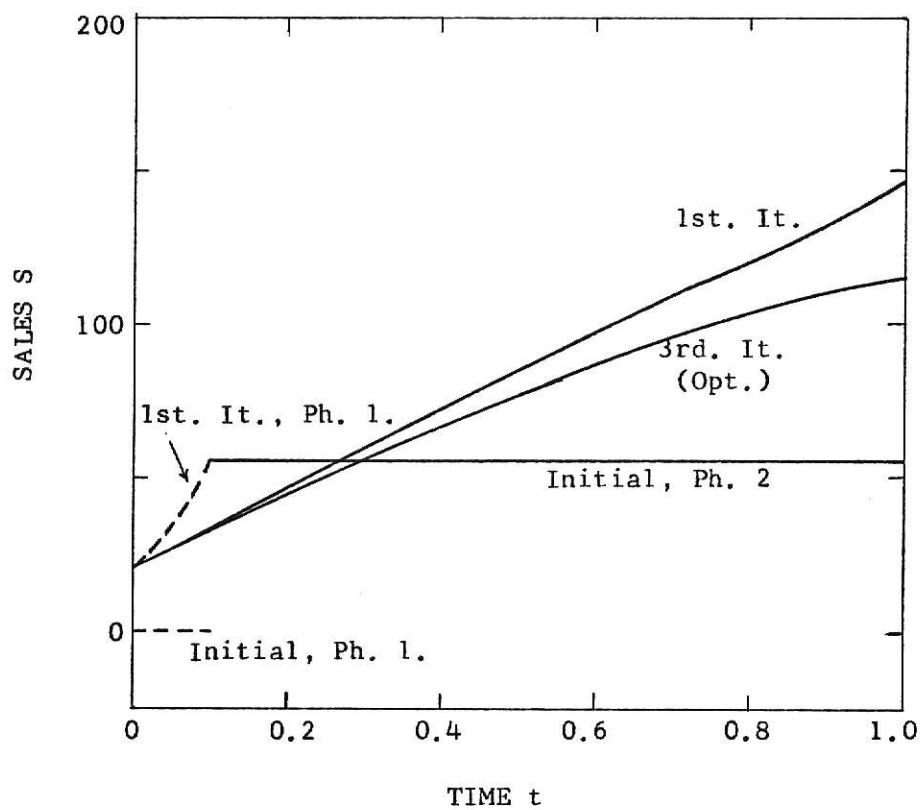


Fig. 4.3. Convergence Rate of Sales. Set 11.  
1 iteration in phase 1.

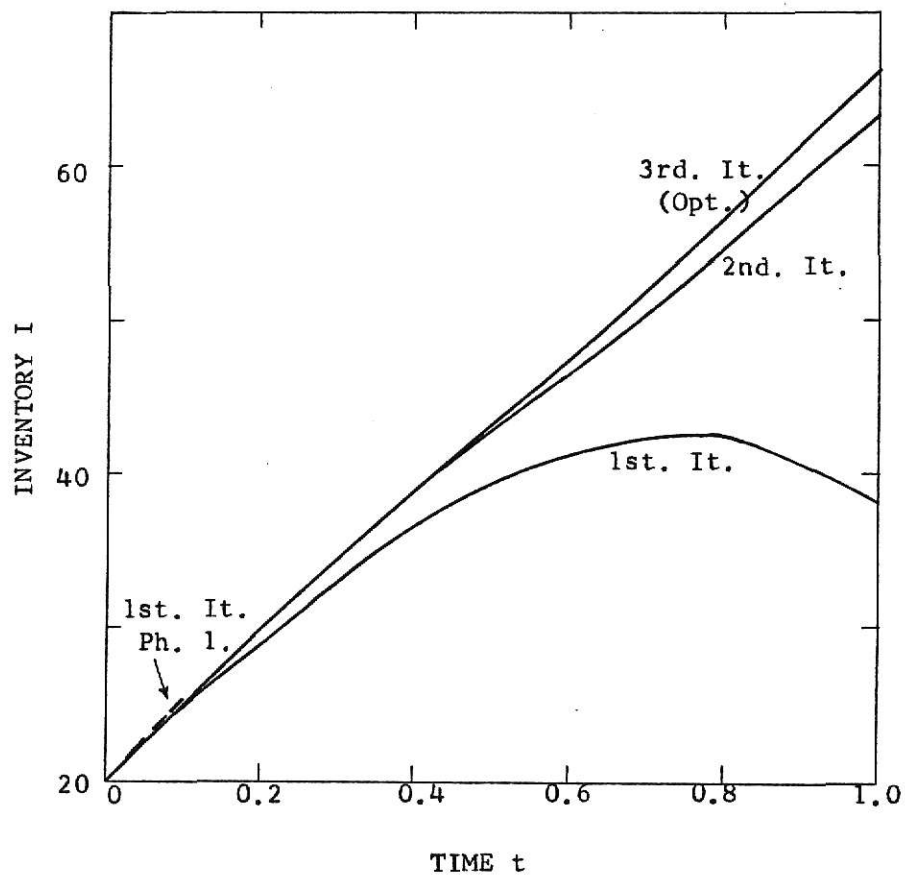


Fig. 4.4. Convergence Rate of Inventory. Set 8.  
1 iteration in phase 1.

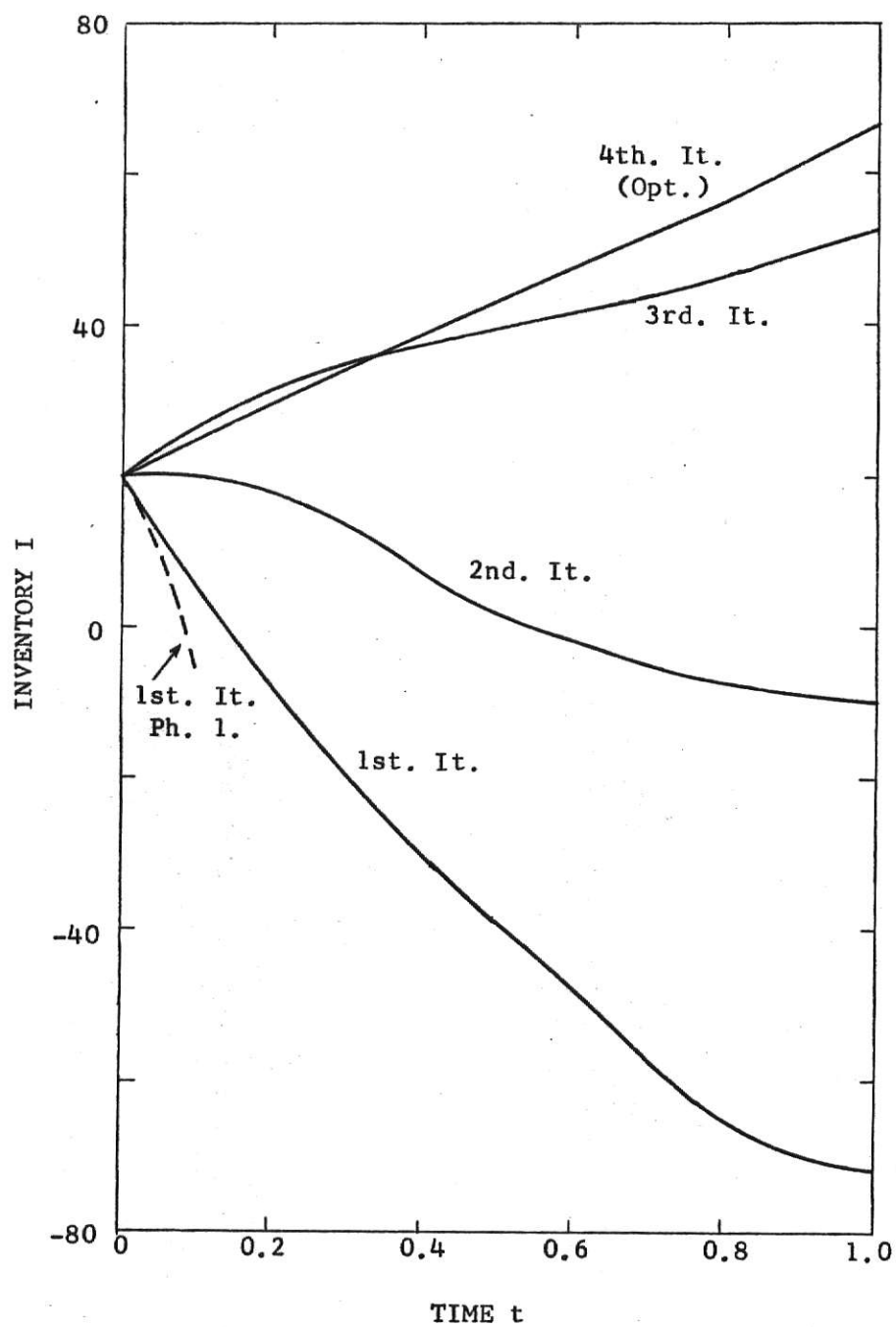


Fig. 4.5. Convergence Rate of Inventory. Set 3.  
1 iteration in phase 1.



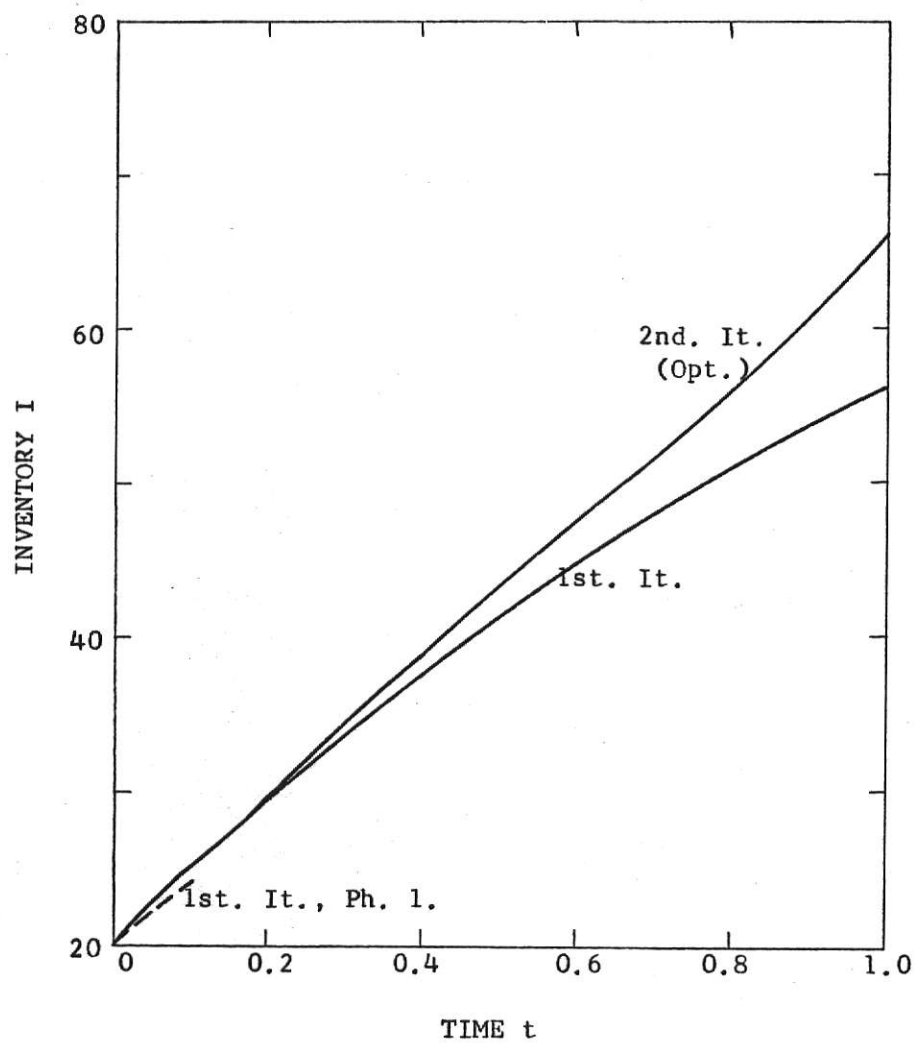


Fig. 4.6. Convergence Rate of Inventory. Set 11.  
1 iteration in phase 1.

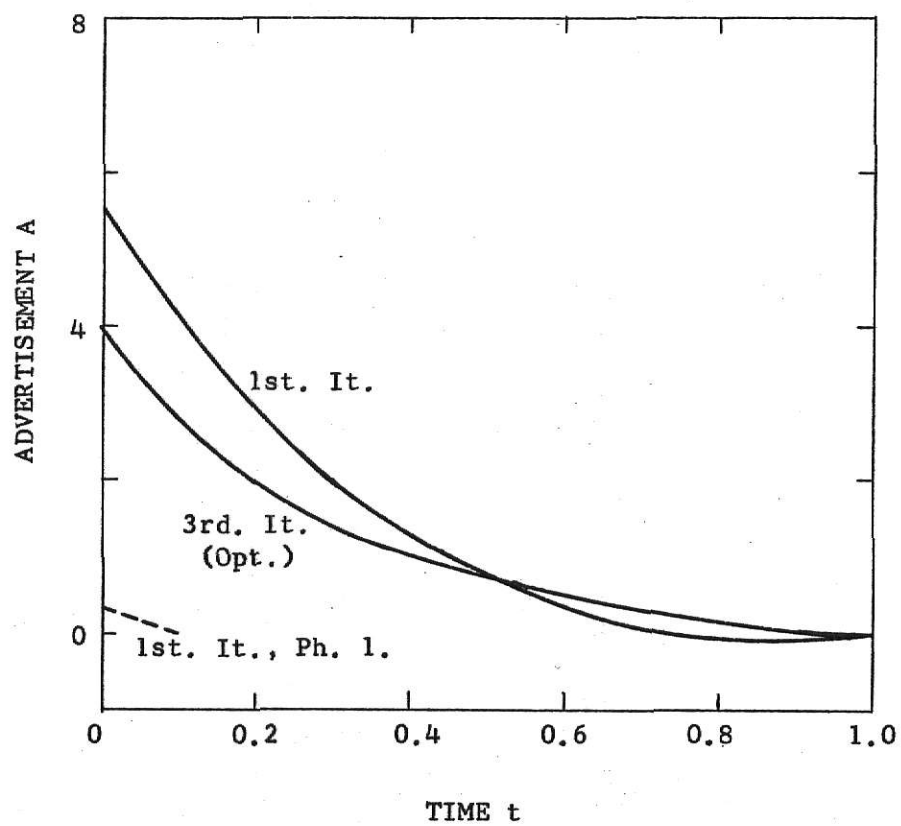


Fig. 4.7. Convergence Rate of Advertisement. Set 8.  
1 iteration in phase 1.

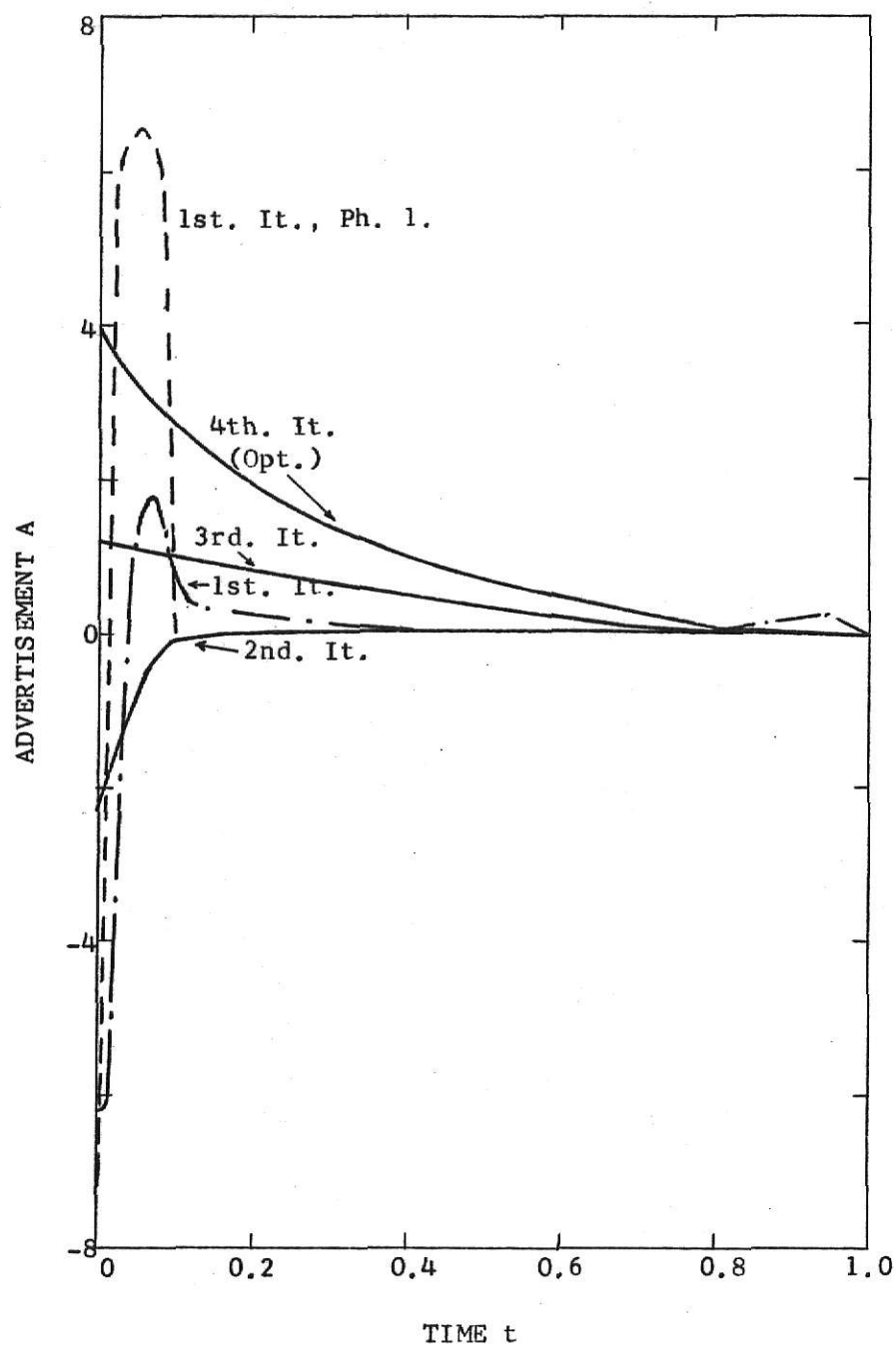


Fig. 4.8. Convergence Rate of Advertisement. Set 3.  
1 iteration in phase 1.

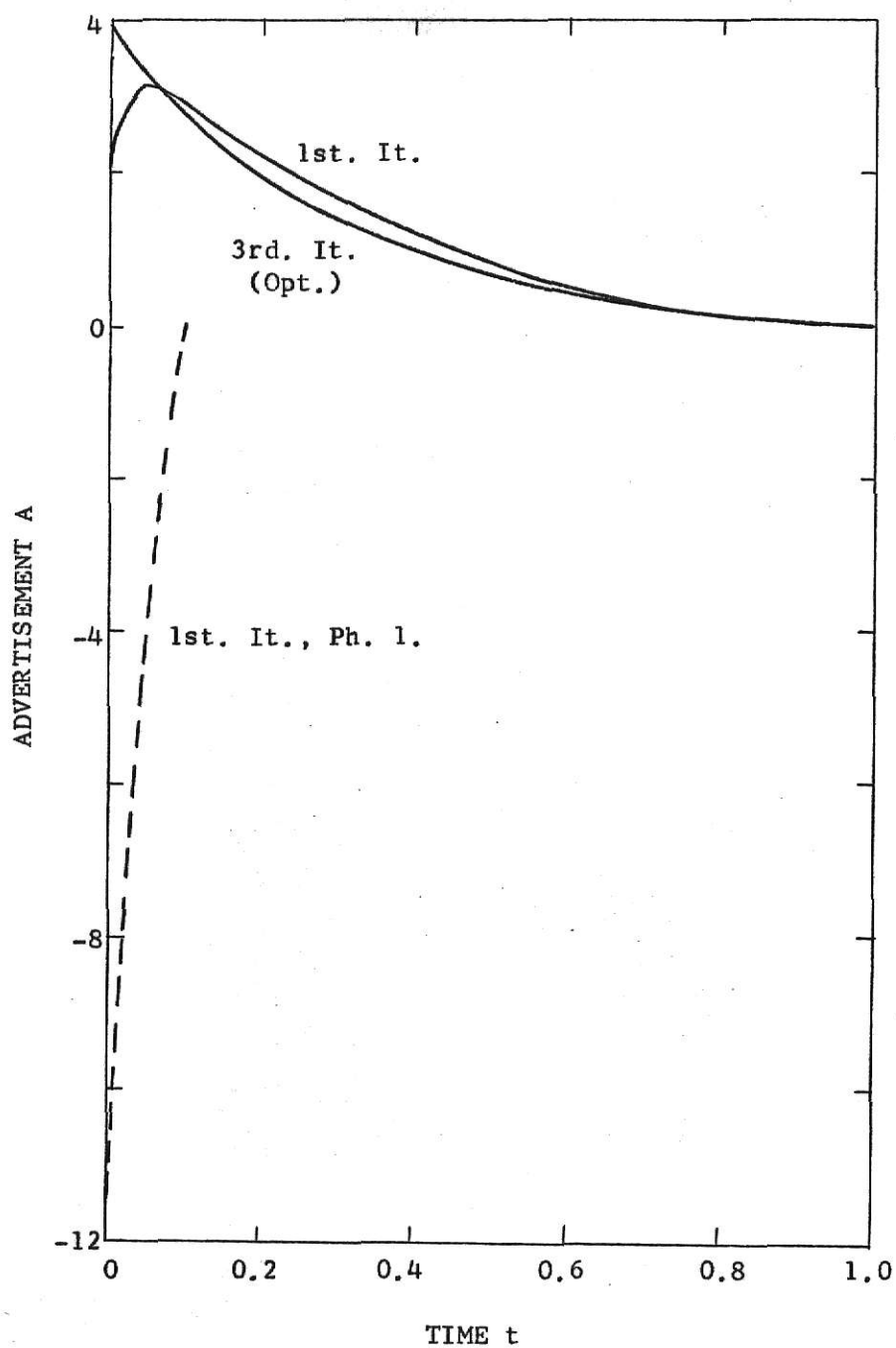


Fig. 4.9. Convergence Rate of Advertisement. Set 11.  
1 iteration in phase 1.

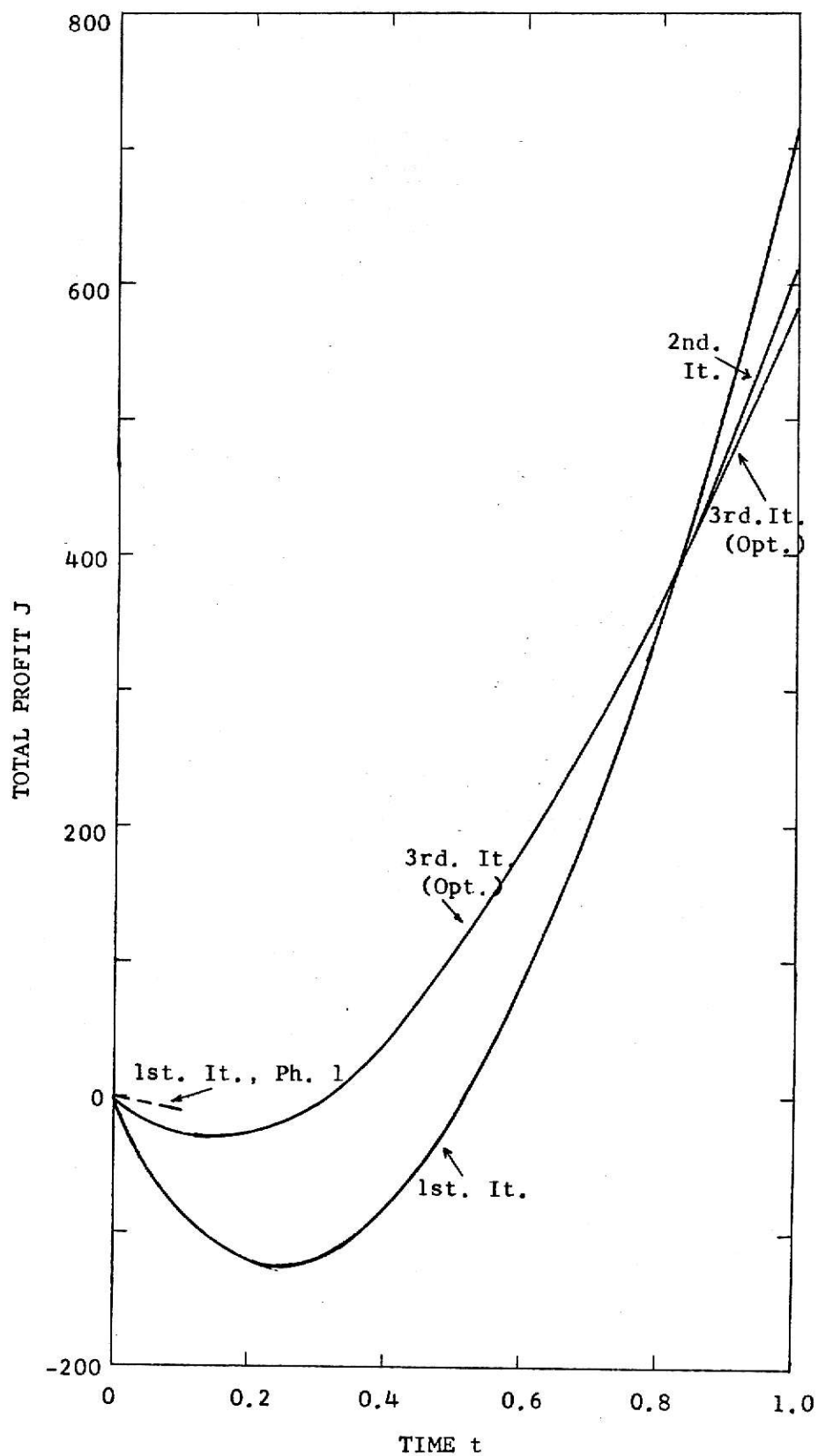


Fig. 4.10. Convergence Rate of Profit. Set 8.  
1 iteration in phase 1.

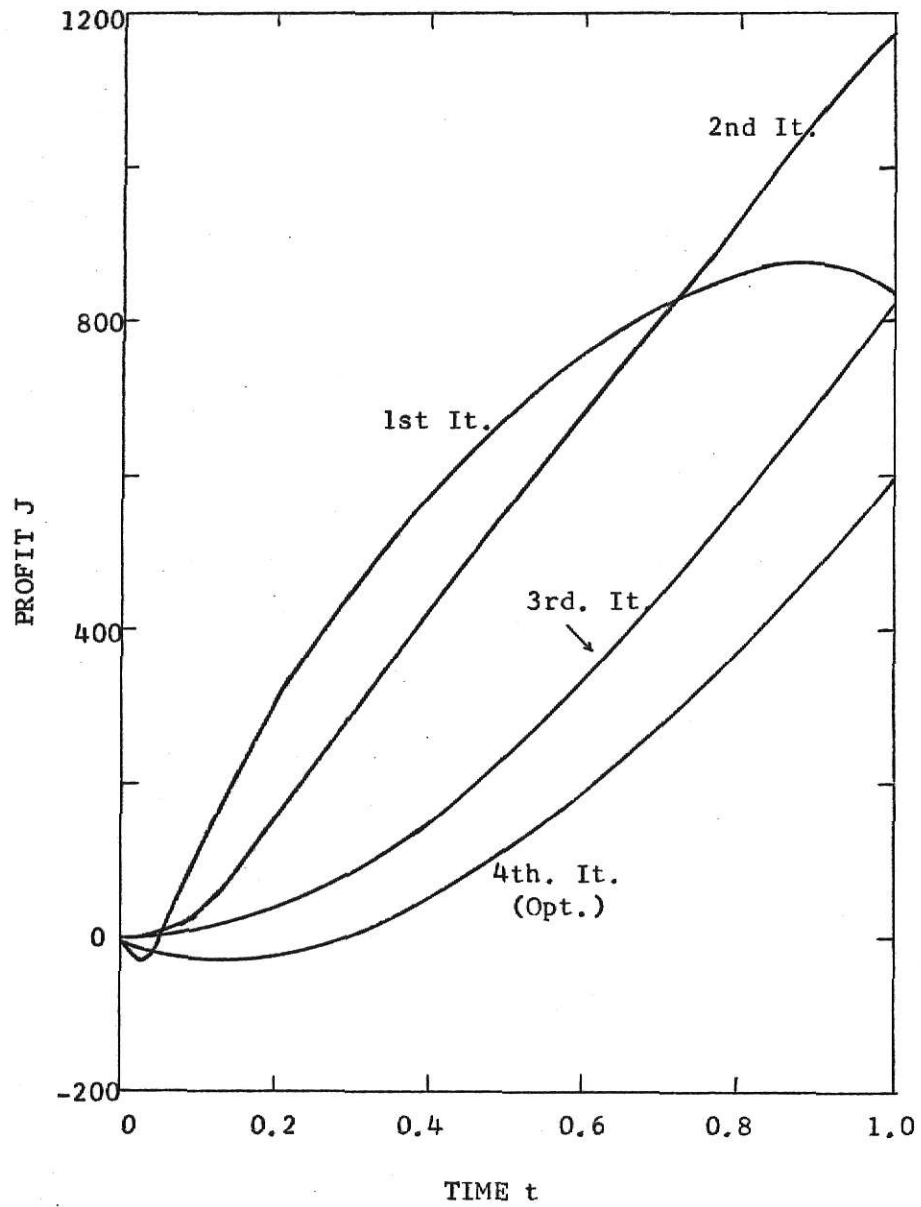


Fig. 4.11. Convergence Rate of Profit. Set 3.  
1 iteration in phase 1.

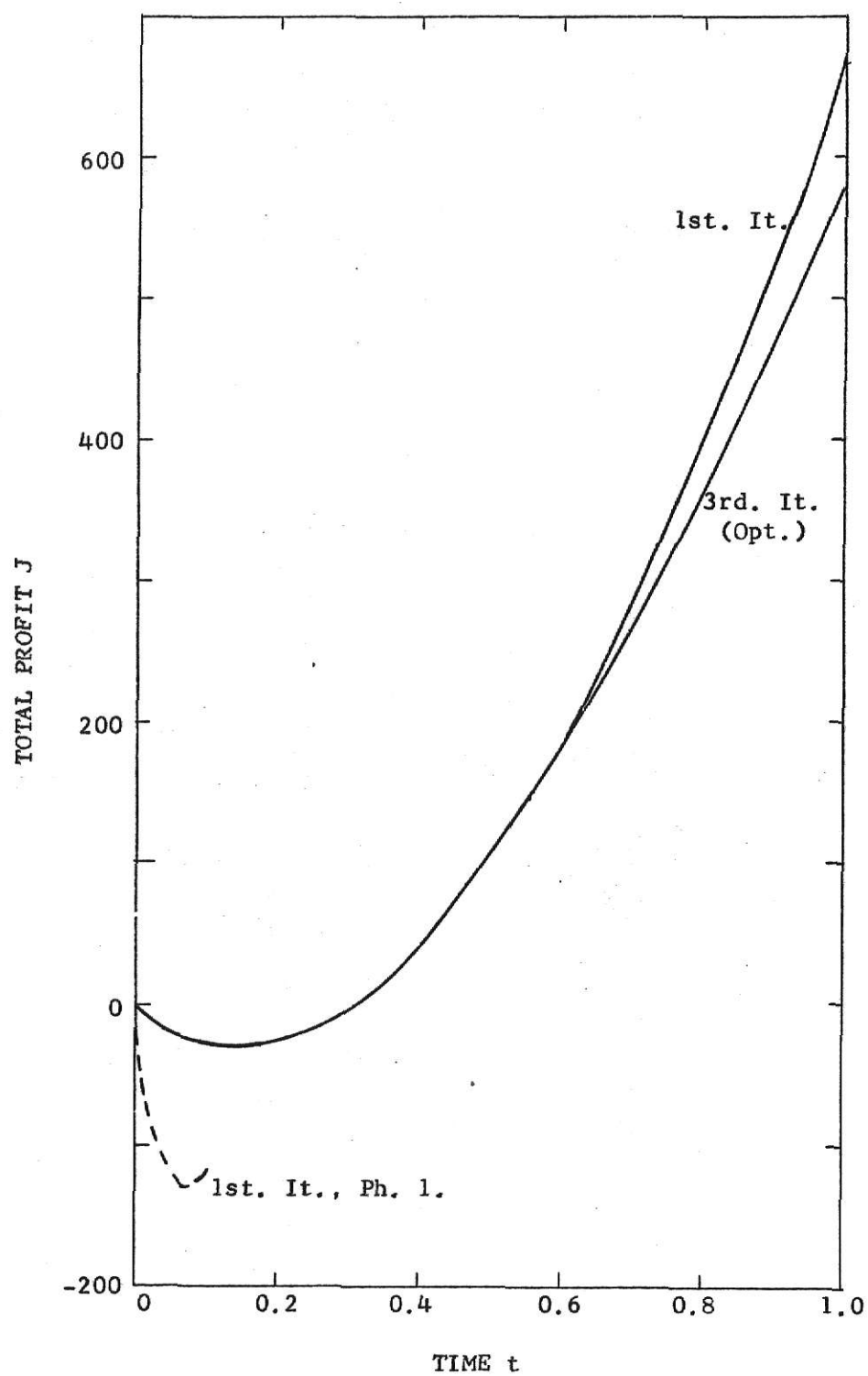


Fig. 4.12. Convergence Rate of Profit. Set 11.  
1 iteration in phase 1.

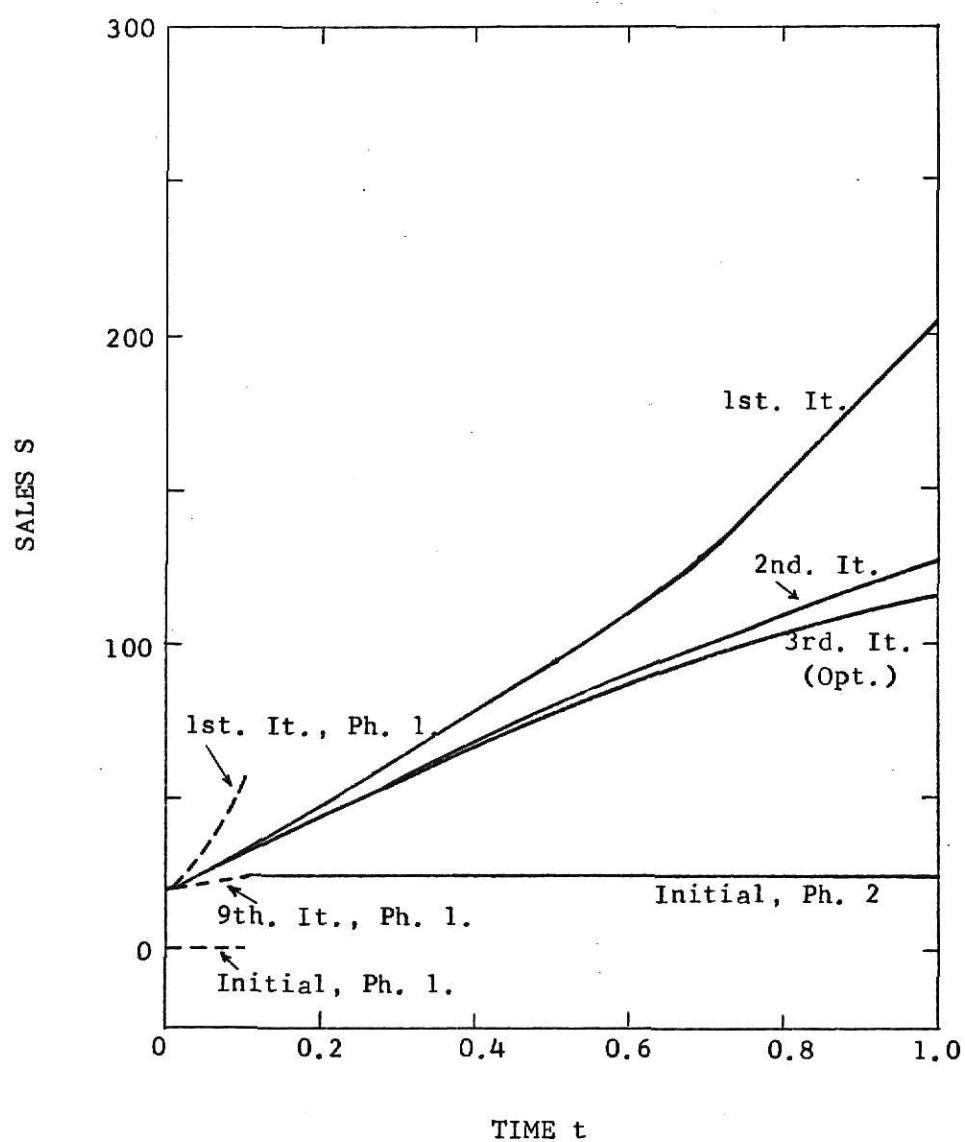


Fig. 4.13. Convergence Rate of Sales. Set 11.  
9 iterations in phase 1.



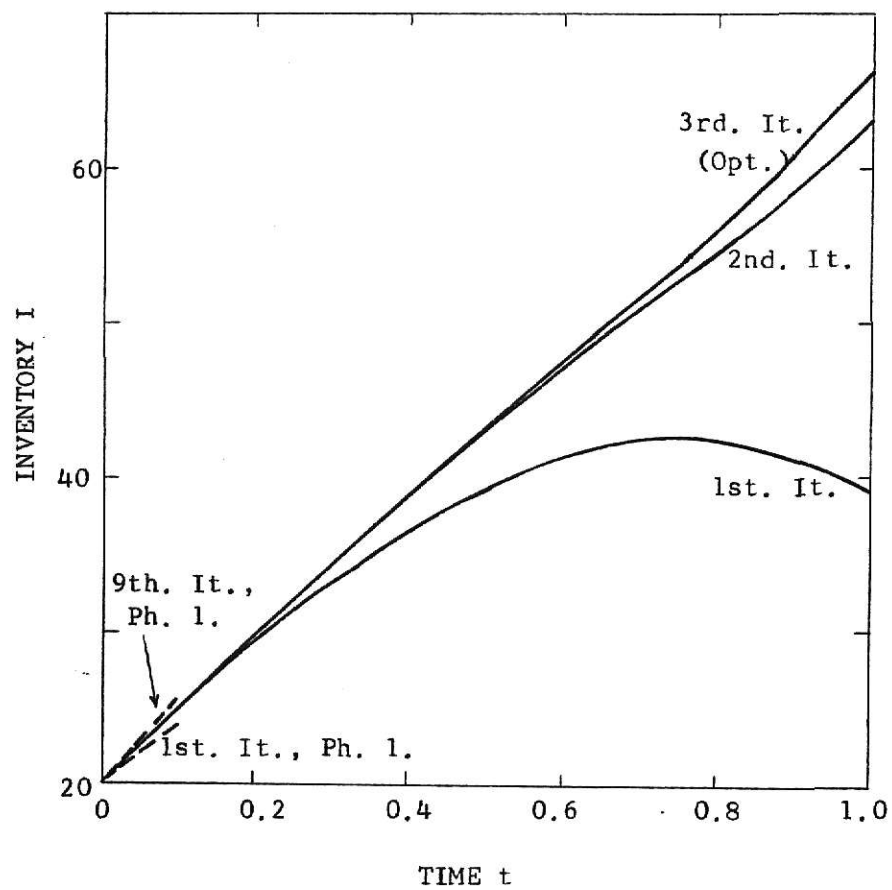


Fig. 4.14. Convergence Rate of Inventory. Set 11.  
9 iterations in phase 1.

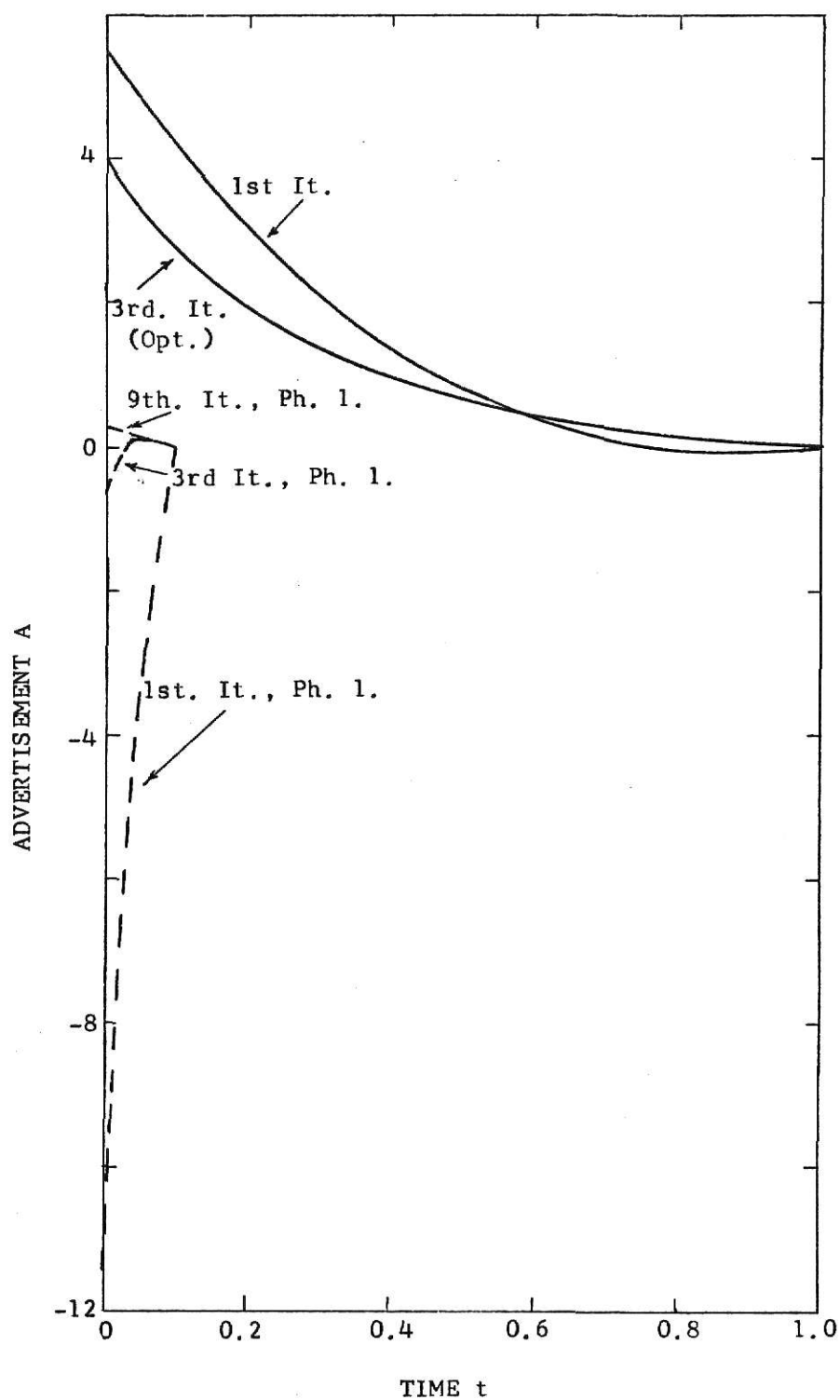


Fig. 4.15. Convergence Rate of Advertisement. Set 11.

9 iterations in phase 1.

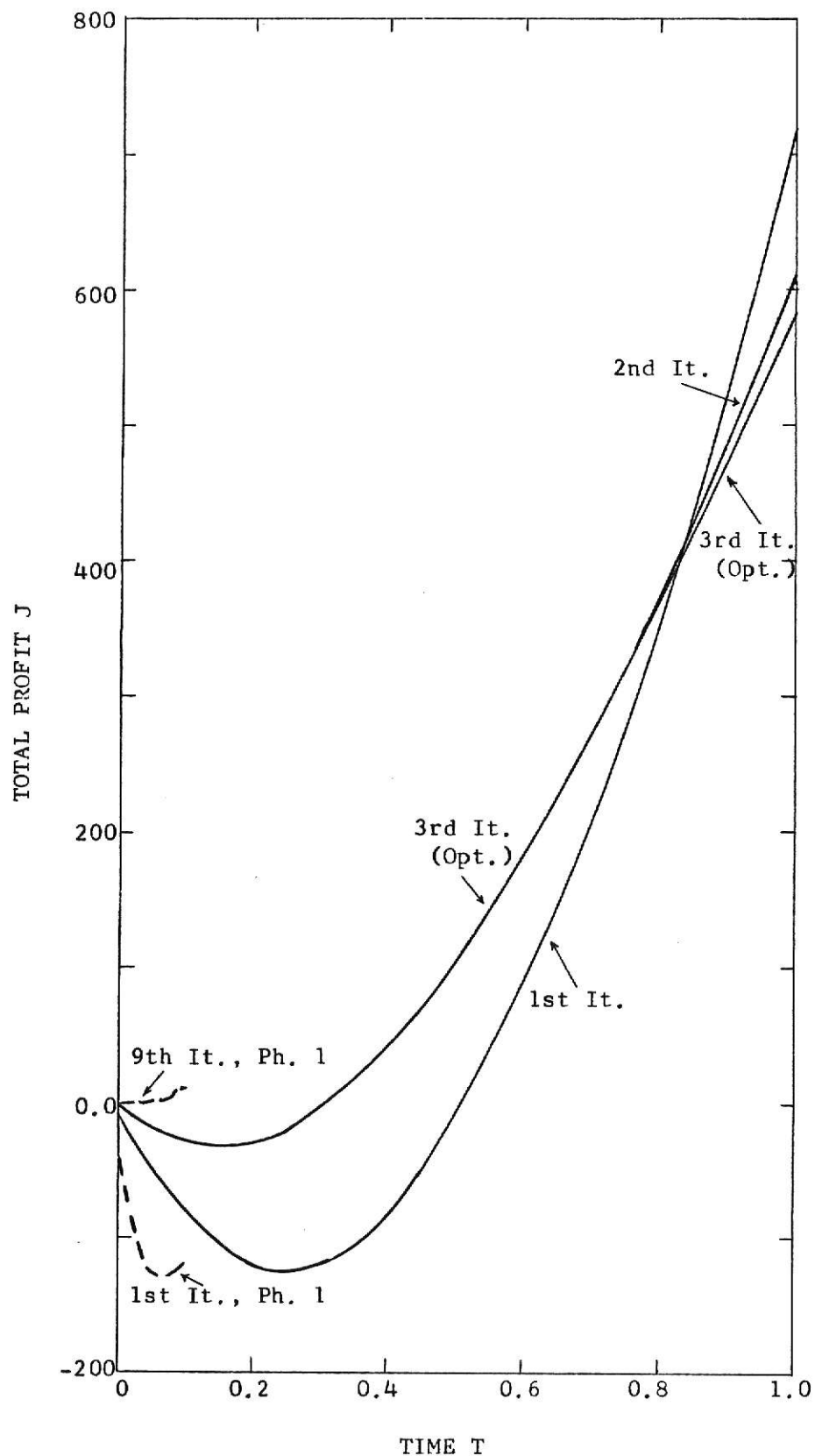


Fig. 4.16. Convergence Rate of Profit Function. Set 11.  
9 iterations in phase 1.

## CHAPTER 5

### APPLICATION TO AN ADVERTISEMENT AND PRODUCTION SCHEDULING PROBLEM

#### 5.1 Introduction

To show the effectiveness of the two-phase method in solving unstable boundary-value problems with large dimension, an advertisement and production scheduling problem will be solved. This problem has six state variables and three control variables. The object is to determine the control variables so that the profit function is maximized. In section 2, the model will be developed. The problem will be defined in section 3, and then formulated into a two-point boundary-value problem in section 4. The non-linear differential equations will be linearized in section 5. Numerical and computational aspects will be treated in section 6. Results and discussion are given in sections 7 and 8, respectively.

#### 5.2 Development of the Model

A firm making two products A and C, has decided to produce and to market a new product B. Manufacturing process of all the three products is shown in Fig. 5.1. The following consecutive reaction takes place in two chemical reactors



Both are first order reactions. The component B is the desired product and C is a waste product. It is assumed that raw material containing A, B and C is available in unlimited quantity from another part of the manufacturing process. Production of B is the most desirable reaction

and production of C is the most undesirable one.

Since B is a new product, advertisement is needed. Again a diffusion model is used for the sales rate of product B. Furthermore, to protect against fluctuations in demand, an inventory will be assumed for B. Products A and C are assumed to have unlimited market at fixed price and are sold as soon as manufactured. Thus no inventory for the raw material and products A and C will be considered.

The problem is to determine the advertising rate and the production rate such that the profit is maximized. A non-linear chemical process will be used for the production of B. This will test the new approach vigorously. The amount of B produced and the amount decomposed can be controlled by the reaction temperature.

The performance of the system is determined by the following equations.

Material balance equations:

$$\left( \begin{array}{c} \text{Amount of} \\ \text{A in} \end{array} \right) = \left( \begin{array}{c} \text{Amount of} \\ \text{A out} \end{array} \right) + \left( \begin{array}{c} \text{Amount of A} \\ \text{Transformed to B} \end{array} \right) \quad (5.1)$$

$$\left( \begin{array}{c} \text{Amount of} \\ \text{B in} \end{array} \right) = \left( \begin{array}{c} \text{Amount of} \\ \text{B out} \end{array} \right) + \left( \begin{array}{c} \text{Amount of B} \\ \text{Transformed to C} \end{array} \right) - \left( \begin{array}{c} \text{Amount of B} \\ \text{Transformed from A} \end{array} \right) \quad (5.2)$$

Inventory balance equation:

$$\left( \begin{array}{c} \text{Rate of change} \\ \text{of inventory} \end{array} \right) = \left( \begin{array}{c} \text{Production} \\ \text{Rate} \end{array} \right) - \left( \begin{array}{c} \text{Sales} \\ \text{Rate} \end{array} \right) \quad (5.3)$$

The performance equation for sales and advertisement is the same as Equation (4.12).

The management wishes to maximize

$$\text{profit} = \int_{t_i}^{t_f} (\text{revenue of B} + \text{revenue of A} + \text{revenue of C} - \text{inventory cost} - \text{advertisement cost} - \text{manufacturing cost}) dt \quad (5.4)$$

To express the model mathematically, let

$x_1, x_2$  = concentration of A in reactors 1 and 2, respectively.

$y_1, y_2$  = concentration of B in reactors 1 and 2, respectively.

$V_1, V_2$  = volume of reactors 1 and 2, respectively.

$k_{a1}, k_{a2}$  = reaction rate constants for the production of B in reactors  
1 and 2, respectively.

$k_{b1}, k_{b2}$  = reaction rate constants for the decomposition of B in reactors  
1 and 2, respectively.

$q$  = flow rate.

$x_0, y_0$  = raw material concentrations of A and B, respectively.

$G_a, G_b$  = frequency constants of the first and second reactions, respectively.

$E_a, E_b$  = activation energies of the first and second reactions, respectively.

$R$  = gas constant.

$T_1, T_2$  = temperature in reactors 1 and 2, respectively.

Equation (5.1) gives, under unsteady state conditions

$$V_1 \frac{dx_1}{dt} = q (x_0 - x_1) - V_1 k_{a1} x_1 \quad (5.5)$$

Similarly, Equation (5.2) gives

$$V_1 \frac{dy_1}{dt} = q (y_0 - y_1) - V_1 k_{b1} y_1 + V_1 k_{a1} x_1 \quad (5.6)$$

With similar arguments, the kinetics of the reactions in the second reactor can be written as

$$V_2 \frac{dx_2}{dt} = q (x_1 - x_2) - V_2 k_{a2} x_2 \quad (5.7)$$

and

$$V_2 \frac{dy_2}{dt} = q (y_1 - y_2) - V_2 k_{b2} y_2 + V_2 k_{a2} x_2 \quad (5.8)$$

The reaction rate constants are expressed as

$$\begin{aligned}
k_{a1} &= G_a \exp \left( -\frac{E_a}{RT_1} \right), & k_{b1} &= G_b \exp \left( -\frac{E_b}{RT_1} \right) \\
k_{a2} &= G_a \exp \left( -\frac{E_a}{RT_2} \right), & k_{b2} &= G_b \exp \left( -\frac{E_b}{RT_2} \right)
\end{aligned} \tag{5.9}$$

Equation (5.3) leads to

$$\frac{dI}{dt} = q y_2 - S \tag{5.10}$$

where  $I$  is the inventory,  $S$  is the sales rate, and  $q y_2$  is the production rate of  $B$ .

According to Equation (4.12), the performance equation for advertisement can be written as

$$\frac{dS}{dt} = S(c_c + A) \left( 1 - \frac{S}{N} \right) \tag{5.11}$$

where  $c_c$  is the constant coefficient,  $A$  is the advertisement, and  $N$  represents the total number of people in the group.

The profit function is obtained from Equation (5.4).

$$\begin{aligned}
J = \int_{t_i}^{t_f} & (c_1 c_q S + c_2 q x_2 + c_3 q (1 - x_2 - y_2) - c_I (I_m - I)^2 \\
& - c_A A^2 S^2 - c_T ((T_{1m} - T_1)^2 + (T_1 - T_2)^2)) dt
\end{aligned} \tag{5.12}$$

Equations (5.5) through (5.8), (5.10) and (5.11) represent the system. There are six state variables,  $x_1$ ,  $x_2$ ,  $y_1$ ,  $y_2$ ,  $I$ , and  $S$ , and three control variables,  $T_1$ ,  $T_2$ , and  $A$ .

### 5.3 Definition of the Problem

Maximize the functional

$$\begin{aligned}
J = \int_{t_i}^{t_f} & (c_1 c_q S + c_2 q x_2 + c_3 q (1 - x_2 - y_2) - c_I (I_m - I)^2 \\
& - c_A A^2 S^2 - c_T ((T_{1m} - T_1)^2 + (T_1 - T_2)^2)) dt
\end{aligned} \tag{5.13}$$

subject to the constraints

$$v_1 \frac{dx_1}{dt} = q(x_o - x_1) - v_1 k_{a1} x_1 \quad (5.14)$$

$$v_1 \frac{dy_1}{dt} = q(y_o - y_1) - v_1 k_{b1} y_1 + v_1 k_{a1} x_1 \quad (5.15)$$

$$v_2 \frac{dx_2}{dt} = q(x_1 - x_2) - v_2 k_{a2} x_2 \quad (5.16)$$

$$v_2 \frac{dy_2}{dt} = q(y_1 - y_2) - v_2 k_{b2} y_2 + v_2 k_{a2} x_2 \quad (5.17)$$

$$\frac{dI}{dt} = qy_2 - S \quad (5.18)$$

$$\frac{dS}{dt} = S(c_c + A) \left(1 - \frac{S}{N}\right) \quad (5.19)$$

with boundary conditions

$$\begin{aligned} x_1(t_i) &= x_1^o & y_2(t_i) &= y_2^o \\ y_1(t_i) &= y_1^o & I(t_i) &= I^o & I(t_f) &= I^1 \\ x_2(t_i) &= x_2^o & S(t_i) &= S^o \end{aligned} \quad (5.20)$$

#### 5.4 Formulation of the Problem

The problem defined above, becomes a two-point boundary-value type when calculus of variation is applied. Once formulated, the non-linear equations are converted into a set of linear equations by using the quasilinearization method. Lagrange Multipliers  $\lambda_i$ ,  $i=1,2,\dots$ , are introduced to define the following functions

$$\begin{aligned} F = & \lambda_1 \left( \dot{x}_1 - \frac{q}{v_1} (x_o - x_1) + k_{a1} x_1 \right) + \lambda_2 \left( \dot{y}_1 - \frac{q}{v_1} (y_o - y_1) + k_{b1} y_1 \right. \\ & \left. - k_{a1} x_1 \right) + \lambda_3 \left( \dot{x}_2 - \frac{q}{v_2} (x_1 - x_2) + k_{a2} x_2 \right) + \lambda_4 \left( \dot{y}_2 \right. \\ & \left. - \frac{q}{v_2} (y_1 - y_2) + k_{b2} y_2 - k_{a2} x_2 \right) + \lambda_5 \left( \dot{I} - qy_2 + S \right) \\ & + \lambda_6 \left( \dot{S} - c_s S - AS + \frac{cS^2}{N} + \frac{AS^2}{N} \right) + c_1 c_q S + c_2 q x_2 \end{aligned}$$



$$\begin{aligned}
& + c_3 q (1 - x_2 - y_2) - c_I (I_m - I)^2 - c_A A^2 S^2 - c_T ((T_{lm} - T_1)^2 \\
& + (T_1 - T_2)^2)
\end{aligned} \tag{5.21}$$

and

$$\begin{aligned}
G = & \theta_1 (x_1(t_i) - x_1^0) + \theta_2 (y_1(t_i) - y_1^0) + \theta_3 (x_2(t_i) - x_2^0) \\
& + \theta_4 (y_2(t_i) - y_2^0) + \theta_5 (I(t_i) - I^0) \\
& + \theta_6 (I(t_f) - I^1) + \theta_7 (S(t_i) - S^0)
\end{aligned} \tag{5.22}$$

where  $(\dot{\phantom{x}})$  represents  $(\frac{d}{dt})$ .

The Euler-Lagrange equations, Equation (4.19) with  $i=1,2,\dots,6$  and  $j=1,2,3$ , can be applied to Equation (5.21) to obtain the following adjoint equations.

$$\frac{d\lambda_1}{dt} = q \left( \frac{\lambda_1}{v_1} - \frac{\lambda_3}{v_2} \right) + (\lambda_1 - \lambda_2) k_{a1} \tag{5.23}$$

$$\frac{d\lambda_2}{dt} = q \left( \frac{\lambda_2}{v_1} - \frac{\lambda_4}{v_2} \right) + \lambda_2 k_{b1} \tag{5.24}$$

$$\frac{d\lambda_3}{dt} = \frac{\lambda_3 q}{v_2} + (\lambda_3 - \lambda_4) k_{a2} + q(c_2 - c_3) \tag{5.25}$$

$$\frac{d\lambda_4}{dt} = \frac{\lambda_4 q}{v_2} + \lambda_4 k_{b2} - q(c_2 + \lambda_5) \tag{5.26}$$

$$\frac{d\lambda_5}{dt} = 2c_I I_m - 2c_I I \tag{5.27}$$

$$\frac{d\lambda_6}{dt} = c_I + \lambda_5 - c \lambda_6 - A \lambda_6 + \frac{2cS \lambda_6}{N} + \frac{2AS \lambda_6}{N} - 2c_A A^2 S \tag{5.28}$$

and the equations for the control variables are

$$A = \frac{\lambda_6}{2c_A} \left( \frac{1}{N} - \frac{1}{S} \right) \tag{5.29}$$

$$\frac{(\lambda_1 - \lambda_2) k_{a1} E_a x_1}{RT_1^2} + \frac{\lambda_2 k_{b1} E_b y_1}{RT_1^2} - 2c_T (2T_1 - T_2 - T_m) = 0 \quad (5.30)$$

$$\frac{(\lambda_3 - \lambda_4) k_{a2} E_a x_1}{RT_2^2} + \frac{\lambda_4 k_{b2} E_b y_2}{RT_2^2} + 2c_T (T_1 - T_2) = 0 \quad (5.31)$$

Equation (5.29) gives explicit relation of the control variable  $A$  in terms of the adjoint variables and the sales. Using this relation,  $A$  can be eliminated from all the performance equations.  $A$  appears only in Equations (5.19) and (5.28). After eliminating  $A$ , these two equations can be rewritten as

$$\frac{dS}{dt} = cS - \frac{cS^2}{N} + \frac{S\lambda_6}{c_A N} - \frac{\lambda_6}{2c_A} - \frac{S^2\lambda_6}{2c_A N^2} \quad (5.32)$$

$$\frac{d\lambda_6}{dt} = c_1 + \lambda_5 - c\lambda_6 - \frac{\lambda_6^2}{2c_A N} + \frac{2cS\lambda_6}{N} + \frac{S\lambda_6^2}{2c_A N^2} \quad (5.33)$$

The control variables  $T_1$  and  $T_2$  cannot be eliminated from the performance equations because these appear implicitly in Equations (5.30) and (5.31).

Equation (5.14) through (5.18), (5.23) through (5.27), (5.32) and (5.33) represent the system. Of these, only Equations (5.32) and (5.33) are non-linear. For the 12 differential equations, only seven boundary conditions are known. The remaining five boundary conditions are obtained by applying the transversality condition as given in Equation (4.27).

These boundary conditions are

$$\begin{aligned} \lambda_1(t_f) &= 0 & \lambda_4(t_f) &= 0 \\ \lambda_2(t_f) &= 0 & \lambda_6(t_f) &= 0 \\ \lambda_3(t_f) &= 0 & & \end{aligned} \quad (5.34)$$

The original problem has been formulated into a 12 dimensional boundary-value type.

### 5.5 Quasilinearization

The linearization procedure described in Chapter 3 is used to linearize the Equations (5.32) and (5.33). The linearized equations are

$$\frac{dx_{1,n+1}}{dt} = \frac{q}{V_1} (x_0 - x_{1,n+1}) - G_a \exp\left(-\frac{E_a}{RT_1}\right) x_{1,n+1} \quad (5.35)$$

$$\begin{aligned} \frac{dy_{1,n+1}}{dt} = & \frac{q}{V_1} (y_0 - y_{1,n+1}) - G_b \exp\left(-\frac{E_b}{RT_1}\right) y_{1,n+1} \\ & + G_a \exp\left(-\frac{E_b}{RT_1}\right) x_{1,n+1} \end{aligned} \quad (5.36)$$

$$\frac{dx_{2,n+1}}{dt} = \frac{q}{V_2} (x_{1,n+1} - x_{2,n+1}) - G_a \exp\left(-\frac{E_a}{RT_2}\right) x_{2,n+1} \quad (5.37)$$

$$\begin{aligned} \frac{dy_{2,n+1}}{dt} = & \frac{q}{V_2} (y_{1,n+1} - y_{2,n+1}) - G_b \exp\left(-\frac{E_b}{RT_2}\right) y_{2,n+1} \\ & + G_a \exp\left(-\frac{E_a}{RT_1}\right) x_{2,n+1} \end{aligned} \quad (5.38)$$

$$\frac{dI_{n+1}}{dt} = qy_{2,n+1} - S_{n+1} \quad (5.39)$$

$$\begin{aligned} \frac{dS_{n+1}}{dt} = & cS_n - \frac{cS_n^2}{N} + \frac{S_n \lambda_{6,n}}{c_A N} - \frac{\lambda_{6,n}}{2c_A} - \frac{S_n^2 \lambda_{6,n}}{2c_A N^2} \\ & + (S_{n+1} - S_n) \left( c - \frac{2cS_n}{N} + \frac{\lambda_{6,n}}{c_A N} - \frac{S_n \lambda_{6,n}}{c_A N^2} \right) \\ & + (\lambda_{6,n+1} - \lambda_{6,n}) \left( \frac{S_n}{c_A N} - \frac{\lambda_1}{2c_A} - \frac{S_n^2}{2c_A N^2} \right) \end{aligned} \quad (5.40)$$

$$\frac{d\lambda_{1,n+1}}{dt} = q \left( \frac{\lambda_{1,n+1}}{V_1} - \frac{\lambda_{3,n+1}}{V_2} + (\lambda_{1,n+1} - \lambda_{2,n+1}) G_a \exp\left(-\frac{E_a}{RT_1}\right) \right) \quad (5.41)$$

$$\frac{d\lambda_{2,n+1}}{dt} = q \left( \frac{\lambda_{2,n+1}}{V_1} - \frac{\lambda_{4,n+1}}{V_2} \right) + \lambda_{2,n+1} G_b \exp\left(-\frac{E_b}{RT_1}\right) \quad (5.42)$$

$$\begin{aligned} \frac{d\lambda_{3,n+1}}{dt} = & \frac{q\lambda_{3,n+1}}{v_2} + (\lambda_{3,n+1} - \lambda_{4,n+1}) G_a \exp\left(-\frac{E_a}{RT_1}\right) \\ & + q(c_2 - c_3) \end{aligned} \quad (5.43)$$

$$\frac{d\lambda_{4,n+1}}{dt} = \frac{q\lambda_{4,n+1}}{v_2} + \lambda_{4,n+1} G_b \exp\left(-\frac{E_b}{RT_2}\right) - c_3 q - q\lambda_{5,n+1} \quad (5.44)$$

$$\frac{d\lambda_{5,n+1}}{dt} = 2c_I I_m - 2c_I I_{n+1} \quad (5.45)$$

$$\begin{aligned} \frac{d\lambda_{6,n+1}}{dt} = & c_1 + \lambda_{5,n+1} - c\lambda_{6,n} - \frac{\lambda_{6,n}^2}{2c_A N} + \frac{2cS_n \lambda_{6,n}}{N} \\ & + \frac{S_n \lambda_{6,n}^2}{2c_A N^2} + (S_{n+1} - S_n) \left[ \frac{2c\lambda_{6,n}}{N} + \frac{\lambda_{6,n}^2}{2c_A N^2} \right] \\ & + (\lambda_{6,n+1} - \lambda_{6,n}) \left[ \frac{2cS_n}{N} - c - \frac{\lambda_{6,n}}{c_A N} + \frac{S_n \lambda_{6,n}}{c_A N^2} \right] \end{aligned} \quad (5.46)$$

The boundary conditions are given by Equations (5.20) and (5.34).

This linear boundary-value problem can be solved fairly routinely on digital computers. The general solution can be written as

$$x_1(t) = x_{1p}(t) + \sum_{k=1}^6 A_k x_{1,kh}(t) \quad (5.47)$$

$$y_1(t) = y_{1p}(t) + \sum_{k=1}^6 A_k y_{1,kh}(t) \quad (5.48)$$

$$x_2(t) = x_{2p}(t) + \sum_{k=1}^6 A_k x_{2,kh}(t) \quad (5.49)$$

$$y_2(t) = y_{2p}(t) + \sum_{k=1}^6 A_k y_{2,kh}(t) \quad (5.50)$$

$$I(t) = I_p(t) + \sum_{k=1}^6 A_k I_{kh}(t) \quad (5.51)$$

$$S(t) = S_p(t) + \sum_{k=1}^6 A_k S_{kh}(t) \quad (5.52)$$

$$\lambda_i(t) = \lambda_{ip}(t) + \sum_{k=1}^6 A_k \lambda_{i,kh}(t) \quad i=1, \dots, 6 \quad (5.53)$$

Runge-Kutta method is used to obtain the solution for each set of initial conditions. The integration constant is determined by applying the final conditions to the Equations (5.51) and (5.53). Equation (5.53) for  $i=5$  is not considered.

## 5.6 Numerical and Computational Aspects

The model discussed above has a condition on inventory at the final point. Henceforth, this model will be called Model A. Four other models without final point condition on inventory will also be discussed. These are models B, C, D and E. In these models, function  $F$ , Equation (5.21), remains the same, but the function  $G$  is modified to:

$$G = \theta_1(x_1(t_i) - x_1^0) + \theta_2(y_1(t_i) - y_1^0) + \theta_3(x_2(t_i) - x_2^0) \\ + \theta_4(y_2(t_i) - y_2^0) + \theta_5(I(t_i) - I^0) + \theta_6(S(t_i) - S^0) \quad (5.54)$$

The only change is in the conditions obtained by applying the transversality relations to Equation (5.54). Now the remaining six boundary conditions are:

$$\begin{array}{lll} \lambda_1(t_f) = 0 & \lambda_3(t_f) = 0 & \lambda_5(t_f) = 0 \\ \lambda_2(t_f) = 0 & \lambda_4(t_f) = 0 & \lambda_6(t_f) = 0 \end{array} \quad (.55)$$

Proper modifications were made in the computer program to solve models B, C, D, and E. The values of the various parameters for each model are given in Table 5.1. The boundary conditions for each model are given in Table 5.2.

There were only two non-linear differential equations, hence only two initial approximations,  $S_o(t)$  and  $\lambda_{6,o}(t)$ , were required. Initial approximations for  $T_1$  and  $T_2$  are needed, because the equations for these two control variables are implicit. The various sets of initial approximations used for each model are given in Table 5.3. The initial values

used for the particular and the homogeneous solutions are given in Table 5.4.

To solve for six integration constants, the matrix inversion subroutine SIMQ supplied by IBM was used. Appendix (3) gives the print out of this subroutine. Using these integration constants, Equations (5.47) through (5.53) gave the final solution. The following approximation was used to calculate the total profit.

$$J = \sum_0^{t_f} \left[ c_1 c_q S + c_2 q x_2 + c_3 q (1 - x_2 - y_2) - c_I (I_m - I)^2 - c_A A^2 S^2 - c_T ((T_{1m} - T_1)^2 + (T_1 - T_2)^2) \right] \Delta t \quad (5.56)$$

To compute  $T_1$  and  $T_2$ , the Newton-Raphson root finding method was used. Lee<sup>17</sup> and Bellman et al.<sup>3</sup> have discussed this method. In the first iteration, at all the grid points the initial approximations for  $T_1$  and  $T_2$  were taken as given in Table 5.3. For other iterations, solutions of  $T_1$  and  $T_2$  at the same grid point of previous iteration were used as the initial approximations. The accuracy for this root finding method was kept as 0.1.

The above computations were used in both phases. The length of the period used in each phase is given with the results. The computer program is given in Appendix (2).

## 5.7 Results

The results will be discussed separately for each model. Tables 5.5 through 5.13 and Figures 5.2 through 5.32 show some of the results. The iteration which gives the optimal profile is the one at which it becomes difficult to differentiate between this profile and that of the next iteration.

For each model, different initial profiles are used. All the sets are solved with two-phase method. In general, only one iteration is allowed in phase 1.

#### Model A

The period in phase 1 used was 0 to 0.5 and in phase 2 was 0 to 1.0. Seven sets of initial profiles were used. These are listed in Table 5.3. The first six sets converged to the solution of the problem. Set 7A faced difficulty in the first iteration of phase 2, because of overflow in computer calculations. By giving 2 iterations in phase 1, it was found that overflow occurred in the second iteration of phase 1. It was tried to solve the problem by taking the duration of phase 1 as 0.1, 0.2, 0.3, 0.4, and 0.5. For each duration 2 iterations were allowed in phase 1. But overflow occurred in iteration 2 of phase 1 for all the durations and no convergence was obtained. Set 2A did not converge with 1 iteration in phase 1. However it converged with 2 iterations in phase 1.

The convergence rate of the variables and the profit function is shown in Figs. 5.2 through 5.12 for Set 3A. The profiles for  $T_1$  and  $T_2$  are very similar in nature. Therefore the profiles for  $T_1$  are shown in detail, and only the initial and optimal profiles are shown for  $T_2$ . Tables 5.11, and 5.12 show the convergence rate for Set 3A. The optimal results of this model can be summarized as

$$\begin{array}{llll}
 x_1(1.0) = 0.47 & x_2(1.0) = 0.41 & y_1(1.0) = 0.46 & y_2(1.0) = 0.48 \\
 I(1.0) = 10.0 & S(1.0) = 34.78 & A(o) = 371.0 & T_1(o) = 362.4 \\
 T_1(1.0) = 340.1 & T_2(o) = 365.6 & T_2(1.0) = 340.1 & J = 79.78
 \end{array}$$

In phase 2, 10 iterations were allowed. In general, six iterations were sufficient to give optimal profiles with 3 digit accuracy.

Model B

The optimal results are summarized in the following.

$$\begin{array}{llll}
 x_1(1.0) = 0.49 & x_2(1.0) = 0.43 & y_1(1.0) = 0.45 & y_2(1.0) = 0.48 \\
 I(1.0) = 2.46 & S(1.0) = 48.4 & A(o) = 481.5 & T_1(o) = 362.79 \\
 T_1(1.0) = 339.9 & T_2(o) = 365.7 & T_2(1.0) = 339.9 & J = 95.83
 \end{array}$$

Three different sets of initial profiles were used. These are listed in Table 5.3. All the sets converged to the optimal solution of the problem with duration 0.5 in phase 1. The optimal profiles for the six state variables, the three control variables and total profit are given in Figs. 5.13 through 5.15. Here, in general, five iterations in phase two are sufficient to give optimal results with 3 digit accuracy.

Model C

The optimal results are summarized in the following.

$$\begin{array}{llll}
 x_1(1.0) = 0.48 & x_2(1.0) = 0.42 & y_1(1.0) = 0.45 & y_2(1.0) = 0.48 \\
 I(1.0) = 3.77 & S(1.0) = 82.68 & A(o) = -5637 & T_1(o) = 362.4 \\
 T_1(1.0) = 339.9 & T_2(o) = 365.5 & T_2(1.0) = 339.9 & J = 113.85
 \end{array}$$

Two different sets of initial profiles were used. These are listed in Table 5.3. According to Shah<sup>29</sup>, both the problems of this model did not converge. He observed that in problem 1C the computer experienced exponential overflow in the first iteration, and in problem 2C the computation stopped in iteration 5. According to Shah the reason of such behaviour is the Newton-Raphson convergence problem.

Using the 2-phase method both the problems converged to the optimal solution. Problem 2C was found easier to solve. Phase 1 had period 0 to 0.5 and phase 2 had period 0 to 1.0. Problem 1C was difficult to solve. It did not converge when phase 1 had a period of 0 to 0.5 or of 0 to 0.2. However, the convergence was obtained with the two-phase



method and with phase 1 having a period of 0 to 0.3.

Another peculiarity of both of the problems was the discontinuity of the advertisement curve as shown in Fig. 5.18. During the initial stages advertisement goes negative besides showing discontinuity. Both the temperature profiles shown in Figs. 5.16 and 5.17 are similar to Models A and B. Other optimal results are shown in Figs. 5.19 and 5.20. The strange behaviour of advertisement and sales can be due to a very small value of  $c_A$  which comes in the denominator of Equations (5.40) and (5.46).

At the most 6 iterations are sufficient to obtain a 3 digit accuracy.

#### Model D

The optimal results are summarized in the following.

$$\begin{array}{llll}
 x_1(1.0) = 0.48 & x_2(1.0) = 0.42 & y_1(1.0) = 0.45 & y_2(1.0) = 0.48 \\
 I(1.0) = 12.9 & S(1.0) = 44.87 & A(o) = 11.69 & T_1(o) = 363.3 \\
 T_1(1.0) = 340.5 & T_2(o) = 365.8 & T_2(1.0) = 340.5 & J = 66.32
 \end{array}$$

Three different sets of initial profiles were used. These are given in Table 5.3. All the sets converged with duration 0.5 in phase 1. Figs. 21 through 30 show the convergence rate and the optimal profiles of problem 1D. All the problems converged to the optimal solution with the two-phase method. According to Shah<sup>29</sup> problem 1D did not converge when solved in one phase only. All the variables behaved in a manner similar to those in models A and B. One exception was the inventory profile because in model A there was a final condition on inventory whereas in the other models there was no final condition on inventory.

In general, 5 iterations in phase 2 give optimal results with 3 digit accuracy.

Model E

The optimal results are summarized in the following.

$$\begin{array}{llll}
 x_1(1.0) = 0.48 & x_2(1.0) = 0.42 & y_1(1.0) = 0.45 & y_2(1.0) = 0.48 \\
 I(1.0) = 13.0 & S(1.0) = 45.02 & A(o) = 164.8 & T_1(o) = 363.3 \\
 T_1(1.0) = 339.6 & T_2(o) = 365.8 & T_2(1.0) = 339.6 & J = 66.04
 \end{array}$$

Only one set of initial profiles was used. It converged to the optimal solution with duration 0.5 in phase 1. Figs. 31 and 32 show the optimal profiles of all the variables and the total profit. Only 5 iterations in phase 2 are enough to give optimal result with 3 digit accuracy.

On the average, for 1 iteration in phase 1 and 10 iterations in phase 2, IBM 360/50 computer took 16 minutes with FORTRAN 4, G LEVEL compiler.

### 5.8 Discussion

The 2-phase method of quasilinearization appears to be more effective than the 1-phase method. Table 5.5 gives a comparison of the two methods in solving 13 problems. It can be seen that 4 of these 13 problems did not converge to the optimal solution by the one-phase method. All the problems converged to the optimal solution by the 2-phase method. It is found that in most of the problems only 1 iteration in phase 1 and only 6 iterations in phase 2 are enough to give optimal results. Only problem 2A required two iterations in phase 1.

The results show that only problem 1C gave trouble when solved with period 0 to 0.5 in phase 1. Moreover, it did not give results when the period in phase 1 was reduced to 0.2. This shows that for very unstable problem, there is an optimal range for the period in phase 1. The period above or below this range will not give convergence.

Now the influence of the number of iterations in phase 1 on the

speed of convergence will be discussed. For this purpose all the models were again solved with 5 iterations in phase 1. The set 3A was solved with 1 through 5 iterations in phase 1. Table 5.13 shows that it is advantageous to allow less number of iterations in phase 1. It may so happen, as in set 2A, that by giving 1 iteration in phase 1, the problem may not converge at all in phase 2. In such cases more iterations are given in phase 1. The results clearly show that the number of iterations in phase 1 should be small to save computation time. Briefly it will suffice to say that there is no advantage in getting the optimal profile in phase 1, when 1 or 2 iterations are enough.

Models 1A, 2A and 3A were also solved by using the backward phasing scheme. In this scheme phase 1 consists of period 0.5 to 1.0. The initial profile given in Table 5.3 is used to start the process. The first profile is computed for this period. This profile is used as an initial profile in phase 2. In the period 0 to 0.5 of phase 2, the initial profile is constant and is equal to the value of the variable in first profile at time 0.5 in phase 1. In the period .5 to 1.0 of phase 2, the initial profile is the first profile found in phase 1. Thus the initial profile for the whole period is determined in phase 2. The setting of initial profile in phase 2 is the only difference between the forward phasing and the backward phasing. For Model 1A, 7 iterations were required in backward phasing, and 8 iterations were required in forward phasing, to achieve convergence up to 3 decimal phases. For Model 2A, the forward phasing method failed with 1 iteration in phase 1, but backward phasing method worked successfully. For model 3A, 8 iterations were required in backward phasing, and 7 iterations were required in forward phasing to achieve convergence. At times, backward phasing method turns

out to be better.

Tables 5.6 through 5.10 show the effect of initial profile on convergence rate. It is found that for the given parameters, the convergence rate is independent of the initial profiles.

In studying the optimal profile it can be said that for the six state variables and the two control variables, the profiles do not show great slopes in the initial stages. Only the advertisement A decreases very fast in the initial stages. In model C, it shows a peculiar tendency of being discontinuous on two occasions in the initial stages. Survival of the 2-phase method in such peculiar conditions is the point worth noting.

Detailed analysis of the models shows that the influence of the various parameters on the optimal profiles of the concentrations and the temperatures is insignificant. The advertisement rate, sales rate, and the inventory were significantly different for the models. The management can study in advance, the different situations arising by changing the parameters.

It was thought to use the 3-phase method for the problems which do not converge in phase 2. To find such a problem, the initial temperature profiles of set 3A were increased 5°F at a time until 370°F. With this change the problem did not give any trouble in the 2-phase method. Then keeping the initial profiles for the temperatures at 370°F, the initial profile for sales rate was taken as 50, 70, and 90 in the sets 5A, 6A, and 7A, respectively. The sets 5A and 6A did not give any trouble in the 2-phase method. The set 7A did not converge in phase 1. No problem could be found which faced difficulty in phase 2. Thus, the 3-phase method could not be applied to determine if it works effectively.

TABLE 5.1 PARAMETERS FOR THE ADVERTISING AND PRODUCTION SCHEDULING PROBLEM.

Parameters common to all the models:

|                                       |                           |
|---------------------------------------|---------------------------|
| $G_a = 0.535 \times 10^{11}$ per min. | $N = 100$                 |
| $G_b = 0.461 \times 10^{18}$ per min. | $c = 1$                   |
| $E_a = 18,000$ cal/mole               | $t_i = 0.0$               |
| $E_b = 3,000$ cal/mole                | $c_I = 1.0\$/\text{gal.}$ |
| $R = 2$ cal/mole $^{\circ}\text{K}$   | $c_1 = 5.0\%$             |
| $q = 60$ gal/min.                     | $c_2 = c_3 = 0.0\%$       |
| $V_1 = V_2 = 12$ gal.                 | $c_q = 1.0$               |
| $T_{lm} = 340^{\circ}\text{K}$        | $\Delta t = 0.01$         |
| $x_o(t) = 0.53$                       | $y_o(t) = 0.43$           |

| Other Parameters: | $I_m$<br>in gallons | $c_A$<br>in \$ | $c_T$<br>in $\$/^{\circ}\text{K}$ |
|-------------------|---------------------|----------------|-----------------------------------|
| Model A and B     | 10                  | 0.01           | 0.001                             |
| Model C           | 20                  | 0.0002         | 0.0005                            |
| Model D and E     | 20                  | 0.01           | 0.0005                            |

TABLE 5.2 BOUNDARY CONDITIONS FOR THE ADVERTISING  
AND PRODUCTION SCHEDULING PROBLEM

Conditions common to all the models:

---

|                 |                 |
|-----------------|-----------------|
| $x_1(o) = 0.53$ | $y_1(o) = 0.43$ |
| $x_2(o) = 0.53$ | $y_2(o) = 0.43$ |

---

| Other Conditions: | I(o) | S(o) | I(1) |
|-------------------|------|------|------|
| Model A           | 1.0  | 0.1  | 10.0 |
| Model B           | 1.0  | 0.1  | ---- |
| Model C           | 8.0  | 0.1  | ---- |
| Model D           | 8.0  | 1.0  | ---- |
| Model E           | 8.0  | 0.1  | ---- |

---

TABLE 5.3 INITIAL PROFILES FOR PHASE 1 OF THE  
ADVERTISING AND PRODUCTION SCHEDULING  
PROBLEM.

| Set            | $T_{1,o}(t)$ | $T_{2,o}(t)$ | $S_o(t)$ | $\lambda_{6,o}(t)$ |
|----------------|--------------|--------------|----------|--------------------|
| <u>Model A</u> |              |              |          |                    |
| 1A             | 345          | 345          | 1        | 0                  |
| 2A             | 345          | 345          | 50       | -0.5               |
| 3A             | 345          | 345          | 30       | -1.0               |
| 4A             | 330          | 330          | 1        | 0                  |
| 5A             | 370          | 370          | 50       | -1.0               |
| 6A             | 370          | 370          | 70       | -1.0               |
| 7A             | 370          | 370          | 90       | -1.0               |
| <u>Model B</u> |              |              |          |                    |
| 1B             | 330          | 330          | 1        | 0                  |
| 2B             | 340          | 340          | 1        | 0                  |
| 3B             | 345          | 345          | 1        | 0                  |
| <u>Model C</u> |              |              |          |                    |
| 1C             | 330          | 330          | 1        | 0                  |
| 2C             | 345          | 345          | 1        | 0                  |
| <u>Model D</u> |              |              |          |                    |
| 1D             | 355          | 355          | 20       | -0.5               |
| 2D             | 345          | 345          | 50       | 0.0                |
| 3D             | 350          | 350          | 25       | -0.25              |
| <u>Model E</u> |              |              |          |                    |
| 1E             | 345          | 345          | 50       | 0                  |





TABLE 5.5 COMPARISON OF ONE-PHASE METHOD  
WITH TWO-PHASE METHOD.

| Problem | One Phase Method | Two Phase Method |
|---------|------------------|------------------|
| 1A      | Converged        | Converged        |
| 2A      | Converged        | Converged        |
| 3A      | Not Converge     | Converged        |
| 4A      | Converged        | Converged        |
| 1B      | Converged        | Converged        |
| 2B      | Converged        | Converged        |
| 3B      | Converged        | Converged        |
| 1C      | Not Converge     | Converged*       |
| 2C      | Not Converge     | Converged        |
| 1D      | Not Converge     | Converged        |
| 2D      | Converged        | Converged        |
| 3D      | Converged        | Converged        |
| 1E      | Converged        | Converged        |

\*The duration of phase 1 is 0 to 0.3.

TABLE 5.6 CONVERGENCE RATE OF  $T_1(o)$ 

| Problem*  | 1A    | 3A    | 4A    | 5A    | 6A    | 1B    | 2B    |
|-----------|-------|-------|-------|-------|-------|-------|-------|
| Iteration |       |       |       |       |       |       |       |
| 0         | 345.0 | 345.0 | 330.0 | 370.0 | 370.0 | 330.0 | 340.0 |
| 1 (PH.1)  | 366.5 | 366.7 | 367.5 | 359.8 | 359.7 | 367.1 | 366.6 |
| 1 (PH.2)  | 366.3 | 360.4 | 367.6 | 364.0 | 363.7 | 360.9 | 361.3 |
| 2         | 360.0 | 362.9 | 358.9 | 361.6 | 361.6 | 363.2 | 363.0 |
| 3         | 363.3 | 362.2 | 363.7 | 362.7 | 362.8 | 362.6 | 362.7 |
| 4         | 361.9 | 362.4 | 361.7 | 362.2 | 362.1 | 362.8 | 362.7 |
| 5         | 362.6 | 362.4 | 362.7 | 362.5 | 362.5 | 362.7 |       |
| 6         | 362.3 |       | 362.2 | 362.3 | 362.3 | 362.7 |       |
| 7         | 362.4 |       | 362.4 | 362.4 | 362.4 |       |       |
|           | 3B    | 1C    | 2C    | 1D    | 2D    | 3D    | 1E    |
| 0         | 345.0 | 330.0 | 345.0 | 355.0 | 345.0 | 350.0 | 345.0 |
| 1 (PH.1)  | 366.2 | 367.9 | 367.2 | 366.0 | 367.1 | 366.6 | 367.1 |
| 1 (PH.2)  | 361.6 | 361.7 | 360.7 | 362.6 | 361.8 | 362.1 | 361.8 |
| 2         | 362.9 | 362.1 | 362.9 | 363.4 | 363.6 | 363.5 | 363.6 |
| 3         | 362.7 | 362.7 | 362.3 | 363.3 | 363.2 | 363.2 | 363.2 |
| 4         | 362.7 | 362.3 | 362.5 | 363.3 | 363.3 | 363.3 | 363.3 |
| 5         |       | 362.5 | 362.4 | 363.3 | 363.3 | 363.3 | 363.3 |
| 6         |       | 362.4 | 362.4 |       |       |       |       |

\*Iterations for set 2A are not shown. Two iterations were required in phase 1 to solve this problem.









TABLE 5.11 CONVERGENCE RATES OF THE STATE VARIABLES  
FOR SET 3A, (PHASE 2)

| $t_k$       | $x_1(t_k)$ | $x_2(t_k)$ | $y_1(t_k)$ | $y_2(t_k)$ | $I(t_k)$ | $S(t_k)$ |
|-------------|------------|------------|------------|------------|----------|----------|
| Iteration 1 |            |            |            |            |          |          |
| 0.0         | 0.530      | 0.530      | 0.430      | 0.430      | 1.00     | 0.10     |
| 0.2         | 0.462      | 0.430      | 0.456      | 0.465      | 5.79     | 8.82     |
| 0.4         | 0.451      | 0.389      | 0.461      | 0.476      | 8.74     | 16.60    |
| 0.6         | 0.483      | 0.417      | 0.453      | 0.473      | 10.43    | 23.40    |
| 0.8         | 0.502      | 0.457      | 0.447      | 0.466      | 10.80    | 29.18    |
| 1.0         | 0.508      | 0.478      | 0.445      | 0.461      | 10.00    | 34.43    |
| Iteration 2 |            |            |            |            |          |          |
| 0.0         | 0.530      | 0.530      | 0.430      | 0.430      | 1.00     | 0.10     |
| 0.2         | 0.488      | 0.462      | 0.454      | 0.466      | 5.55     | 8.78     |
| 0.4         | 0.473      | 0.436      | 0.461      | 0.480      | 8.61     | 17.32    |
| 0.6         | 0.458      | 0.415      | 0.464      | 0.486      | 10.20    | 24.29    |
| 0.8         | 0.448      | 0.393      | 0.464      | 0.487      | 10.63    | 30.17    |
| 1.0         | 0.462      | 0.386      | 0.461      | 0.484      | 10.00    | 34.73    |
| Iteration 3 |            |            |            |            |          |          |
| 0.0         | 0.530      | 0.530      | 0.430      | 0.430      | 1.00     | 0.10     |
| 0.2         | 0.480      | 0.452      | 0.456      | 0.467      | 5.55     | 8.81     |
| 0.4         | 0.471      | 0.424      | 0.461      | 0.481      | 8.62     | 17.36    |
| 0.6         | 0.471      | 0.418      | 0.462      | 0.485      | 10.23    | 24.32    |
| 0.8         | 0.471      | 0.418      | 0.462      | 0.487      | 10.63    | 29.94    |
| 1.0         | 0.482      | 0.423      | 0.459      | 0.486      | 10.00    | 34.75    |
| Iteration 4 |            |            |            |            |          |          |
| 0.0         | 0.530      | 0.530      | 0.430      | 0.430      | 1.00     | 0.10     |
| 0.2         | 0.482      | 0.455      | 0.455      | 0.466      | 5.55     | 8.82     |
| 0.4         | 0.470      | 0.427      | 0.462      | 0.481      | 8.61     | 17.37    |
| 0.6         | 0.465      | 0.416      | 0.463      | 0.486      | 10.23    | 24.35    |
| 0.8         | 0.463      | 0.409      | 0.464      | 0.488      | 10.62    | 29.98    |
| 1.0         | 0.475      | 0.411      | 0.460      | 0.487      | 10.00    | 34.79    |

TABLE 5.11 (CONT.)

| $t_k$       | $x_1(t_k)$ | $x_2(t_k)$ | $y_1(t_k)$ | $y_2(t_k)$ | $I(t_k)$ | $S(t_k)$ |
|-------------|------------|------------|------------|------------|----------|----------|
| Iteration 5 |            |            |            |            |          |          |
| 0.0         | 0.530      | 0.530      | 0.430      | 0.430      | 1.00     | 0.10     |
| 0.2         | 0.482      | 0.454      | 0.456      | 0.467      | 5.55     | 8.81     |
| 0.4         | 0.471      | 0.427      | 0.461      | 0.481      | 8.61     | 17.37    |
| 0.6         | 0.468      | 0.417      | 0.463      | 0.486      | 10.23    | 24.34    |
| 0.8         | 0.466      | 0.413      | 0.463      | 0.488      | 10.62    | 29.97    |
| 1.0         | 0.477      | 0.416      | 0.460      | 0.487      | 10.00    | 34.78    |
| Iteration 6 |            |            |            |            |          |          |
| 0.0         | 0.530      | 0.530      | 0.430      | 0.430      | 1.00     | 0.10     |
| 0.2         | 0.482      | 0.454      | 0.456      | 0.467      | 5.55     | 8.81     |
| 0.4         | 0.471      | 0.427      | 0.461      | 0.481      | 8.61     | 17.37    |
| 0.6         | 0.467      | 0.417      | 0.463      | 0.486      | 10.23    | 24.35    |
| 0.8         | 0.465      | 0.411      | 0.464      | 0.488      | 10.62    | 29.97    |
| 1.0         | 0.477      | 0.414      | 0.460      | 0.487      | 10.00    | 34.78    |
| Iteration 7 |            |            |            |            |          |          |
| 0.0         | 0.530      | 0.530      | 0.430      | 0.430      | 1.00     | 0.10     |
| 0.2         | 0.482      | 0.454      | 0.455      | 0.467      | 5.55     | 8.81     |
| 0.4         | 0.471      | 0.427      | 0.461      | 0.481      | 8.61     | 17.37    |
| 0.6         | 0.467      | 0.417      | 0.463      | 0.486      | 10.23    | 24.35    |
| 0.8         | 0.465      | 0.412      | 0.463      | 0.488      | 10.62    | 29.97    |
| 1.0         | 0.477      | 0.414      | 0.460      | 0.487      | 10.00    | 34.78    |



TABLE 5.12 CONVERGENCE RATES OF THE CONTROL  
VARIABLES FOR SET 3A, (PHASE 2)

| $t_k$       | $T_1(t_k)$ | $T_2(t_k)$ | $A(t_k)$ | $J(t_k)$ |
|-------------|------------|------------|----------|----------|
| Iteration 1 |            |            |          |          |
| 0.0         | 360.4      | 364.8      | 348.9    | -0.93    |
| 0.2         | 357.6      | 358.8      | 4.6      | -8.56    |
| 0.4         | 360.2      | 357.8      | 1.6      | 0.35     |
| 0.6         | 363.3      | 360.8      | 0.6      | 19.68    |
| 0.8         | 363.5      | 363.3      | 0.1      | 45.91    |
| 1.0         | 340.0      | 340.0      | 0.0      | 77.80    |
| Iteration 2 |            |            |          |          |
| 0.0         | 362.9      | 365.9      | 368.8    | -0.94    |
| 0.2         | 359.5      | 361.3      | 4.6      | -8.71    |
| 0.4         | 357.9      | 359.3      | 1.7      | 0.41     |
| 0.6         | 357.3      | 358.0      | 0.6      | 20.32    |
| 0.8         | 357.0      | 357.6      | 0.2      | 47.24    |
| 1.0         | 340.1      | 340.0      | 0.0      | 79.76    |
| Iteration 3 |            |            |          |          |
| 0.0         | 362.2      | 365.6      | 370.8    | -0.94    |
| 0.2         | 359.2      | 360.6      | 4.6      | -8.73    |
| 0.4         | 359.1      | 358.9      | 1.7      | 0.40     |
| 0.6         | 359.7      | 358.9      | 0.6      | 20.35    |
| 0.8         | 359.5      | 359.5      | 0.2      | 47.33    |
| 1.0         | 340.0      | 340.0      | 0.0      | 79.73    |
| Iteration 4 |            |            |          |          |
| 0.0         | 362.4      | 365.7      | 371.0    | -0.94    |
| 0.2         | 359.2      | 360.8      | 4.6      | -8.73    |
| 0.4         | 358.6      | 359.0      | 1.7      | 0.41     |
| 0.6         | 358.9      | 358.6      | 0.6      | 20.36    |
| 0.8         | 358.6      | 358.9      | 0.2      | 47.36    |
| 1.0         | 340.0      | 340.0      | 0.0      | 79.78    |

TABLE 5.12 (CONT.)

| $t_k$       | $T_1(t_k)$ | $T_2(t_k)$ | $A(t_k)$ | $J(t_k)$ |
|-------------|------------|------------|----------|----------|
| Iteration 5 |            |            |          |          |
| 0.0         | 362.4      | 365.6      | 370.9    | -0.94    |
| 0.2         | 359.2      | 360.7      | 4.6      | -8.73    |
| 0.4         | 358.8      | 359.0      | 1.7      | 0.41     |
| 0.6         | 359.2      | 358.7      | 0.6      | 20.36    |
| 0.8         | 358.9      | 359.1      | 0.2      | 47.36    |
| 1.0         | 340.0      | 340.0      | 0.0      | 79.78    |
| Iteration 6 |            |            |          |          |
| 0.0         | 362.4      | 365.6      | 371.0    | -0.94    |
| 0.2         | 359.2      | 360.7      | 4.6      | -8.73    |
| 0.4         | 358.7      | 359.0      | 1.7      | 0.41     |
| 0.6         | 359.1      | 358.6      | 0.6      | 20.36    |
| 0.8         | 358.8      | 359.0      | 0.2      | 47.36    |
| 1.0         | 340.0      | 340.0      | 0.0      | 79.78    |
| Iteration 7 |            |            |          |          |
| 0.0         | 362.4      | 365.6      | 371.0    | -0.94    |
| 0.2         | 359.2      | 360.8      | 4.6      | -8.73    |
| 0.4         | 358.7      | 359.0      | 1.7      | 0.41     |
| 0.6         | 359.1      | 358.7      | 0.6      | 20.36    |
| 0.8         | 358.8      | 359.1      | 0.2      | 47.36    |
| 1.0         | 340.0      | 340.0      | 0.0      | 79.78    |

TABLE 5.13 THE EFFECT OF THE NUMBER OF ITERATIONS  
IN PHASE 1 ON THE CONVERGENCE RATE IN  
PHASE 2.

| Number of Iterations required in<br>Phase 2 for 3 Digit Accuracy. |                            |                          |
|-------------------------------------------------------------------|----------------------------|--------------------------|
| Set                                                               | 1 Iteration in<br>Phase 1. | 5 Iterations in Phase 1. |
| 1A                                                                | 7                          | 7                        |
| 2A                                                                | 6*                         | 6                        |
| 3A                                                                | 6                          | 6                        |
| 4A                                                                | 8                          | 6                        |
| 1B                                                                | 5                          | 6                        |
| 2B                                                                | 5                          | 6                        |
| 3B                                                                | 6                          | 6                        |
| 1C                                                                | 6                          | 6                        |
| 2C                                                                | 5                          | 6                        |
| 1D                                                                | 6                          | 6                        |
| 2D                                                                | 5                          | 6                        |
| 3D                                                                | 5                          | 5                        |
| 1E                                                                | 5                          | 6                        |

\*Two iterations were used in phase 1.

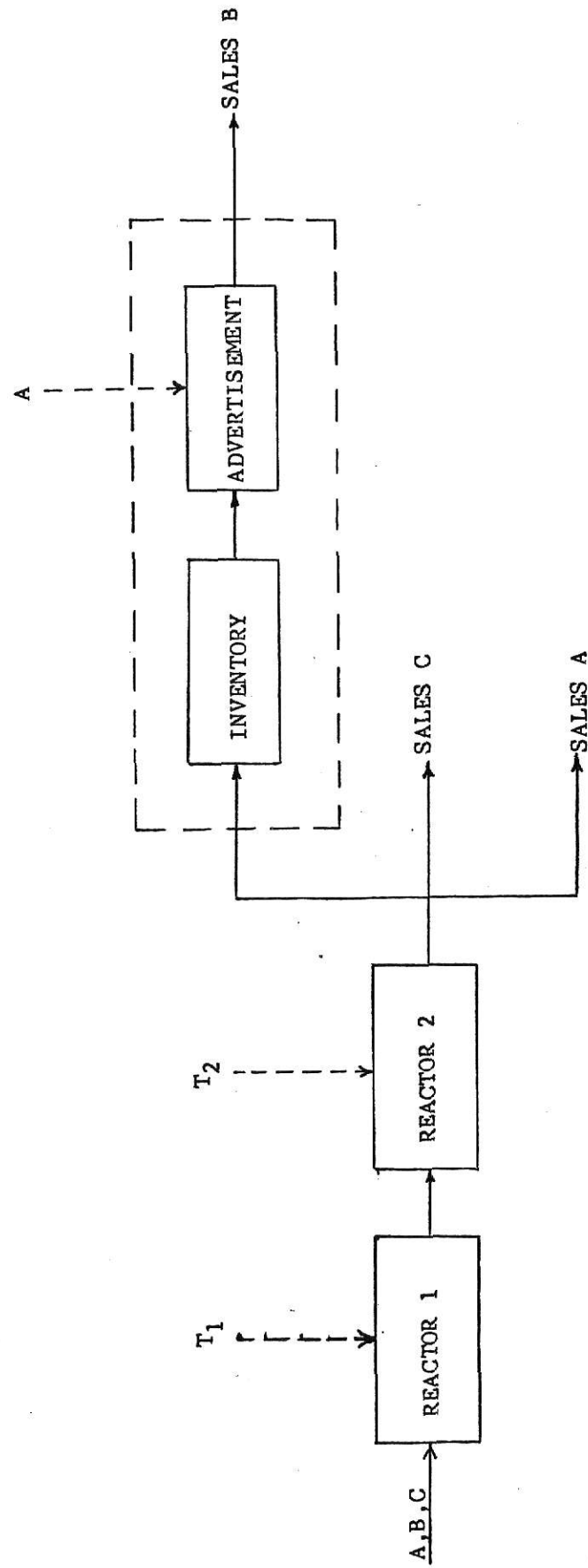


Fig. 5.1 Advertisement and Production Scheduling Model.

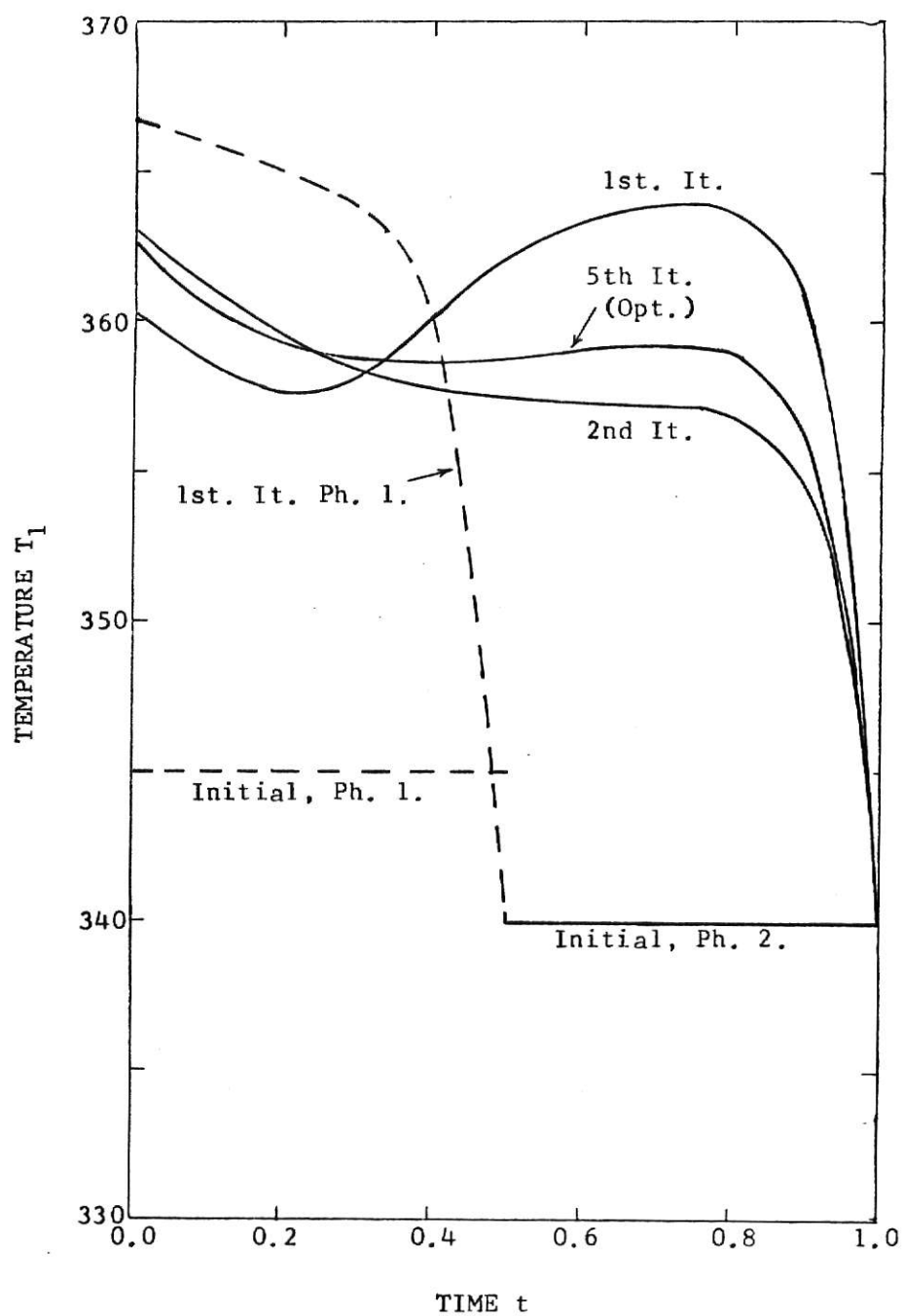


Fig. 5.2 Convergence Rate of Temperature  $T_1$ , Problem 3A.

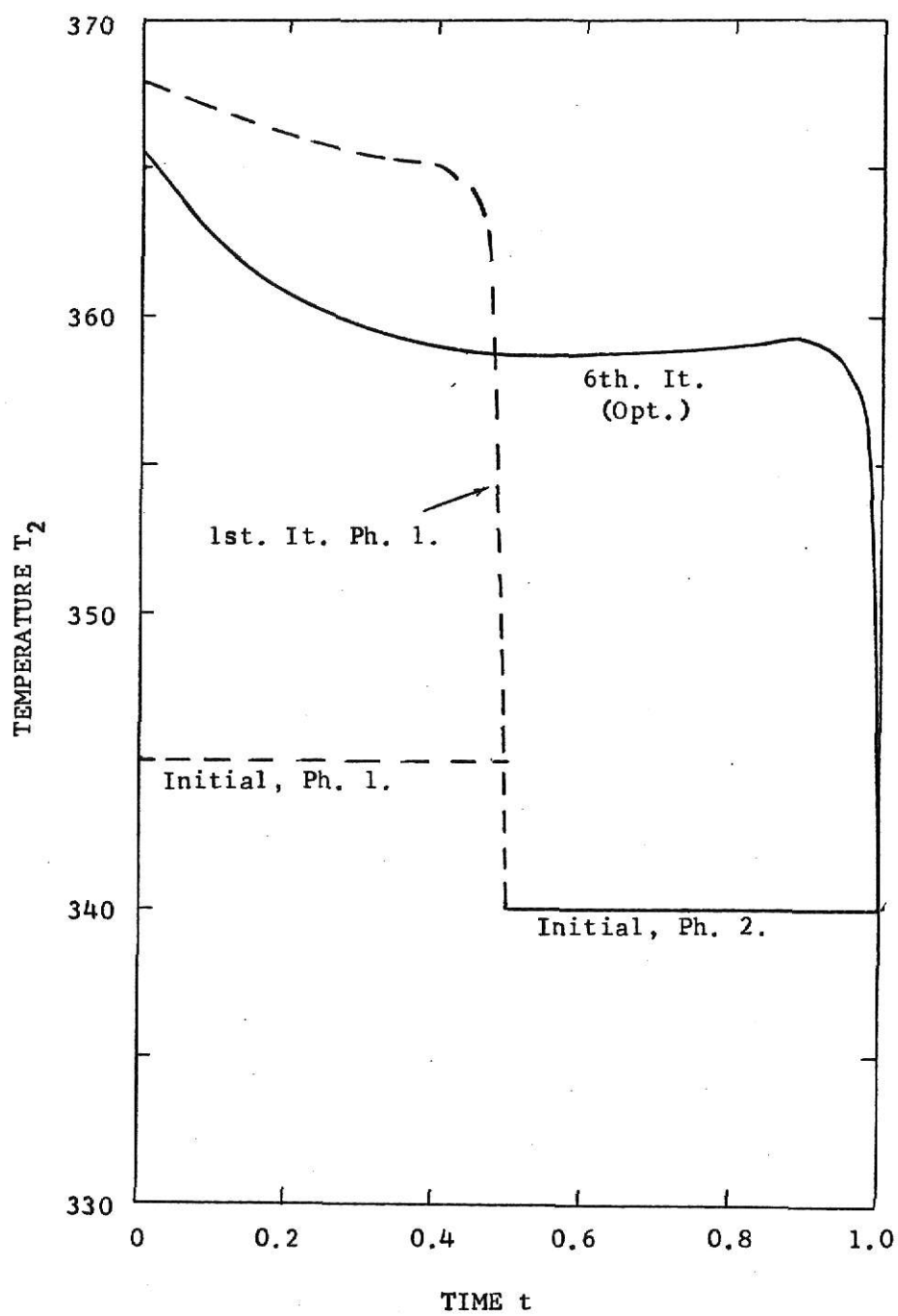


Fig. 5.3 Initial and Optimal Profile of Temperature  $T_2$ , Problem 3A.

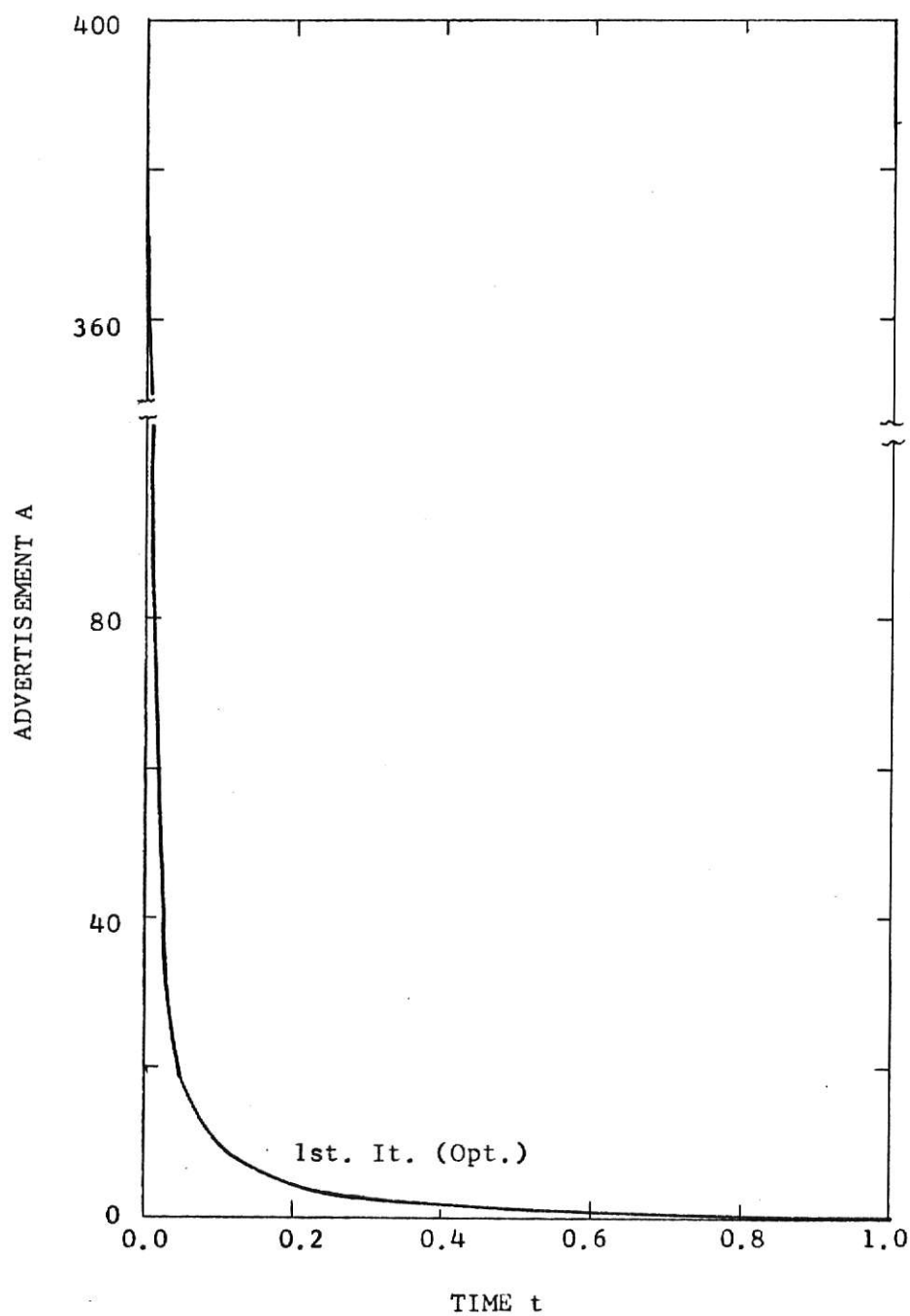


Fig. 5.4 Convergence Rate of Advertisement A, Problem 3A.

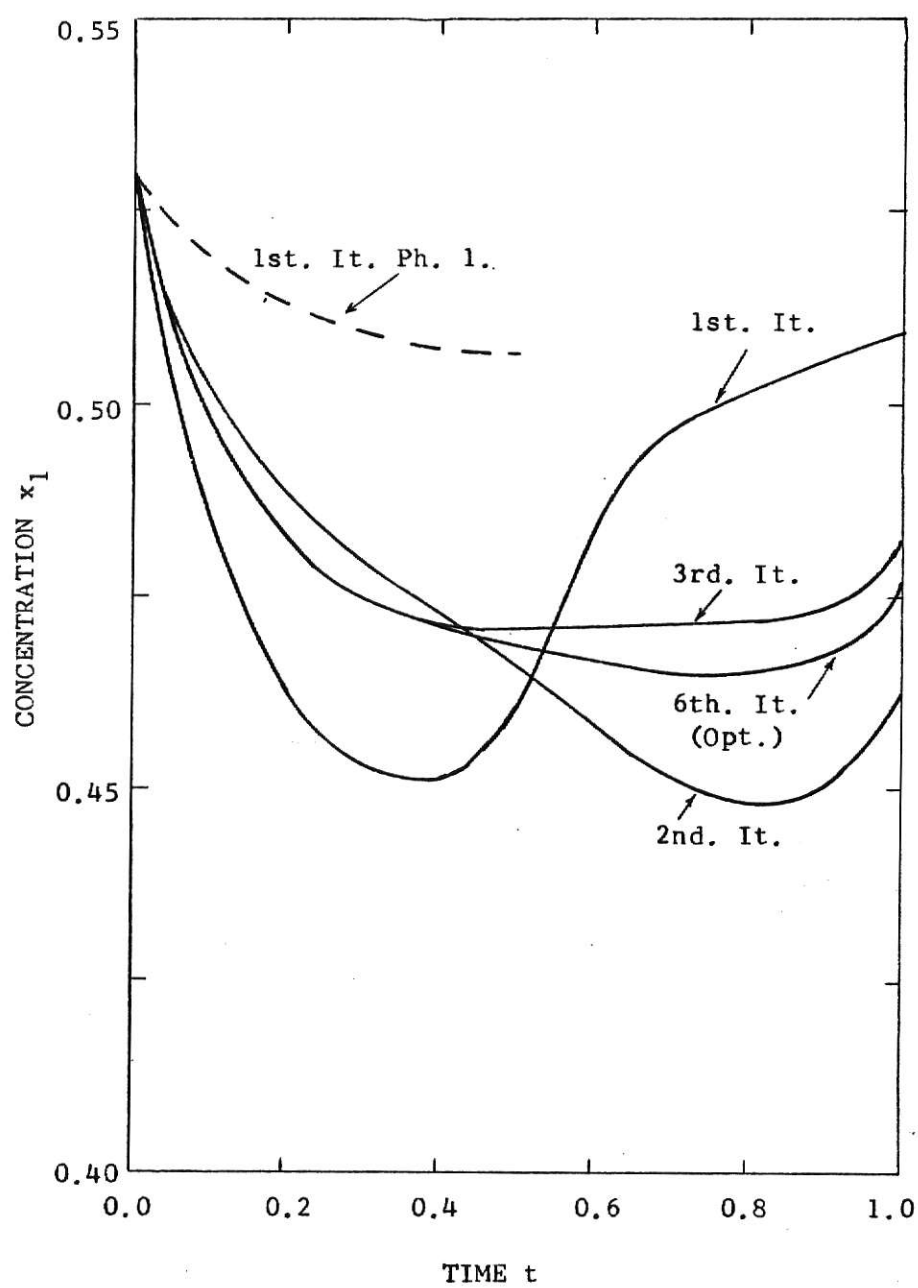


Fig. 5.5 Convergence Rate of Concentration  $x_1$ , Problem 3A.



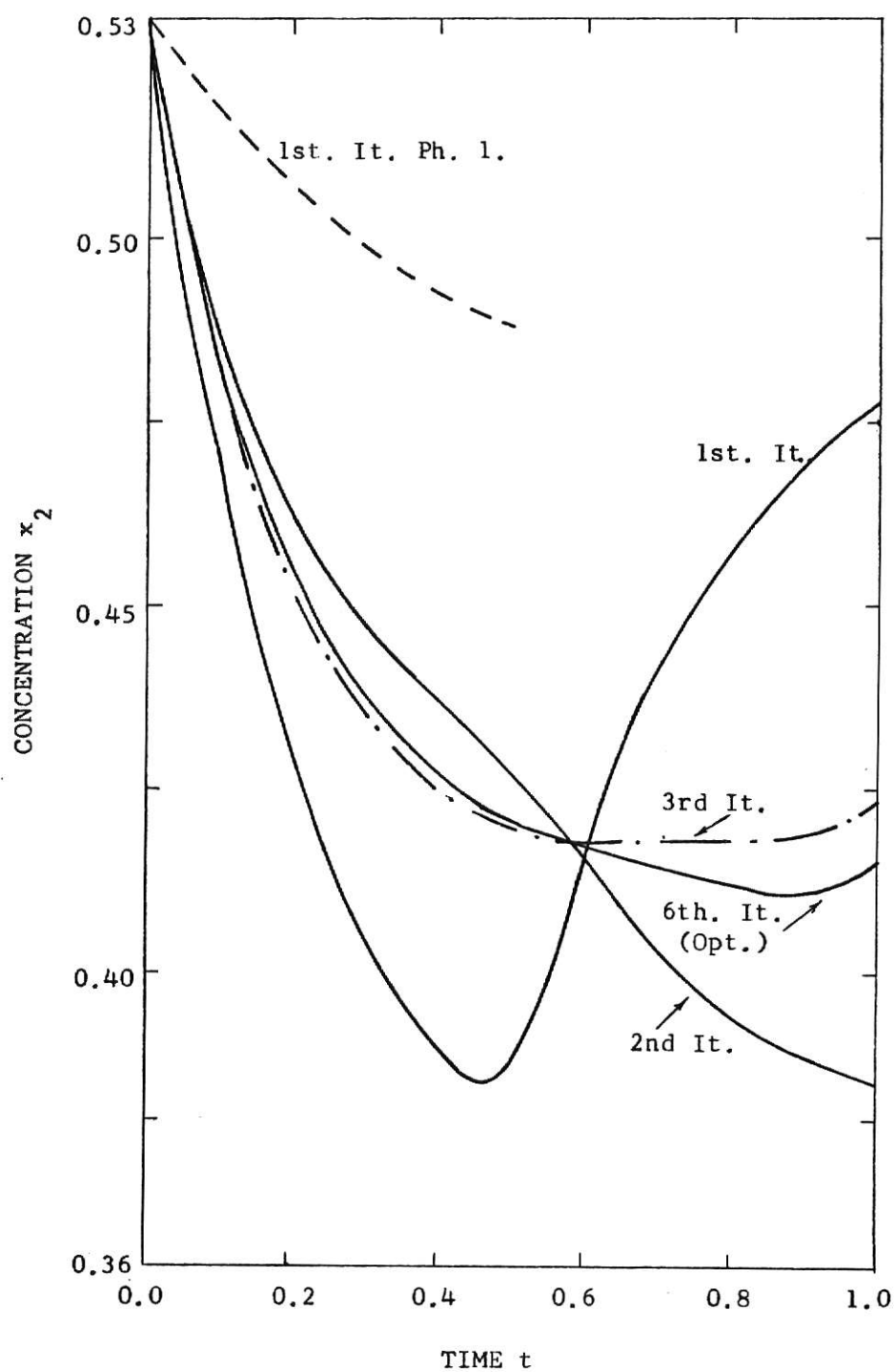


Fig. 5.6 Convergence Rate of Concentration  $x_2$ , Problem 3A.

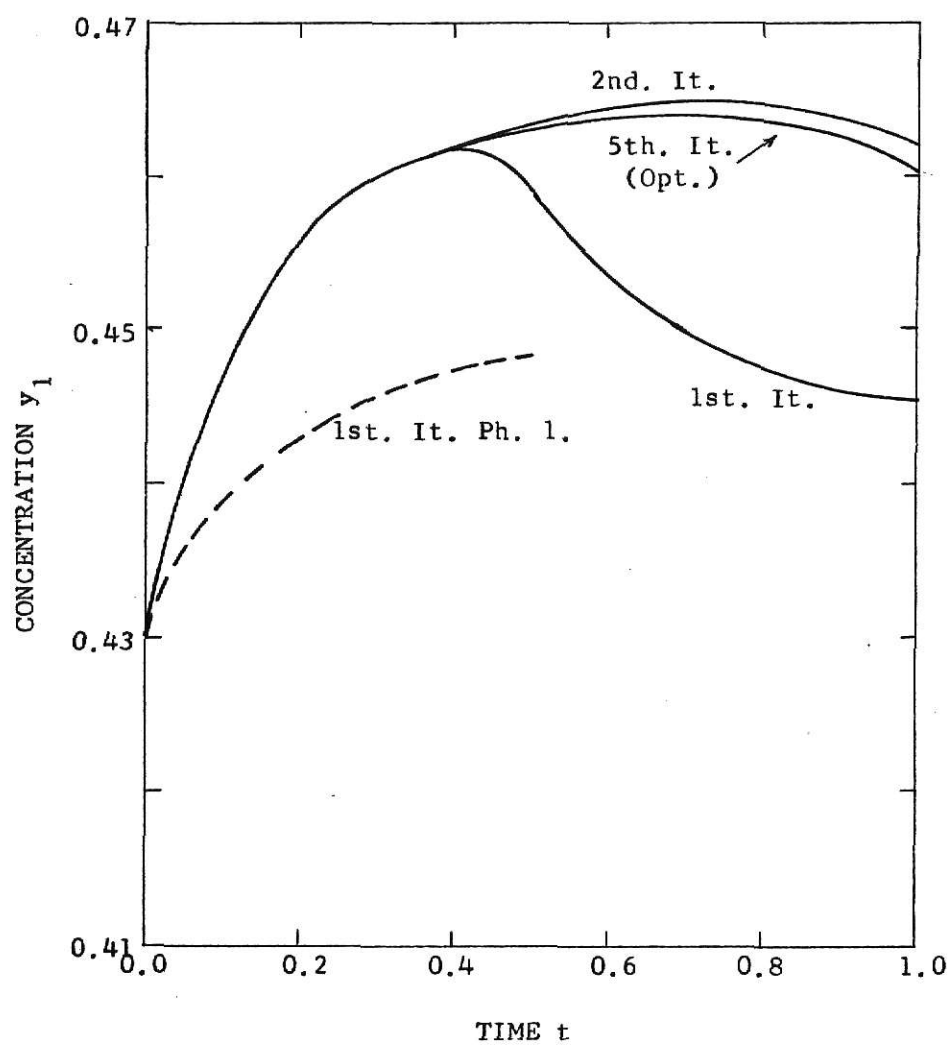


Fig. 5.7 Convergence Rate of Concentration  $y_1$ , Problem 3A.

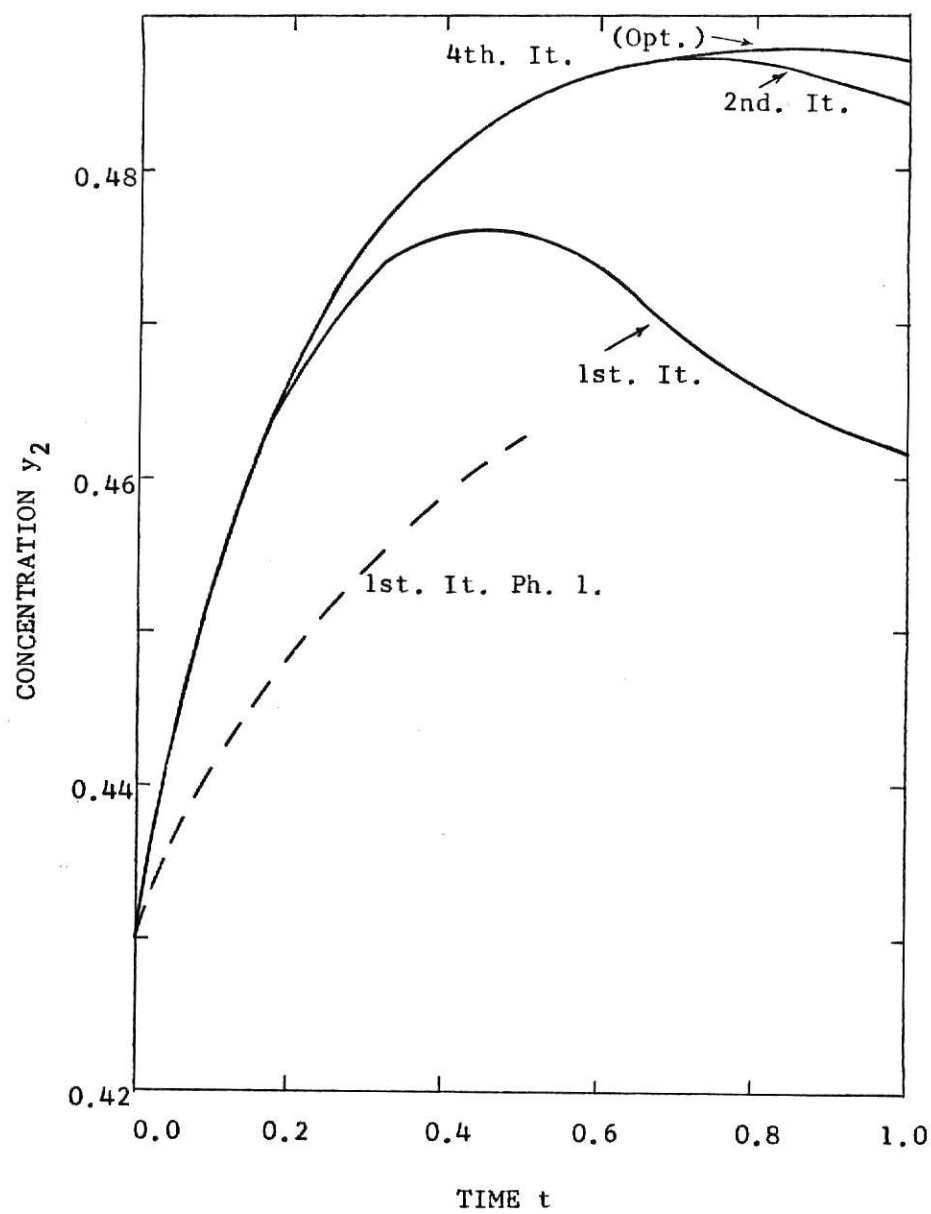


Fig. 5.8 Convergence Rate of Concentration  $y_2$ , Problem 3A.

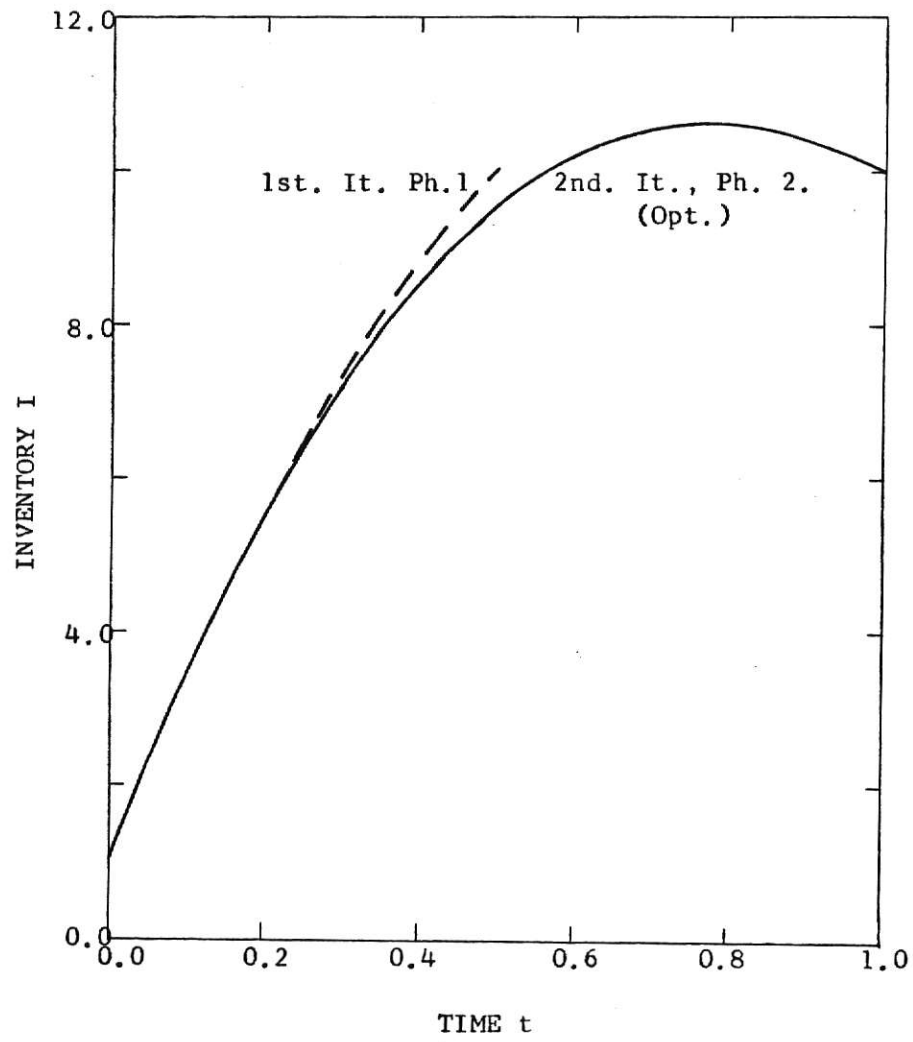


Fig. 5.9 Convergence Rate of Inventory I, Problem 3A.

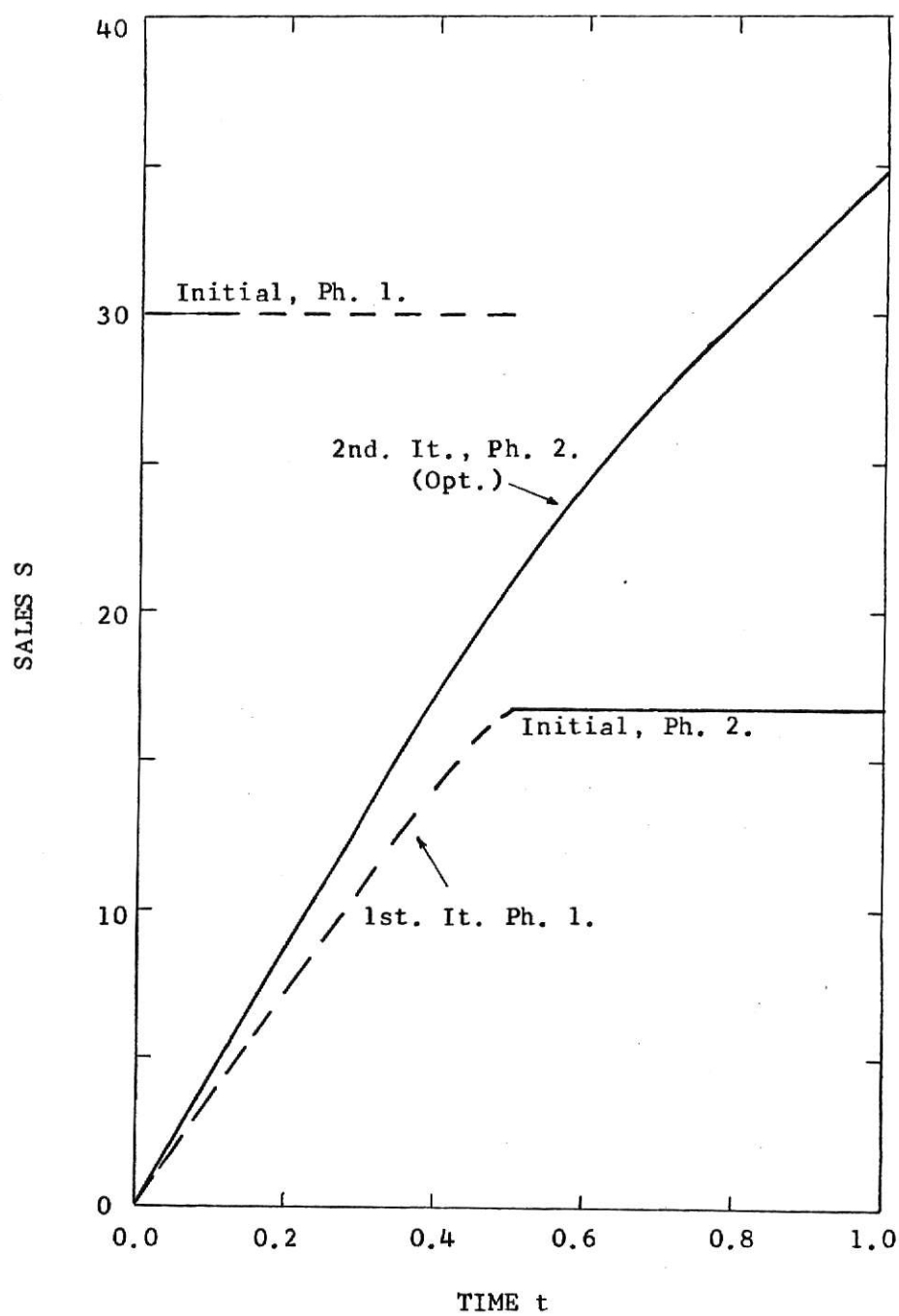


Fig. 5.10 Convergence Rate of Sales  $S$ , Problem 3A.

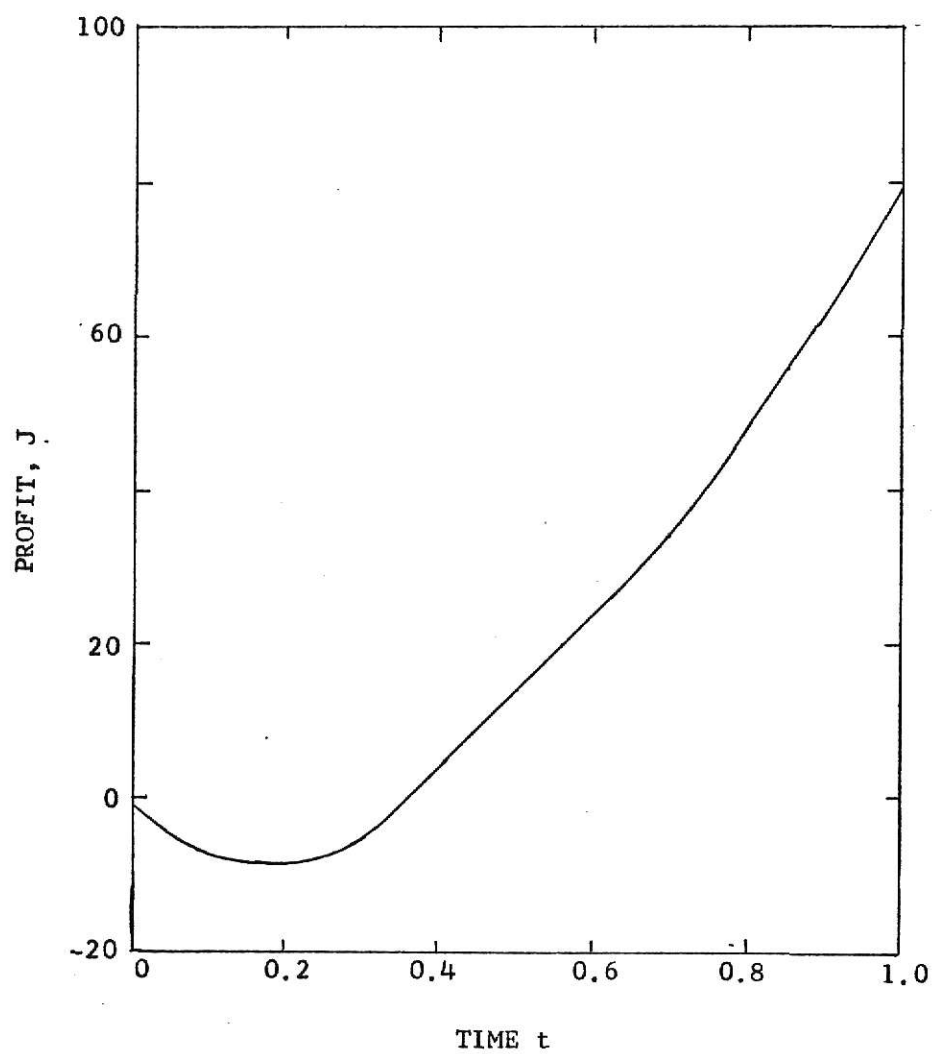


Fig. 5.11. Optimal Profit Curve, Problem 3A.

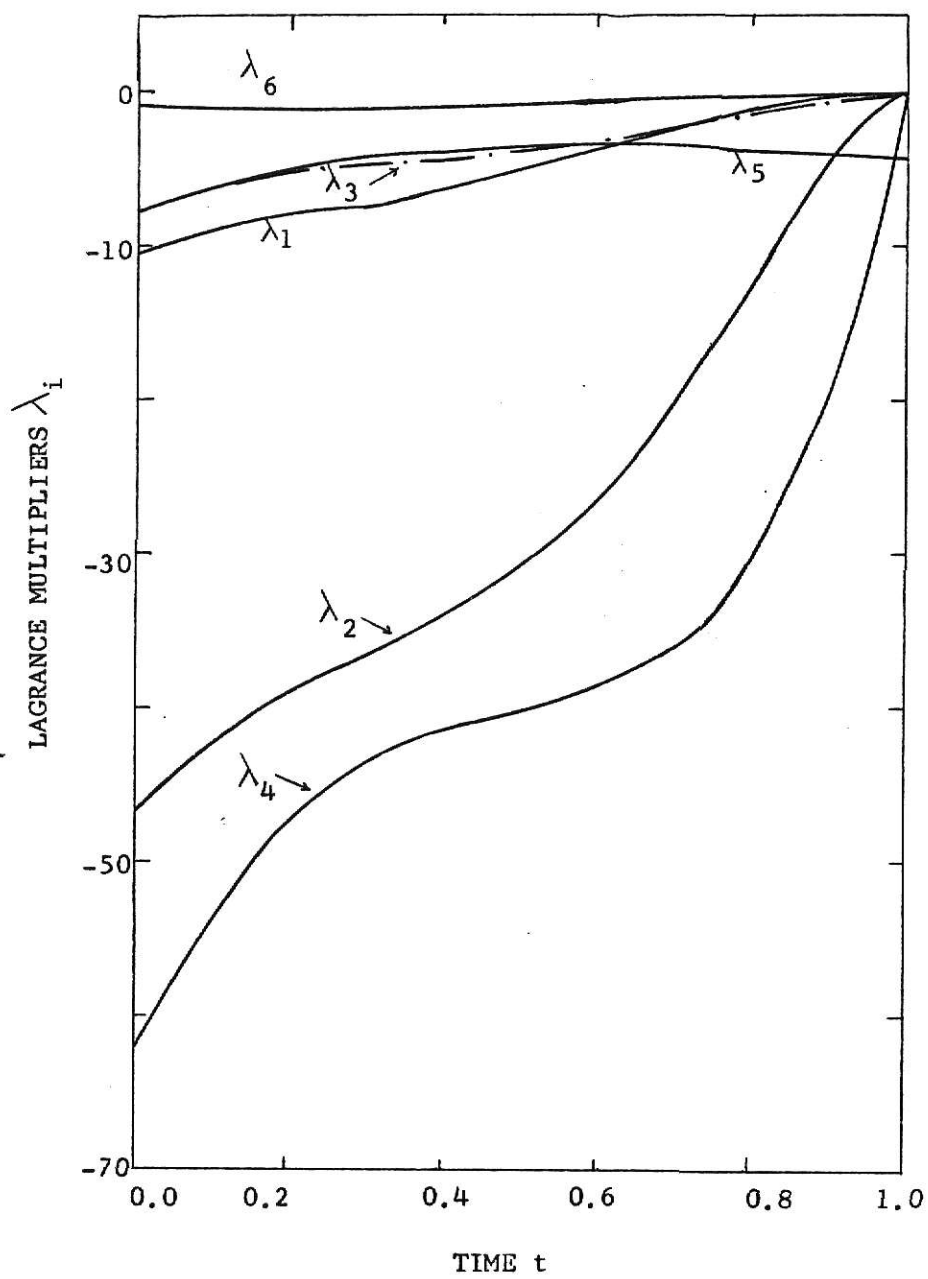


Fig. 5.12. Optimal Profiles of  $\lambda_1$  through  $\lambda_6$ , Problem 3A.

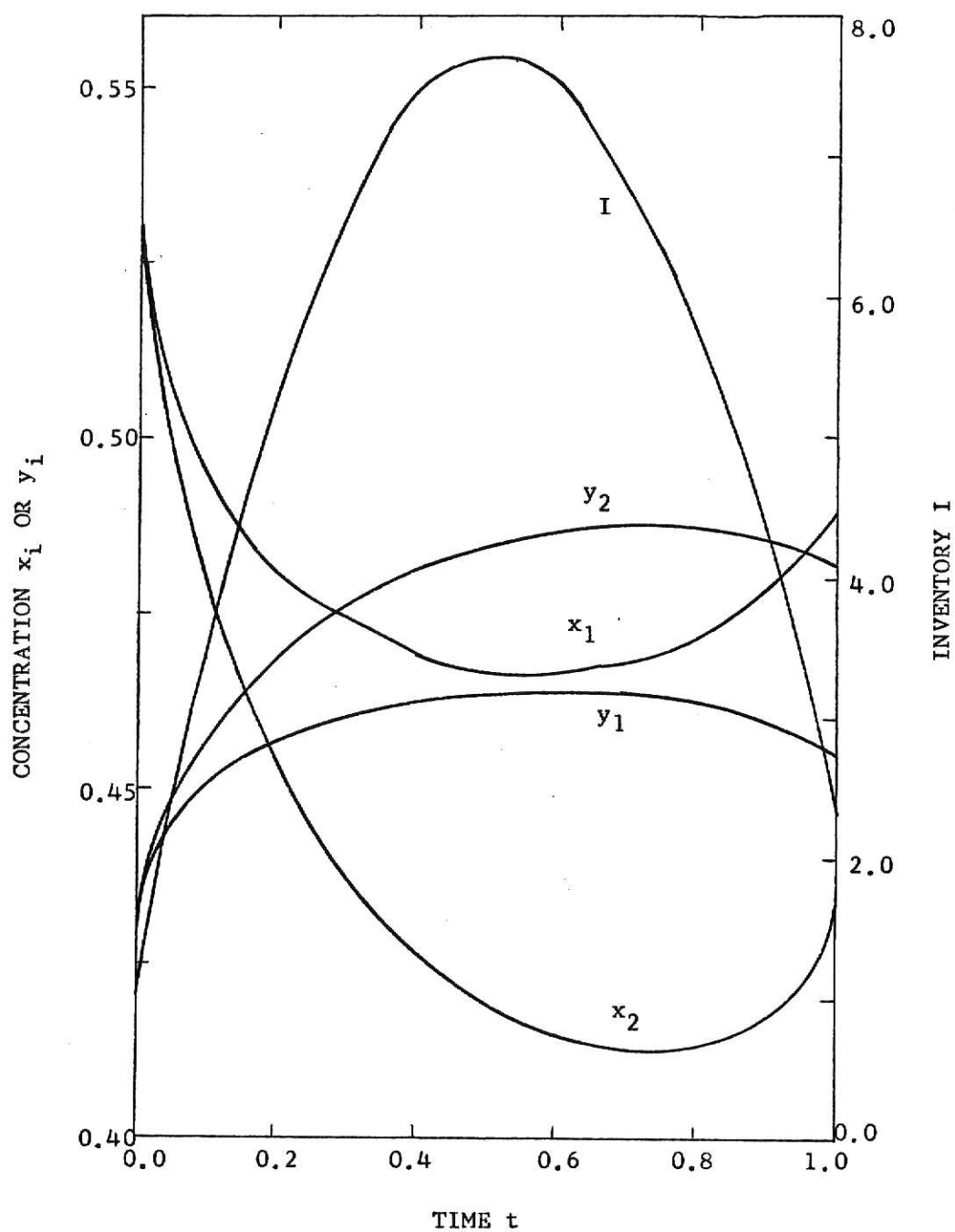


Fig. 5.13 Optimal Solutions of  $x_1$ ,  $x_2$ ,  $y_1$ ,  $y_2$ , and  $I$ , Problem B.



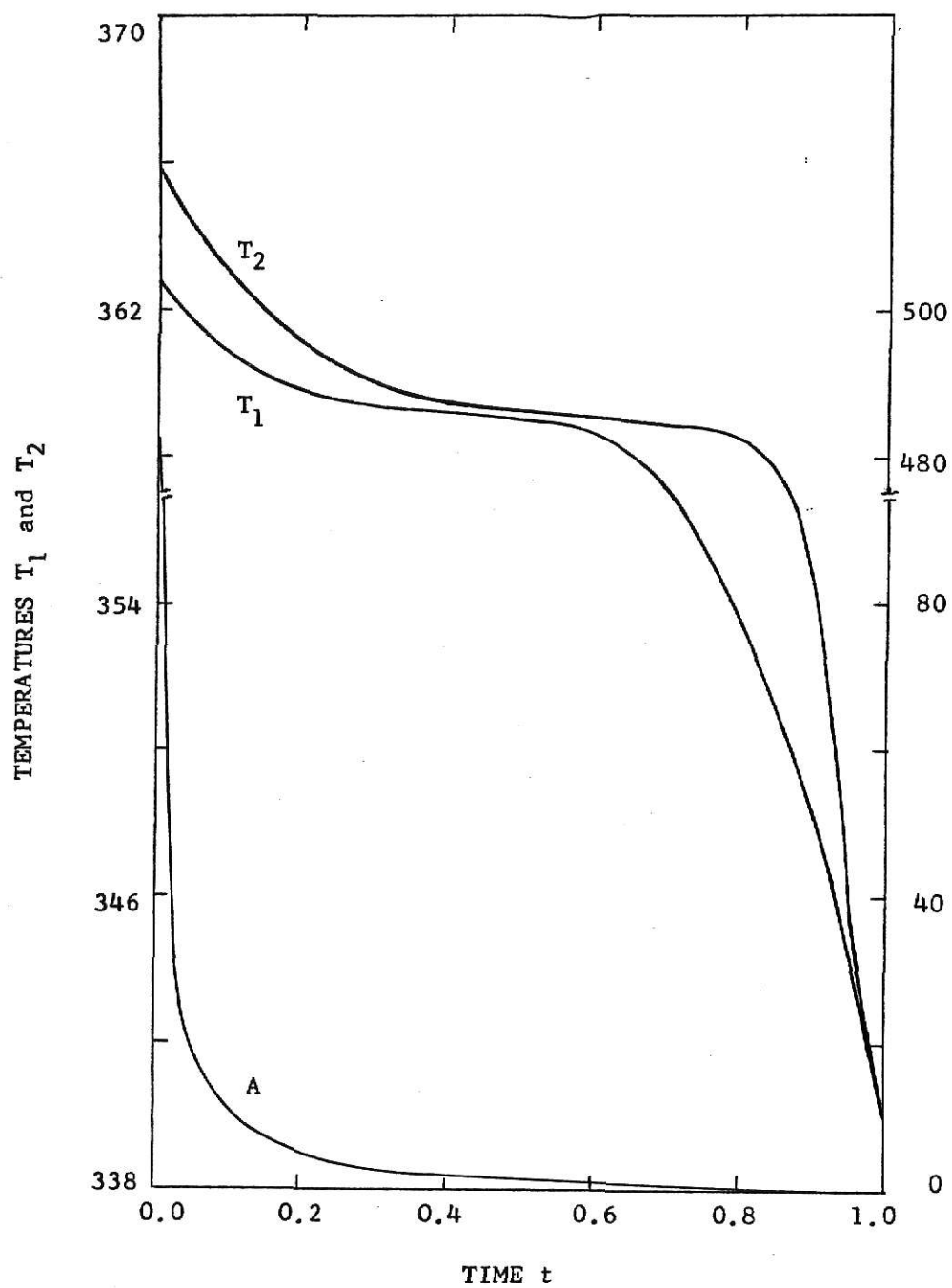


Fig. 5.14 Optimal Profiles of  $T_1$ ,  $T_2$ , and  $A$ , Problem B.

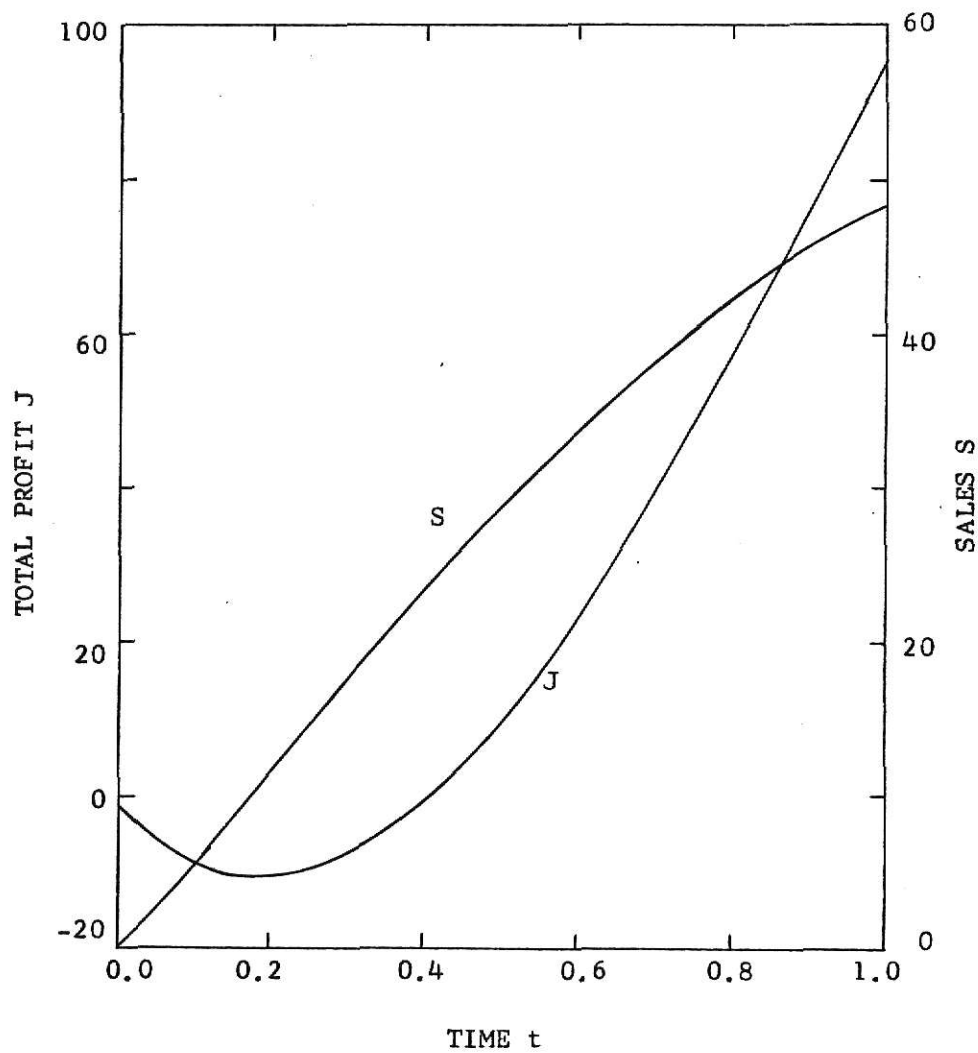


Fig. 5.15 Optimal Profile of S and J, Problem B.

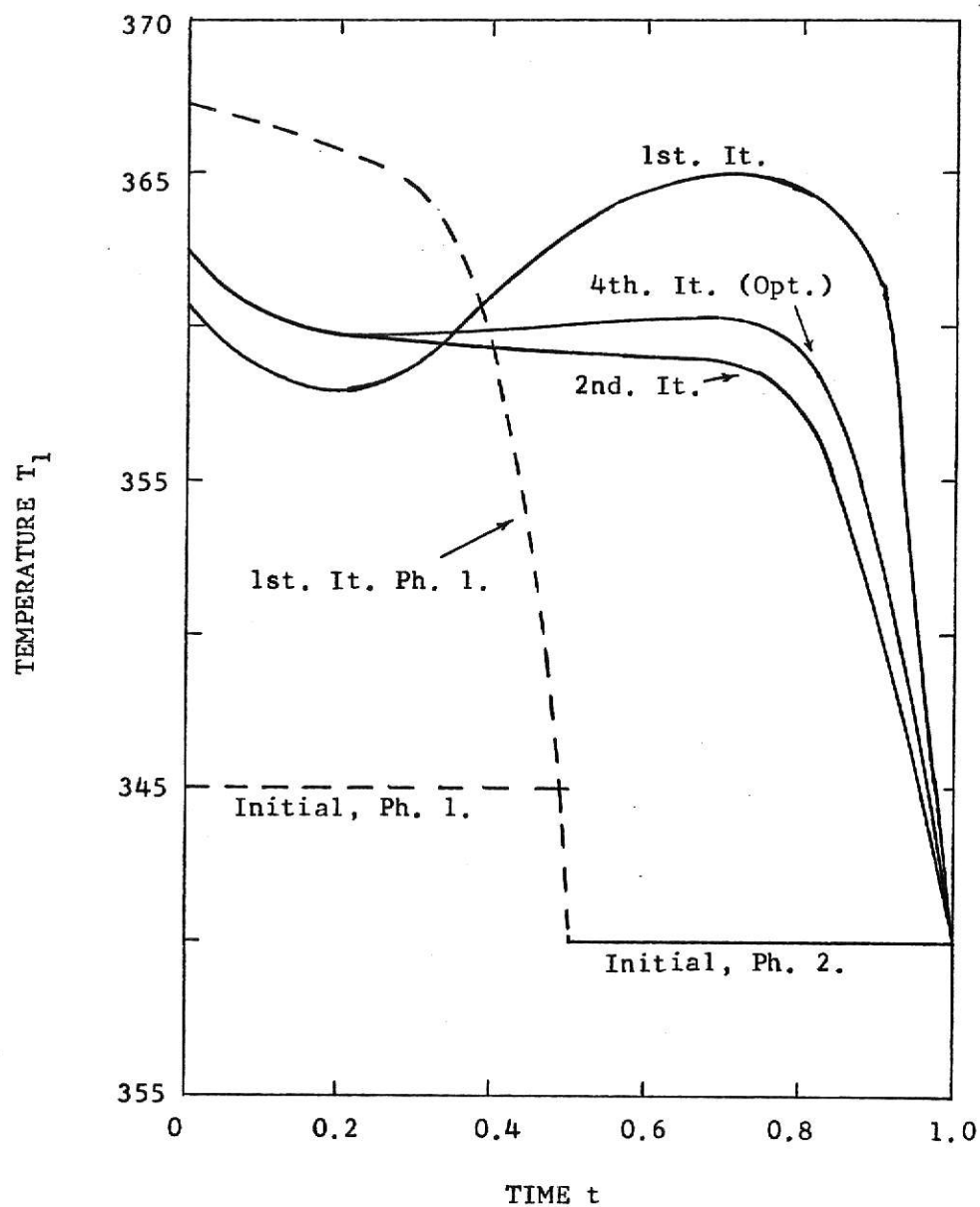


Fig. 5.16 Convergence Rate of Temperature,  $T_1$ , Problem 2C.

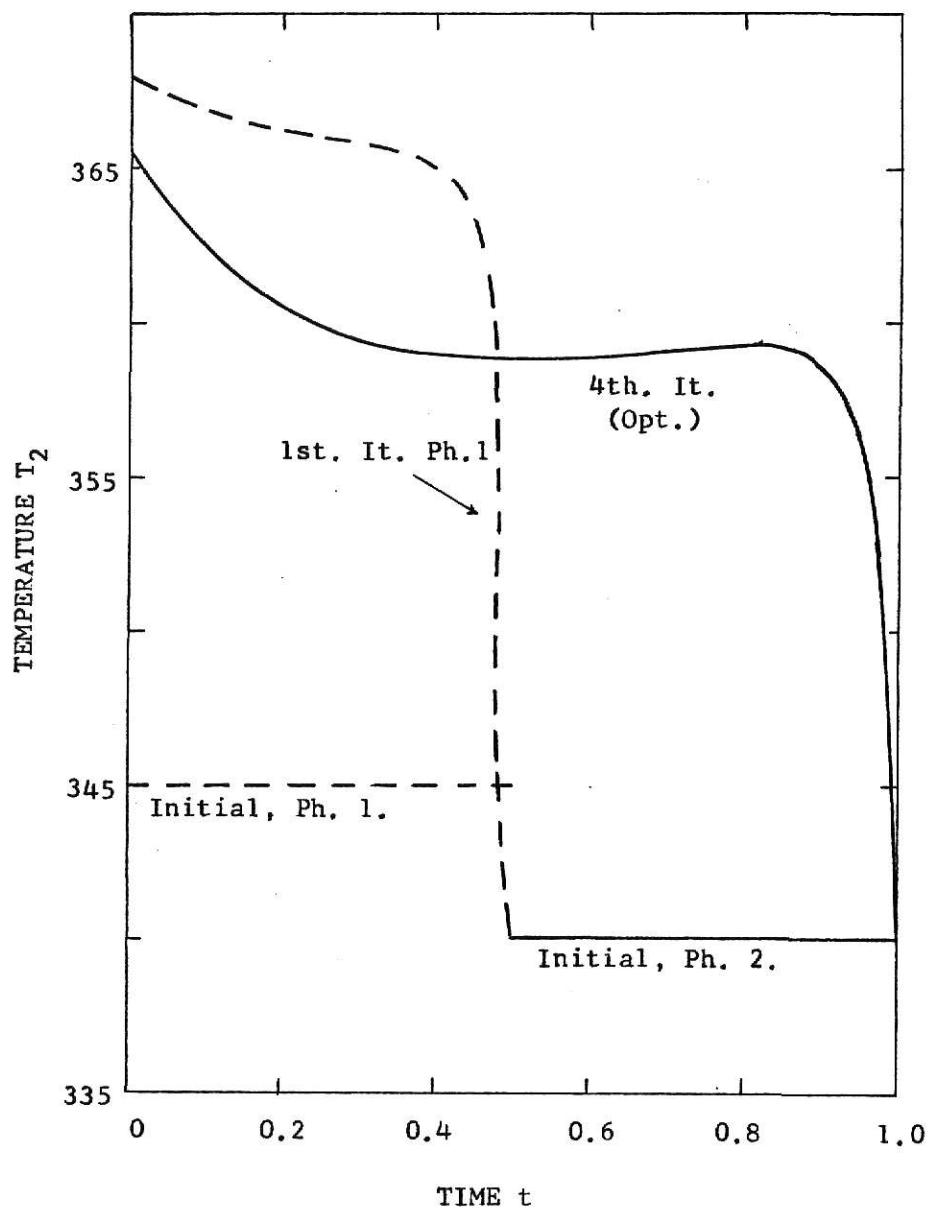


Fig. 5.17 Initial and Optimum Profiles of Temperature,  $T_2$ , Problem 2C.

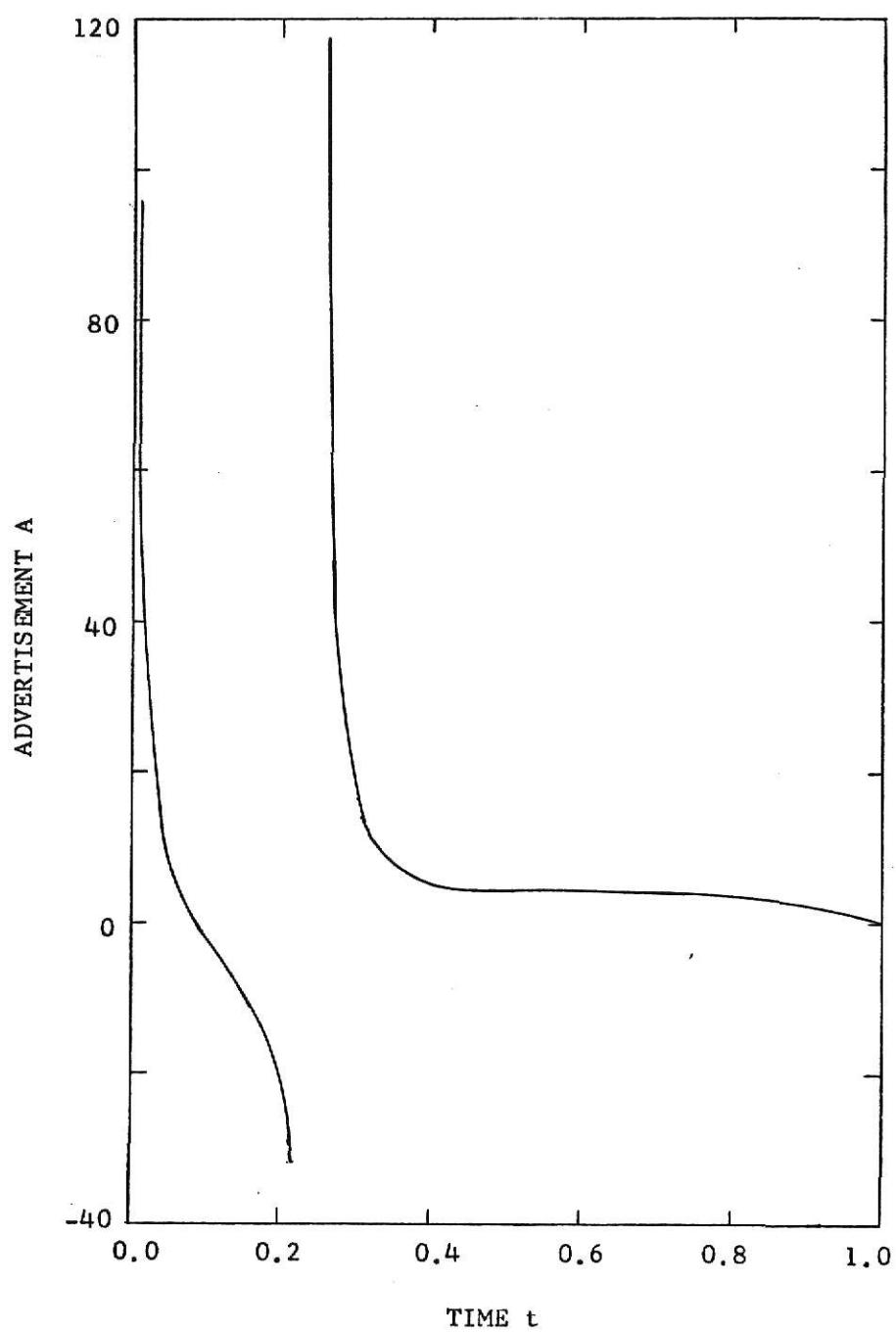


Fig. 5.18 Optimum Profile of Advertisement, A, Problem 2C.

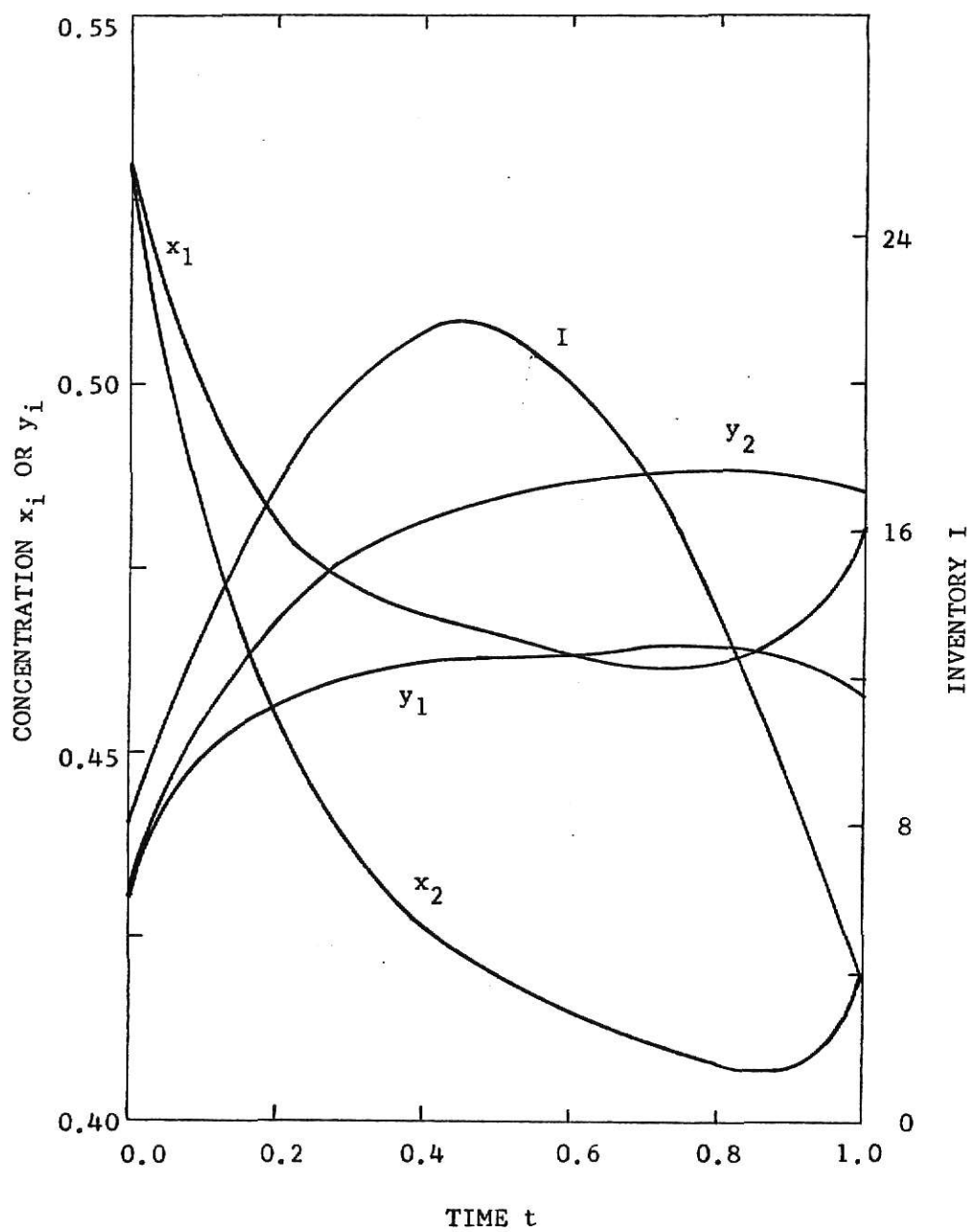


Fig. 5.19 Optimal Profiles of  $x_1$ ,  $x_2$ ,  $y_1$ ,  $y_2$ , and  $I$ , Problem 2C.

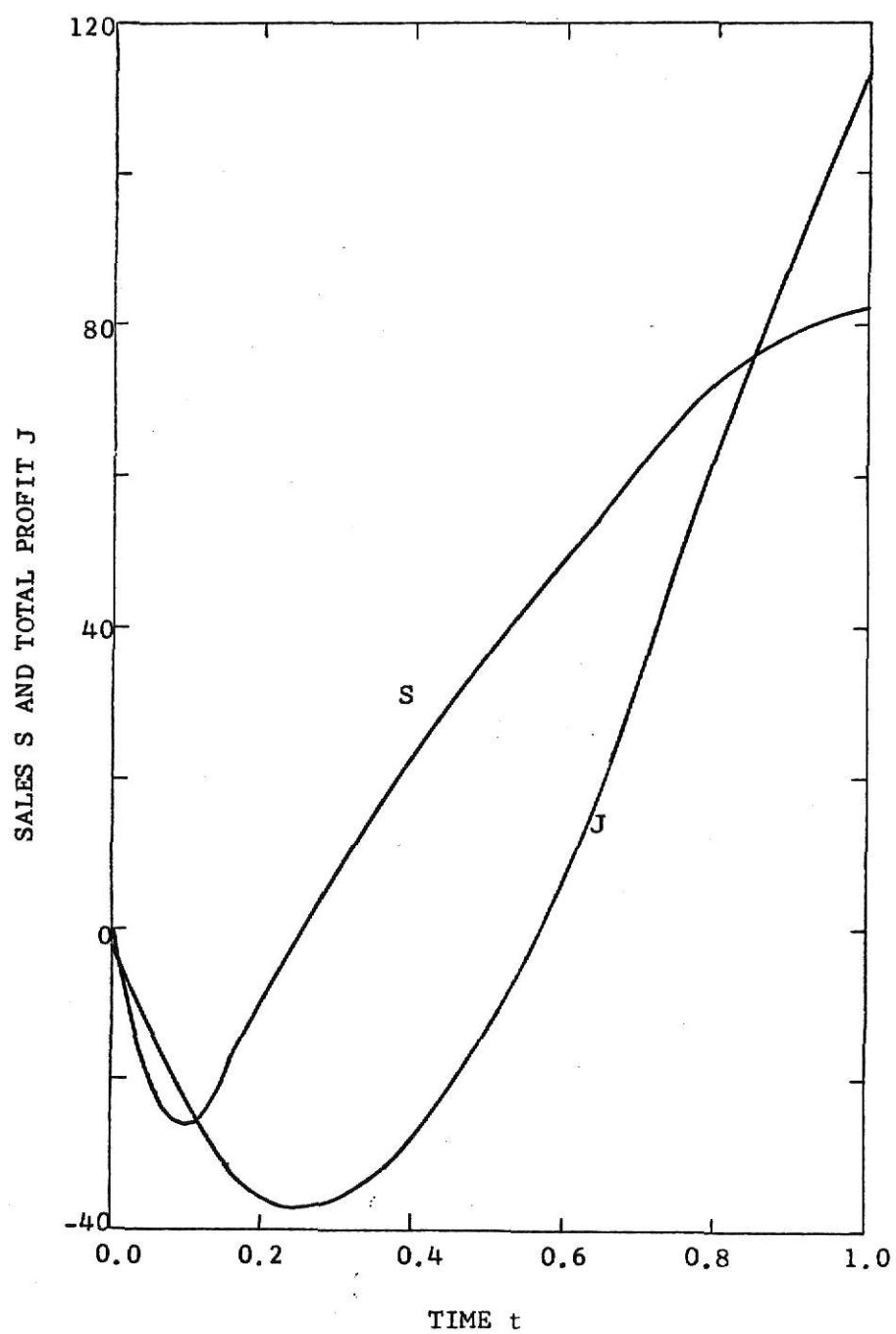


Fig. 5.20 Optimal Profile of  $S$  and  $J$ , Problem 2C.

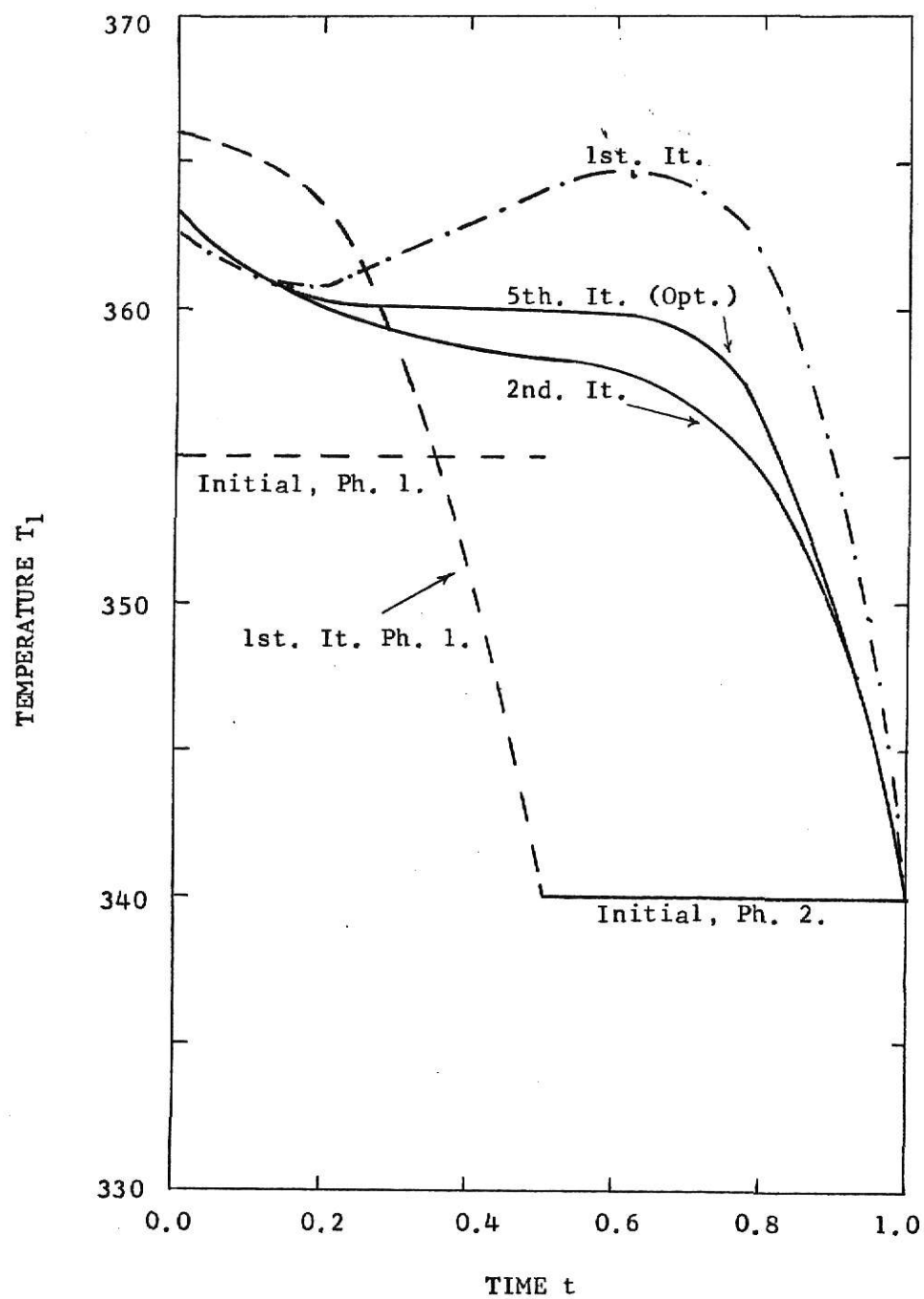


Fig. 5.21 Convergence Rate of Temperature  $T_1$ , Problem 1D.  
1 Iteration in Phase 1.



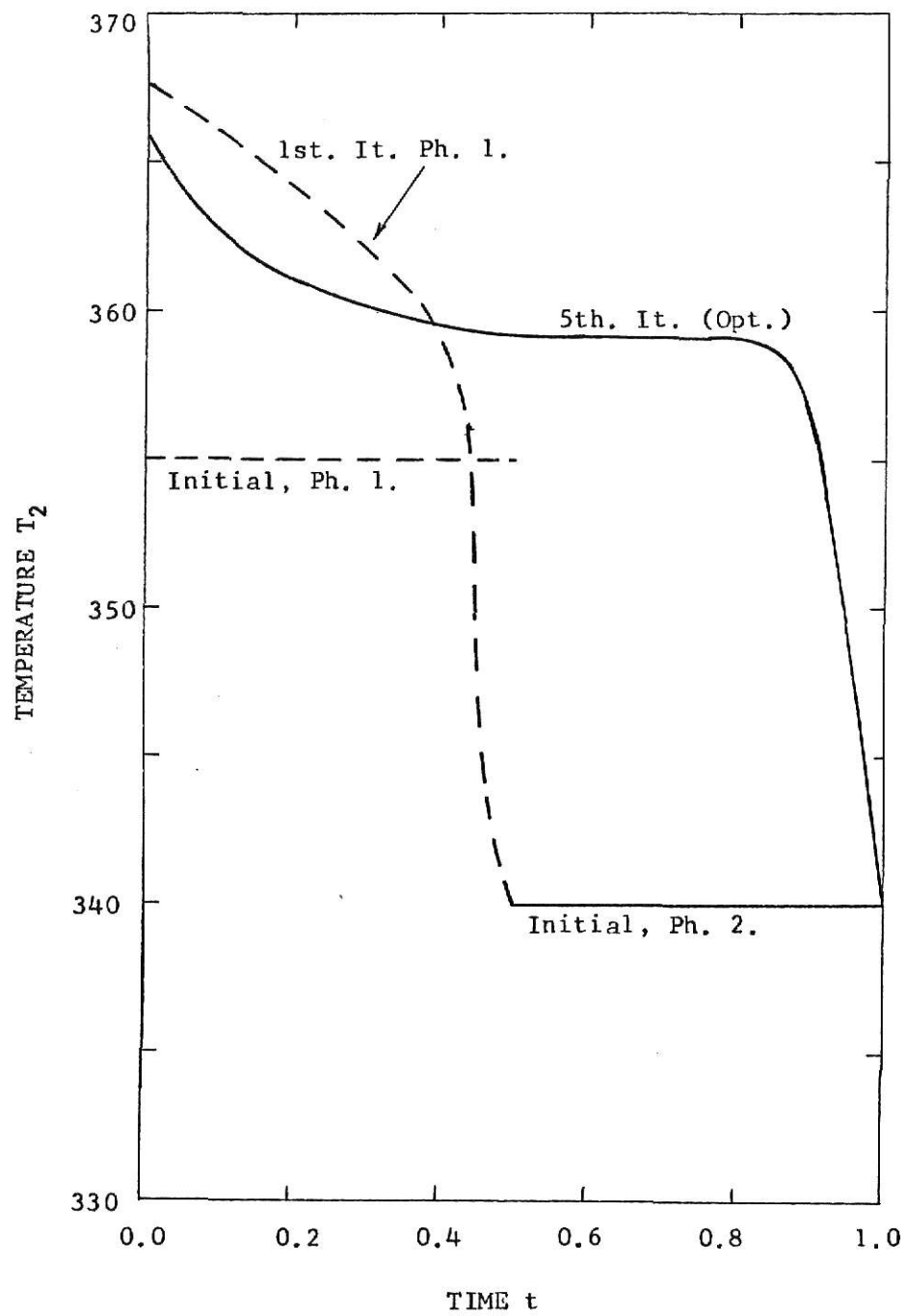


Fig. 5.22. Initial and Optimal Profile of Temperature  $T_2$ , Problem 1D.  
1 Iteration in Phase 1

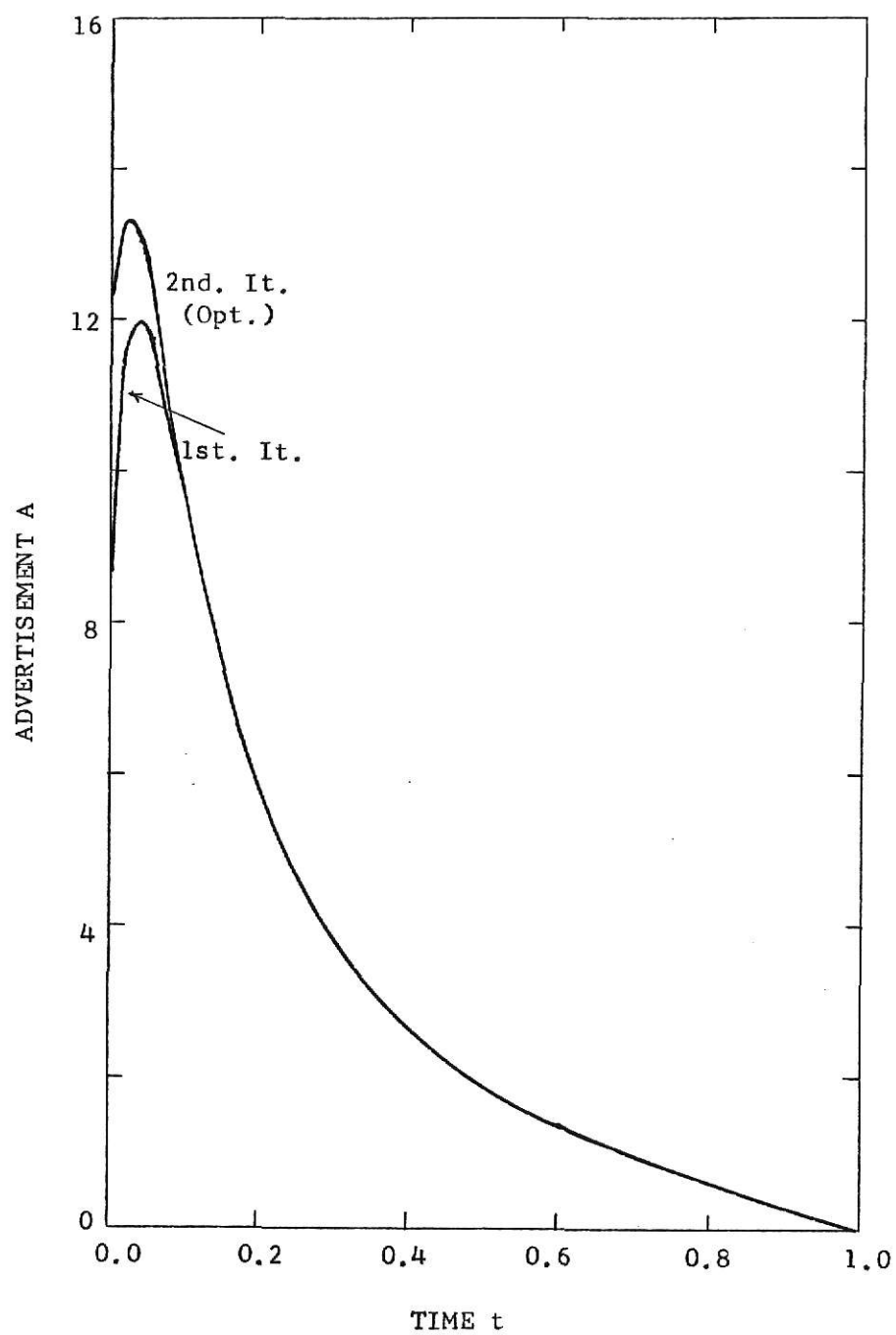


Fig. 5.23 Convergence Rate of Advertisement A, Problem 1D.  
(Only for Phase 2)

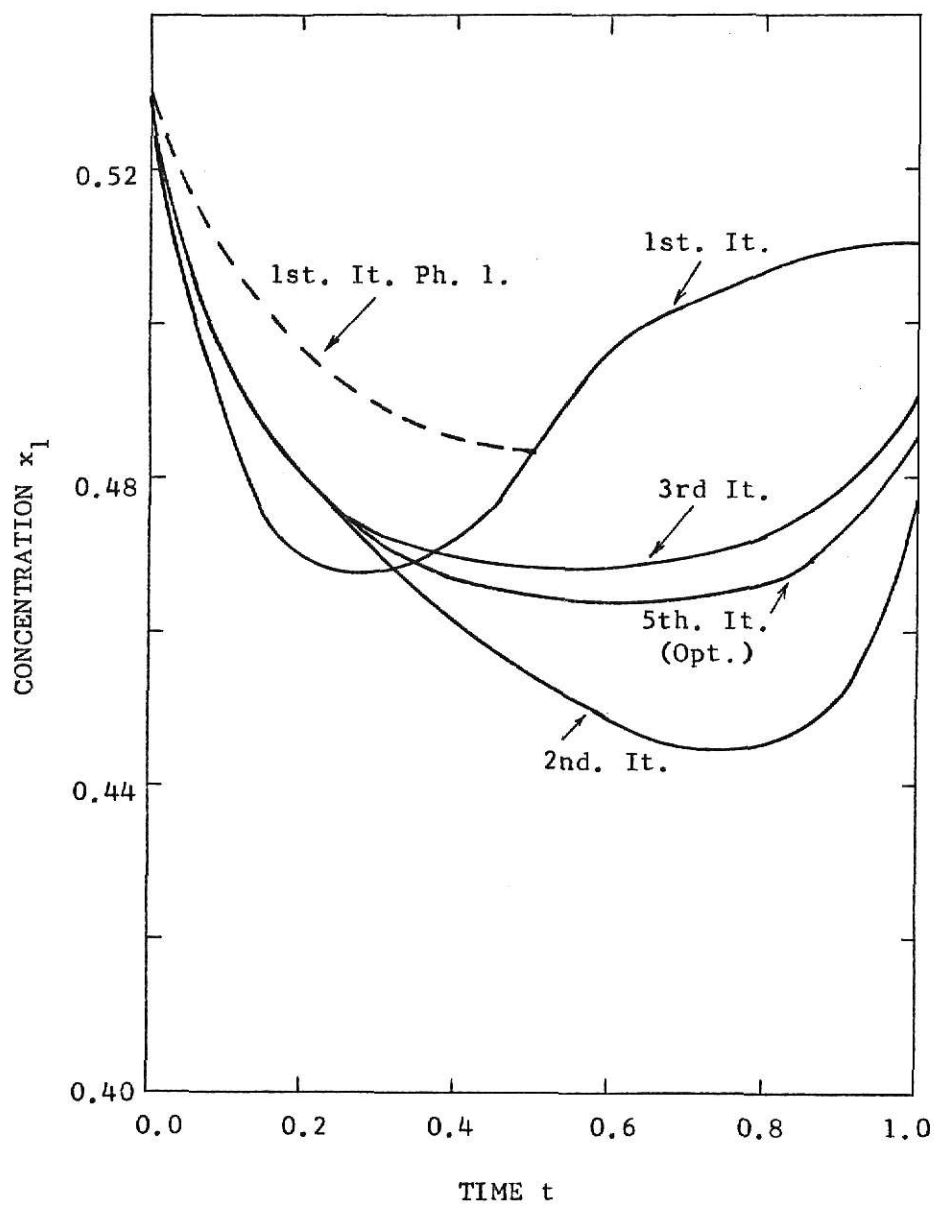


Fig. 5.24 Convergence Rate of Concentration  $x_1$ , Problem 1D.  
1 Iteration in phase 1

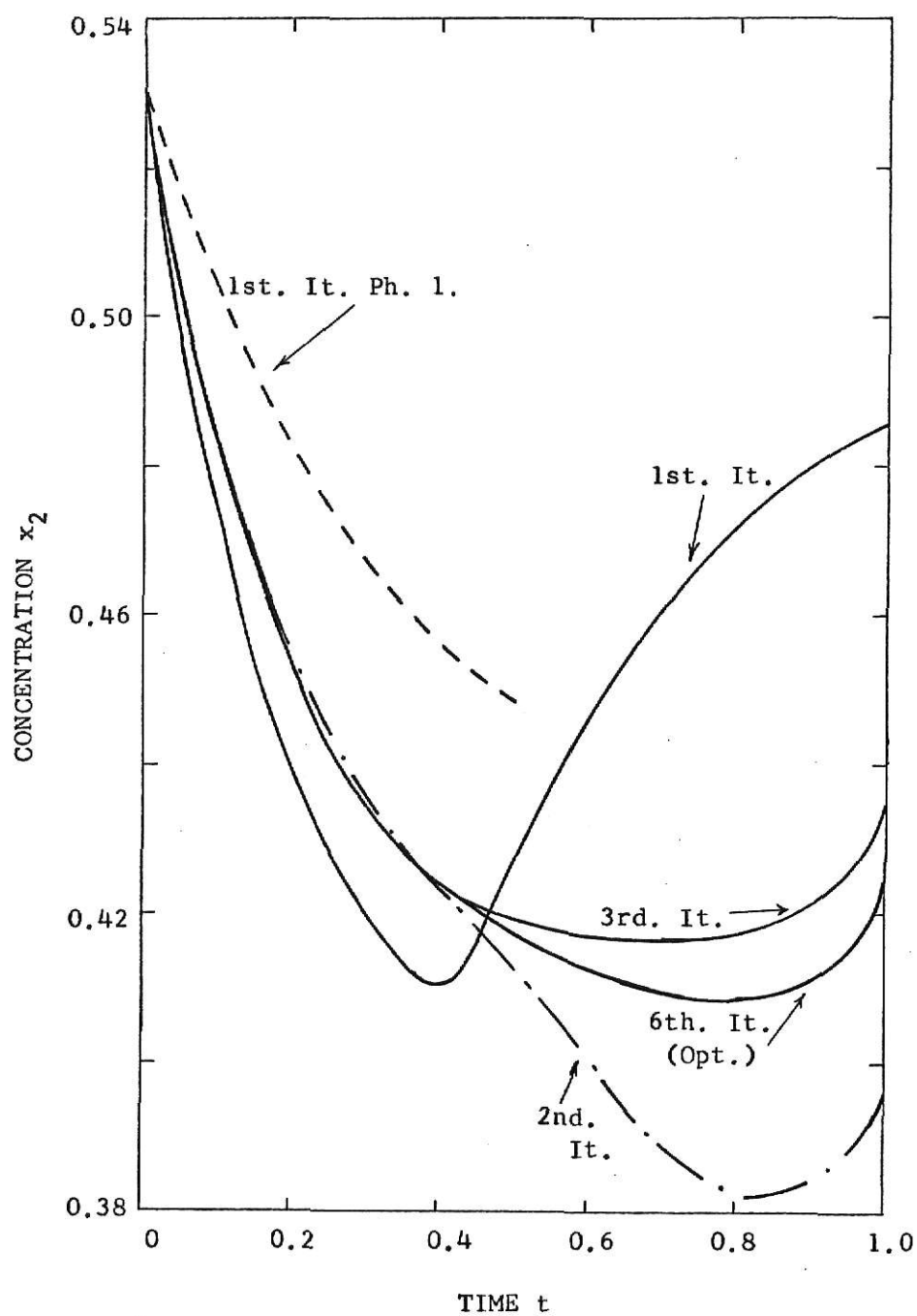


Fig. 5.25 Convergence Rate of Concentration  $x_2$ , Problem 1D.  
1 Iteration in Phase 1

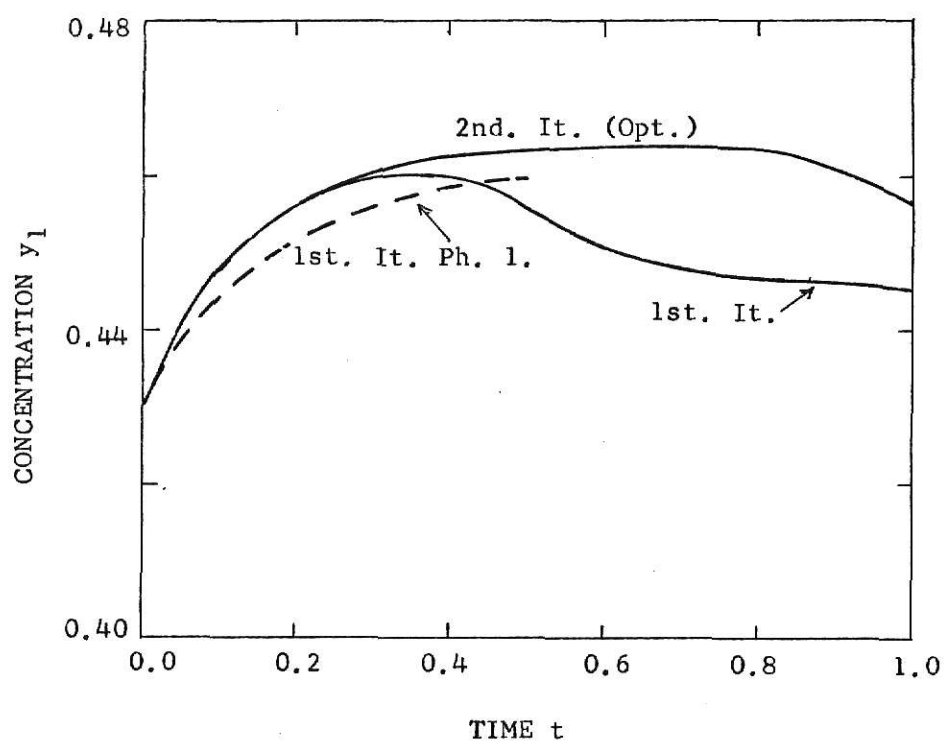


Fig. 5.26 Convergence Rate of Concentration  $y_1$ , Problem 1D.  
1 Iteration in Phase 1

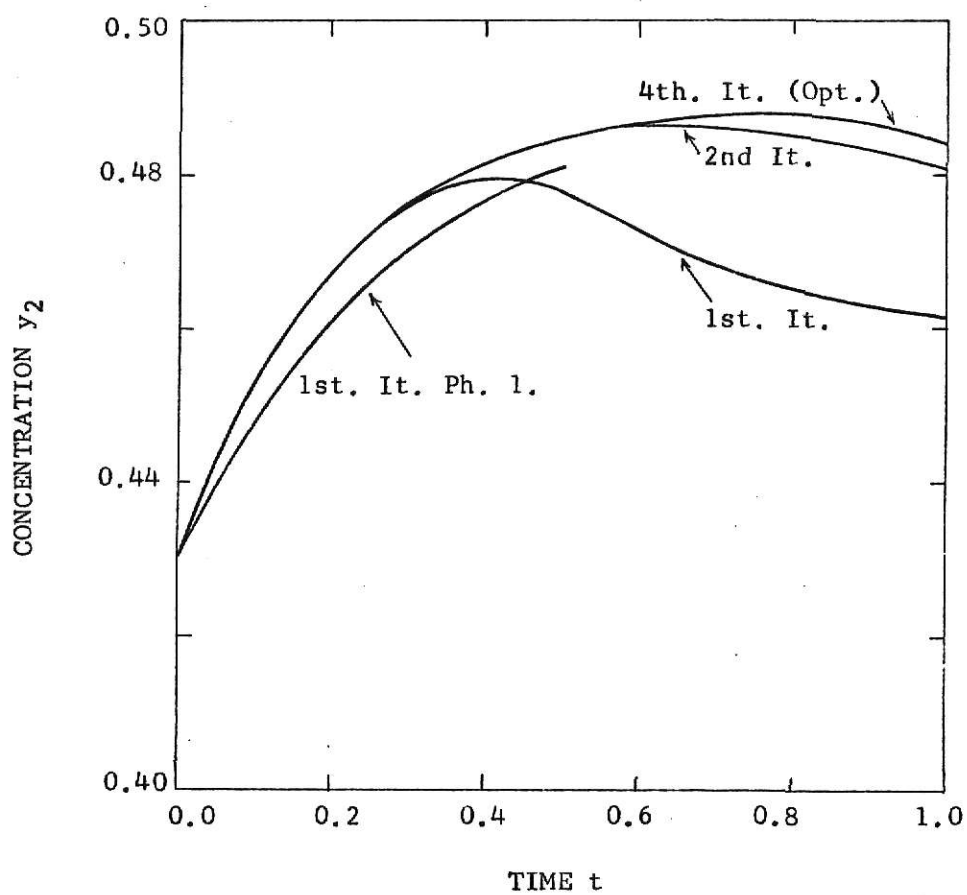


Fig. 5.27 Convergence Rate of Concentration  $y_2$ , Problem 1D.  
1 Iteration in Phase 1

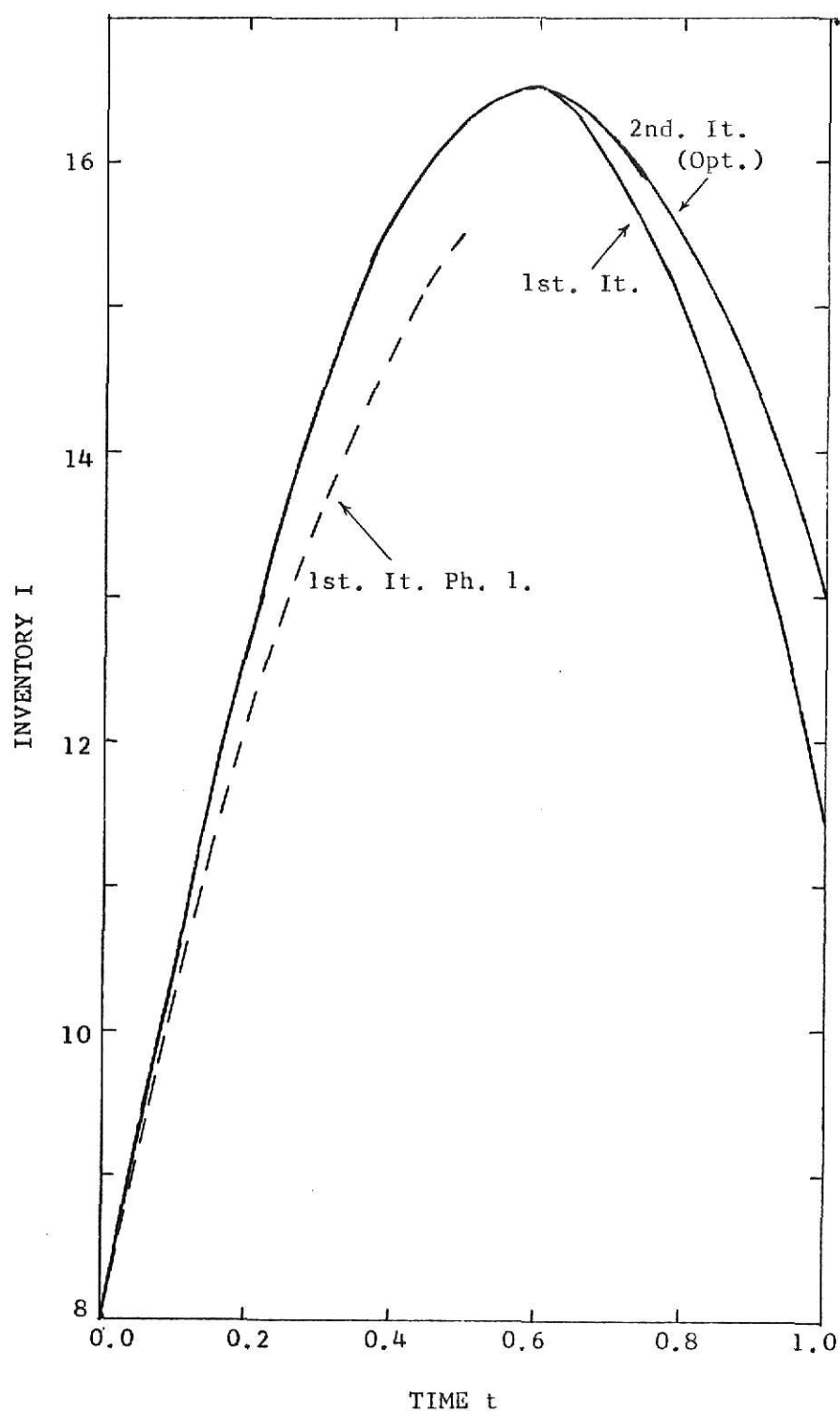


Fig. 5.28 Convergence Rate of Inventory I, Problem 1D.  
1 Iteration in Phase 1

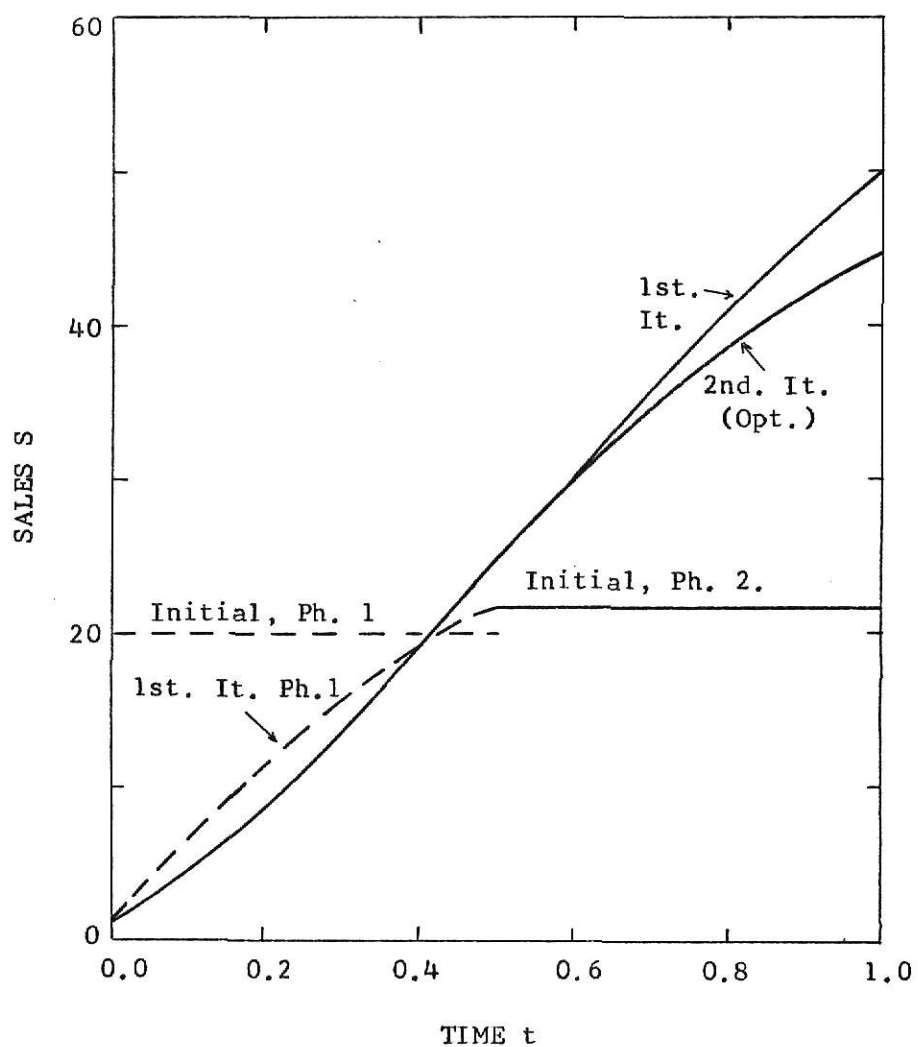


Fig. 5.29 Convergence Rate of Sales  $S$ , Problem 1D.  
1 Iteration in Phase 1



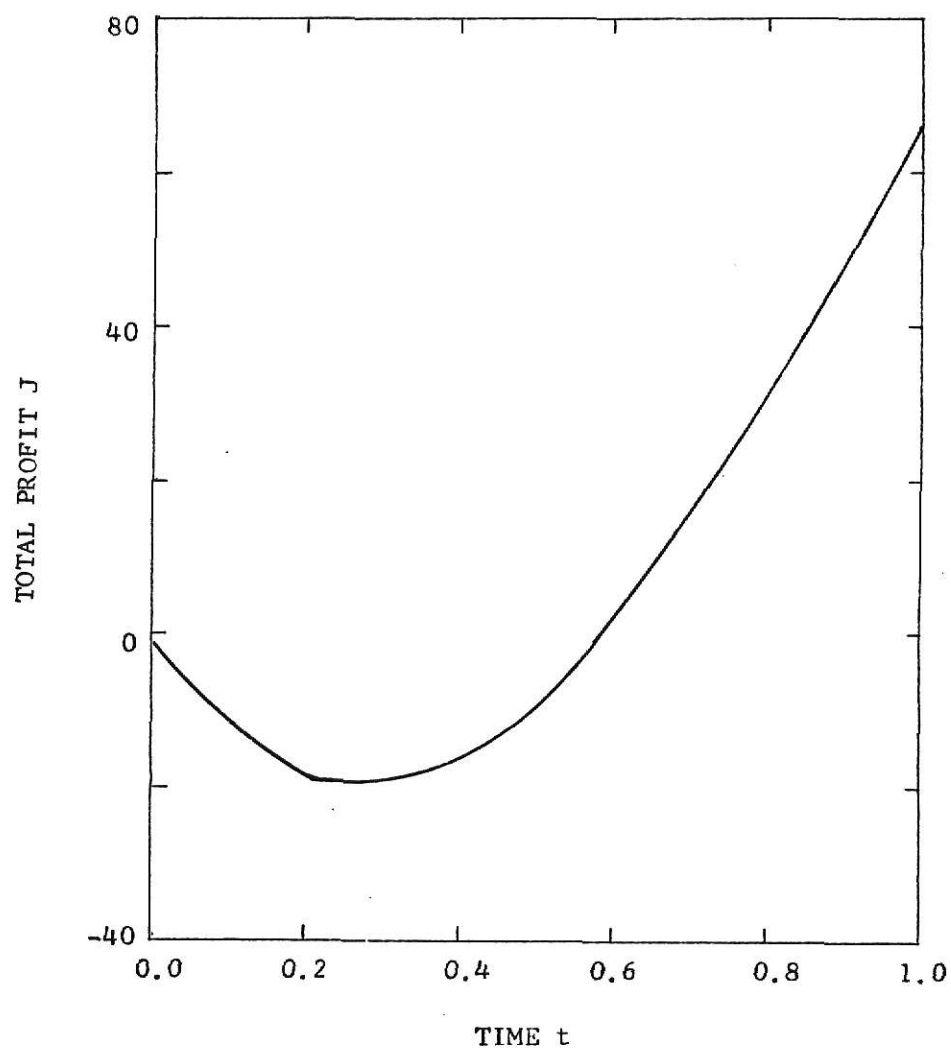


Fig. 5.30 Optimal Total Profit Curve, Problem 1D.

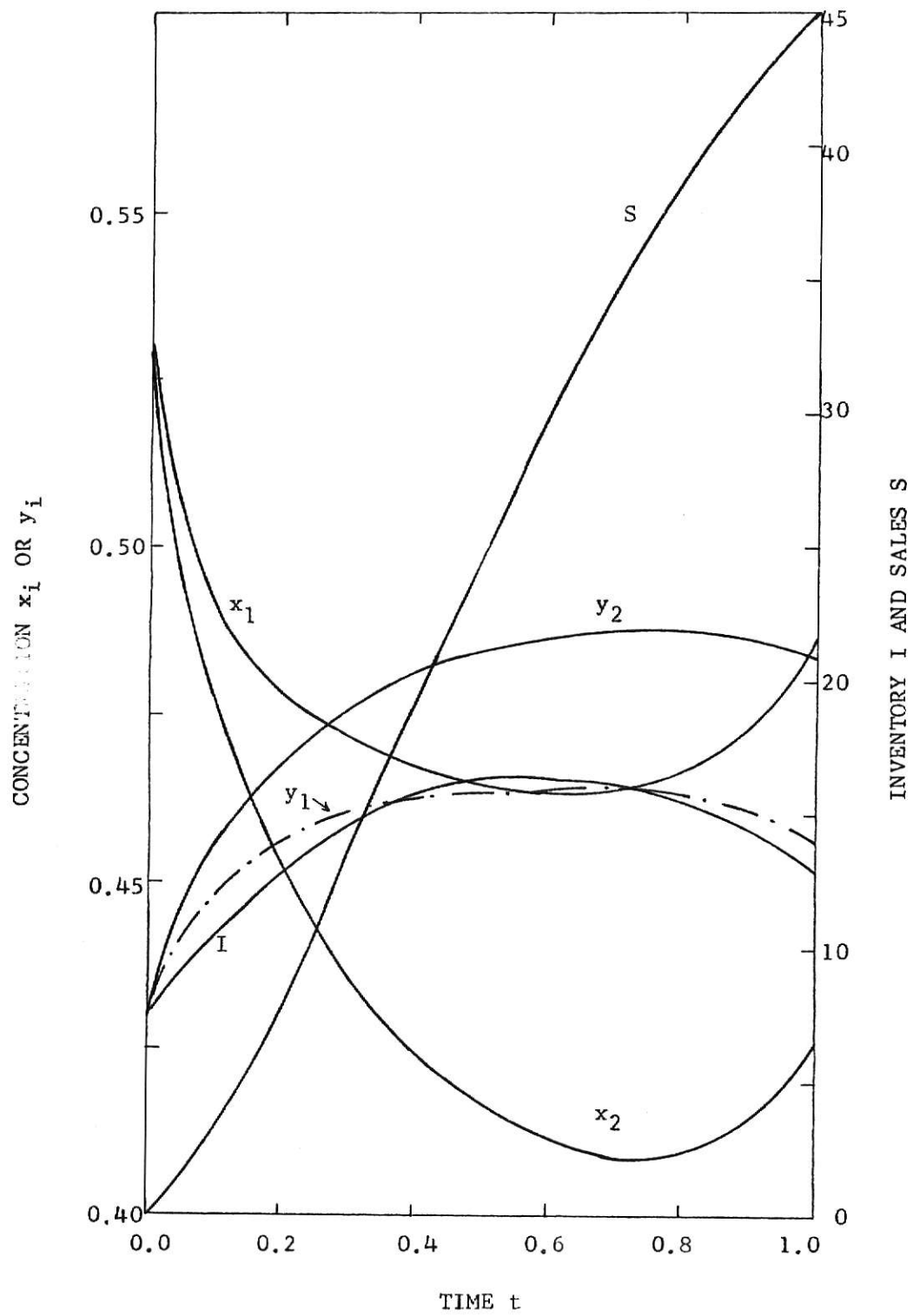


Fig. 5.31 Optimal Solution of  $x_1$ ,  $x_2$ ,  $y_1$ ,  $y_2$ ,  $I$  and  $S$ , Problem E.

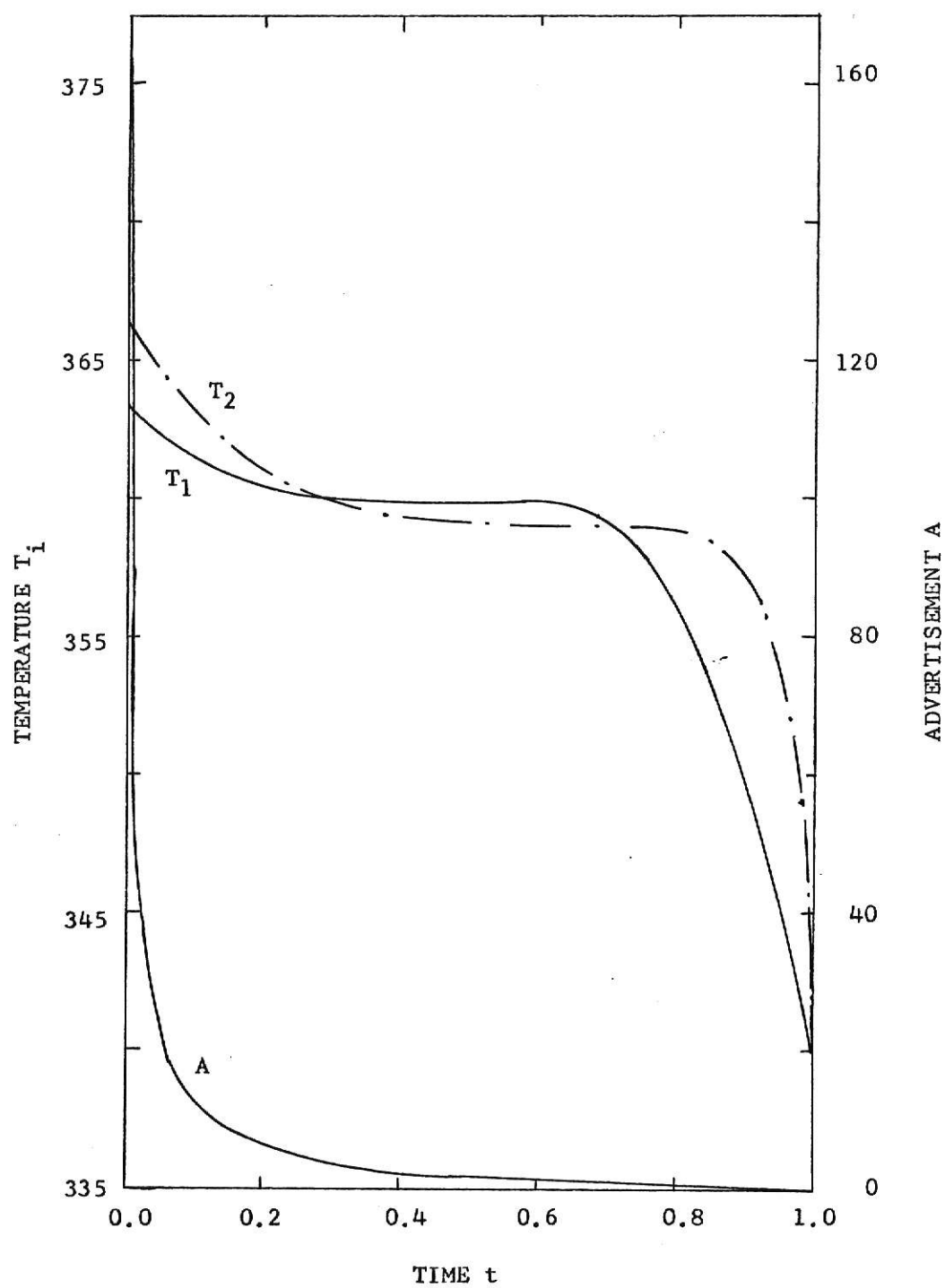


Fig. 5.32 Optimal Profiles of  $A$ ,  $T_1$ , and  $T_2$ , Problem E.

## CHAPTER 6

### CONCLUSION

The 2-phase method of quasilinearization has proved to be better than 1-phase method of quasilinearization used by Shah<sup>29</sup>. It is clear that the range of convergence increases tremendously by using this method. In the inventory and advertising scheduling problem discussed in Chapter 4, the range of convergence is doubled with the 2-phase method. In the advertising and production scheduling problem discussed in Chapter 5, the range of convergence increases for all the models. Few of these models are quite unstable and did not converge to the solution by 1-phase method. Shah<sup>29</sup> observed that, at times, the Newton-Raphson root finding method also failed. The 2-phase method works extremely well even when the control variables cannot be eliminated completely from the performance equations.

It is found that there is no use in getting optimal solution in phase 1. In general, only one iteration in phase 1 is sufficient to bring the profiles within the convergence range of phase 2. For the two examples discussed in this work, it has been found that the convergence range is not decreased if one iteration is used in phase 1. Besides this, the computation time of phase 1 is very small compared to the total computation time. Thus the net increase in computational cost is very small when one iteration is used in phase 1.

Even in the two-phase method of quasilinearization the convergence is obtained in almost a fixed number of iterations, irrespective of the

initial profile chosen. This is one of the advantages of the quasilinearization method.

Backward phasing was compared with forward phasing. Sets 1A, 2A and 3A of the production and advertisement scheduling problem were used to test which one is faster in convergence. Backward phasing appeared to be slightly better, at times.

Generally, in quasilinearization method, the control variables are eliminated when formulating the problem. This makes the method superior to other techniques such as dynamic programming and the gradient techniques. Besides, it is found that the two-phase method of quasilinearization works effectively even if the control variables are not eliminated.

The two-phase method of quasilinearization is better than the differential sensitivity analysis. Kumar<sup>16</sup> solved Model B of advertisement and production scheduling problem using differential sensitivity analysis. It is found that the convergence range is larger, and the computer time taken is less for the two-phase method.

The problems converged to the optimal solution by using two phases. More phases were not required. In the two management systems discussed in this work, in cases of very crude initial approximations, the method failed in phase 1 itself. So there was no opportunity to introduce a third phase. Phase 3 could have been introduced only if the problem fails to converge in phase 2. The author feels that in such cases, the three-phase method will be helpful.

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## APPENDIX 1

COMPUTER PROGRAM FOR INVENTORY AND ADVERTISING  
SCHEDULING PROBLEM.

```

C      ADVERTISING AND INVENTORY SCHEDULING PROBLEM.
C
C      COMMON SX2,SX4,P,DT,A,B,CC,CA,AN,CI,AIM,C,J,X1,X2,X3,X4,ITF
C      DIMENSION X1(105),X2(105),X3(105),X4(105),SX1(10,102),
C      1SX2(10,102),SX3(10,102),SX4(10,102),PX1(102),PX2(102),
C      2PX3(102),PX4(102),H1X1(102),H1X2(102),H1X3(102),H1X4(102),
C      3H2X1(102),H2X2(102),H2X3(102),H2X4(102),PR1(105),PR2(105),
C      4PR3(105),PRFT(105),ADVT(105)
C
C      READING IN DATA
C
C      READ 100,A,B,C,CC,CA,AN,CI,AIM
C      READ 101,DT,W1,W2
C
C      PRINTING OUT DATA
C
C      PRINT 110
C      PRINT 120,A,B,C,CC,CA,AN,CI,AIM,W1,W2,DT
C
C      INITIAL PROFILE PRODUCED IN PJASE ONE
C
C      ITF=11
C      KAT=2
C      DO 130 I=1,ITF
C      SX2(1,I)=W1
C      SX4(1,I)=W2
130  CONTINUE
C      READ 150,XRA,XSA,XRB,XSB
C
C      THESE FOUR VALUES ARE TO BE USED FOR THE INITIAL CONDITIONS
C      WHEN FINDING PARTICULAR AND HOMOGENEOUS SOLUTION
C
C      DO 600 IJK=1,2
C      IF(IJK-1) 400,500,400
400  IPK=ITF
C
C      INITIAL PROFILE PRODUCED IN PHASE TWO
C
C      DO 410 I=1,IPK
C      SX2(1,I)=SX2(KAT,I)
410  SX4(1,I)=SX4(KAT,I)
C      ITF=101
C      DO 420 I=IPK,ITF
C      SX2(1,I)=SX2(KAT,IPK)
420  SX4(1,I)=SX4(KAT,IPK)
C
C      QUASILINEARIZATION ITERATIONS START
C
C      KAT=15
500  DO 300 J=2,KAT

```

C PARTICULAR SOLUTION

C

```

PRINT 160
P=1.
X1(1)=XRA
X2(1)=XSA
X3(1)=XSB
X4(1)=XSB
PRINT 140,X1(1),X2(1),X3(1),X4(1)
CALL RKT
DO 170 I=1,ITF
PX1(I)=X1(I)
PX2(I)=X2(I)
PX3(I)=X3(I)
PX4(I)=X4(I)
170 CONTINUE
PRINT 141,(I,X1(I),X2(I),X3(I),X4(I),I=1,ITF,10)

```

C

C

C

HOMOGENEUS SOLUTION FIRST SET

```

P=0.
PRINT 180
X1(1)=XSB
X2(1)=XSB
X3(1)=XRB
X4(1)=XSB
PRINT 140,X1(1),X2(1),X3(1),X4(1)
CALL RKT
DO 190 I=1,ITF
H1X1(I)=X1(I)
H1X2(I)=X2(I)
H1X3(I)=X3(I)
H1X4(I)=X4(I)
190 CONTINUE
PRINT 141,(I,X1(I),X2(I),X3(I),X4(I),I=1,ITF,10)

```

C

C

C

HOMOGENEUS SOLUTION SECOND SET

```

P=0.
PRINT 200
X1(1)=XSB
X2(1)=XSB
X3(1)=XSB
X4(1)=XRB
PRINT 140,X1(1),X2(1),X3(1),X4(1)
CALL RKT
DO 210 I=1,ITF
H2X1(I)=X1(I)
H2X2(I)=X2(I)
H2X3(I)=X3(I)

```

```

      H2X4(I)=X4(I)
210 CONTINUE
      PRINT 141,(I,X1(I),X2(I),X3(I),X4(I),I=1,ITF,10)
C
C      SOLUTION OF INTEGRATION CONSTANTS
      BB1=XSB
      BB2=XSE
      B1=BB1-PX3(ITF)
      B2=BB2-PX4(ITF)
      DET=H1X3(ITF)*H2X4(ITF)-H1X4(ITF)*H2X3(ITF)
      A1=(B1*H2X4(ITF)-B2*H2X3(ITF))/DET
      A2=(B2*H1X3(ITF)-B1*H1X4(ITF))/DET
      PRINT 230,A1,A2
C
C      RECOVERY OF SOLUTION          SUPERPOSITION PRINCIPLE
C
      DO 250 I=1,ITF
      SX1(J,I)=PX1(I)+A1*H1X1(I)+A2*H2X1(I)
      SX2(J,I)=PX2(I)+A1*H1X2(I)+A2*H2X2(I)
      SX3(J,I)=PX3(I)+A1*H1X3(I)+A2*H2X3(I)
      SX4(J,I)=PX4(I)+A1*H1X4(I)+A2*H2X4(I)
250 CONTINUE
      JJ=J-1
      PRINT 260, JJ
      PRINT 265
C
C      CALCULATION OF CONTROL VARIABLE AND PROFIT
C
      P1R=0.
      P2R=0.
      P3R=0.
      DO 280 I=1,ITF
      ADVT(I)=(SX4(J,I)/(2.*CA))*(SX2(J,I)/AN-1.)
      PR1(I)=P1R+DT*(C*SX2(J,I))
      P1R=PR1(I)
      PR2(I)=P2R+DT*(CI*((AIM-SX1(J,I))**2))
      P2R=PR2(I)
      PR3(I)=P3R+DT*(CA*SX2(J,I)*(ADVT(I)**2))
      P3R=PR3(I)
      PRFT(I)=PR1(I)-PR2(I)-PR3(I)
280 CONTINUE
      PRINT 270,(I,SX1(J,I),SX2(J,I),SX3(J,I),SX4(J,I),ADVT(I),
1PRFT(I),I=1,ITF)
      PRINT 290,PRFT(ITF),PR1(ITF),PR2(ITF),PR3(ITF)
300 CONTINUE
600 CONTINUE
C
100 FORMAT(8F9.4)
101 FORMAT(3F9.4)
110 FORMAT('1VALUE OF THE CONSTANTS')

```

```

120 FORMAT(' A=',F8.3,' B=',F8.3,' C=',F8.3,' CC=',F8.3,' CA=',
1F7.3,' AN=',F8.3,' CI=',F7.4,' AIM=',F9.4,' S,D(T)=',F9.4,
2'L6,D(T)=',F9.4,' DT=',F6.3)
140 FORMAT(1H ,20X,' X1(0)=',F10.4,' X2(0)=',F10.4,' X3(0)=',F10
1' X4(0)=',F10.4)
141 FORMAT(1H ,9X,I4,2X,4E20.7)
150 FORMAT(4F10.4)
160 FORMAT('-PARTICULAR SOLUTION')
180 FORMAT('-HOMOGENEOUS SOLUTION FIRST SET')
200 FORMAT('-HOMOGENEOUS SOLUTION SECOND SET')
230 FORMAT(1H ,A1=',F9.4,' A2=',F9.4)
260 FORMAT(' FINAL SOLUTION ITERATION NO ',I4,/)
265 FORMAT(' STAGE',9X,' INVENTORY',10X,' SALES',14X,' LAGRANGE
1MULTIPLIERS',9X,' ADVERTISEMENT',9X,' PROFIT')
270 FORMAT(1H ,I4,2X,4E19.7,6X,E15.5,6X,E15.5)
290 FORMAT('-TOTAL PROFIT ',F15.3,15X,3F10.3)
STOP
END

```

```

SUBROUTINE RKT
COMMON SX2,SX4,P,DT,A,B,CC,CA,AN,CI,AIM,C,J,X1,X2,X3,X4,ITF
DIMENSION X1(105),X2(105),X3(105),X4(105),A1(105),
1A2(105),A3(105),A4(105),B1(105),B2(105),B3(105),B4(105),C1(10
2C2(105),C3(105),C4(105),D1(105),D2(105),D3(105),D4(105),
3SX2(10,102),SX4(10,102)

```

C

```

DO 500 I=1,ITF
V=SX2(J-1,I)
W=SX4(J-1,I)
TA=I-1
T=TA*DT
A1(I)=DT*(P*A+P*B*T-X2(I))
B1A=CC*V+W*(V**2)/(CA*AN)-W*V/(2.*CA)-CC*(V**2)/AN-
1W*(V**3)/(2.*CA*(AN**2))
B2A=CC+2.*V*W/(CA*AN)-W/(2.*CA)-2.*CC*V/AN-3.*W*(V**2)
1/(2.*CA*(AN**2))
B3A=V**2/(CA*AN)-V/(2.*CA)-V**3/(2.*CA*(AN**2))
B1(I)=DT*(P*B1A-P*V*B2A-P*W*B3A+X2(I)*B2A+X4(I)*B3A)
C1(I)=DT*(P*2.*CI*AIM-2.*CI*X1(I))
D1A=C-W*CC+3.*(W**2)*(V**2)/(4.*CA*(AN**2))+W**2/(4.*CA)
1-(W**2)*V/(CA*AN)+2.*CC*V*W/AN

```

```

D2A=3.*V*(W**2)/(2.*CA*(AN**2))-W**2/(CA*AN)+2.*CC*W/AN
D3A=W/(2.*CA)-CC+3.*W*(V**2)/(2.*CA*(AN**2))-2.*W*V/
1(CA*AN)+2.*CC*V/AN
D1(I)=DT*(X3(I)+P*D1A-P*V*D2A-P*W*D3A+X2(I)*D2A+X4(I)*
1D3A)
A2(I)=DT*(P*A+P*B*(T+DT/2.)-(X2(I)+B1(I)/2.))
B2(I)=DT*(P*B1A-P*V*B2A-P*W*B3A+(X2(I)+B1(I)/2.)*B2A+
1(X4(I)+D1(I)/2.)*B3A)
C2(I)=DT*(P*2.*CI*AIM-2.*CI*(X1(I)+A1(I)/2.))
D2(I)=DT*((X3(I)+C1(I)/2.)+P*D1A-P*V*D2A-P*W*D3A+(X2(I)
1+B1(I)/2.)*D2A+(X4(I)+D1(I)/2.)*D3A)
A3(I)=DT*(P*A+P*B*(T+DT/2.)-(X2(I)+B2(I)/2.))
B3(I)=DT*(P*B1A-P*V*B2A-P*W*B3A+(X2(I)+B2(I)/2.)*B2A+
1(X4(I)+D2(I)/2.)*B3A)
C3(I)=DT*(P*2.*CI*AIM-2.*CI*(X1(I)+A2(I)/2.))
D3(I)=DT*((X3(I)+C2(I)/2.)+P*D1A-P*V*D2A-P*W*D3A+(X2(I)
1+B2(I)/2.)*D2A+(X4(I)+D2(I)/2.)*D3A)
A4(I)=DT*(P*A+P*B*(T+DT)-(X2(I)+B3(I)))
B4(I)=DT*(P*B1A-P*V*B2A-P*W*B3A+(X2(I)+B3(I))*B2A+
1(X4(I)+D3(I))*B3A)
C4(I)=DT*(P*2.*CI*AIM-2.*CI*(X1(I)+A3(I)))
D4(I)=DT*((X3(I)+C3(I))+P*D1A-P*V*D2A-P*W*D3A+(X2(I)+
1B3(I))*D2A+(X4(I)+D3(I))*D3A)
X1(I+1)=X1(I)+(A1(I)+2.*A2(I)+2.*A3(I)+A4(I))/6.
X2(I+1)=X2(I)+(B1(I)+2.*B2(I)+2.*B3(I)+B4(I))/6.
X3(I+1)=X3(I)+(C1(I)+2.*C2(I)+2.*C3(I)+C4(I))/6.
X4(I+1)=X4(I)+(D1(I)+2.*D2(I)+2.*D3(I)+D4(I))/6.
500 CONTINUE
RETURN
END

```

## APPENDIX 2

COMPUTER PROGRAM FOR ADVERTISEMENT AND PRODUCTION  
SCHEDULING PROBLEM.

```

C      PRODUCTION AND ADVERTISEMENT SCHEDULING PROBLEM.
C
C
C      MAIN PROGRAM
COMMON X1,Y1,X2,Y2,AI,Q,Z1,Z2,Z3,Z4,Z5,Z6,DT,X0,Y0,
1TP,TQ,GA,GB,EA,EB,R,AQ,V1,V2,AN,CA,CI,AIM,CB,CC,CQ,C,
2AC,CT,P,TEM1,TEM2,W1,W2,W3,W4,J,SAQ,SZ6,ITF
  DIMENSION X1(105),X2(105),Y1(105),Y2(105),AI(105),Q(105),
1Z1(105),Z2(105),Z3(105),Z4(105),Z5(105),Z6(105),
2SX1(10,102),SX2(10,102),SY1(10,102),SY2(10,102),
3SAI(10,102),SAQ(10,102),SZ1(10,102),SZ2(10,102),
4SZ3(10,102),SZ4(10,102),SZ5(10,102),SZ6(10,102),
5PX1(102),PX2(102),PY1(102),PY2(102),PI(102),PQ(102),
6PZ1(102),PZ2(102),PZ3(102),PZ4(102),PZ5(102),PZ6(102),
7XD(105),YD(105),TEM1(10,102),TEM2(10,102),HX1(6,102),
8HX2(6,102),HY1(6,102),HY2(6,102),HAI(6,102),HQ(6,102),
9HZ1(6,102),HZ2(6,102),HZ3(6,102),HZ4(6,102),HZ5(6,102),
2HZ6(6,102),TM1(202),TM2(202),ADV(105),PRFT(105),
5PR3(105),PR4(105),PR5(105),PR6(105),PR1(105),PR2(105),
1B(6),A(36),BB(6)
C
C      READING IN DATA
READ 100,GA,GB,EA,EB,TP,TQ,R,AQ,V1,V2,AIM,AN,C
READ 110,CT,DT,CA,AC,CB,CC,CQ,CI,W1,W2,W3,W4,TIM,TF
C
C      THESE ARE SIX INITIAL CONDITIONS GIVEN IN THE PROBLEM
C      TO BE USED FOR GETTING PARTICULARSOLUTION
C
  READ 410,SOL1,SOL2,SOL3,SOL4,SOL5,SOL6
  READ 410,(BB(I),I=1,6)
C
C      THESE ARE SIX FINAL CONDITIONS OBTAINED FOR LAGRANGE
C      MULTIPLIERS BY APPLYING TRANSVERSALITY CONDITION AND WILL
C      BE USED TO DETERMINE INTEGRATION CONSTANTS
C
C
C      PRINTING OUT THE DATA USED
C
  PRINT 120
  PRINT 130,GA,GB,EA,EB,R
  PRINT 135,AQ,V1,V2,AIM,TIM
  PRINT 140,AN,C,CT,DT,CA,AC,CB,CC,CQ,CI,W1,W2
  PRINT 145,TP,TQ,W3,W4
C
C      INPUTING INITIAL PROFILE
C
  DO 900 IJK=1,2
  IF(IJK-2) 190,200,200
190  ITF=51
  DO 210 I=1,ITF

```



```

    TEM1(1,I)=TP
    TEM2(1,I)=TQ
    SAQ(1,I)=W3
210  SZ6(1,I)=W4
    KAT=2
    NAT=KAT
    GO TO 300
200  IPK=ITF
    DO 230 I=1,IPK
        TEM1(1,I)=TEM1(KAT,I)
        TEM2(1,I)=TEM2(KAT,I)
        SAQ(1,I)=SAQ(KAT,I)
230  SZ6(1,I)=SZ6(KAT,I)
    ITF=101
    DO 240 I=IPK,ITF
        TEM1(1,I)=TEM1(KAT,IPK)
        TEM2(1,I)=TEM2(KAT,IPK)
        SAQ(1,I)=SAQ(KAT,IPK)
240  SZ6(1,I)=SZ6(KAT,IPK)
    KAT=11
C
C    START OF QUASILINEARIZATION ITERATIONS
C
300  DO 800 J=2,KAT
C
C    FINDING PARTICULAR SOLUTION
C
    PRINT 310
    P=1.
    X1(1)=SOL1
    X2(1)=SOL2
    Y1(1)=SOL3
    Y2(1)=SOL4
    AI(1)=SOL5
    Q(1)=SOL6
    Z1(1)=0.
    Z2(1)=0.
    Z3(1)=0.
    Z4(1)=0.
    Z5(1)=0.
    Z6(1)=0.
    PRINT 150,X1(1),X2(1),Y1(1),Y2(1),AI(1),Q(1),Z1(1),
1  Z2(1),Z3(1),Z4(1),Z5(1),Z6(1)
    CALL RKT
    PRINT 160
    DO 320 I=1,ITF
        PX1(I)=X1(I)
        PX2(I)=X2(I)
        PY1(I)=Y1(I)
        PY2(I)=Y2(I)

```

```

    PI(I)=AI(I)
    PQ(I)=Q(I)
    PZ1(I)=Z1(I)
    PZ2(I)=Z2(I)
    PZ3(I)=Z3(I)
    PZ4(I)=Z4(I)
    PZ5(I)=Z5(I)
320 PZ6(I)=Z6(I)
    PRINT 170,(I,X1(I),X2(I),Y1(I),Y2(I),AI(I),Q(I),Z1(I),
    LZ2(I),Z3(I),Z4(I),Z5(I),Z6(I),I=1,ITF,10)

```

C

C

```

    SIX SETS OF HOMOGENEOUS SOLUTIONS
    NSET=0
    DO 400 IHCM=1,6
    NSET=NSET+1
    PRINT 180,NSET
    X1(1)=0.
    X2(1)=0.
    Y2(1)=0.
    Y1(1)=0.
    AI(1)=0.
    Q(1)=0.
    Z1(1)=0.
    Z2(1)=0.
    Z3(1)=0.
    Z4(1)=0.
    Z5(1)=0.
    Z6(1)=0.
    IF(IHCM-2) 330,340,350
330 Z1(1)=1.
    GO TO 360
340 Z2(1)=1.
    GO TO 360
350 IF(IHCM-4) 351,352,353
351 Z3(1)=1.
    GO TO 360
352 Z4(1)=1.
    GO TO 360
353 IF(IHCM-6) 354,355,355
354 Z5(1)=1.
    GO TO 360
355 Z6(1)=1.
360 P=0.
    PRINT 150,X1(1),X2(1),Y1(1),Y2(1),AI(1),Q(1),Z1(1),
    LZ2(1),Z3(1),Z4(1),Z5(1),Z6(1)
    CALL RKT
    DO 370 I=1,ITF
    HX1(IHOM,I)=X1(I)
    HX2(IHOM,I)=X2(I)
    HY2(IHOM,I)=Y2(I)

```

```

HY1(IHOM,I)=Y1(I)
HAI(IHOM,I)=AI(I)
HQ(IHCM,I)=Q(I)
HZ1(IHOM,I)=Z1(I)
HZ2(IHOM,I)=Z2(I)
HZ3(IHOM,I)=Z3(I)
HZ4(IHOM,I)=Z4(I)
HZ5(IHOM,I)=Z5(I)
370 HZ6(IHOM,I)=Z6(I)
      PRINT 170,(I,X1(I),X2(I),Y1(I),Y2(I),AI(I),Q(I),Z1(I),
      1Z2(I),Z3(I),Z4(I),Z5(I),Z6(I),I=1,ITF,10)
400 CONTINUE

```

C  
C  
C

VALUE OF INTEGRATION CONSTANTS

```

N=6
PRINT 420,(BB(I),I=1,6)
B(1)=BB(1)-PZ1(ITF)
B(2)=BB(2)-PZ2(ITF)
B(3)=BB(3)-PZ3(ITF)
B(4)=BB(4)-PZ4(ITF)
B(5)=BB(5)-PI(ITF)
B(6)=BB(6)-PZ6(ITF)
KM=1
DO 425 I=1,36,6
  A(I)=HZ1(KM,ITF)
  A(I+1)=HZ2(KM,ITF)
  A(I+2)=HZ3(KM,ITF)
  A(I+3)=HZ4(KM,ITF)
  A(I+4)=HAI(KM,ITF)
  A(I+5)=HZ6(KM,ITF)
425 KM=KM+1
KS=0
CALL SIMQ(A,B,N,KS)
PRINT 426
PRINT 430,(B(I),I=1,6)

```

C  
C  
C

APPLICATION OF SUPERPOSITION PRINCIPLE

```

DO 440 I=1,ITF
SHX1=C.
SHX2=C.
SHY2=C.
SHY1=C.
SHAI=C.
SHAQ=C.
SHZ1=C.
SHZ2=C.
SHZ3=C.
SHZ4=C.

```

```

SHZ5=C.
SHZ6=C.
DO 435 KM=1,6
SHX1=SHX1+B(KM)*HX1(KM,I)
SHX2=SHX2+B(KM)*HX2(KM,I)
SHY2=SHY2+B(KM)*HY2(KM,I)
SHY1=SHY1+B(KM)*HY1(KM,I)
SHAI=SHAI+B(KM)*HAI(KM,I)
SHZ1=SHZ1+B(KM)*HZ1(KM,I)
SHZ2=SHZ2+B(KM)*HZ2(KM,I)
SHZ3=SHZ3+B(KM)*HZ3(KM,I)
SHZ4=SHZ4+B(KM)*HZ4(KM,I)
SHZ5=SHZ5+B(KM)*HZ5(KM,I)
SHZ6=SHZ6+B(KM)*HZ6(KM,I)
SHAQ=SHAQ+B(KM)*HQ(KM,I)
435 CONTINUE
SX1(J,I)=PX1(I)+SHX1
SX2(J,I)=PX2(I)+SHX2
SY2(J,I)=PY2(I)+SHY2
SY1(J,I)=PY1(I)+SHY1
SAI(J,I)=PI(I)+SHAI
SAQ(J,I)=PQ(I)+SHAQ
SZ1(J,I)=PZ1(I)+SHZ1
SZ2(J,I)=PZ2(I)+SHZ2
SZ3(J,I)=PZ3(I)+SHZ3
SZ4(J,I)=PZ4(I)+SHZ4
SZ5(J,I)=PZ5(I)+SHZ5
SZ6(J,I)=PZ6(I)+SHZ6
440 CONTINUE
C
C   FINAL SOLUTION
C
JJ=J-1
PRINT 450,JJ
PRINT 460
PRINT 470,(I,SX1(J,I),SX2(J,I),SY1(J,I),SY2(J,I),
1SAI(J,I),SAQ(J,I),I=1,ITF)
PRINT 480
PRINT 470,(I,SZ1(J,I),SZ2(J,I),SZ3(J,I),SZ4(J,I),
1SZ5(J,I),SZ6(J,I),I=1,ITF)
C
C   CALCULATION OF CONTROL VARIABLES AND TOTAL PROFIT
C
P1R=C.
P2R=C.
P3R=C.
P4R=C.
P5R=C.
P6R=C.
DO 500 I=1,ITF

```

```

ADV(I)=(SZ6(J,I)/(2.*CA))*((1./AN)-(1./SAQ(J,I)))
PR1(I)=P1R+AC*CQ*SAQ(J,I)*DT
P1R=PR1(I)
PR2(I)=P2R+CB*AQ*SX2(J,I)*DT
P2R=PR2(I)
PR3(I)=P3R+CC*AQ*(1.-SX2(J,I)-SY2(J,I))*DT
P3R=PR3(I)
PR4(I)=P4R+CI*((AIM-SAI(J,I))**2)*DT
P4R=PR4(I)
PR5(I)=P5R+CA*(ADV(I)**2)*(SAQ(J,I)**2)*DT
P5R=PR5(I)
PR6(I)=P6R+CT*((TIM-TEM2(J-1,I))**2+(TEM1(J-1,I)
1-TEM2(J-1,I))**2)*DT
P6R=PR6(I)
PRFT(I)=PR1(I)+PR2(I)+PR3(I)-PR4(I)-PR5(I)-PR6(I)
500 CONTINUE
PRINT 510,PRFT(ITF),PR1(ITF),PR2(ITF),PR3(ITF),PR4(ITF),
1PR5(ITF),PR6(ITF)
PRINT 520
C
C      NEWTON RAPHSON METHOD FOR CALCULATING TEMP 1 AND TEMP 2
C
IF(IJK-2) 525,530,530
525 TM1(1)=TEM1(J-1,1)
TM2(1)=TEM2(J-1,1)
GO TO 545
530 IF(J-2) 535,535,525
535 TM1(1)=TEM1(NAT,1)
TM2(1)=TEM2(NAT,1)
545 DO 800 I=1,ITF
DO 550 N=1,201
DATAA=((SZ1(J,I)-SZ2(J,I))*GA*SX1(J,I)*EA)/(R*(TM1(N)**4))
DATAB=(EA/R-2.*TM1(N))*EXP(-EA/(R*TM1(N)))
DATAC=(SZ2(J,I)*GB*SY1(J,I)*EB)/(R*(TM1(N)**4))
DATAC=(EB/R-2.*TM1(N))*EXP(-EB/(R*TM1(N)))
DATA=CATAA*DATAB+DATAC*DATAC-4.*CT
DATB=2.*CT
DBTA=2.*CT
DBTBA=((SZ3(J,I)-SZ4(J,I))*GA*SX2(J,I)*EA)/(R*(TM2(N)**4))
DBTBB=(EA/R-2.*TM2(N))*EXP(-EA/(R*TM2(N)))
DBTBC=(SZ4(J,I)*GB*SY2(J,I)*EB)/(R*(TM2(N)**4))
DBTBD=(EB/R-2.*TM2(N))*EXP(-EB/(R*TM2(N)))
DBTB=DBTBA*DBTBB+DBTBC*DBTBC-2.*CT
FUN1A=((SZ1(J,I)-SZ2(J,I))*GA*SX1(J,I)*EA)/(R*(TM1(N)**2))
FUN1B=EXP(-EA/(R*TM1(N)))
FUN1C=(SZ2(J,I)*GB*SY1(J,I)*EB)/(R*(TM1(N)**2))
FUN1D=EXP(-EB/(R*TM1(N)))
FUN1E=2.*CT*(2.*TM1(N)-TM2(N)-TIM)
FUN1=FUN1A*FUN1B+FUN1C*FUN1D-FUN1E
FUN2A=((SZ3(J,I)-SZ4(J,I))*GA*SX2(J,I)*EA)/(R*(TM2(N)**2))

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```

FUN2B=EXP(-EA/(R*TM2(N)))
FUN2C=(SZ4(J,I)*GB*SY2(J,I)*EB)/(R*(TM2(N)**2))
FUN2D=EXP(-EB/(R*TM2(N)))
FUN2E=2.*CT*(TM1(N)-TM2(N))
FUN2=FUN2A*FUN2B+FUN2C*FUN2C+FUN2E
RHS1=TM1(N)*DATA+TM2(N)*DATB-FUN1
RHS2=TM1(N)*DBTA+TM2(N)*DBTB-FUN2
DETR=DATA*DBTB-CATB*DBTA
TM1(N+1)=(RHS1*DBTB-RHS2*DAB)/DETR
TM2(N+1)=(RHS2*DATA-RHS1*DBTA)/DETR
DEL1=TM1(N+1)-TM1(N)
DEL2=TM2(N+1)-TM2(N)
TMNW1=TM1(N+1)
TMNW2=TM2(N+1)
IF(ABS(DEL1).GT.0.1) GO TO 550
IF(ABS(DEL2).LE.0.1) GO TO 560
550 CONTINUE
PRINT 570,N,DEL1,DEL2
560 TEM1(J,I)=TMNW1
TEM2(J,I)=TMNW2
IF(J.GE.3) GO TO 580
TM1(1)=TEM1(J,I)
TM2(1)=TEM2(J,I)
GO TO 600
580 IF(I-ITF) 590,600,600
590 TM1(1)=TEM1(J-1,I+1)
TM2(1)=TEM2(J-1,I+1)
600 CONTINUE
PRINT 620,I,ADV(I),TEM1(J,I),TEM2(J,I),PRFT(I)
800 CONTINUE
C
C      END OF ONE ITERATION
C
900 CONTINUE
C
100 FORMAT(2E10.3,2F7.1,2F5.1,5F4.1,F5.1,F3.1)
110 FORMAT(F7.5,F4.2,F7.5,5F5.3,4F6.3,2F5.1)
410 FORMAT(6F5.2)
120 FORMAT(' VALUE OF THE CONSTANTS USED')
130 FORMAT(' GA=',E10.3,' GB=',E10.3,' EA=',F7.1,' EB=',
1F7.1,' R=',F4.1)
135 FORMAT(' FLOW RATE=',F5.1,' V1=',F4.1,
2' V2=',F4.1,' MEAN INVENTORY=',F4.1,' MEAN TEMP=',F6.2)
140 FORMAT(' N=',F5.1,' C=',F5.1,' CT= ',F9.5,' DT=',F6.2,
1'CA=',F9.5,'C1=',F7.3/' C2=',F7.3,'C3=',F7.3,
2'CC=',F7.3,'CI=',F7.3,'XO(T)=',F8.3,'YO(T)=',F8.3)
145 FORMAT(' T1=',E12.4,'T2=',E12.4,'QC(T)=',E10.2,'L6(T)=',
1E10.2)
150 FORMAT(' X1(0)=',F6.2,'X2(0)=',F6.2,'Y1(0)=',F6.2,
1'Y2(0)=',F6.2,'I(0)=',F6.2,'Q(0)=',F6.2/' L1(0)=',F6.2,

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2'L2(0)=' ,F6.2,'L3(0)=' ,F6.2,'L4(0)=' ,F6.2,'L5(0)=' ,
4F6.2,'L6(0)=' ,F6.2)
160 FORMAT(// ' STAGE' ,2X,'CONCENTRATION OF A' ,2X,
1'CONCENTRATION OF B' ,3X,'INVENTORY' ,2X,'SALES' ,3X,
3'LAGRANGE MULTIPLIERS'//)
170 FORMAT(I4,2X,12E10.2)
180 FORMAT(' HOMOGENEOUS SOLUTION SET #' ,I4)
310 FORMAT(' PARTICULAR SOLUTION')
420 FORMAT(' FINAL CONDITIONS FOR LAGRANGE MULTIPLIERS' ,
15X/6F10.6)
426 FORMAT(' INTEGRATION CONSTANTS-' )
430 FORMAT(' A1=' ,F15.5,'A2=' ,F15.5,'A3=' ,F15.5,
1'A4=' ,F15.5,/' A5=' ,F15.5,'A6=' ,F15.5)
450 FORMAT(' FINAL SOLUTION ITERATION #' ,I5)
460 FORMAT(// ' STAGE' ,11X,'CONCENTRATION OF A' ,18X,
1'CONCENTRATION OF B' ,16X,'INVENTORY' ,11X,'SALES'//)
470 FORMAT(I4,3X,6E18.5)
480 FORMAT(' LAGRANGE MULTIPLIERS LAMDA 1 TO LAMDA 6')
510 FORMAT(' TOTAL PROFIT=' ,F15.7,10X,6F15.5)
520 FORMAT(' ADVERT FOR PREVIOUS ITERATION' ,21X,'TEMP' ,1X,
1' FOR CURRENT ITERATION' ,12X,'PROFIT FOR CURRENT ITERATION')
570 FORMAT(70X,'TEMP DID NOT CONVERGE' ,2X,I3,' DEL1=' ,F9.3,
1' DEL2=' ,F9.3)
620 FORMAT(1H ,I4,5X,'ADVT=' ,E12.4,17X,'T1=' ,E14.6,3X,'T2=' ,
1E14.6,12X,'TOTAL PROFIT' ,F10.5)
STOP
END

```

# SUBROUTINE RKT

C  
C  
C  
C

RUNGE KUTTA METHOD HAS BEEN USED TO INTEGRATE 12 LINEAR  
EQUATIONS

```

COMMON X1,Y1,X2,Y2,AI,Q,Z1,Z2,Z3,Z4,Z5,Z6,DT,X0,Y0,
1TP,TQ,GA,GB,EA,EB,R,AQ,V1,V2,AN,CA,CI,AIM,CB,CC,CQ,C,
2AC,CT,P,TEM1,TEM2,W1,W2,W3,W4,J,SAQ,SZ6,ITF
DIMENSION X1(105),X2(105),Y1(105),Y2(105),AI(105),Q(105),
1Z1(105),Z2(105),Z3(105),Z4(105),Z5(105),Z6(105),
2A1(105),B1(105),C1(105),D1(105),E1(105),F1(105),G1(105),
3A2(105),B2(105),C2(105),D2(105),E2(105),F2(105),G2(105),
4A3(105),B3(105),C3(105),D3(105),E3(105),F3(105),G3(105),

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5A4(105),B4(105),C4(105),D4(105),E4(105),F4(105),G4(105),
6H1(105),R1(105),S1(105),T1(105),U1(105),XO(105),
7H2(105),R2(105),S2(105),T2(105),U2(105),YO(105),
8H3(105),R3(105),S3(105),T3(105),U3(105),TEM1(10,102),
9H4(105),R4(105),S4(105),T4(105),U4(105),TEM2(10,102),
1SAQ(10,102),SZ6(10,102)
SP=AQ/V1
SQ=AQ/V2
DO 130 I=1,ITF
TT1=R*TEM1(J-1,I)
TT2=R*TEM2(J-1,I)
XO(I)=W1
YO(I)=W2
V=SAQ(J-1,I)
W=SZ6(J-1,I)
A1(I)=DT*(P*SP*XO(I)-SP*X1(I)-GA*EXP(-EA/TT1)*X1(I))
B1(I)=DT*(P*SP*YO(I)-SP*Y1(I)-GB*EXP(-EB/TT1)*Y1(I)+
1GA*EXP(-EA/TT1)*X1(I))
C1(I)=DT*(SQ*(X1(I)-X2(I))-GA*EXP(-EA/TT2)*X2(I))
D1(I)=DT*(SQ*(Y1(I)-Y2(I))-GB*EXP(-EB/TT2)*Y2(I)+
1GA*EXP(-EA/TT2)*X2(I))
E1(I)=DT*(AQ*Y2(I)-CQ*Q(I))
F1A=(C*V-C*(V**2)/AN+V*W/(CA*AN)-W/(2.*CA)-W*(V**2)/
1(2.*CA*(AN**2)))
F2A=(C-2.*C*V/AN+W/(CA*AN)-W*V/(CA*(AN**2)))
F3A=(V/(CA*AN)-1./(2.*CA)-V**2/(2.*CA*(AN**2)))
F1(I)=DT*(P*F1A-P*V*F2A-P*W*F3A+Q(I)*F2A+Z6(I)*F3A)
G1(I)=DT*(SP*Z1(I)-SQ*Z3(I)+(Z1(I)-Z2(I))*GA*EXP(-EA/TT1))
H1(I)=DT*(SP*Z2(I)-SQ*Z4(I)+Z2(I)*EXP(-EB/TT1)*GB)
R1(I)=DT*(SQ*Z3(I)+(Z3(I)-Z4(I))*GA*EXP(-EA/TT2)+P*AQ*(CB-CC)
S1(I)=DT*(SQ*Z4(I)+Z4(I)*GB*EXP(-EB/TT2)-Z5(I)*AQ-P*CC*AQ)
T1(I)=DT*(P*2.*CI*AIM-2.*CI*AI(I))
U1A=(AC*CQ-C*W+2.*C*V*W/AN+(W**2)*V/(2.*CA*(AN**2))
1-W**2/(2.*CA*AN))
U2A=(2.*C*W/AN+W**2/(2.*CA*(AN**2)))
U3A=(2.*C*V/AN-C+W*V/(CA*(AN**2))-W/(CA*AN))
U1(I)=DT*(CQ*Z5(I)+P*U1A-P*V*U2A-P*W*U3A+Q(I)*U2A+Z6(I)
1*U3A)
A2(I)=DT*(P*SP*XO(I)-SP*(X1(I)+A1(I)/2.))-GA*EXP(-EA/TT1)
1*(X1(I)+A1(I)/2.))
B2(I)=DT*(P*SP*YO(I)-SP*(Y1(I)+B1(I)/2.))-GB*EXP(-EB/TT1)
1*(Y1(I)+B1(I)/2.))+GA*EXP(-EA/TT1)*(X1(I)+A1(I)/2.))
C2(I)=DT*(SQ*((X1(I)+A1(I)/2.))-(X2(I)+C1(I)/2.))-
1GA*EXP(-EA/TT2)*(X2(I)+C1(I)/2.))
D2(I)=DT*(SQ*((Y1(I)+B1(I)/2.))-(Y2(I)+D1(I)/2.))-GB*EXP(-EB
1/TT2)*(Y2(I)+B1(I)/2.))+GA*EXP(-EA/TT2)*(X2(I)+C1(I)/2.))
E2(I)=DT*(AQ*(Y2(I)+B1(I)+D1(I)/2.))-CQ*(Q(I)+F1(I)/2.))
F2(I)=DT*(P*F1A-P*V*F2A-P*W*F3A+(Q(I)+F1(I)/2.))*F2A+
1(Z6(I)+U1(I)/2.))*F3A)
G2(I)=DT*(SP*(Z1(I)+G1(I)/2.))-SQ*(Z3(I)+R1(I)/2.))+((

```



```

1Z1(I)+G1(I)/2.)-(Z2(I)+H1(I)/2.))*GA*EXP(-EA/TT1))
H2(I)=DT*(SP*(Z2(I)+H1(I)/2.)-SQ*(Z4(I)+S1(I)/2.))+
1(Z2(I)+H1(I)/2.)*EXP(-EB/TT1)*GB)
R2(I)=DT*(SQ*(Z3(I)+R1(I)/2.))+((Z3(I)+R1(I)/2.)-(Z4(I)
1+S1(I)/2.))*GA*EXP(-EA/TT2)+P*AQ*(CB-CC))
S2(I)=DT*(SQ*(Z4(I)+S1(I)/2.))+((Z4(I)+S1(I)/2.))*GB*
1EXP(-EB/TT2)-(Z5(I)+T1(I)/2.)*AQ-P*CC*AQ)
T2(I)=DT*(P*2.*CI*AIM-2.*CI*(AI(I)+E1(I)/2.))
U2(I)=DT*(CQ*(Z5(I)+T1(I)/2.))+P*U1A-P*V*U2A-P*W*U3A+
1(Q(I)+F1(I)/2.)*U2A+(Z6(I)+U1(I)/2.)*U3A)
A3(I)=DT*(P*SP*XO(I)-SP*(X1(I)+A2(I)/2.))-GA*EXP(-EA/TT1)
1*(X1(I)+A2(I)/2.))
B3(I)=DT*(P*SP*YO(I)-SP*(Y1(I)+B2(I)/2.))-GB*EXP(-EB/TT1)
1*(Y1(I)+B2(I)/2.))+GA*EXP(-EA/TT1)*(X1(I)+A2(I)/2.))
C3(I)=DT*(SQ*((X1(I)+A2(I)/2.)-(X2(I)+C2(I)/2.)))-GA*EXP(-EA/
1TT2)*(X2(I)+C2(I)/2.))
D3(I)=DT*(SQ*((Y1(I)+B2(I)/2.)-(Y2(I)+D2(I)/2.)))-GB*EXP(-EB/
1TT2)*(Y2(I)+B2(I)/2.))+GA*EXP(-EA/TT2)*(X2(I)+C2(I)/2.))
E3(I)=DT*(AQ*(Y2(I)+D2(I)/2.))-CQ*(Q(I)+F2(I)/2.))
F3(I)=DT*(P*F1A-P*V*F2A-P*W*F3A+(Q(I)+F2(I)/2.))*F2A+
1(Z6(I)+U2(I)/2.))*F3A)
G3(I)=DT*(SP*(Z1(I)+G2(I)/2.))-SQ*(Z3(I)+R2(I)/2.))+((
1Z1(I)+G2(I)/2.)-(Z2(I)+H2(I)/2.))*GA*EXP(-EA/TT1))
H3(I)=DT*(SP*(Z2(I)+H2(I)/2.))-SQ*(Z4(I)+S2(I)/2.))+
1(Z2(I)+H2(I)/2.)*EXP(-EB/TT1)*GB)
R3(I)=DT*(SQ*(Z3(I)+R2(I)/2.))+((Z3(I)+R2(I)/2.)-(Z4(I)
1+S2(I)/2.))*GA*EXP(-EA/TT2)+P*AQ*(CB-CC))
S3(I)=DT*(SQ*(Z4(I)+S2(I)/2.))+((Z4(I)+S2(I)/2.))*GB*
1EXP(-EB/TT2)-(Z5(I)+T2(I)/2.)*AQ-P*CC*AQ)
T3(I)=DT*(P*2.*CI*AIM-2.*CI*(AI(I)+E2(I)/2.))
U3(I)=DT*(CQ*(Z5(I)+T2(I)/2.))+P*U1A-P*V*U2A-P*W*U3A+
1(Q(I)+F2(I)/2.)*U2A+(Z6(I)+U2(I)/2.)*U3A)
A4(I)=DT*(P*SP*XO(I)-SP*(X1(I)+A3(I)))-GA*EXP(-EA/TT1)*
1(X1(I)+A3(I)))
B4(I)=DT*(P*SP*YO(I)-SP*(Y1(I)+B3(I)))-GB*EXP(-EB/TT1)*
1(Y1(I)+B3(I))+GA*EXP(-EA/TT1)*(X1(I)+A3(I)))
C4(I)=DT*(SQ*((X1(I)+A3(I))-(X2(I)+C3(I)))-GA*EXP(-EA/
1TT2)*(X2(I)+C3(I)))
D4(I)=DT*(SQ*((Y1(I)+B3(I))-(Y2(I)+D3(I)))-GB*EXP(-EB/
1TT2)*(Y2(I)+B3(I))+GA*EXP(-EA/TT2)*(X2(I)+C3(I)))
E4(I)=DT*(AQ*(Y2(I)+D3(I)))-CQ*(Q(I)+F3(I)))
F4(I)=DT*(P*F1A-P*V*F2A-P*W*F3A+(Q(I)+F3(I))*F2A+(Z6(I)
1+U3(I))*F3A)
G4(I)=DT*(SP*(Z1(I)+G3(I)))-SQ*(Z3(I)+R3(I))+((Z1(I)+
1G3(I))-(Z2(I)+H3(I)))*GA*EXP(-EA/TT1))
H4(I)=DT*(SP*(Z2(I)+H3(I)))-SQ*(Z4(I)+S3(I))+((Z2(I)+
1H3(I))*EXP(-EB/TT1)*GB)
R4(I)=DT*(SQ*(Z3(I)+R3(I))+((Z3(I)+R3(I))-(Z4(I)+S3(I)
1))*GA*EXP(-EA/TT2)+P*AQ*(CB-CC))
S4(I)=DT*(SQ*(Z4(I)+S3(I))+((Z4(I)+S3(I))*GB*EXP(-EB/

```

```

1 TT2)-(Z5(I)+T3(I))*AQ-P*CC*AQ)
  T4(I)=DT*(P*2.*CI*AIM-2.*CI*(AI(I)+E3(I)))
  U4(I)=DT*(CQ*(Z5(I)+T3(I))+P*U1A-P*V*U2A-P*W*U3A+(Q(I)
1+F3(I))*U2A+(Z6(I)+U3(I))*J3A)
  X1(I+1)=X1(I)+(A1(I)+2.*A2(I)+2.*A3(I)+A4(I))/5.
  Y1(I+1)=Y1(I)+(B1(I)+2.*B2(I)+2.*B3(I)+B4(I))/5.
  Y2(I+1)=Y2(I)+(D1(I)+2.*D2(I)+2.*D3(I)+D4(I))/5.
  X2(I+1)=X2(I)+(C1(I)+2.*C2(I)+2.*C3(I)+C4(I))/5.
  AI(I+1)=AI(I)+(E1(I)+2.*E2(I)+2.*E3(I)+E4(I))/5.
  Z1(I+1)=Z1(I)+(G1(I)+2.*G2(I)+2.*G3(I)+G4(I))/5.
  Z2(I+1)=Z2(I)+(H1(I)+2.*H2(I)+2.*H3(I)+H4(I))/5.
  Z3(I+1)=Z3(I)+(R1(I)+2.*R2(I)+2.*R3(I)+R4(I))/5.
  Z4(I+1)=Z4(I)+(S1(I)+2.*S2(I)+2.*S3(I)+S4(I))/5.
  Z5(I+1)=Z5(I)+(T1(I)+2.*T2(I)+2.*T3(I)+T4(I))/5.
  Z6(I+1)=Z6(I)+(U1(I)+2.*U2(I)+2.*U3(I)+U4(I))/5.
  Q(I+1)=Q(I)+(F1(I)+2.*F2(I)+2.*F3(I)+F4(I))/5.
130 CONTINUE
  RETURN
  END

```

### APPENDIX 3

SUBROUTINE SIMQ TO SOLVE FOR THE  
INTEGRATION CONSTANTS.

```

SUBROUTINE SIMO(A,B,N,KS)
DIMENSION A(1),B(1)
C
C      FORWARD SOLUTION
C
TOL=0.0
KS=0
JJ=-N
DO 65 J=1,N
JY=J+1
JJ=JJ+N+1
BIGA=0
IT=JJ-J
DO 30 I=J,N
C
C      SEARCH FOR MAXIMUM COEFFICIENT IN COLUMN
C
IJ=IT+I
IF(ABS(BIGA)-ABS(A(IJ))) 20,30,30
20 BIGA=A(IJ)
IMAX=I
30 CONTINUE
C
C      TEST FOR PIVOT LESS THAN TOLERANCE (SINGULAR MATRIX)
C
IF(ABS(BIGA)-TOL) 35,35,40
35 KS=1
RETURN
C
C      INTERCHANGE ROWS IF NECESSARY
C
40 I1=J+N*(J-2)
IT=IMAX-J
DO 50 K=J,N
I1=I1+N
I2=I1+IT
SAVE=A(I1)
A(I1)=A(I2)
A(I2)=SAVE
C
C      DIVIDE EQUATION BY LEADING COEFFICIENT
C
SAVE=B(IMAX)
C
50 A(I1)=A(I1)/BIGA
B(IMAX)=B(J)
B(J)=SAVE/BIGA
C
C      ELIMINATE NEXT VARIABLE
C
IF(J-N) 55,70,55

```

```

55 IQS=N*(J-1)
   DO 65 IX=JY,N
     IXJ=IQS+IX
     IT=J-IX
     DO 60 JX=JY,N
       IXJX=N*(JX-1)+IX
       JJX=IXJX+IT
60  A(IXJX)=A(IXJX)-(A(IXJ)*A(JJX))
65  B(IX)=B(IX)-(B(J)*A(IXJ))

```

C  
C  
C

BACK SOLUTION

```

70 NY=N-1
   IT=N*N
   DO 80 J=1,NY
     IA=IT-J
     IB=N-J
     IC=N
     DO 80 K=1,J
       B(IB)=B(IB)-A(IA)*B(IC)
     IA=IA-N
80  IC=IC-1
   RETURN
END

```

OPTIMIZATION OF MANAGEMENT SYSTEMS  
USING THE TWO-PHASE METHOD OF  
QUASILINEARIZATION

by

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Past studies have shown the effectiveness of quasilinearization in dealing with large dimensional problems and in overcoming the two-point boundary-value difficulty. For unstable problems, the quasilinearization method fails if the initial guess is not a good one. Recent studies are oriented towards increasing the range of convergence for the quasilinearization method. The two-phase method of quasilinearization helps in serving this purpose.

This work investigates the computational features of the two-phase method of quasilinearization in solving two complex management problems. First, the two-phase method is described. Then it is applied to an inventory and advertising scheduling problem having two state variables and one control variable. Finally, it is applied to an advertisement and production scheduling problem having six state variables and three control variables.

The number of iterations in phase 1, the period of phase 1, the phasing scheme are a few of the items discussed in this work. It was concluded that:

- 1) The range of convergence increases tremendously in both of the problems studied.
- 2) There is no advantage in obtaining the optimum profile in phase 1. In general, no more than one iteration is needed.
- 3) Shorter periods in phase 1 are helpful.
- 4) Backward phasing might be helpful in cases where the profiles have large slopes in the final stages.
- 5) In general, convergence is achieved in the same number of iterations for all the initial approximations.

Based on these two problems, it can be said that the two-phase method of quasilinearization is more effective than both, the ordinary quasilinearization method and the differential sensitivity analysis.