THE EFFECT OF WALL ELECTRICAL CONDUCTANCE ON THE FULLY DEVELOPED VELOCITY DISTRIBUTION FOR LAMINAR, STEADY, TWO-DIMENSIONAL, MAGNETOHYDRODYNAMIC CHANNEL FLOW WITH HEAT TRANSFER

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NOMENCLATURE

A = dimensionless temperature gradient, equation (5-18b)

a = $(\mu_e H_y^2/\rho)^{\frac{1}{2}}$ = Alfven wave speed, meter/second

B_{ap} = B_y = applied magnetic field density, weber/square meter

C_p = specific heat at constant pressure, Joules/kilogram $^\circ$K

D = defined by equation (5-18a)

$\underline{E}$ = electric field intensity vector, volt/meter

$\underline{E}_{ap}$ = applied electric field intensity in z-direction, volt/meter

$\underline{E}_{ind}$ = induced electric field intensity vector, volt/meter

F = $F(Y)$ = defined by equation (5-26a)

$\underline{F}_{ind}$ = induced Lorentz force, Ampere-turn

$\underline{H}$ = magnetic field intensity vector, ampere-turn/meter

H_x, H_y = x-, y-components of $\underline{H}$, ampere-turn/meter

$\overline{H}$ = H_x/H_y, dimensionless number

H_o = H_y, ampere-turn/meter

h = wall thickness, meter

I = total current in Z-direction, ampere

I^* = $I/(\sigma_f u_m B_y L)$, dimensionless current

$\underline{J}$ = electric current density vector, ampere/square meter

$\underline{J}_{ind}$ = induced current density vector, ampere/square meter

k = thermal conductivity, Joules/second meter $^\circ$K

L = half depth of the channel, meter

M = $\mu_e H_o L\sqrt{\sigma/\rho\nu}$ = $aL/(\eta^{\frac{1}{2}}\nu^{\frac{1}{2}})$ = Hartmann number

P = $-\frac{\partial p}{\partial x}$ = pressure gradient, Newton/cubic meter

$\overline{P}$ = $-(L^3/\rho\nu^2)(\partial p/\partial x)$ = dimensionless pressure gradient

p = pressure, Newton/square meter

P_r = $\nu/\alpha = \frac{C_p \mu}{k}$ = Prandtl number

P_{re} = η/ν = electromagnetic Prandtl number

Q_1 = $\frac{q_1 L}{\mu u_m^2}$ $Q_2 = \frac{q_2 L}{\mu u_m^2}$ dimensionless heat-flow number

q = $-k(\frac{\partial T}{\partial y})_w$ = wall heat flux, Joules/second square meter

R = defined by equation (5-35)

R_e = $u_m L/\nu$ = Reynolds number

R_M = aL/ν = magnetic Reynolds number

r = ϕ/ϕ_{open}, voltage ratio

T = temperature, degree Kelvin

T_b = bulk temperature, equation (5-32a), $^\circ K$

T_o = initial bulk temperature, $^\circ K$

t = time, second

U = u/u_m, equation (5-1)

u = velocity component in x-direction, meter/second

$\bar{u}$ = u/a dimensionless velocity

u_m = $\frac{1}{2L}\int_{-L}^{L} u dy$ = mean velocity, meter/second

$\underline{V}$ = fluid velocity vector, meter/second

X = x/L, dimensionless longitudinal coordinate

Y = y/L, dimensionless transverse coordinate

α = $k/\rho C_p$ = thermal diffusivity, square meter/second

η = $1/\sigma\mu_e$ = electromagnetic diffusivity, square meter/second

ρ = density, kilogram/cubic meter

μ = voscosity, kilogram/meter second

μ_e = magnetic permeability, Henry/meter

ν = μ/ρ = kinematic viscosity, meter^2/second

σ = electrical conductivity Mho/meter

Φ = $E_z / u_m B_y$, dimensionless voltage number

Φ open= $(\Phi)_{I=0}$, equation (5-11)

Φ_1, Φ_2 = dimensionless parameters, equation (4-18), (4-19)

φ_1 = $\sigma_1 h_1/\sigma_f L$, $\varphi_2 = \sigma_2 h_2/\sigma_f L$

θ = $T/(\mu u_m^2/k)$ = dimensionless temperature

θ_b = dimensionless bulk temperature

θ_o = initial dimensionless bulk temperature

Subscripts

1, 2 = lower, upper wall

w = wall

f = fluid

INTRODUCTION

Magnetohydrodynamics (MHD) is the science of the motion of an electrically conducting incompressible fluid in the presence of a magnetic field. It is a special case of the study of plasma phenomena. Many names have been applied to the phenomenon. The subject of MHD includes five classes of the engineering sciences, namely, fluid-mechanics, thermodynamics, mechanics, materials, and the electrical science.

Consider an electrically conducting fluid with a velocity $\underline{V}$ [1]. Perpendicular to this a magnetic field with field density $\underline{B}_{ap}$ is applied (Fig. 1). Assume that steady flow conditions have been attained. Because of the interaction of these two fields, an electric field $\underline{E}_{ind}$ is induced perpendicularly to both $\underline{V}$ and $\underline{B}_{ap}$.

This electric field is given by the following equation:

$$\underline{E}_{ind} = \underline{V} \times \underline{B}_{ap} \tag{1-1}$$

For simplification assume that the electrical conductivity, σ , is constant in spite of the magnetic field.* By Ohm's law the current density induced in the conducting fluid, and denoted by $\underline{J}_{ind}$, is:

$$\underline{J}_{ind} = \sigma \underline{E}_{ind} \tag{1-2}$$

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*This is a simplification. For a sufficiently strong magnetic field, the conductivity becomes a tensor quantity.

Simultaneously occurring with the induced current is the induced Lorentz force $\underline{F}_{ind}$ which is given by the following:

$$\underline{F}_{ind} = \underline{J}_{ind} \times \underline{B}_{ap} \tag{1-3}$$

The force $\underline{F}_{ind}$ occurs because the conducting fluid cuts the lines of the magnetic field. Because the vector product of equation (1-3) yields a vector perpendicular to both $\underline{J}_{ind}$ and $\underline{B}_{ap}$, the induced force is parallel to $\underline{V}$ but opposite in direction.

For the more general case, consider an electric field $\underline{E}_{ap}$ perpendicular to both $\underline{B}_{ap}$ and $\underline{V}$, but opposite in direction to $\underline{J}_{ind}$. The current density due to this applied electric field is $\underline{J}_{cond}$. The net current $\underline{J}$ through the conducting fluid is then

$$\begin{aligned}\underline{J} &= \sigma(\underline{E}_{ap} + \underline{V} \times \underline{B}_{ap}) \\ &= \sigma(\underline{E}_{ap} + \underline{E}_{ind})\end{aligned} \tag{1-4}$$

The ponderomotive or Lorentz force associated with this current is then

$$\underline{F} = \underline{J} \times \underline{B}_{ap} = \sigma(\underline{E}_{ap} + \underline{V} \times \underline{B}_{ap}) \times \underline{B}_{ap} \tag{1-5}$$

If in equation (1-5) $\underline{E}_{ap} > \underline{V} \times \underline{B}_{ap}$ the system is an accelerator (or a pump) which may be used as a thrust-producing device. If $\underline{E}_{ap} < \underline{V} \times \underline{B}_{ap}$, it is a generator.

The purpose of this report is to present the detailed as well as critical review of some mathematical and physical aspects of the MHD flow. This review should benefit the beginning investigators in this field.

First, the details of the pioneering work by Hartmann and Lazarus on the subject of channel flow [2] are described. The modified Hartmann flow with the electrical conductance of walls investigated by Chang and Lundgren [4], and by Chang and Yen [5] are then reviewed. The effect of electrical conductance of the walls for both the thermally and hydrodynamically fully developed region was investigated very recently by Alpher [7], Yen [8], and Snyder [9]. While their treatments are reviewed generally, only Snyder's approach [9] is presented in detail.

THE FUNDAMENTAL EQUATIONS OF MAGNETOHYDRODYNAMICS

The equations of MHD of continuous fluid media are the ordinary electromagnetic and hydrodynamic equations, modified to take account of the interaction between the fluid motion and the magnetic field.

On the assumptions that: (1) the fluid is incompressible, (2) the displacement current is negligible (as in most electromagnetic problems), i.e., no oscillations of very high frequency occur, (3) the permeability and conductivity are constant scalar quantities, and (4) the Lorentz force is the only body force on the fluid, the MHD equations are [3]

Maxwell's equation in MKS units system:

$$\text{curl } \underline{H} = \underline{J} \tag{2-1}$$

$$\text{div } \underline{J} = 0 \tag{2-2}$$

$$\text{curl } \underline{E} = \mu_e \frac{\partial \underline{H}}{\partial t} \tag{2-3}$$

$$\text{div } \underline{H} = 0; \tag{2-4}$$

Ohm's law for a moving fluid:

$$\underline{J} = \sigma(\underline{E} = \underline{V} \times \mu_e \underline{H}); \tag{2-5}$$

the equation of continuity:

$$\text{div } \underline{V} = 0; \tag{2-6}$$

and the modified Navier-Stokes equation:

$$\frac{\partial \underline{V}}{\partial t} + (\underline{V} \cdot \text{grad})\underline{V} = -\frac{1}{\rho}\text{grad } p + \nu\nabla^2\underline{V} + \frac{1}{\rho}(\underline{J} \times \mu_e \underline{H}) \tag{2-7}$$

The usual procedure followed to obtain equations for $\underline{H}$ and $\underline{V}$ is to eliminate the electric field $\underline{E}$ and the current density $\underline{J}$ among equations (2-1), (2-3), (2-5) and (2-7). Making use of the divergence relations as represented by equation (2-4) and (2-6), one obtains the resulting equation in the forms (c.f. Appendix I):

$$\frac{\partial \underline{H}}{\partial t} - \text{curl } (\underline{V} \times \underline{H}) = \eta \nabla^2 \underline{H} \tag{2-8}$$

$$\frac{\partial \underline{V}}{\partial t} + (\underline{V} \cdot \text{grad}) \underline{V} - \frac{\mu_e}{\rho} (\underline{H} \cdot \text{grad})\underline{H} = - \text{grad}(\frac{p}{\rho} + \frac{\mu_e}{2\rho} \underline{H}^2) + \nu\nabla^2\underline{V} \tag{2-9}$$

In equation (2-8), η is written for $1/\sigma\mu_e$. These equations, together with equation (2-6), are sufficient to determine all the variables $\underline{V}$, $\underline{H}$, and p.

HARTMANN FLOW

The proposed investigation is mainly concerned with the transport and rate processes in steady, two-dimensional, laminar, magnetohydrodynamic flow of incompressible fluid media between two parallel plates. Therefore, the solution of the flow equation for this type of flow, originally obtained by Hartmann [2] and well summarized in Cowling [3], is presented in detail.

Referring to Fig. 2, the two electrically non-conducting infinite parallel plates are at rest at $y = \pm L$. It is then permissible to assume that

$$\underline{V} = (u, o, o) \tag{3-1}$$

$$\underline{H} = (H_x, H_y, o) \tag{3-2}$$

are functions of y only. In other words the flow has only the x-direction component, and a uniform magnetic field H_o is imposed perpendicular to the bounding walls, i.e., H_o is parallel to y-axis. Because the fluid near the median plane y = 0 moves faster than that near the walls, it tends to pull out the lines of force in its direction of motion. Thus, the field acquires a component H_x parallel to the motion.

Applying equations (3-1), and (3-2) to equations (2-1) through (2-7), and remembering that u = 0 on the walls, the velocity profile is obtained as [3] (c.f. Appendix II)

$$u = \frac{PM}{\mu_e^2 \sigma H_o^2} \left(\frac{\cosh M - \cosh (\frac{My}{L})}{\sinh M} \right) \tag{3-3}$$

The velocity profile is presented in Fig. 3. The average value of u between y = ± L is u_m where:

$$u_m = \frac{P}{\mu_e^2 \sigma H_o^2} (M \coth M - 1) \tag{3-4}$$

Then, the dimensionless velocity is,

$$U = \frac{u}{u_m} = M \left(\frac{\cosh M - \cosh (M \frac{y}{L})}{M \cosh M - \sinh M} \right) \tag{3-5}$$

The corresponding value of H_x is found to be

$$H_x = \frac{PL}{\mu_e H_o} \left(\frac{\sinh (\frac{My}{L})}{\sinh M} - \frac{y}{L} \right) \tag{3-6}$$

where

$M = \mu_e H_o L \sqrt{\frac{\sigma}{\rho\nu}}$, Hartmann number, a dimensionless number.

$P = -\frac{\partial p}{\partial x}$, a constant which is the pressure gradient in the x-direction.

MODIFIED HARTMANN FLOW WITH THE EFFECT OF WALL ELECTRICAL CONDUCTANCE

Extending the work of Hartmann, who investigated the case of walls that were electrically non-conducting, consider the steady flow of conducting fluids through ducts with electrically conducting walls under transverse magnetic field [4] [5].

An electrically conducting incompressible fluid flows in the x-direction through a duct (Fig. 4) with constant wall thicknesses, h_1, h_2, and electrical conductivities, σ_1, σ_2. The conductivity σ_3 is zero outside of the duct. The system is composed of the flowing fluid and the conducting walls. Equations (2-6) and (2-7) are valid in the fluid only. The other equations (2-1) to (2-5) are valid everywhere in the complete system. As in Hartmann flow the assumptions are: (1) the flow is fully developed, (2) the magnetic field in the y-direction is assumed to be constant, $H_y = H_o$, and (3) the magnetic field in the z-direction is zero, i.e., $H_z = 0$. Then equation (2-7) will be reduced to the following equation which corresponds to equation (AII-8):

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \nu \frac{\partial^2 u}{\partial y^2} + \frac{\mu_e}{\rho} H_o \frac{\partial H_x}{\partial y} \qquad (4\text{-}1)$$

Taking the curl of equation (2-5), one obtains

$$\nabla \times \underline{J} = \nabla \times \sigma(\underline{E} + \underline{V} \times \mu_e \underline{H}) \qquad (4\text{-}2)$$

$$\underline{V} \times \mu_e \underline{H} = \mu_e \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u & 0 & 0 \\ H_x & H_o & 0 \end{vmatrix} = \mu_e H_o u \underline{k}$$

$$\nabla \times \sigma(\underline{E} + \underline{V} \times \mu_e \underline{H}) = \sigma \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & (E_z + \mu_e H_o u) \end{vmatrix} \qquad (4\text{-}3)$$

From equation (2-1)

$$\underline{J} = \nabla \times \underline{H} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & 0 \end{vmatrix} = (0 + 0 - \frac{\partial H_x}{\partial y} \underline{k})$$

$$\nabla \times \underline{J} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & -\frac{\partial H_x}{\partial y} \end{vmatrix} \qquad (4\text{-}4)$$

Substituting equations (4-3) and (4-4) into equation (4-2) one obtains for the x-component,

$$-\frac{\partial^2 H_x}{\partial y^2} = \sigma\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \mu_e H_o \frac{\partial u}{\partial y}\right) \qquad (4\text{-}5)$$

From equation (2-3),

$$\text{curl } \underline{E} = -\mu_e \frac{\partial H}{\partial t} = 0 \text{ for steady motion}$$

$$\nabla \times \underline{E} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = 0$$

The x-component gives

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0$$

Therefore, equation (4-5) becomes

$$\frac{\partial^2 H_x}{\partial y^2} = \sigma \mu_e H_o \frac{\partial u}{\partial y} \tag{4-6}$$

Introducing the following dimensionless variables,

$$\bar{u} = \frac{u}{a}$$

$$\bar{H}_x = \frac{H_x}{H_y}$$

$$\bar{P} = -\left(\frac{1}{\rho \nu^2 L^{-3}}\right)\left(\frac{dp}{dx}\right)$$

$$Y = \frac{y}{L}$$

equations (4-1) and (4-6) become

$$\frac{d^2\bar{u}}{dY^2} + R_M\left(\frac{d\bar{H}_x}{dY}\right) = -\frac{\bar{P}}{R_M} \tag{4-7}$$

$$P_{re}\frac{d^2\bar{H}_x}{dy^2} + R_M\left(\frac{d\bar{u}}{dy}\right) = 0 \tag{4-8}$$

where

$$R_M = aL/\nu$$

$$P_{re} = \eta/\nu$$

$$a = (\mu_e H_y^2/\rho)^{1/2} \text{, Alfven velocity}$$

$$\eta = \frac{1}{\sigma \mu_e}$$

Within the wall (outside the fluid region), $\bar{u} = 0$ and $\frac{d\bar{u}}{dy} = 0$, and thus equation (4-8) becomes

$$\frac{d^2\bar{H}_x}{dY^2} = 0 \qquad (4\text{-}9)$$

At the boundary between the fluid and the wall,

$$\underline{E} = \underline{V} \times \underline{B} = \underline{0} \times \underline{B} = 0$$
$$\underline{E} = \frac{J}{\sigma} = 0 \qquad (4\text{-}10)$$

From equation (2-1)

$$\underline{J} = \nabla \times \underline{H} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & 0 \end{vmatrix} = -\frac{\partial H_x}{\partial y}\underline{k}$$

Substituting this into equation (4-10),

$$\frac{1}{\sigma}\frac{\partial H_x}{\partial y} = 0 \qquad (4\text{-}11)$$

Therefore, at the boundary between the fluid and upper wall, where $y = +L$, equation (4-11) becomes

$$\frac{1}{\sigma_f}\left(\frac{dH_x}{dy}\right)_f - \frac{1}{\sigma_2}\left(\frac{dH_x}{dy}\right)_2 = 0 \qquad (4\text{-}12)$$

or

$$\frac{1}{\sigma_f}\left(\frac{d\bar{H}_x}{dY}\right)_f - \frac{1}{\sigma_2}\left(\frac{d\bar{H}_x}{dY}\right)_2 = 0 \qquad (4\text{-}12a)$$

From equation (4-9)

$$\frac{d\bar{H}_x}{dY} = \text{constant}$$

Therefore, $\bar{H}_x$ varies linearly across the duct wall from $(\bar{H}_x)_{Y = (1+\frac{h2}{L})} = 0$ at the outer boundary of the wall to $(\bar{H}_x)_{Y=1} > 0$ at the inner wall boundary. Then

$$\left(\frac{d\bar{H}_x}{dy}\right)_2 = - \frac{\bar{H}_x}{h_2/L} = - \frac{L(\bar{H}_x)_{Y=1}}{h_2}$$

where h_2 is the thickness of the upper duct wall. Substituting this into equation (4-12a), one obtains

$$\frac{1}{\sigma_f}\left(\frac{d\bar{H}_x}{dY}\right)_f - \frac{1}{\sigma_2}\left(- \frac{L\bar{H}_x}{h_2}\right) = 0$$

$$\frac{d\bar{H}_x}{dY} + \frac{1}{\frac{\sigma_2 h_2}{\sigma_f L}} \bar{H}_x = 0$$

or

$$\frac{d\bar{H}_x}{dY} + \frac{1}{\varphi_2} \bar{H}_x = 0 \quad \text{at} \quad Y = +1 \qquad (4\text{-}13)$$

where

$$\varphi_2 = \frac{\sigma_2 h_2}{\sigma_f L}$$

Similarly, for the lower duct wall

$$\frac{d\bar{H}_x}{dY} - \frac{1}{\varphi_1} \bar{H}_x = 0 \quad \text{at} \quad Y = -1 \qquad (4\text{-}14)$$

where

$$\varphi_1 = \frac{\sigma_1 h_1}{\sigma_f L}$$

The solution of equations (4-7) and (4-8) for the boundary conditions represented by equations (4-13) and (4-14), and for which $u = 0$, at $Y = \pm 1$, can be written as [5] (c.f. Appendix III)

$$\left(\frac{u}{\nu L^{-1}}\right)\left(\frac{1}{\bar{P}}\right) = \left(\frac{\Phi_1}{M}\right)\left(\coth M - \frac{\cosh MY}{\sinh M}\right) \qquad (4\text{-}15)$$

$$\frac{H_x}{H_y}\frac{1}{\bar{P}} = \frac{1}{R_M^2}\left[\frac{\Phi_1 \sinh MY}{\sinh M} + \Phi_2\left(\frac{1}{M} - \coth M\right) - Y\right] \qquad (4\text{-}16)^*$$

$$J_z = \frac{J_z}{a\sigma\mu_e H_y} = \left(\frac{\bar{P}}{M^2 R_M}\right)\left(1 - \frac{\Phi_1 M \cosh MY}{\sinh M}\right) \qquad (4\text{-}17)$$

where $M = aL(\gamma\nu)^{-\frac{1}{2}} = \mu_e H_o L \sqrt{\frac{\sigma}{\rho\nu}}$, Hartmann number

$$\Phi_1 = \frac{\varphi_2 + \varphi_1 + 2}{(\varphi_2+\varphi_1)\, M \coth M + 2} \qquad (4\text{-}18)$$

$$\Phi_2 = \frac{(\varphi_2 - \varphi_1)\, M}{(\varphi_2+\varphi_1)\, M \coth M + 2} \qquad (4\text{-}19)$$

From the above solutions the following conclusion can be made:

1. From equations (4-15) and (4-17), the velocity profiles and the electrical current distribution are symmetrical with respect to the center line. They depend not on the individual values of φ_1 and φ_2, but only on the sum $\varphi_1 + \varphi_2$ and on the Hartmann number M.

2. An increase in the sum of φ_1 and φ_2, or an increase in the Hartmann number, flattens the velocity profile and increases the electrical current (See Figs. 5 and 6).

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*Equation (4-16) is given in References [5] and [8] as follows:

$$\frac{H_x}{H_y}\frac{1}{\bar{P}} = \frac{1}{R_M^2}\left[\frac{\Phi_1 \sinh MY}{\sinh M + \Phi_2}\left(\frac{1}{M} - \coth M) - Y\right)\right]$$

However, it obviously contains a typographical error (c.f. Appendix III).

3. When $\varphi_1 = \varphi_2 = 0$, equation (4-15) reduces to equation (3-3), which is the Hartmann velocity profile.

THE INFLUENCE OF WALL CONDUCTANCE ON MAGNETOHYDRODYNAMIC CHANNEL-FLOW HEAT TRANSFER

There have been increasing efforts, in recent years, to investigate the heat transfer in channel MHD flow. The literature is well reviewed by Romig [6]. The effects of the electrical conductance of walls on the heat transfer were investigated by Alpher [7], Yen [8], and Snyder [9]. Because of the mathematical complexity, the assumptions that the flow is thermally and hydrodynamically fully developed were made in their investigations.

Alpher [7] considered only the open circuit conditon. Since the effects of Ohmic heating of the walls and viscous dissipation of the flow on the temperature distribution were neglected, his analysis does not adequately account for the wall influence.

Yen [8] considered the same problem as Alpher but included viscous dissipation and unequal wall electrical conductivities. However, Yen's analysis is also based on the open-circuit case.

Snyder [9] attempted to generalize the analysis of Alpher and Yen. His analysis accounts for the effect of Joule heating whithin the walls on the temperature distribution within the fluid. It is also based on arbitrary external electrical loading and is valid for both the generator and accelerator modes of operation. Snyder's analysis is followed in this section. However, the author of this report has found some points in the physical interpretation of the phenomena that are in disagreement with Snyder's.

Assume the flow (Fig. 7) to be thermally and hydrodynamically fully developed, with constant heat flux boundary conditions applied at the outer surfaces of the walls and with the applied magnetic field perpendicular to the walls. Constant properties of the fluid and walls are assumed [9].

Referring to the section of the modified Hartmann flow, the velocity and electrical current are given as

$$\left(\frac{u}{V_L - 1}\right)\left(\frac{1}{\bar{P}}\right) = \left(\frac{\Phi_1}{M}\right)\left(\coth M - \frac{\cosh MY}{\sinh M}\right) \qquad (4\text{-}15)$$

$$\left(\frac{J_z}{a\sigma\mu_e H_y}\right) = \left(\frac{\bar{P}}{M^2 R_M}\right)\left(1 - \frac{\Phi_1 M \cosh MY}{\sinh M}\right) \qquad (4\text{-}17)$$

Define the dimensionless velocity profile U as follows:

$$U = \frac{u}{u_m}$$

$$\text{where } u_m = \frac{1}{2L}\int_{-L}^{L} u \, dy$$

Then, from equation (4-15),

$$U = \frac{\dfrac{(V_L - 1)\bar{P}\Phi_1}{M}\left(\coth M - \dfrac{\cosh MY}{\sinh M}\right)}{\dfrac{1}{2L}\displaystyle\int_{-L}^{L} \dfrac{(V_L - 1)\bar{P}\Phi_1}{M}\left(\coth M - \dfrac{\cosh MY}{\sinh M}\right) dy}$$

$$= \frac{2L\left(\coth M - \frac{\cosh MY}{\sinh M}\right)}{\left[y \coth M - \frac{L \sinh MY}{M \sinh M}\right]_{-L}^{L}}$$

$$= \frac{2L\,(\coth M)\left(1 - \frac{\cosh MY}{\sinh M \coth M}\right)}{L \coth M - \frac{L \sinh M}{M \sinh M} + L \coth M - \frac{L \sinh M}{M \sinh M}}$$

$$= \frac{2L(\coth M)\left(1 - \frac{\cosh MY}{\cosh M}\right)}{2L\left(\coth M - \frac{1}{M}\right)}$$

$$= \left(\frac{M}{M - \tanh M}\right)\left(1 - \frac{\cosh MY}{\cosh M}\right)$$

(5-1)

The energy equation is

$$\rho\, C_p u\, \frac{\partial T}{\partial x} = k\, \frac{\partial^2 T}{\partial y^2} + \frac{J_z^2}{\sigma} + \mu \left(\frac{du}{dy}\right)^2 \qquad (5\text{-}2)$$

where $J_z = \sigma(E_z + uB_y)$

The three terms on the right side of equation (5-2) are contributed by outside heat flux, Joule heating, and viscous dissipation.

Applying equation (5-2) into the lower and upper walls, where $u=0$, and $J_z = \sigma E_z$

$$k_1 \frac{\partial^2 T_1}{\partial y^2} + \sigma_1 E_z^2 = 0 \qquad (5\text{-}3)$$

$$k_2 \frac{\partial^2 T}{\partial y^2} + \sigma_2 E_z^2 = 0 \tag{5-4}$$

The boundary conditions are

$$\text{at } y = -L,\ u = 0,\ T = T_1\ k \frac{\partial T}{\partial y} = k_1 \frac{\partial T_1}{\partial y} \tag{5-5}$$

$$y = L,\ u = 0,\ T = T_2\ k \frac{\partial T}{\partial y} = k_2 \frac{\partial T_2}{\partial y} \tag{5-6}$$

$$y = -(L + h_1) \qquad k_1 \frac{\partial T_1}{\partial y} = q_1 \tag{5-7}$$

$$y = L + h_2 \qquad k_2 \frac{\partial T_2}{\partial y} = -q_2 \tag{5-8}$$

where q_1 and q_2 are positive if the walls are being cooled. Introducing the following dimensionless variables for convenience,

$$X = \frac{x}{L},\ Y = \frac{y}{L},\ M = B_y L \sqrt{\frac{\sigma}{\mu}}$$

$$\phi = \frac{E_z}{u_m B_y},\ \theta = \frac{T}{\frac{\mu u_m^2}{k}},\ Q_1 = \frac{q_1 L}{\mu u_m^2},\ Q_2 = \frac{q_2 L}{\mu u_m^2} \tag{5-9}$$

the total current per unit length in x-direction becomes

$$I = \sigma_1 \int_{-(L+h_1)}^{-L} E_z dy + \sigma_f \int_{-L}^{L} (E_z + uB_y) dy + \sigma_2 \int_{L}^{L+h_2} E_z dy \tag{5-10a}$$

$$= \sigma_1 E_z h_1 + \sigma_f (2LE_z + B_y 2Lu_m) + \sigma_2 E_z h_2$$

$$\frac{I}{\sigma_f B_y u_m L} = 2 + \frac{E_z}{\sigma_f B_y u_m L} (2\sigma_f L + \sigma_1 h_1 + \sigma_2 h_2)$$

$$\frac{I}{\sigma_f B_y u_m L} = 2 + \frac{E_z}{B_y u_m} \left(2 + \frac{\sigma_1 h_1}{\sigma_f L} + \frac{\sigma_2 h_2}{\sigma_f L}\right)$$

or

$$I^* = 2 + \Phi\,(2 + \varphi_1 + \varphi_2) \qquad (5\text{-}10b)$$

where $I^* = \dfrac{I}{\sigma_f u_m B_y L}$ $\varphi_1 = \dfrac{\sigma_1 h_1}{\sigma_f L}$, $\varphi_2 = \dfrac{\sigma_2 h_2}{\sigma f L}$

Under the open circuit condition, $I^* = 0$, and from equation (5-10b)

$$\Phi_{open} = - \frac{2}{2 + \varphi_1 + \varphi_2} \qquad (5\text{-}11)$$

Defining a voltage ration as,

$$r = \frac{\Phi}{\Phi_{open}} \qquad (5\text{-}12)$$

then

$$\Phi = - \frac{2\,r}{2 + \varphi_1 + \varphi_2} \qquad (5\text{-}13)$$

Integrating equations (5-3) and (5-4), one obtains

$$k_1 \frac{\partial T_1}{\partial y} + \sigma_1 E_z^2 y = C_1 \qquad (5\text{-}14a)$$

$$k_2 \frac{\partial T_2}{\partial y} + \sigma_2 E_z^2 y = C_2 \qquad (5\text{-}14b)$$

Substitution of equation (5-7) into equation (5-14a) gives

$$q_1 + \sigma_1 E_z^2\,(-L-h_1) = C_1$$

$$k_1 \frac{\partial T_1}{\partial y} + \sigma_1 E_z^2 y = q_1 - \sigma_1 E_z^2\,(L + h_1)$$

Substituting $y = -L$ into the above equation, one obtains

$$k_1 \frac{\partial T_1}{\partial y} = q_1 - \sigma_1 E_z^2 h_1$$

$$\frac{L}{\mu u_m^2} k_1 \frac{\partial T_1}{\partial y} = \frac{L}{\mu u_m^2} q_1 - \frac{L}{\mu u_m^2} \sigma_1 E_z^2 h_1$$

$$\frac{\partial \theta}{\partial Y})_{Y=-1} = Q_1 - \frac{E_z^2}{u_m^2 B_y^2} \frac{L^2 \sigma_f B_y^2}{\mu} \frac{\sigma_1 h_1}{\sigma_f L}$$

$$\frac{\partial \theta}{\partial Y})_{Y=-1} = (Q_1 - \phi^2 M^2 \omega_1) \tag{5-15}$$

Similarly

$$\frac{\partial \theta}{\partial Y})_{Y=1} = -(Q_2 - \phi^2 M^2 \omega_2) \tag{5-16}$$

Equations (5-15) and (5-16) are the boundary conditions for equation (5-2). The thermally fully developed flow implies that $\frac{\partial T}{\partial x}$ = constant, which is determined from the following over-all energy balance;

$$2L\rho C_p u_m \frac{\partial T}{\partial x} = \int_{-L}^{L} \left[\sigma(E_z + uB_y)^2 + \mu(\frac{du}{dy})^2\right] dy - (q_1 - \sigma_1 E_z^2 h_1) - (q_2 - \sigma_2 E_z^2 h_2) \tag{5-17}$$

The left side of equation (5-17) expresses the rate at which the fluid acquires energy in the x-direction. The first term of the right side of equation (5-17) is the total rate of thermal energy generation in the fluid which includes the Joule heating and viscous dissipation. The last two terms express the difference of the heat flux and the Joule heating in the walls. Dividing equation (5-17) by $2\rho C_p \mu u_m^3/k$, and

$$\frac{1}{\frac{\mu u_m^2}{kL}}\frac{\partial T}{\partial x} = \frac{1}{2\,\frac{\mu C_p}{k}\,\frac{\rho u_m L}{\mu}}\left[\left\{\int_{-L}^{L}\frac{L\sigma}{\mu}\left(\frac{E_z}{u_m}+\frac{uBy}{u_m}\right)^2 + \frac{L}{u_m^2}\left(\frac{du}{dy}\right)^2\right\}dy + \frac{L}{\mu u_m^2}\left(\sigma_1 E_z^2 h_1 + \sigma_2 E_z^2 h_2\right) - \frac{L}{\mu u_m^2}\left(q_1 + q_2\right)\right]$$

In dimensionless form, the above equation is

$$\frac{\partial\theta}{\partial X} = A = \text{constant}$$

$$= \frac{1}{2\,P_r R_e}\left[\int_{-1}^{1} B_y^2 L^2\,\frac{\sigma}{\mu}\left(\frac{E_z}{u_m B_y}+\frac{u}{u_m}\right)^2 + \frac{L^2}{u_m^2}\left(\frac{du}{dy}\right)^2\right\}dY + \frac{E_z^2}{u_m^2 B_y^2}\,\frac{B_y^2 L^2\sigma}{\mu}\left(\frac{\sigma_1 h_1}{\sigma_f L}+\frac{\sigma_2 h_2}{\sigma_f L}\right) - (Q_1 + Q_2)\right]$$

Introducing

$$D = M^2(\Phi + U)^2 + \left(\frac{dU}{dY}\right)^2 \tag{5-18a}$$

the temperature gradient equation is

$$\frac{\partial\theta}{\partial X} = A = \frac{\int_{-1}^{1} D dy + \Phi^2 M^2 (\varphi_1 + \varphi_2) - (Q_1 + Q_2)}{2\ Pr\ Re} \tag{5-18b}$$

where

$$Re = \frac{\rho u_m L}{\mu}, \qquad Pr = \frac{\mu C_p}{k} \tag{5-18c}$$

Equation (5-2) can be written as

$$Pr\ Re\ U\,\frac{\partial\theta}{\partial X} = \frac{\partial^2\theta}{\partial Y^2} + M^2(\Phi + U)^2 + \left(\frac{dU}{dY}\right)^2 \tag{5-19}$$

From equation (5-18b)

$$\frac{\partial\theta}{\partial X} = A = \text{constant} \tag{5-20}$$

and thus

$$\theta - \theta_0 = AX + f(Y) \tag{5-21}$$

Substitution of equation (5-21) into equation (5-19) gives

$$\frac{d^2f}{dY^2} = A\ Re\ Pr\ U - D \tag{5-22}$$

Integrating equation (5-22), one obtains

$$\frac{df}{dY} = \int_0^Y (A\ Re\ Pr\ U - D)\,dY + C_1 \tag{5-23}$$

Using equations (5-15) and (5-16), and recalling that

$\frac{df}{dY} = \frac{\partial\theta}{\partial Y}$, the following is obtained

$$-(Q_2 - \Phi^2M^2\varphi_2) = \int_0^1 (A\ Re\ Pr\ U - D)dY + C_1 \tag{5-24a}$$

$$(Q_1 - \Phi^2M^2\varphi_1) = \int_0^{-1} (A\ Re\ Pr\ U - D)dY + C_1 \tag{5-24b}$$

Adding equation (5-24a) and 5-24b), and noting that (A Re Pr U - D) is an even function of Y, one obtains

$$C_1 = \frac{1}{2}\left[(Q_1 - Q_2) + \Phi^2M^2(\varphi_2 - \varphi_1)\right] \tag{5-25}$$

The integration of equation (5-23), by parts gives

$$d\,f(Y) = \int_0^Y (A\ Re\ Pr\ U - D)dY\ dY + C_1 dY$$

$$f(Y) = Y\int_0^Y (A\ Re\ Pr\ U - D)dY - \int_0^Y Y d\left[\int_0^Y (A\ Re\ Pr\ U - D)\,dY\right]$$

$$= C_1\ dY$$

$$f(Y) = Y\int_0^Y (A\ Re\ Pr\ U - D)dY - \int_0^Y Y\,(A\ Re\ Pr\ U - D)dY + C_1Y + C_2$$

Changing the variable Y to S results in

$$f(Y) = Y\int_0^Y (ARePrU - D)dS - \int_0^Y S\,(ARePrU - D)dS + C_1Y + C_2$$

$$= \int_0^Y (Y - S)\ (A\ Re\ Pr\ U - D)dS + C_1Y + C_2 \tag{5-26}$$

$$f(Y) = F(Y) + C_2$$

where

$$F(Y) = \int_0^Y (Y - S)(A \text{ Re Pr } U - D)dS + C_1 Y \qquad (5\text{-}26a)$$

The substitution of equation (5-26) into equation (5-21) gives

$$\theta - \theta_0 = A\,X + F(Y) + C_2 \qquad (5\text{-}27)$$

C_2 is evaluated from an overall energy balance between the points x = 0 and x = x, the equation for which is

$$\rho C_p \int_{-L}^{L} u(T-T_o)dy = x \int_{-L}^{L} \left[\sigma(E_z + uB_y)^2 + \mu\left(\frac{du}{dy}\right)^2\right]dy + x\left[- k \frac{\partial T}{\partial y})_{-L} + k \frac{\partial T}{\partial y})_L\right] \qquad (5\text{-}28)$$

or, dimensionless form,

$$R_e P_r \int_{-1}^{1} U(\theta - \theta_0)dY = X\left[\int_{-1}^{1} DdY + \Phi^2 M^2(\varphi_1 + \varphi_2) - (Q_1 + Q_2)\right] \qquad (5\text{-}29)$$

Substituting equation (5-18b) into (5-29), one obtains

$$\text{RePr} \int_{-1}^{1} U(\theta - \theta_0)dY = X(2 \text{ Pr Re } A)$$

$$\int_{-1}^{1} U(\theta - \theta_0)dY = 2\,A\,X \qquad (5\text{-}29a)$$

Substituting equation (5-27) into (5-29a) and

$$\int_{-1}^{1} U(A\,X + F(Y) + C_2)dY = 2AX$$

$$A\,X \int_{-1}^{1} UdY + \int_{-1}^{1} UFdY + C_2 \int_{-1}^{1} UdY = 2AX$$

where

$$\int_{-1}^{1} UdY = \frac{M}{M - \tanh M} \int_{-1}^{1} \left(1 - \frac{\cosh MY}{\cosh M}\right)dY$$

$$= 2$$

Hence

$$C_2 = -\frac{1}{2}\int_{-1}^{1} UFdY \tag{5-30}$$

Finally, the temperature distribution is

$$\theta - \theta_0 = AX + F(Y) - \frac{1}{2}\int_{-1}^{1} UFdY \tag{5-31}$$

Because of the algebraic complexity involved in this equation, it is left in implicit form.

Define the fluid bulk temperature as

$$T_b - T_0 = \frac{\int_{-L}^{L} u(T-T_0)dy}{\int_{-L}^{L} udy} \tag{5-32a}$$

or

$$\theta_b - \theta_0 = \frac{1}{2}\int_{-1}^{1} U(\theta - \theta_0)dY \tag{5-32b}$$

Recalling equation (5-29a) one obtains

$$\theta_b - \theta_0 = A\,X \tag{5-32c}$$

From equations (5-31) and (5-32c), one obtains the difference between the fluid bulk temperature and the wall temperature at the interfaces as

$$\theta_b - \theta(1) = \frac{1}{2}\int_{-1}^{1} UFdY - F(1) \tag{5-33a}$$

$$\theta_b - \theta(-1) = \frac{1}{2}\int_{-1}^{1} UFdY - F(-1) \tag{5-33b}$$

Numberical presentation:

One of the most important heat-transfer quantities is the difference between the fluid bulk temperature and the wall temperature. Considering the case with equal and constant heat fluxes at the outer surfaces of the two walls and equal wall conductances $\varphi_1 = \varphi_2$, one has,

$C_1 = 0$ (from equation 5-25)

$F(1) = F(-1)$

$$\theta_b - \theta_w = \frac{1}{2}\int_{-1}^{1} UFdY - F(1) \tag{5-33}$$

From this it is noted that there are four independent external parameters to be specified; (1) the Hartmann number M, (2) the voltage ratio r, (3) the wall heat fluxes Q_1 and Q_2, and (4) the wall conductances φ_1 and φ_2. It is interesting that for the case of equal and constant heat fluxes, and equal wall conductances, the wall heat fluxes appear as the sum $Q_1 + Q_2$, and likewise the conductances appear as the sum $\varphi_1 + \varphi_2$.

From equation (5-18b),

$$A \text{ Re Pr} = \frac{1}{2}\left[\int_{-1}^{1} DdY + \Phi^2 M^2 (\varphi_1 + \varphi_2) - (Q_1 + Q_2)\right] \tag{5-34}$$

The first two terms on the right side of equation (5-34) represent the total energy dissipation, consisting of viscous dissipation and Joule heating in the fluid and Joule heating in the walls. The last term represents the heat transfer at the outer surfaces of the two walls. It is convenient to define the ratio R as

$$R = \frac{Q_1 + Q_2}{\int_{-1}^{1} DdY + \Phi^2 M^2 (\varphi_1 + \varphi_2)} \tag{5-35}$$

Equation (5-34) becomes

$$A \text{ Re Pr} = \frac{1}{2}(1-R)\left[\int_{-1}^{1} DdY + \Phi^2 M^2 (\varphi_1 + \varphi_2)\right] \tag{5-36}$$

If the total heat transfer equals the total dissipation (R=1) then A = 0 and the temperature is uniform in the x-direction.

Using equations (5-1), (5-13), (5-18a), (5-26a), (5-33), and (5-36), one obtains the evaluation of $(\theta_b - \theta_w)$. $(\theta_b - \theta_w)$ is plotted as the ordinate of Fig. 8 for the following numerical data:

$M = 10$, $R = 1$, $R = 10$, $u_m = 100 \text{ft/sec}$.

$\mu = 2.6 \times 10^{-5}\ lb_f\ sec/ft^2$ $k = 6$ Btu/hr ft F.

The value of μ and k chosen correspond to mercury at 200 deg F. then

$\varphi_1 + \varphi_2$	$T_b - T_w$ deg F
0	8.2
1	32.8
10	72.6

In Fig. 8, the range $r > 1$ corresponds to the accelerator mode. The range $r < 1$ corresponds to the generator mode. The influence of the wall conductivity on the heat transfer is opposite for the generator and accelerator modes of operation.

From equation (5-10a) it can be seen that when the net current is equal to zero, i.e., when I = 0, $E_{open} \equiv (E_z)_{I=0}$ and $|E_{open}| \leqslant u_m B_y$, where E_{open} is in the negative z-direction (Fig. 9). Furthermore, the equality relation in $|E_{open}| \leqslant u_m B_y$ holds when $\varphi_1 + \varphi_2 = 0$, and the inequality relation holds when $(\varphi_1 + \varphi_2) > 0$, noting that $\varphi_1 \geqslant 0$ and $\varphi_2 \geqslant 0$.

With the above relationship the following correspondences between the range of voltage ratio and the operating mode exist. (I). For the generator mode of operation for which the conditions that E_z is in negative direction and that $|E_z| < |E_{open}|$, the following relation is obtained with the help of equation (5-9), (See Fig. 10).

$$\frac{E_z}{E_{open}} = \frac{\Phi}{\Phi_{open}} < 1 \quad \text{but} > 0$$

$$\therefore \; 0 < r < 1$$

which agrees with that of Snyder [9].

(II). For the generator mode of operation under the condition that E_z is in positive direction (Fig. 11)

$$\frac{E_z}{E_{open}} = \frac{\Phi}{\Phi_{open}} < 0$$

$$\therefore \; r < 0$$

which Snyder [9] states is for the accelerator mode.

(III). For the accelerator mode of operation under the conditions that E_z is in negative direction and $|E_z| > |E_{open}|$ (Fig. 12)

$$\frac{E_z}{E_{open}} = \frac{\Phi}{\Phi_{open}} > 1$$

$$\therefore \; r > 1$$

which Snyder [9] states as corresponding "to the application of an external voltage larger than the induced open circuit voltage and in the same direction as the induced voltage". Thus the author of this report is not in entire agreement with Snyder's [9] conclusions, and has co-authored a discussion [10] on this disagreement.

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APPENDIX

I. Derivation of equations (2-8) and (2-9)

From equation (2-5), one obtains

$$\underline{E} = \frac{1}{\sigma} (\underline{J} - \sigma \underline{V} \times \mu_e \underline{H}) \qquad \text{(AI-1)}$$

Substituting (AI-1) into the left side of equation (2-3), one obtains:

$$\text{curl} \left[\frac{1}{\sigma} (\underline{J} - \sigma V \times \mu_e \underline{H}) \right] = - \mu_e \frac{\partial \underline{H}}{\partial t} \qquad \text{(AI-2)}$$

From equation (2-1), one obtains

$$\underline{J} = \text{curl } \underline{H} \qquad \text{(AI-3)}$$

Then substituting this into equation (AI-2)

$$\text{curl} \left[\frac{1}{\sigma} (\text{curl } \underline{H} - \sigma \underline{V} \times \mu_e \underline{H}) \right] = - \mu_e \frac{\partial \underline{H}}{\partial t}$$

$$\frac{\partial \underline{H}}{\partial t} - \text{curl } (\underline{V} \times \underline{H}) = - \frac{1}{\sigma \mu_e} \text{curl curl } \underline{H} \qquad \text{(AI-4)}$$

Since

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{H}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{H}) - \underline{\nabla} \cdot \underline{\nabla} \underline{H}$$

and using equation (2-4), one obtains equation (2-8):

$$\frac{\partial \underline{H}}{\partial t} - \text{curl } (\underline{V} \times \underline{H}) = \frac{1}{\sigma \mu_e} \underline{\nabla}^2 \underline{H} \qquad \text{(2-8)}$$

where

$$\frac{1}{\sigma \mu_e} = \eta$$

Substituting equation (AI-3) into the last term of equation (2-7), one obtains equation (2-9):

$$\frac{1}{\rho} (\underline{J} \times \mu_e \underline{H}) = \frac{1}{\rho} (\text{curl } \underline{H}) \times \mu_e \underline{H}$$

$$= \frac{\mu_e}{\rho} (\text{curl } \underline{H}) \times \underline{H}$$

$$= \frac{\mu_e}{\rho} \left\lfloor (\underline{H} \cdot \text{grad})\underline{H} - \frac{1}{2} \underline{\nabla}\, \underline{H}^2 \right\rfloor$$

in which the following vector indentifications are employed:

$$\underline{\nabla}(\underline{U} \cdot \underline{V}) = \underline{U} \cdot \underline{\nabla}\, \underline{V} + \underline{V} \cdot \underline{\nabla}\, \underline{U} + \underline{U} \times (\underline{\nabla} \times \underline{V}) + \underline{V} \times (\underline{\nabla} \times \underline{U})$$

Let $\underline{U} = \underline{V} = \underline{H}$ then

$$\underline{\nabla}\, (\underline{H}^2) = 2(\underline{H} \cdot \underline{\nabla})\underline{H} + 2\underline{H} \times (\underline{\nabla} \times \underline{H})$$

$$2(\underline{\nabla} \times \underline{H}) \times \underline{H} = 2(\underline{H} \cdot \underline{\nabla})\underline{H} - \underline{\nabla}(\underline{H}^2)$$

II. Derviation of equations (3-3), (3-4), (3-5) and (3-6).

From equation (2-4)

$$\text{div } \underline{H} = 0$$

$$(\underline{i} \frac{\partial}{\partial x} + \underline{j} \frac{\partial}{\partial y} + \underline{k} \frac{\partial}{\partial z}) \cdot (\underline{i}\, H_x + \underline{j}\, H_y + 0) = \frac{\partial H_y}{\partial y} = 0$$

Therefore

$$H_y = \text{constant} = H_o \qquad \text{(AII-1)}$$

It is assumed that the induced magnetic field, H_x, is independent of the x-direction, i.e., = 0. From equation (2-1)

$$\underline{J} = \text{curl } \underline{H} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_o & 0 \end{vmatrix} = - \frac{\partial H_x}{\partial y} \underline{k}$$

According to Ohm's law, equation (2-5)..

$$\underline{J} = \sigma(\underline{E} + \mu_e \underline{V} \times \underline{H})$$

$$\underline{E} = - E_o \underline{k}$$

$$\underline{V} \times \underline{H} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u & 0 & 0 \\ H_x & H_0 & 0 \end{vmatrix} = u\, H_0\, \underline{k}$$

Therefore

$$\underline{J} = -\frac{\partial H_x}{\partial y}\, \underline{k} = -\sigma(E_0 - \mu_e\, u\, H_0)\underline{k} \qquad \text{(AII-2)}$$

From equation (2-7)

$$\frac{\partial \underline{V}}{\partial t} + (\underline{V} \cdot \text{grad})\underline{V} = -\frac{1}{\rho}\, \text{grad } p + \nu \nabla^2 \underline{V} + \frac{1}{\rho}\, (\underline{J} \times \mu_e \underline{H})$$

$$\frac{\partial \underline{V}}{\partial t} = 0 \qquad \text{for steady motion} \qquad \text{(AII-3)}$$

$$(\underline{V} \cdot \text{grad})\underline{V} = \text{grad } \frac{1}{2}\, \underline{V}^2 - \underline{V} \times \text{curl } \underline{V} = u\, \frac{\partial u}{\partial y}\, \underline{j} - u\, \frac{\partial u}{\partial y} \qquad \text{(AII-4)}$$

$$\underline{j} = 0$$

$$\text{grad } p = (\underline{i}\, \frac{\partial}{\partial x} + \underline{j}\, \frac{\partial}{\partial y} + \underline{k}\, \frac{\partial}{\partial z})p = \underline{i}\, \frac{\partial p}{\partial x} + \underline{j}\, \frac{\partial p}{\partial y} \qquad \text{(AII-5)}$$

$$\nabla^2\, \underline{V} = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})(\underline{i}u) = \underline{i}\, \frac{\partial^2 u}{\partial y^2} \qquad \text{(AII-6)}$$

$$\underline{J} \times \mu_e\, \underline{H} = \mu_e \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & -\frac{\partial H_x}{\partial y} \\ H_x & H_0 & 0 \end{vmatrix}$$

$$= \mu_e\, (\underline{i}\, H_0\, \frac{\partial H_x}{\partial y} - \underline{j}\, H_x\, \frac{\partial H_x}{\partial y}) \qquad \text{(AII-7)}$$

Substituting equations (AII-3) through (AII-7) into equation (2-7) and equalizing the components, one obtains

$$0 = -\frac{1}{\rho}\, \frac{\partial p}{\partial x} + \nu\, \frac{\partial^2 u}{\partial y^2} + \frac{\mu_e}{\rho}\, H_0\, \frac{\partial H_x}{\partial y} \qquad \text{(AII-8)}$$

$$0 = -\frac{1}{\rho}\, \frac{\partial p}{\partial y} - \frac{\mu_e}{\rho}\, H_x\, \frac{\partial H_x}{\partial y} \qquad \text{(AII-9)}$$

Taking the first derivative of equation (AII-8) with respect to x, one obtains

$$\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right) = 0$$

$$\frac{\partial p}{\partial x} = \text{constant} = -P \qquad \text{(AII-10)}$$

$$p = -P\,x + \text{constant} \qquad \text{(AII-10a)}$$

Integrating equation (AII-9) with respect to y, one obtains

$$p = -\frac{\mu_e H_x^2}{2} + f(x)$$

A comparison of equation (AII-10a) with the above equation gives

$$p = -P\,x - \frac{\mu_e H_x^2}{2}$$

The pressure gradient in the x-direction is a constant -P, but the pressure, which is different from the case of ordinary viscous flow, has its change in the y-direction. (The pressure gradient in the y-direction is balanced by the value of the y-component in the Lorentz force.)

Substituting equations (AII-10), (AII-2) into (AII-8), one obtains

$$-\frac{P}{\rho} = \nu\frac{d^2u}{dy^2} + \frac{\sigma\mu_e}{\rho} H_0(E_0 - \mu_e u H_0)$$

or

$$\frac{d^2u}{dy^2} - \frac{\sigma\mu_e^2 H_0^2}{\nu\rho} u = -\frac{P + \sigma\mu_e H_0 E_0}{\rho\nu} \qquad \text{(AII-11)}$$

with boundary conditions

B.C. (1) $u = 0$ at $y = \pm L$,

B.C. (2) $\frac{du}{dy} = 0$ at $y = 0$

The complementary function, u_c, and a particular solution, u_p, of equation (AII-11) are

$$u_c = A_1 \cosh \frac{\sigma \mu_e^2 H_0^2}{\rho} y + A_2 \sinh \frac{\sigma \mu_e^2 H_0^2}{\rho} y$$

$$u_p = \frac{P + \sigma \mu_e H_0 E_0}{\sigma \mu_e^2 H_0^2}$$

The general solution is

$$u = A_1 \cosh \frac{\sigma \mu_e^2 H_0^2}{\rho} y + A_2 \sinh \frac{\sigma \mu_e^2 H_0^2}{\rho} y$$

$$+ \frac{P + \sigma \mu_e H_0 E_0}{\sigma \mu_e^2 H_0^2} \qquad \text{(AII-12)}$$

Applying boundary condition (2), one obtains: $A_2 = 0$. Let $M = \mu_e H_0 L \sqrt{\frac{\sigma}{\sigma \nu}}$ = Hartmann number, which is a dimensionless number. Applying boundary condition (1), one obtains

$$0 = A_1 \cosh M + \left(\frac{P}{\sigma \mu_e^2 H_0^2} + \frac{E_0}{\mu_e H_0}\right)$$

$$A_1 = - \frac{1}{\cosh M} \left(\frac{P}{\sigma \mu_e^2 H_0^2} + \frac{E_0}{\mu_e H_0}\right)$$

Therefore, the solution of equation (AII-11) with the given boundary conditions is

$$u = \left(\frac{P}{\sigma \mu_e^2 H_0^2} + \frac{E_0}{\mu_e H_0}\right) \left(1 - \frac{\cosh\left(\frac{My}{L}\right)}{\cosh M}\right) \qquad \text{(AII-13)}$$

For open circuit condition, it is assumed that the uniform field E_0 in equation (AII-13) is adjusted so that the total electric current $\int J_z dy$ flowing between y = -L and y = +L will vanish. Then

$$0 = \int_{-L}^{L} J_z dy = - \int_{-L}^{L} \sigma \, (E_0 - u_e u H_0) dy$$

$$0 = \int_{-L}^{L} \sigma E_0 dy - \sigma \mu_e H_0 \int_{-L}^{L} \left(\frac{E_0}{\mu_e H_0} + \frac{P}{\sigma \mu_e^2 H_0^2}\right)\left(1 - \frac{\cosh \frac{My}{L}}{\cosh M}\right) dy$$

$$= \sigma E_0 \, 2L - \sigma \mu_e H_0 \left(\frac{E_0}{\mu_e H_0} + \frac{P}{\sigma \mu_e^2 H_0^2}\right) \left[y - \frac{L}{M \cosh M} \sinh \left(\frac{My}{L}\right)\right]_{-L}^{L}$$

$$= \sigma E_0 \, 2L - \sigma \mu_e H_0 \left(\frac{E_0}{\mu_e H_0} + \frac{P}{\sigma u_e^2 H_0^2}\right) \left(2L - \frac{2L}{M \cosh M} \sinh M\right)$$

Multiplying both sides by $\frac{M \mu_e H_0}{2L}$, one obtains

$$\sigma \, \mu_e H_0 E_0 M - (\sigma \, \mu_e H_0 E_0 + P) \, (M - \tanh M) = 0$$

$$(\sigma \, \mu_e H_0 E_0 + P) \tanh M = PM$$

$$E_0 = \frac{P(M - \tanh M)}{\sigma \, \mu_e H_0 \tanh M} \qquad \text{(AII-14)}$$

Substituting equation (AII¨14) into equation (AII-13), one obtains

$$u = \left(\frac{P}{\sigma u_e^2 H_0^2} + \frac{P(M - \tanh M)}{\sigma \mu_e^2 H_0^2 \tanh M}\right) \left(1 - \frac{\cosh \left(\frac{My}{L}\right)}{\cosh M}\right)$$

or

$$u = \frac{PM}{\sigma\mu_e^2 H_0^2} \left(\frac{\cosh M - \cosh \frac{My}{L}}{\sinh M} \right) \tag{3-3}$$

The average value of u between $y = \pm L$, i.e., u_m, is given as:

$$u_m = \frac{\int_{-L}^{L} u dy}{\int_{-L}^{L} dy} = \frac{1}{2L} \frac{PM}{\sigma \mu_e^2 H_0^2} \left(\frac{\cosh M}{\sinh M} 2L - \frac{2L}{M \sinh M} \sinh M \right)$$

$$= \frac{P}{\sigma \mu_e^2 H_0^2} (M \coth M - 1) \tag{3-4}$$

$$U = \frac{u}{u_m} = M \left[\frac{\cosh M - \cosh (\frac{My}{L})}{M \cosh M - \sinh M} \right] \tag{3-5}$$

Substituting equation (AII-14) and (3-3) into equation (AII-2), and integrating between the interval of $\pm$ L, results in

$$- \int \frac{dHx}{dy} dy = - \int \sigma(E_0 - \mu_e u H_0) dy$$

$$H_x = \int \sigma \left[\frac{P(M - \tanh M)}{\sigma \mu_e H_0 \tanh M} - \frac{PM}{\sigma u_e H_0} \left(\frac{\cosh M - \cos \frac{My}{L}}{\sinh M} \right) \right] dy$$

$$= \frac{P(M - \tanh M)}{\mu_e H_0 \tanh M} y - \frac{PM \cosh M}{u_e H_0 \sinh M} y + \frac{LPM \sinh \frac{My}{L}}{M \mu_e H_0 \operatorname{xinh} M} + C$$

Since $H_x = 0$ at $y = L$, $C = 0$

and

$$H_x = \frac{PL}{\mu_e H_0} \left(\frac{\sinh (\frac{My}{L})}{\sinh M} - \frac{y}{L} \right) \tag{3-6}$$

III Derivation of Equations (4-15), (4-16), and (4-17)

Integrating equations (4-7) and (4-8), one obtains

$$\frac{d\bar{u}}{dy} + R_m \bar{H}_x = - \frac{\bar{P}}{R_m} Y + C_1 \tag{AIII-1}$$

$$P_{re} \frac{d\bar{H}_x}{dy} + R_m \bar{u} = C_2 \tag{AIII-2}$$

or

$$\bar{H}_x = - \frac{1}{R_m} \frac{d\bar{u}}{dy} - \frac{\bar{P}}{R_m^2} Y + \frac{C_1}{R_m} \tag{AIII-1a}$$

$$\frac{d\bar{H}_x}{dY} = - \frac{R_m}{P_{re}} \bar{u} + \frac{C_2}{P_{re}} \tag{AIII-2a}$$

Substituting equations (AIII-1a) and (AIII-2a) into equations (4-13) and (4-14), and recalling that $\bar{u}=0$ at $Y=\pm 1$, one obtains

$$\frac{C_2}{P_{re}} + \frac{1}{\varphi_2} \left[- \frac{I}{R_m} \left(\frac{d\bar{u}}{dy}\right)_{Y=1} - \frac{\bar{P}}{R_m^2} + \frac{C_1}{R_m} \right] = 0 \tag{AIII-3}$$

$$\frac{C_2}{P_{re}} - \frac{1}{\varphi_1} \left[- \frac{1}{R_m} \left(\frac{d\bar{u}}{dY}\right)_{Y=-1} + \frac{\bar{P}}{R_m^2} + \frac{C_1}{R_m} \right] = 0 \tag{AIII-4}$$

From equations (AIII-3) and (AIII-4) one obtains

$$\left(\frac{1}{\varphi_2} + \frac{1}{\varphi_1}\right) \frac{C_1}{R_m} = \frac{1}{R_m} \left(\frac{1}{\varphi_2} \frac{d\bar{u}}{dY}\right)_{Y=1} + \frac{1}{R_m} \left(\frac{1}{\varphi_1} \frac{d\bar{u}}{dY}\right)_{Y=-1}$$

$$+ \frac{\bar{P}}{R_m^2} \left(\frac{1}{\varphi_2} - \frac{1}{\varphi_1}\right) \tag{AIII-5}$$

Substituting C_1, given by equation (AIII-5), into equation (AIII-3), one obtains

$$\frac{C_2}{P_{re}} = \frac{1}{R_M\varphi_2}\frac{d\bar{u}}{dY})_{Y=1} + \frac{\bar{P}}{R_M^2\varphi_2} - \frac{1}{2(\frac{1}{\varphi_2}+\frac{1}{\varphi_1})}\left[\frac{1}{R_M}(\frac{1}{\varphi_2}\frac{d\bar{u}}{dY})_{Y=1}\right.$$

$$\left. + \frac{1}{R_M}(\frac{1}{\varphi_1}\frac{d\bar{u}}{dY})_{Y=-1} + \frac{\bar{P}}{R_M^2}(\frac{1}{\varphi_2}-\frac{1}{\varphi_1})\right] \qquad \text{(AIII-6)}$$

Substituting equation (AIII-2a) into equation (4-7), one obtains

$$\frac{d^2\bar{u}}{dY^2} + R_M\left(-\frac{R_M}{P_{re}}\bar{u} + \frac{C_2}{P_{re}}\right) = -\frac{\bar{P}}{R_M}$$

$$\frac{d^2\bar{u}}{dY^2} - \frac{R_M^2}{P_{re}}\bar{u} = -\frac{R_M}{P_{re}}C_2 - \frac{\bar{P}}{R_M}$$

where

$$\frac{R_M^2}{P_{re}} = \frac{a^2L^2\nu}{\nu^2\eta} = a^2L^2/\eta\nu = M^2$$

$$\therefore \frac{d^2\bar{u}}{dY^2} - M^2\bar{u} = -\frac{R_M}{P_{re}}C_2 - \frac{\bar{P}}{R_M} \qquad \text{(AIII-7)}$$

The complementary function, $\bar{u}_c$, and a particular solution, $\bar{u}_p$, of equation (AIII-7) are

$$\bar{u}_c = A_1\cosh MY + A_2\sinh MY$$

$$\bar{u}_p = \frac{R_M C_2}{M^2 P_{re}} + \frac{\bar{P}}{M^2 R_M}$$

The general solution is

$$\bar{u} = A_1 \cosh MY + A_2 \sinh MY + \frac{R_M C_2}{M^2 P_{re}} + \frac{\bar{P}}{M^2 R_M}$$

with boundary conditions $\bar{u} = 0$ at $Y = \pm 1$, one obtains

$$0 = A_1 \cosh M + A_2 \sinh M + \frac{R_M C_2}{M^2 P_{re}} + \frac{\bar{P}}{M^2 R_M}$$

$$0 = A_1 \cosh M - A_2 \sinh M + \frac{R_M C_2}{M^2 P_{re}} + \frac{\bar{P}}{M^2 R_M}$$

Hence,

$$A_2 = 0$$

$$A_1 = - \frac{1}{\cosh M} \left(\frac{R_M C_2}{M^2 P_{re}} + \frac{\bar{P}}{M^2 R_M} \right)$$

Therefore, the solution of equation (AIII-7) is:

$$\bar{u} = - \frac{\cosh MY}{\cosh M} \left(\frac{R_M C_2}{M^2 P_{re}} + \frac{\bar{P}}{M^2 R_M} \right) + \frac{R_M C_2}{M^2 P_{re}} + \frac{\bar{P}}{M^2 R_M}$$

$$= \left(\frac{R_M C_2}{M^2 P_{re}} + \frac{\bar{P}}{M^2 R_M} \right) \left(1 - \frac{\cosh MY}{\cosh M} \right) \qquad \text{(AIII-8)}$$

$$\frac{d\bar{u}}{dY} = - \frac{M \sinh MY}{\cosh M} \left(\frac{R_M C_2}{M^2 P_{re}} + \frac{\bar{P}}{M^2 R_M} \right) \qquad \text{(AIII-9)}$$

Substituting $Y = \pm 1$ into equation (AIII-9), one obtains

$$\left.\frac{d\bar{u}}{dY}\right)_{Y=1} = - \frac{M \sinh M}{\cosh M} \left(\frac{R_M C_2}{M^2 Pr_e} + \frac{\bar{P}}{M^2 R_M} \right) \qquad \text{(AIII-10)}$$

$$\left.\frac{d\bar{u}}{dY}\right)_{Y=-1} = \frac{M \sinh M}{\cosh M} \left(\frac{R_M C_2}{M^2 Pr_e} + \frac{\bar{P}}{M^2 R_M} \right) \qquad \text{(AIII-11)}$$

Hence,

$$\left.\frac{d\bar{u}}{dY}\right)_{Y=1} = - \left.\frac{d\bar{u}}{dY}\right)_{Y=-1} \qquad \text{(AIII-12)}$$

Substituting equation (AIII-12) into equation (AIII-6) to evaluate C_2, one obtains

$$\frac{C_2}{Pr_e} = \frac{1}{R_M\varphi_2}\left.\frac{d\bar{u}}{dY}\right)_{Y=1} + \frac{\bar{P}}{R_M^2\varphi_2} - \frac{1}{(\varphi_1 + \varphi_2)}\left[\frac{1}{R_M}\left(\frac{1}{\varphi_2} - \frac{1}{\varphi_1}\right)\left.\frac{d\bar{u}}{dY}\right)_{Y=1} + \frac{\bar{P}}{R_M^2}\left(\frac{1}{\varphi_2} - \frac{1}{\varphi_1}\right)\right]$$

$$= \frac{1}{\varphi_2}\left[\frac{1}{R_M}\left.\frac{d\bar{u}}{dY}\right)_{Y=1} + \frac{\bar{P}}{R_M^2}\right] - \frac{\varphi_1}{(\varphi_1 + \varphi_2)}\left(\frac{1}{\varphi_2} - \frac{1}{\varphi_1}\right)\left[\frac{1}{R_M}\left.\frac{d\bar{u}}{dY}\right)_{Y=1} + \frac{\bar{P}}{R_M^2}\right]$$

$$= \left[\frac{1}{R_M}\left.\frac{d\bar{u}}{dY}\right)_{Y=1} + \frac{\bar{P}}{R_M^2}\right]\left[\frac{1}{\varphi_2} - \frac{\varphi_1}{\varphi_1 + \varphi_2}\left(\frac{1}{\varphi_2} - \frac{1}{\varphi_1}\right)\right]$$

$$= \frac{1}{\varphi_2}\left[\frac{1}{R_M}\left.\frac{d\bar{u}}{dY}\right)_{Y=1} + \frac{\bar{P}}{R_M^2}\right]\left[1 - \frac{\varphi_1 - \varphi_2}{\varphi_1 + \varphi_2}\right]$$

$$= \frac{2}{\varphi_1 + \varphi_2}\left[\frac{1}{R_M}\left.\frac{d\bar{u}}{dY}\right)_{Y=1} + \frac{\bar{P}}{R_M^2}\right]$$

Substitution of equation (AIII-10) into this equation yields

$$\frac{C_2}{P_{re}} = \frac{2}{\varphi_1 + \varphi_2}\left[-\frac{M \sinh M}{R_M \cosh M}\left(\frac{R_M C_2}{M^2 P_{re}} + \frac{\bar{P}}{M^2 R_M}\right) + \frac{\bar{P}}{R_M^2}\right]$$

$$= -\frac{2M \sinh M}{(\varphi_1 + \varphi_2)R_M \cosh M}\frac{R_M C_2}{M^2 P_{re}} + \left(1 - \frac{M \sinh M}{M^2 \cosh M}\right)\frac{2}{(\varphi_1 + \varphi_2)}\frac{\bar{P}}{R_M^2}$$

$$\frac{C_2}{P_{re}}\left(1 + \frac{2}{(\varphi_1 + \varphi_2)M \coth M}\right) = \frac{2M \coth M - 2}{(\varphi_1 + \varphi_2)M \coth M}\frac{\bar{P}}{R_M^2}$$

$$\frac{C_2}{P_{re}} = \frac{2M \coth M - 2}{(\varphi_1 + \varphi_2)M \coth M + 2}\frac{\bar{P}}{R_M^2} \qquad \text{(AIII-13)}$$

Substituting this result into equation (AIII-8), there results

$$\bar{u} = \left(\frac{R_M}{M^2}\ \frac{2\ M \coth M - 2}{(\varphi_1 + \varphi_2)M \coth M + 2}\ \frac{\bar{P}}{R_M^2} + \frac{\bar{P}}{M^2 R_M}\right)\left(1 - \frac{\cosh MY}{\cosh M}\right)$$

$$= \frac{\bar{P}}{M^2 R_M}\left(\frac{2M \coth M - 2 + (\varphi_1 + \varphi_2)M \coth M + 2}{(\varphi_1 + \varphi_2)M \coth M + 2}\right)\left(1 - \frac{\cosh MY}{\cosh M}\right)$$

$$= \frac{\bar{P}}{M^2 R_M}\ \frac{(\varphi_1 + \varphi_2 + 2)M \coth M}{(\varphi_1 + \varphi_2)M \coth M + 2}\ \left(1 - \frac{\cosh MY}{\cosh M}\right)$$

$$= \frac{\bar{P}}{M\ R_M}\ \frac{\varphi_1 + \varphi_2 + 2}{(\varphi_1 + \varphi_2)M \coth M + 2}\ \left(\coth M - \frac{\cosh MY}{\sinh M}\right)$$

$$= \frac{\bar{P}}{M\ R_M}\ \Phi_1\ \left(\coth M - \frac{\cosh MY}{\sinh M}\right)$$

$$\therefore \left(\frac{u}{\nu L^{-1}}\right)\left(\frac{1}{\bar{P}}\right) = \frac{\Phi_1}{M}\ \left(\coth M - \frac{\cosh MY}{\sinh M}\right) \qquad \text{(AIII-14)}$$

$$\text{(4-15)}$$

where

$$\Phi_1 = \frac{\varphi_1 + \varphi_2 + 2}{(\varphi_1 + \varphi_2)M \coth M + 2} \qquad \text{(4-19)}$$

The derivative of equation (AIII-14) is

$$\frac{d\bar{u}}{dY} = \frac{-\ \bar{P}\ \Phi_1\ \sinh MY}{R_M\ \sinh M} \qquad \text{(AIII-15)}$$

Substituting equation (AIII-12) into equation (AIII-5), one obtains

$$\left(\frac{1}{\varphi_2} + \frac{1}{\varphi_1}\right)\frac{C_1}{R_M} = \left(\frac{1}{\varphi_2} - \frac{1}{\varphi_1}\right)\frac{1}{R_M}\left.\frac{d\bar{u}}{dY}\right)_{Y=1} + \frac{\bar{P}}{R_M^2}\left(\frac{1}{\varphi_2} - \frac{1}{\varphi_1}\right)$$

$$\frac{\varphi_1 + \varphi_2}{\varphi_1 - \varphi_2}\ \frac{C_1}{R_M} = \left.\frac{d\bar{u}}{dY}\right)_{Y=1}\frac{1}{R_M} + \frac{\bar{P}}{R_M^2}$$

Substituting equation (AIII-10) into this, one obtains

$$\frac{\varphi_1 + \varphi_2}{\varphi_1 - \varphi_2}\frac{C_1}{R_M} = -\frac{M \sinh M}{R_M \cosh M}\left(\frac{R_M}{M^2}\frac{2 M \coth M - 2}{(\varphi_1 + \varphi_2) M \coth M + 2}\frac{\bar{P}}{R_M^2} + \frac{\bar{P}}{M^2 R_M}\right)$$

$$+\frac{\bar{P}}{R_M^2}$$

$$= -\frac{\bar{P}}{M R_M^2 \coth M}\frac{(\varphi_1 + \varphi_2 + 2) M \coth M}{(\varphi_1 + \varphi_2) M \coth M + 2} + \frac{\bar{P}}{R_M^2}$$

$$= -\Phi_1 \frac{\bar{P}}{R_M^2} + \frac{\bar{P}}{R_M^2}$$

$$\therefore \frac{C_1}{R_M} = \frac{\varphi_1 - \varphi_2}{\varphi_1 + \varphi_2}(1 - \Phi_1)\frac{\bar{P}}{R_M^2}$$

$$= \frac{\varphi_1 - \varphi_2}{\varphi_1 + \varphi_2}\frac{(\varphi_1 + \varphi_2) M \coth M + 2 - (\varphi_1 + \varphi_2 + 2)}{(\varphi_1 + \varphi_2) M \coth M + 2}\frac{\bar{P}}{R_M^2}$$

$$= \frac{\varphi_1 - \varphi_2}{\varphi_1 + \varphi_2}\frac{(M \coth M - 1)(\varphi_1 + \varphi_2)}{(\varphi_1 + \varphi_2) M \coth M + 2}\frac{\bar{P}}{R_M^2}$$

$$= \frac{-(\varphi_1 - \varphi_2) M}{(\varphi_1 + \varphi_2) M \coth M + 2}\left(\frac{M \coth M - 1}{M}\right)\frac{\bar{P}}{R_M^2}$$

$$= \Phi_2 \left(\frac{1}{M} - \coth M\right)\frac{\bar{P}}{R_M^2} \qquad \text{(AIII-16)}$$

where

$$\Phi_2 = \frac{(\varphi_1 - \varphi_2) M}{(\varphi_1 + \varphi_2) M \coth M + 2}$$

Substitute equations (AIII-15) and (AIII-16) into equation (AIII-1a), one obtains

$$\bar{H}_x = -\frac{1}{R_M}\left(\frac{-\bar{P}\Phi_1 \sinh MY}{R_M \sinh M}\right) - \frac{\bar{P}}{R_M^2} Y$$

$$+ \Phi_2 \left(\frac{1}{M} - \coth M\right) \frac{\bar{P}}{R_M^2}$$

$$= \frac{\bar{P}}{R_M^2}\left[\frac{\Phi_1 \sinh MY}{\sinh M} + \Phi_2 \left(\frac{1}{M} - \coth M\right) - Y\right]$$

$$\therefore \frac{H_x}{H_y}\frac{1}{\bar{P}} = \frac{1}{R_M^2}\left[\frac{\Phi_1 \sinh MY}{\sinh M} + \Phi_2 \left(\frac{1}{M} - \coth M\right) - Y\right] \qquad \text{(AIII-17)}$$

(4-16)

If $\varphi_1 = \varphi_2 = \varphi$ i.e., $\Phi_2 = 0$, then equation (AIII-16) becomes

$$\frac{H_x}{H_y}\frac{1}{\bar{P}} = \frac{1}{R_M^2}\left[\frac{\Phi_1 \sinh MY}{\sinh M} - Y\right]$$

where $\Phi_1 = \dfrac{2\varphi + 2}{2\varphi M \coth M + 2} = \dfrac{\varphi + 1}{\varphi M \coth M + 1}$

$$\frac{H_x}{H_y}\frac{1}{\bar{P}} = \frac{1}{R_M^2}\left[\frac{(\varphi+1)\sinh MY}{(\varphi M \coth M+1)\sinh M} - Y\right] \qquad \text{(AIII-18)}$$

Equation (AIII-18) corresponds to equation (15) in Chang and Lundgren's paper [4].

From equation (2-1)

$$\underline{J} = \text{curl } \underline{H} = -\frac{\partial H_x}{\partial y} \underline{K} \tag{AIII-19}$$

From equation (AIII-17)

$$\frac{d\,\bar{H}_x}{dY} = \frac{\bar{P}}{R_M^2}\left[\frac{-M\Phi_1 \cosh MY}{\sinh M} - 1\right]$$

$$\frac{dH_x}{dy} = \frac{H_y\, d\,\bar{H}_x}{L\, dY} = \frac{H_y\, \bar{P}}{L\, R_M^2}\left[\frac{-M\Phi_1 \cosh MY}{\sinh M} - 1\right] \tag{AIII-20}$$

Substituting equation (AIII-20) into equation (AIII-19), one obtains

$$J_z = \frac{H_y\, \bar{P}}{L\, R_M^2}\left[1 - \frac{\Phi_1 M \cosh MY}{\sinh M}\right]$$

or

$$\bar{J}_z = \frac{J_z}{a\, \sigma\, \mu_e H_y} = \frac{1}{a\, \sigma\, \mu_e H_y}\, \frac{H_y\, \bar{P}}{L\, R_M^2}\left[1 - \frac{\Phi_1\, M \cosh MY}{\sinh M}\right]$$

$$= \frac{1}{a\, L\, \sigma\mu_e\, \frac{aL}{\nu}}\, \frac{\bar{P}}{R_M}\left(1 - \frac{\Phi_1\, M \cosh MY}{\sinh M}\right)$$

$$= \frac{1}{a^2 L^2/\eta\nu}\, \frac{\bar{P}}{R_M}\left(1 - \frac{\Phi_1\, M \cosh MY}{\sinh M}\right)$$

$$= \frac{\bar{P}}{M^2 R_M}\left(1 - \frac{\Phi_1\, M \cosh MY}{\sinh M}\right) \tag{4-17}$$

FIGURES

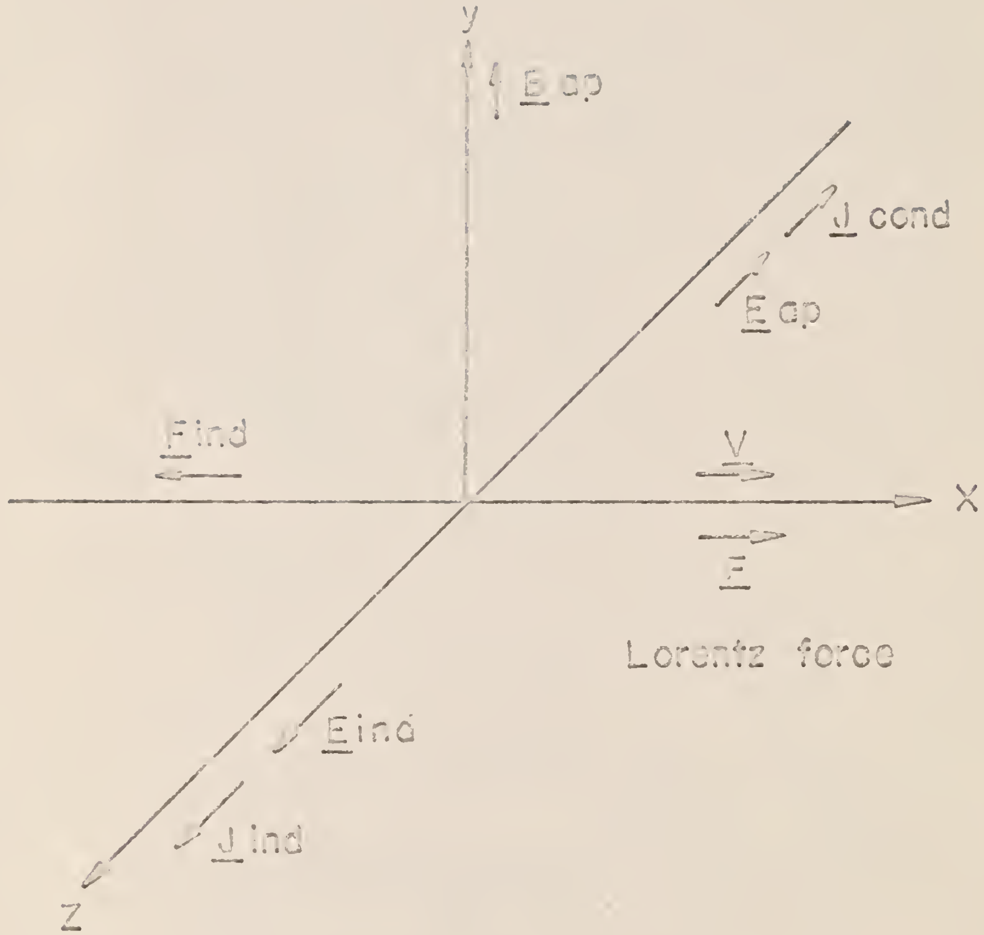


Fig. I. Vector diagram of MHD.

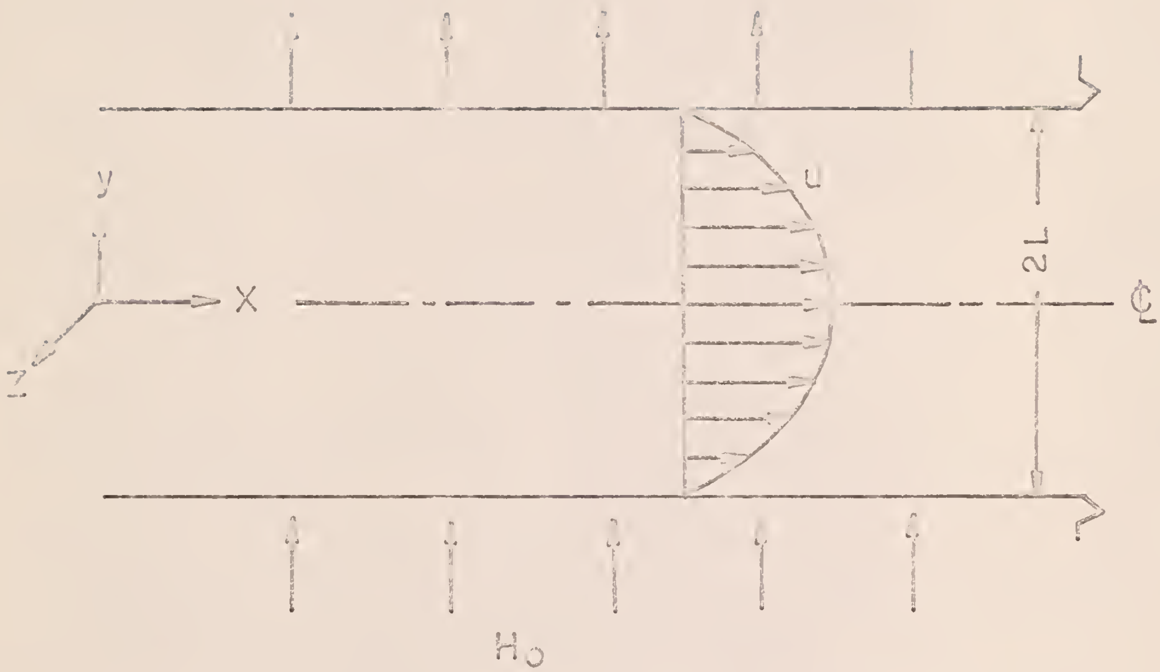


Fig. 2. Steady, fully developed, laminar, incompressible flow between two parallel, non-conducting plates.

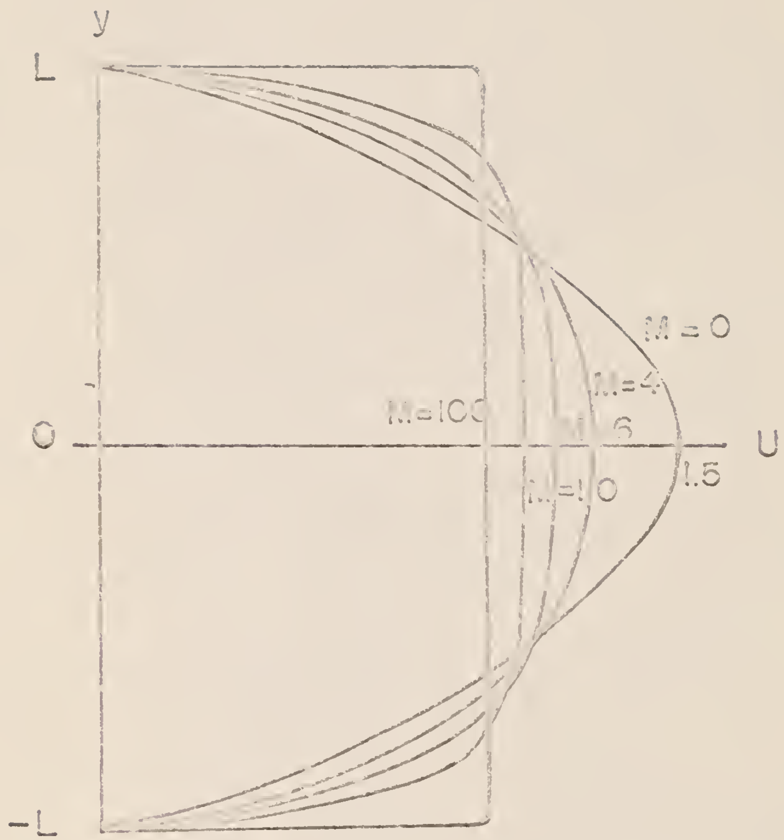


Fig. 3. Velocity profiles in Hartmann flow for different values of M. The profiles are based on equation (3.5). For $M = 0$, $U = \frac{3}{2}\left\{1 - \left(\frac{y}{L}\right)^2\right\}$, Poiseuille flow.

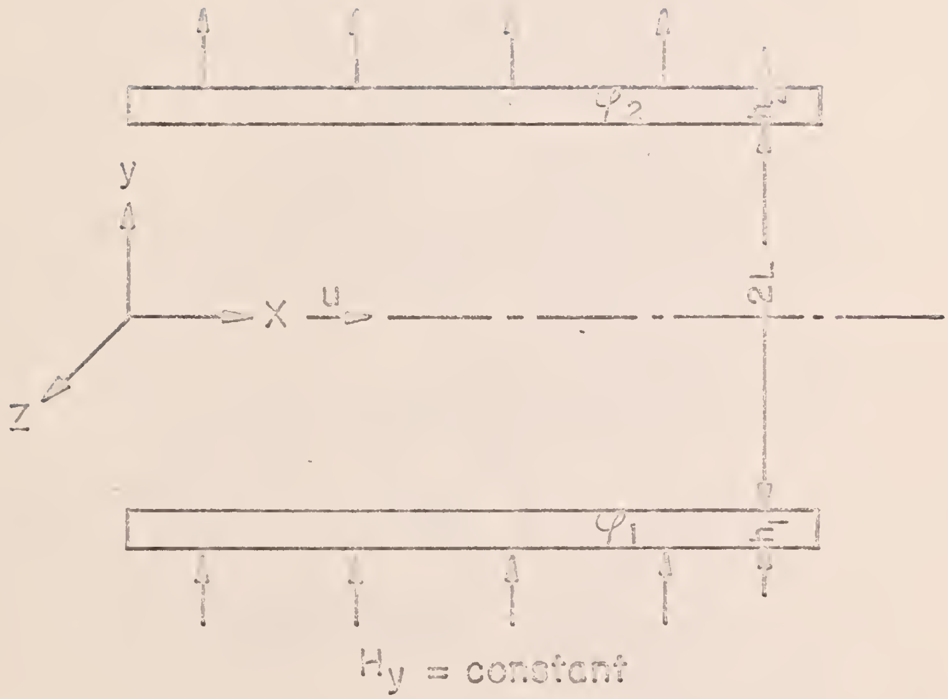


Fig. 4. Modified Hartmann flow with the effect of wall electrical conductance.

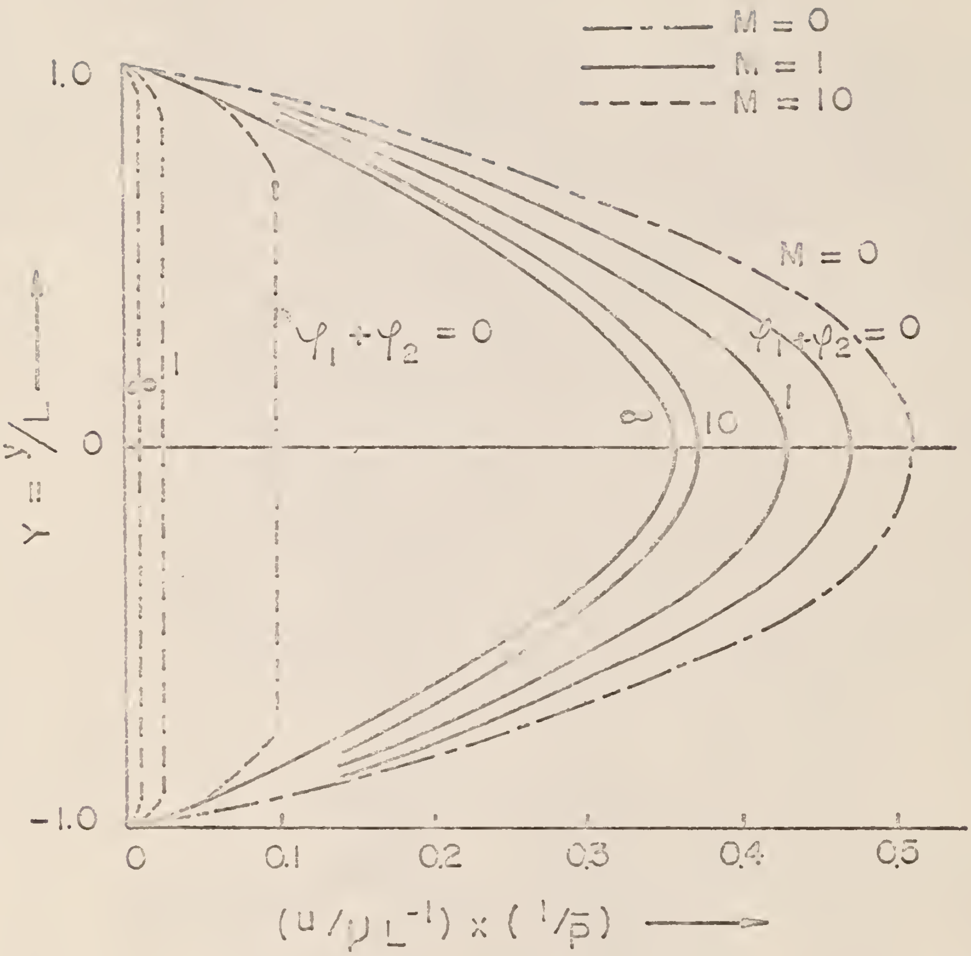


Fig. 5. Velocity profiles as influenced by wall conditions and Hartmann number M. These profiles are based on equation (4.15). [5]

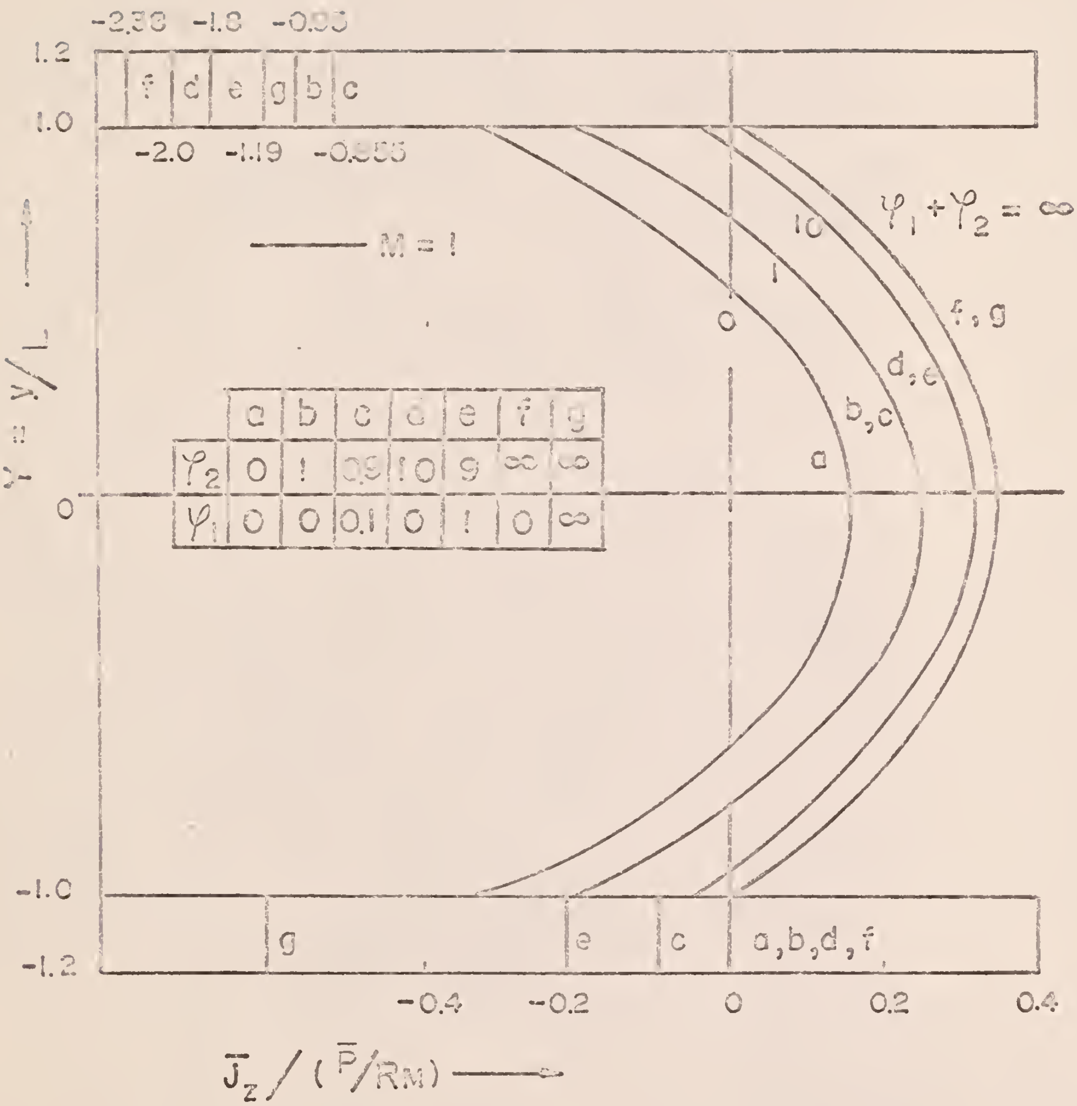


Fig. 6. Electrical current profiles in the fluid and in the walls. These profiles are based on equation (4.17). [5]

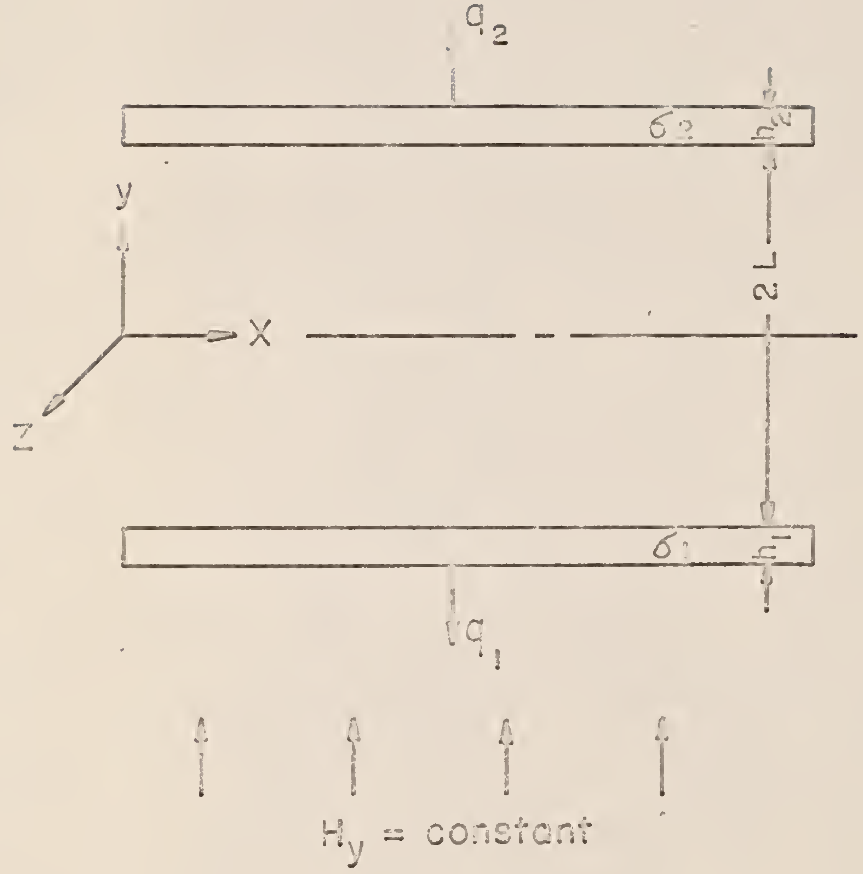


Fig. 7. Channel flow geometry.

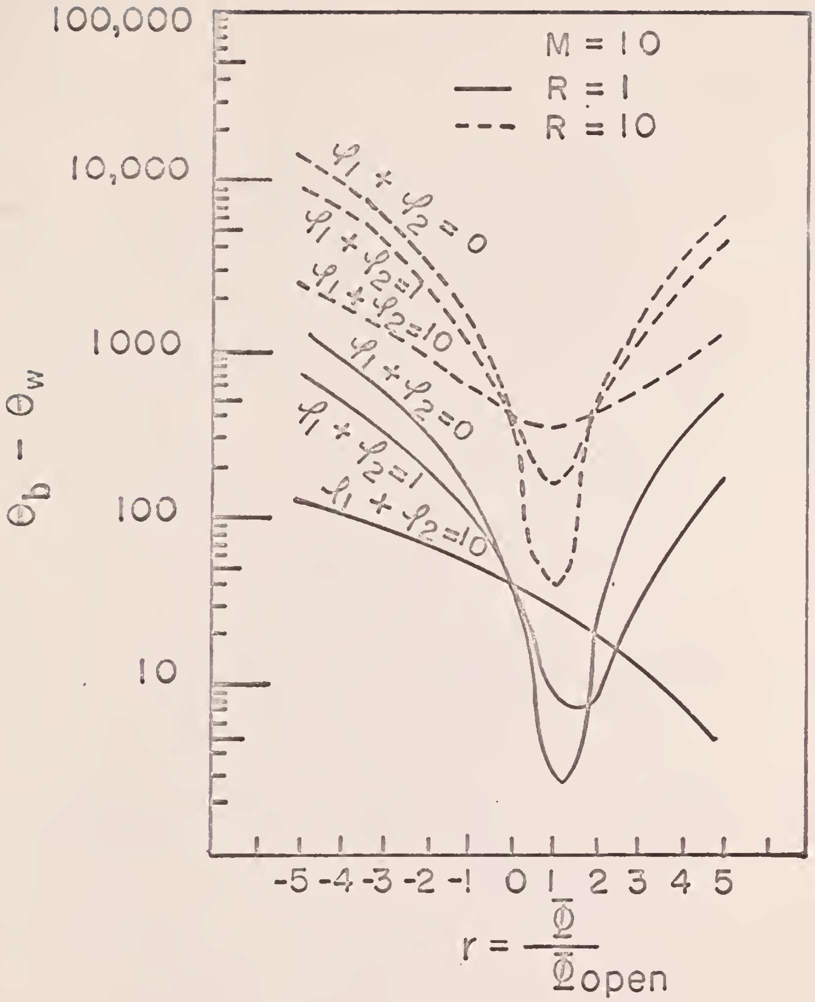


Fig. 8. Bulk to wall temperature difference for M = 10. [9]

u_m=100 ft/sec.

μ =2.6 x 10^{-5} lb_f sec/ft^2

k =6 Btu/hr ft F.

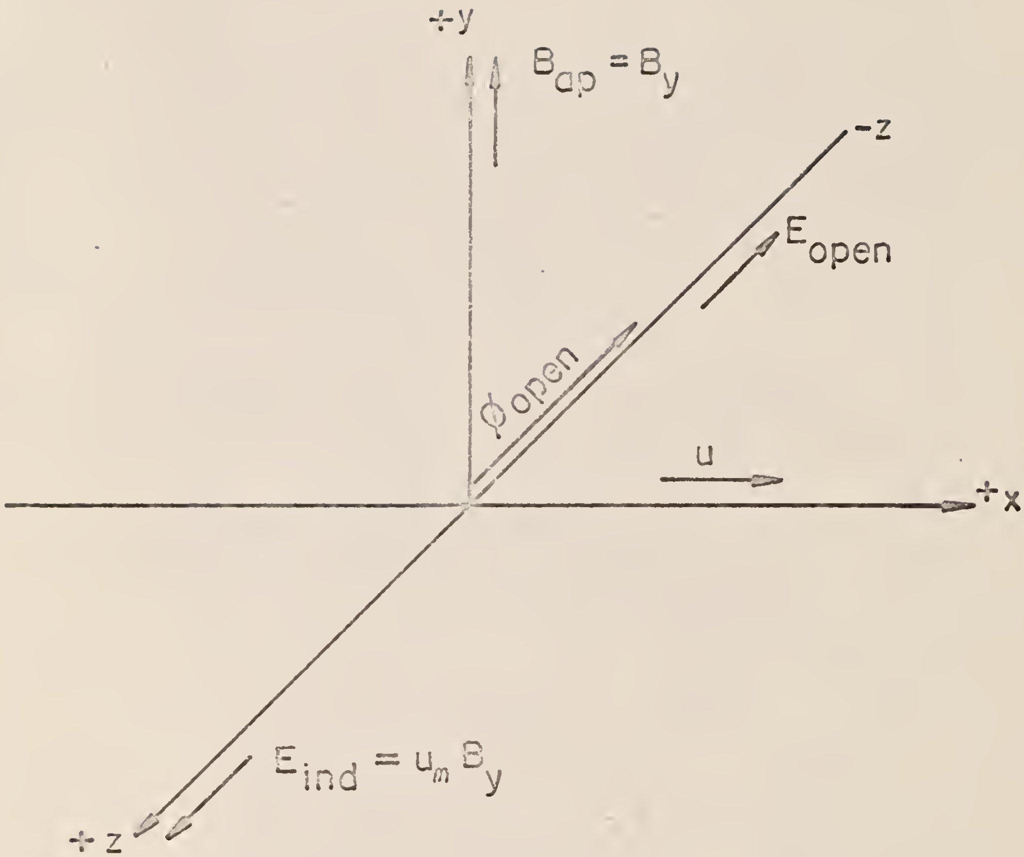
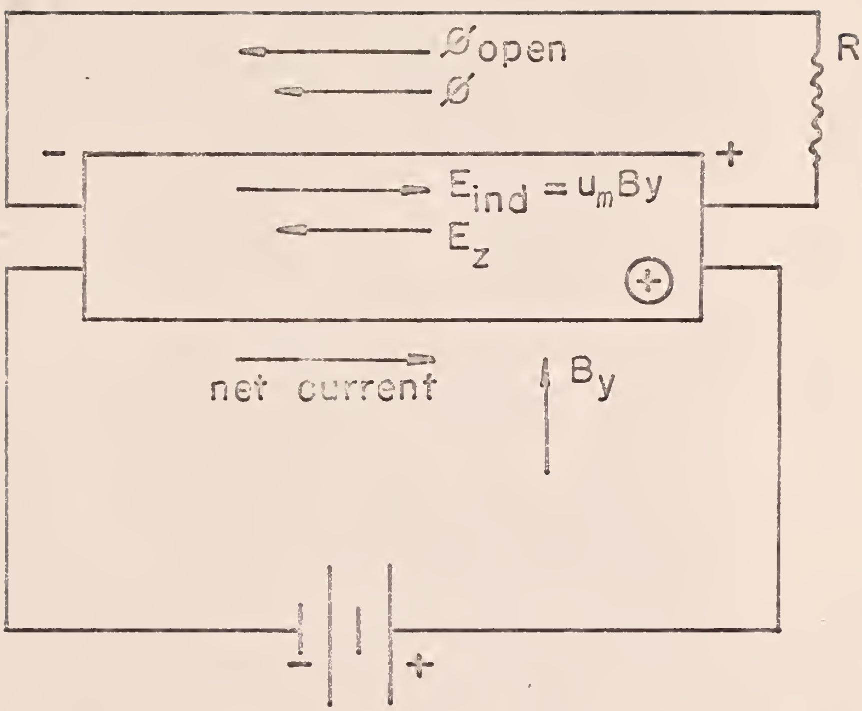


Fig. 9. Vector diagram of open-circuit condition.



⊕ indicates the direction of fluid flow

Fig. 10. The generator mode of operation with $0 < r < 1$.

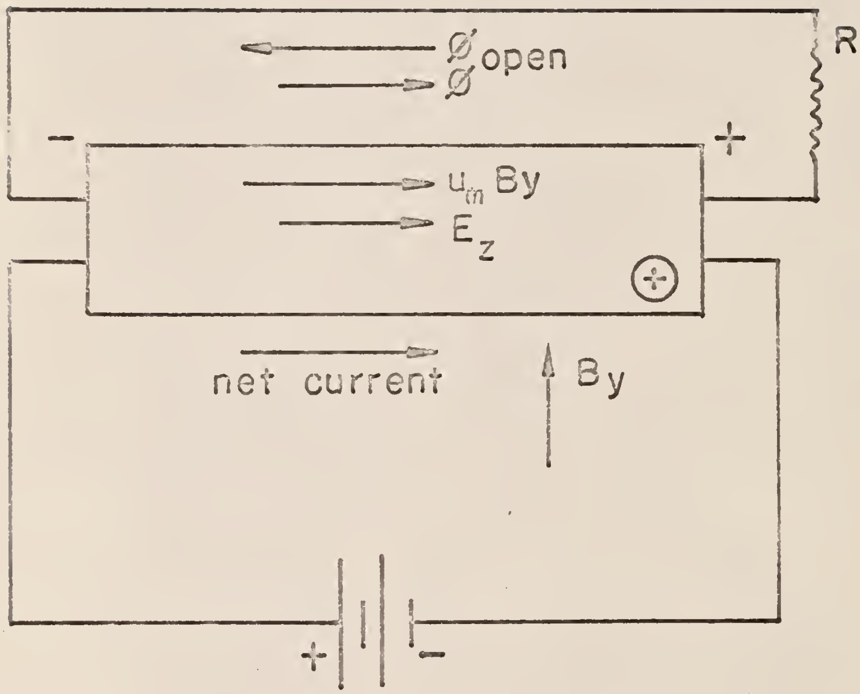


Fig. 1.1. The generator mode of operation with $r < o$.

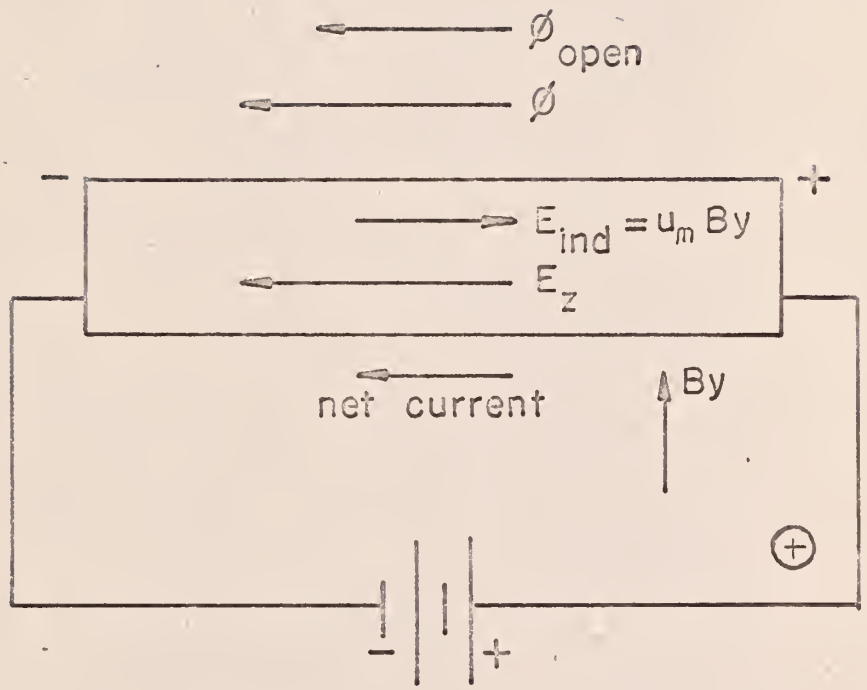


Fig. 12. The accelerator mode of operation with $r > 1$.

THE EFFECT OF WALL ELECTRICAL CONDUCTANCE ON THE FULLY DEVELOPED VELOCITY DISTRIBUTION FOR LAMINAR, STEADY TWO-DIMENSIONAL, MAGNETOHYDRODYNAMIC CHANNEL FLOW WITH HEAT TRANSFER

by

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Magnetohydrodynamics (MHD) is the science of the motion of an electrically conducting incompressible fluid in the presence of a magnetic field. The fundamental equations of MHD are the modified electromagnetic and hydrodynamic equations. Hartmann first considered the steady, laminar, fully developed flow of an incompressible electrically conducting fluid between two electrically non-conducting infinite parallel plates. Perpendicular to these plates a magnetic field is applied. Constant properties of the fluid and walls are assumed. Modified Hartmann flow is the flow with the effect of wall electrical conductance. The velocity profile is solved with the modified Navier-Stokes equation and the Ohm's law for a moving fluid.

The heat transfer problems in walls and in the fluid are solved simultaneously with appropriate matching conditions at the fluid-wall interfaces. The influence of finite wall electrical conductivity is considered also. The flow is assumed to be thermally and hydrodynamically fully developed and constant heat flux boundary conditions are applied at the outer surfaces of the walls. It is shown that the influence of the wall conductivity on the heat transfer is opposite for the generator and accelerator modes of operation.