

A POWER STUDY FOR ONE-SIDED MULTIVARIATE TESTS

by

KUO-HSING CHANG

B.S., National Cheng-Kung University, Taiwan, 1974

M.S., The University of Iowa, 1985

.....

A DISSERTATION

submitted in partial fulfillment of the

requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Statistics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1992

Approved by:

Jane J. Higgins
Major Professor

LD
2668
.D5
STAT
1992
C53

ABSTRACT



A11212647526

In this study, parametric and nonparametric tests for two classes of multivariate alternatives called tailored and omnibus alternatives are considered. Included among the tests are UMP parametric tests and their nonparametric counterparts. From a simulation study it is shown that the proposed tests of tailored alternatives have substantially greater power than the omnibus tests under the tailored alternatives. Conditions under which omnibus tests perform well are given.

The asymptotic properties of the proposed tailored tests are investigated. For the multivariate case, unlike the univariate case, the ARE of the Wilcoxon statistic relative to the t test depends not only on the population distribution but also on such factors as dimension, covariance structure, and correlation. The robustness of tests of tailored alternatives to misspecification of the form of the tailored alternative is investigated. Modest model misspecification has only small effects on efficiencies.

Small sample test procedures are discussed with the focus being on the situation of a smaller sample size than the number of measurements. Permutation tests of one-sided tailored alternatives are considered. The normal approximation to the permutation distribution is shown to be useful even for small sample size. A power study is conducted

to investigate factors that affect the relative performances of the proposed test.

ACKNOWLEDGEMENTS

I would like to express my sincere thanks to my major professor, Dr. James Higgins, for suggesting this area of research, his great assistance with advice and action, his availability for consultation and his encouragement every step of the way. Special thanks to Dr. Shian K. Perng for a timely suggestion and assistance. Appreciation is also extended to the members of my committee, Dr. George A. Milliken, and Dr. Shing I Chang. I am grateful to Dr. John E. Boyer, Jr and Dr. James Schwenke for the extensive use of their P.C.'s.

On a more personal level, I would like to thank my family throughout these many years. In particular, I want to thank my dear wife, Debra, has been a faithful and steadfast friend, for her patience and support in every way.

TABLE OF CONTENTS

CHAPTER	Page
I. INTRODUCTION	1
1-1 Motivation.....	1
1-2 Review of Literature.....	2
1-3 Problem Settings and Assumptions.....	3
1-4 Description of Test Hypotheses.....	6
1-5 Scope of the Study.....	9
II. PROPOSED TEST STATISTICS AND RELATED DISTRIBUTIONAL PROPERTIES.....	11
2-1 Definition of Terms.....	11
2-2 Test Statistics and their Properties.....	14
2-3 Properties of Population Distribution.....	33
III. COMPARISON OF TESTS OF TAILORED ALTERNATIVES TO TESTS OF OMNIBUS ALTERNATIVES.....	36
3-1 Introduction.....	36
3-2 Purposes of the Simulation.....	36
3-3 Power Advantage of Tests for Tailored Alternatives.....	42
3-4 Power Advantage of Tests for Omnibus Alternatives.....	50
3-5 General Factors Affecting both Tailored and Omnibus Tests and Conclusion.....	52
IV. ASYMPTOTIC RELATIVE EFFICIENCY AND ROBUSTNESS OF TEST PROCEDURES.....	53
4-1 Introduction.....	53
4-2 Definition of Efficacy of the Test and Asymptotic Relative Efficiency of Two Competing Tests.....	54
4-3 Numerical Computation of ARE.....	76
4-4 Investigation of Robustness of Test Procedure $a'W$	82
V. SMALL SAMPLE TEST PROCEDURE.....	94
5-1 Introduction.....	94
5-2 Permutation Test.....	94
5-3 Permutation Test and Normal Approximation to Permutation Distribution.....	98
5-4 Numerical Comparison of Permutation Test and its Normal Approximation.....	108
5-5 Simulation Study.....	111
VI. SUMMARIES AND IMPLICATIONS FOR FUTURE RESEARCH....	124
APPENDIX A.....	127
REFERENCES	139

Chapter 1

Introduction

1-1 Motivation

It is typical in medical research and other areas for many response variables to be measured on each subject. For instance, electrical activity of the brain may be measured at different times or different locations on the scalp. It is often possible to anticipate the direction that multivariate responses will go if a treatment is effective. For instance, if a visual or verbal stimulus elicits a certain response, a researcher might expect that electrodes placed on the head to respond with a positive voltage. The question is, can we take this additional knowledge into account to construct tests that would be more powerful than standard tests such as the χ^2 test or Hotelling T^2 ?

In the case when the population is univariate normal, the problem can be solved by an ordinary one-sided test using the univariate t test or an appropriate nonparametric test. However, when we have a multivariate responses instead, can we apply a multivariate analog of the one-sided t or Wilcoxon signed rank tests? If so, how do these test perform in terms

of the ability to detect true differences?

1-2 Review of Literature

Kudo (1963) considered the problem of performing one-sided multivariate tests under normal theory. His solution centered on the derivation of a likelihood ratio test (LRT) for the hypothesis that all coordinate means are nonnegative with at least one strictly positive. Nuesch (1965) extended Kudo's work to the case where Σ , the covariance matrix of a multivariate distribution, is unknown. However, neither of them investigated the power of the LRT. Shorack (1967) modified the Kruskal-Wallis test and Friedman's χ^2 test and applied them to ordered treatment mean problems using a process to obtain the best estimators of the location parameters under ordered restrictions, as proposed by Bartholomew (1959). Boyett and Shuster (1977) noticed that Hotelling's T^2 is not sensitive to one sided hypotheses and proposed using a permutation test based on the maximum among the p univariate t -statistics obtained from a p -dimensional sample. Their results were given for a two-sample completely randomized design and a two-sample matched-pair design. Dietz (1989) generalized Jonckheere's univariate statistics to a multivariate version. The multivariate test statistic is a

function of coordinate-wise Jonckheere statistics which tests an ordered means alternative. Raz (1989) discussed the drawbacks of ordinary analyses of repeated measurement design for not exploiting information contained in the ordering of the data points over time and suggested using randomization tests to overcome this disadvantage. Blair, Higgins, Karniski and Kromrey (1992) proposed a multivariate permutation test for dealing with applications in the field of physiological research and showed the one-sided tests have significant power advantages over the Hotelling's T^2 . Blair and Karniski (1992) proposed an alternative distribution-free test based on the permutation principle to compare and show the advantage over the test proposed by Guthrie and Buchwald (1991), a Bonferroni-type procedure for testing the difference between two treatments using the multivariate responses measured from subjects.

1-3 Problem Settings and Assumptions

We will consider the one-sample multivariate location problem. Let $\underline{X}_1, \underline{X}_2, \dots, \underline{X}_n$ be a random sample size of n from a p -dimensional distribution with location parameter $\underline{\mu}$ and positive definite dispersion matrix Σ . Assume the p -variate cumulative distribution function (cdf) F is absolutely continuous with marginal cdfs $F_1(t_1 - \mu_1), \dots, F_p(t_p - \mu_p)$ and

bivariate marginal cdfs $F_{ij}(t_i - \mu_i, t_j - \mu_j)$, $i, j = 1, \dots, p$. Also assume that the joint pdf $f(t_1, \dots, t_p)$ is diagonally symmetry about $\underline{\mu}$, i.e. it satisfies $f(t_1 + \mu_1, \dots, t_p + \mu_p) = f(\mu_1 - t_1, \dots, \mu_p - t_p)$ for all $\underline{t}$. This results in symmetric marginal densities $f_1, \dots, f_p$ where the location vector $\underline{\mu}' = (\mu_1, \dots, \mu_p)$ represents the centers of symmetry of the marginal distributions.

There are two distributions used as population models in this study. One is the well known multivariate normal distribution and one is a multivariate t distribution. We define these as follows.

Definition 1.3.1 Let $\underline{X}$ be a $p \times 1$ vector. If $\underline{X}$ has a p.d.f of the form given by

$$p(\underline{X}, \underline{\mu}, \Sigma) = (2\pi)^{-\frac{p}{2}} |\Sigma|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (\underline{X} - \underline{\mu})' \Sigma^{-1} (\underline{X} - \underline{\mu}) \right\},$$

then the random vector $\underline{X}$ has a multivariate normal distribution with mean $\underline{\mu}$ and covariance matrix Σ , denoted by $N_p(\underline{\mu}, \Sigma)$.

Definition 1.3.2 If $\underline{Z}$ is an p -dimensional multivariate normal vector with $E(\underline{Z}) = \underline{0}$ and a nonsingular covariance matrix Σ , Y is a χ^2 random variable with v degree of freedom, and $\underline{Z}$ and Y are independent, then the random vector

$$\underline{X} = \frac{\sqrt{v} \underline{Z}}{\sqrt{Y}} + \underline{\mu}$$

has an p-dimensional t-distribution with v degrees of freedom, location vector $\underline{\mu}$ and dispersion matrix Σ . This distribution of $\underline{X}$ is denoted by $MVT_p(\underline{\mu}, \Sigma, v)$. The smallest degrees of freedom in which the second moments exist is $v = 3$. This is the one used in our study.

1-4 Description of Test Hypotheses

In this study we deal with a class of one-sided multivariate alternatives we call tailored alternatives.

Definition 1.4.1 A multivariate tailored alternative is one in which each of the means μ_i can be expressed as a function of a single parameter θ . That is

$$\mu_i = \psi_i(\theta)$$

where ψ_i is known.

Specifically we consider simple linear functions of θ given by

$$(i) \quad H_a : \mu_i = a_i \theta, \text{ for all } i = 1, \dots, p$$

where the a_i 's are known constants. A special case of interest in this study is

$$(ii) \quad H_a : \mu_1 > 0, \mu_2 = \dots = \mu_p = 0$$

Here $a_1 > 0, a_2 = \dots = a_p = 0$. An hypothesis is one-sided tailored alternative if the hypothesis may be

expressed as

$$H_a : \theta > 0 \quad (\text{ or } H_a : \theta < 0)$$

A simple example of such a tailored alternative would occur when the expected response of a subject at time t_i after a treatment has been applied is a sine wave function given by

$$\mu_i = \theta \cdot \sin(t_i)$$

where θ is the amplitude of the sine wave. More generally one might observe a signal, as carried from an electrode on brain, of the form

$$\mu_i = \theta \cdot f(t_i)$$

where $f(\cdot)$ is a known wave function which may be determined empirically from information obtained in previously performed experiments.

In contrast to a tailored alternative, we will also be interested in omnibus alternatives.

Definition 1.4.2 An multivariate omnibus alternative is one in which the μ_i 's cannot be expressed as function of a single

parameter.

The one-sided omnibus alternative of interest in this study is the one studied by Kudo (1963),

$$\begin{aligned} \text{(iii)} \quad H_a : & \mu_i \geq 0 \text{ for all } i = 1, \dots, p. \\ & \mu_j > 0 \text{ for some } j = 1, \dots, p. \end{aligned}$$

1-5 Scope of the study

In chapter 2, we define the parametric and nonparametric tests for tailored and omnibus alternatives used in this study and discuss their properties. Included among the tests are UMP parametric tests and their nonparametric counterparts. In chapter 3, we conduct a simulation study to demonstrate the advantages and disadvantages of tests for tailored alternatives in comparison to tests for the omnibus alternatives. In chapter 4, we consider the asymptotic properties of the proposed tailored tests. We obtain efficacies of test procedures and compare Pitman relative efficiencies (i.e asymptotic relative efficiency) for the parametric tests and their nonparametric counterparts. The comparisons include the weighted Wilcoxon signed-rank tests against the weighted coordinate t test, and different weighted Wilcoxon signed-rank tests versus each other under various parameter configurations. In chapter 5, we discuss small sample nonparametric tests. We focus especially on the situation of a smaller sample size than the number of measurements where ordinary tests, such as Hotelling T^2 , are not valid. We discuss permutation tests in the multivariate setting for one-sided tailored alternatives, and investigate a normal approximation to the permutation distribution. Furthermore, we demonstrate how well this approximation works

through a power study. We then conduct a simulation study to investigate the power advantages and disadvantages of using actual observations in permutation tests as compared to using the corresponding ranks. Factors that affect the relative performances of the proposed tests are investigated.

Chapter 2

Proposed Test Statistics and Related Distributional Properties

2-1 Definition of terms

Assume that $\underline{X}_1, \underline{X}_2, \dots, \underline{X}_n$ comprise a random sample size of n from a p -variate symmetric distribution with mean vector $\underline{\mu}$ and positive definite covariance matrix Σ , where

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & . & . & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & . & . & \sigma_{2p} \\ . & . & . & . & . \\ \sigma_{p1} & \sigma_{p2} & . & . & \sigma_{pp} \end{bmatrix} .$$

(1) $\underline{Z}$ vector

Let $\underline{Z}$ be a $p \times 1$ column vector such that $\underline{Z} = (Z_1, \dots, Z_p)'$ where

$$Z_j = \frac{\sqrt{n} \bar{X}_j}{\sqrt{\sigma_{jj}}}, \quad j = 1, \dots, p$$

and

$$\bar{X}_j = \frac{1}{n} \sum_{i=1}^n X_{ij}, \quad j = 1, \dots, p.$$

(2) W vector

Let W be a $p \times 1$ column vector such that $\underline{W} = (W_1, \dots, W_p)'$ where W_j is the Wilcoxon signed rank statistics of j^{th} coordinate defined by

$$W_j = \sum_{i=1}^n \text{Rank}(|X_{ij}|) \text{Sgn}(X_{ij}) \quad \forall j = 1, \dots, p,$$

$\text{Rank}(|X_{ij}|)$ is the rank of $|X_{ij}|$ among $\{|X_{1j}|, \dots, |X_{nj}|\}$, $j = 1, \dots, p$, and

$$\text{Sgn}(X_{ij}) = \begin{cases} 1 & \text{if } X_{ij} > 0 \\ 0 & \text{if } X_{ij} = 0 \\ -1 & \text{if } X_{ij} < 0 \end{cases}.$$

(3) R matrix

Let ρ_{ij} denote the correlation between X_{ki} and X_{kj} , $k=1, \dots, n$. Define R to be a $p \times p$ correlation matrix of X where

$$R = \begin{bmatrix} 1 & \rho_{12} & \cdot & \cdot & \rho_{1p} \\ \rho_{21} & 1 & \cdot & \cdot & \rho_{2p} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \rho_{p1} & \rho_{p2} & \cdot & \cdot & 1 \end{bmatrix}.$$

(4) S matrix

Let s_{ij} denote the sample covariance between the i^{th} and j^{th} variates. Define S to be the sample covariance matrix of X given by

$$S = \begin{bmatrix} s_{11} & s_{12} & \cdot & \cdot & s_{1p} \\ s_{21} & s_{22} & \cdot & \cdot & s_{2p} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{p1} & s_{p2} & \cdot & \cdot & s_{pp} \end{bmatrix}.$$

(5) V matrix

Let V be a $p \times p$ matrix where the ij^{th} element is defined by

$$v_{ij} = \begin{cases} \frac{1}{3} & i=j \\ 4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_i(x) F_j(y) dF_{ij}(x,y) - 1 & i \neq j \end{cases},$$

$F_i(x)$ and $F_j(x)$ are the marginal cumulative distribution functions for the i^{th} and j^{th} variates, respectively, and $F_{ij}(x,y)$ is the joint cumulative distribution function of i^{th} and j^{th} variates. V is the asymptotic covariance matrix for the vector $(1/n)^{\frac{1}{2}} \cdot (1/(n+1))^2 \underline{W}$ as given by Puri and Sen (1971).

(6) V^* matrix

Let V^* be a $p \times p$ matrix where ij^{th} element is defined by

$$v_{ij}^* = \frac{1}{n} \sum_{k=1}^n \frac{\text{Rank}(|x_{ki}|)}{n+1} \frac{\text{Rank}(|x_{kj}|)}{n+1} \text{Sgn}(x_{ki}) \cdot \text{Sgn}(x_{kj})$$

and $\text{Rank}(|x_{ki}|)$ and $\text{Sgn}(x_{ki})$ are defined as in (2).

(7) $\phi(x)$

$\phi(x)$ is a nonrandomized decision rule based on observed data x . One rejects H_0 when $\phi(x) = 1$ and does not reject when $\phi(x) = 0$.

2-2 Test Statistics and their Properties

There are two categories of test statistics considered in this chapter. One consists of parametric tests, and the other consists of rank tests which are analogous to the parametric tests. We begin with the parametric test statistics.

2-2-1 Parametric test statistics.

(1) Weighted $\underline{Z}$ type

Let $\underline{Z}$ be given as in (1). The weighted $\underline{Z}$ type are given by

$$(1-a) \tau_1(\underline{Z}) = \underline{a}'\underline{Z}$$

and

$$(1-b) \tau_2(\underline{Z}) = \underline{j}'\underline{R}^{-1}\underline{Z}$$

where $\underline{a}' = (a_1, a_2, \dots, a_p)$ is a vector of constants. The statistics $\tau_1(\underline{Z})$ and $\tau_2(\underline{Z})$ are linear combinations of coordinate Z statistic.

Now we will discuss the optimal properties of a class of tests based on $\underline{Z}$. We start with the following theorem.

Theorem 2.2.1 The UMP test statistic for testing the hypothesis $H_0 : \underline{\mu} = \underline{0}$ verse $H_a : \mu_i = a_i\theta$, for all $i = 1, 2, \dots, p$ and $\theta > 0$ under the assumption of multivariate normality is

$$\tau_1(Z) = \underline{a}^{*'} R^{-1} Z,$$

and the decision rule is

$$\phi_{\tau_1}(\tau_1) = \begin{cases} 1 & \text{if } \tau_1 > k_1 \\ 0 & \text{otherwise} \end{cases}$$

where

$$\underline{a}^{*'} = \frac{1}{\sqrt{n}} \underline{a}' \text{Diag}\left(\frac{1}{\sqrt{\sigma_{11}}}, \frac{1}{\sqrt{\sigma_{22}}}, \dots, \frac{1}{\sqrt{\sigma_{pp}}}\right)$$

and $\text{Diag}(d_1, \dots, d_p)$ is a diagonal matrix with $d_1, \dots, d_p$ in main diagonal terms and zeros in off-diagonal entries.

PROOF: To show the test is UMP, we need to form a ratio of two likelihood functions with respect to θ_1 and θ_2 respectively. Under the normality assumption,

$$p(x_i, \mu, \Sigma) = (2\pi)^{\frac{-p}{2}} |\Sigma|^{\frac{-1}{2}} \exp \left\{ \frac{-1}{2} (x_i - \mu)' \Sigma^{-1} (x_i - \mu) \right\}$$

and its likelihood function is

$$\begin{aligned} \prod_{i=1}^n p(x_i, \mu, \Sigma) &= \prod_{i=1}^n (2\pi)^{\frac{-p}{2}} |\Sigma|^{\frac{-1}{2}} \exp \left\{ \frac{-1}{2} (x_i - \mu)' \Sigma^{-1} (x_i - \mu) \right\} \\ &= (2\pi)^{\frac{-np}{2}} |\Sigma|^{\frac{-n}{2}} \exp \left\{ \frac{-1}{2} \sum_{i=1}^n (x_i - \mu)' \Sigma^{-1} (x_i - \mu) \right\} \end{aligned}$$

Therefore, for any two values θ_1 and θ_2 such that $\theta_1 > \theta_2 \geq 0$ and $\mu_1' = a'\theta_1$, $\mu_2' = a'\theta_2$

$$\frac{P(X, \mu_1, \Sigma)}{P(X, \mu_2, \Sigma)} = k \exp \left\{ n (\theta_1 - \theta_2)' a' \Sigma^{-1} \bar{X} \right\}$$

It can be seen from the above equation that the ratio $p(X, \theta_1, \Sigma) / p(X, \theta_2, \Sigma)$ depends on the data matrix X only through the value of the statistic $a'\Sigma^{-1}\bar{X}$ and that this ratio is an increasing function of $a'\Sigma^{-1}\bar{X}$. Therefore, $p(X, \mu_1, \Sigma)$ has a monotone likelihood ratio in the statistics $a'\Sigma^{-1}\bar{X}$. Since

$$R = \text{Diag} \left(\frac{1}{\sqrt{\sigma_{11}}}, \dots, \frac{1}{\sqrt{\sigma_{pp}}} \right) \Sigma \text{Diag} \left(\frac{1}{\sqrt{\sigma_{11}}}, \dots, \frac{1}{\sqrt{\sigma_{pp}}} \right)$$

and

$$\bar{X} = \frac{1}{\sqrt{n}} \text{Diag}(\sqrt{\sigma_{11}}, \dots, \sqrt{\sigma_{pp}}) Z'$$

it is straightforward to show

$$a' \Sigma^{-1} \bar{X} = a' \frac{1}{\sqrt{n}} \text{Diag} \left(\frac{1}{\sqrt{\sigma_{11}}}, \dots, \frac{1}{\sqrt{\sigma_{pp}}} \right) R^{-1} Z = a^{*'} R^{-1} Z$$

Thus, the proof is complete.

Corollary 2.2.1 If $a_i = (\sigma_{ii})^{\frac{1}{2}}$ for $i = 1, 2, \dots, p$ then the UMP test statistic is

$$\tau_2(Z) = j' R^{-1} Z$$

with decision rule

$$\phi_{\tau_2}(\tau_2) = \begin{cases} 1 & \text{if } \tau_2 > k_2 \\ 0 & \text{otherwise} \end{cases}$$

PROOF: This is a special case of Theorem 2.2.1.

Corollary 2.2.2 If $a_i = (\sigma_{ii})^{\frac{1}{2}}$ and $\rho_{ij} = 0$ for $i \neq j$ where $i, j = 1, 2, \dots, p$, then the UMP test statistic is

$$\tau_1(Z) = j' Z$$

with decision rule

$$\phi_{\tau_1}(\tau_1) = \begin{cases} 1 & \text{if } \tau_1 > k_3 \\ 0 & \text{otherwise} \end{cases}$$

The proof is a special case of Corollary 2.2.1.

Now consider the other type of parametric test statistics as follows.

(2) Kudo's parametric type

Kudo (1963) considered the omnibus test of hypothesis, (iii) of Section 1-4,

$$H_0 : \mu_i = 0 \text{ for all } i = 1, \dots, p.$$

$$H_a : \mu_i \geq 0 \text{ for all } i = 1, \dots, p.$$

$$\mu_j > 0 \text{ for some } j = 1, \dots, p.$$

He applied the likelihood ratio test and obtained the following statistic:

$$\bar{\chi}^2 = n \bar{X}' \Sigma^{-1} \bar{X} - \min_{\mu_i \geq 0 \forall i} (\bar{X} - \mu)' \Sigma^{-1} (\bar{X} - \mu) .$$

Theorem 2.2.2 (Kudo's Theorem) Under the assumption that $\underline{X}$ has a multivariate normal distribution, the probability that the value of χ^2 exceeds c is given by

$$Pr (\bar{\chi}^2 > c) = \sum_{\emptyset \subset M \subset K} Pr (\chi_{n(M)}^2 \geq c) P \{ (\Sigma_M)^{-1} \} P \{ \Sigma_{M,M'} \}$$

where the summation runs over all the subsets M of

$K = \{1, \dots, p\}$, and $n(M)$ is the number of elements in M , M^C is the complement of M , Σ_M is the covariance matrix of X_i , $i \in M$, $\Sigma_{M;M^C}$ is the covariance matrix under the condition $X_j = 0$ where

$j \in M$, $P(\Sigma)$ is the probability that the variables are all positive, $\chi^2_{(n(M))}$ is random variable with Chi-square distribution with $n(M)$ degrees of freedom, and

$$\chi^2_{n(\emptyset)} = 0, \quad P\{\Sigma_{\emptyset, K}\} = 1, \quad P\{(\Sigma_{K^C})^{-1}\} = P\{\Sigma_{\emptyset}^{-1}\} = 1.$$

For computational purposes, we will simplify Kudo's statistic. Before we proceed, let us consider the following optimization problem. Suppose we want to minimize $f(\underline{x}) = 2\underline{q}'\underline{x} + \underline{x}'A\underline{x}$ and subject to $x_i \geq 0$ for all $i = 1, \dots, p$ where $\underline{q}$ is a $p \times 1$ vector of constants and A is a positive definite symmetric matrix of dimension $p \times p$. This is equivalent to maximizing

$$\begin{cases} g(X) = -2 \underline{q}' X - X' A X \\ \text{subject to } X_j \geq 0 \quad \forall j=1, \dots, p \end{cases} \quad (2.2.1)$$

The following lemmas provide the properties of the solution of the above maximization problem.

Lemma 2.2.4 $\underline{x}$ is the solution which maximizes (2.2.1) if and only if the following two conditions hold.

$$(1) \quad q_j + a'_j \hat{x} \geq 0 \quad \forall j=1, \dots, p,$$

and

$$(2) \quad \text{if } q_j + a'_j \hat{x} > 0 \Rightarrow \hat{x}_j = 0 \quad \forall j=1, \dots, p.$$

where q_j is the j^{th} element of vector q and a_j' is the j^{th} row of matrix A .

PROOF: By using Karush-Kuhn-Tucker (KKT) conditions, (1951, Kuhn & Tucker), we have (i) & (ii) below.

$$(i) \quad \left. \frac{\partial g(x)}{\partial x} \right|_{x=\hat{x}} \leq 0$$

where $\hat{x}$ is maximizes (2.2.1). Thus

$$\left. \frac{\partial g(x)}{\partial x_j} \right|_{x_j=\hat{x}_j} = - [2 q_j + 2 a'_j \hat{x}] \leq 0 \quad \forall j = 1, \dots, p,$$

so

$$\left. \frac{\partial g(x)}{\partial x} \right|_{x=\hat{x}} = - 2 q - 2 A \hat{x} \leq 0,$$

and therefore

$$- [2 q_j + 2 a'_j \hat{x}] \leq 0 \quad \forall j = 1, \dots, p$$

Thus

$$q_j + a'_j \hat{x} \geq 0 \quad \forall j = 1, \dots, p.$$

Furthermore, we have

$$(ii) \quad \hat{x}_j \cdot \frac{\partial g(\underline{x})}{x_j} \Big|_{\underline{x}=\hat{\underline{x}}} = 0 \quad \text{at } \underline{x} = \hat{\underline{x}}$$

so

$$\hat{x}_j [- 2 (q_j + \underline{a}_j' \hat{\underline{x}})] = 0$$

Then if

$$q_j + \underline{a}_j' \hat{\underline{x}} > 0$$

then

$$\hat{x}_j = 0 \quad \forall j=1, \dots, p.$$

Therefore the proof is complete.

Lemma 2.2.5 If $\underline{x}$ maximizes (2.2.1), then

$$g(\hat{\underline{x}}) = \hat{\underline{x}}' \underline{A} \hat{\underline{x}}.$$

PROOF: From Lemma 2.2.4, if

$$q_j + \underline{a}_j' \hat{\underline{x}} > 0$$

this implies

$$\hat{x}_j (q_j + \underline{a}_j' \hat{\underline{x}}) = 0,$$

and if

$$q_j + a_j' \hat{x} = 0$$

this implies

$$\hat{x}_j (q_j + a_j' \hat{x}) = 0 .$$

Therefore,

$$\sum_{j=1}^p \hat{x}_j (q_j + a_j' \hat{x}) = 0 .$$

It follows that

$$q' \hat{x} + \hat{x}' A \hat{x} = 0 .$$

This implies

$$q' \hat{x} = -\hat{x}' A \hat{x} .$$

Therefore the proof is complete.

Now we show that the Kudo's statistic can be simplified as in the following theorem.

Theorem 2.2.5 If $\underline{\mu}^* \in \{ \underline{\mu} : \mu_i \geq 0 , \text{ for all } i=1, \dots, p \}$ and satisfies the following equation,

$$(\bar{X} - \underline{\mu}^*)' \Sigma^{-1} (\bar{X} - \underline{\mu}^*) = \min_{\mu_i \geq 0 \forall i} (\bar{X} - \underline{\mu})' \Sigma^{-1} (\bar{X} - \underline{\mu}) ,$$

then the Kudo's statistic can be simplified as

$$\bar{\chi}^2 = n \underline{\mu}^{*'} \Sigma^{-1} \underline{\mu}^* .$$

PROOF: The value $\underline{\mu}^*$, which minimizing the following expression

$$\min_{\mu_i \geq 0 \forall i} (\bar{X} - \underline{\mu})' \Sigma^{-1} (\bar{X} - \underline{\mu})$$

is equivalent to the one which minimizing the expression

$$\min_{\mu_i \geq 0 \forall i} \{ -2 \bar{X}' \Sigma^{-1} \underline{\mu} + \underline{\mu}' \Sigma^{-1} \underline{\mu} \} .$$

By applying the Lemma 2.2.5, we let $\underline{q}' = -\bar{X}' \Sigma^{-1}$ and $A = \Sigma^{-1}$. Then the desired result follows immediately.

Theorem 2.2.6 The equivalent form of the Kudo's LRT test statistic is

$$\tau_k(\underline{\mu}^*) = n \underline{\mu}^{*'} \Sigma^{-1} \underline{\mu}^*$$

with decision rule

$$\phi_k(\tau_k) = \begin{cases} 1 & \text{if } \tau_k > k \\ 0 & \text{otherwise} \end{cases}$$

PROOF: This is a straightforward application of the previous theorem.

Now we consider the following simplified form of Kudo's test statistic.

(2-a) Kudo's statistic:

$$\tau_k = n \underline{\mu}^{*'} \Sigma^{-1} \underline{\mu}^*$$

where $\underline{\mu}^* \in \{ \underline{\mu} : \mu_i \geq 0, \text{ for all } i = 1, \dots, p \}$ and minimizes the following equation,

$$\min_{\mu_i \geq 0 \forall i} (\bar{X} - \underline{\mu})' \Sigma^{-1} (\bar{X} - \underline{\mu}) .$$

We also consider a special version of the Kudo's test statistic with a different restriction imposed on the alternative hypotheses.

(2-b) UMPT test statistic:

$$\tau_u = n \underline{\mu}^{**'} \Sigma^{-1} \underline{\mu}^{**}$$

where $\underline{\mu}^{**} \in \{ \underline{\mu} : \mu_1 \geq 0, \mu_i = 0, \text{ for all } i \neq 1 \}$ and minimizes the following equation,

$$\min_{\substack{\mu_1 \geq 0 \\ \mu_i = 0, \forall i \neq 1}} (\bar{X} - \underline{\mu})' \Sigma^{-1} (\bar{X} - \underline{\mu}) .$$

Theorem 2.2.7 The UMPT test in (2-b) above is the uniformly most powerful test for testing $H_0 : \underline{\mu} = \underline{0}$ verse $H_a : \mu_1 > 0$ and $\mu_2 = \dots = \mu_p = 0$, under the assumption of multivariate normality.

PROOF: It is sufficient to show that the test τ_u can be put in the form of the test $\tau_1(\underline{Z})$ in Theorem 2.2.1 when $a_1 = 1$ and $a_2 = \dots = a_p = 0$. Now to find the LRT, we need to find the test statistic given by the likelihood ratio

$$\lambda(\underline{x}) = \frac{\max \{ P(\underline{x}, \underline{\mu}, \Sigma) : \underline{\mu} \in \Omega_1 \}}{\max \{ P(\underline{x}, \underline{\mu}, \Sigma) : \underline{\mu} \in \Omega_0 \}}$$

where $\Omega_1 = \{ \underline{\mu} : \mu_1 > 0, \mu_2 = \dots = \mu_p = 0 \}$ and $\Omega_0 = \{ \underline{\mu} : \underline{\mu} = \underline{0} \}$. Under the normality assumption, the likelihood function is given by

$$\begin{aligned} \prod_{i=1}^n p(\underline{x}_i, \underline{\mu}, \Sigma) &= \prod_{i=1}^n (2\pi)^{-\frac{p}{2}} |\Sigma|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (\underline{x}_i - \underline{\mu})' \Sigma^{-1} (\underline{x}_i - \underline{\mu}) \right\} \\ &= (2\pi)^{-\frac{np}{2}} |\Sigma|^{-\frac{n}{2}} \exp \left\{ -\frac{1}{2} \sum_{i=1}^n (\underline{x}_i - \underline{\mu})' \Sigma^{-1} (\underline{x}_i - \underline{\mu}) \right\}. \end{aligned}$$

Now we will find the maxima of $p(\underline{x}, \underline{\mu}, \Sigma)$ with respect to $\underline{\mu} \in \Omega_1$ and $\underline{\mu} \in \Omega_0$ respectively. It is obvious

$$\max \{ p(\underline{x}, \underline{\mu}, \Sigma) : \underline{\mu} \in \Omega_0 \} = (2\pi)^{-\frac{np}{2}} |\Sigma|^{-\frac{n}{2}} \exp \left\{ -\frac{1}{2} \sum_{i=1}^n \underline{x}_i' \Sigma^{-1} \underline{x}_i \right\}.$$

Let

$$\bar{x} = [\bar{x}_1 : \bar{x}_2']'$$

and

$$\Sigma^{-1} = \begin{bmatrix} \sigma^{11} & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22} \end{bmatrix}$$

where x_1 is a scalar, x_2 is a vector of $1 \times (p-1)$, σ^{11} is a scalar, and Σ^{12} is a vector of $1 \times (p-1)$. It is easy to show that the m.l.e. of $\underline{\mu}$ under H_a is $\underline{\mu} = (\mu_1, 0, \dots, 0)$ and the m.l.e of μ_1 is $\mu_1 = \max \{ x_1 + (\Sigma^{12}/\sigma^{11}) \cdot x_2, 0 \}$. Thus

$$\max\{p(X, \underline{\mu}, \Sigma, \underline{\mu} \in \Omega_1)\} = (2\pi)^{-\frac{np}{2}} |\Sigma|^{-\frac{n}{2}} \exp \left\{ \frac{-1}{2} \sum_{i=1}^n (X_i - \underline{\mu})' \Sigma^{-1} (X_i - \underline{\mu}) \right\}$$

Then after algebraic manipulations, we obtain

$$\lambda(x) = \begin{cases} 1 & \text{if } \hat{\mu}_1 = 0 \\ e^{n\sigma^{11}\hat{\mu}_1^2} & \text{if } \hat{\mu}_1 > 0 \end{cases}.$$

This is a monotone function of μ_1 . For simplicity, let $\hat{\mu}_1^* = \sigma^{11} \hat{\mu}_1$. Then the LRT test has the form given by

$$\tau_u(\hat{\mu}_1^*) = \begin{cases} 1 & \text{if } \hat{\mu}_1^* > k \\ 0 & \text{otherwise} \end{cases}.$$

Since

$$\bar{X} = \frac{1}{\sqrt{n}} \text{Diag}(\sqrt{\sigma_{11}}, \dots, \sqrt{\sigma_{pp}}) Z' ,$$

so

$$\begin{aligned} \beta_1^* &= \sigma^{11} \bar{X}_1 + \Sigma^{12} \bar{X}_2 = \frac{1}{\sqrt{n}} [\sigma^{11}, \Sigma^{12}] \text{Diag}(\sqrt{\sigma_{11}}, \dots, \sqrt{\sigma_{pp}}) Z \\ &= \frac{1}{\sqrt{n}} (1, 0, \dots, 0) \text{Diag}\left(\frac{1}{\sqrt{\sigma_{11}}}, \dots, \frac{1}{\sqrt{\sigma_{pp}}}\right) R^{-1} Z \\ &= a^{*'} R^{-1} Z . \end{aligned}$$

Thus, the LRT has the same form as that of the UMP test. The proof is complete.

For completeness, we also consider two well-known parametric omnibus test statistics as follows.

(3) χ^2 statistic:

$$\chi^2 = n \bar{X}' \Sigma^{-1} \bar{X}$$

where Σ^{-1} is the inverse of covariance matrix of X , and $\bar{X}$ is defined as before.

(4) Hotelling T^2 statistics:

$$T^2 = n \bar{X}' S^{-1} \bar{X}$$

where $\bar{X}$ is defined as above, and S^{-1} is the inverse of matrix S , where S is the sample covariance matrix of $p \times p$.

2-2-2 Nonparametric Statistics based on Ranks

There are two types of rank test statistics in which we are interested.

(1) Weighted Wilcoxon rank type

Let $\underline{W}$ be given as in (2) of Section 2-1. The weighted $\underline{W}$ type are given by

$$(1-a) \tau_3(\underline{W}) = \underline{a}' \underline{W}_n$$

and

$$(1-b) \tau_4(\underline{W}) = \underline{j}' \underline{V}^{-1} \underline{W}_n$$

where $\underline{a}$ is a known column vector of constants and $\underline{W}_n$ is the column vector of the Wilcoxon signed-rank statistics. The statistic $\tau_3(\underline{W})$ and $\tau_4(\underline{W})$ are linear combinations of coordinate Wilcoxon rank sum statistic.

(2) Kudo's rank type

We consider the following two tests as signed rank versions of Kudo's tests. That is, we use signed ranks instead of original observations to apply Kudo's LRT test statistics.

(2-a) Kudo's rank statistics:

$$\tau_{kr} = n \underline{\omega}^{*'} \underline{V}^{-1} \underline{\omega}^*$$

where $\underline{\omega}^* \in \{ \underline{\omega} : \omega_i \geq 0, \text{ for all } i=1, \dots, p \}$ and minimizes the following equation,

$$\min_{\omega_i \geq 0 \forall i} (\bar{W} - \underline{\omega})' V^{-1} (\bar{W} - \underline{\omega}) .$$

(2-b) UMPT rank statistic:

$$\tau_{ur} = n \underline{\omega}^{**'} V^{-1} \underline{\omega}^{**}$$

where $\underline{\omega}^{**} \in \{ \underline{\omega} : \omega_1 \geq 0, \omega_i = 0, \text{ for all } i \neq 1 \}$ and minimizes the following equation,

$$\min_{\substack{\omega_1 \geq 0 \\ \omega_i = 0, \forall i \neq 1}} (\bar{W} - \underline{\omega})' V^{-1} (\bar{W} - \underline{\omega}) .$$

2-3 Properties of Population Distribution

As we pointed out in the Chapter 1, the multivariate student t distribution plays an important role in our simulation study. We continue with a discussion of distributional properties of a multivariate student t random vector.

Lemma 2.3.1 If $\underline{X}$ is $MVT_p(\underline{\mu}, \underline{\Sigma}, \nu)$ with $\nu \geq 3$, then $E(\underline{X}) = \underline{\mu}$ and $Var(\underline{X}) = [\nu / (\nu - 2)] \cdot \underline{\Sigma}$.

PROOF: It is straightforward and can be shown by using the definition.

Lemma 2.3.2 If $\underline{X}$ is $MVT_p(\underline{\mu}, \underline{\Sigma}, \nu)$ and let $\underline{T} = \underline{A} \underline{X} + \underline{b}$, where $\underline{A}$ is a $q \times p$ matrix of rank q and $\underline{b}$ is a $q \times 1$ vector of constants. Then $\underline{T}$ is $MVT_q(\underline{A}\underline{\mu} + \underline{b}, \underline{A}\underline{\Sigma}\underline{A}', \nu)$.

PROOF: By Definition 1.3.2

$$\underline{X} = \frac{\sqrt{\nu} \underline{Z}}{\sqrt{Y}} + \underline{\mu} \quad (2.3.1)$$

where $\underline{Z}$ is $N_p(0, \underline{\Sigma})$, Y is $\chi^2(\nu)$ and $\underline{Z}$, Y are independent.

We substitute the right-hand side of (2.3.1) into the equation below for $\underline{X}$,

$$\underline{T} = \underline{A} \underline{X} + \underline{b}.$$

Then

$$T = A \left\{ \frac{\sqrt{v} Z}{\sqrt{Y}} + \mu \right\} + b$$

$$= \frac{\sqrt{v} [AZ]}{\sqrt{Y}} + [A\mu + b]$$

Hence AZ is $N_q(0, A\Sigma A')$ and by Definition 1.3.2, the proof is complete.

Lemma 2.3.3 If $\underline{X}$ is $MVT_p(0, \Sigma, v)$, then $\underline{X} \Sigma^{*-1} \underline{X}$ is a multiple central F distribution where $\Sigma^* = [v/(v-2)] \cdot \Sigma$.

PROOF: By Definition 1.3.2, we have

$$\underline{X} = \frac{\sqrt{v} Z}{\sqrt{Y}}$$

and

$$\underline{X}' \Sigma^{*-1} \underline{X} = \frac{\underline{Z}'}{\sqrt{\frac{Y}{v}}} \left[\frac{v}{v-2} \Sigma \right]^{-1} \frac{\underline{Z}}{\sqrt{\frac{Y}{v}}}$$

$$= \frac{p \cdot (v-2)}{v} \cdot \left[\frac{\frac{Z' \Sigma^{-1} Z}{p}}{\frac{Y}{v}} \right]$$

It is straightforward to see that the random variable $\underline{X} \Sigma^{*-1}$
 $\underline{X}$ is a multiple F distribution with degrees of freedom p and
 v , respectively, and the multiple is $p(v-2)/v$.

Chapter 3

Comparison of Tests of Tailored Alternatives to Tests of Omnibus Alternatives

3-1 Introduction

With the discussion of the properties of the test statistics and hypotheses in the last chapter, now we will investigate how well the tests of tailored and omnibus alternatives perform through a Monte Carlo study.

3-2 Purposes of the simulation

The purpose of the following study is to compare the power of tests for tailored alternatives to that of tests for omnibus alternatives. Factors that may affect power comparison are investigated. Included among these factors are: number of variates, sample size, distribution, and correlation.

3-2-1 Description of the simulation

For all the simulation work, we use the GAUSS program and the DPROQ routine in the International Mathematical and Statistical Library. One thousand random samples of size n were generated from the underlying p -variate distribution F with various values of ρ in an equi-correlation structure and various H_a . An equi-correlation structure is the one in which $\rho_{ij} = \rho$ for all $i \neq j$.

We consider the following factors, each with two levels:

- (1) Number of variates, $p = 2$ and 8 .
- (2) Sample size, $n = 10$ and 20 .
- (3) Underlying distribution, F , p -variate normal and t distribution with 3 degrees of freedom.
- (4) Magnitude of correlation between pair of variates, $\rho = 0.2$ and 0.8 .
- (5) Alternative hypothesis H_a , (I) one in which only a single coordinate mean changes and (II) one in which all coordinate means change the same magnitude.

We randomly selected sixteen different random seeds to initiate the simulation. Each one of them was then assigned to an individual case of $1/2$ of 2^5 factorial arrangement of the levels of the factors above. Within each arrangement we conducted the power study for all 9 tests (including the χ^2 test) using the same random number generator. Thus, the design is a split-plot, repeated measures, type of

experimental design. We have 144 replicates totally. For each replicate, we generated 1000 sets of data for each parameter configuration and then computed the empirical power for each of the test procedures using .05 critical points. The exact distributions under H_0 of the weighted $\underline{Z}$ tests, Hotelling T^2 and χ^2 test statistics with normality assumption are well known results. In addition, the exact distribution of the χ^2 statistic under the multivariate t assumption is as a multiple of the F distribution. Therefore, we obtained the critical values for those test statistics directly from the standard tables. For the rest of the statistics we considered, we acquired the critical value of each test using the simulation of 5000 replicates under the null hypothesis. While we could have obtained critical values from Kudo's test analytically under the normal distribution, numerical problems were of sufficient complexity that we also simulated the critical values for this test.

3-2-2 Data Analysis

To compare power functions, we used the method proposed by Higgins and McNulty (1987). Let $\theta_1, \dots, \theta_k$ be a set of values of the parameter. For each of these values θ_i , the power of size α tests S and T are estimated by simulation, resulting in a set of estimates $(\theta_1, \pi_{S1}), \dots, (\theta_k, \pi_{Sk})$ and $(\theta_1, \pi_{T1}), \dots, (\theta_k, \pi_{Tk})$. Let S denotes a standard test and T denotes the competing test. A smooth estimate of the power function of each test is obtained by fitting a cubic polynomial to the k estimated power values. Let P_S and P_T denote the fitted power functions. The average gain in power of the competitor T over the standard S is then measured by computing the average area between the two estimated power functions over a specific range; that is, we define the average gain by the following integral

$$Avg_{power} = \frac{1}{k-h} \int_h^k \{ P_T(\theta) - P_S(\theta) \} d\theta$$

where h and k are chosen such that $P_S(h)=.1$ and $P_S(k)=.9$ using the guideline in Higgins and McNulty (1987). These values result in evaluation of the average power gain over the range in which power functions generally change most rapidly. In this study, we use the χ^2 test as a standard test and then calculate the average power gain with respect to each one of

the other 8 tests throughout the various model configurations.

PROC GLM in SAS was applied to the data to decompose the total sum of squares into its components. Each of the 15 independent effects in the wholeplot has one degree of freedom, and each of the 16 components in the subplot has 7 degrees of freedom. Since the primary focus is the effect of the various factors on the test statistics, the subplot analysis provides the primary information of interest.

An analysis of variance table is given in Table 3.1. Sums of squares and mean squares are presented. Since the average gain in power was determined from power functions which were estimated using 1000 replications, experimental error in determining the average power gain is negligible. Thus the relevant information in determining the most important factors is contained in the sums of squares and mean squares.

Table 3.1

Decomposition of Total SS
Dependent Variable: Average Power gain

Source	DF	Type III SS	Mean Square	
Wholeplot:			% of total wholeplot sum of squares	
H	1	0.8891845	0.8891845	31.4
p	1	0.5994797	0.5994797	21.1
n	1	0.1193442	0.1193442	4.2
F	1	0.3753137	0.3753137	13.2
p	1	0.0158692	0.0158692	0.5
H*p	1	0.3151525	0.3151525	11.1
H*n	1	0.0007654	0.0007654	0.0
H*F	1	0.2196817	0.2196817	7.7
H*p	1	0.0178860	0.0178860	0.6
p*n	1	0.0669524	0.0669524	2.3
p*F	1	0.0054842	0.0054842	0.3
p*p	1	0.0435981	0.0435981	1.5
n*F	1	0.0003726	0.0003726	0.0
n*p	1	0.1700801	0.1700801	5.9
F*p	1	0.0013124	0.0013124	0.2
Subplot			% of total subplot sum of squares	
test	7	0.3792351	0.0541764	8.0
test*H	7	2.8785087	0.4112155	61.0
test*p	7	0.1947405	0.0278201	4.0
test*n	7	0.2680858	0.0382980	5.6
test*F	7	0.1162477	0.0166068	2.5
test*p	7	0.1016666	0.0145238	2.2
test*H*p	7	0.2800099	0.0400014	5.9
test*H*n	7	0.0132794	0.0018971	0.3
test*H*F	7	0.0968539	0.0138363	2.0
test*H*p	7	0.0861874	0.0123125	1.8
test*p*n	7	0.1232147	0.0176021	2.5
test*p*F	7	0.0097881	0.0013983	0.2
test*p*p	7	0.0700719	0.0100103	1.5
test*n*F	7	0.0360632	0.0051519	0.8
test*n*p	7	0.0455953	0.0065136	0.9
test*F*p	7	0.0172411	0.0024630	0.3

Note: H denotes hypothesis, p is dimension, n is sample size,
F is distribution, and p is correlation.

3-3 Power Advantage of Tests for Tailored Alternatives

To investigate the importance of matching the test with the corresponding tailored alternative, we examine the Test*H sum of squares in Table 3.1 and all sum of squares involving the Test*H term. We express each sum of squares, which are based on 7 degrees of freedom, as a percentage of the total of these sums of squares. Results are shown in Table 3.2.

Table 3.2

Decomposition of Sum of Squares
involving test*hypothesis term

Source	df	Sum of Squares	Mean squares	% of totals
test*H	7	2.8785087	0.4112155	86.0
test*H*p	7	0.2800099	0.0400014	8.3
test*H*F	7	0.0968539	0.0138363	2.8
test*H* ρ	7	0.0861874	0.0123125	2.5
test*H*n	7	0.0132794	0.0018971	0.4

Here we see that test*H contributes 86% to the total whereas the largest among the rest of the factors contributes no more than 9%. Moreover, among the mean squares of all the factors, test*H is relatively large. This suggests that if the test is matched to the appropriate hypothesis, other factors such as distribution, correlation, sample size, and dimension have relatively small effects on relative power

comparison. To further examine the implications of the analysis of variance, we present in Tables 3.4 -- 3.8, two- and three-way means of the response variable, average gain in power over the χ^2 test.

Most of the tests tailored to alternatives, except the rank version of the UMPT test under hypothesis (I), perform better than the χ^2 test and Kudo's omnibus test. Moreover, the gains in power may be substantial. For instance, for hypothesis (II) and $p = 8$ in Table 3.5, the tailored test $j'W$ has a 31.4% average gain in power over the χ^2 test and a 23.1% gain over Kudo's test. However, the tailored tests perform poorly for the alternatives for which the tests are not tailored. For instance, under hypothesis (I), the UMPT test performs well while the test $j'W$ performs considerably worse. On the contrary, the $j'W$ test performs well while the UMPT test performs relatively poorly under hypothesis (II).

Now let us consider some special cases. From the table of two-way means, Table 3.4, we see under the hypothesis (I) the UMPT tailored test performs well having an 8.4% average power gain over the χ^2 test while Kudo's test has an average power gain of 5.3% over the χ^2 test. Similarly under hypothesis (II) the tailored test $j'W$ performs well, having a 25.2% average power gain over the χ^2 test and a 17% average power gain over Kudo's test. Moreover, as Table 3.5 shows,

with dimension $p = 2$ and hypothesis (I) the UMPT test which is tailored for this hypothesis has a 6% average power gain over Kudo's test while most other tests have negative average power gain. In particular, $j'W$ test performs poorly relative to χ^2 test (21.6% less) under hypothesis (I). However, under hypothesis (II) the average power gain of Kudo's test is 10% less than that of the test $j'W$ which is tailored for hypothesis (II).

The general pattern of superiority of one test over the other carries over to the case $p = 8$ and also for the other three-way tables. However, magnitudes of advantages may be affected somewhat by the various factors. For instance, under hypothesis (II), the tailored test $j'Z$ has about a 7% advantage over Kudo's test when $p = 2$ but has about a 19% advantage when $p = 8$.

Now we will consider the parametric tests versus their nonparametric counterparts. First we note that the rank version of Kudo's test performs poorly and therefore is not recommended. In most cases the rank version of the UMPT test also performs poorly. However, the tests $j'W$ and $j'V^{-1}W$ will under appropriate conditions outperform the parametric test, but the relative performance appears to depend on whether the nonparametric tests have been correctly tailored for the alternative being tested. For example, $j'W$ performs worse than $j'Z$ under hypothesis (I) for both the normal and t

distributions. However, under hypothesis (II), for which these tests are correctly tailored, $j'Z$ performs slightly better than $j'W$ in the normal case whereas $j'W$ is substantially better than $j'Z$ under the t distribution.

Thus we conclude that using a test procedure that is tailored to the alternative will result in significant gains in power for a wide range of conditions. Using an omnibus test when it is known that a particular tailored alternative is appropriate may result in a significant loss in power both for the nonparametric and parametric tests.

Table 3.3

Average Gain in Power

One-way Mean

Test Procedures

$j'Z$	UMPT	$j'W$	$j'V^{-1}W$	UMPT(R)	K(O)	T^2	K(R)
-.052	-.048	-.097	.015	-.108	.068	-.058	-.065

Note: $j'Z$: Weighted Z type, defined as (1-a) in Section 2-2-1 when $\underline{a} = j$
UMPT : UMP tailored test, defined as (2-b) in Section 2-2-1
 $j'W$: Weighted W type, defined as (1-a) in Section 2-2-2 when $\underline{a} = j$
 $j'V^{-1}W$: Weighted W type, defined as (1-b) in Section 2-2-2
UMPT(R) : Rank version of UMPT, defined as (2-b) in Section 2-2-2
K(O) : Kudo's statistics, defined as (2-a) in Section 2-2-1
 T^2 : Hotelling T^2 , defined as (4) in Section 2-2-1
K(R) : Rank version of Kudo's, defined as (2-a) in Section 2-2-2

Table 3.4

Average Gain in Power

Two-way Means

Test by Hypothesis

H_a	$j'Z$	UMPT	$j'W$	$j'V^{-1}W$	UMPT(R)	K(O)	T^2	K(R)
I	-.316	.084	-.446	-.116	-.064	.053	-.058	-.151
II	.212	-.180	.252	.145	-.153	.084	-.059	-.021

Test by Dimension

p	$j'Z$	UMPT	$j'W$	$j'V^{-1}W$	UMPT(R)	K(O)	T^2	K(R)
2	-.018	.013	-.013	.050	.017	.086	.002	.065
8	-.085	-.109	-.181	-.020	-.234	.049	-.119	-.194

Test by Sample Size

n	$j'Z$	UMPT	$j'W$	$j'V^{-1}W$	UMPT(R)	K(O)	T^2	K(R)
10	-.075	-.002	-.112	-.080	-.117	.075	-.134	-.145
20	-.028	-.094	-.082	.110	-.100	.061	.017	.015

Test by Distribution

F	$j'Z$	UMPT	$j'W$	$j'V^{-1}W$	UMPT(R)	K(O)	T^2	K(R)
N	-.039	-.114	-.169	-.045	-.187	.042	-.117	-.149
T	-.064	.019	-.025	.075	-.030	.094	.001	.019

Test by Correlation

ρ	$j'Z$	UMPT	$j'W$	$j'V^{-1}W$	UMPT(R)	K(O)	T^2	K(R)
.2	.012	-.021	-.058	.005	-.119	.081	-.081	-.076
.8	-.116	-.075	-.136	.024	-.097	.055	-.036	-.054

Table 3.5

Average Gain in Power

Three-way Means
Test by Hypothesis by Dimension

$H_a = I$		Test Procedures						
p	$j'Z$	UMPT	$j'W$	$j'V^{-1}W$	UMPT(R)	K(O)	T^2	K(R)
2	-.188	.150	-.216	-.066	.110	.091	.009	.039
8	-.444	.018	-.675	-.165	-.240	.016	-.124	-.340

$H_a = II$		Test Procedures						
p	$j'Z$	UMPT	$j'W$	$j'V^{-1}W$	UMPT(R)	K(O)	T^2	K(R)
2	.151	-.123	.189	.166	-.079	.082	-.004	.089
8	.273	-.237	.314	.124	-.226	.083	-.113	-.047

Table 3.6

Average Gain in Power

Three-way Means
Test by Hypothesis by Correlation

$H_a = I$		Test Procedures						
ρ	$j'Z$	UMPT	$j'W$	$j'V^{-1}W$	UMPT(R)	K(O)	T^2	K(R)
.2	-.188	.101	-.360	-.116	-.076	.059	.088	-.160
.8	-.444	.067	-.531	-.115	-.052	.048	-.027	-.141

$H_a = II$		Test Procedures						
ρ	$j'Z$	UMPT	$j'W$	$j'V^{-1}W$	UMPT(R)	K(O)	T^2	K(R)
.2	.213	-.144	.244	.127	-.163	.103	-.073	.009
.8	.212	-.217	.259	.164	-.143	.063	-.044	-.033

Table 3.7

Average Gain in Power
Three-way Means
Test by Hypothesis by Distribution

$H_a = I$		Test Procedures						
F	$\bar{j}'Z$	UMPT	$\bar{j}'W$	$\bar{j}'V^{-1}W$	UMPT(R)	K(O)	T^2	K(R)
N	-.254	.032	-.484	-.105	-.092	.012	-.049	-.173
t	-.378	.136	-.407	-.126	-.036	.094	-.066	-.127

$H_a = II$		Test Procedures						
F	$\bar{j}'Z$	UMPT	$\bar{j}'W$	$\bar{j}'V^{-1}W$	UMPT(R)	K(O)	T^2	K(R)
N	.175	-.262	.146	.016	-.281	.072	-.186	-.124
t	.249	-.098	.357	.275	-.024	.093	.068	-.166

Table 3.8

Average Gain in Power
Three-way Means
Test by Hypothesis by Sample Size

$H_a = I$		Test Procedures						
n	$\bar{j}'Z$	UMPT	$\bar{j}'W$	$\bar{j}'V^{-1}W$	UMPT(R)	K(O)	T^2	K(R)
10	-.353	.129	-.463	-.208	-.089	.078	-.142	-.228
20	-.279	.040	-.428	-.023	-.039	.028	-.026	-.074

$H_a = II$		Test Procedures						
N	$\bar{j}'Z$	UMPT	$\bar{j}'W$	$\bar{j}'V^{-1}W$	UMPT(R)	K(O)	T^2	K(R)
10	.203	-.132	.239	.047	-.144	.072	-.126	-.062
20	.222	-.228	.264	.243	-.161	.093	.008	-.104

3-4 Power Advantage of Tests for Omnibus Alternatives

Suppose alternatives are of the form $\mu_i = a_i\theta$ where a_i 's are not known, but regarded as being selected at random prior to experimentation. In such a case, a specific test statistic tailored to a fixed set of a_i ' might not be expected to do well whereas an omnibus test might, on average, perform better. By examining the power gains averaged over the two alternatives in this study, we can investigate the potential advantage of omnibus tests. Averaging in this way is equivalent to assuming that with probability $1/2$ the alternative is either H_I or H_{II} .

We begin by examining sums of squares associated with Test and all factors excluding H. Clearly, the test statistic will have an effect on power and additionally other factors such as N and P and to a lesser extent F and R will affect the power when averaged over these two sets of alternatives. To be more specific we examine one way and two way tables, Tables 3.3 to 3.4, involving the test statistics. Here we find that Kudo's test performs well in relationship to the other tests. For instances, as Table 3.4 shows, Kudo's test has the largest power gain among those tests for both cases of $p = 2$ (8.6%) and $p = 8$ (4.9%).

Moreover, among all the tests, on average, Kudo's test has the largest power gain for both cases of the normal (4.2%) and the t distribution (9.4%). It is also interesting to note that many of the test statistics perform poorly relative to the χ^2 test as indicated by the negative average power gains whereas Kudo's test always has positive gains in power over the χ^2 test.

Thus Kudo's test shows promise when alternatives are not tailored. However, problems may limit the usefulness of Kudo's procedure. As we mentioned before, Kudo's test is based on normal theory and requires a known correlation. Moreover, the computation of the test statistic requires quadratic programming to carry it out.

3-5 General Factors Affecting both Tailored and Omnibus Tests and Conclusion

Finally, we look briefly at the factors affecting the performance of the tests overall. From the analysis of variance table (3.1) we see that again the hypothesis is the dominant factor. For instance, the term H and $H \cdot P$, $H \cdot F$ make up more than 50% of the whole plot total sum of squares. This again suggests that there are significant gains to be made in multivariate testing by taking advantage of prior knowledge of how treatments may affect mean responses.

The remainder of the study will focus on tailored alternatives.

Chapter 4

Asymptotic Relative Efficiency and Robustness of Test Procedures

4-1 Introduction

In this chapter we consider large sample properties of tests for tailored alternatives. We consider two aspects of asymptotic relative efficiency (ARE):

(1) ARE of the parametric tailored tests to their nonparametric counterparts,

(2) ARE of tailored nonparametric tests with respect to different weight functions.

By finding the ARE in (1) above, we are able to compare the power performance between parametric tests and their nonparametric counterparts under different probability models and parameter configurations. For a multivariate case, unlike the univariate case where the ARE of Wilcoxon relative to t depends on distribution only, other factors such as dimension, covariance structure and the correlation may affect ARE as well. In (2) above, we investigate the sensitivity of tests of tailored alternatives to model misspecification.

4-2 Definition of efficacy of the test and asymptotic relative efficiency of two competing tests

Before we define the efficacy of the test, we need the asymptotic theory of multivariate vectors consisting of t statistics and Wilcoxon statistics computed on each coordinate. The conditions and assumptions are as stated in Section 1-3 of Chapter 1.

4-2-1 Asymptotic Distribution of Weighted Z_n or T_n

A sequence of random variables $\{ X_n \}$ is asymptotically normal with mean $\mu_n(\theta)$ and variance $\sigma_n^2(\theta)$ if $\sigma_n(\theta) > 0$ for all n sufficiently large and

$$\frac{X_n - \mu_n(\theta)}{\sigma_n(\theta)} \xrightarrow{d} N(0,1)$$

We write " X_n is $AN(\mu_n(\theta), \sigma_n^2(\theta))$ ".

Theorem 4.2.1 (Multivariate Central Theorem) (Wald and Wolfowitz 1944) Suppose that $\underline{X}_n = (X_{n1}, X_{n2}, \dots, X_{np})$ is a sequence of i.i.d p-dimensional random vectors with $E(\underline{X}_n) = \underline{\mu}$ and $\text{Var}(\underline{X}_n) = \Sigma$. Define the sequence of random vectors

$$\bar{\underline{X}}_n = (\bar{X}_{n1}, \dots, \bar{X}_{np})$$

where

$$\bar{X}_{nj} = \frac{1}{n} \sum_{i=1}^n X_{ij}.$$

Then

$$\bar{\underline{X}}_n \text{ is } AN_p(\underline{\mu}, \frac{1}{n} \Sigma) .$$

Theorem 4.2.2 (Cramer-Wold Theorem) The random vector $\underline{X}_n$ converges in distribution to $\underline{X}$ if and only if each linear combination of the components of $\underline{X}_n$ converges in distribution to the corresponding linear combination of the components of $\underline{X}$.

Theorem 4.2.2 Let $\underline{Z}_n = (Z_{n1}, Z_{n2}, \dots, Z_{np})'$ be a sequence of random vectors, where Z_{nj} 's are defined as in the Chapter 2 for each n . Then

$$\underline{a}' \underline{Z}_n \text{ is } AN_1 \left(\sqrt{n} \sum_{i=1}^p \frac{a_i \theta_i}{\sqrt{\sigma_{ii}}}, \underline{a}' R \underline{a} \right)$$

PROOF: Let $\underline{a}'$ be a non-null vector of constants, and write

$$\underline{a}' \underline{Z}_n = \underline{a}' \text{Diag} \left(\frac{1}{\sqrt{\sigma_{11}}}, \dots, \frac{1}{\sqrt{\sigma_{pp}}} \right) \sqrt{n} \bar{\underline{X}}_n .$$

Let

$$g(\bar{\underline{X}}_n) = \underline{a}' \text{Diag} \left(\frac{1}{\sqrt{\sigma_{11}}}, \dots, \frac{1}{\sqrt{\sigma_{pp}}} \right) \bar{\underline{X}}_n = \sum_{i=1}^p \frac{a_i \bar{X}_{ni}}{\sqrt{\sigma_{ii}}} .$$

By the multivariate central limit theorem, we have

$$\bar{\underline{X}}_n \text{ is } AN_p \left(\underline{\theta}, \frac{1}{n} \Sigma \right) .$$

Applying Theorem 4.2.1, the desired result follows immediately. Thus the proof is complete.

Corollary 4.2.2 The random variable $\underline{j}' R^{-1} \underline{Z}_n$ is $AN_1(h_n(\underline{\theta}), \underline{j}' R^{-1} \underline{j})$ where

$$h_n(\underline{\theta}) = \sqrt{n} \underline{j}' \text{Diag}(\sqrt{\sigma_{11}}, \dots, \sqrt{\sigma_{pp}}) \Sigma^{-1} \underline{\theta}$$

PROOF: This is a special case of Theorem 4.2.2 when $\underline{a}' = \underline{j}' R^{-1}$.

Theorem 4.2.3. Let $T_{nj} = n^{1/2} (X_{nj}/S_{nj})$ for $j = 1, \dots, p$. Let S_{nj} denote the sample standard deviation based on the j^{th} column of X_n , and let $\underline{T}_n = (T_{n1}, T_{n2}, \dots, T_{np})'$. Then $\underline{a}' \underline{T}_n$ has the same limiting distribution as that of $\underline{a}' \underline{Z}_n$.

PROOF: Let $\underline{a}'$ be a non-null vector of constants, and write

$$\begin{aligned} \underline{a}' \underline{T}_n &= \underline{a}' \text{Diag} \left(\frac{1}{S_{n1}}, \dots, \frac{1}{S_{np}} \right) \sqrt{n} \bar{X}_n \\ &= \underline{a}' \left[\text{Diag} \left(\frac{1}{S_{n1}}, \dots, \frac{1}{S_{np}} \right) - \text{Diag} \left(\frac{1}{\sqrt{\sigma_{11}}}, \dots, \frac{1}{\sqrt{\sigma_{pp}}} \right) \right] \sqrt{n} \bar{X}_n \\ &\quad + \underline{a}' \text{Diag} \left(\frac{1}{\sqrt{\sigma_{11}}}, \dots, \frac{1}{\sqrt{\sigma_{pp}}} \right) \sqrt{n} \bar{X}_n \\ &= \underline{a}' \left[\text{Diag} \left(\frac{1}{S_{n1}}, \dots, \frac{1}{S_{np}} \right) - \text{Diag} \left(\frac{1}{\sqrt{\sigma_{11}}}, \dots, \frac{1}{\sqrt{\sigma_{pp}}} \right) \right] \sqrt{n} \bar{X}_n \\ &\quad + \underline{a}' \underline{Z}_n \end{aligned}$$

Since S_{ni} converges to $(\sigma_{ii})^{1/2}$ in probability, $1/S_{ni}$ converges

to $1/(\sigma_{ii})^{\frac{1}{2}}$ in probability. Therefore

$$[Diag(\frac{1}{s_{n1}}, \dots, \frac{1}{s_{np}}) - Diag(\frac{1}{\sqrt{\sigma_{11}}}, \dots, \frac{1}{\sqrt{\sigma_{pp}}})] \xrightarrow{P} 0$$

By applying the Slutsky Theorem (Hettmansperger 1984), $\underline{a}'\underline{T}_n$ has the same limiting distribution as that of $\underline{a}'\underline{Z}_n$. Therefore the proof is complete.

Corollary 4.2.3 The random variable $\underline{j}'\underline{T}_n$ is $AN_1(h_n(\underline{\theta}), \underline{j}'\underline{R}\underline{j})$ where

$$h_n(\underline{\theta}) = \sqrt{n} \sum_{i=1}^p \frac{\theta_i}{\sqrt{\sigma_{pp}}}$$

PROOF: This is a special case of Theorem 4.2.3 when $\underline{a} = \underline{j}$.

Theorem 4.2.4 Let γ_n denote the sample correlation matrix based on data X_n , and γ_n^{-1} denote the inverse matrix of γ_n . Then $\underline{j}'\gamma_n^{-1}\underline{T}_n$ has the same limiting distribution as that of $\underline{j}'\underline{R}^{-1}\underline{Z}_n$.

PROOF: Write

$$\underline{j}'\gamma_n^{-1}\underline{T}_n = \underline{j}'(\gamma_n^{-1} - \underline{R}^{-1})\underline{T}_n + \underline{j}'\underline{R}^{-1}\underline{T}_n$$

$$\begin{aligned}
&= j' (\gamma_n^{-1} - R^{-1}) T_n + j' R^{-1} \text{Diag} \left[\frac{1}{s_{n1}}, \dots, \frac{1}{s_{np}} \right] \sqrt{n} \bar{X}_n \\
&= j' (\gamma_n^{-1} - R^{-1}) T_n + j' R^{-1} \left\{ \text{Diag} \left[\frac{1}{s_{n1}}, \dots, \frac{1}{s_{np}} \right] \right. \\
&\quad \left. - \text{Diag} \left[\frac{1}{\sqrt{\sigma_{11}}}, \dots, \frac{1}{\sqrt{\sigma_{pp}}} \right] \right\} \sqrt{n} \bar{X}_n + j' R^{-1} \text{Diag} \left[\frac{1}{\sqrt{\sigma_{11}}}, \dots, \frac{1}{\sqrt{\sigma_{pp}}} \right] \sqrt{n} \bar{X}_n .
\end{aligned}$$

Since the first two terms on the right hand side of the above equation converge to 0 in probability, and the last term is

$$j' R^{-1} \text{Diag} \left[\frac{1}{\sqrt{\sigma_{11}}}, \dots, \frac{1}{\sqrt{\sigma_{pp}}} \right] \sqrt{n} \bar{X}_n = j' R^{-1} Z_n ,$$

the desired result follows immediately after applying Corollary 4.2.2 and the Slutsky Theorem.

Now in order to derive the asymptotic relative efficiency, we need the asymptotic distribution of the statistic $\tau_1 = \underline{a}'T_n$, under the sequence of alternatives defined by

$$\underline{\theta}_n = n^{-1/2} \underline{\theta} \quad (4.2.1)$$

where $\underline{\theta} = (\theta_1, \dots, \theta_p)'$ is a $p \times 1$ vector of real and finite constants. From theorems 4.2.2 and 4.2.3, then under (4.2.1) we have

$$\underline{a}'T_n \xrightarrow{d} N\left(\sum_{i=1}^p \frac{a_i \theta_i}{\sqrt{\sigma_{ii}}}, \underline{a}'R\underline{a}\right)$$

Similiary, from Corollary 4.2.2 and Theorem 4.2.4, then under (4.2.1) we have

$$\underline{j}'\gamma_n^{-1}T_n \xrightarrow{d} N(\underline{j}'\text{Diag}(\sigma_1, \dots, \sigma_p)\Sigma^{-1}\underline{\theta}, \underline{j}'R^{-1}\underline{j})$$

4-2-2 Asymptotic Distribution of Weighted $\underline{W}_n$

Next, we discuss the asymptotic distribution of multivariate Wilcoxon signed rank statistics. Let $R_{ij}(\mu_j)$ be the rank of $|x_{ij} - \mu_j|$ among $\{ |x_{1j} - \mu_j|, \dots, |x_{nj} - \mu_j| \}$, $\forall j = 1, \dots, p$, and let the rank statistic for the j^{th} coordinate be defined as

$$W_j^*(\mu_j) = \frac{1}{n} \sum_{i=1}^n \frac{R_{ij}(\mu_j)}{n+1} \text{Sgn}(X_{ij} - \mu_j)$$

for any fixed μ_j .

Let $\underline{W}^*(\underline{\mu})$ denote the vector $(W_1^*(\mu_1), \dots, W_p^*(\mu_p))'$ and note that $\underline{W}^*$ denote is the special case when $\underline{\mu} = \underline{0}$. Also note that $\underline{W}^* = 1/(n(n+1))\underline{W}$, where $\underline{W}$ is defined as in the chapter 2. The inclusion of constant $1/(n(n+1))$ enables us to get a result from Puri and Sen (1971) directly.

Now we find the asymptotic distribution of the vector of statistics, $\underline{W}_n^*$, under the sequence of alternatives defined by (4.2.1). We define the $p \times p$ diagonal matrix Λ with diagonal elements

$$2\lambda_j = \int_{-\infty}^{\infty} \frac{d}{dx} [2F_j(x) - 1] dF_j(x)$$

$$\begin{aligned}
&= 2 \int_{-\infty}^{\infty} f_j(x) dF_j(x) \\
&= 2 \int_{-\infty}^{\infty} f_j^2(x) dx \quad (4.2.2)
\end{aligned}$$

where $F_j(x)$ and $f_j(x)$ are defined in Chapter 1. The asymptotic distribution of $\underline{W}_n^*$ under (4.2.1) follows directly from Corollary 4.4.3.1 of Puri and Sen (1971), so we simply just state the result.

Theorem 4.2.5 Under the sequence of alternative hypotheses as (4.2.1) then $n^{1/2} \underline{W}_n^*$ has an asymptotically p-variate normal distribution with mean vector $\Lambda \underline{\theta}$ and dispersion matrix V , where

$$\Lambda \underline{\theta} = (2\theta_1 \lambda_1, \dots, 2\theta_p \lambda_p)', \quad (4.2.3)$$

λ_j is defined as (4.2.2), and the ij^{th} element of V is given by

$$\gamma_{ij} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [2F_i(x) - 1] [2F_j(y) - 1] dF_{ij}(x, y) .$$

This can be simplified as the follows:

$$V = [\gamma_{ij}] = \begin{cases} \frac{1}{3} & i=j \\ 4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_i(x) F_j(y) dF_{ij}(x, y) - 1 & i \neq j \end{cases} \quad (4.2.4) .$$

Now we can find the distribution of the proposed test statistics.

$$\text{Corollary 4.2.6} \quad \sqrt{n} \mathbf{a}' \mathbf{W}_n^* \xrightarrow{d} N(\mathbf{a}' \mathbf{\Lambda} \mathbf{\theta}, \mathbf{a}' \mathbf{V} \mathbf{a})$$

where $\mathbf{\Lambda} \mathbf{\theta}$ and $\mathbf{V}$ are defined as (4.2.3) and (4.2.4) respectively.

PROOF: By applying Theorem 4.2.2, we obtain the desired result provided $\mathbf{a}$ is not a null vector.

$$\text{Corollary 4.2.7} \quad \sqrt{n} \mathbf{j}' \mathbf{W}_n^* \xrightarrow{d} N(\mathbf{j}' \mathbf{\Lambda} \mathbf{\theta}, \mathbf{j}' \mathbf{V} \mathbf{j})$$

where $\mathbf{\Lambda} \mathbf{\theta}$ and $\mathbf{V}$ are defined as (4.2.3) and (4.2.4) respectively.

PROOF: Apply Corollary 4.2.6 when $\mathbf{a} = \mathbf{j}$.

$$\text{Corollary 4.2.8} \quad \sqrt{n} \mathbf{j}' \hat{\mathbf{V}}_n^{-1} \mathbf{W}_n^* \xrightarrow{d} N(\mathbf{j}' \mathbf{V}^{-1} \mathbf{\Lambda} \mathbf{\theta}, \mathbf{j}' \mathbf{V}^{-1} \mathbf{j})$$

where $\mathbf{\Lambda} \mathbf{\theta}$ is defined as (4.2.3), $\mathbf{V}^{-1}$ is the inverse of $\mathbf{V}$, and $\mathbf{V}_n^{-1}$ is the inverse of sample covariance of ranks $\mathbf{V}_n$ given by

$$\hat{v}_{jk} = \frac{1}{n} \sum_{i=1}^n \frac{R_{ij}}{n+1} \frac{R_{ik}}{n+1} \text{sgn}(X_{ij}) \text{sgn}(X_{ik}) \quad \forall j, k=1, \dots, p.$$

PROOF: v_{jk} converge to v_{jk} in probability for all j, k (Hettmansperger, 1984). Thus, we can write

$$\sqrt{n} j' \hat{V}_n^{-1} W_n^* = \sqrt{n} j' [\hat{V}_n^{-1} - V^{-1}] W_n^* + \sqrt{n} j' V^{-1} W_n^*.$$

Since the first term on the right hand side of the equation converges to 0 in probability, we then apply Corollary 4.2.6 and have

$$\sqrt{n} j' V^{-1} W_n^* \xrightarrow{d} N(j' V^{-1} \Lambda \theta, j' V^{-1} j).$$

By using the Slutsky Theorem, the proof is complete.

We now define the efficacy of a test. At first, we state a set of regularity conditions, which are due to Pitman and given in Randles and Wolfe (1979). Suppose that ξ_n is a statistic for testing

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_a : \theta > \theta_0$$

with critical function such that rejects H_0 if $\xi_n \geq K_n$.

1. ξ_n is a consistent test.

2. There exist sequences $\{ \mu_n(\theta) \}$ and $\{ \sigma_n(\theta) \}$ such that

$(\xi_n - \mu_n(\theta)) / \sigma_n(\theta)$ is asymptotically $n(0,1)$.

3. $\frac{d}{d\theta} \mu_n(\theta) |_{\theta=\theta_0} = \mu_n'(\theta_0)$ exists.

4. For a sequence $\{ \theta_n \}$ such that $\theta_n \rightarrow \theta_0$, as $n \rightarrow \infty$

$$\frac{\sigma_n(\theta_n)}{\sigma_n(\theta_0)} \rightarrow 1 \quad \text{and} \quad \frac{\mu_n'(\theta_n)}{\mu_n'(\theta_0)} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

$$5. \quad \frac{\mu_n'(\theta_0)}{\sqrt{n} \sigma_n(\theta_0)} \rightarrow e > 0.$$

Then the quantity e is so called the efficacy of the test procedure.

Assume $\theta_i = c_i \theta$ for all $i = 1, \dots, p$ where θ is a true parameter. Under the sequence of alternatives defined as (4.2.1), we let $\underline{\theta} = \theta \underline{c}$, where $\underline{c} = (c_1, \dots, c_p)'$

Now we derive efficacies of the tailored tests.

$$(1) \tau_1 = \underline{a}' \underline{T}_n \text{ (or } \underline{a}' \underline{Z}_n)$$

From Corollary 4.2.3, we have

$$\mu_n(\theta) = \sqrt{n} \theta \left[\sum_{i=1}^p \frac{a_i c_i}{\sqrt{\sigma_{ii}}} \right],$$

$$\sigma_n(\theta) = \{ \underline{a}' R \underline{a} \}^{\frac{1}{2}} = \left\{ \sum_{i=1}^p \sum_{j=1}^p a_i a_j \rho_{ij} \right\}^{\frac{1}{2}},$$

and

$$\mu_n'(\theta) = \sqrt{n} \left[\sum_{i=1}^p \frac{a_i c_i}{\sqrt{\sigma_{ii}}} \right].$$

It is easy to show the test satisfies these Pitman regularity conditions. Thus,

$$\begin{aligned}
e_{\underline{a}' T_n} &= \lim_{n \rightarrow \infty} \frac{\sqrt{n} \left(\sum_{i=1}^p \frac{a_i c_i}{\sqrt{\sigma_{ii}}} \right)}{\sqrt{n} \{ \underline{a}' R \underline{a} \}^{\frac{1}{2}}} \\
&= \frac{\sum_{i=1}^p \frac{a_i c_i}{\sqrt{\sigma_{ii}}}}{\{ \underline{a}' R \underline{a} \}^{\frac{1}{2}}} \\
&= \frac{\sum_{i=1}^p \frac{a_i c_i}{\sqrt{\sigma_{ii}}}}{\left(\sum_{i=1}^p \sum_{j=1}^p a_i a_j \rho_{ij} \right)^{\frac{1}{2}}} . \tag{4.2.5}
\end{aligned}$$

If $\underline{a}' = \underline{j}'$ and $\sigma_i = \sigma = 1$, then

$$e_{\underline{j}' T_n} = \frac{\sum_{i=1}^p c_i}{\left(\sum_{i=1}^p \sum_{j=1}^p \rho_{ij} \right)^{\frac{1}{2}}} \tag{4.2.6}$$

$$(2) \tau_2 = \underline{j}' r_n^{-1} \underline{T}_n \text{ (or } \underline{j}' R^{-1} \underline{Z}_n \text{)}$$

From Theorem 4.2.4 and Corollary 4.2.2, the asymptotic mean and variance of $\underline{j}' r_n^{-1} \underline{T}_n$ or $\underline{j}' R^{-1} \underline{Z}_n$ can be obtained as follows. We have

$$\mu_n(\theta) = \sqrt{n} \theta j' \text{Diag}(\sqrt{\sigma_{11}}, \dots, \sqrt{\sigma_{pp}}) \Sigma^{-1} \underline{c}$$

and $\sigma_n^2(\theta) = j' R^{-1} j$. Thus we have

$$\mu_n'(\theta) = \sqrt{n} j' \text{Diag}(\sqrt{\sigma_{11}}, \dots, \sqrt{\sigma_{pp}}) \Sigma^{-1} \underline{c}$$

and by definition of the efficacy, we have

$$\begin{aligned} e_{j' R^{-1} j}^{-1} T_n &= \lim_{n \rightarrow \infty} \frac{\sqrt{n} j' \text{Diag}(\sqrt{\sigma_{11}}, \dots, \sqrt{\sigma_{pp}}) \Sigma^{-1} \underline{c}}{\sqrt{n} \{ j' R^{-1} j \}^{\frac{1}{2}}} \\ &= \frac{j' R^{-1} \left[\frac{c_1}{\sqrt{\sigma_{11}}}, \dots, \frac{c_p}{\sqrt{\sigma_{pp}}} \right]'}{\{ j' R^{-1} j \}^{\frac{1}{2}}} . \end{aligned} \quad (4.2.7)$$

$$(3) \tau_3 = \underline{a}' \underline{W}_n$$

Under (4.2.1) and from Corollary 4.2.6, we know statistic $n^{\frac{1}{2}} \underline{a}' \underline{W}_n^*$ is asymptotically normally distributed with asymptotic mean

$$\mu_n(\theta) = \sqrt{n} \underline{a}' (2\theta_1 \lambda_1, \dots, 2\theta_p \lambda_p)'$$

$$= 2\sqrt{n}\theta \sum_{i=1}^p a_i c_i \lambda_i$$

and the asymptotic variance $\sigma_n^2(\theta) = \underline{a}'V \underline{a}$ with V defined as (4.2.4). Let

$$\Delta_{ij} = 4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_i(x) F_j(y) dF_{ij}(x, y) - 1 \quad .$$

Then

$$\sigma_n^2(\theta) = \frac{1}{3} \sum_{i=1}^p a_i^2 + 2 \sum_{i < j}^p a_i a_j \Delta_{ij} \quad .$$

Therefore

$$\mu_n'(0) = \sqrt{n} \sum_{i=1}^p 2 a_i c_i \lambda_i \quad .$$

Thus the efficacy of $\tau_3 = \underline{a}'W_n$ given by

$$\begin{aligned} e_{\underline{a}'W_n} &= \lim_{n \rightarrow \infty} \frac{\sqrt{n} \left(\sum_{i=1}^p 2 a_i c_i \lambda_i \right)}{\sqrt{n} \left\{ \frac{1}{3} \sum_{i=1}^p a_i^2 + 2 \sum_{i < j}^p a_i a_j \Delta_{ij} \right\}^{\frac{1}{2}}} \\ &= \frac{\left(\sum_{i=1}^p 2 a_i c_i \lambda_i \right)}{\left\{ \frac{1}{3} \sum_{i=1}^p a_i^2 + 2 \sum_{i < j}^p a_i a_j \Delta_{ij} \right\}^{\frac{1}{2}}} \end{aligned} \quad (4.2.8)$$

where λ_i is defined as (4.2.2) for $i=1, \dots, p$. Hence, the efficacy of the test procedure $j'W_n$, for $a = j$ becomes

$$e_{j'W_n} = \frac{\left(\sum_{i=1}^p 2 c_i \lambda_i \right)}{\left\{ \frac{1}{3} p + 2 \sum_{i < j}^p \Delta_{ij} \right\}^{\frac{1}{2}}} \quad (4.2.9)$$

$$(4) \tau_4 = j'V_n^{-1}W_n \text{ (or } j'V^{-1}W_n \text{)}$$

By Corollary 4.2.8, the statistic $j'V_n^{-1}W_n$ is asymptotically normal distributed with

$$\begin{aligned} \mu_n(\theta) &= \sqrt{n} j'V^{-1} (2\theta_1 \lambda_1, \dots, 2\theta_p \lambda_p)' \\ &= 2\sqrt{n} \sum_{i=1}^p j' Y^{(i)} \theta_i \lambda_i \\ &= 2\sqrt{n} \theta \sum_{i=1}^p j' Y^{(i)} c_i \lambda_i \end{aligned}$$

where $\underline{y}^{(i)}$ is i^{th} column of V^{-1} , the inverse matrix of V , and

$$\sigma_n^2(\theta) = j' V^{-1} j = \sum_{i=1}^p \sum_{j=1}^p \Delta^{(ij)}$$

where $\Delta^{(ij)}$ is ij^{th} entry of V^{-1} , for all i, j .

Therefore

$$\mu_n'(0) = 2\sqrt{n} \sum_{i=1}^p j' Y^{(i)} C_i \lambda_i .$$

Thus, the efficacy of $j'V_n^{-1}W_n$ given by

$$e_{j'V_n^{-1}W_n} = \frac{2 \sum_{i=1}^p j' Y^{(i)} C_i \lambda_i}{\left\{ \sum_{i=1}^p \sum_{j=1}^p \Delta^{(ij)} \right\}^{\frac{1}{2}}} . \quad (4.2.10)$$

Now we investigate the asymptotic relative efficiency for pairs of the competing tests. Let us define the asymptotic relative efficiency of test $\xi_n^{(1)}$ relative to test $\xi_n^{(2)}$.

Definition: Let $\xi_n^{(i)}$, $i = 1, 2$, provide two tests of $H_0: \theta = 0$, versus $H_a: \theta > 0$, suppose both tests satisfy conditions 1 - 5 stated in the previous section. Then the asymptotic relative efficiency of $\xi_n^{(1)}$ relative to $\xi_n^{(2)}$ is defined as

$$ARE(\xi_n^{(1)}, \xi_n^{(2)}) = \frac{e_1^2}{e_2^2}$$

where e_i is the efficacy of test $\xi_n^{(i)}$, for $i = 1, 2$.

Thus, with this definition, we are able to compute the asymptotic relative efficiency for the following pairs of tests of interest to us.

$$(1) \tau_3 = \underline{a}'\underline{W}_n \text{ vs. } \tau_1 = \underline{a}'\underline{T}_n \text{ (or } \underline{a}'\underline{Z}_n \text{)}$$

$$\begin{aligned} ARE(\underline{a}'\underline{W}_n, \underline{a}'\underline{T}_n) &= \frac{\left[\frac{(2 \sum_{i=1}^p a_i c_i \lambda_i)^2}{(\frac{1}{3} \sum_{i=1}^p a_i^2 + 2 \sum_{i < j}^p a_i a_j \Delta_{ij})} \right]}{\left[\frac{(\sum_{i=1}^p \frac{a_i c_i}{\sqrt{\sigma_{ii}}})^2}{(\sum_{i=1}^p a_i^2 + 2 \sum_{i < j}^p a_i a_j \rho_{ij})} \right]} \\ &= \left[\frac{2 \sum_{i=1}^p a_i c_i \lambda_i}{\sum_{i=1}^p \frac{a_i c_i}{\sqrt{\sigma_{ii}}}} \right]^2 \left[\frac{\sum_{i=1}^p a_i^2 + 2 \sum_{i < j}^p a_i a_j \rho_{ij}}{\frac{1}{3} \sum_{i=1}^p a_i^2 + 2 \sum_{i < j}^p a_i a_j \Delta_{ij}} \right] \end{aligned} \quad (4.2.11)$$

If $\underline{a} = \underline{j}$, then ARE of $\underline{j}'\underline{W}_n$ vs. $\underline{j}'\underline{Z}_n$ becomes

$$ARE(\underline{j}'\underline{W}_n, \underline{j}'\underline{T}_n) = \left[\frac{2 \sum_{i=1}^p c_i \lambda_i}{\sum_{i=1}^p \frac{c_i}{\sqrt{\sigma_{ii}}}} \right]^2 \left[\frac{P + 2 \sum_{i < j}^p \rho_{ij}}{\frac{1}{3} P + 2 \sum_{i < j}^p \Delta_{ij}} \right] \quad (4.2.12)$$

Now let's consider the special cases. (i) Suppose $c_i = 1$ for all $i = 1, \dots, p$ and $f_1 = f_2 = \dots = f_p = f$. Then (4.2.12) reduces to

$$ARE(j'W_n, j'T_n) = \left[\frac{2P\lambda_1}{\sum_{i=1}^p \frac{1}{\sqrt{\sigma_{ii}}}} \right]^2 \left[\frac{P + 2 \sum_{i < j}^p \rho_{ij}}{\frac{1}{3} P + 2 \sum_{i < j}^p \Delta_{ij}} \right] \quad (4.2.13)$$

Furthermore, (ii) if $p = 1$, then (4.2.13) reduces to

$$ARE(j'W_n, j'T_n) = \left[\frac{2\lambda}{\frac{1}{\sigma}} \right]^2 \left[\frac{1}{\frac{1}{3}} \right] = 12\sigma^2 \left[\int_{-\infty}^{\infty} f^2(x) dx \right]^2 \quad (4.2.14)$$

That is, the efficiency becomes that of the Mann-Whitney test with respect to the univariate Z (or T) test. (iii) If Σ and V are diagonal matrices with $f_1 = f_2 = \dots = f_p = f$, and $c_i = 1$ for $i=1, \dots, p$, then (4.2.12) reduces to

$$ARE(j'W_n, j'T_n) = \left[\frac{2P\lambda}{\frac{P}{\sigma}} \right]^2 \left[\frac{P}{\frac{P}{3}} \right] = 12\sigma^2 \left[\int_{-\infty}^{\infty} f^2(x) dx \right]^2 \quad (4.2.15)$$

Thus, the efficiency again reduces to (4.2.14). (iv) If F is a p -variate normal distribution with covariance matrix $\Sigma = \sigma^2[(1-p)I + pJ]$ and $c_i = 1$ for all $i=1, \dots, p$, and λ is defined by

$$\lambda = \int_{-\infty}^{\infty} f^2(x) dx = \frac{1}{2\sigma\sqrt{\pi}} \quad (4.2.16)$$

then

$$\begin{aligned}
ARE(j'W_n, j'Z_n) &= \left[\frac{2P\lambda}{\frac{P}{\sigma}} \right]^2 \left[\frac{P + P(P-1)\rho}{\frac{1}{3}P + P(P-1)\Delta} \right] \\
&= 12\sigma^2 \left[\frac{1}{2\sigma\sqrt{\pi}} \right]^2 \left[\frac{1 + (P-1)\rho}{1 + 3(P-1)\Delta} \right] \\
&= \frac{3[1 + (P-1)\rho]}{\pi[1 + 3(P-1)\Delta]} \quad (4.2.17)
\end{aligned}$$

$$(2)\tau_4 = j'V_n^{-1}W_n \text{ vs. } \tau_2 = j'R^{-1}Z_n \text{ (or } j'V_n^{-1}T_n \text{)}$$

From (4.2.7) and (4.2.10), we have

$$\begin{aligned}
ARE(j'\hat{V}_n^{-1}W_n, j'R^{-1}Z_n) &= \frac{\left[\frac{(2 \sum_{i=1}^p j'Y^{(i)} c_i \lambda_i)^2}{j'V^{-1}j} \right]}{\left[\frac{(j'R^{-1} [\frac{c_1}{\sqrt{\sigma_{11}}}, \dots, \frac{c_p}{\sqrt{\sigma_{pp}}}]')^2}{j'R^{-1}j} \right]} \\
&= \left[\frac{2 \sum_{i=1}^p j'Y^{(i)} c_i \lambda_i}{j'R^{-1} [\frac{c_1}{\sqrt{\sigma_{11}}}, \dots, \frac{c_p}{\sqrt{\sigma_{pp}}}]'} \right]^2 \left[\frac{j'R^{-1}j}{j'V^{-1}j} \right] \quad (4.2.18)
\end{aligned}$$

Now we consider the following special cases. (i) Suppose $c_i = 1$ for $i = 1, \dots, p$, $f_1 = f_2 = \dots = f_p = f$, and $\sigma_i = \sigma$ for $i = 1, \dots, p$. Then (4.2.18) reduces to

$$\begin{aligned}
ARE (j' \hat{V}_n^{-1} W_n, j' R^{-1} Z_n) &= \left[\frac{2 \lambda j' V^{-1} j}{\frac{1}{\sigma} j' R^{-1} j} \right]^2 \left[\frac{j' R^{-1} j}{j' V^{-1} j} \right] \\
&= \frac{4 \sigma^2 \lambda^2 [j' V^{-1} j]}{j' R^{-1} j} . \quad (4.2.19)
\end{aligned}$$

Furthermore, (ii) if F is a normal distribution, then from (4.2.16) we have

$$\lambda = \frac{1}{2 \sigma \sqrt{\pi}} .$$

Thus (4.2.18) reduces to

$$ARE (j' \hat{V}_n^{-1} W_n, j' R^{-1} Z_n) = \frac{j' V^{-1} j}{\pi [j' R^{-1} j]} . \quad (4.2.20)$$

4-3 Numerical Computation of Asymptotic Relative Efficiency

In last section, we derived the efficacies of the test procedures and the asymptotic relative efficiency of the competing tests based on asymptotic theory. In this section, we compute the ARE's for parametric vs. nonparametric tests. Specifically, we compute the ARE's of $\underline{a}'\underline{W}_n$ relative to $\underline{a}'\underline{T}_n$ for different weight functions $\underline{a}$'s, and the ARE's of the $\underline{j}'\underline{V}_n^{-1}\underline{W}_n$ relative to $\underline{j}'\underline{\gamma}_n^{-1}\underline{T}_n$. These computations are made across various parameter configurations, including dimension p from 1 to 10, correlation $\rho = 0.2$ and 0.8 , covariance structure of equi-correlation and AR(1) correlation, and the distributions normal and t .

For mathematical simplicity, we consider three different weight functions as follows: (1) uniform denoted by $\underline{j}$, (2) triangle denoted by $\underline{t}$, and (3) spike denoted by $\underline{s}$. More precisely, the uniform weight places equal weight on each coordinate. The triangle weight is one in which the weights on each coordinate are placed in the manner of an isocoles triangle. The spike weight is the one in which all the nonnegative weight is placed on a single coordinate. In the practical problem of monitoring electrical impulses on the scalp, a uniform weight would occur when all electrodes respond the same amount to a stimulus. A triangle weight could occur when the response to a stimulus is somewhat

localized with larger responses tending to occur more in one location than another. A spike response would represent a highly localized response.

First consider a comparison of $\underline{a}'\underline{W}$ to $\underline{a}'\underline{T}$ in the equi-correlation case. Efficiencies are shown in Table 4.3.1. As Table 4.3.1 indicates, efficiency decreases with increasing dimension and increases with increasing correlation. For the normal case, efficiencies range from 0.811 to 0.955 where 0.955 is the efficiency of the Wilcoxon with respect to the t in the univariate case. We see that dimension decreases the relative efficiency. The efficiency improves from the uniform case to the triangle case, and from the triangle case to the spike case as well. For instance, when $p=10$ and $\rho=0.2$, the efficiencies are 0.811 in the uniform case, 0.821 in the triangle case, and 0.927 in the spike case. As in the univariate case for a heavier-tailed distributions like the p -variate student t , the nonparametric test is substantially more efficient than the parametric test. The range of efficiencies is between 1.613 and 1.899.

Next consider a comparison of $\underline{a}'\underline{W}$ to $\underline{a}'\underline{T}$ under the AR(1) correlation structure whose efficiencies are given in Table 4.3.2. We observe the same general pattern as in Table 4.3.1 with efficiency decreasing with increasing dimension and increasing with increasing correlation for both normal and t

distributions. Under the normal distribution, the efficiency ranges from 0.698 to 0.955 while it ranges from 1.389 to 1.899 in the t distribution. Furthermore, we observe that the efficiency for each case in Table 4.3.2 is smaller (except for $p=1$) than that of the corresponding one in Table 4.3.1. This indicates that the efficiency is also affected by the correlation structure. For instance, in a 5-variate normal distribution with a uniform weight on each coordinate mean and $\rho=0.8$, the efficiency is 0.904 in the case of the AR(1) correlation structure while it is 0.910 in the equi-correlation case.

Next consider a comparison of $j'V_n^{-1}\underline{W}_n$ to $j'\gamma_n^{-1}\underline{T}_n$ in Table 4.3.3. Note that the weight functions for these tests are not the same as in Tables 4.3.1 and 4.3.2. Their weight functions are $j'V_n^{-1}$ for test $\underline{W}_n$ and $j'\gamma_n^{-1}$ for $\underline{T}_n$ respectively. We consider the uniform weight on each coordinate mean. We observe the same dependence on correlation and dimension in Table 4.3.3 as before. Under the normal distribution, the range of the efficiency is between 0.811 to 0.955 in an equi-correlation structure while it is between 0.697 to 0.955 in an AR(1) correlation structure. Moreover, under the t distribution, the range of the efficiency is between 1.613 to 1.899 in an equi-correlation structure while it is between 1.387 to 1.899 in an AR(1) correlation structure.

Table 4.3.1

Asymptotic Relative efficiency of $\underline{a}'\underline{W}_n$ to $\underline{a}'\underline{T}_n$

p-variate normal distribution with equi-correlation

$$\Sigma = \sigma^2 [(1-\rho)I + \rho J]$$

		<u>a</u> = uniform		<u>a</u> = triangle		<u>a</u> = spike	
P		$\rho=0.2$	$\rho = 0.8$	$\rho=0.2$	$\rho = 0.8$	$\rho=0.2$	$\rho = 0.8$
1		.955	.955	.	.	.	.
2		.913	.928	.	.	.	.
3		.885	.918	.893	.921	.939	.942
4		.865	.913	.873	.915	.937	.941
5		.851	.910	.861	.912	.936	.940
6		.839	.908	.849	.910	.933	.939
7		.830	.906	.841	.908	.933	.938
8		.822	.905	.833	.907	.930	.937
9		.816	.904	.827	.906	.929	.936
10		.811	.904	.821	.905	.927	.935

p-variate t distribution (df=3) with equi-correlation

$$\Sigma = \sigma^2 [(1-\rho)I + \rho J]$$

		<u>a</u> = uniform		<u>a</u> = triangle		<u>a</u> = spike	
P		$\rho=0.2$	$\rho = 0.8$	$\rho=0.2$	$\rho = 0.8$	$\rho=0.2$	$\rho = 0.8$
1		1.899	1.899	.	.	.	.
2		1.816	1.846	.	.	.	.
3		1.761	1.827	1.777	1.832	1.867	1.874
4		1.721	1.817	1.736	1.820	1.864	1.872
5		1.692	1.810	1.713	1.815	1.861	1.870
6		1.669	1.806	1.688	1.810	1.857	1.867
7		1.651	1.803	1.673	1.807	1.855	1.866
8		1.636	1.801	1.657	1.804	1.850	1.863
9		1.624	1.799	1.646	1.802	1.849	1.862
10		1.613	1.797	1.634	1.800	1.844	1.860

Table 4.3.2

Asymptotic Relative efficiency of $\underline{a}'\underline{W}_n$ to $\underline{a}'\underline{T}_n$

p-variate normal distribution with AR(1) correlation

$$\Sigma = [\sigma_{ij}] = \sigma^2 \rho^{|j-i|}$$

<u>a</u> = uniform			<u>a</u> = triangle		<u>a</u> = spike	
P	$\rho=0.2$	$\rho = 0.8$	$\rho=0.2$	$\rho = 0.8$	$\rho=0.2$	$\rho = 0.8$
1	.955	.955	.	.	.	.
2	.913	.928	.	.	.	.
3	.878	.917	.890	.920	.939	.942
4	.847	.910	.861	.913	.937	.941
5	.818	.904	.842	.909	.934	.940
6	.791	.899	.817	.904	.931	.939
7	.765	.894	.799	.901	.930	.938
8	.742	.889	.778	.897	.927	.936
9	.720	.884	.761	.894	.926	.935
10	.698	.879	.742	.890	.923	.933

p-variate t distribution with AR(1) correlation

$$\Sigma = [\sigma_{ij}] = \sigma^2 \rho^{|j-i|}$$

<u>a</u> = uniform			<u>a</u> = triangle		<u>a</u> = spike	
P	$\rho=0.2$	$\rho = 0.8$	$\rho=0.2$	$\rho = 0.8$	$\rho=0.2$	$\rho = 0.8$
1	1.899	1.899	.	.	.	.
2	1.816	1.846	.	.	.	.
3	1.747	1.824	1.771	1.831	1.867	1.874
4	1.685	1.810	1.713	1.816	1.862	1.872
5	1.627	1.798	1.674	1.808	1.858	1.870
6	1.573	1.788	1.626	1.798	1.853	1.866
7	1.522	1.778	1.590	1.792	1.850	1.865
8	1.475	1.768	1.547	1.784	1.844	1.861
9	1.431	1.758	1.514	1.777	1.842	1.860
10	1.389	1.749	1.476	1.770	1.836	1.856

Table 4.3.3

Asymptotic Relative efficiency of $j'V_n^{-1}W_n$ to $j'Y_n^{-1}T_n$
p-variate normal distribution

$\Sigma = \sigma^2 [(1-\rho)I + \rho J]$			$\Sigma = [\sigma_{ij}] = \sigma^2 \rho j-i $	
P	$\rho=0.2$	$\rho = 0.8$	$\rho=0.2$	$\rho = 0.8$
1	.955	.955	.955	.955
2	.913	.928	.913	.928
3	.885	.918	.878	.917
4	.865	.913	.846	.913
5	.851	.910	.817	.909
6	.839	.908	.790	.904
7	.830	.906	.764	.898
8	.823	.905	.740	.892
9	.816	.904	.718	.886
10	.811	.904	.697	.880

p-variate t distribution (df=3)

$\Sigma = \sigma^2 [(1-\rho)I + \rho J]$			$\Sigma = [\sigma_{ij}] = \sigma^2 \rho j-i $	
P	$\rho=0.2$	$\rho = 0.8$	$\rho=0.2$	$\rho = 0.8$
1	1.899	1.899	1.899	1.899
2	1.816	1.846	1.816	1.846
3	1.761	1.827	1.747	1.826
4	1.721	1.817	1.683	1.816
5	1.692	1.810	1.625	1.809
6	1.669	1.806	1.571	1.799
7	1.651	1.803	1.520	1.786
8	1.636	1.801	1.473	1.774
9	1.624	1.799	1.428	1.762
10	1.613	1.797	1.387	1.750

4-4 Investigation of Robustness of Test Procedure $\underline{a}'\underline{W}_n$

In the last section, we discussed the asymptotic relative efficiency of nonparametric test procedures such as $\underline{a}'\underline{W}_n$ and $\underline{j}'\underline{V}^{-1}\underline{W}_n$ with respect to their associated parametric tests. For each one of these nonparametric tests, as we have discussed, the statistic is a linear combination of coordinate Wilcoxon signed rank statistics. The right weight to impose on each coordinate statistic is dependent on knowing the correct weights in the tailored hypothesis $\mu_i = a_i\theta$. However, we may not know the weights exactly. Thus, we need to investigate how robust one procedure, for example $\underline{a}'\underline{W}_n$, is when we have misspecified the weights in the tailored hypothesis.

Assume $\theta_i = c_i\theta$ for $i = 1, \dots, p$ where θ is a true parameter. Under the sequence of alternatives defined as (4.2.1), we have $\underline{\theta}' = \underline{c}'\theta$. If we use the test procedure $\underline{a}'\underline{W}_n$, where $\underline{a} \neq \underline{0}$, then from (4.2.8) we have

$$e_{\underline{a}'\underline{W}_n} = \frac{\left(\sum_{i=1}^p 2a_i c_i \lambda_i \right)}{\left\{ \frac{1}{3} \sum_{i=1}^p a_i^2 + \sum_{i < j}^p a_i a_j \Delta_{ij} \right\}^{\frac{1}{2}}}$$

Likewise, the efficacy of test procedure $\underline{b}'\underline{W}_n$ is

$$e_{b'W_n} = \frac{\left(\sum_{i=1}^p 2b_i c_i \lambda_i \right)}{\left\{ \frac{1}{3} \sum_{i=1}^p b_i^2 + 2 \sum_{i < j}^p b_i b_j \Delta_{ij} \right\}^{\frac{1}{2}}}$$

Therefore, the asymptotic relative efficiency of $\underline{a}'W_n$ relative to $\underline{b}'W_n$ with respect to the true mean $\underline{c}'\theta$ is given,

$$ARE(\underline{a}'W_n, \underline{b}'W_n) = \left[\frac{\sum_{i=1}^p a_i c_i \lambda_i}{\sum_{i=1}^p b_i c_i \lambda_i} \right]^2 \left[\frac{\frac{1}{3} \sum_{i=1}^p b_i^2 + 2 \sum_{i < j}^p b_i b_j \Delta_{ij}}{\frac{1}{3} \sum_{i=1}^p a_i^2 + 2 \sum_{i < j}^p a_i a_j \Delta_{ij}} \right]. \quad (4.4.1)$$

If $\underline{b} = \underline{c}$ then (4.4.1) becomes

$$ARE(\underline{a}'W_n, \underline{b}'W_n) = \left[\frac{\sum_{i=1}^p a_i c_i \lambda_i}{\sum_{i=1}^p c_i^2 \lambda_i} \right]^2 \left[\frac{\frac{1}{3} \sum_{i=1}^p c_i^2 + 2 \sum_{i < j}^p c_i c_j \Delta_{ij}}{\frac{1}{3} \sum_{i=1}^p a_i^2 + 2 \sum_{i < j}^p a_i a_j \Delta_{ij}} \right], \quad (4.4.2)$$

or if $\underline{a} = \underline{c}$ then (4.4.1) becomes

$$ARE(\underline{a}'W_n, \underline{b}'W_n) = \left[\frac{\sum_{i=1}^p c_i^2 \lambda_i}{\sum_{i=1}^p b_i c_i \lambda_i} \right]^2 \left[\frac{\frac{1}{3} \sum_{i=1}^p b_i^2 + 2 \sum_{i < j}^p b_i b_j \Delta_{ij}}{\frac{1}{3} \sum_{i=1}^p c_i^2 + 2 \sum_{i < j}^p c_i c_j \Delta_{ij}} \right]. \quad (4.4.3)$$

Now we consider several special cases as follows.

(i) If $f_1 = f_2 = \dots = f_p = f$ and $f_{ij} = f_{kl}$ for all i, j, k, l then (4.4.1) reduces to

$$ARE(\underline{a}'W_n, \underline{b}'W_n) = \left[\frac{\sum_{i=1}^p a_i c_i}{\sum_{i=1}^p b_i c_i} \right]^2 \left[\frac{\frac{1}{3} \sum_{i=1}^p b_i^2 + 2\Delta \sum_{i < j}^p b_i b_j}{\frac{1}{3} \sum_{i=1}^p a_i^2 + 2\Delta \sum_{i < j}^p a_i a_j} \right] \quad (4.4.4)$$

where Δ is defined as (4.2.4) without the i, j subscript. Furthermore, if (ii) $\underline{c} = \underline{1}$ (i.e uniform weight on the coordinate mean) then (4.4.1) reduces to

$$ARE(\underline{a}'W_n, \underline{b}'W_n) = \left[\frac{\sum_{i=1}^p a_i}{\sum_{i=1}^p b_i} \right]^2 \left[\frac{\frac{1}{3} \sum_{i=1}^p b_i^2 + 2\Delta \sum_{i < j}^p b_i b_j}{\frac{1}{3} \sum_{i=1}^p a_i^2 + 2\Delta \sum_{i < j}^p a_i a_j} \right]. \quad (4.4.5)$$

If (iii) Σ and V are diagonal matrices then (4.4.1) becomes

$$ARE(\underline{a}'W_n, \underline{b}'W_n) = \left[\frac{\sum_{i=1}^p a_i}{\sum_{i=1}^p b_i} \right]^2 \left[\frac{\sum_{i=1}^p b_i^2}{\sum_{i=1}^p a_i^2} \right] \quad (4.4.6)$$

In addition, if (iv) $\underline{1}'\underline{a} = \underline{1}'\underline{b}$ then (4.4.1) reduces to

$$ARE(a'W_n, b'W_n) = \frac{\sum_{i=1}^p b_i^2}{\sum_{i=1}^p a_i^2} = \frac{b' b}{a' a} \quad (4.4.7)$$

This is simply just a square of the ratio of two norms.

4-4-1 Numerical Study of the Robustness of test $a'W_n$

We investigate the robustness of nonparametric tailored tests with three different weight functions and discuss how those tests behave with respect to a different true coordinate means. As we discussed in the Section 4-3, the weight functions we consider are uniform, triangle, and spike weights. We compute the asymptotic relative efficiency with all combinations of the three weights across several different models. These include multivariate normal or student t distribution with equi-correlation or AR(1) covariance structure. For convenience, we label the uniform weight as $\underline{u}$, the triangle weight as $\underline{t}$, and the spike weight as $\underline{s}$.

We now summarize the results in Tables 4.4.1 - 4.4.3.

True mean with $\underline{u}$ weight (Table 4.4.1)

The efficiency using the test procedure with $\underline{t}$ weight increases with increasing correlation and increases from normal to a heavier-tailed t distribution. Under a 10-variate normal distribution, the range of the efficiency is between 0.818 and 0.974 in an equi-correlation structure while it is between 0.818 and 0.879 in an AR(1) correlation structure. Similarly, under a 10-variate t distribution, the range of the

efficiency is between 0.818 and 0.978 in an equi-correlation structure while it is between 0.818 and 0.889 in an AR(1) correlation structure.

The efficiency using the test procedure with s weight also increases with increasing correlation. Under a 10-variate normal distribution, the range of the efficiency is between 0.165 and 0.624 in an equi-correlation structure while it is between 0.165 and 0.476 in an AR(1) correlation structure. Similarly, under a 10-variate t distribution, the range of the efficiency is between 0.165 and 0.660 in an equi-correlation structure while it is between 0.165 and 0.511 in an AR(1) correlation structure.

True mean with t weight (Table 4.4.2)

The efficiency using the test procedure with u weight decreases with increasing correlation and increases from an equi-correlation structure to a AR(1) structure. For instance, the efficiency, under a normal case with a equi-correlation structure, is 0.729 when $\rho=0.2$ while it is 0.686 when $\rho=0.8$. The efficiency, under a normal case with $\rho=0.4$, is 0.707 in a equi-correlation structure while it is 0.802 in a AR(1) correlation structure. Under a 10-variate normal distribution, the range of the efficiency is between 0.686 and

0.818 in an equi-correlation structure while it is between 0.760 and 0.818 in an AR(1) correlation structure. Similarly, under a 10-variate t distribution, the range of the efficiency is between 0.684 and 0.818 in an equi-correlation structure while it is between 0.752 and 0.818 in an AR(1) correlation structure.

The efficiency using the test procedure with $\underline{s}$ weight also increases with increasing correlation and decreases from an equi-correlation to AR(1) correlation structure. Under a 10-variate normal distribution, the range of the efficiency is between 0.304 and 0.967 in an equi-correlation structure while it is between 0.304 and 0.816 in an AR(1) correlation structure. Similarly, under a 10-variate t distribution, the range of the efficiency is between 0.304 and 1.018 in an equi-correlation structure while it is between 0.304 and 0.868 in an AR(1) correlation structure.

True mean with $\underline{s}$ weight (Table 4.4.3)

The loss of efficiency due to using the test procedures with $\underline{u}$ and $\underline{t}$ weights are considerably large. The efficiency is as low as 0.041 and never higher than 0.304.

Table 4.4.1

Asymptotic Relative Efficiency of $\underline{a}'\underline{W}_n$ vs. $\underline{b}'\underline{W}_n$

case : (1) $\underline{a}$ = triangle vs. $\underline{b}$ = uniform

(2) $\underline{a}$ = spike vs. $\underline{b}$ = uniform

Σ : (1) equi-correlation structure

(2) AR(1) correlation structure

F : N = 10-variate normal

t = 10-variate t with df = 3

true mean: uniform weights on coordinate mean

Case	F	Σ	ρ	ARE	Case	F	Σ	ρ	ARE
1	N	1	0.0	0.818	1	t	1	0.0	0.818
			0.2	0.917				0.2	0.929
			0.4	0.946				0.4	0.953
			0.6	0.963				0.6	0.968
			0.8	0.974				0.8	0.978
2	N	1	0.0	0.165	2	t	1	0.0	0.165
			0.2	0.327				0.2	0.365
			0.4	0.436				0.4	0.472
			0.6	0.534				0.6	0.569
			0.8	0.624				0.8	0.660
1	N	2	0.0	0.818	1	t	2	0.0	0.818
			0.2	0.842				0.2	0.867
			0.4	0.834				0.4	0.856
			0.6	0.840				0.6	0.856
			0.8	0.879				0.8	0.889
2	N	2	0.0	0.165	2	t	2	0.0	0.165
			0.2	0.236				0.2	0.277
			0.4	0.281				0.4	0.320
			0.6	0.352				0.6	0.390
			0.8	0.476				0.8	0.511

Table 4.4.2

Asymptotic Relative Efficiency of $\underline{a}'\underline{W}_n$ vs. $\underline{b}'\underline{W}_n$

case : (1) $\underline{a}$ = uniform vs. $\underline{b}$ = triangle
 (2) $\underline{a}$ = spike vs. $\underline{b}$ = triangle
 Σ : (1) equi-correlation structure
 (2) AR(1) correlation structure
 F : N = 10-variate normal
 t = 10-variate t with df = 3

true mean: triangle weights on coordinate mean

Case	F	Σ	ρ	ARE	Case	F	Σ	ρ	ARE
1	N	1	0.0	0.818	1	t	1	0.0	0.818
			0.2	0.729				0.2	0.720
			0.4	0.707				0.4	0.701
			0.6	0.694				0.6	0.691
			0.8	0.686				0.8	0.684
2	N	1	0.0	0.304	2	t	1	0.0	0.304
			0.2	0.539				0.2	0.594
			0.4	0.696				0.4	0.748
			0.6	0.838				0.6	0.888
			0.8	0.967				0.8	1.018
1	N	2	0.0	0.818	1	t	2	0.0	0.818
			0.2	0.794				0.2	0.771
			0.4	0.802				0.4	0.781
			0.6	0.796				0.6	0.780
			0.8	0.760				0.8	0.752
2	N	2	0.0	0.304	2	t	2	0.0	0.304
			0.2	0.423				0.2	0.482
			0.4	0.508				0.4	0.565
			0.6	0.633				0.6	0.687
			0.8	0.816				0.8	0.868

Table 4.4.3

Asymptotic Relative Efficiency of $\underline{a}'\underline{W}_n$ vs. $\underline{b}'\underline{W}_n$

case : (1) $\underline{a}$ = uniform vs. $\underline{b}$ = spike

(2) $\underline{a}$ = triangle vs. $\underline{b}$ = spike

Σ : (1) equi-correlation structure

(2) AR(1) correlation structure

F : N = 10-variate normal

t = 10-variate t with df = 3

true mean: spike weights on coordinate mean

Case	F	Σ	ρ	ARE	Case	F	Σ	ρ	ARE
1	N	1	0.0	0.165	1	t	1	0.0	0.165
			0.2	0.083				0.2	0.074
			0.4	0.062				0.4	0.058
			0.6	0.051				0.6	0.048
			0.8	0.044				0.8	0.041
2	N	1	0.0	0.304	2	t	1	0.0	0.304
			0.2	0.172				0.2	0.156
			0.4	0.133				0.4	0.124
			0.6	0.111				0.6	0.104
			0.8	0.096				0.8	0.091
1	N	2	0.0	0.165	1	t	2	0.0	0.165
			0.2	0.115				0.2	0.098
			0.4	0.097				0.4	0.085
			0.6	0.077				0.6	0.070
			0.8	0.057				0.8	0.053
2	N	2	0.0	0.304	2	t	2	0.0	0.304
			0.2	0.219				0.2	0.192
			0.4	0.182				0.4	0.164
			0.6	0.146				0.6	0.135
			0.8	0.114				0.8	0.107

4-4-2 Conculsion

We conclude this chapter with the following comments.

(1) As one would expect, under normal distribution the loss of efficiency from using nonparametric test is negligible when p is relatively small. This loss increases with an increasing dimension. However, under a heavier-tailed distribution as t , the nonparametric test is considerably more efficient than the competing parametric test. The ARE of the nonparametric tests relative to their parametric counterparts decreases as dimension increases and increases as correlation increases. The magnitude of the efficiency decreases only slightly from the equi-correlation structure to $AR(1)$ structure.

(2) For robustness of the weighted type nonparametric tests relative to misspecification of weights, it appears that moderate misspecification will not greatly affect efficiencies. This is evident, for instance, in the small loss in efficiencies due to using triangular weights when the true model has uniform weights on the means. However, larger misspecifications in weights can cause considerable loss in efficiency. For instance, neither the tests with uniform or triangular weights did well when the true model for the means was a spike function. In spite of the fact that no single weight function will serve the researcher in all situations, we believe in many areas it is common for researchers to have

prior knowledge which would enable them to select correct or nearly correct weights for the problems they deal with.

Chapter 5

Small Sample Test Procedure

5-1 Introduction

In this chapter, we discuss permutation versions of tailored tests in the multivariate setting. We investigate a normal approximation to the permutation distribution and demonstrate how well this approximation works through a power study. We then conduct a Monte Carlo study to investigate power performance of the proposed tailored tests and investigate the effect of misspecification of weights on the power of these tests.

5-2 Permutation Tests

In this section we define a permutation test, after describing such tests with an example.

5-2-1 Example

We illustrate the details involved in conducting a permutation test by considering a specific example with a bivariate case. Suppose we randomly select a sample of size 3 from a bivariate population symmetric with respect to their location parameter $\theta' = (\theta_1, \theta_2)$. The values are $(-1.0, 0.4)$, $(-0.1, 1.5)$ and $(.2, .3)$. We wish to test $H_0 : (\theta_1, \theta_2) = (0, 0)$ against $H_1: \theta_1 > 0, \theta_2 > 0$, using a statistic $S((x_1, y_1), (x_2, y_2), (x_3, y_3))$, a function of sample. For illustration, use

$$S[(x_1, y_1), (x_2, y_2), (x_3, y_3)] = \bar{x} + \bar{y}$$

where

$$\bar{X} = \frac{1}{3} \sum_{i=1}^3 X_i \quad \text{and} \quad \bar{Y} = \frac{1}{3} \sum_{i=1}^3 Y_i$$

A list of the $2^3 = 8$ possible sets of observation and their corresponding S values is shown below

Sample value	S
(-1 , 0.4) (-.1 , 1.5) (.2 , .3)	0.4333
(-1 , 0.4) (-.1 , 1.5) (-.2 , -.3)	0.1000
(-1 , 0.4) (.1 , -1.5) (.2 , .3)	-0.5000
(1 , -0.4) (-.1 , 1.5) (.2 , .3)	0.8333
(-1 , 0.4) (.1 , -1.5) (-.2 , -.3)	-0.8333
(1 , -0.4) (-.1 , 1.5) (-.2 , -.3)	0.5000
(1 , -0.4) (.1 , -1.5) (.2 , .3)	-0.1000
(1 , -0.4) (.1 , -1.5) (-.2 , -.3)	-0.4333

The associated conditional null H_0 distribution of S , given the observation, then assigns probability $\frac{1}{n}$ to each of the values in above table. Since large values of S are indicative of the alternative H_a , the critical regions for the corresponding permutation test would contain an appropriate number of the large values. Thus, for example, an $\alpha = 0.125$ level critical region would be $C_\alpha = \{ 0.8333 \}$ and the corresponding $\alpha = 0.25$ level critical region is $C_\alpha = \{ 0.5, 0.8333 \}$. For our observed samples the value of S is 0.4333. Thus, the smallest level at which this test would reject H_0 in favor H_a is $\alpha = 0.375$.

5-2-2 Definition

Now, we can formally define the permutation test, given the observed sample, $X = (\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n)$ where $\underline{x}_i = (x_{i1}, \dots, x_{ip})$ $i=1, \dots, n$. Under the null hypothesis, we have

$$Pr \{ X_1 = (-1)^{K_1} \underline{x}_1, \dots, X_n = (-1)^{K_n} \underline{x}_n \} = (1/2)^n$$

where

$$K_i = \begin{cases} 1 & \text{w.p. } 1/2 \\ 0 & \text{w.p. } 1/2 \end{cases} \quad \forall i = 1, \dots, n$$

Let $S(\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n)$ be a selected test statistic given

the observed sample. The exact probability distribution of the statistic S then can be obtained by computing the values of S based on each one of the 2^n permuted samples. If large values of S indicate that H_a is true, the observed significance level is the fraction of the value of S among the 2^n permutation that are greater than or equal to the values of S computed from the observed sample.

5-3 Permutation Test and Normal Approximation Procedure to Permutation Distribution

In this section, we focus on the test statistics based on the linear combinations of coordinate t statistics and Wilcoxon signed rank statistics. It is easy to find the exact permutation distribution of a test statistic for small n . However, it becomes more difficult as sample size increases. For instance, if sample size is 20, one needs to permute the sample 2^{20} times to obtain the permutation distribution. Thus, we propose a test using the normal approximation to the permutation distribution. Furthermore, we demonstrate and compare these two methods through a Monte Carlo study.

5-3-1 The Univariate Case

In the univariate case if X is a random variable with cdf F which is symmetric about 0, then the sign of X is positive or negative with equal probability, $\frac{1}{2}$. Let $q(X)$ be a nonnegative function of X and let $X_1, X_2, \dots, X_n$ be the observations.

If we put

$$Y_i = S_i \cdot q(X_i) \quad (5.3.1)$$

where S_i 's are i.i.d random variables with $P(S_i=1) = P(S_i=-1) = \frac{1}{2}$. Then given $|X_i| = |x_i|$, Y_i has a two-point distribution with $\Pr[Y_i = -q(x_i)] = \Pr[Y_i = +q(x_i)] = \frac{1}{2}$.

Suppose, now that we have n independent observations $X_1, X_2, \dots, X_n$ and define $Y_1, Y_2, \dots, Y_n$ according to (5.3.1). Then $\Pr[Y_1 = y_1, \dots, Y_n = y_n \mid |X_1|=|x_1|, \dots, |X_n|=|x_n|] = (\frac{1}{2})^n$. Let

$$V(\underline{X}, \theta) = \sum_{i=1}^n Y_i \quad (5.3.2)$$

It is easy, though tedious for large n , to find the exact conditional distribution of this statistic. However, it would be useful to be able to approximate the conditional distribution of V by a normal approximation. Since conditionally V is the sum of independent random variables, such an approximation seems plausible. It is clear that conditionally

$$\begin{aligned} E(V(\underline{X}) \mid |X_1|=|x_1|, \dots, |X_n|=|x_n|) \\ &= \sum_{i=1}^n E\{S_i \cdot q(X_i) \mid q(X_1) = q(x_1), \dots, q(X_n) = q(x_n)\} \\ &= \sum_{i=1}^n \left\{ q(x_i) \cdot \frac{1}{2} + (-q(x_i)) \cdot \frac{1}{2} \right\} = 0 \end{aligned}$$

and

$$\text{Var}(V(\underline{X}) \mid |X_1|=|x_1|, \dots, |X_n|=|x_n|)$$

$$\begin{aligned}
&= \sum_{i=1}^n \text{Var} \{ S_i \cdot q(X_i) \mid |X_1|, \dots, |X_n| \} \\
&= \sum_{i=1}^n [q(X_i)]^2 \cdot \text{Var} (S_i) \\
&= \sum_{i=1}^n [q(x_i)]^2 \\
&= \sum_{i=1}^n q_i^2
\end{aligned}$$

where $q_i = q(X_i)$.

Before we show the normality of the test statistic, we state the following theorem due to Hajek and Sidak (1967).

Theorem 5.3.1 Suppose $S_1, S_2, \dots, S_n$ are i.i.d. with $E(S) = 0$ and $\text{Var}(S) = \sigma^2$. Let V be defined as

$$V = \sum_{i=1}^n q_i S_i$$

If

$$\frac{\max_i q_i^2}{\sum_{i=1}^n q_i^2} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

then

$$\frac{V}{\sqrt{\text{Var}(V)}} \xrightarrow{d} N(0, 1)$$

Now consider the following two special cases of $q(X)$, (i) $q_i = |X_i|$ and (ii) $q_i = \text{Rank}(|X_i|) = R_i$. In case (i), $V(\underline{X}) = \sum_{i=1}^n |X_i| \cdot S_i$, and in case (ii), $V(\underline{X}) = \sum_{i=1}^n R_i \cdot S_i$, where S_i is defined as (5.3.1). In case (i), since $q_i = |X_i|$ and $q_i^2 = X_i^2$, the conditional variance is

$$\text{Var}(V(\underline{X}) \mid |X_i| = |x_i|, \forall i=1, \dots, n) = \sum_{i=1}^n x_i^2$$

let

$$T^* = \frac{V}{\sqrt{\text{Var}(V)}} = \frac{\sum_{i=1}^n |x_i| \cdot S_i}{\sqrt{\sum_{i=1}^n x_i^2}} \quad (5.3.3)$$

The distribution of T^* is that of the normalized permutation test based on actual observations.

Theorem 5.3.2 The distribution function of random variable T^* , defined as in (5.3.3), converges to an $N(0,1)$ distribution function as $n \rightarrow \infty$, for almost all sequence $X_{(1)}, X_{(2)}, \dots$.

PROOF: Let $X_{(n)}^2 = \max \{ X_1^2, \dots, X_n^2 \}$. Since from Barndorff-Nielsen (1963) we have

$$\frac{X_{(n)}^2}{n} \xrightarrow{\text{as}} 0$$

From the strong law of large number

$$\frac{\sum_{i=1}^n X_i^2}{n} \xrightarrow{as} \mu^2 + \sigma^2 .$$

Thus

$$\frac{X_{(n)}^2}{\sum_{i=1}^n X_i^2} = \frac{X_{(n)}^2}{n} \cdot \frac{1}{\frac{1}{n} \sum_{i=1}^n X_i^2} \xrightarrow{as} 0$$

From Theorem 5.3.1

$$T^* \xrightarrow{d} N(0, 1)$$

for almost all sequence $X_{(1)}, X_{(2)}, \dots$.

Likewise, in case (ii) $q_i = R_i$, $V(\underline{X}) = \sum R_i \cdot S_i$. Let $W=V(\underline{X})$, then under the null hypothesis,
 $E \{ W(\underline{X}) \mid |X_1|, \dots, |X_n| \} = 0$ and

$$\begin{aligned} \text{Var} (W(\underline{X}) \mid |X_1|, \dots, |X_n|) &= \sum_{i=1}^n \text{Var} \{ S_i \cdot R_i \mid |X_1|, \dots, |X_n| \} \\ &= \sum_{i=1}^n R_i^2 \cdot \text{Var} (S_i) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^n i^2 \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

It is easy to show the condition of theorem 5.3.1 holds, since

$$\frac{\max_i R_i^2}{\sum_{i=1}^n R_i^2} = \frac{n^2}{\frac{n(n+1)(2n+1)}{6}} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

Let

$$W^* = \frac{W}{\sqrt{\frac{n(n+1)(2n+1)}{6}}}$$

Then for large n , by applying theorem 5.3.1 we have

$$W^* \xrightarrow{d} N(0, 1)$$

5-3-2 The Multivariate Case

Let us consider the multivariate version of the problem now. We denote the sample point by $X_n = (X_1, \dots, X_n)$, and $X_i = (X_{i1}, \dots, X_{ip})'$, $i = 1, \dots, n$, and the sample space by X_n . Under the null hypothesis, conditionally given X_n , each one of the all 2^n sample points has an equal conditional probability of $(\frac{1}{2})^n$.

Now, employing the notation we have used in the univariate case, we put $\underline{Y}_j = (Y_{1j}, \dots, Y_{nj})'$, where Y_{ij} is defined as in (5.3.1) for $j = 1, \dots, p$, and let $\underline{q}_j = (q_{1j}, \dots, q_{nj})'$, where $q_{ij} = q_j(X_{ij})$, $i = 1, \dots, n$, and $q_j(X)$ is a nonnegative function of X .

We consider a vector of the statistics, $\underline{V} = (V_1, \dots, V_p)'$ where V_j is defined as in (5.3.2) for $j = 1, \dots, p$. As an univariate case, conditionally on $\{ |X_{ij}|, \forall i = 1, \dots, n, \text{ and } j = 1, \dots, p. \}$, then $E \{ \underline{V} \} = \underline{0}$ and

$$\text{Var}[V_j] = \sum_{i=1}^n q_{ij}^2 \quad (5.3.4)$$

The covariance between variates V_j and V_k is

$$\text{Cov} \left[\sum_{i=1}^n S_{ij} \cdot q_{ij}, \sum_{i=1}^n S_{ik} \cdot q_{ik} \right]$$

conditionally on $\{ |X_{ij}|, \forall i = 1, \dots, n, \text{ and } j = 1, \dots, p. \}$. Under the permutation scheme, the product of the $S_{ij} \cdot S_{ik} = \text{Sgn}(x_{ij}) \cdot \text{Sgn}(x_{ik})$ remains constant within the same i and

$$S_{ij} \cdot S_{mk} = \begin{cases} 1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases} \quad \forall i \neq m$$

For fixed j and k , we then have

$$\begin{aligned} & \text{Cov} \left[\sum_{i=1}^n S_{ij} \cdot q_{ij}, \sum_{i=1}^n S_{ik} q_{ik} \mid |X_{ij}|, \forall i=1, \dots, n, j=1, \dots, p. \right] \\ &= \sum_{i=1}^n S_{ij} \cdot S_{ik} \cdot q_{ij} \cdot q_{ik} \\ &+ \sum_{i=1}^n \sum_{l \neq i} \{ S_{ij} \cdot S_{lk} q_{ij} \cdot q_{lk} \cdot \text{Pr}(S_{ij} \cdot S_{lk} = 1) \\ &+ S_{ij} \cdot S_{lk} q_{ij} \cdot q_{lk} \cdot \text{Pr}(S_{ij} \cdot S_{lk} = -1) \} \\ &= \sum_{i=1}^n \text{Sgn}(x_{ij}) \cdot \text{Sgn}(x_{ik}) \cdot q_{ij} \cdot q_{ik} \end{aligned} \quad (5.3.5)$$

Now consider the special cases. (i) If $q_{ij} = |X_{ij}|$ for $i = 1, \dots, n$ and $j = 1, \dots, p$, then from (5.3.4), we have

$$\text{Var}(V_j) = \sum_{i=1}^n x_{ij}^2 \quad \forall j = 1, \dots, p.$$

also from (5.3.5) we then have

$$\text{Cov}(V_j, V_k) = \sum_{i=1}^n x_{ij} x_{ik} \quad \forall j \neq k \text{ and } j, k = 1, \dots, p.$$

Similarly, (ii) if $q_{ij} = R(|X_{ij}|) = R_{ij}$. We can easily show that $E\{\underline{V}\} = \underline{0}$ and from (5.3.4)

$$\begin{aligned} \text{Var}\left[\sum_{i=1}^n S_{ij} \cdot R_{ij}\right] &= \sum_{i=1}^n R_{ij}^2 \\ &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

The covariance between j^{th} and k^{th} Wilcoxon signed rank statistics, then from (5.3.5)

$$\begin{aligned} \text{Cov}\left[\sum_{i=1}^n S_{ij} \cdot R_{ij}, \sum_{i=1}^n S_{ik} \cdot R_{ik} \mid |X_{ij}|, \forall i=1, \dots, n, j=1, \dots, p.\right] \\ = \sum_{i=1}^n \text{Sgn}(x_{ij}) \cdot \text{Sgn}(x_{ik}) \cdot R_{ij} \cdot R_{ik} \end{aligned}$$

$\forall j, k$ and $j, k = 1, \dots, p$.

Thus, for the multivariate case, unlike the univariate case, its $\text{Cov}(W_j, W_k)$ depends on the sample configuration. Hence the statistic based on the multivariate Wilcoxon signed ranks is not distribution free.

If we wish to use test statistics which are the linear combination of the coordinate T statistics, $\underline{a}'\underline{T}$, or coordinate Wilcoxon signed rank statistics, $\underline{a}'\underline{W}$, to test $H_0 : \theta_1 = \dots = \theta_p = 0$ against $H_a : \theta_1 = a_1\theta, \dots, \theta_p = a_p\theta$ then the conditional variances of $\underline{a}'\underline{T}$ and $\underline{a}'\underline{W}$ are given by

$$\text{Var}(\underline{a}'\underline{T} | X) = \sum_{j=1}^p a_j^2 \sum_{i=1}^n x_{ij}^2 + 2 \sum_{k=1}^p \sum_{l=1}^p a_k a_l \sum_{i=1}^n x_{ik} x_{il}$$

and

$$\text{Var}(\underline{a}'\underline{W} | R) = c \sum_{i=1}^n a_i^2 + 2 \sum_{k=1}^p \sum_{l=1}^p a_k a_l \sum_{i=1}^n \text{Sgn}(x_{ik}) \text{Sgn}(x_{il}) R_{ik} R_{il}$$

where

$$c = \frac{n(n+1)(2n+1)}{6}$$

The tests we propose using the normal approximation are as follows:

$$\text{Reject } H_0 \quad \text{if} \quad \frac{\underline{a}'\underline{T}}{\{ \text{Var}(\underline{a}'\underline{T} | X) \}^{1/2}} > z_\alpha$$

and

$$\text{Reject } H_0 \quad \text{if} \quad \frac{\underline{a}'\underline{W}}{\{ \text{Var}(\underline{a}'\underline{W} | R) \}^{1/2}} > z_\alpha$$

where z_α is the value of upper- α quantile from the standard normal distribution.

5-4 Numerical Comparison between Permutation Test and Its Normal Approximation Test

In this section we consider normal approximations to permutation tests. In particular we compare power functions of permutation tests to the power functions of tests based on normal approximations. The two classes of test compared in this study are $\underline{a}'\underline{T}$ and $\underline{a}'\underline{W}$, linear combinations of the T and W statistics, respectively. We obtained critical values for these statistics using the permutation principle (PP) and the corresponding normal approximations (NA). Power comparisons were then made for a sample of size $n = 5$ and dimension $p = 10$ under the normal and t distributions across several combinations of the parameters including the covariance structure Σ , the correlation ρ , and the weight function $\underline{a}$. These parameter combinations used are summarized in Table 5.4.1. The table provides the numbers of the tables in Appendix A containing the simulation results for the corresponding parameter configuration. Thus, Table C-2 in Appendix 1 would contain the results of the simulations using the sample size of 5 from a 10-variate t distribution with a triangle weight on the coordinate mean, and an equi-correlation structure of $\rho = 0.2$ and 0.8 .

Table 5.4.1

List of Tables of Simulation Results in Appendix A.

p = 10 n = 5		MVN			MVT		
		<u>a</u>			<u>a</u>		
Σ	ρ	u	t	s	u	t	s
EQ	0.2	A-1	A-2	A-3	C-1	C-2	C-3
EQ	0.8	A-1	A-2	A-3	C-1	C-2	C-3
AR	0.2	B-1	B-2	B-3	D-1	D-2	D-3
AR	0.8	B-1	B-2	B-3	D-1	D-2	D-3

note: a is a weight function applied to coordinate mean
u is uniform, t is triangle, and s is spike
EQ denotes the equi-correlation structure and AR
denotes AR(1) correlation structure.

Table 5.4.2 contains a summary of the maximum of the absolute difference of the simulated power between the exact permutation test and its normal approximation. This table shows that the normal approximation is generally quite good even though the sample size is quite small. The largest difference observed in the table is 0.072. In most cases the difference is less than 0.05. Thus, the power based on the normal approximation to the permutation test is comparable to that of the exact permutation test. The importance of using the normal approximation is that one can save considerable computing time in a simulation study. For a power study using the permutation test even with a moderate sample size of $n = 10$, one needs to permute the sample 2^{10} times just for a

single run of the simulation. Taking this into consideration, we decided to use the normal approximation to the permutation test in the simulation study in the next section.

Table 5.4.2

The Maximum of the Absolute Difference Between
Power Functions Using PP and NA Approaches

F	Σ	$\underline{a}$	R	Max(Abs)	
				$\underline{a}'T$	$\underline{a}'W$
N	EQ	$\underline{u}$	0.2	0.036	0.024
			0.8	0.023	0.037
		$\underline{t}$	0.2	0.026	0.035
			0.8	0.012	0.045
		$\underline{s}$	0.2	0.023	0.024
			0.8	0.034	0.027
N	AR(1)	$\underline{u}$	0.2	0.061	0.017
			0.8	0.030	0.020
		$\underline{t}$	0.2	0.072	0.064
			0.8	0.014	0.057
		$\underline{s}$	0.2	0.020	0.022
			0.8	0.052	0.054
T	EQ	$\underline{u}$	0.2	0.044	0.024
			0.8	0.034	0.026
		$\underline{t}$	0.2	0.026	0.027
			0.8	0.037	0.061
		$\underline{s}$	0.2	0.019	0.024
			0.8	0.055	0.031
T	AR(1)	$\underline{u}$	0.2	0.025	0.047
			0.8	0.009	0.053
		$\underline{t}$	0.2	0.038	0.049
			0.8	0.015	0.055
		$\underline{s}$	0.2	0.020	0.033
			0.8	0.032	0.058

5-5 A Simulation Study

The objectives of this study are essentially the same as those in Chapter 4. That is we compare power performances of the tests based on actual observation, i.e those of the $a'T$ type, to their counterparts based on ranks, i.e those of the $a'W$ type. We will also consider the effect of model misspecification in selecting the weights for the tailored tests. Here, of course, the focus will be on small samples typically found in medical research.

5-5-1 Description of the simulation

One thousand random samples of size n were generated from the underlying p -variate distribution F with various values of ρ and various covariance structures Σ .

We consider the following factors, each with two levels:

- (1) Number of variates, $p = 10$ and 20 .
- (2) Sample size, $n = 5$ and 10 .
- (3) Underlying distribution, F , a p -variate normal and t distribution with three degrees of freedom.

- (4) Magnitude of correlation between pair of variates,
 $\rho = 0.2$ and 0.8 .
- (5) Covariance structures, $\Sigma = \sigma^2 [(1-\rho)I + \rho J]$ and
 $\Sigma = [\sigma]_{ij} = \sigma^2 [\rho |j-i|]$

Like the simulation study in chapter 2, we randomly selected sixteen different random seeds to initiate the simulation. Each one of them was then assigned to an individual case of $1/2$ of 2^5 factorial arrangement of treatment levels. Within each case we then conducted the power study for all combinations of 7 tests and 3 weight functions on coordinate means. The tests are linear combinations of $\underline{T}$ and $\underline{W}$ with 3 different weight functions (uniform, triangular, and spike) and χ^2 test. The weights on coordinate mean are uniform, triangular, and spike. Thus, again this is a split-plot type of experimental design. The treatment structure for the whole plot is a $1/2$ of 2^5 factorial and a two way treatment structure (7 tests and 3 weights on coordinate mean) for the subplot.

A total of 336 separate simulation replicates were carried out. Each individual replicate consisted of a sequence of simulation runs to calculate the empirical power. We generated 1000 sets of data for each parameter configuration and then computed the empirical power for each of the test procedures using the normal approximate to the permutation distribution. We used the 0.0625 critical point

since this is the nearest critical point to 0.05 that can be achieved by the exact permutation test with $n = 5$.

5-5-2 Data Analyses

We analyzed the data using average power gain as a response variable as in Chapter 2 and using the χ^2 test as a standard test for the purpose of comparison.

PROC GLM in SAS was applied to the data to decompose the total sum of squares into its components. Each of the 15 independent effects in the wholeplot has one degree of freedom. The degrees of freedom for each of the 48 components in the subplot is shown as in Table 5.5.1. Since the focal point is the effect of the various factors on the test statistics, the subplot analysis provides the primary information of interest. An analysis of variance table is given in Table 5.5.1. Sums of squares and mean squares are presented. As described in the Chapter 2, the experimental error in determining the average power gain is negligible. Thus, the relevant information in determining the most important factor, is contained in the sums of squares and mean squares.

Table 5.5.1

Analysis of Variance

Average power gain over χ^2 test

Source	DF	Anova SS	Mean Square	

Wholeplot:				% of total wholeplot sum of squares
p	1	0.1643336	0.1643336	7.4
n	1	0.3458278	0.3458278	15.6
p	1	0.9292730	0.9292730	42.0
F	1	0.2838472	0.2838472	12.8
Σ	1	0.2468714	0.2468714	11.1
p*n	1	0.0103913	0.0103913	0.4
p*p	1	0.0828401	0.0828401	3.7
p*F	1	0.0078816	0.0078816	0.3
p* Σ	1	0.0432175	0.0432175	1.9
n*p	1	0.0005897	0.0005897	0.2
n*F	1	0.0424089	0.0424089	1.9
n* Σ	1	0.0092007	0.0092007	0.4
p*F	1	0.0001012	0.0001012	0.0
p* Σ	1	0.0505657	0.0505657	2.3
F* Σ	1	0.0004129	0.0004129	0.0

Subplot:				% of total subplot sum of squares
w	2	7.7266511	3.8633255	40.3
t	5	1.5502215	0.3100443	8.1
w*t	10	5.1773896	0.5177390	27.6
w*p	2	0.1907219	0.0953609	1.0
w*n	2	0.1668933	0.0834466	0.9
w*p	2	1.4317953	0.7158977	7.5
w*F	2	0.2102198	0.1051099	1.1
w* Σ	2	0.8714457	0.4357229	4.5
w*p*n	2	0.0014338	0.0007169	0.0
w*p*p	2	0.0998293	0.0499147	0.5
w*p*F	2	0.0035609	0.0017805	0.0
w*p* Σ	2	0.0638657	0.0319329	0.3
w*n*p	2	0.0235693	0.0117846	0.1
w*n*F	2	0.0072634	0.0036317	0.0
w*n* Σ	2	0.0292635	0.0146318	0.1
w*p*F	2	0.0355821	0.0177911	0.2
w*p* Σ	2	0.2839854	0.1419927	1.5
w*F* Σ	2	0.0152951	0.0076476	0.0
t*p	5	0.0345020	0.0069004	0.2
t*n	5	0.0125605	0.0025121	0.0
t*p	5	0.3667863	0.0733573	2.0
t*F	5	0.0425886	0.0085177	0.2
t* Σ	5	0.1544201	0.0308840	0.8
t*p*n	5	0.0106075	0.0021215	0.0

Source	DF	Anova SS	Mean Square	% of total subplot sum of squares
t*p*p	5	0.0149273	0.0029855	0.0
t*p*F	5	0.0080173	0.0016035	0.0
t*p*Σ	5	0.0189849	0.0037970	0.1
t*n*p	5	0.0392375	0.0078475	0.2
t*n*F	5	0.0347463	0.0069493	0.2
t*n*Σ	5	0.0060800	0.0012160	0.0
t*p*F	5	0.0191171	0.0038234	0.1
t*p*Σ	5	0.0326217	0.0065243	0.2
t*F*Σ	5	0.0063788	0.0012758	0.0
w*t*p	10	0.0959251	0.0095925	0.5
w*t*n	10	0.0214413	0.0021441	0.1
w*t*p	10	0.0487046	0.0048705	0.3
w*t*F	10	0.0271923	0.0027192	0.2
w*t*Σ	10	0.0139990	0.0013999	0.1
w*t*p*n	10	0.0154101	0.0015410	0.1
w*t*p*p	10	0.0116123	0.0011612	0.1
w*t*p*F	10	0.0211177	0.0021118	0.1
w*t*p*Σ	10	0.0519384	0.0051938	0.3
w*t*n*p	10	0.0779600	0.0077960	0.4
w*t*n*F	10	0.0294043	0.0029404	0.2
w*t*n*Σ	10	0.0076738	0.0007674	0.0
w*t*p*F	10	0.0321853	0.0032185	0.2
w*t*p*Σ	10	0.0095038	0.0009504	0.0
w*t*F*Σ	10	0.0076747	0.0007675	0.0

To investigate the robustness of the tests to the different weight functions, we examine the w*t (i.e. test*weight) sum of squares in Table 5.5.1 and all sum of squares involving the w*t term. We express each sum of squares, which are based on 10 degrees of freedom, as a percentage of the total of these sums of squares. Results are shown in Table 5.5.2.

Table 5.5.2

Decomposition of Sum of Squares
involving test*weight term

Source	df	Sum of Squares	Mean squares	% of totals
test*weight	10	5.1773896	0.5177389	96.2
test*weight*p	10	0.0959251	0.0095925	1.8
test*weight*n	10	0.0214413	0.0021441	0.4
test*weight*p	10	0.0487046	0.0048705	0.9
test*weight*F	10	0.0271923	0.0027192	0.5
test*weight* χ^2	10	0.0139990	0.0013999	0.2

Here we see that test*weight contributes 96.2% to the total whereas the largest among the rest of the factors contributes no more than 2%. Moreover, among the mean squares of all the factors, test*weight is relatively large. This suggests that if the test is matched to the appropriate weight function, other factors such as distribution, correlation, sample size, dimension, and covariance structure have relatively small effects on relative power comparison. To further examine the implications of the analysis of variance, we present in Tables 5.5.3 -- 5.5.8 two- and three-way means of the response variable, average gain in power over the χ^2 test. The effect of the weights on the small sample permutation tests with large dimensional variates is similar to that observed in the asymptotic study in Chapter 4. Modest missepecification of weights appears to have only small effect on the power of the tests. For instance from Table 5.5.3,

when the true weight on the coordinate means is uniform, the uniform and triangular weights on the T and W statistics all provide power gains over the χ^2 between 0.30 and 0.35. On the other hand the spike weight has a power gain of only about 0.20. The use of uniform or triangular weights on the T and W statistics is ineffective when the true weight function on the coordinate means is a spike function. Similar pattern are observed in the other tables.

Table 5.5.3

Two-way mean

Average Power Gain over χ^2 Test

test by weight

weight	<u>u'</u> T	<u>t'</u> T	<u>s'</u> T	<u>u'</u> W	<u>t'</u> W	<u>s'</u> W
<u>u</u>	0.339	0.305	0.196	0.350	0.314	0.202
<u>t</u>	0.104	0.207	0.189	0.120	0.225	0.205
<u>s</u>	-.216	-.167	0.173	-.213	-.166	0.162

test by distribution

Distribution	<u>u'</u> T	<u>t'</u> T	<u>s'</u> T	<u>u'</u> W	<u>t'</u> W	<u>s'</u> W
normal	-.028	0.012	0.081	-.029	0.009	0.058
t	.180	0.217	0.291	.200	0.239	0.320

test by correlation structure

Σ	<u>u'</u> T	<u>t'</u> T	<u>s'</u> T	<u>u'</u> W	<u>t'</u> W	<u>s'</u> W
Equi	0.021	0.069	0.157	0.030	0.079	0.171
AR(1)	0.130	0.160	0.215	0.141	0.169	0.208

test by correlation

ρ	<u>u'</u> T	<u>t'</u> T	<u>s'</u> T	<u>u'</u> W	<u>t'</u> W	<u>s'</u> W
0.2	0.170	0.217	0.247	0.177	0.224	0.257
0.8	-.019	0.013	0.125	-.006	0.025	0.122

test by sample size

n	<u>u'</u> T	<u>t'</u> T	<u>s'</u> T	<u>u'</u> W	<u>t'</u> W	<u>s'</u> W
5	0.062	0.097	0.170	0.069	0.104	0.166
10	0.090	0.132	0.201	0.102	0.145	0.213

test by dimension

P	<u>u'</u> T	<u>t'</u> T	<u>s'</u> T	<u>u'</u> W	<u>t'</u> W	<u>s'</u> W
10	0.075	0.104	0.177	0.084	0.112	0.166
20	0.077	0.126	0.195	0.087	0.136	0.213

Let us consider the comparison of tests using the actual observations and ranks. As Table 5.5.3 shows, there is no advantage of using proposed tests either with the actual observations or ranks. The largest difference been observed, in between tests $\underline{t}'\underline{T}$ and $\underline{t}'\underline{W}$, under the triangle weight, is only 0.018. Likewise, the power performances of the tests either using actual observations or ranks are the same in general as shown in the Tables of three-way means Tables 5.5.4 -- 5.5.8. For instance, the largest differences are 0.037 in Table 5.5.4, 0.043 in Table 5.5.5, 0.038 in Table 5.5.6, and 0.046 in Table 5.5.7. This suggests that there is essentially no difference in using tests based on either the actual observations or ranks for the ranges of the factors we are considering in this study.

It should be noted that rank tests will generally perform better than tests based on actual observations when samples become larger and are selected from heavy-tailed distribution such as the t distribution. This is suggested by the asymptotic relative efficiencies in Chapter 4 where the rank tests are considerably more efficient than the tests based on actual observations when samples are selected from the t distribution. It is also shown in Chapter 3 where the nonparametric tests performed better than the parametric tests under the t distribution and samples of size 10 and 20. However, whether we use actual observations or ranks, the most

important factor in gaining power over a standard test such as the χ^2 test is the weight function.

Table 5.5.4

Three-way mean

Average Power Gain over χ^2 Test

F = normal distribution

weight	<u>u'</u> T	<u>t'</u> T	<u>s'</u> T	<u>u'</u> W	<u>t'</u> W	<u>s'</u> W
<u>u</u>	0.283	0.250	0.140	0.278	0.242	0.132
<u>t</u>	-0.015	0.093	0.075	-0.013	0.094	0.069
<u>s</u>	-.353	-.307	0.027	-.351	-.308	-.026

F = t distribution

weight	<u>u'</u> T	<u>t'</u> T	<u>s'</u> T	<u>u'</u> W	<u>t'</u> W	<u>s'</u> W
<u>u</u>	0.394	0.359	0.251	0.423	0.387	0.272
<u>t</u>	0.223	0.321	0.303	0.253	0.356	0.340
<u>s</u>	-.079	-.028	0.319	-.075	-.025	0.349

Table 5.5.5

Three-way mean

Average Power Gain over χ^2 Test

Σ = Equi-correlation

weight	<u>u'</u> T	<u>t'</u> T	<u>s'</u> T	<u>u'</u> W	<u>t'</u> W	<u>s'</u> W
<u>u</u>	0.322	0.313	0.235	0.332	0.322	0.240
<u>t</u>	-.012	0.099	0.110	0.004	0.119	0.126
<u>s</u>	-.247	-.206	0.126	-.244	-.203	0.147

Σ = AR(1)-correlation

weight	<u>u'</u> T	<u>t'</u> T	<u>s'</u> T	<u>u'</u> W	<u>t'</u> W	<u>s'</u> W
<u>u</u>	0.355	0.295	0.157	0.369	0.307	0.163
<u>t</u>	0.220	0.315	0.269	0.237	0.331	0.283
<u>s</u>	-.184	-.129	0.220	-.183	-.129	0.177

Table 5.5.6

Three-way mean

Average Power Gain over χ^2 Test

Sample size $n = 5$

weight	<u>u'</u> T	<u>t'</u> T	<u>s'</u> T	<u>u'</u> W	<u>t'</u> W	<u>s'</u> W
<u>u</u>	0.322	0.291	0.181	0.329	0.297	0.188
<u>t</u>	0.089	0.188	0.168	0.103	0.204	0.184
<u>s</u>	-.227	-.187	0.163	-.225	-.189	0.125

Sample size $n = 10$

weight	<u>u'</u> T	<u>t'</u> T	<u>s'</u> T	<u>u'</u> W	<u>t'</u> W	<u>s'</u> W
<u>u</u>	0.355	0.318	0.211	0.371	0.332	0.216
<u>t</u>	0.119	0.226	0.210	0.138	0.246	0.225
<u>s</u>	-.204	-.148	0.183	-.202	-.143	0.198

Table 5.5.7

Three-way mean

Average Power Gain over χ^2 Test

Dimension $p=10$

weight	<u>u'</u> T	<u>t'</u> T	<u>s'</u> T	<u>u'</u> W	<u>t'</u> W	<u>s'</u> W
<u>u</u>	0.336	0.314	0.184	0.348	0.325	0.186
<u>t</u>	0.168	0.238	0.194	0.186	0.256	0.205
<u>s</u>	-.280	-.242	0.152	-.282	-.245	0.106

Dimension $p=20$

weight	<u>u'</u> T	<u>t'</u> T	<u>s'</u> T	<u>u'</u> W	<u>t'</u> W	<u>s'</u> W
<u>u</u>	0.341	0.295	0.208	0.352	0.303	0.217
<u>t</u>	0.040	0.176	0.185	0.055	0.194	0.204
<u>s</u>	-.151	-.093	0.193	-.145	-.088	0.218

Table 5.5.8

Three-way mean

Average Power Gain over χ^2 Test

$\rho = 0.2$

weight	<u>u'</u> T	<u>t'</u> T	<u>s'</u> T	<u>u'</u> W	<u>t'</u> W	<u>s'</u> W
<u>u</u>	0.369	0.321	0.139	0.379	0.331	0.143
<u>t</u>	0.237	0.352	0.256	0.248	0.362	0.267
<u>s</u>	-.096	-.022	0.344	-.095	-.021	0.360

$\rho = 0.8$

weight	<u>u'</u> T	<u>t'</u> T	<u>s'</u> T	<u>u'</u> W	<u>t'</u> W	<u>s'</u> W
<u>u</u>	0.308	0.288	0.252	0.322	0.298	0.261
<u>t</u>	-.029	0.062	0.122	-.008	0.088	0.142
<u>s</u>	-.335	-.312	0.002	-.332	-.311	-.037

Chapter 6

Summaries and Implications for Future Research

Tailored alternatives and their corresponding tests provide the researcher a simple way for making use of prior information in a multivariate setting without appeal Bayesian methodology. The tailored tests are substantially more powerful than such omnibus tests as the χ^2 and Hotelling T^2 which fail to take into account the one sided nature of the tailored hypotheses. The tailored tests also are substantially more powerful than the one-sided omnibus test proposed by Kudo. The disadvantage of tailored tests is the potential sensitivity to model misspecification. However, modest model misspecification appears to cause little loss in efficiency or power of such tests. Tailored tests can be implemented using the permutation principle, or as shown by a simulation, the tests may be carried out using a normal approximation to the permutation distribution. Moreover these tests can be used to deal with situations, common in medical research, where the number of subjects is small but the dimensions of the multivariate responses is large.

Tailored tests may be based on either actual observations or on ranks of observations. Asymptotic theory was developed

to compare efficiencies of the tailored tests based either on actual observations or ranks. The efficiencies generally follow the pattern that one finds in the univariate case. That is, the rank tests perform better for heavy-tailed distributions but worse for lighter-tailed distribution. However, it appears that factors present in the multivariate case, but not present in the univariate case, may somewhat diminish the importance of rank tests in certain multivariate situations. For instance, the efficiency of rank tests relative to tests based on actual observations tends to decrease as dimension of the multivariate vector increases. In the case of the normal distribution, this may cause significant loss of efficiency of the rank tests. In the case of heavy-tailed distribution, this may result in less of an advantage of the rank tests.

For future study, a Bayesian estimation of the mean function of multivariate distributions could be pursued. Of particular interest would be inferences for non-normal distributions with restrictions on ranges of parameters being accounted for by restrictions on the support of the prior distribution. However, computational complexity may make such an approach unattractive to the practitioner. The situations in which omnibus one-sided tests perform well could be investigated more thoroughly. In particular, one could assume that the multivariate mean is selected from a multivariate

distribution $g(\mu_1, \dots, \mu_p)$ and investigate the power of the omnibus tests in this situation. Also, because of the difficulty in implementing Kudo's test, one could propose a permutation version of this test and investigate its properties.

APPENDIX A

Table A-1

Power Comparisons Between the Normal Approximation
(NA) and the Permutation Principle (PP)

multivariate normal distribution

$p = 10$, $n = 5$, $\Sigma = \sigma^2[(1-\rho)I + \rho J]$, $\rho = 0.2$

$\underline{a}$ is an uniform weight type

θ	$\underline{a}'\underline{T}$		$\underline{a}'\underline{W}$	
	PP	NA	PP	NA
0	0.068	0.067	0.071	0.063
1.5	0.156	0.172	0.172	0.178
3.0	0.307	0.329	0.316	0.319
4.5	0.516	0.543	0.512	0.530
6.0	0.674	0.710	0.692	0.706
9.0	0.936	0.935	0.949	0.925
12.0	0.994	0.998	0.999	0.992

multivariate normal distribution

$p = 10$, $n = 5$, $\Sigma = \sigma^2[(1-\rho)I + \rho J]$, $\rho = 0.8$

$\underline{a}$ is an uniform weight type

θ	$\underline{a}'\underline{T}$		$\underline{a}'\underline{W}$	
	PP	NA	PP	NA
0	0.070	0.069	0.071	0.074
1.5	0.100	0.105	0.097	0.116
3.0	0.162	0.178	0.165	0.187
4.5	0.285	0.287	0.286	0.309
6.0	0.378	0.401	0.396	0.414
9.0	0.643	0.649	0.647	0.668
12.0	0.826	0.849	0.825	0.862

Table A-2

Power Comparisons Between the Normal Approximation
(NA) and the Permutation Principle (PP)

multivariate normal distribution
 $p = 10, n = 5, \Sigma = \sigma^2[(1-\rho)I + \rho J] \rho = 0.2$
 $\underline{a}$ is a triangle weight type

θ	$\underline{a}'\underline{T}$		$\underline{a}'\underline{W}$	
	PP	NA	PP	NA
0	0.067	0.068	0.064	0.064
1.5	0.189	0.190	0.202	0.219
3.0	0.380	0.398	0.369	0.398
4.5	0.622	0.648	0.621	0.656
6.0	0.796	0.813	0.840	0.830
9.0	0.983	0.982	0.952	0.970
12.0	0.998	0.997	0.999	1.000

multivariate normal distribution
 $p = 10, n = 5, \Sigma = \sigma^2[(1-\rho)I + \rho J] \rho = 0.8$
 $\underline{a}$ is an triangle weight type

θ	$\underline{a}'\underline{T}$		$\underline{a}'\underline{W}$	
	PP	NA	PP	NA
0	0.067	0.073	0.069	0.072
1.5	0.117	0.124	0.101	0.114
3.0	0.198	0.203	0.157	0.189
4.5	0.351	0.340	0.283	0.294
6.0	0.473	0.468	0.464	0.498
9.0	0.755	0.743	0.722	0.767
12.0	0.925	0.915	0.917	0.931

Table A-3

Power Comparisons Between the Normal Approximation
(NA) and the Permutation Principle (PP)

multivariate normal distribution
 $p = 10, n = 5, \Sigma = \sigma^2[(1-\rho)I + \rho J] \rho = 0.2$
 $\underline{a}$ is a spike weight type

θ	$\underline{a}'\underline{T}$		$\underline{a}'\underline{W}$	
	PP	NA	PP	NA
0	0.070	0.068	0.067	0.063
0.5	0.212	0.223	0.215	0.219
1.0	0.441	0.453	0.444	0.448
1.5	0.706	0.699	0.695	0.731
2.0	0.901	0.878	0.884	0.907
3.0	0.991	0.989	0.990	0.995
4.5	1.000	1.000	1.000	1.000

multivariate normal distribution
 $p = 10, n = 5, \Sigma = \sigma^2[(1-\rho)I + \rho J] \rho = 0.8$
 $\underline{a}$ is a spike weight type

θ	$\underline{a}'\underline{T}$		$\underline{a}'\underline{W}$	
	PP	NA	PP	NA
0	0.070	0.072	0.071	0.071
0.5	0.189	0.194	0.197	0.205
1.0	0.364	0.354	0.356	0.389
1.5	0.607	0.573	0.592	0.617
2.0	0.800	0.725	0.747	0.774
3.0	0.979	0.950	0.962	0.950
4.5	1.000	0.999	0.999	0.998

Table B-1

**Power Comparisons Between the Normal Approximation
(NA) and the Permutation Principle (PP)**

multivariate normal distribution
 $p=10, n=5, \Sigma = [\sigma]_{ij} = \sigma^2 [\rho^{ij}]^{-1} \rho = 0.2$
a is an uniform weight type

θ	<u>a'T</u>		<u>a'W</u>	
	PP	NA	PP	NA
0	0.063	0.062	0.069	0.066
1.5	0.193	0.210	0.241	0.246
3.0	0.415	0.441	0.463	0.461
4.5	0.671	0.732	0.730	0.747
6.0	0.863	0.911	0.897	0.911
9.0	0.986	0.996	0.985	0.985
12.0	0.999	1.000	0.999	0.997

multivariate normal distribution
 $p=10, n=5, \Sigma = [\sigma]_{ij} = \sigma^2 [\rho^{ij}]^{-1} \rho = 0.8$
a is an uniform weight type

θ	<u>a'T</u>		<u>a'W</u>	
	PP	NA	PP	NA
0	0.072	0.069	0.071	0.073
1.5	0.140	0.145	0.137	0.140
3.0	0.202	0.220	0.228	0.242
4.5	0.332	0.346	0.359	0.363
6.0	0.477	0.507	0.496	0.518
9.0	0.756	0.773	0.767	0.785
12.0	0.930	0.943	0.925	0.947

Table B-2

Power Comparisons Between the Normal Approximation
(NA) and the Permutation Principle (PP)

multivariate normal distribution
 $p = 10, n = 5, \Sigma = [\sigma]_{ij} = \sigma^2 [\rho^{j-i}] \rho = 0.2$
 $\underline{a}$ is a triangle weight type

θ	$\underline{a}'\underline{T}$		$\underline{a}'\underline{W}$	
	PP	NA	PP	NA
0	0.063	0.062	0.064	0.072
1.5	0.234	0.238	0.242	0.279
3.0	0.466	0.507	0.469	0.513
4.5	0.743	0.815	0.798	0.815
6.0	0.907	0.928	0.901	0.924
9.0	0.995	0.999	0.972	0.991
12.0	1.000	1.000	0.999	1.000

multivariate normal distribution
 $p = 10, n = 5, \Sigma = [\sigma]_{ij} = \sigma^2 [\rho^{j-i}] \rho = 0.8$
 $\underline{a}$ is an triangle weight type

θ	$\underline{a}'\underline{T}$		$\underline{a}'\underline{W}$	
	PP	NA	PP	NA
0	0.076	0.073	0.069	0.071
1.5	0.157	0.170	0.171	0.174
3.0	0.248	0.258	0.237	0.270
4.5	0.422	0.436	0.399	0.456
6.0	0.575	0.578	0.546	0.575
9.0	0.859	0.859	0.832	0.884
12.0	0.998	0.967	0.977	0.998

Table B-3

Power Comparisons Between the Normal Approximation
(NA) and the Permutation Principle (PP)

multivariate normal distribution
 $p=10, n=5, \Sigma = [\sigma]_{ij} = \sigma^2 [\rho^{j-i}] \rho = 0.2$
a is a spike weight type

θ	<u>a'T</u>		<u>a'W</u>	
	PP	NA	PP	NA
0	0.064	0.066	0.067	0.073
0.5	0.222	0.242	0.213	0.223
1.0	0.456	0.473	0.454	0.480
1.5	0.743	0.758	0.736	0.758
2.0	0.899	0.906	0.898	0.913
3.0	0.999	0.998	0.995	0.999
4.5	1.000	1.000	1.000	1.000

multivariate normal distribution
 $p=10, n=5, \Sigma = [\sigma]_{ij} = \sigma^2 [\rho^{j-i}] \rho = 0.8$
a is a spike weight type

θ	<u>a'T</u>		<u>a'W</u>	
	PP	NA	PP	NA
0	0.074	0.070	0.071	0.073
0.5	0.210	0.213	0.217	0.238
1.0	0.394	0.379	0.408	0.414
1.5	0.642	0.621	0.616	0.670
2.0	0.847	0.795	0.797	0.832
3.0	0.984	0.973	0.958	0.970
4.5	1.000	1.000	0.999	0.999

Table C-1

Power Comparisons Between the Normal Approximation
(NA) and the Permutation Principle (PP)

multivariate t distribution with $\nu = 3$
 $p = 10, n = 5, \Sigma = \sigma^2[(1-\rho)I + \rho J], \rho = 0.2$
 $\underline{a}$ is an uniform weight type

θ	$\underline{a}'\underline{T}$		$\underline{a}'\underline{W}$	
	PP	NA	PP	NA
0	0.064	0.075	0.068	0.070
1.5	0.151	0.154	0.160	0.172
3.0	0.286	0.284	0.301	0.302
4.5	0.401	0.423	0.420	0.444
6.0	0.512	0.556	0.564	0.572
9.0	0.773	0.780	0.796	0.789
12.0	0.895	0.897	0.897	0.910

multivariate t distribution with $\nu = 3$
 $p = 10, n = 5, \Sigma = \sigma^2[(1-\rho)I + \rho J], \rho = 0.8$
 $\underline{a}$ is an uniform weight type

θ	$\underline{a}'\underline{T}$		$\underline{a}'\underline{W}$	
	PP	NA	PP	NA
0	0.068	0.064	0.070	0.065
1.5	0.119	0.116	0.124	0.136
3.0	0.173	0.164	0.173	0.199
4.5	0.205	0.208	0.237	0.240
6.0	0.301	0.291	0.326	0.332
9.0	0.513	0.479	0.521	0.526
12.0	0.659	0.653	0.672	0.671
15.0	0.785	0.753	0.792	0.777

Table C-2

Power Comparisons Between the Normal Approximation
(NA) and the Permutation Principle (PP)

multivariate t distribution with $\nu = 3$
 $p = 10, n = 5, \Sigma = \sigma^2[(1-\rho)I + \rho J] \rho = 0.2$
 $\underline{a}$ is a triangle weight type

θ	$\underline{a}'\underline{T}$		$\underline{a}'\underline{W}$	
	PP	NA	PP	NA
0	0.068	0.072	0.069	0.073
1.5	0.169	0.173	0.176	0.174
3.0	0.330	0.323	0.324	0.351
4.5	0.491	0.501	0.501	0.528
6.0	0.629	0.627	0.629	0.656
9.0	0.850	0.846	0.852	0.854
12.0	0.932	0.958	0.927	0.945

multivariate t distribution with $\nu = 3$
 $p = 10, n = 5, \Sigma = \sigma^2[(1-\rho)I + \rho J] \rho = 0.8$
 $\underline{a}$ is an triangle weight type

θ	$\underline{a}'\underline{T}$		$\underline{a}'\underline{W}$	
	PP	NA	PP	NA
0	0.071	0.057	0.069	0.067
1.5	0.129	0.097	0.132	0.116
3.0	0.202	0.196	0.201	0.204
4.5	0.267	0.292	0.260	0.321
6.0	0.376	0.379	0.365	0.403
9.0	0.617	0.587	0.588	0.601
12.0	0.767	0.730	0.734	0.761
15.0	0.866	0.856	0.838	0.852

Table C-3

Power Comparisons Between the Normal Approximation
(NA) and the Permutation Principle (PP)

multivariate t distribution with $\nu = 3$
 $p = 10, n = 5, \Sigma = \sigma^2[(1-\rho)I + \rho J] \rho = 0.2$
 $\underline{a}$ is a spike weight type

θ	$\underline{a}'\underline{T}$		$\underline{a}'\underline{W}$	
	PP	NA	PP	NA
0	0.071	0.066	0.067	0.076
0.5	0.174	0.159	0.181	0.199
1.0	0.359	0.351	0.363	0.373
1.5	0.565	0.549	0.565	0.575
2.0	0.719	0.700	0.725	0.746
3.0	0.881	0.869	0.867	0.891
4.5	0.978	0.966	0.967	0.976
6.0	0.985	0.979	0.981	0.985
9.0	0.999	0.994	1.000	0.998

multivariate t distribution with $\nu = 3$
 $p = 10, n = 5, \Sigma = \sigma^2[(1-\rho)I + \rho J] \rho = 0.8$
 $\underline{a}$ is a spike weight type

θ	$\underline{a}'\underline{T}$		$\underline{a}'\underline{W}$	
	PP	NA	PP	NA
0	0.069	0.068	0.068	0.071
0.5	0.178	0.171	0.184	0.203
1.0	0.323	0.295	0.335	0.349
1.5	0.470	0.474	0.480	0.508
2.0	0.629	0.621	0.594	0.625
3.0	0.845	0.790	0.785	0.791
4.5	0.956	0.894	0.913	0.900
6.0	0.979	0.972	0.955	0.966
9.0	0.991	0.996	0.986	0.981

Table D-1

**Power Comparisons Between the Normal Approximation
(NA) and the Permutation Principle (PP)**

multivariate t distribution with $v = 3$
 $p = 10, n = 5, \Sigma = [\sigma]_{ij} = \sigma^2 [\rho^{j-i}] \rho = 0.2$
a is an uniform weight type

θ	<u>a'T</u>		<u>a'W</u>	
	PP	NA	PP	NA
0	0.073	0.068	0.068	0.071
1.5	0.177	0.181	0.198	0.202
3.0	0.386	0.403	0.401	0.413
4.5	0.569	0.596	0.589	0.602
6.0	0.693	0.718	0.694	0.741
9.0	0.884	0.896	0.896	0.904
12.0	0.954	0.973	0.967	0.975

multivariate t distribution with $v = 3$
 $p = 10, n = 5, \Sigma = [\sigma]_{ij} = \sigma^2 [\rho^{j-i}] \rho = 0.8$
a is an uniform weight type

θ	<u>a'T</u>		<u>a'W</u>	
	PP	NA	PP	NA
0	0.071	0.073	0.070	0.072
1.5	0.132	0.128	0.124	0.142
3.0	0.204	0.203	0.173	0.226
4.5	0.278	0.284	0.237	0.301
6.0	0.382	0.371	0.376	0.391
9.0	0.581	0.578	0.621	0.615
12.0	0.763	0.772	0.772	0.785
15.0	0.839	0.835	0.848	0.856

Table D-2

**Power Comparisons Between the Normal Approximation
(NA) and the Permutation Principle (PP)**

multivariate t distribution with $\nu = 3$
 $p = 10, n = 5, \Sigma = [\sigma]_{ij} = \sigma^2 [\rho |j^{-1}|] \rho = 0.2$
a is a triangle weight type

θ	<u>a'T</u>		<u>a'W</u>	
	PP	NA	PP	NA
0	0.064	0.069	0.069	0.059
1.5	0.201	0.209	0.216	0.214
3.0	0.441	0.443	0.424	0.455
4.5	0.597	0.613	0.601	0.650
6.0	0.741	0.779	0.769	0.794
9.0	0.916	0.927	0.925	0.918
12.0	0.966	0.977	0.972	0.963

multivariate t distribution with $\nu = 3$
 $p = 10, n = 5, \Sigma = [\sigma]_{ij} = \sigma^2 [\rho |j^{-1}|] \rho = 0.8$
a is an triangle weight type

θ	<u>a'T</u>		<u>a'W</u>	
	PP	NA	PP	NA
0	0.075	0.074	0.069	0.078
1.5	0.144	0.134	0.132	0.155
3.0	0.240	0.230	0.201	0.256
4.5	0.340	0.330	0.360	0.368
6.0	0.447	0.433	0.465	0.471
9.0	0.666	0.651	0.688	0.675
12.0	0.821	0.816	0.834	0.825
15.0	0.899	0.881	0.892	0.896

Table D-3

**Power Comparisons Between the Normal Approximation
(NA) and the Permutation Principle (PP)**

multivariate t distribution with $\nu = 3$
 $p = 10, n = 5, \Sigma = [\sigma]_{ij} = \sigma^2 [\rho |j-i|]$ $\rho = 0.2$
 $\underline{a}$ is a spike weight type

θ	$\underline{a}'\underline{T}$		$\underline{a}'\underline{W}$	
	PP	NA	PP	NA
0	0.068	0.058	0.067	0.076
0.5	0.183	0.181	0.194	0.205
1.0	0.375	0.362	0.376	0.388
1.5	0.580	0.560	0.565	0.585
2.0	0.729	0.725	0.739	0.760
3.0	0.887	0.890	0.867	0.900
4.5	0.980	0.977	0.967	0.979
6.0	0.986	0.985	0.981	0.989
9.0	0.999	0.996	1.000	0.998

multivariate t distribution with $\nu = 3$
 $p = 10, n = 5, \Sigma = [\sigma]_{ij} = \sigma^2 [\rho |j-i|]$ $\rho = 0.8$
 $\underline{a}$ is a spike weight type

θ	$\underline{a}'\underline{T}$		$\underline{a}'\underline{W}$	
	PP	NA	PP	NA
0	0.068	0.067	0.068	0.071
0.5	0.179	0.166	0.190	0.208
1.0	0.332	0.304	0.341	0.370
1.5	0.495	0.463	0.480	0.538
2.0	0.651	0.635	0.630	0.644
3.0	0.854	0.822	0.804	0.842
4.5	0.959	0.936	0.913	0.949
6.0	0.982	0.973	0.955	0.970
9.0	0.993	0.986	0.986	0.996

REFERENCES

1. Barndoreff-Nielsen, Ole. (1963), "On the Limit Behaviour of Extreme Order Statistics", Annals of Mathematical Statistics, 34, 992-1002.
2. Blair, R.C., Higgins, J.J., Karliniski, W., and Kromrey, J.D. (1992), "Proposed Multivariate Permutation Tests As Replacements For Hotelling's T^2 Test in Prescribed Circumstances", submit for publication.
3. Blair, R.C., and Karliniski, W. (1992), "An Alternative Method for Significance Testing of Waveform Difference Potentials", submit for publication.
4. Boyett, J. M., and Shuster, J. J. (1977), "Nonparametric One-Sided Tests in Multivariate Analysis with Medical Applications", Journal of the American Statistical Association, 72, 665-668.
5. Bradley, J.V. (1968) Distribution-free Statistical Tests. Prentice Hall, Englewood Cliffs.
6. Dietz, E. J. (1989), "Multivariate Generalizations of Jonckheere's Test for Ordered Alternatives," Communications in Statistics: Theory and Method, 18, 3763-3783.
7. Gibbons, J.D. (1971). Nonparametric Statistical Inference. McGraw-Hill, New York.
8. Hajek, J. and Sidak, Z. (1967). Theory of Rank Tests. Academic press, New York.
9. Hettmansperger, T.P. (1984) Statistical Inference Based on Ranks. Wiley, New York.
10. Keller-McNulty, S. and Higgins, J. (1987) "Effect of Tail Weight and Outliers on Power and Type-I Error of Robust Permutation Tests for Location" Communications in Statistics: Simulation and Computation, Vol.16,1, 17-36.
11. Kudo, Akio (1963). " A Multivariate Analogue of the One-sided Test, " Biometrika, 50, 3 and 4. 403-418.
12. Lehmann, E.L. (1975) Nonparametrics: Statistical Methods based on Ranks. Holden-Day, San Francisco.

13. Nuesch, P. E. (1966). " On the Problem of Testing Location in Multivariate Populations for Restricted Alternatives," Annals of Mathematical Statistics, 37, 113-119.
14. Puri, M. L. and Sen, P.K. (1971) Nonparametric Methods in Multivariate Analysis. Wiley, New York.
15. Ranald, H.R. and Wolfe, D.A. (1979) Introduction to The Theory of Nonparametric Statistics. Wiley, New York.
16. Raz, Jonathan. (1989), "Analysis of Repeated Measurements Using Nonparametric Smoothers and Randomization Tests," Biometrics, 45, 851-871.
17. Shorack, Galen R. (1967), "Testing Against Ordered Alternatives in Model I Analysis of Variance; Normal Theory and Nonparametric," Annals of Mathematical Statistics, 38, 1740-1752.