ON INFINITE PRODUCTS

by

ROSE KORDONOWY SHAW

B. S., Dickinson State College, 1964

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1966

Approved by:

R. G. Sanger
Major Professor

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INTRODUCTION

Many functions can be represented in many different ways. They can be represented by Fourier series, orthogonal polynomials, infinite series, and infinite products. Infinite products will be considered as the background for the development of expansion of functions. The infinite product representations of sin x and cos x will be considered.

The theory of infinite products is very closely related to that of infinite series; so closely, in fact, that the test for convergence of such products will actually be reduced to the test for the convergence of certain infinite series.

The object of this paper is to begin with the basic definition of an infinite product and to take the study of infinite products to the point where a basic understanding of infinite products and how they can be handled is reached.

BASIC DEFINITIONS

An infinite product is of the form

$$a_1 \cdot a_2 \cdots a_k \cdots,$$

where a_i is a complex number, $i = 1, 2, \ldots, n, \ldots$, and is denoted by

$$\prod_{k=1}^{\infty} a_k = a_1\, a_2 \cdots a_k \cdots$$

The partial products p_k are defined in the following manner:

$$p_1 = a_1$$
$$p_2 = a_1 \cdot a_2$$
$$\vdots \qquad \vdots$$
$$p_k = a_1 \cdot a_2 \ldots a_k$$

where $k = 1, 2, 3, \ldots$.

With these preliminary comments on the form of infinite products, convergence, divergence, and oscillation of an infinite product will be defined.

<u>Definition 1</u>. The infinite product $\prod_{k=1}^{\infty} a_k$ converges if and only if there exists a K such that

$$\lim_{g\to\infty} \prod_{k=K}^{K+g} a_k = A \neq 0.$$

If the infinite product converges, its value is $A \prod_{k=1}^{K-1} a_k$.

<u>Definition 2</u>. An infinite product $\prod_{k=1}^{\infty} a_k$ is said to be divergent if A in definition 1 is equal to zero for all K or when A is infinite.

<u>Definition 3</u>. An infinite product is oscillating when A does not approach any definite limit. An oscillating product is not convergent.

The following examples will be presented in an attempt to clarify the meanings of definitions 1, 2, and 3.

<u>Example 1</u>. The infinite product $0 \cdot 2 \cdot 0 \cdot 2 \ldots$ where

$$a_{2k-1} = 0 \text{ and } a_{2k} = 2 \text{ for } k = 1, 2, \ldots$$

is divergent because it contains an infinite number of zeros, i.e., there does not exist a K such that

$$\lim_{g\to\infty} \prod_{k=K}^{K+s} a_k = A \neq 0.$$

The infinite product

$$\prod_{k=1}^{\infty} \frac{1}{k} = 1 \cdot 1/2 \cdot 1/3 \ldots 1/k \ldots$$

is divergent because, although it does not contain any zeros,

$$\lim_{k\to\infty} p_k = 0.$$

The infinite product, $k = 1 \cdot 2 \cdot 3 \ldots k \ldots$, is divergent because $\lim\limits_{k\to\infty} p_k = \infty$.

The infinite product $\prod_{k=1}^{\infty} (-1)^k = -1 \cdot 1 \cdot -1 \ldots (-1)^k \ldots$ is oscillating because the product does not approach any definite limit. Consider the partial products of $\prod_{k=1}^{\infty} (-1)^k$.

$$p_1 = -1$$
$$p_2 = -1$$
$$p_3 = 1$$
$$p_4 = 1$$
$$\vdots$$

It is seen that the infinite product is not approaching any definite limit.

The infinite product $0 \cdot 0 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \ldots$ where

$$a_k = 0 \text{ for } k = 1, 2$$

and $$a_k = 1 \text{ for } k = 3, 4, 5, \ldots$$

is convergent because $\lim_{g\to\infty} \prod_{k=3}^{3+g} a_k = 1$. The value of the infinite product is $1 \cdot \prod_{k=1}^{2} a_k = 0$.

It can be seen from the above definitions and examples that a convergent infinite product can contain a finite number of a_i's equal to zero. If $\prod_{k=1}^{\infty} a_k = 0$, it is not known whether a finite number of a_i's are equal to zero and the infinite product is convergent or if all a_i's are not equal to zero and $\prod_{k=1}^{\infty} a_k = 0$ and the product is divergent. It can also be seen that oscillating infinite products occur when some $a_i < 0$. To simplify the theory to be presented, it will be assumed that $a_i > 0$ for every $i = 1, 2, \ldots$.

CONVERGENCE OF INFINITE PRODUCTS

At this point, the characteristics of convergent infinite products will be considered.

<u>Theorem</u> <u>1a</u>. The sequence of the factors in a convergent infinite product always tends to approach one.

The proof is as follows:

$$p_{k-1} = a_1 \cdot a_2 \ldots a_{k-1}$$

$$p_k = a_1 \cdot a_2 \ldots a_k$$

since p_{k-1} approaches A and p_k approaches A and $A \neq 0$

$$\frac{p_k}{p_{k-1}} = \frac{a_1 \cdot a_2 \cdots a_{k-1} \cdot a_k}{a_1 \cdot a_2 \cdots a_{k-1}} = a_k \rightarrow 1 .$$

The following theorem is closely related to theorem 1a, and hence will be termed theorem 1b.

<u>Theorem</u> <u>1b</u>. For the infinite product $\prod_{k=1}^{\infty} a_k$ to converge, it is necessary and sufficient that for every $\epsilon > 0$, a $K > 0$ exists such that for every $k > K$ and for every integer $r \geqq 1$

$$\left| a_{k+1} \; a_{k+2} \cdots a_{k+r} - 1 \right| < \epsilon .$$

The proof of theorem 1b is as follows.

If $\prod_{k=1}^{\infty} a_k$ is convergent, then the sequence of the factors in a convergent infinite product always tends to approach one. Let

$$q_k = \frac{p_{k+r}}{p_k} .$$

Because p_k converges to a limit P,

$$\lim_{k \to \infty} q_k = 1$$

which implies that

$$\left| q_k - 1 \right| < \epsilon \qquad \text{for } k > K.$$

But

$$q_k = a_{k+1} \; a_{k+2} \cdots a_{k+r}$$

and hence

$$\left| a_{k+1} \; a_{k+2} \cdots a_{k+r} - 1 \right| < \epsilon$$

and the necessity part of the proof is complete.

If

$$q_k = \frac{p_{k+r}}{p_k}$$

and if the condition $\left| q_k - 1 \right| < \epsilon$ is satisfied, then

$$1 - \epsilon < q_k < \epsilon + 1$$

and

$$q_k = \frac{p_{k+r}}{p_k} \to 1$$

and

$$p_{k+r} \to p_k$$

which implies that for every $\epsilon > 0$,

$$\left| p_{k+r} - p_k \right| < \epsilon \ .$$

This is the Cauchy condition for the convergence of a sequence. Therefore

$$\prod_{k=1}^{\infty} a_k$$

is convergent, and the theorem is proved.

This theorem gives the general convergence principle. On the basis of this theorem (let $r = 1$ and $k + 1 = n$), it is necessary that the

$$\lim_{n\to\infty} a_n = 1 \ .$$

Therefore set the factors of the product equal to $(1 + u_n)$ and the following form of the infinite product arises.

$$\prod_{n=1}^{\infty} (1 + u_n) = (1 + u_1)(1 + u_2) \ \ldots \ (1 + u_n) \ \ldots \ .$$

Because the assumption was made that every $a_i > 0$, every

$(1 + u_i) > 0$, and $u_i > -1$.

The partial product P_n is defined as

$$P_n = (1 + u_1)(1 + u_2) \ldots (1 + u_n) \quad \text{for } n = 1, 2, 3, \ldots$$

The infinite product $\prod_{n=1}^{\infty} (1 + u_n)$ is convergent if and only if there exists a finite $P \neq 0$ such that $\lim_{n\to\infty} P_n = P$ exists; otherwise the infinite product is said to be divergent.

If the infinite product $\prod_{n=1}^{\infty} (1 + a_n)$ is convergent, then by the general condition of convergence of a sequence

$$\left| P_{n+p} - P_n \right| < \epsilon \text{ , } n > N, \ p = 1, 2, 3, \ldots ,$$

and also there exists a g such that

$$\left| P_n \right| > g$$

for every n, where g is a positive constant. The condition $\left| P_n \right| > g$ implies that $P_n \neq 0$ for every n for a convergent product.

Sufficient background has been developed so that the following theorem for convergence can be considered.

<u>Theorem</u> <u>2</u>. A necessary and sufficient condition for the convergence of the infinite product

$$\prod_{n=1}^{\infty} (1 + u_n)$$

is that, for any arbitrary small $\epsilon > 0$, an $N > 0$ can be found such that for $n \geqq N$,

$$\left| \frac{P_{n+p}}{P_n} - 1 \right| < \epsilon$$

where p is a positive integer.

Suppose first that the product is convergent, then since $|P_n| > g$, $|P_{n+p} - P_n| < \epsilon$ can be divided by $|P_n|$ getting

$$\left| \frac{P_{n+p}}{P_n} - 1 \right| \quad \frac{\epsilon}{g} = \epsilon_1$$

The condition is therefore necessary.

If now the condition

$$\left| \frac{P_{n+p}}{P_n} - 1 \right| < \epsilon$$

is satisfied, for any assigned ϵ, the inequality can be written

$$\left| (1 + u_{n+1})(1 + u_{n+2}) \ldots (1 + u_{n+p}) - 1 \right| < \epsilon$$

for $n > N$, and for every p, and by theorem 1b the infinite product is convergent.

If in the above theorem p is set equal to 1,

$$\frac{P_{n+1}}{P_n} = 1 + u_{n+1}$$

and $|u_{n+1}| < \epsilon$, $n > N$, which gives the next theorem.

Theorem 3. A necessary condition for convergence of the infinite product $\prod_{n=1}^{\infty} (1 + u_n)$ is that $\lim_{n \to \infty} u_n = 0$.

This condition is not in general a sufficient condition. This is shown by the example of $u_n = \pm \frac{1}{n}$, for

$$\prod_{i=1}^{n} (1 + \frac{1}{i}) = (1 + \frac{1}{1})(1 + \frac{1}{2})(1 + \frac{1}{3}) \ldots (1 + \frac{1}{n})$$

$$= 2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \ldots \frac{n+1}{n} = n + 1$$

and

$$\prod_{i=2}^{n} \frac{1}{i} = (1 - \frac{1}{2})(1 - \frac{1}{3}) \ldots (1 - \frac{1}{n})$$

$$= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \ldots \frac{n - 1}{n} = \frac{1}{n}$$

so that both infinite products are divergent, since for the first $P_n \to \infty$ and for the second $P_n \to 0$, and yet $\lim_{n \to \infty} u_n = 0$ for both.

RELATING CONVERGENCE OF INFINITE PRODUCTS TO CONVERGENCE OF INFINITE SERIES

Definitions and theorems on the convergence of infinite products have been given. The next step is to relate the convergence of infinite products to the convergence of an infinite series. To develop an equivalent series, consider the partial product P_n with the additional condition that $P_0 = 1$ and the recursion formula

$$P_i = (1 + u_i)P_{i-1} = P_{i-1} + u_i P_{i-1} .$$

It follows that

$$\begin{aligned} P_n &= P_{n-1} + u_n P_{n-1} \\ &= P_{n-2} + u_{n-1} P_{n-2} + u_n P_{n-1} \\ &= P_{n-3} + u_{n-2} P_{n-3} + u_{n-1} P_{n-2} + u_n P_{n-1} \\ & \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \end{aligned}$$

Therefore

$$\text{(I)} \quad P_n = 1 + \sum_{i=1}^{n} u_i P_{i-1} .$$

If the infinite product converges, then $\lim_{n \to \infty} P_n$ exists, and hence the series

$$1 + \sum_{i=1}^{\infty} u_i P_{i-1}$$

is convergent with sum equal to the value of the product. Conversely, if the series is convergent, then $\lim_{n \to \infty} P_n$ exists and the product is convergent with value equal to the sum of the series:

$$\prod_{n=1}^{\infty} (1 + u_n) = 1 + \sum_{n=1}^{\infty} u_n P_{n-1} ,$$

therefore

<u>Definition 4</u>, the series

$$1 + \sum_{n=1}^{\infty} u_n P_{n-1}$$

is called the series equivalent to the infinite product.

By means of the formula

$$\prod_{n=1}^{\infty} (1 + u_n) = 1 + \sum_{n=1}^{\infty} u_n P_{n-1} \tag{1}$$

an infinite product can be transformed into an infinite series. Conversely, the partial sum U_n of a series $U_n = u_1 + u_2 + \dots + u_n$ can be transformed into a product. From the identity

$$1 = \frac{u_i}{U_{i-1}} = \frac{U_i}{U_{i-1}} , \quad i > 1$$

it follows that

$$U_n = u_1 \cdot \prod_{i=2}^{n} (1 + \frac{u_i}{U_{i-1}}) \tag{2}$$

Formulas (1) and (2) are not in general practically applicable, since it would be difficult in most cases to express the partial products P_n and the partial sums U_n in simple form. However, they are of theoretical value.

Before the theorem which reduces the problem of the convergence of infinite products to the convergence of infinite series is stated, a theorem on inequalities due to Weierstrass must be considered to aid in the proof of the theorem.

Theorem 4. Weierstrass' Theorem on Inequalities.

If $u_1, u_2, \ldots, u_n, \ldots$ are all positive and < 1, $n = 1, 2, \ldots$, and P_1 is excluded,

(a) $$(1 + u_1)(1 + u_2)(1 + u_3) \ldots (1 + u_n) > 1 + (u_1 + u_2 + \ldots + u_n),$$

(b) $$(1 - u_1)(1 - u_2)(1 - u_3) \ldots (1 - u_n) > 1 - (u_1 + u_2 + \ldots + u_n),$$

and if $$\sum_{i=1}^{n} u_i < 1,$$

(c) $$(1 + u_1)(1 + u_2) \ldots (1 + u_n) \left[1 - (u_1 + u_2 + \ldots + u_n)\right]^{-1},$$

(d) $$(1 - u_1)(1 - u_2) \ldots (1 - u_n) \left[1 + (u_1 + u_2 + \ldots + u_n)\right]^{-1}.$$

Part (a) can be proved by using mathematical induction.

For $n = 2$, (a) becomes

$$(1 + u_1)(1 + u_2) = 1 + u_1 + u_2 + u_1u_2 > 1 + (u_1 + u_2)$$

Assume that (a) is true for $n = k$.

(I) $$(1 + u_1)(1 + u_2) \ldots (1 + u_k) > 1 + (u_1 + u_2 + \ldots + u_k)$$

Then (a) must be true for $n = k + 1$.

(II) $$(1 + u_1)(1 + u_2) \ldots (1 + u_{k+1}) > 1 + (u_1 + u_2 + \ldots + u_{k+1}).$$

Multiply both sides of (I) by $(1 + u_{k+1})$.

$$(1 + u_1)(1 + u_2)\ldots(1 + u_k)(1 + u_{k+1}) > (1 + u_1 + u_2 + \ldots + u_k)(1 + u_{k+1})$$

$$= 1 + u_1 + u_2 + \ldots + u_k + u_{k+1} + u_1u_{k+1} + u_2u_{k+1} + \ldots + u_ku_{k+1} > 1 + (u_1 + u_2 + \ldots + u_{k+1})$$

which is (II).

This completes the proof of (a) for all $n > 2$.

Similarly, part (b) can be proven as follows.

For $n = 2$,

$$(1 - u_1)(1 - u_2) = 1 - u_1 - u_2 + u_1u_2 > 1 - (u_1 + u_2)$$

Assume (b) is true for $n = k$,

(III) $(1 - u_1)(1 - u_2)\ldots(1 - u_k) > 1 - (u_1 + u_2 + \ldots + u_k)$.

Then (b) must be true for $n = k + 1$,

(IV) $(1 - u_1)(1 - u_2)\ldots(1 - u_{k+1}) > 1 - (u_1 + u_2 + \ldots + u_{k+1})$.

Multiply both sides of (III) by $(1 - u_{k+1})$,

$$(1 - u_1)(1 - u_2)\ldots(1 - u_k)(1 - u_{k+1}) > \left[1 - (u_1 + u_2 + \ldots + u_k)\right](1 - u_{k+1})$$

$$= \left[1 - u_1 - u_2 - \ldots u_k\right]\left[1 - u_{k+1}\right] = (1 - u_1 - u_2 - \ldots u_k - u_{k+1} + u_1u_{k+1} + u_2u_{k+1} + \ldots + u_ku_{k+1})$$

$$> 1 - (u_1 + u_2 + \ldots + u_{k+1})$$

which completes the proof.

Now for part (c)

$$1 + u_i = \frac{1 - u_i^2}{1 - u_i} < \frac{1}{1 - u_i}$$

hence

$$(1 + u_1)(1 + u_2)\ldots(1 + u_n) < \frac{1}{(1 - u_1)(1 - u_2)\ldots(1 - u_n)},$$

and since $\sum_{i=1}^{n} u_i < 1$, using part (b),

$$(1 + u_1)(1 + u_2)\ldots(1 + u_n) < \Big[1 - (u_1 + u_2 + \ldots + u_n)\Big]^{-1} .$$

Finally, for part (d),

$$1 - u_i = \frac{1 - u_i^2}{1 + u_i} < \frac{1}{1 + u_i}$$

Hence

$$(1 - u_1)(1 - u_2)\ldots(1 - u_n) < \Big[(1 + u_1)\ldots(1 + u_n)\Big]^{-1} ,$$

and using part (a)

$$(1 - u_1)(1 - u_2)\ldots(1 - u_n) < \Big[1 + (u_1 + u_2 + \ldots + u_n)\Big]^{-1}$$

Combining these results and taking limits, the following inequalities are obtained:

$$\text{(e)} \quad (1 - \sum_{n=1}^{\infty} u_n)^{-1} > \prod_{n=1}^{\infty} (1 + u_n) > 1 + \sum_{n=1}^{\infty} u_n ,$$

$$\text{(f)} \quad (1 + \sum_{n=1}^{\infty} u_n)^{-1} > \prod_{n=1}^{\infty} (1 - u_n) > 1 - \sum_{n=1}^{\infty} u_n .$$

In addition to the Weierstrass' Inequalities needed in the proof of the theorem relating the convergence of infinite products to that of convergence of infinite series, the following definition and theorem are needed.

Definition 5. A real sequence $\{u_n\}$ each of whose terms is greater than the preceding, that is, such that $u_n < u_{n+1}$ for every n, is called an increasing sequence. If $u_n > u_{n+1}$ for every n, the sequence is called a decreasing sequence.

If $u_n \leqq u_{n+1}$ for every n, the sequence is called monotonic increasing. If $u_n \geqq u_{n+1}$ for every n, the sequence is called monotonic decreasing.

Theorem 5. A bounded monotonic sequence is always convergent.

The proof will be restricted to the case of a monotonic increasing sequence, since just changing the signs of a monotonic increasing sequence will give a monotonic decreasing sequence.

Suppose, then, that $\{u_n\}$ is monotonic increasing and bounded from above. Let U be the least upper bound of the numbers u_n. Then if $\epsilon > 0$, $U - \epsilon < u_n$ for some n, say $n = N$, and $u_n \leqq U$ for every n. Since $u_N \leqq u_n$ for every $n \geqq N$, by virtue of the assumption that $u_n \leqq u_{n+1}$, it can be seen that $U - \epsilon < u_n \leqq U$ if $N \leqq n$. Thus by definition, $\lim_{n\to\infty} u_n = U$.

All the material needed to present the theorem relating the convergence of infinite products to that of convergence of the associated infinite series has now been given.

<u>Theorem 6</u>. If $u_1, u_2, \ldots, u_n, \ldots$ all lie between 0 and 1, then the necessary and sufficient condition for the convergence of the infinite products

$$\prod_{n=1}^{\infty} (1 + u_n) \text{ and } \prod_{n=1}^{\infty} (1 - u_n)$$

is the convergence of the series $\sum\limits_{n=1}^{\infty} u_n$.

There are two ways to prove theorem 6. The proofs will be presented as proof (1) and proof (2).

<u>Proof (1)</u>. Set

$$P_n = (1 + u_1)(1 + u_2)\ldots(1 + u_n),$$

$$Q_n = (1 - u_1)(1 - u_2)\ldots(1 - u_n).$$

It is evident that the sequence $\{P_n\}$ is increasing and the sequence $\{Q_n\}$ is decreasing since $u_1, u_2, \ldots, u_n$, lie between 0 and 1.

First assume that $\sum_{n=1}^{\infty} u_n$ is convergent and then prove that the infinite products $\prod_{n=1}^{\infty} (1 + u_n)$ and $\prod_{n=1}^{\infty} (1 - u_n)$ are therefore convergent.

Since $\sum_{n=1}^{\infty} u_n$ is convergent, an N can be found such that if

$$s_m = u_{m+1} + u_{m+2} + \dots > 1$$

then

$$\frac{1}{1 - s_m} > \frac{1}{1 - (u_{m+1} + \dots + u_n)} ,$$

but

$$\frac{P_n}{P_m} = (1 + u_{m+1})(1 + u_{m+2})\dots(1 + u_n). \qquad (n > m) .$$

Now using the Weierstrass Inequality (c) of theorem 5, it follows that

$$\frac{P_n}{P_m} < \frac{1}{1 - (u_{m+1} + \dots + u_n)}$$

and therefore

$$\frac{P_n}{P_m} < \frac{1}{1 - s_m} ,$$

or

$$\text{(g)} \qquad P_n < \frac{P_m}{1 - s_m} \qquad (n > m).$$

Similarly,

$$\frac{Q_n}{Q_m} > 1 - (u_{m+1} + \dots + u_n) \quad 1 - s_m ,$$

or (h) $\quad Q_n > Q_m(1 - s_m)$.

Now $\{P_n\}$ is increasing and (g) shows that it is bounded from above, and $\{Q_n\}$ is decreasing and is bounded below by (h); therefore both these sequences have limits

$$P \leqq \frac{P_m}{1 - s_m} , \qquad Q \geqq Q_m(1 - s_m)$$

and by theorem 5 are always convergent. Therefore the infinite products $\prod_{n=1}^{\infty} (1 + u_n)$ and $\prod_{n=1}^{\infty} (1 - u_n)$ are convergent.

Next suppose $\sum_{n=1}^{\infty} u_n$ is divergent, then an m can be found such that for any positive number G, no matter how large, $u_1 + u_2 + \dots + u_n > G$. $(n > m)$. Remembering that

$$P_n = (1 + u_1)(1 + u_2)\dots(1 + u_n),$$

by inequality (a) of the Weierstrass Inequalities theorem,

$$(1 + u_1)(1 + u_2)\dots(1 + u_n) > 1 + (u_1 + u_2 + \dots + u_n)$$

or

$$P_n > 1 + G.$$

Also remembering that

$$Q_n = (1 - u_1)(1 - u_2)\dots(1 - u_n),$$

by inequality (d) of theorem 4,

$$(1 - u_1)(1 - u_2)\dots(1 - u_n) < \frac{1}{1 + (u_1 + u_2 + \dots + u_n)}$$

or

$$Q_n < \frac{1}{1 + G} \qquad (n > m)$$

Hence $\lim_{n \to \infty} P_n =$ and $\lim_{n \to \infty} Q_n = 0$

so that both products are divergent.

It still remains to show that if

$$\prod_{n=1}^{\infty} (1 + u_n) \quad \text{and} \quad \prod_{n=1}^{\infty} (1 - u_n)$$

are convergent, the series $\sum_{n=1}^{\infty} u_n$ is convergent.

First, assume that $\prod_{n=1}^{\infty} (1 + u_n)$ is convergent. Since $\prod_{n=1}^{\infty} (1 + u_n)$ is convergent, $\lim_{n \to \infty} P_n = P$, where $P_n = (1 + u_1)(1 + u_2)\ldots(1 + u_n)$. Since $0 < u_i < 1$, this is an increasing sequence, and hence

$$(1 + u_1)(1 + u_2)\ldots(1 + u_n) < P.$$

By inequality (a) of theorem 4,

$$(1 + u_1)(1 + u_2)\ldots(1 + u_n) > 1 + (u_1 + u_2 + \ldots + u_n).$$

Hence

$$1 + (u_1 + u_2 + \ldots + u_n) < P,$$

or

$$\sum_{i=1}^{\infty} u_i < P - 1 = R,$$

which says

$$\sum_{i=1}^{n} u_n < R.$$

Now since $0 < u_i < 1$ for all i, the series $\sum_{i=1}^{n} u_i$ is monotonic, and since it is bounded from above the series is convergent.

A similar result can be obtained if $\prod_{n=1}^{\infty} (1 - u_n)$ is considered to be convergent.

Proof 2. The second way to prove this theorem is by using the equivalent series.

Assume first that $\sum_{n=1}^{\infty} u_n$ is convergent, then when n increases, P_n can only increase. To prove that P_n approaches a finite limit, the only thing that must be done is to prove that P_n remains less than a fixed number.

It is known that

$$P_n = (1 + u_1)(1 + u_2)\ldots(1 + u_n) = 1 + \sum_{i=1}^{n} u_i P_{i-1}$$ (see equation I).

Putting

$$U_n = u_1 + u_2 + \ldots + u_n,$$

and

$$U = \lim_{n \to \infty} U_n = \sum_{i=1}^{\infty} u_i,$$

$$P_n = 1 + \frac{U_n}{1!} + \frac{U_n^2}{2!} + \ldots + \frac{U_n^n}{n!} < e^{U_n} < e^U,$$

hence P_n approaches a limit and the product is convergent.

Conversely, if the product is convergent, P_n remains finite, and

$$P_n = 1 + \sum_{i=1}^{n} u_i P_{i-1} = 1 + M \sum_{i=1}^{n} u_i$$

where M is the least upper bound of the P_i's; therefore P_n can only be convergent when the series $\sum_{i=1}^{n} u_i$ is convergent.

At this point it will be helpful to look at several examples to illustrate theorem 6. The following are two common examples.

Example 1. The infinite product $\prod_{n=2}^{\infty} (1 - 1/n^2)$ is convergent, since $\sum_{n=1}^{\infty} 1/n^2$ is convergent. This may be verified since

$$1 = \frac{1}{n^2} = \frac{(n-1)(n+1)}{n^2}$$

and Q_{n-1} can be written

$$Q_{n-1} = \frac{1 \cdot 3}{2^2} \cdot \frac{2 \cdot 4}{3^2} \cdot \frac{3 \cdot 5}{4^2} \cdots \frac{(n-1)(n+1)}{n^2} = \frac{1}{2} \cdot \frac{n+1}{n}$$

so that $\lim_{n\to\infty} Q_{n-1} = \frac{1}{2}$.

Similarly, $\prod_{n=1}^{\infty} (1 + 1/n^2)$ is convergent, although its value is not so easily calculated.

Example 2. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, the products $\prod_{n=1}^{\infty} (1 + 1/n)$ and $\prod_{n=1}^{\infty} (1 - 1/n)$ will also diverge.

For $\prod_{n=1}^{\infty} (1 + 1/n)$,

$$P_n = 2 \cdot 3/2 \cdot 4/3 \cdot 5/4 \ldots (n+1)/n = (n+1)/2$$

so that $\lim_{n\to\infty} P_n = \infty$.

Consider Q_n for the infinite product $\prod_{n=1}^{\infty} (1 - 1/n)$

$$Q_n = 1/2 \cdot 2/3 \cdot 3/4 \ldots (n-1)/n = 1/n$$

and

$$\lim_{n \to \infty} Q_n = 0.$$

This shows that they are divergent by the definition of divergence of infinite products.

Theorem 7 deals with the rearrangement of infinite products.

Theorem 7. When $u_1, u_2, u_3, \ldots$ are between 0 and 1, the values of the two infinite products $P = (1 + u_1)(1 + u_2) \ldots$ and $Q = (1 - u_1)(1 - u_2) \ldots$, are both independent of the order of the factors.

Proof.

Let the two rearrangements of P and Q be P' and Q', respectively. Now factors of P' and Q' must be taken in order to include the first n factors of P and Q. Then

$$P \geqq P_r' > P_n, \; Q \leqq Q_r' < Q_n, \text{ if } r \geqq p.$$

Now n can be taken large enough to bring P_n and Q_n as close to P and Q, respectively, as desired.

Consequently,

$$\lim_{r \to \infty} P_r' = P, \; \lim_{r \to \infty} Q_r' = Q.$$

In like manner, if P diverges to infinity, so does P'; and if Q diverges to 0, so does Q'.

In all the previous work the assumption has been made that $u_1, u_2, \ldots, u_n, \ldots$ are all less than one. This assumption can be made without any loss of generality, for there can only be a finite number of u_i with value greater than one. If this were not true, the product would diverge. Therefore these factors can be omitted without affecting the convergence of the product.

ABSOLUTE CONVERGENCE

An absolutely convergent infinite product is not a product $\prod_{k=1}^{\infty} a_k$ for which $\prod_{k=1}^{\infty} |a_k|$ converges. Such a definition would be of no value because all infinite products would then be absolutely convergent. The definition of absolute convergence is as follows.

Definition 6. If the infinite product $\prod_{n=1}^{\infty} (1 + |u_n|)$ converges, then $\prod_{n=1}^{\infty} (1 + u_n)$ is said to be absolutely convergent.

The definition of an absolutely convergent infinite product leads to the consideration of the following theorem.

Theorem 8. If the infinite series $\sum_{n=1}^{\infty} |u_n|$ converges, (i.e., if $\sum_{n=1}^{\infty} u_n$ converges absolutely), then $\prod_{n=1}^{\infty} (1 + |u_n|)$ converges, and thus $\prod_{n=1}^{\infty} (1 + u_n)$ converges absolutely. The converse theorem also holds.

The proof of theorem 8 is as follows.

It follows directly from theorem 6 that if $\sum_{n=1}^{\infty} |u_n|$ converges, then $\prod_{n=1}^{\infty} (1 + |u_n|)$ converges.

Since $\prod_{n=1} (1 + |u_n|)$ converges, by definition 6, $\prod_{n=1}^{\infty} (1 + u_n)$ converges absolutely.

The proof of the converse theorem is very similar and therefore will not be shown.

It can be proved that if an infinite product is absolutely convergent, then its factors can be reordered without affecting the value of the product. The following theorem deals with this case.

<u>Theorem 9</u>. If the infinite product $\prod_{n=1}^{\infty} (1 + u_n)$ is absolutely convergent, its value is independent of the order of its factors.

<u>Proof</u>. Consider

$$\text{(I.)} \qquad \prod_{n=1}^{\infty} (1 + u_n)$$

which is absolutely convergent. By theorem 8, the absolute convergence of (I.) implies that $\sum_{n=1}^{\infty} u_n$ converges.

Let $\sum_{n=1}^{\infty} v_n$ be a series obtained by rearranging the terms of $\sum_{n=1}^{\infty} u_n$ in any order, being sure to include all the terms.

Write $|u_n| = a_n$ and $|v_n| = b_n$, then

$U_n = (1 + u_1)\ldots(1 + u_n)$, $V_n = (1 + v_1)\ldots(1 + v_n)$

$A_n = (1 + a_1)\ldots(1 + a_n)$, $B_n = (1 + b_1)\ldots(1 + b_n)$.

Then choose $p > n$ so that U_p contains the whole of V_n and consequently A_p contains the whole of B_n; on multiplying out it is evident that $\frac{A_p}{B_n} - 1$ contains every term in $\frac{U_p}{V_n} - 1$, but with the signs made positive. Hence

$$\left| \frac{U_p}{V_n} - 1 \right| \leqq \frac{A_p}{B_n} - 1$$

and

$$V_n \leqq B_n$$

also

$$\left| \frac{U_p - V_n}{V_n} \right| \leqq \frac{A_p - B_n}{B_n}$$

so that

$$|U_p - V_n| \leqq A_p - B_n .$$

Now, as explained in theorem 7,

$$\lim_{n\to\infty} B_n = \lim_{n\to\infty} A_n = A \text{ for } 0 < u_i < 1,\ i = 1, 2, \ldots, n.$$

Consequently an n_0 can be found so great that

$$A > A_p > B_n > A - \frac{1}{2}\epsilon \text{ , if } p > n > n_0.$$

Hence

$$A_p - B_n < \frac{1}{2}\epsilon \text{ , if } n > n_0 \text{ ,}$$

and so

$$|U_p - V_n| < \frac{1}{2} \quad \text{, if } n > n_0.$$

But $\lim_{p\to\infty} U_p = U$, and therefore if $p > n > n_1, \left| U - U_p \right| < \frac{1}{2}\epsilon$.

Thus if $n' > n_0$ and $n' > n_1$,

$$\left| U - V_n \right| < \epsilon \text{, if } n > n',$$

that is, $\lim_{n\to\infty} V_n = U$, and the theorem is proved.

As in the case of infinite series, a non-absolutely convergent infinite product may be made to converge to any value, or to diverge, by altering the order of the factors.

THE ASSOCIATE LOGARITHMIC SERIES

Attention will now be turned to relating the theory of infinite products to the theory of infinite series by taking logarithms. The following definition is needed before this can be done.

Definition 7. The series $\sum_{n=1}^{\infty} \ln(1 + u_n)$ is called the associate logarithmic series of the infinite product $\prod_{n=1}^{\infty} (1+u_n)$.

Now that this basic definition has been stated, the following theorem will be considered.

Theorem 10. The positive infinite product $\prod_{n=1}^{\infty} (1 + u_n)$ and associate logarithmic series $\sum_{n=1}^{\infty} \ln(1 + u_n)$ converge or diverge simultaneously.

Proof.

Assume the convergence of $\prod_{n=1}^{\infty} (1 + u_n)$. Then the sequence

$$\prod_{i=1}^{n} (1 + u_i) = P_n$$

approaches a finite nonzero limit, say P. Put

$$\lambda_n = \sum_{i=1}^{n} \ln(1 + u_i)$$

then

$$\ln P_n = \lambda_n$$

$$P_n = e^{\lambda_n} .$$

Because $\prod_{n=1}^{\infty} (1 + u_n)$ is convergent, P_n approaches the finite nonzero limit, P, and hence λ_n approaches a limit, say λ, and $P = e^{\lambda}$. Therefore $\sum_{n=1}^{\infty} \ln(1 + u_n)$ converges.

If $\sum_{n=1}^{\infty} \ln(1 + u_n)$ is assumed to be convergent, then

$$\lambda_n = \sum_{i=1}^{n} \ln(1 + u_i)$$

approaches λ, a nonzero, finite limit. Then, because $\sum_{n=1}^{\infty} \ln(1 + u_n)$ is the associate logarithmic series of

$$\prod_{n=1}^{\infty} (1 + u_n), \quad \ln P_n = \lambda_n$$

and because

$$\lim_{n \to \infty} \lambda_n = \lambda , \quad \lim_{n \to \infty} P_n = P$$

and

$$\ln P = \lambda.$$

The proofs involving the divergence of the product and its associate logarithmic series follow readily. For this reason they will be omitted.

The next theorem relates the convergence and divergence of the associate logarithmic series to the convergence and divergence of the associate infinite series.

Theorem 11. The positive series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} \ln(1 + u_n)$ converge or diverge simultaneously.

Proof.

The power series expansion

$$\ln(1 + u_n) = u_n - \frac{u_n^2}{2} + \frac{u_n^3}{3} + \dots$$

is valid for $\left| u_n \right| < 1$. If $0 \quad u_n < 1$, from the theory of alternating series it is known that

$$u_n > \ln(1 + u_n) > u_n - \frac{u_n^2}{2} = u_n(1 - \frac{u_n}{2}) > \frac{u_n}{2} .$$

Therefore the two series, $\sum_{n=1}^{\infty} u_n$, $\sum_{n=1}^{\infty} \ln(1 + u_n)$ converge or diverge together, by well known comparison tests.

The problem involved in the practical determination of the convergence of a series of the form $\sum_{n=1}^{\infty} \ln(1 + u_n)$ is usually very difficult. The next theorem is a step in the direction of reducing this difficulty.

Theorem 12. If the series $\sum_{n=1}^{\infty} u_n$ is convergent, and the series $\sum_{n=1}^{\infty} u_n^2$ is convergent, then $\sum_{n=m+1}^{\infty} \ln(1 + u_n)$ and with it the infinite product $\prod_{n=m+1}^{\infty} (1 + u_n)$ is convergent.

Consider the identity

$$\ln(1 + u_n) = u_n - \frac{1}{2} u_n^2 + \frac{1}{3} u_n^3 - \frac{1}{4} u_n^4 + \dots \ .$$

Let

$$K_n = \frac{1}{2} - \frac{1}{3} u_n + \frac{1}{4} u_n^2 + \dots$$

which converges if $|u_n| < 1$. Then

$$\ln(1 + u_n) = u_n - K_n u_n^2 \ .$$

Because $\sum_{n=1}^{\infty} u_n$ has been assumed to be convergent, $\lim_{n\to\infty} u_n = 0$ and because $K_n \to \frac{1}{2}$ as $u_n \to 0$, K_n has an upper bound, K.

Therefore for every integer $p > 0$,

$$\sum_{n=m+1}^{m+p} \ln(1 + u_n) = \sum_{n=m+1}^{m+p} u_n - \sum_{n=m+1}^{m+p} K \cdot u_n^2 \ .$$

Because $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} u_n^2$ have been assumed to be convergent and because the difference of two convergent series is convergent, $\sum_{n=m+1}^{\infty} \ln(1 + u_n)$ in convergent.

If $\sum_{n=m+1}^{\infty} \ln(1 + u_n)$ is convergent, then $\prod_{n=m+1}^{\infty} (1 + u_n)$ is convergent and the theorem is proved.

UNIFORM CONVERGENCE

The concept of uniform convergence of infinite products is easily defined by analogy with infinite series or sequences in general.

Definition 8. If

$$\prod_{i=1}^{n} \left[1 + u_i(x)\right] = P_n(x)$$

and

$$\prod_{n=1}^{\infty} \left[1 + u_n(x)\right] = P(x)$$

$P_n(x)$ converges uniformly to $P(x)$ in a region R if, given any $\epsilon > 0$, there exists an N, depending only on ϵ and not on the particular value of x in R, such that

$$\left| P_n(x) - P(x) \right| < \epsilon$$

for all $n > N$.

The next theorem associates uniform convergence of infinite products with the uniform convergence of the associate logarithmic series.

Theorem 13. The infinite product

$$\prod_{n=1}^{\infty} \left[1 + u_n(x)\right] = P(x)$$

is uniformly convergent in a region R if and only if the series

$$\lambda(x) = \sum_{n=1}^{\infty} \ln\left[1 + u_n(x)\right]$$

is uniformly convergent in R.

The proof is as follows. If the associate logarithmic series $\sum_{n=1}^{\infty} \ln 1 + u_n(x)$ is uniformly convergent, then in a region R if, given any $\epsilon > 0$, there exists an N depending on and not on the particular value of x in R, such that

$$\left| \lambda_n(x) - \lambda(x) \right| < \epsilon \quad \text{for } n > N.$$

Put

$$\lambda(x) = \lambda_n(x) + \mu_n(x)$$

so that

$$\lambda(x) - \lambda_n(x) = \mu_n(x)$$

and there exists an N and an $\epsilon > 0$ such that for $n > N$

$$\left| \mu_n(x) \right| < \epsilon$$

for every x in R. Then, because

$$\ln P(x) = \lambda(x) \quad \text{and} \quad \ln P_n(x) = \lambda_n(x)$$

$$P(x) = e^{\lambda(x)} = e^{\lambda_n(x) + \mu_n(x)} = e^{\lambda_n(x)} \cdot e^{\mu_n(x)}$$

$$= P_n(x) \cdot e^{\mu_n(x)}$$

so that

$$P_n(x) = P(x) \cdot e^{-\mu_n(x)}$$

and

$$P(x) + P_n(x) + P(x) - P(x) \cdot e^{-\mu_n(x)}$$

and

$$\left| P(x) - P_n(x) \right| = \left| P(x) \right| \left| 1 - e^{-\mu_n(x)} \right|$$

Since there exists an N and an $\epsilon > 0$, such that for $n > N$

$$\left| \mu_n(x) \right| < \epsilon$$

for every x in R, $\mu_n(x)$ approaches its limit zero uniformly as n approaches infinity, hence $e^{-\mu_n(x)}$ approaches its limit one uniformly as n approaches infinity.

Since $P(x)$ is finite and $e^{-\mu_n(x)}$ approaches its limit one uniformly as n approaches infinity

$$\left| P(x) - P_n(x) \right| < \epsilon'$$

and the product

$$\prod_{n=1}^{\infty} \left[1 + u_n(x) \right]$$

must be uniformly convergent in R.

Conversely, if the product is uniformly convergent in R, there exists an N and an $\epsilon > 0$ such that for $n > N$

$$\left| P(x) - P_n(x) \right| < \epsilon$$

for every x in R. Put

$$P(x) = P_n(x) \left[1 + \phi_n(x) \right] = P_n(x) + P_n(x)\phi_n(x)$$

then

$$\left| P(x) - P_n(x) \right| = \left| P_n(x) \right| \left| \phi_n(x) \right| < \epsilon$$

Since $\left| P_n(x) \right|$ is finite and greater than zero, there exists an N and an $\epsilon_2 > 0$ such that for $n > N$, $\left| \phi_n(x) \right| < \epsilon_2$ for every x in R. Then, because

$$\ln P(x) = \lambda(x) \quad \text{and} \quad \ln P_n(x) = \lambda_n(x)$$

on taking logarithms

$$P(x) = P_n(x) \left[1 + \phi_n(x) \right]$$

becomes

$$\lambda(x) = \lambda_n(x) + \ln \left[1 + \phi_n(x) \right]$$

and

$$\lambda(x) - \lambda_n(x) = \ln \left[1 + \phi_n(x) \right] .$$

Since there exists an N and an $\epsilon_2 > 0$ such that for $n > N$,

$$\left| \phi_n(x) \right| < \epsilon_2$$

for every x in R, $\phi_n(x)$ approaches its limit zero uniformly as n approaches infinity and

$$\ln\left[1 + \phi_n(x)\right]$$

approaches its limit zero uniformly as n approaches infinity, which implies that

$$\left|\ln\left[1 + \phi_n(x)\right]\right| < \epsilon_3$$

for $n > N$, $\epsilon_3 > 0$ for all x in R and

$$\left|\lambda(x) - \lambda_n(x)\right| < \epsilon_3$$

which proves that the series

$$\sum_{n=1}^{\infty} \ln\left[1 + u_n(x)\right]$$

converges uniformly in R and the theorem is proved.

Uniform convergence of the infinite product

$$\prod_{n=1}^{\infty}\left[1 + u_n(x)\right]$$

will be related to the convergence of the series

$$\sum_{n=1}^{\infty} u_n(x)$$

in the following theorem.

Theorem 14. If the series of positive terms

$$\sum_{n=1}^{\infty}\left|u_n(x)\right|$$

is uniformly convergent in a region R, the infinite product

$$\prod_{n=1}^{\infty}\left[1 + u_n(x)\right]$$

converges uniformly in R, provided none of the functions $u_n(x)$ takes the value -1 as x ranges over R.

The proof of this theorem makes use of the identity

$$\ln\left[1 + u_n(x)\right] = u_n(x) \cdot k_n$$

where k_n is given in the following identity.

$$\ln(1 + u_n(x)) = u_n(x) - \frac{u_n(x)^{2}}{2} + \frac{u_n(x)^{3}}{3} - \cdots$$

$$+ (-1)^{n-1} \frac{u_n(x)^{n}}{n} + \cdots$$

$$= u_n(x) \quad 1 - \frac{u_n(x)}{2} + \frac{u_n(x)^{2}}{3} - \cdots$$

$$+ (-1)^{n-1} \frac{u_n(x)^{n-1}}{n} + \cdots$$

$$= u_n(x) \cdot k_n$$

Because $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent, $\sum_{n=1}^{\infty} u_n(x)$ is convergent, therefore $\lim_{n \to \infty} u_n(x) = 0$. Also, k_n converges for $0 < u_i < 1$, $i = 1, 2, \ldots, n$. As $\mu_n \to 0$, $k_n \to 1$, therefore k_n has an upper bound for each n. Call the greatest of these upper bounds G. Then

$$\left| \sum_{i=n+1}^{n+p} \ln\left[1 + u_i(x)\right] \right| \leqq |G| \cdot \sum_{i=n+1}^{n+p} |u_i(x)| .$$

Therefore $\sum_{i=1}^{\infty} \ln\left[1 + u_n(x)\right]$ is uniformly convergent and by theorem 13, $\prod_{n=1}^{\infty} \left[1 + u_n(x)\right]$ is uniformly convergent in R.

Theorem 15 is the analogue of Weierstrass' M-test for infinite series.

<u>Theorem 15</u>. If $\sum_{n=1}^{\infty} M_n$ is a convergent series of positive constants, and if for every n and for every x in a region R,

$$\left| u_n(x) \right| \leqq M_n$$

the infinite product $\prod_{n=1}^{\infty} \left[1 + u_n(x) \right]$ will be uniformly convergent for every x in R.

The proof will use the Weierstrass M-test for infinite series. If $\sum_{n=1}^{\infty} M_n$ is a convergent series of positive constants, and if for every n and for every x in a region R,

$$\left| u_n(x) \right| \leqq M_n$$

the infinite series $\sum_{n=1}^{\infty} u_n(x)$ will be uniformly convergent for every x in R.

Because $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent, by theorem 14,

$$\prod_{n=1}^{\infty} \left[1 + u_n(x) \right]$$

is uniformly convergent for every x in R and the theorem is proved.

DEVELOPMENT OF TRIGONOMETRIC FUNCTIONS AS INFINITE PRODUCTS

Sin x and cos x will be the trigonometric functions to be developed in this section. The development of sin x is as follows. The identity

$$\sin nx = a_0 \sin^n x + a_1 \sin^{n-1}(x) + \ldots + a_{n-1} \sin x$$

where n is an odd integer and where the coefficients a_i are integers will be used for developing an infinite product representation of sin x.

If sin x = t,

$$\sin nx = F_n(t) = a_0 + {}^{n} + a_1 + {}^{n-1} + \ldots + a_{n-1} + ,$$

n finite.

Since $F_n(t)$ is a polynomial of degree n, by the fundamental theorem of Algebra, $F_n(t)$ must have n roots, which correspond to the values of x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ which make sin nx = 0, for n finite, namely,

$$0, \pm \sin \frac{\pi}{n}, \pm \sin \frac{2\pi}{n}, \ldots, \pm \sin \frac{n-1}{2} \cdot \frac{\pi}{n}.$$

$F_n(t)$ can now be written,

$$F_n(t) = a_0 t\left(t - \sin \frac{\pi}{n}\right)\left(t + \sin \frac{\pi}{n}\right)\left(t - \sin \frac{2\pi}{n}\right) \ldots$$

$$= a_0 t\left(t^2 - \sin^2 \frac{\pi}{n}\right)\left(t^2 - \sin^2 \frac{2\pi}{n}\right)\ldots\left(t^2 - \sin^2 \frac{n-1}{2} \cdot \frac{\pi}{n}\right)$$

$$= a_0 \sin x\left(\sin^2 x - \sin^2 \frac{\pi}{n}\right)\left(\sin^2 x - \sin^2 \frac{2\pi}{n}\right) \ldots$$

$$\left(\sin^2 x - \sin^2 \frac{n-1}{2} \cdot \frac{\pi}{n}\right)$$

Multiplying and dividing the right side through by

$$\sin^2 \frac{\pi}{n} \cdot \sin^2 \frac{2\pi}{n} \ldots \sin^2 \frac{n-1}{2} \cdot \frac{\pi}{n}$$

yields

$$\sin nx = K \cdot \sin x \left\{1 - \frac{\sin^2 x}{\sin^2 \frac{\pi}{n}}\right\}\left\{1 - \frac{\sin^2 x}{\sin^2 \frac{2\pi}{n}}\right\} \ldots$$

$$\left\{1 - \frac{\sin^2 x}{\sin^2 \frac{n-1}{2} \cdot \frac{\pi}{n}}\right\} \qquad (1)$$

where K is a constant.

Dividing (1) by sin x, and observing that

$$\lim_{x \to 0} \frac{\sin nx}{\sin x} = n$$

K is found to be equal to n. Now replace x by x/n, and insert K = n.

$$\sin x = n \cdot \sin^{x/n} \cdot \prod_{r=1}^{n-1/2} \left\{1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}}\right\}$$

Put

$$P(x) = \prod_{r=1}^{\infty} \left(1 - \frac{x^2}{r^2\pi^2}\right)$$

$$P(\sin x,\, n) = \prod_{r=1}^{\infty} \left\{ 1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}} \right\}$$

$$\lambda(x) = \ln P(x) = \sum_{r=1}^{\infty} \ln \left(1 - \frac{x^2}{r^2\pi^2} \right)$$

and

$$\lambda(\sin x,\, n) = \ln P(\sin x, n) = \sum_{r=1}^{\infty} \ln \left\{ 1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}} \right\}$$

Write

$$\lambda(\sin x, n) = \lambda_m(\sin x, n) + \overline{\lambda}_m(\sin x, n)$$

where

$$\lambda_m(\sin x, n) = \sum_{r=1}^{m} \ln \left\{ 1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}} \right\}$$

and

$$\overline{\lambda}_m(\sin x, n) = \sum_{r=m+1}^{\infty} \ln \left\{ 1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}} \right\}$$

Using the same notation, write

$$\lambda(x) = \lambda_m(x) + \overline{\lambda}_m(x).$$

Then

$$\left|\lambda(\sin x,n) - \lambda(x)\right| \leqq \left|\lambda_m(\sin x,n) - \lambda_m(x)\right| + \left|\bar{\lambda}_m(\sin x,n)\right| + \left|\bar{\lambda}_m(x)\right| .$$

The next step is to prove that if $0 \quad x \quad \frac{\pi}{2}$, then

$$\frac{x}{2} < \sin x < x$$

which is the same as proving that $\frac{1}{2} < \frac{\sin x}{x} < 1$.

Let $f(x) = \frac{\sin x}{x}$. Consider the derivative of $f(x)$.

$$f'(x) = \frac{x \cdot \cos x - \sin x}{x^2} = \frac{\cos x(x - \tan x)}{x^2} < 0$$

Because $f'(x) < 0$, $f(x)$ is decreasing; therefore $f(x)$ reaches its maximum as x approaches 0 and its minimum as x approaches $\frac{\pi}{2}$. As x approaches 0, $f(x)$ approaches 1, and as x approaches $\frac{\pi}{2}$, $f(x)$ approaches $\frac{2}{\pi} \cdot \frac{2}{\pi} > \frac{1}{2}$; therefore

$$\frac{1}{2} < \frac{\sin x}{x} < 1$$

and

$$\frac{x}{2} < \sin x < x .$$

So if $0 < x < \frac{\pi}{2}$, $\frac{x}{2} < \sin x < x$; therefore if r is greater than some m,

$$\frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}} > \frac{\frac{x^2}{4n^2}}{\frac{r^2\pi^2}{n^2}} = \frac{x^2}{4r^2\pi^2}$$

therefore for m_1 such that $\frac{x}{m_1} < 1$,

$$\text{(I)} \quad \ln \left\{ 1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}} \right\} \quad \ln \left(1 - \frac{x^2}{4\pi^2 r^2} \right), \quad r > m.$$

Next, the inequality

$$-\ln(1 - a) < a + Ka^2$$

will be proved for $0 < a < 1$, K a constant.

Consider the identity

$$-\ln(1 - a) = a + \frac{a^2}{2} + \frac{a^3}{3} + \frac{a^4}{4} + \ldots$$

$$= a + \frac{a^2}{2} \left[\frac{1}{2} + \frac{a}{3} + \frac{a^2}{4} + \ldots \right]$$

Let

$$K_n = \frac{1}{2} + \frac{a}{3} + \frac{a^2}{4} + \ldots$$

which is convergent for a $|a| < 1$. Then, as was shown in theorem 12, K_n has an upperbound, K, and

$$-\ln(1 - a) < a + Ka^2$$

for K a constant, $0 < a < 1$. Therefore

$$-\sum_{r=m_1}^{\infty} \ln\left(1 - \frac{x^2}{4\pi^2 r^2}\right) < \sum_{r=m_1}^{\infty} \frac{x^2}{4\pi^2 r^2} + K \sum_{r=m_1}^{\infty} \left(\frac{x^2}{4\pi^2 r^2}\right)^2$$

$$< \frac{x^2}{4\pi^2} \sum_{r=m_1}^{\infty} \frac{1}{r^2} + K \cdot \frac{x^4}{16\ \pi^4} \sum_{r=m_1}^{\infty} \frac{1}{r^4}$$

But $\sum_{r=m_1}^{\infty} \frac{1}{r^2}$ and $\sum_{r=m_1}^{\infty} \frac{1}{r^4}$ are convergent, hence

$$\sum_{r=m_1}^{\infty} \ln\left(1 - \frac{x^2}{4\pi^2 r^2}\right)$$

is convergent, which implies that

$$\prod_{r=m_1}^{\infty} \left(1 - \frac{x^2}{4\pi^2 r^2}\right)$$

is convergent, and because of inequality (I),

$$\lambda(\sin x, n) = \sum_{r=m_1}^{\infty} \left\{ \ln \left\{ 1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}} \right\} \right\}$$

is convergent. Therefore if m is sufficiently large, it follows that

$$\left| \bar{\lambda}_m(\sin x, n) \right| < \frac{\epsilon}{3} \quad \text{and} \left| \bar{\lambda}_m(x) \right| < \frac{\epsilon}{3}$$

The next step deals with showing that

(II) $\left| \lambda_m(\sin x, n) - \lambda_m(x) \right| < \epsilon/3$.

Consider

$$\text{(III)} \qquad \lim_{n\to\infty} \frac{\left\{1 - \dfrac{\sin^2 \frac{x}{n}}{\sin^2 \frac{\pi}{n}}\right\} \cdots \left\{1 - \dfrac{\sin^2 \frac{x}{n}}{\sin^2 \frac{m\pi}{n}}\right\}}{(1 - \frac{x^2}{\pi^2}) \quad \cdots \quad (1 - \frac{x^2}{\pi^2 m^2})}$$

Looking at

$$\lim_{n\to\infty} \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{\pi}{n}} = \lim_{n\to\infty} \frac{\sin^2 \frac{x}{n} \cdot \frac{x^2}{n^2} \cdot \frac{\pi^2}{n^2}}{\sin^2 \frac{\pi^2}{n^2} \cdot \frac{x^2}{n^2} \cdot \frac{\pi^2}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\sin^2 \frac{x}{n}}{\frac{x^2}{n^2}} \cdot \frac{\frac{\pi^2}{n^2}}{\sin^2 \frac{\pi}{n}} \cdot \frac{\frac{x^2}{n^2}}{\frac{\pi^2}{n^2}} \to \frac{x^2}{\pi^2}$$

and it can be seen that (III) approaches zero as n approaches infinity. But (III) is equal to

$$\lim_{n\to\infty} \left[\lambda_m(\sin x,n) - \lambda_m(x)\right] \longrightarrow 0$$

which implies that

$$\left|\lambda_m(\sin x,n) - \lambda_m(x)\right| < \epsilon/3$$

which is (II). Then

$$\left|\lambda(\sin x,n) - \lambda(x)\right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

so that

$$\lim_{n\to\infty} \lambda(\sin x,n) = \lambda(x)$$

It follows that

$$\lim_{n\to\infty} P(\sin x,n) = \lim_{n\to\infty} e^{\lambda(\sin x,n)} = e^{\lambda(x)} = P(x)$$

Then

$$\lim_{n\to\infty} (n \sin \frac{x}{n}) = \lim_{n\to\infty} x \cdot \frac{\sin \frac{x}{n}}{x/n} = x$$

hence

$$\sin x = x \cdot \prod_{r=1}^{\infty} (1 - \frac{x^2}{r^2\pi^2})$$

The infinite product representation of cos x can be derived from that of sin x. The identity $\sin 2x = 2 \sin x \cos x$ will be used.

$$\cos x = \frac{1}{2} \cdot \frac{2x}{x} \cdot \frac{\prod_{n=1}^{\infty} (1 - \frac{4 x^2}{n^2\pi^2})}{\prod_{n=1}^{\infty} (1 - \frac{x^2}{n^2\pi^2})}$$

$$= \frac{\prod_{m=1}^{\infty} (1 - \frac{x^2}{m^2\pi^2})}{\prod_{n=1}^{\infty} (1 - \frac{x^2}{n^2\pi^2})} \cdot \prod_{m=1}^{\infty} (1 - \frac{4 x^2}{(2m-1)^2 \pi^2})$$

Therefore

$$\cos x = \prod_{n=1}^{\infty} (1 - \frac{4 x^2}{(2n-1)^2 \pi^2})$$

There are many more things that could be said about infinite products. More time could be spent on all the material covered in this paper; discussions of the work done by Euler and Weierstrass, discussion of the Gamma function, Beta function, Bessel function, Neumann function, and Zeta function could fill many pages; but the purpose of this paper was to develop infinite products from the definition and to bring them to the point where enough background is laid so that work involving them can be understood.

ACKNOWLEDGMENT

The author wishes to express her thanks and appreciation to Professor R. G. Sanger for his helpful suggestions, comments, and careful checking of this report.

BIBLIOGRAPHY

1. Bromwich, T. J.
An Introduction to the Theory of Infinite Series, London: Macmillan Company, 1908.

2. Chrystal, G.
Algebra, Edinburgh: Adam and Charles Black, 1889.

3. Ritt, J. F.
Theory of Functions, New York: King's Crown Press, 1947.

4. Smail, Lloyd L.
Elements of the Theory of Infinite Processes, New York: McGraw-Hill Book Company, Inc., 1923.

5. Taylor, Angus E.
Advanced Calculus, Ginn and Company, 1955.

6. Titchmarsh, E. C.
The Theory of Functions, Oxford: Oxford Clarendon Press, 1932.

7. Watson, G. N., and E. T. Whittaker.
A Course of Modern Analysis, Cambridge: Cambridge University Press, 1915.

8. Wilson, Edwin Bidwell.
Advanced Calculus, Boston: Ginn and Company, 1912.

ON INFINITE PRODUCTS

by

ROSE KORDONOWY SHAW

B. S., Dickinson State College, 1964

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1966

The purpose of this report was to make a basic study of infinite products. The study started with the definition of an infinite product and continued on to definitions of convergence, divergence, and oscillation of such a product. Examples followed these basic definitions in an attempt to clarify their meaning. Restrictions were introduced so that only infinite products dealing with positive real numbers were to be dealt with in the report.

Necessary and sufficient conditions for the convergence of an infinite product were considered next. One of these theorems was used to change the form of the infinite product to the standard form.

The next step was to relate the testing for convergence of an infinite product to that of certain infinite series. Before the theorem which gives this reduction could be given, two things had to be developed. The first was the development of the equivalent series, and the second was the Weierstrass theorem on inequalities.

As in the case of infinite series, some infinite products are absolutely convergent. Hence the definition of absolute convergence and theorems which made this definition meaningful were considered.

The convergence and divergence of infinite products were reduced to the theory of convergence and divergence of infinite series by taking logarithms. The theory in this section was useful in the study of uniform convergence.

The last topic considered was the development of the sine and cosine in terms of infinite products. This last section served as a demonstration of the use of infinite products in the development of the expansion of functions.