VARIANCE ESTIMATE OF THE MARGINAL PRODUCTIVITY FOR COBB-DOUGLAS FUNCTION

by

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TABLE OF CONTENTS

	Page
ACKNOWLEDGMENTS	i
CHAPTER	
I INTRODUCTION	1
II CURRENT VARIANCE ESTIMATE FOR MARGINAL PRODUCTIVITY AND ITS SHORTCOMING	6
III DERIVATION OF VARIANCE FORMULAS.	9
One Variate Case	9
K-Variate Case	11
IV SIMPLE NUMERICAL EXAMPLE	14
V DISCUSSION AND SUMMARY	18
APPENDIX	22
REFERENCES	23

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CHAPTER I

INTRODUCTION

The statistical investigations into laws of production by C. W. Cobb and Paul H. Douglas are among the most celebrated in the history of econometrics. Statisticians have proposed the general function

$$Y = AX_1^{B_1} X_2^{B_2} X_3^{B_3}\epsilon , \qquad (1)$$

where Y = output,

X_i = inputs; $(i = 1, 2, 3)$,

ϵ = random variation,

as a fairly universal law of production; and have estimated it in numerous samples from manufacturing industries throughout the world. This exponential type of production function has no more claim to general validity as a decription of technology than other mathematical functions. However, it does have many properties that make it a very convenient choice and it graduates data on output and input well. This function is almost always referred to as the "Cobb-Douglas production function".

The Cobb-Douglas function has constant elasticities of output variation with respect to its various inputs, where B_i equals the elasticity with respect to an input X_i. This function has a non-linear relationship. For constant levels of X_2 and X_3, the input (X_1) - output relationship is shown in Figure 1.

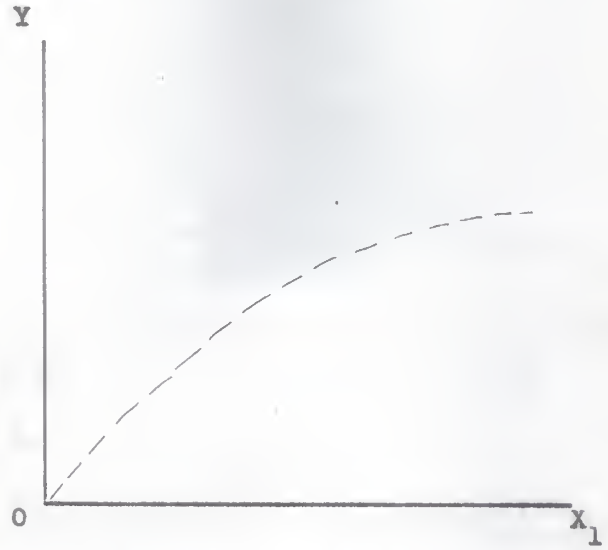

Fig. 1. A Cobb-Douglas Production Function

If any input is zero, output is zero. Thus all inputs must be non-zero. The curvature of the function is such that marginal productivity falls as input grows if each elasticity is less than unity. There is no asymptotic level of output (or ceiling) beyond which production cannot grow, but the rate of increase decreases at high levels of input.

Although the function is non-linear, it can be transformed into a linear function by converting all variables to logarithms. Thus,

$$\log Y = \log A + B_1 \log X_1 + B_2 \log X_2 + B_3 \log X_3 + \log \epsilon$$

or

$$Y' = A' + B_1 X'_1 + B_2 X'_2 + B_3 X'_3 + \epsilon' .$$

Since the B_i's are elasticity coefficients, they are pure numbers and can easily be compared among different samples using varied units of measurements. Therefore, this function is very convenient in inter-industry or international comparisons. In a sense, one is able to capture the flavor of essential nonlinearities of the production process and yet benefit from the simplifications of calculation from linear relationship by transforming to logarithms.

The parameters (B_i's) of equation (1), in addition to being elasticities, possess other attributes important in economic analysis. The sum of the exponents shows the degree of "returns to scale" in production. Thus,

$\sum_{i=1}^{3} B_i < 1$ indicates decreasing returns to scale,

$\sum_{i=1}^{3} B_i = 1$ indicates constant returns to scale,

$\sum_{i=1}^{3} B_i > 1$ indicates increasing returns to scale.

Suppose that each input is increased by r per cent

$$X_i \text{ increased to } X_i\left(1 + \frac{r}{100}\right),\ i = 1, 2, 3.$$

Thus the output is increased by less than r per cent, by r per cent, or by more than r per cent, depending on whether there are decreasing, constant, or increasing returns to scale.

Marginal productivity of any factor is the slope of the function when all other inputs are held constant. It can also be noted that the marginal productivity changes as the levels of factor input change. However, the Cobb-Douglas function takes on a linear form when expressed in logarithms instead of arithmetic units. Therefore, with inputs X_2 and X_3 being held constants,

$$B_1 = \frac{\text{Change in log of output}}{\text{Change in log of } X_1} .$$

The change in natural logarithm of some variable is the same thing as the percentage change, we, therefore, can write

$$B_1 = \frac{\text{Percentage change in } Y}{\text{Percentage change in } X_1}$$

$$= \frac{\Delta Y}{Y + \Delta Y} \bigg/ \frac{\Delta X_1}{X_1 + \Delta X_1}$$

$$= \frac{X_1 + \Delta X_1}{Y + \Delta Y} \cdot \frac{\Delta Y}{\Delta X_1}$$

$$= \frac{X_1 \text{ total input}}{\text{total output}} \cdot MP_{X_1} .$$

Hence, an important property of the function is: marginal and average productivity are proportional, where the factor of proportionality is the associated exponent.

$$MP_{X_1} = B_1 \cdot \frac{\text{total output}}{X_1 \text{ total input}}$$

$$= B_1 \cdot (\text{average productivity of } X_1) .$$

The same relationships hold for X_2 and X_3.

CHAPTER II

CURRENT VARIANCE ESTIMATE FOR MARGINAL PRODUCTIVITY AND ITS SHORTCOMING

For the purpose of simplicity, consider a one-variate Cobb-Douglas function

$$Y_i = AX_i^B \epsilon_i , \quad i = 1, 2, \ldots, n. \qquad (2)$$

The usual procedure of estimating these constants A and B is to apply least-squares methed to the logarithms of (2) resulting in the regression equation.

$$\log Y_i = \log A + B\log X_i + \log \epsilon_i \quad ,$$

or

$$Y_i' = A' + BX_i' + \epsilon_i'$$

where $\quad Y_i' = \log Y_i$

$A' = \log A$

$X_i' = \log X_i$

and $\quad \epsilon_i' = \log \epsilon_i \quad .$

The usual assumption is that the ϵ_i' are independent error variables with equal variances.

The least-square estimates of A', B and A are

$$b = \frac{\Sigma(Y_i' - \bar{Y}')(X_i' - \bar{X}')}{\Sigma(X' - \bar{X}')^2}$$

$$a' = \bar{Y}' - b\bar{X}'$$

and $\quad a = e^{a'}$

For a given fixed input value X, thus the estimated output is given by

$$\hat{Y} = aX^b \tag{3}$$

or $$\hat{Y}' = a' + bX' .$$

Thus, the marginal productivity at X is estimated by

$$d\hat{Y}/dX = abX^{b-1} = (b \cdot \hat{Y})/X \tag{4}$$

To find the variance of the marginal productivity, the current method is obtained from (4) by regarding both X and Y as constants. Therefore, the current formula for variance is

$$V\left(\frac{b\,\hat{Y}}{X}\right) = \left(\frac{\hat{Y}}{X}\right)^2 \cdot V(b) \tag{5}$$

where $V(b) = s^2/\Sigma(X_i' - \bar{X}')^2$

and $$s^2 = \frac{\Sigma(Y_i' - \bar{Y}')^2 - b^2\Sigma(X_i' - \bar{X}')^2}{n - 2}$$

However, one must examine the assumptions being made above. The assumption of fixed input level X is, of course, reasonable and generally used by statisticians in regression theory. But, the assumption that $\hat{Y}$ is also fixed is clearly unrealistic since it is known to be computed from formula (3), which involves the estimates of 'a' and 'b', and hence will vary from sample to sample even though X is held constant. It is also deemed unsuitable to regard 'a' and 'b' as constants. It is the purpose of this report to reach a more realistic variance estimate for the marginal productivity which considers $\hat{Y}$ and both 'a' and 'b' as variables with non-zero variances and covariances, and only X being held as a constant. In other words, the current variance formula for the marginal productivity may underestimate the real variability in some cases, and overestimate the true variability in others. A numerical example will be given later to illustrate the difference of the two formulas. It is also the purpose of this report to identify, if possible, those conditions under which equation (5) would under or over estimate the true variance.

CHAPTER III

DERIVATION OF VARIANCE FORMULAS

One Variate Case

The variance of X and Y can be written

$$\begin{aligned} V(XY) &= E\{(XY)^2\} - \{E(XY)\}^2 \\ &= \mu_X^2\,\mu_Y^2\left\{\sigma_X^2/\mu_X^2 + \sigma_Y^2/\mu_Y^2 + 2\mathrm{cov}(XY)/\mu_X\mu_Y\right\} \\ &= \mu_Y^2\,\sigma_X^2 + \mu_X^2\,\sigma_Y^2 + 2\mu_X\mu_Y\mathrm{cov}(XY)\ . \end{aligned}$$

From formula (4), but considering only X as a constant, the estimated variance for the marginal productivity will be

$$\begin{aligned} V\left(\frac{b\,\hat{Y}}{X}\right) &= \frac{1}{X^2}\,V(b\cdot\hat{Y}) \\ &= \frac{1}{X^2}\left\{E(b\cdot\hat{Y})^2 - \left[E(b\cdot\hat{Y})\right]^2\right\} \\ &= \frac{1}{X^2}\left\{B^2V(\hat{Y}) + \bar{Y}^2V(b) + 2B\bar{Y}\ \mathrm{cov}(b\cdot\hat{Y})\right\}\ , \qquad (6) \end{aligned}$$

where $\bar{Y} = E(\hat{Y})$

and $B = E(b)$.

Furthermore, from regression theory, it is known that

$$V(b) = s^2/\Sigma(X' - \bar{X}')^2 \qquad (7)$$

and $$V(\hat{Y}') = \sigma^2_{Y'\cdot X'}\left[\frac{1}{n} + \frac{(X' - \bar{X}')^2}{\Sigma(X_i' - \bar{X}')^2}\right]\ ,$$

where $X' = \log X$ corresponding to the value at that point the marginal productivity is evaluated.

The covariance of b and $\hat{Y}$ is given by

$$\mathrm{cov}(b\hat{Y}') = \sigma^2_{Y'.X'}\left[\frac{(X' - \bar{X}')}{\Sigma(X_i' - \bar{X}')^2}\right]^1.$$

Since equation (3) may also be written as

$$\hat{Y} = aX^b = e^{\hat{Y}'},$$

then, $$d\hat{Y} = e^{\hat{Y}'}\cdot d\hat{Y}' = \hat{Y}\cdot d\hat{Y}'$$

From this an approximate relationship becomes

$$V(\hat{Y}) = \bar{Y}^2 V(\hat{Y}')$$

$$= \bar{Y}^2\sigma^2_{Y'.X'}\left[\frac{1}{n} + \frac{(X' - \bar{X}')^2}{\Sigma(X_i' - \bar{X}')^2}\right], \qquad (8)$$

and $$\mathrm{cov}(b\cdot\hat{Y}) = \bar{Y}\ \mathrm{cov}(b\cdot\hat{Y}')$$

$$= \bar{Y}\ \sigma^2_{Y'.X'}\left[\frac{(X' - \bar{X}')}{\Sigma(X_i' - \bar{X}')^2}\right]. \qquad (9)$$

Since in most cases, $\bar{Y}$, B and $\sigma^2_{Y'.X'}$ are unknown, use $\hat{Y}$, b and s^2 as their estimates respectively. By substituting (7), (8) and (9) into (6) the equation becomes

$$V\left(\frac{b\hat{Y}}{X}\right) = \frac{1}{X^2}\left\{b^2\hat{Y}^2 s^2\left[\frac{1}{n} + \frac{(X' - \bar{X}')^2}{\Sigma(X_i' - \bar{X}')^2}\right]\right.$$

$$\left. + Y^2\frac{s^2}{\Sigma(X_i' - \bar{X}')^2} + \frac{2b\hat{Y}^2 s^2(X'-\bar{X}')}{\Sigma(X_i' - \bar{X}')^2}\right\}$$

$$= \left(\frac{\hat{Y}}{X}\right)^2 s^2\left\{\frac{b^2}{n} + \frac{1+2b(X'-\bar{X}')+b^2(X'-\bar{X}')^2}{\Sigma(X_i' - \bar{X}')^2}\right\}$$

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[1]See Appendix

$$= \left(\frac{\hat{Y}}{X}\right)^2 s^2 \left\{ \frac{b^2}{n} + \frac{[1 + b(X' - \bar{X}')]^2}{\Sigma(X_i' - \bar{X}')^2} \right\}. \quad (10)$$

K-Variate Case

The k-variate Cobb-Douglas function can be written as

$$Y = AX_1^{B_1} X_2^{B_2} \ldots X_k^{B_k} \epsilon , \quad (11)$$

where is again the random variation. The parameters A, B_1, B_2, . . . , B_k are usually estimated by least-squares solutions for the logarithms of (11):

$$\log Y = \log A + B_1 \log X_1 + \ldots + B_k \log X_k + \log \epsilon$$

or $$Y' = A' + B_1 X_1' + \ldots + B_k X_k' + \epsilon' , \quad (12)$$

where $Y' = \log Y$,

$A' = \log A$,

$X_i' = \log X_i \qquad (i = 1, 2, \ldots , k)$

and $\epsilon' = \log \epsilon$.

The coefficients of this normal equations can be expressed in a matrix form, that is,

$$A = \begin{bmatrix} \Sigma x_1^2 & \Sigma x_1 x_2 & \ldots & \Sigma x_1 x_k \\ \Sigma x_2 x_1 & \Sigma x_2^2 & \ldots & \Sigma x_2 x_k \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \Sigma x_k x_1 & \Sigma x_k x_2 & \ldots & \Sigma x_k^2 \end{bmatrix}$$

The estimated regression equation of (12) will be

$$\hat{Y} = a' + b_1 X_1' + \ldots + b_k X_k'$$

For any given input value of X_i $(i = 1, 2, \ldots, k)$, the estimated output is given by

$$\hat{Y} = aX_1^{b_1} X_2^{b_2} \ldots X_k^{b_k} = e^{\hat{Y}'} ,$$

where $\quad a = e^{a'}$,

$$\hat{Y}' = a' + b_1 X_1' + \ldots + b_k X_k' .$$

The marginal productivity of a particular X, say X_i, is estimated by

$$\frac{\partial Y}{\partial X_i} = \frac{b_i}{X_i} \hat{Y} \tag{13}$$

If both X and $\hat{Y}$ are considered to be fixed, then the variance estimate is given by

$$V\left(\frac{b_i \hat{Y}}{X}\right) = \left(\frac{\hat{Y}}{X_i}\right)^2 V(b_i) , \tag{14}$$

where $\quad V(b_i) = c_{ii}\sigma^2_{Y'.X'}$ (15)

$$C = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1k} \\ c_{21} & c_{22} & \ldots & c_{2k} \\ \vdots & \vdots & & \vdots \\ c_{k1} & c_{k2} & & c_{kk} \end{bmatrix} = A^{-1}$$

In order to get the analogue of equation (10) by considering $\hat{Y}$ and b_i as random variables, again equation (6) is used. The standard formulas needed are[1]

$$V(\hat{Y}') = \sigma^2_{Y'.X'}\left[\frac{1}{n}+\sum_{i=1}^{k} c_{ii}(X_i'-\bar{X}_i')^2 + 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} c_{ij}(X_i'-\bar{X}_i')(X_j'-\bar{X}_j')\right]$$

and

$$\text{cov}(b_i\hat{Y}') = \sigma^2_{Y'.X'}\sum_{j=1}^{k} c_{ij}(X_j' - \bar{X}_j')$$

analogous to (8) and (9), are

$$\begin{aligned} V(\hat{Y}) &= \bar{Y}^2 V(\hat{Y}') \\ &= \bar{Y}^2\sigma^2_{Y'.X'}\left[\frac{1}{n} + \sum_{i=1}^{k} c_{ii}(X_i' - \bar{X}_i')^2\right. \\ &\quad \left. + 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} c_{ij}(X_i' - \bar{X}_i')(X_j' - \bar{X}_j')\right] \end{aligned} \qquad (16)$$

and

$$\begin{aligned} \text{cov}(b_i\hat{Y}) &= \bar{Y}\ \text{cov}(b_i\hat{Y}') \\ &= \bar{Y}\ \sigma^2_{Y'.X'}\sum_{j=1}^{k} c_{ij}(X_j' - \bar{X}_j') \ . \end{aligned} \qquad (17)$$

Estimating $\bar{Y}$, B_i and $\sigma^2_{Y'.X'}$ by $\hat{Y}$, b_i and $s^2_{Y'.X'}$ respectively, then by substituting (15), (16) and (17) into (6), the equation becomes

$$\begin{aligned} V\left(\frac{b_iY}{X_i}\right) &= \left(\frac{Y}{X_i}\right)^2 s^2_{Y'.X'}\left\{\frac{b_i^2}{n} + b_i^2\left[\sum_{i=1}^{k} c_{ii}(X_i' - \bar{X}_i')^2\right.\right. \\ &\quad \left. + 2\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(X_i' - \bar{X}_i')(X_j' - \bar{X}_j')\ c_{ij}\right] \\ &\quad \left. + 2b_i\sum_{j=1}^{k} c_{ij}(X_j' - \bar{X}_j') + c_{ii}\right\}. \end{aligned} \qquad (18)$$

[1]See Brownlee, K.A., Statistical Theory and Methodology in Science and Engineering, New York: John Wiley and Sons, Inc., 1960, pp. 480 - 485.

CHAPTER IV

SIMPLE NUMERICAL EXAMPLE

To illustrated the difference between the variance estimate for marginal productivity obtained by current formula and the one obtained by above formula (either (10) or (18), a very simple example was computed here to serve this purpose.

A time-series sample (1934 - 1953) of U. S. aggregate production and labor employment was taken and computed as follows.[1] (See Table 4.1)

Y - aggregate production (billion)

X - labor employment (million)

$Y' = \log Y$ and $X' = \log X$

$\Sigma Y' = 94.1061$ $\qquad$ $\Sigma X' = 80.4177$

$\Sigma Y'^2 = 443.9961$ $\qquad$ $\Sigma X'^2 = 323.7631$

$\bar{Y}' = 4.7049$ $\qquad$ $X' = 4.0209$

$\Sigma X'Y' = 378.9802$ $\qquad$ $\Sigma(X_i' - \bar{X}')^2 = 0.39$

$$b = \frac{\Sigma(X' - \bar{X}')(Y' - \bar{Y}')}{\Sigma(X_i' - \bar{X}')^2}$$

$$= 1.5138$$

$$a' = \bar{Y}' - b\bar{X}'$$

$$= -1.38156$$

or $\quad a = 0.2513$

Hence, the Cobb-Douglas function is given by

$$Y = 0.2513X^{1.5138} \tag{19}$$

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[1] Source of data: Klein, L. R., An Introduction to Econometrics, New York: Prentice-Hall, Inc., 1962, pp. 103.

Table 4.1 U. S. Aggregate Production and Labor Employment (1934 - 1953)

Year	Y	X	Natural Logarithmic Values			
			Y'	X'	$\hat{Y}'$	$Y' - \hat{Y}'$
1934	64.4	42.7	4.16511	3.75420	4.30155	-.13644
35	75.4	44.2	4.32281	3.78872	4.35380	-.03099
36	85.0	47.1	4.44265	3.85227	4.45000	-.00735
37	92.7	48.2	4.52937	3.87536	4.48496	.04441
38	85.4	46.4	4.44735	3.83730	4.42734	.02001
39	92.3	47.8	4.52504	3.86703	4.47235	.05269
40	101.2	49.6	4.61512	3.90399	4.52830	.08682
41	113.3	54.1	4.72390	3.99083	4.65976	.06414
42	107.8	59.1	4.68213	4.07923	4.79358	-.11145
43	105.2	64.9	4.65396	4.17285	4.93530	-.28134
44	107.2	66.0	4.67283	4.18965	4.96073	-.28790
45	108.8	64.4	4.69135	4.16511	4.92358	-.23223
46	131.4	58.9	4.87520	4.07584	4.78845	.08675
47	130.9	59.3	4.87520	4.08261	4.79870	.07630
48	134.7	60.2	4.89784	4.09767	4.82149	.07635
49	129.1	58.7	4.85981	4.07244	4.78330	.07651
50	147.8	60.0	4.99043	4.09434	4.81645	.17398
51	152.1	63.8	5.02388	4.15575	4.90941	.11447
52	154.3	64.9	5.03695	4.17285	4.93530	.10165
53	159.9	66.0	5.07517	4.18965	4.96073	.11444

or $$Y' = -1.38156 + 1.5138X' \tag{20}$$

The marginal productivity of X, when X = 60 million, is obtained by differentiating (19) and evaluating at X = 60. That is

$$\frac{dY}{dX} = (0.2513)(1.5138)X^{0.5138}$$

$$= 0.3804(60)^{0.5138}$$

$$= 3.115 \quad \text{(thousand dollars)}$$

or $3,115.00

$$S^2_{Y'X'} = \frac{\Sigma(Y' - \hat{Y}')^2}{n - 2}$$

$$= 0.35665/18$$

$$= 0.019814$$

$$S_{Y'X'} = 0.14076 \quad \text{(or 1.15 billion)}$$

$$V(b) = S^2_b = \frac{S^2_{Y'X'}}{\Sigma(X'-\bar{X}')^2}$$

$$= 0.019814/0.39$$

$$= 0.05081$$

$$S_b = 0.22540 \qquad \text{(or \$1,250.00)}$$

Thus, the variance estimate of the marginal productivity of labor at 60 million by the current formula will be

$$V(\frac{b\ Y}{X}) = (\ \frac{\bar{Y}}{\bar{X}}\ )^2\ V(b)$$

$$= (4.81645/4.09434)^2(0.05081)$$

$$= (1.38385)(0.05081)$$

$$= 0.07031$$

$$\sqrt{V\left(\frac{b\,Y}{X}\right)} = S_{mp}$$

$$= 0.26516 \qquad \text{(or \$1,305.00)} \qquad (21)$$

The variance of the marginal productivity of labor at 60 million estimated by equation (10) will be

$$V\left(\frac{b\,Y}{X}\right) = (1.38385)(0.019814)\left\{\frac{(1.5138)^2}{20} + \frac{[1+(1.5138)(0.07346)]^2}{0.39}\right\}$$

$$= (0.02741)\left\{0.11458 + \frac{1.23477}{.39}\right\}$$

$$= 0.00314 + 0.86782$$

$$= 0.87096$$

$$S_{mp} = 0.93325 \qquad \text{(or \$2,545.00)} \qquad (22)$$

Thus, in this particular case, the variance of the marginal productivity at X = 60 million by the current formula would underestimate the true variation, since the variance for mp_X from equation (5) is approximately one-half of that obtained from equation (10).

CHAPTER V

DISCUSSION AND SUMMARY

After obtaining the variance of marginal productivity for X_1, one could easily compute approximate 95 or 99 per cent confidence interval for MP_{X_1} . For example, the 95 per cent confidence interval for MP_{X_1} would be

$$mp_{X_1} - 1.96 S_{mp_{X_1}} \leq MP_{X_1} \leq mp_{X_1} + 1.96 S_{mp_{X_1}}$$

for large n,

where mp_{X_1} = marginal productivity evaluated either from (4) or (13) at a given value of X_1 ;

$S_{mp_{X_1}} = \sqrt{V(\frac{b\ Y}{X_1})}$ = standard error of mp_{X_1} .

By using the example above, the 95% confidence interval for mp_X (at X = 60 million) would be:

a. By the result of (21)

$$3{,}115 - 2.093(1{,}305) \leq MP_X \leq 3{,}115 + 2.093(1{,}305)$$

$$\text{or} \qquad 383.63 \leq MP_X \leq 5{,}846.37 \qquad (23)$$

b. By the result of (22)

$$3{,}115 - 2.093(2{,}545) \leq MP_X \leq 3{,}115 + 2.093(2{,}545)$$

$$\text{or} \qquad -2{,}211.69 \leq MP_X \leq 5{,}326.69 \qquad (24)$$

Since the current formula in this case underestimates the variance for the marginal productivity, (23) would underestimate the length of 95 per cent confidence interval, and the result of (24) is safer then that of (23).

In addition to this, some exact tests of certain hypotheses concerning the marginal productivity were developed by Fuller[1] as quoted below:

1. "To test the hypothesis that $MP_{X_1} = 0$ is equivalent to test the $B_1 = 0$, since $e^{\hat{Y}'}$ can never be zero.

2. To test the hypothesis that $MP_{X_1} = k \cdot MP_{X_2}$ at a given point, while k is a constant. The hypothesis would be

$$\frac{b_1}{X_1} e^{\hat{Y}'} - k \frac{b_2}{X_2} e^{\hat{Y}'} = 0$$

or

$$\left(\frac{b_1}{X_1} - k \frac{b_2}{X_2}\right) e^{\hat{Y}'} = 0 .$$

Since $e^{\hat{Y}'}$ can never be zero, the test becomes a test of

$$\frac{b_1}{X_1} - \frac{kb_2}{X_2} = 0$$

which could be performed by constructing the unbiased estimate of the latter quantity as DED'

where $D = (0, \frac{1}{X_1}, -\frac{k}{X_2})$

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[1] Fuller, Wayne A., "Estimating the Reliability of Quantities Derived from Empirical Production Functions", Journal of Farm Economics, Vol. XLIV, No. 1, February 1962, 19p.

D' = transpose of D

and

$$E = \begin{bmatrix} \frac{(\log 10)^2}{n} & 0 & 0 \\ 0 & c_{11} & c_{12} \\ 0 & c_{21} & c_{22} \end{bmatrix} S^2$$

Where c_{ij} is the element of ith row and jth column of A^{-1} which was defined on equation (15) and $S^2 = V(\hat{Y}')$ as given in Chapter III."

In many cases, b is small (close to zero) and sample size (n) is large, then b^2/n will be a very small number. If b^2/n can be ignored and set (or evaluate) $X' = \bar{X}'$, then formula (10) would be identically the same as formula (5) which assumes both X and $\hat{Y}$ are fixed. This also holds for the k-variate case.

By observing formula (5) and (10), it would be found that the variance obtained from (10) would be no less than the variance obtained from (5), if the inequality

$$b(X' - \bar{X}') \geqslant 0$$

holds. In other words, if b and $(X' - \bar{X}')$ have the same sign, the variance obtained from current formula would underestimate the real variance for the marginal productivity. If b and $(X' - \bar{X}')$ have different sign, then the inequality

$$\frac{b^2}{n} + \frac{[1 + b(X' - \bar{X}')]^2}{\Sigma(X_i' - \bar{X}')^2} \leqslant \frac{1}{\Sigma(X_i' - \bar{X}')^2} \tag{25}$$

might sometimes be true. In this case, the variance obtained from (5) would be greater than that obtained from (10). The direction of inequatity in (25) depends upon the magnitudes of n, b, $(X' - \bar{X}')$ and $\Sigma(X_i' - \bar{X}')^2$.

In all, formulas (10) and (18) are based on assumptions which were considered to be more realistic than those underlying the variance formula (5) and (14) which, in many cases, may lead to considerable errors in estimation.

The biggest advantage of the current formula over equations (10) and (18) is ease in computation. If b and $(X' - \bar{X}')$ have different sign, and the inequality (25) holds, then it is still desirable to use current methods for computing the variance of marginal productivity.

APPENDIX

Given: $Y = a + bX$, $a = \bar{Y} - b\bar{X}$

and $V(a) = \sigma^2 \left[\frac{1}{n} + \frac{X^2}{\Sigma(X_i - \bar{X})^2} \right]$

$$\mathrm{cov}(b\ Y) = E(b\ Y) - E(b)\ E(Y)$$
$$= E(b\ Y) - B\ \bar{Y}$$

where $E(b\ Y) = E[b(a + bX)]$

$$= E(ab) + X\ E(b^2)$$

while $E(ab) = E[(Y - bX)b] = \bar{Y}\ E(b) - \bar{X}\ E(b^2)$

$$= Y\ B - X \cdot E\left[\left(\frac{\Sigma(X_i - \bar{X})(Y_i - \bar{Y})}{\Sigma(X_i - \bar{X})^2}\right)^2\right]$$

$$= \bar{Y}\ B - \frac{\bar{X}\sigma^2}{\Sigma(X_i - \bar{X})^2}$$

and

$$X\ E(b^2) = \frac{X\ \sigma^2}{\Sigma(X_i - \bar{X})^2}$$

thus,

$$E(b\ Y) = \bar{Y}\ B - \frac{\bar{X}\ \sigma^2}{\Sigma(X_i - \bar{X})^2} + \frac{X\ \sigma^2}{\Sigma(X_i - \bar{X})^2}$$

$$= \bar{Y}\ B + \frac{\sigma^2(X - \bar{X})}{\Sigma(X_i - \bar{X})^2}$$

Hence,

$$\mathrm{cov}(b\ Y) = \bar{Y}\ B + \frac{\sigma^2(X - \bar{X})}{\Sigma(X_i - \bar{X})^2} - \bar{Y}\ B$$

$$= \frac{\sigma^2(X - \bar{X})}{\Sigma(X_i - \bar{X})^2}$$

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VARIANCE ESTIMATE OF THE MARGINAL PRODUCTIVITY
FOR COBB-DOUGLAS FUNCTION

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ABSTRACT

The Cobb-Douglas production function, with its interesting properties, has become one of the major tools in doing statistical investigations into many manufacturing fields throughout the world.

The current formula of the marginal productivity for the Cobb-Douglas function is based on the assumption that both independent variables (X_i) and dependent variables (Y) are constants. This is not realistic at all. The purpose of this report is to try to reach a formula for the variance of the marginal productivity of the Cobb-Douglas function, based on more realistic assumptions which consider Y, A, and B_i as variables with only X_i being held fixed. Also it is intended, if possible, to determine the conditions for which the current formula will over or under estimate the true variability.

In this report, the procedure used to develop the formula is based on the statistical rules concerning the variance of constants and variables, with the aid of those known results from regression theory.

The results show that the current formulas for the marginal productivity of this particular production function are only a special case of the new formulas. The current formulas would underestimate the true variance in some cases, and overestimate it in others: If b and $(X' - \bar{X}')$ have the same sign, then current formula would underestimate the variance and in case

of having different signs, then it is possible for the current formula to either underestimate or overestimate the variance. But difficult in compution is the big disadvantadge of the new formula over current formulas.