BUCKLING OF RIGID FRAMES

by

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B. S., University of Wyoming, 1961

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Civil Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1963

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SYNOPSIS

It is considered that elastic stability is a problem of great importance in the modern use of steel and high-strength alloys in engineering structures, especially in tall buildings, bridges and aircrafts.

This paper presents an analysis of buckling of frames based on the well-known slope-deflection procedure. The stability of one-story and multi-story plane frames is studied for the anti-symmetrical mode of buckling. Other methods for calculating the buckling load of frames are also discussed. Typical examples are solved using the direct analytical procedure, and the results obtained are compared with the results obtained through the use of the moment-distribution method. It was found that both solutions give results which are in very good agreement.

It is felt that the slope-deflection method is simpler and more direct than the moment-distribution method in solving stability problems.

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INTRODUCTION

The modern use of steel and high-strength alloys in engineering structures, especially in tall buildings, bridges, ships, and aircrafts, has made elastic instability a problem of great importance. Urgent practical requirements have given rise in recent years to extensive investigations, both theoretical and experimental, of the conditions governing the stability of such structural elements as bars, plates, ...

The first problem of elastic instability, concerning lateral buckling of compressed members, was solved about 200 years ago by L. Euler.[1] Under forces of practical interest, the problem of lateral buckling of columns, originated by Euler, has been extensively investigated theoretically and experimentally, and the limits within which the theoretical formulas can be applied have been established. However, lateral buckling of compressed members is only a particular case of elastic instability, which is very important in the field of structural engineering. B. W. James[2] adapted the moment-distribution procedure to include the effects of direct loads of the column; the work of James was extended by Lundquist[3] to determine the elastic collapse loads or the critical loads within the elastic range of plane frameworks.

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[1] L. Euler, "Elastic Curves," translated and annotated by W. A. Oldfather, C. A. Ellis, and D. M. Brown, 1933.

[2] B. W. James, "Principal Effects of Axial Loads on Moment Distribution Analysis of Rigid Structures," N.A.C.A. Tech. Note 534, 1935.

[3] E. E. Lundquist, "Stability of Structure Members Under Axial Load," N.A.C.A. Tech. Note 617, 1937.

In what is probably the best existing treatise on critical loads of elastic structures, Chandler[4] makes the observation that Lundquist's work was "....the foundation stone in the concept of stability, but in regard to the numerical evaluation of critical loads left much to be desired."

Recently, several authors such as S. Hansbo[5] had extended the moment-distribution method for solving the buckling load of frames to include the multi-story structures. The energy method had also been used by some authors to solve the buckling load of frames, but it has been considered that the result of this method gives an upper bound of the critical load of the frame which is too conservative for the purpose of structural design.

This paper presents the slope-deflection method to solve the buckling load for one-story and multi-story structures. The fundamental slope-deflection formula relates the bending at one end of a member, such as a column or a girder, to the end slopes and relative transverse displacement of these ends. In applying this formula to problems of transversely loaded frames, the equation expressing the equilibrium of moments at the joint and the equilibrium of shear at each story can be obtained. Generally, it is assumed that the axial strains in the members may be disregarded, that is, the horizontal or transverse displacements of all joints in a given frame at a given floor or level will be

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[4] D. B. Chandler, "The Prediction of Critical Loads of Elastic Structures," Ph.D. Thesis, Manchester University, 1955.

[5] S. Hansbo, "The Critical Load of Rectangular Frames Analyzed by Convergence Methods," Transaction of Chalmers University of Technology, Vol. 164-181, 1956.

the same. Thus, only one relative transverse displacement need be defined for all the columns of one story. However, plane frames usually do not exist singly. A building structure normally consists of a set of such frames which are connected by floors, roofs, and some bracing systems. Usually, floors and roofs may provide an additional rigidity against the lurching mode of buckling.

Plane frames may buckle in either an anti-symmetric mode or a symmetric mode which involves or does not involve lateral displacement of the frame, respectively.

METHODS OF CALCULATING THE BUCKLING LOAD

Several methods[6] of calculating the buckling load of frames had been worked out. Among these methods, three essentially different approaches are discussed:

1. The energy method.
2. The moment-distribution method.
3. The direct analytical solution based on slope-deflection procedure.

The energy method is based on the condition that if a frame in stable equilibrium is given a small distortion, it will always strive to return to its original position. If, on the other hand, it is in unstable equilibrium, the distortion will increase to infinity. This fact forms the energy buckling criterion. Buckling will occur when the work done by the external forces in a virtual displacement equals the change in strain energy.

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[6] Loc. cit.

The stability of a framework with rigid joints can be investigated by using the moment-distribution method. In the use of this method, a particular set of values of the external loads is assumed, and the corresponding axial forces in the bars are determined, assuming that the frame has pin joints. Then an arbitrary moment is applied to one of the joints of the frame, and the moments in the frame are distributed in the usual way. If the moment-distribution computations converge to finite values for the final end moments, the frame is, in general, stable. The entire process is then repeated using increased loads on the structure but maintaining the loads in the same proportion. If the loads are above the critical value, the moment-distribution computations will not converge, in general, to finite values of the end moments in the columns. Thus by successive applications of this procedure, the critical load is determined.

The direct analytical solution based on slope-deflection procedure consists of setting up a system of equations, expressing the relations between the joint displacements and the joint rotations which occur due to distortion of the frame. These relations form a set of linear homogeneous equations, usually called the stability equations. Generally, the unknowns (displacements, rotations and moments) in the stability equations are equal to zero, if the determinant of the coefficient is different from zero, which means that no distortion of the frame is taking place, and the frame is in a stable condition. The stability criterion is obtained from setting the determinant of the unknowns equal to zero, and this usually yields the value of the buckling load.

THE FUNDAMENTAL THEORY FOR THE DIRECT ANALYTICAL PROCEDURE[7]

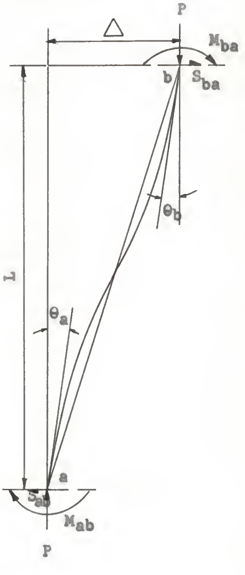

Fig. 1. Applied forces and distortion of a column.

[7] J. E. Goldberg, "Buckling of One-Story Frames and Buildings," Journal of the Structural Division, A.S.C.E., Vol. 86, Oct. 1960, P. 53.

If a number ab is subjected to an axial compressive load P, the end b displaces with respect to the other end a, as shown in Fig. 1. The moments at both ends of the member, by applying the slope-deflection method, can be found in terms of the angular and transverse displacements at those ends in the following form:

$$M_{ab} = K_{ab} \left\{ A_{ab}\, \theta_a + B_{ab}\, \theta_b - (A_{ab} + B_{ab}) \frac{\Delta_{ab}}{L_{ab}} \right\} \text{ - - - - } (1)$$

and

$$M_{ba} = K_{ab} \left\{ A_{ab}\, \theta_b + B_{ab}\, \theta_a - (A_{ab} + B_{ab}) \frac{\Delta_{ab}}{L_{ab}} \right\} \text{ - - - - } (2)$$

where

$$K_{ab} = \frac{EI}{L_{ab}} ;$$

E is the appropriate modulus of elasticity;

I denotes the moment of inertia of the cross section of the member;

L is the length of the member;

θ is the angular displacement at the ends of the member;

Δ refers to relative transverse displacement of ends.

A and B are constants which depend upon the sign and magnitude of the axial load and may be expressed completely as functions of the ratio of the axial load to the Euler load of the member. They are given by the following formulas:

(a) For compressive axial load

$$A = \frac{\sin pL - pL \cos pL}{\frac{2}{pL}(1 - \cos pL) - \sin pL} \text{ - - - - - - - - - - - - } (3a)$$

$$B = \frac{pL - \sin pL}{\frac{2}{pL}(1 - \cos pL) - \sin pL} \quad \text{- - - - - - - - - - - - - - -} \quad (3b)$$

(b) For tensile axial load

$$A = \frac{pL \cosh pL - \sinh pL}{\frac{2}{pL}(1 - \cosh pL) + \sinh pL} \quad \text{- - - - - - - - - - - - -} \quad (4a)$$

$$B = \frac{\sinh pL - pL}{\frac{2}{pL}(1 - \cosh pL) + \sinh pL} \quad \text{- - - - - - - - - - - - -} \quad (4b)$$

in which

$$pL = \pi\sqrt{|\rho|} \quad \text{- -} \quad (5a)$$

and

$$\rho = \frac{P}{Pe} = \frac{P}{EI\frac{\pi^2}{L^2}} \quad \text{- - - - - - - - - - - - - - - - -} \quad (5b)$$

where P is an axial load, denoted as positive while the member is under compression. The value of A and B may be taken from Table 1 or Figs. 2 and 3. It can also be found from Table 1 that A and B will have the values 4 and 2, respectively, when the axial load is zero, which agrees with the well-known slope-deflection equations.

The shear equation may be found by taking the moment about either end of the column, as shown in Fig. 1.

$$S_{ab} = \frac{-1}{L_{ab}}(M_{ab} + M_{ba} + P_{ab}\Delta_{ab}) \quad \text{- - - - - - - - - - - - - -} \quad (6)$$

Substitution of Eqs. (1) and (2) into Eq. (6) yields:

$$S_{ab} = \frac{-K_{ab}}{L_{ab}}(A_{ab} + B_{ab})\left(\theta_a + \theta_b - 2\frac{\Delta_{ab}}{L_{ab}}\right) - P_{ab}\frac{\Delta_{ab}}{L_{ab}} \quad \text{- - -} \quad (7)$$

Table 1. Slope deflection coefficients A and B for various values of load ratio ρ.

ρ	A	B	ρ	A	B
3.9	-78.34	78.56	0.3	3.589	2.109
3.8	-39.05	39.54	0.2	3.730	2.070
3.7	-24.69	25.39			
3.6	-17.87	18.79	0.1	3.865	2.033
3.5	-13.73	14.86	0	4.000	2.000
3.4	-10.91	12.24	0	4.000	2.000
3.3	- 8.86	10.40	-0.1	4.131	1.968
3.2	- 7.30	9.02	-0.2	4.255	1.938
3.1	- 6.05	7.96	-0.3	4.384	1.910
3.0	- 5.03	7.12	-0.4	4.502	1.883
2.8	- 3.449	5.884	-0.5	4.619	1.857
2.6	- 2.252	5.019	-0.6	4.736	1.834
2.5	- 1.749	4.678	-0.7	4.849	1.811
2.4	- 1.300	4.383	-0.8	4.959	1.789
2.2	- 0.519	3.901	-0.9	5.069	1.769
2.0	0.143	3.521	-1.0	5.175	1.749
1.8	0.717	3.224	-1.2	5.383	1.713
1.6	1.224	2.980	-1.4	5.583	1.681
1.5	1.457	2.873	-1.6	5.777	1.651
1.4	1.673	2.778	-1.8	5.964	1.623
1.2	2.090	2.610	-2.0	6.147	1.598
1.0	2.468	2.468	-2.5	6.580	1.544
0.9	2.645	2.404	-3.0	6.990	1.499
0.8	2.816	2.346	-4.0	7.75	1.43
0.7	2.981	2.291	-5.0	8.42	1.38
0.6	3.140	2.241	-7.0	9.62	1.30
0.5	3.295	2.194	-9.0	10.69	1.26
0.4	3.444	2.150			

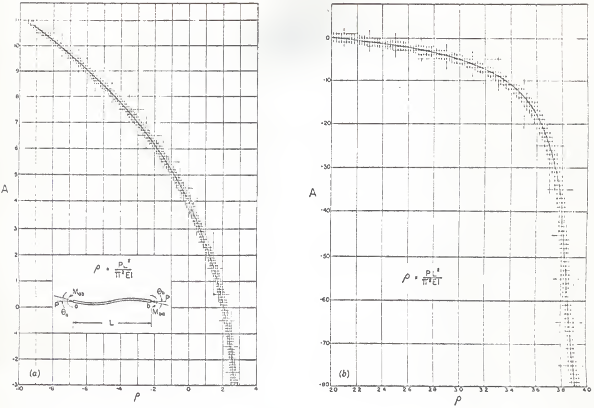

FIG. 2.—VALUES OF A

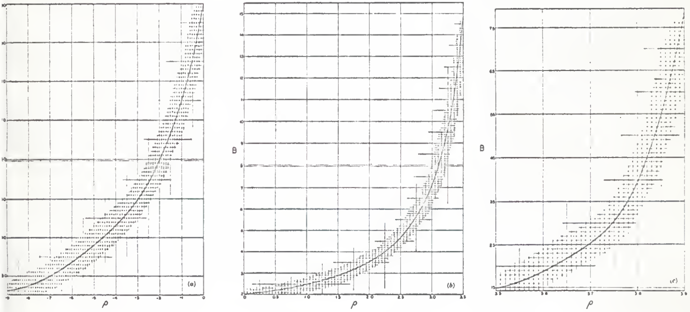

FIG. 3.—VALUES OF B

Because of the various conditions at the column base, the lower end a, is assumed to be elastically restrained by a rotational spring having a spring rate Q_a, then the amount at the end a, due to this spring will be:

$$M_{ab} = -Q_a \theta_a \quad \text{- -} \quad (8)$$

which can be equated to Eq. (1) to yield θ_a or θ_b:

$$-Q_a \theta_a = K_{ab} \left\{ A_{ab}\theta_a + B_{ab}\theta_b - (A_{ab} + B_{ab}) \frac{\Delta_{ab}}{L_{ab}} \right\} \quad \text{- - -} \quad (9)$$

Thus,

$$\theta_a = \frac{1}{A_{ab} + \frac{Q_a}{K_{ab}}} \left\{ - B_{ab}\theta_b + (A_{ab} + B_{ab}) \frac{\Delta_{ab}}{L_{ab}} \right\} \quad \text{- - - - -} (10)$$

Substitution of Eq. (10) into Eqs. (2) and (7) yields:

$$M_{ba} = K_{ab} \left\{ \left(A_{ab} - \frac{B_{ab}^2}{A_{ab} + \frac{Q_a}{K_{ab}}}\right) \theta_b - (A_{ab} + B_{ab}) \right.$$

$$\left. \times \left(1 - \frac{B_{ab}}{A_{ab} + \frac{Q_a}{K_{ab}}}\right) \frac{\Delta_{ab}}{L_{ab}} \right\} \quad \text{- - - - - - - - - - - - - - -} (11)$$

and

$$S_{ab} = \frac{-K_{ab}}{L_{ab}} (A_{ab} + B_{ab}) \left\{ \theta_b \left(1 - \frac{B_{ab}}{A_{ab} + \frac{Q_a}{K_{ab}}}\right) - \left(2 - \frac{A_{ab} + B_{ab}}{A_{ab} + \frac{Q_a}{K_{ab}}}\right) \right.$$

$$\left. \times \frac{\Delta_{ab}}{L_{ab}} \right\} - P_{ab} \frac{\Delta_{ab}}{L_{ab}} \quad \text{- - - - - - - - - - - - - - - - - - -} (12)$$

For convenience, it is assumed that:

$$C_{ab} = A_{ab} + B_{ab} \quad \text{- -} \quad (13a)$$

$$A^{1}_{ab} = A_{ab} - \frac{B^{2}_{ab}}{A_{ab} + \frac{Q_a}{K_{ab}}} \text{ - - - - - - - - - - - - - - - - - - } (13b)$$

$$c_{ab} = 1 - \frac{B_{ab}}{A_{ab} + \frac{Q_a}{K_{ab}}} \text{ - - - - - - - - - - - - - - - - - - } (13c)$$

$$c^{1}_{ab} = 2 - \frac{A_{ab} + B_{ab}}{A_{ab} + \frac{Q_a}{K_{ab}}} \text{ - - - - - - - - - - - - - - - - } (13d)$$

Substitution of Eqs. (13a) to (13d) into Eqs. (11) and (12), yields the following relations:

$$M_{ba} = K_{ab} \left(A^{1}_{ab}\, \theta_b - C_{ab}\, c_{ab} \frac{\Delta_{ab}}{L_{ab}}\right) \text{ - - - - - - - - - - - - - } (14)$$

and

$$S_{ab} = -\frac{K_{ab}}{L_{ab}} C_{ab} \left(c_{ab}\, \theta_b - c^{1}_{ab} \frac{\Delta_{ab}}{L_{ab}}\right) - P_{ab} \frac{\Delta_{ab}}{L_{ab}} \text{ - - - - - - } (15)$$

For the two different conditions of column bases, namely, pinned end and built-in end, the corresponding value of Q will be zero and infinity, respectively. Thus, Eqs. (14) and (15) become:

(a) for pinned end condition:

$$M_{ba} = K_{ab} \frac{C_{ab}}{A_{ab}} (A_{ab} - B_{ab}) \left(\theta_b - \frac{\Delta_{ab}}{L_{ab}}\right) \text{ - - - - - - - - - - } (16)$$

and

$$S_{ab} = \frac{-K_{ab}}{L_{ab}} \frac{C_{ab}}{A_{ab}} (A_{ab} - B_{ab}) \left(\theta_b - \frac{\Delta_{ab}}{L_{ab}}\right) - P_{ab} \frac{\Delta_{ab}}{L_{ab}} \text{ - - - } (17)$$

For this case, Eqs. (13b) to (13d) reduce to:

$$A^{1}_{ab} = \frac{C_{ab}}{A_{ab}} (A_{ab} - B_{ab}) \text{ - - - - - - - - - - - - - - - - - - - } (18a)$$

$$c_{ab} = \frac{A_{ab} - B_{ab}}{A_{ab}} \text{ -(18b)}$$

$$c^{1}_{ab} = 2 - \frac{A_{ab} + B_{ab}}{A_{ab}} = c_{ab} \text{ - - - - - - - - - - - - - - - - - -(18c)}$$

(b) for built-in end condition:

$$M_{ba} = K_{ab} \left(A_{ab}\, \theta_b - C_{ab} \frac{\triangle_{ab}}{L_{ab}}\right) \text{ - - - - - - - - - - - - - - - (19)}$$

and

$$S_{ab} = - \frac{K_{ab}}{L_{ab}} C_{ab} \left(\theta_b - 2 \frac{\triangle_{ab}}{L_{ab}}\right) - P_{ab} \frac{\triangle_{ab}}{L_{ab}} \text{ - - - - - - - - (20)}$$

For this case, Eqs. (13b) to (13d) become:

$$A^{1}_{ab} = A_{ab} \text{ -(21a)}$$

$$c_{ab} = 1 \text{ -(21b)}$$

$$c^{1}_{ab} = 2 \text{ -(21c)}$$

The above mentioned formulas will be used to evaluate the buckling load of various frames.

SYMMETRICAL BUCKLING OF FRAMES

The frame buckles in symmetrical mode if lateral displacement is not allowed to occur. The theory is applied to a simple fixed-end portal frame, as shown in Fig. 4. From symmetry, it follows that:

$$\triangle_{ab} = \triangle_{dc} = \triangle_{bc} = 0$$

$$\theta_b = -\theta_c$$

The equilibrium condition of joints states that:

$$\sum M_{@b} = 0$$

Therefore, $M_{ba} + M_{bc} = 0$ - - - - - - - - - - - - - - - - -(22a) or

$$K_{ab} A_{ab} \theta_b + 2K_{bc} \theta_b = 0 \text{ - - - - - - - - - - - - - - - - - - - } (22b)$$

and

$$A_{ab} = - \frac{2K_{bc}}{K_{ab}} \text{ - } (22c)$$

To find the critical load for the above problem: A_{ab} can be computed from Eq. (22c) and ρ will be obtained from either Table 1 or Figs. 2 and 3. The critical load Pcr. is found from the Equation

$$P_{cr.} = \rho P_e$$

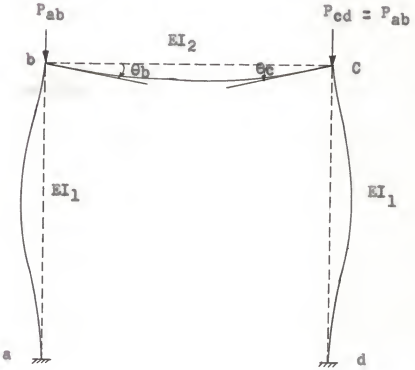

Fig. 4. One-story single bay rectangular frame.

ANTI-SYMMETRICAL BUCKLING OF FRAMES

The frame, in general, buckles in anti-symmetrical mode if lateral displacement is allowed to occur.

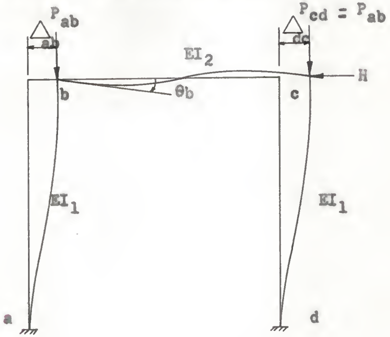

Fig. 5. Simple portal frame.

I. In case of portal frame as shown in Fig. 5. Since it is symmetric, then only two equilibrium equations need be written in order to find the critical load. They are:

(1) $\sum M_{@b} = 0$ or $\sum M_{@c} = 0$

(2) The fictitious force H must vanish.

For anti-symmetric case, the following relations hold:

$$\theta_b = \theta_c$$

$$\Delta_{ab} = \Delta_{dc}$$

$$\Delta_{bc} = 0$$

Thus, from

$$\sum M_{@b} = 0$$

Then

$$M_{ba} + M_{bc} = 0 \text{ - } (23a)$$

and

$$K_{ab} \left(A_{ab}\,\theta_b - C_{ab}\frac{\Delta_{ab}}{L_{ab}}\right) + 6\,K_{bc}\,\theta_b = 0 \text{ - - - - - - - - - } (23b)$$

or

$$\theta_b = \frac{K_{ab}\,C_{ab}\,\Delta_{ab}}{L_{ab}\,(K_{ab}\,A_{ab} + 6K_{bc})} \text{ - - - - - - - - - - - - - - - } (23c)$$

Also, the force H must vanish.

Therefore,

$$-S_{ab} - S_{dc} = H = 0 \text{ - } (23d)$$

Since $S_{ab} = S_{dc}$, it follows that

$$-S_{ab} = \frac{H}{2} = 0 \text{ - } (23e)$$

Substitution of Eq. (20) into Eq. (23e) yields:

$$\frac{K_{ab}}{L_{ab}}\,C_{ab}\left(\theta_b - 2\,\frac{\Delta_{ab}}{L_{ab}}\right) + P_{ab}\frac{\Delta_{ab}}{L_{ab}} = \frac{H}{2} = 0 \text{ - - - - - - - } (23f)$$

For this problem, a trial and error procedure for the solution of Eq. (23) and the determination of the critical load will be found to be convenient and relatively rapid. Noting that, for a frame such as shown in Fig. 5, the critical load must be less than the Euler load computed with an appropriate modulus, a trial value is selected for P_{ab} and the coefficients evaluated. Assuming Δ_{ab} to have unit magnitude, the value of θ_b is found from Eq. (23c). Substitution of this value of θ_b and the unit value of Δ_{ab} into the Eq. (23f) yields the value of H that would be required to hold the frame in the deflected position with the prescribed column loads. The sign of the computed value of H determines whether the axial loads are greater or less than the critical loads. If H is positive, the assumed column loads exceed the critical load, since the force H is now supporting the frame against further deflection. If H is negative, the assumed column loads are less than the critical loads, since the direction of H now implies that the frame has "reserve stiffness." If H vanishes, the assumed column loads are equal to the critical loads. In the trial-and-error procedure, if H is not zero, its sign clearly indicates whether the next trial value for the axial load should be larger or smaller.

II. Portal frame with one fixed-end and one pinned-end column base, as shown in Fig. 6.

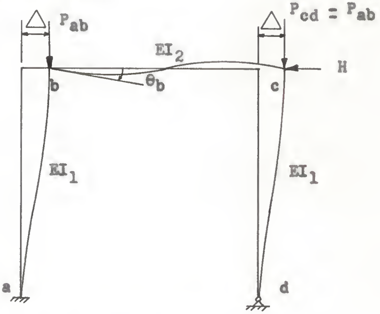

Fig. 6. Simple portal frame.

Assuming an anti-symmetric mode of buckling, it follows that:

$$\theta_b = \theta_c$$

$$\triangle_{ab} = \triangle_{dc} = 1$$

$$\triangle_{bc} = 0$$

$$P_{ab} = P_{cd} = P$$

and

$$\sum M_{@b} = 0 \qquad \text{or} \qquad \sum M_{@c} = 0$$

then

$$M_{ba} + M_{bc} = 0 \text{ -------------------- } (24a)$$

and

$$K_{ab} \left(A_{ab}\, \theta_b - C_{ab} \frac{\triangle_{ab}}{L_{ab}}\right) + K_{bc} \left\{ 4\theta_b + 2(\theta_b) \right\} = 0 \text{ ---- } (24b)$$

or

$$\theta_b = \frac{K_{ab} C_{ab} \dfrac{\triangle_{ab}}{L_{ab}}}{(K_{ab} A_{ab} + 6\, K_{bc})} \text{ ------------------ } (24c)$$

Writing the equilibrium of shear in the horizontal direction yields:

$$-S_{ab} - S_{dc} = H = 0 \text{ ------------------ } (24d)$$

Putting Eqs. (17) and (20) into Eq. (24d) and simplified:

$$\left\{ \frac{K_{ab}}{L_{ab}} C_{ab} + \frac{K_{cd}}{L_{cd}} \frac{C_{cd}}{A_{cd}} (A_{cd} - B_{cd}) \right\} \theta_b - \left\{ 2 \frac{K_{ab}}{L_{ab}} C_{ab} + \frac{K_{cd} C_{cd}}{L_{cd} A_{cd}} (A_{cd} - B_{cd}) \right\} \frac{\triangle}{L_{cd}} + 2 \frac{P_{ab} \triangle}{L_{ab}} = H = 0 \text{ ------ } (24e)$$

The critical load of this problem can be obtained by using the procedure described previously.

III. Three-story single bay structural frame.

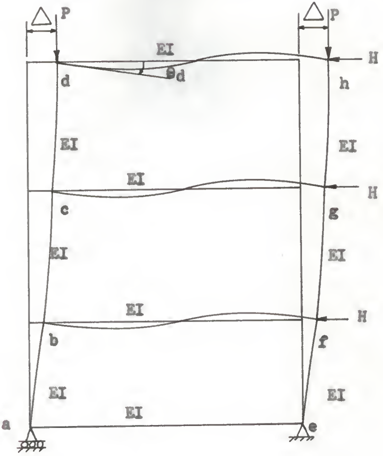

Fig. 7. Three-story single bay frame.

Assuming an anti-symmetric mode of buckling for the frame shown in Fig. 7, it follows that:

$$\theta_d = \theta_h$$

$$\theta_c = \theta_g$$

$$\theta_b = \theta_f$$

$$\theta_a = \theta_e$$

$$P_{cd} = P_{gh} = P$$

$$\Delta_{cd} = \Delta_{gh}$$

$$\triangle_{bc} = \triangle_{fg}$$

$$\triangle_{ab} = \triangle_{ef}$$

and

$$(\sum M)_{\text{at } d,\ c,\ b,\ h,\ g,\ \text{or } f} = 0$$

It follows that the expression for the rotations at the joints are:

$$\theta_d = \frac{K_{cd}\, C_{cd} \triangle_{cd}}{L_{cd}\,(K_{cd}\, A_{cd} + 6\, K_{dh})} \quad \text{-------------------(25a)}$$

$$\theta_c = \frac{K_{cd}\, C_{cd} \frac{\triangle_{cd}}{L_{cd}} + K_{cb}\, C_{cb} \frac{\triangle_{cb}}{L_{cb}}}{(K_{cb}\, A_{cb} + K_{cd}\, A_{cd} + 6\, K_{cg})} \quad \text{------------ (25b)}$$

$$\theta_b = \frac{K_{bc}\, C_{bc} \frac{\triangle_{bc}}{L_{bc}} + K_{ab} \frac{C_{ab}}{A_{ab}} (A_{ab} - B_{ab}) \frac{\triangle_{ab}}{L_{ab}}}{K_{bc}A_{bc} + 6K_{bf} + K_{ba} \frac{C_{ab}}{A_{ab}} (A_{ab} - B_{ab})} \quad \text{---- (25c)}$$

and the shearing equations are:

$$\frac{K_{cd}}{L_{cd}} C_{cd} \left(\theta_d - 2 \frac{\triangle_{cd}}{L_{cd}}\right) + P_{cd} \frac{\triangle_{cd}}{L_{cd}} = \frac{H}{2} = 0 \quad \text{-------(25d)}$$

$$\frac{K_{bc}}{L_{bc}} C_{bc} \left(\theta_c - 2 \frac{\triangle_{bc}}{L_{bc}}\right) + P_{bc} \frac{\triangle_{bc}}{L_{bc}} = H = 0 \quad \text{-------(25e)}$$

$$\frac{K_{ab}}{L_{ab}} C_{ab} \left(\theta_b - 2 \frac{\triangle_{ab}}{L_{ab}}\right) + P_{ab} \frac{\triangle_{ab}}{L_{ab}} = \frac{3}{2} H = 0 \quad \text{------(25f)}$$

The critical load of this problem can be obtained by using trial-and-error procedure as outlined previously.

IV. Three-story three-bay building frame.

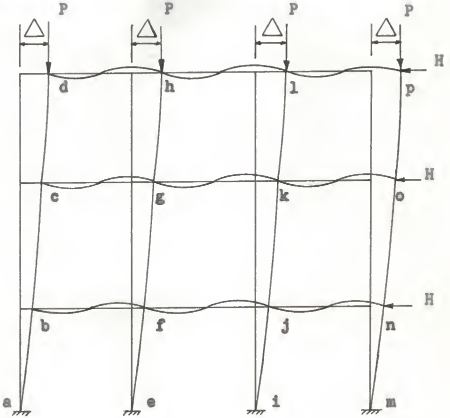

Fig. 8. Three-story three-bay frame.

Assuming an anti-symmetric mode of buckling for the frame shown in Fig. 8, it follows that:

$$\theta_d = \theta_p$$

$$\theta_c = \theta_o$$

$$\theta_b = \theta_n$$

$$\theta_l = \theta_h$$

$$\theta_k = \theta_g$$

$$\theta_j = \theta_f$$

$$\Delta_{cd} = \Delta_{gh} = \Delta_{kl} = \Delta_{op}$$

$$\Delta_{bc} = \Delta_{fg} = \Delta_{jk} = \Delta_{no}$$

$$\Delta_{ab} = \Delta_{ef} = \Delta_{ij} = \Delta_{mn}$$

and

$\sum M = 0$ at all joints of the frame.

It follows that the expressions for the rotations at the joints are:

$$(4K_{hd} + 6K_{hl} + K_{hg}\,A_{hg})\,\theta_h + 2K_{hd}\,\theta_d - K_{hg}\,C_{hg}\frac{\Delta_{hg}}{L_{hg}} = 0 \text{ --(26a)}$$

$$(4K_{dh} + K_{dc}\,A_{dc})\,\theta_d + 2K_{dh}\,\theta_h - K_{dc}\,C_{dc}\frac{\Delta_{dc}}{L_{dc}} = 0 \text{ - - - -(26b)}$$

$$(K_{cd}\,A_{cd} + 4K_{cg} + K_{cb}\,A_{cb})\,\theta_c + 2K_{cg}\,\theta_g - K_{cd}\,C_{cd}\frac{\Delta_{cd}}{L_{cd}}$$

$$- K_{cb}C_{cb}\frac{\Delta_{cb}}{L_{cb}} = 0 \text{ - - - - - - - - - - - - - - - - - - (26c)}$$

$$(4K_{gc} + K_{gh}\,A_{gh} + 6K_{gk} + K_{gf}\,A_{gf})\,\theta_g + 2K_{gc}\,\theta_c - K_{gh}C_{gh}\frac{\Delta_{gh}}{L_{gh}}$$

$$- K_{gf}\,C_{gf}\frac{\Delta_{gf}}{L_{gf}} = 0 \text{ - - - - - - - - - - - - - - - - - (26d)}$$

$$(K_{bc}\,A_{bc} + 4K_{bf} + K_{ba}\,A_{ba})\,\theta_b + 2K_{bf}\,\theta_f - K_{bc}\,C_{bc}\frac{\Delta_{bc}}{L_{bc}}$$

$$- K_{ba}\,C_{ba}\frac{\Delta_{ba}}{L_{ba}} = 0 \text{ - - - - - - - - - - - - - - - - - (26e)}$$

$$(K_{fg}\,A_{fg} + 4K_{fb} + 6K_{fj} + K_{fe}\,A_{fe})\,\theta_f + 2K_{fb}\,\theta_b$$

$$-K_{fg}C_{fg}\frac{\Delta_{fg}}{L_{fg}} - K_{fe}\,C_{fe}\frac{\Delta_{fe}}{L_{fe}} = 0 \text{ - - - - - - - - - - (26f)}$$

and the shearing equations are:

$$\frac{K_{cd}}{L_{cd}} C_{cd} \left(\theta_d - 2\frac{\Delta_{cd}}{L_{cd}}\right) + \frac{K_{gh}}{L_{gh}} C_{gh} \left(\theta_h - 2\frac{\Delta_{gh}}{L_{gh}}\right) + P_{cd}\frac{\Delta_{cd}}{L_{cd}}$$

$$+ P_{gh}\frac{\Delta_{gh}}{L_{gh}} = \frac{H}{2} = 0 \text{ - - - - - - - - - - - - - - - - (26g)}$$

$$\frac{K_{bc}}{L_{bc}} C_{bc} \left(\theta_c - 2\frac{\Delta_{bc}}{L_{bc}}\right) + \frac{K_{fg}}{L_{fg}} C_{fg} \left(\theta_g - 2\frac{\Delta_{fg}}{L_{fg}}\right) + P_{bc}\frac{\Delta_{bc}}{L_{bc}}$$

$$+ P_{fg}\frac{\Delta_{fg}}{L_{fg}} = H = 0 \text{ - - - - - - - - - - - - - - - - - - (26h)}$$

$$\frac{K_{ab}}{L_{ab}} C_{ab} \left(\theta_b - 2\frac{\Delta_{ab}}{L_{ab}}\right) + \frac{K_{ef}}{L_{ef}} C_{ef} \left(\theta_f - 2\frac{\Delta_{ef}}{L_{ef}}\right) + P_{ab}\frac{\Delta_{ab}}{L_{ab}}$$

$$+ P_{ef}\frac{\Delta_{ef}}{L_{ef}} = \frac{3H}{2} = 0 \text{ - - - - - - - - - - - - - - - (26i)}$$

The critical load of this problem can also be obtained as stated previously.

INDEPENDENTLY BRACED FRAMES[8]

As it has been stated before, frames cannot exist singly. In frames which are braced against sidesway, the bracing system of these frames may be considered as external to the frame, and this bracing system may be in the form of sheathing, wall panels, or diagonal tie-rod in the plane of the frame. By applying the slope-deflection theory and assuming that the bracing system is represented by a linear spring, then the critical load of the frame can also be found. The same procedure will be used as to solve the anti-symmetrical buckling of frames, except that the spring

[8] *Loc. cit.*

force must be added into the shearing equilibrium equation.

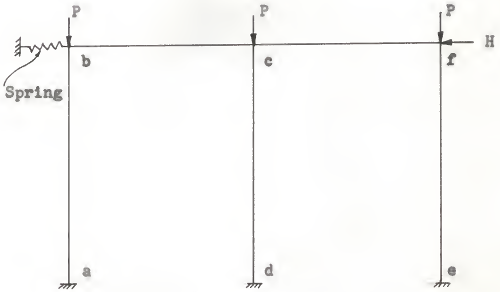

Fig. 9. Representation of an independently braced frame.

For the purpose of illustration, a two-bay single-story frame as shown in Fig. 9, with the spring at joint b, represents the additional or external bracing. It is also assumed that the column bases are fixed. Therefore, the three joint equilibrium equations will be used as described previously. If K is the constant or rate of spring, then $K\Delta_{ab}$ is the spring force which acts in the direction opposite the deflection Δ_{ab}, or it acts in the positive direction of the fictitious force H. Therefore, this force, $K\Delta_{ab}$ must be added to the force H in the shearing equilibrium equation. Thus, from Fig. 9, the following equations may be obtained:

$$-S_{ab} - S_{dc} - S_{ef} = H + K\Delta_{ab} \quad \text{- - - - - - - - - - - - - - -(27a)}$$

or,

$$\frac{K_{ab}}{L_{ab}} C_{ab}\,\theta_b + \frac{K_{cd}}{L_{cd}} C_{cd}\,\theta_c + \frac{K_{ef}}{L_{ef}} C_{ef}\,\theta_f - \left(\frac{2K_{ab}}{L^2_{ab}} C_{ab} + \frac{2K_{cd}}{L^2_{cd}} C_{cd}\right.$$

$$\left. + \frac{2K_{ef}}{L^2_{ef}} C_{ef} - \frac{P_{ab}}{L_{ab}} - \frac{P_{cd}}{L_{cd}} - \frac{P_{ef}}{L_{ef}} + K\right)\triangle = H = 0 \quad ---(27b)$$

The procedure for determining the critical load from the above equation is the same as that outlined in the previous section, which is based on the trial-and-error procedure.

Example 1

The Portal Frame Shown in Fig. 10. (The same frame is shown in Fig. 5).

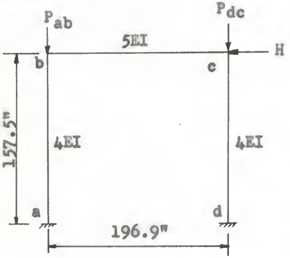

Fig. 10. Simple portal frame.

Assuming that:

$E = 30 \times 10^6$ psi

$I = 20$ in.^{4}

$\theta_b = \theta_c$

$\triangle_{ab} = \triangle_{dc} = 1$

$\triangle_{bc} = 0$

Then,

$P_e = \dfrac{EI\pi^2}{L^2} = 955{,}000$ lb.

$K_{ab} = \dfrac{4EI}{L_{ab}} = 15.22 \times 10^6$ lb - in.

$K_{bc} = \dfrac{5EI}{L_{bc}} = 15.22 \times 10^6$ lb - in.

The trial-and-error procedure is tabulated as follows:

Number of Trials	$\rho = \frac{P}{P_e}$	$P = \rho P_e$ (lb)	A	B	C = A+B	θ_b obtained from Eq. (23c)	H obtained from Eq. (23f) (lb)
1st	0.7	668,500	2.981	2.291	5.272	0.00372	-670
2nd	0.75	716,000	2.899	2.3685	5.2675	0.003755	+ 10
3rd	0.746	712,000	2.9051	2.3163	5.2214	0.003715	0

Therefore,

$$P_{cr.} = 712{,}000\#$$

which agrees with S. Hansbo's "exact" value. From his example, p. 36 of Reference 5, he obtained

$$(KL)_{cr.} = 2.714$$

where

$$K = \sqrt{\frac{P}{EI}}$$

or

$$P_{cr.} = \frac{7.36}{L^2}(4EI)$$

Substitution of $E = 30 \times 10^6$ psi. and $I = 20^{in.^4}$ into the above equation, yields:

$$P_{cr.} = 712{,}000\ \#.$$

Example 2

The Portal Frame Shown in Fig. 11. (The same frame is shown in Fig. 6.)

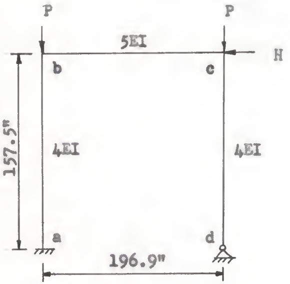

Fig. 11. Simple portal frame.

Assuming that:

$E = 30 \times 10^6$ psi

$I = 20$ in.^{4}

$\theta_B = \theta_c$

Then,

$$P_e = \frac{4EI\pi^2}{L^2} = 955{,}000 \text{ lb.}$$

$$K_{ab} = \frac{4EI}{L_{ab}} = 15.22 \times 10^6 \text{ lb.-in.}$$

$$K_{bc} = \frac{5EI}{L_{bc}} = 15.22 \times 10^6 \text{ lb.-in.}$$

The trial-and-error procedure is tabulated as follows:

Number of trials	ρ	$P = \rho P_e$	A	B	C = A+B	θ_b obtained from Eq. (24c)	H obtained from Eq. (24e)
1st	0.45	430,000	3.3695	2.175	5.5445	0.00376	+ 185
2nd	0.447	427,000	3.374	2.1706	5.5446	0.003758	+ 130
3rd	0.440	420,000	3.3944	2.1676	5.5620	0.003755	0

Therefore,

$$P_{cr.} = 420{,}000 \text{ lb.}$$

which is different by 0.942 per cent from S. Hansbo's "exact" value for the same frame, p. 42 of Reference 5. He obtained:

$$(KL)_{cr.} = 2.10$$

and for $EI = 600 \times 10^6$ lb.-in.^{2}, the critical load becomes:

$$P_{cr.} = 424{,}000 \text{ lb.}$$

Example 3

A Three-story Single Bay Structural Frame as in Fig. 12. (The same frame is shown in Fig. 7.)

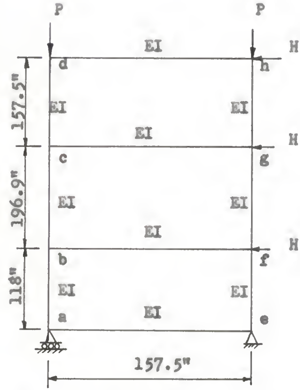

Fig. 12. Three-story single bay frame.

Assuming that:

$E = 30 \times 10^6$ psi.

$I = 100$ in.^{4}

Then,

at column cd

$P_e = 11.93 \times 10^5$ lb.

at column bc

$P_e = 7.65 \times 10^5$ lb.

at column ab

$P_e = 21.3 \times 10^5$ lb.

and

$K_{cd} = K_{gh} = 19.05 \times 10^6$ lb.-in.

$K_{dh} = K_{cg} = K_{bf} = K_{ae} - 19.05 \times 10^6$ lb.-in.

$K_{ba} = K_{ef} = 25.04 \times 10^6$ lb.-in.

$K_{bc} = K_{gf} = 15.22 \times 10^6$ lb.-in.

The trial-and-error procedure is tabulated in the following:

	No. of trials	1st	2nd	3rd
At column cd or gh	ρ	0.407	0.325	0.373
	$P = \rho P_e$ (lb)	0.484×10^6	0.388×10^6	0.446×10^6
	A	3.4336	3.5528	3.483
	B	2.154	2.11925	2.1389
	$C = A + B$	5.5876	5.67205	5.6219
	θ_d from Eq.(25a)	0.00376	0.003765	0.00377
	H from Eq.(25d)	- 6060	- 7320	- 6620
At column bc or fg	ρ	0.633	0.507	0.583
	P (lb)	0.484×10^6	0.388×10^6	0.446×10^6
	A	3.0876	3.28415	3.1665
	B	2.2575	2.19729	2.2322
	$C = A + B$	5.3451	5.48144	5.3987
	θ_c from Eq.(25b)	0.0048	0.00499	0.00479
	H from Eq. (25e)	+ 39	- 330	0
At column ab or ef	ρ	Since it is unstable at column bc for the assumed column load, there is no need to solve $P_{cr.}$ for this column.	0.182	0.209
	P (lb)		0.388×10^6	0.446×10^6
	A		3.7543	3.7173
	B		2.0633	2.0735
	$C = A + B$		5.8176	5.7908
	θ_b from Eq.(25c)		0.004275	0.00426
	H from Eq. (25f)		- 8390	- 7950

Thus, $P_{cr.}$ = 446,000 lb. agrees with S. Hansbo's result, p. 31 of Reference 5. He obtained $P_{cr.} = 148.6 \times 10^{-6}$ EI. Substitution of EI = 3000×10^6 lb.-in.^{2} into $P_{cr.}$ Then, $P_{cr.}$ = 446,000 lb.

Example 4

Three-story Three Bay Building Frame as Shown in Fig. 13. (The same frame is shown in Fig. 8.)

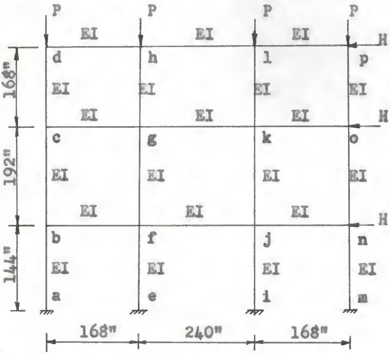

Fig. 13. Three-story three bay frame.

Assuming that:

$E = 30 \times 10^6$ psi

$I = 100$ in.^{4}

$\theta_p = \theta_d$

$\theta l = \theta_h$

$\theta_c = \theta_o$

$\theta_k = \theta_g$

$\theta b = \theta_n$

$\theta j = \theta_f$

and all the horizontal joint displacements equal to unity,

then,

$$K_{cd} = K_{hg} = K_{lk} = K_{po} = 17.86 \times 10^6 \text{ lb-in.}$$

$$K_{cb} = K_{fg} = K_{jk} = K_{no} = 15.62 \times 10^6 \text{ lb-in.}$$

$$K_{ab} = K_{ef} = K_{ij} = K_{mn} = 20.83 \times 10^6 \text{ lb-in.}$$

$$K_{dh} = K_{cg} = K_{bf} = K_{lp} = K_{ko} = K_{jn} = 17.86 \times 10^6 \text{ lb-in.}$$

$$K_{hl} = K_{gk} = K_{fj} = 12.5 \times 10^6 \text{ lb-in.}$$

The trial-and-error procedure is tabulated in the following:

	No. of trials	1st	2nd
At column cd or gh	ρ	0.457	0.496
	$P = \rho P_e$	0.48×10^6	0.52×10^6
	A	3.3591	3.301
	B	2.1751	2.1922
	C = A + B	5.5342	5.4932
	From Eq. (26a) θ_d	0.0039	0.003885
	and Eq. (26b) θ_h	0.00217	0.00217
	H from Eq. (26g)	- 9400	- 8380
At column bc or fg	ρ	0.598	0.648
	$P = \rho P_e$	0.48×10^6	0.52×10^6
	A	3.143	3.0636
	B	2.240	2.265
	C = A + B	5.383	5.3286
	From Eq. (26c) θ_c	0.004985	0.00505
	and Eq. (26d) θ_g	0.003502	0.003308
	H from Eq. (26h)	- 390	+ 0
At column ab or ef	ρ	0.336	0.364
	$P = \rho P_e$	0.48×10^6	0.52×10^6
	A	3.5368	3.4962
	B	2.12375	2.1352
	C = A + B	5.66055	5.6314
	From Eq. (26e) θ_b	0.00577	0.00577
	and Eq. (26f) θ_f	0.003696	0.003905
	H from Eq. (26i)	- 5580	- 5050

Since from the second trial H = 0, therefore, $P_{cr.}$ = 520,000 lb.

In the above examples the actual loading system consisted of forces applied at the joints only, and the effect of primary bending moments was neglected. This is justified by the results of E. F. Masur[9] who presented a method to solve buckling problems including the effect of primary bending moments on the elastic stability of structure of a portal frame similar to that shown in Figs. 5 and 6. The loading system is applied on the beam at a certain distance from the joint. Masur, p. 20 of Reference 9, states,

> ....The unbuckled structure is therefore assumed to be in its virginal state; that is, its members are straight, and no "primary" bending moments are present. The replacement of the actual loading system by one that is (for each member) statically equivalent to it is usually justified by the assumption that very small errors are thus introduced.

Therefore, in the elastic range, primary bending moments affect the stability of structure only very little and can usually be neglected. The stability of a partially plastic structure is certain to be intimately related to the presence of primary bending moments. However, the treatment of this problem is beyond the scope of this report.

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[9] E. F. Masur, I. C. Chang, and L. H. Donnell, "Stability of Frames in the Presence of Primary Bending Moments," <u>Proceedings A.S.C.E.</u>, Vol. 87 No. EM4, August 1961, Part 1.

CONCLUSIONS

This report treats the stability analysis of rectangular rigid frames through the use of the direct analytical procedure based on the slope-deflection method. This procedure is fast and accurate, particularly with the use of digital computers, because the buckling problem of frames will be reduced to the solution of a set of simultaneous equations.

The theory of slope-deflection procedure for solving stability problems of rectangular frames is simple, provided that buckling occurs within the elastic range. It is felt that the slope-deflection procedure is more direct and comprehensive than the moment-distribution method for solving stability problems.

In this report the effect of primary bending moment which occurs from the application of the actual loading system at points along the member rather than at joints of the member is not considered. The replacement of the actual loading system by one that is statically equivalent to it, is usually justified by the assumption that very small errors are thus introduced; but this does not affect the application of this method for engineering purposes.

The comparison of the results obtained using slope-deflection procedure and the results obtained by S. Hansbo, using the moment-distribution method for the same problems, shows that the difference is smaller than 0.942 per cent. This proves that both solutions give results which are in very good agreement.

All the results presented in this report are obtained through the use of a slide rule.

ACKNOWLEDGMENTS

The author wishes to express his appreciation to his major professor, Dr. K. N. Jabbour, for suggesting this subject and for his help and advice throughout this study.

APPENDIX A

Example 2. Multistory frame.

In the three-story single-bay frame according to Fig. 17 all members have constant and equal bending rigidity, EI tm². The columns are centrally loaded by the loads P. Assuming $P = 0.23\,EI$ t as a probable value of the critical load, the following data are obtained:

TABLE 3 a.

Member	kL	c	C	Z	t	p	Σt	Σp	$-ZpL/2$
1—2	1.44	1.24	.558	1.218	.0766	.352	.153	.704	−.643
5—6	1.44	1.24	.558	1.218	.0766	.352			−.643
2—3	2.40	.634	.705	2.14	.0460	.0403	.0920	.0806	−.216
6—7	2.40	.634	.705	2.14	.0460	.0430			.216
3—4	1.92	.870	.614	1.488	.575	.118	.115	.236	−.351
7—8	1.92	.870	.614	1.488	.575	.118			−.351
1 5, 2 6, 3 7, 1 8	0	1	.500						
Multiplier.		EI/m			EI/m^3	EI/m^3	EI/m^3	EI/m^3	EI/m^2

By applying virtual antisymmetrical moments at the joints 2, 6, 3, and 7, adjacent to the critical middle story, it will be necessary to study only one half of the frame.

The moment distribution is carried through as in Ex. 1. Each stage of joint balancing (M_i) is followed by a correction of the shear in the columns of each story through sidesway (S_i).

Table 3 c shows that $P/EI = 0.23/m^2$ is close to the buckling value. Another trial with $P/EI = 0.24/m^2$ will show a rapid divergence of Δ'. Hence $P = 0.23\,EI$ t can be accepted as the critical load P_{cr}.

This result can be checked by Granholm's method.[1]) The frame parts a, b, and c, Fig. 18, buckle simultaneously if the frame transversals are split in accordance with Table 4.

The frame has a critical load slightly larger than $0.23\,EI$ t which agrees with the previous result.

TABLE 3 b. *Balancing procedure.*

	M_{15}	M_{12}	M_{01}	M_{21}	M_{26}	M_{23}	M_{02}	M_{32}	M_{37}	M_{34}	M_{03}	M_{43}	M_{48}	M_{04}
d	.446	.554		.432	.348	.220		.253	.399	.348		.465	.535	
c	.223	.247		.309	.174	.178		.155	.200	.285		.214	.268	
M 1				+4.32	+3.48	+2.20	+10.0	+2.53	+3.99	+3.48	+10.0			
				—4.32	—3.48	—2.20		—2.53	—3.99	—3.48				
		—2.47			—1.74	—1.78		—1.55	—2.00			—2.14		
S 1		—2.47	—2.47		—1.74	—1.78	— 3.52	—1.55	—2.00		— 3.55	—2.14		—2.14
		+1.50		+1.50		+3.56		+3.56		+1.59		+1.59		
M 2		— .97	— .97	+1.50	—1.74	+1.78	+ 1.54	+2.01	—2.00	+1.59	+ 1.60	— .55		— .55
	+.43	+ .54		— .66	— .54	— .34		— .40	— .64	— .56		+ .26	+ .29	
	+.22	— .38		+ .30	— .27	— .28		— .24	— .32	+ .16		— .34	+ .15	
S 2	+.65	— .81	— .16	+1.14	—2.55	+1.16	— .25	+1.37	—2.96	+1.19	— .40	— .63	+ .44	— .19
		+ .13		+ .13		+1.36		+1.36		+ .36		+ .36		
M 3	+.65	— .68	— .03	+1.27	—2.55	+2.52	+ 1.24	+2.73	—2.96	+1.55	+ 1.32	— .27	+ .44	+ .17
	+.01	+ .02		— .54	— .43	— .27		— .33	— .53	— .46		— .08	— .09	
	+.01	— .31		+ .01	— .22	— .24		— .19	— .26	— .05		— .28	— .05	
S 3	+.67	— .97	— .30	+ .74	—3.20	+2.01	— .45	+2.21	—3.75	+1.04	— .50	— .63	+ .30	— .33
		+ .56		+ .56		+1.10		+1.10		+ .64		+ .64		
M 4	+.67	— .41	+ .26	+1.30	—3.20	+3.11	+ 1.21	+3.31	—3.75	+1.68	+ 1.24	+ .01	+ .30	+ .31
	—.12	— .14		— .52	— .42	— .27		— .31	— .50	— .43		— .14	— .17	
	—.06	— .30		— .08	— .21	— .22		— .19	— .25	— .09		— .27	— .08	
S 4	+.49	— .85	— .36	+ .70	—3.83	+2.62	— .51	+2.81	—4.50	+1.16	— .53	— .40	+ .05	— .35
		+ .57		+ .57		+1.07		+1.07		+ .70		+ .70		
M 5	+.40	— .28	+ .21	+1.27	—3.83	+3.69	+ 1.13	+3.88	—4.50	+1.86	+ 1.24	+ .30	+ .05	+ .35
	—.09	— .12		— .40	— .30	— .25		— .31	— .50	— .43		— .16	— .19	
	—.05	— .28		— .06	— .20	— .22		— .18	— .25	— .10		— .27	— .09	
S 5	+.35	— .68	— .33	+ .72	—4.42	+3.22	— .48	+3.39	—5.25	+1.33	— .53	— .13	— .23	— .36
		+ .58		+ .58		+1.02		+1.02		+ .71		+ .71		
M 6	+.35	— .10	+ .25	+1.30	—4.42	+4.24	+ 1.12	+4.41	—5.25	+2.04	+ 1.20	+ .58	— .23	+ .35
	—.11	— .14		— .48	— .39	— .25		— .30	— .48	— .42		— .16	— .19	
	—.06	— .28		— .08	— .19	— .21		— .17	— .24	— .10		— .26	— .09	
S 6	+.18	— .52	— .34	+ .74	—5.00	+3.78	— .48	+3.94	—5.97	+1.52	— .51	+ .16	— .51	— .35
		+ .58		+ .58		+ .99		+ .99		+ .70		+ .70		
M 7	+.18	+ .06	+ .24	+1.23	—5.00	+4.77	+ 1.09	+4.93	—5.97	+2.22	+ 1.18	+ .86	— .51	+ .35
	—.11	— .13		— .47	— .38	— .24		— .30	— .47	— .41		— .16	— .19	
	—.05	— .27		— .07	— .19	— .21		— .17	— .24	— .10		— .25	— .09	
S 7	+.02	— .34	— .32	+ .78	—5.57	+4.32	— .47	+4.46	—6.68	+1.71	— .51	+ .45	— .79	— .34
		+ .58		+ .58		+ .99		+ .99		+ .68		+ .68		
M 8	.02	+ .24	+ .26	+1.36	—5.57	+5.31	+ 1.10	+5.45	—6.68	+2.39	+ 1.16	+1.13	— .79	+ .34
	.12	— .14		— .48	— .38	.24		— .29	— .46	— .41		— .16	— .18	
	-.06	.27		.08	19	.21		.17	— .23	— .10		— .25	— .09	
	— .16	— .17	— .33	+ .80	— 6.14	— 4.86	— .48	+ 4.99	— 7.37	+ 1.88	— .50	+ .72	— 1.06	— .34

TABLE 3 c. *Correction for sidesway.*

	Member	T/EI	Δ	Δ'	M'/EI
S 1	1—2	—.165	0	— .231	— 1.50
	2—3	—.133	0	1.65	3.56
	3—4	—.107	0	.451	1.50
S 2	1—2	+.022	— .234	.020	.13
	2—3	+.101	—1.65	— .63	— 1.36
	3—4	+.028	— .454	— .102	— .36
S 3	1—2	— .022	— .254	— .087	.56
	2—3	.169	—2.28	— .51	1.10
	3—4	— .021	— .566	— .182	— .64
S 4	1—2	—.010	— .341	— .088	— .57
	2—3	+ .217	—2.79	— .50	+ 1.07
	3—4	+.038	— .738	— .199	— .70
S 5	1—2	— .003	— .429	— .090	.58
	2—3	+.265	—3.29	— .47	+1.02
	3—4	+.060	— .937	— .203	+ .71
S 6	1—2	+ .015	— .519	— .091	— .58
	2—3	+.309	—3.76	— .46	— .99
	3—4	+.084	—1.14	— .20	+ .70
S 7	1—2	+.029	— .610	— .091	+ .58
	2—3	+ .351	—4.22	— .46	+ .99
	3—4	+.108	—1.34	— .20	+ .68
S 8	1—2	+.042	— .701	— .092	—
	2—3	—.394	—4.68	— .45	—
	3—4	+.130	—1.54	— .20	—

TABLE 4.

Frame part	$(EI)_2$	$(EI)_1$	m_2	m_1	$(kL)^2$	P_{cr}/EI
a	.25	1.00	1.5	6.0	3.70	.23
b	.91	.75	6.8	5.6	5.75	.23
c	1.00	.09	4.5	.4	2.25	.24
Multiplier:	EI	EI	1	1	1	$1/m^2$

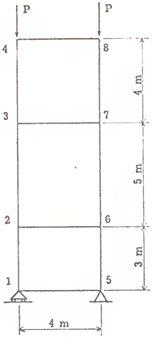

Fig. 17

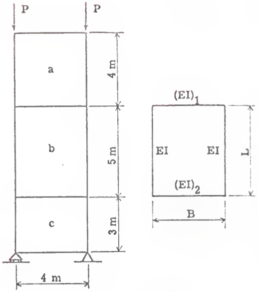

Fig. 18

Matrix analysis

Example 3. Portal frame.

The simple portal frame shown in Fig. 19, fixed at base, is unstabl at a value $(kL)_{cr} = 0.865\,\pi = 2.72$[1]).

Assuming $kL = 2.70$ as a probable critical value, Table 5 a is calculated:

TABLE 5 a.

Member	kL	e	C	Z	$-Zp/2\,\Sigma p$
01, 23	2.70	2.92	.792	3.30	−.825
12	0	4.00	.500		
Multiplier:		EI/m	1	1	1

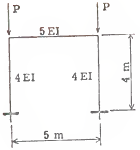

Fig. 19

When the frame is symmetric with respect to loading and bending rigidity as in this case, the value of p will be equal for the columns in each particular story. Hence $p_n/\Sigma_n p_n = 1/N$, where N is the total number of the columns in the story.

The matrix elements are calculated in Fig. 20 and 21. Due to the symmetry of loading and bending rigidity, the matrix r will be symmetric.

Fig. 21 yields

$$r_{12} = r_{21} = \frac{0.335}{1 - 0.624} = 0.89$$

$$r = \begin{bmatrix} 0 & .89 \\ .89 & 0 \end{bmatrix}$$

The dominant eigenvalue λ of this matrix is equal to the matrix elements 0.89. Obviously, the frame is stable for $kL = 2.70$.

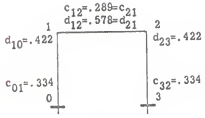

Fig. 20

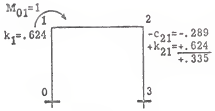

Fig. 21

Next a buckling value of $kL = 2.74$ is guessed at. Table 5 b is calculated:

TABLE 5 b.

Member	kL	c/EI	C	Z	$-Zp/2\Sigma p$
01, 23	2.74	2.88	.806	3.59	-.898
12	0	4.00	.500		
Dimension:		1/m			

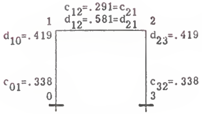

Fig. 22

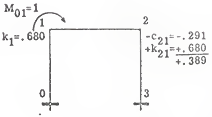

Fig. 23

Fig. 22 to 23 yield

$$r_{12} = r_{21} = \frac{0.389}{1 - 0.680} = 1.21$$

The matrix

$$r = \begin{bmatrix} 0 & 1.21 \\ 1.21 & 0 \end{bmatrix}$$

yields $\lambda = 1.21$.

Thus $kL = 2.74$ is above the critical value. Interpolation between these two values of the dominant eigenvalue gives $(kL)_{cr} = 2.714$. The value obtained by the matrix method is in agreement with the »exact» value.

Example 5. Portal frame having one column fixed at base and the other hinged.

Finally, the matrix method will be used to find the critical load of the portal frame, Fig. 30. As first approximation of the critical load the mean value may be chosen between the critical load for the frame fixed at base and for the frame hinged at base. This gives us a probable value $kL = 2.04$.[1])

Now assuming as the first trial $kL = 2.10$, table 7 is calculated.

TABLE 7.

Member	kL	c	C	Z	p	Σp	$-Zp/\Sigma p$
01	2.10	3.38	.643	1.660	.01047	.00666	—2.60
23	2.10	1.98	0	— .814	—.00381		— .465
12	0	4.00	.500				
Multiplier:		EI/m	1	1	EI/m^3	EI/m^3	1

The matrix elements are evulated in Fig. 31 to 33.

Fig. 32 yields

$$r_{12} = 0.079/(1 - 0.978) = 3.60$$

Likewise. Fig. 33 yields

$$r_{21} = 0.095/(1 - 0.154) = 0.112$$

Thus the matrix

$$r = \begin{bmatrix} 0 & 3.60 \\ 0.112 & 0 \end{bmatrix}$$

The dominant eigenvalue λ of this matrix is

$$\lambda = \sqrt{3.60 \cdot 0.112} = 0.64$$

Hence $KL = 2.10$ is below the critical value.

Next a value $kL = 2.11$ is guessed at. Table 8 is calculated.

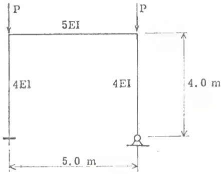

Fig. 30

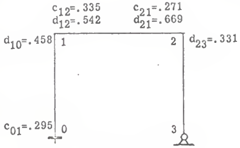

Fig. 31

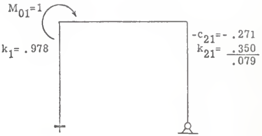

Fig. 32

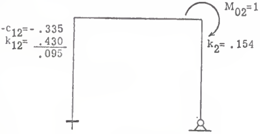

Fig. 33

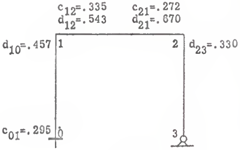

Fig. 34

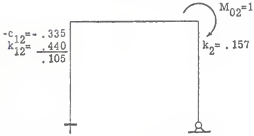

Fig. 35

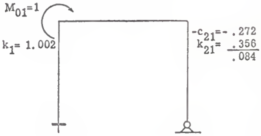

Fig. 36

Table 8.

Member	KL	e	C	Z	p	Σp	$-\frac{Zp}{\Sigma p}$
01	2.11	3.37	0.645	1.671	0.01036	0.00648	-2.67
23	2.11	1.97	0	-0.792	-0.00388		-0.474
12	0	4.00	0.500				
Multiplier		EI/M	1	1	EI/M^3	EI/M^3	1

The matrix elements are evaluated in Figs. 34 to 36. Fig. 36 yields $K_1 > 1$, which proves that KL = 2.11 is above the critical value. Inspection shows that KL = 2.10 is closer to the critical value than is KL = 2.11. Hence, KL = 2.10 can be accepted as the critical value.

APPENDIX B

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BUCKLING OF RIGID FRAMES

by

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B. S., University of Wyoming, 1961

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Civil Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1963

It is considered that elastic stability is a problem of great importance in the modern use of steel and high-strength alloys in engineering structures, especially in tall buildings, bridges and aircrafts.

This paper presents an analysis of buckling of frames based on the well-known slope-deflection procedure. The stability of one-story and multi-story plane frames is studied for the anti-symmetrical mode of buckling. Other methods for calculating the buckling load of frames are also discussed. Typical examples are solved using the direct analytical procedure, and the results obtained are compared with the results obtained through the use of the moment-distribution method. It was found that both solutions give results which are in very good agreement.

It is felt that the slope-deflection method is simpler and more direct than the moment-distribution method in solving stability problems.