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ADVANCED STUDY OF THE COLUMN ANALOGY METHOD

by

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SYNOPSIS

The purpose of this report is to show a general derivation and application of the method of column analogy. This method can be used in finding the moments in third degree statically indeterminate structures. It is also very effective in calculating the stiffness and carry-over factors for nonprismatic members for moment distribution.

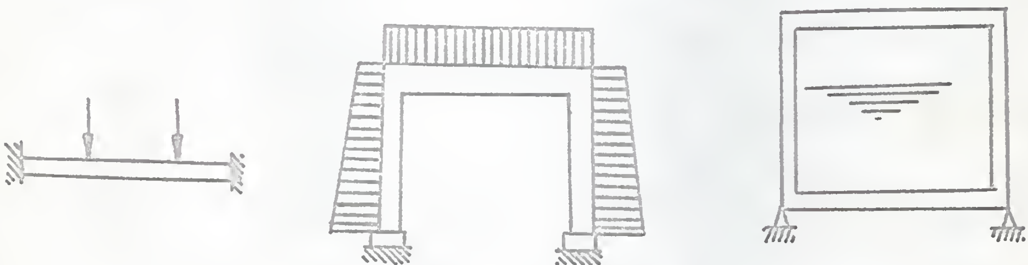
The application section of this report shows how this method can be applied to actual problems of finding moments and stiffness and carry-over factors. A comparison of the results of the column analogy method and the moment distribution method is presented, which demonstrates the fact that these results are essentially identical for the problem solved.

INTRODUCTION

The method of column analogy was first discovered in 1930, by Professor Hardy Cross (1, 6).* This method can be used in computing the moments in rigid frames and the stiffness and carry-over factors of the moment distribution method. Professor Cross stated that, "The column analogy is a mathematical identity between the moments produced by continuity in a beam, bent or arch and the fiber stresses in a short column eccentrically loaded." (1)

The method of column analogy is one of the most important methods used in solving rigid frames or any structures which can be considered to be closed rings. These include pipes, culverts, tanks, arches, bents, and single span fixed ended beams.

In general, the method of column analogy can be applied to the three types of structures shown in Fig. 1. (6)



(a) Single Span Fixed Ended Beam (b) Bent (c) Culvert

Fig. 1

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*Numbers in parentheses refer to corresponding items in the References.

The earth support is considered as a portion of the ring with EI of infinite magnitude. These three types of structures are suggestive of the many structures to which the method is applicable.

The column analogy method is used to compute the fiber stresses in an analogous column loaded with the statically determinate moment diagram from the reduced ring. The axis of the analogous column is the same as that of the ring; the thickness of the analogous column is equal to 1/EI of the ring. E and I can be either constant or variable. The fiber stresses in the analogous column are numerically equal to the moments required to restore continuity to the reduced ring structure. Thus by adding the determinate moment at a section to the fiber stress in the analogous column at that section one obtains the total moment in the actual structure for the section investigated.

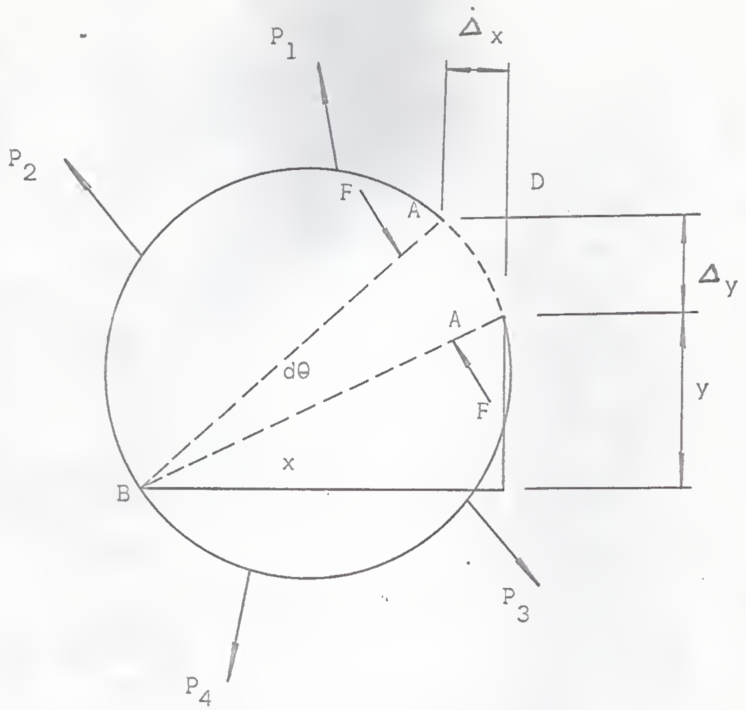
DERIVATION

Formulas for a Symmetrical Cross Section

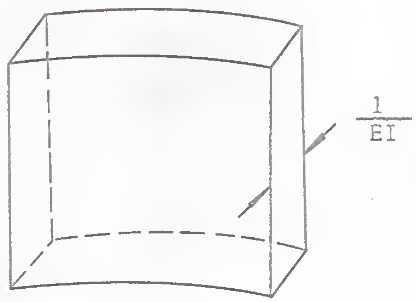
A ring structure as shown in Fig. 2(a) is being considered. The ring is loaded with a set of forces P. Under the action of P, there will exist shear, thrust (or tension) and moment at section A. F is used to indicate the resultant of these shears, thrusts (or tension), and moments on each side of section A.

After the ring is cut at A, the combined effect of the external loads P, and the forces F, would keep it continuous at this section. Assume that these two sets of forces, P and F, are acting separately.

(1) The loads P_1, P_2, P_3............, P_{n-1}, P_n will cause bending of the structure and produce statically determinate moment, M_d,

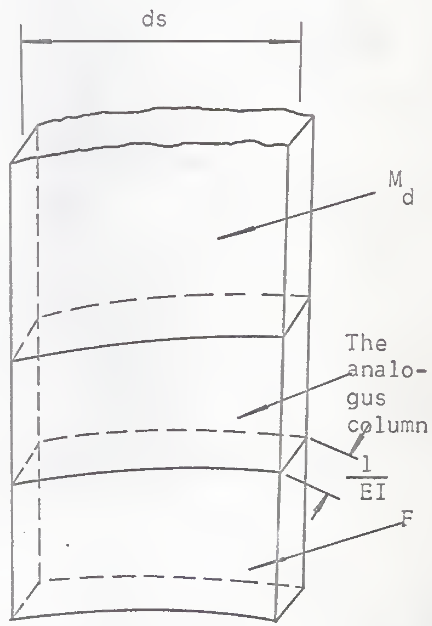


(a) Geometry of closed ring



(b) Portion of the analogous column of the ring

Fig. 2. A portion of the analogous column



(c) A portion of the analogous column loaded with M_d diagram and reaction force F at any section throughout the ring. Owing to this moment there will be relative rotation and relative displacement of both ends at the section where it is cut. The rotation in a small length ds is

$$d\theta_d = \frac{M_d \cdot ds}{EI} \tag{1}$$

In Fig. 2, since both Δ_x and Δ_y are very small, AA' and AB are assumed to be mutually perpendicular, then

$$\Delta ABC \sim \Delta AA'D$$

or,

$$\frac{\Delta_{xd}}{AA'} = \frac{y}{AB} \tag{2}$$

and,

$$\frac{\Delta_{yd}}{AA'} = \frac{x}{AB} \tag{3}$$

Solving equations (2) and (3) for Δ_{yd} and Δ_{xd} yields

$$\Delta_{yd} = \frac{AA'}{AB}(x) \tag{4}$$

and,

$$\Delta_{xd} = \frac{AA'}{AB}(y) \tag{5}$$

Since θ is very small, it can be assumed equal to its own tangent, then

$$\theta = \frac{AA'}{AB} \quad \text{then } \theta_d = \int \frac{M_d \cdot ds}{EI}$$

Substitute θ into equations (4) and (5), then

$$\Delta_{yd} = \int M_d \left(\frac{ds}{EI} \right) x \tag{6}$$

and,

$$\Delta_{xd} = \int M_d \left(\frac{ds}{EI} \right) y \tag{7}$$

(2) As a result of the F forces, which are acting on both sides of the section at A, after the ring is cut, there is a statically indeterminate moment M_i at every section throughout the whole ring. By the same reasoning as found in section (1), the rotation and deformation caused by the indeterminate moment M_i, are

$$\theta_i = \int M_i \left(\frac{ds}{EI} \right) \tag{8}$$

$$\Delta_{yi} = \int M_i \left(\frac{ds}{EI} \right) \tag{9}$$

and,

$$\Delta_{xi} = \int M_i \left(\frac{ds}{EI} \right) \tag{10}$$

Since the forces P and F are assumed to keep the ring continuous, the total rotation and total displacement due to both determinate moment, M_d, and indeterminate moment, M_i, must be equal to zero. Hence,

$$\theta_d + \theta_i = 0 \tag{11}$$

$$\Delta_{yd} + \Delta_{yi} = 0 \tag{12}$$

$$\Delta_{xd} + \Delta_{xi} = 0 \tag{13}$$

or substituting equations (1), (6), (7), (8), (9), and (10) into equations (11), (12), and (13) respectively.

$$\int M_d \left(\frac{ds}{EI}\right) + \int M_i \left(\frac{ds}{EI}\right) = 0 \tag{14}$$

$$\int M_d \left(\frac{ds}{EI}\right)x + \int M_i \left(\frac{ds}{EI}\right)x = 0 \tag{15}$$

$$\int M_d \left(\frac{ds}{EI}\right)y + \int M_i \left(\frac{ds}{EI}\right)y = 0 \tag{16}$$

The above equations are called the equations of moments in a closed ring.

(3) Take a small element ds at some distances x and y along the X and Y axes respectively from the centroid of the analogous column. M_d is the load acting on the top of the analogous column and F is the reaction force acting at the bottom of the column, as shown in Fig. 2(c). ds/EI is the area of the portion of the column. Considering the conditions of equilibrium, the following equations are obtained.

$$\Sigma V = 0$$

or,

$$M_d \left(\frac{ds}{EI}\right) + F\left(\frac{ds}{EI}\right) = 0 \tag{17}$$

and,

$$\Sigma M_{xx} = 0$$

or,

$$M_d \left(\frac{ds}{EI}\right)y + F\left(\frac{ds}{EI}\right)y = 0 \tag{18}$$

and,

$$\Sigma M_{yy} = 0$$

or,

$$M_d \left(\frac{ds}{EI} \right) x + F \left(\frac{ds}{EI} \right) x = 0 \tag{19}$$

Integrating to include the entire column, equations (17), (18), and (19) can be written respectively as

$$\int M_d \left(\frac{ds}{EI} \right) + \int F \left(\frac{ds}{EI} \right) = 0 \tag{20}$$

$$\int M_d \left(\frac{ds}{EI} \right) y + \int F \left(\frac{ds}{EI} \right) y = 0 \tag{21}$$

$$\int M_d \left(\frac{ds}{EI} \right) x + \int F \left(\frac{ds}{EI} \right) x = 0 \tag{22}$$

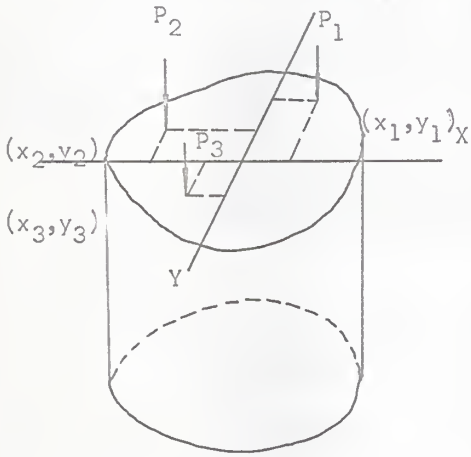
In computing the fiber stresses F of a column, either concentrically loaded or eccentrically loaded, the following formula is used.

$$F = \frac{P}{A} + \frac{M_x \cdot y}{I_x} + \frac{M_y \cdot x}{I_y}$$

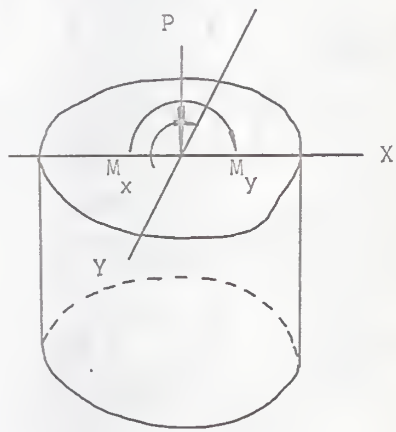
In comparing equations (14), (15), and (16), and equations (20), (21), and (22), M_i is equal to F. The final moment in the ring is equal to $M_d - M_i$, or M_d - F. This is the basic principle of the column analogy method. (6)

Formulas for an Unsymmetrical Cross Section

The formulas for an unsymmetrical cross section have been derived by Chu-Kia Wang as follows (8). Assume a column as shown in Fig. 3a, which is subjected to a set of downward loads P_1, P_2, P_3,...... acting on the points (X_1, Y_1), (X_2, Y_2), and (X_3, Y_3),........ etc. These loads can be transferred to the centroid of the cross section of the column as indicated in Fig. 3b, as a downward concentric load and one bending moment about each axis.



(a) A column loaded with eccentric loads



(b) The transformation of load

Fig. 3. Eccentrically loaded column and its transformation.

According to the principles of statics, after the transfer, the new load P, and the new moments, M_x and M_y, may be written as

$$P = P_1 + P_2 + P_3 \tag{24a}$$

$$M_x = P_1 y_1 + P_2 y_2 + P_3 y_3 \tag{24b}$$

$$M_y = P_1 x_1 + P_2 x_2 + P_3 x_3 \tag{24c}$$

The pressure F at any point can be expressed as

$$F = a + bx + cy \tag{25}$$

where the constants a, b, and c can be determined from the three equations of statics, as follows:

$$P = \int_0^A F dA \tag{26a}$$

$$M_x = \int_0^A F(dA)(y) \tag{26b}$$

$$M_y = \int_0^A F(dA)(x) \tag{26c}$$

Substituting equation (25) into equation (26a), then

$$P = \int_0^A F(dA)$$

$$= \int_0^A (a + bx + cy)dA$$

$$= \int_0^A dA + \int_0^A (x)dA + \int_0^A (y)dA$$

Since

$$\int_0^A dA = A$$

$$\int_0^A (x)dA = 0$$

$$\int_0^A (y)dA = 0$$

Hence,

$$P = a\int_0^A dA$$

$$= aA \tag{27}$$

Substituting equation (25) into equation (26b), then

$$M_x = \int_0^A F(dA)y$$

$$= \int_0^A (a + bx + cy)(dA)y$$

$$= \int_0^A (ay + bxy + cy^2)dA$$

$$= a\int_0^A y(dA) + b\int_0^A (xy)dA + c\int_0^A (y^2)dA$$

$$= 0 + b(I_{xy}) + c(I_x)$$

$$= b(I_{xy}) + c(I_x) \tag{28}$$

Substituting equation (25) into equation (26c), then

$$M_y = \int_0^A F(dA)x$$

$$= \int_0^A (a + bx + cy)(dA)x$$

$$M_y = \int_0^A (ax + bx^2 + cxy)dA$$

$$= a\int_0^A (x)dA + b\int_0^A (x^2)dA + c\int_0^A (xy)dA$$

$$= 0 + b(I_y) + c(I_{xy})$$

$$= b(I_y) + c(I_{xy}) \tag{29}$$

Solving equations (27), (28), and (29) for the constants a, b, and c.

$$a = \frac{P}{A}$$

$$b = \frac{M_y - M_x \dfrac{I_{xy}}{I_x}}{I_y\left(1 - \dfrac{I_{xy}^2}{I_x \cdot I_y}\right)}$$

$$c = \frac{\left(M_x - M_y \dfrac{I_{xy}}{I_y}\right)}{I_x\left(1 - \dfrac{I_{xy}^2}{I_x \cdot I_y}\right)}$$

Let

$$M_x' = M_x - M_y\left(\frac{I_{xy}}{I_y}\right)$$

$$M_y' = M_y - M_x\left(\frac{I_{xy}}{I_x}\right)$$

$$I_x' = I_x\left(1 - \frac{I_{xy}^2}{I_x \cdot I_y}\right)$$

$$I_y' = I_y\left(1 - \frac{I_{xy}^2}{I_x \cdot I_y}\right)$$

then

$$a = \frac{P}{A}$$

$$b = \frac{M_y'}{I_y'}$$

$$c = \frac{M_x'}{I_x'}$$

Substitute the values of a, b, and c into equation (25),

then

$$F = \frac{P}{A} + \frac{M_y'}{I_y'}(x) + \frac{M_x'}{I_x'}(y) \tag{30}$$

SIGN CONVENTION

The sign convention of this report is that used by Chu-Kia Wang. (7) This convention is known by most engineers as the regular beam sign convention.

(1) Loading on the top of column is downward if M_d (statical moment, or moment due to the applied loading on the simple beam AB as determined by the law of statics) is positive, which means that it causes compression on the outside. The definition of the terms inside and outside is as indicated in Fig. 4.

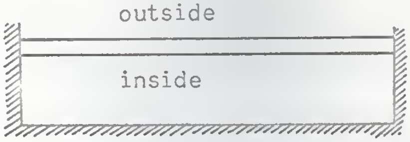


Fig. 4. Simple Ring Type Structure

(2) Upward pressure on the bottom of column, M_i (indeterminate moment, or moment to be determined to satisfy the conditions of geometry), is positive.

(3) The moment at any point in the structure is equal to $M = M_d - M_i$, which is positive if it causes compression on the outside.

(4) The sign of the moment-arms, x and y, is taken from the coordinate system. That is, positive up and to the right from the axis of the coordinates.

GENERAL PROCEDURE

There are five steps in finding moments by the method of the column analogy.

(1) Make the ring (or any structure which can be considered as a ring) statically determinate.

(2) Compute the statically determinate moment M_d, which is produced by the external loads P, according to the laws of statics.

(3) Apply the M_d diagram which is obtained in the second step, as a load, on the top of the analogous column.

(4) Compute the fiber stresses in the analogous column by the laws of statics.

(5) The final moment in the ring (or any structure which can be considered as a ring) is

$$M = M_d - M_i$$

or,

$$M = M_d - F$$

In a manner similar to that for finding moments, the stiffness and carry-over factors for members with either constant or variable cross section can also be found by using the method of column analogy. The details of finding these factors are illustrated in the section, Applications of the Method.

APPLICATION OF THE METHOD

Loaded Gable Frame

Assume a gable frame with one axis of symmetry and loaded with a 15 Kip external concentrated load as indicated in Fig. 5a. Assume that ends A and E of the frame are fixed. The dimensions of each member of the frame are given in Fig. 5a.

The properties of the analogous column of Fig. 5b are (assume EI = 1):

$$A = 2(1/2 \times 36) + 2(30) = 96$$

$$\overline{Y} = \frac{36(10) + 60(35)}{96} = 25.6'$$

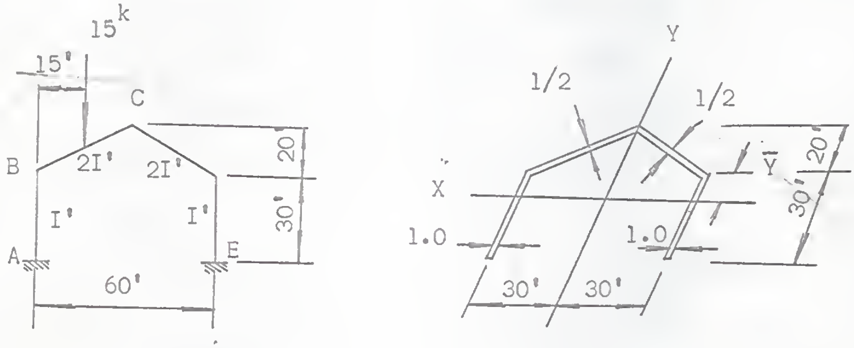
$$I_x = 2 \quad \frac{1}{12}(1)(30)^3 + 30(1)(9.4)^2 \; +$$

$$2\;\frac{1}{12}(1/2)(36)^2(\frac{20}{36}) + 18(15.6)^2 \; = 19.78$$

$$I_y = 2(30)(30)^2 + 2\;\frac{1}{12}(1/2)(36)^2(\frac{30}{36}) + 18(15)^2$$

$$= 54{,}000 + 2(1350 + 4050)$$

$$= 64{,}800.00$$



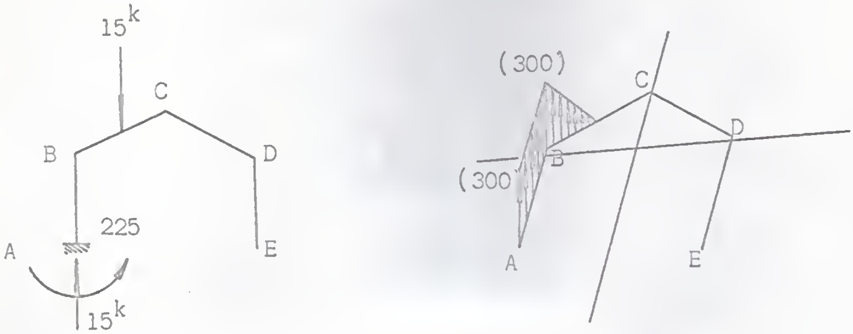
(a) Loaded Gable Frame

(b) Analogous-column section (considering EI = 1)

Fig. 5

Cut the right support, E, of the frame in order to make the whole frame a determinate structure as shown in Fig. 6a. The determinate moment diagram, which is produced by the external load, is treated as a load acting on the top of the analogous column as shown in Fig. 6b.

The total load on the top of the analogous column, and the moments about the axes of the cross section of the analogous column are calculated to be:



(a) Basic determinate structure

(b) Analogous column with M_d diagram as a load

Fig. 6

$$P = 1/2(18)(150) + 30(300)$$

$$= 10350.00$$

$$M_x = 30(300)(9.4) - 1/2(18)(150)(8.9)$$

$$= 84,600 - 12,000$$

$$= 72,600.00 \qquad \text{(clockwise)}$$

$$M_y = 30(300)(30) + 1/2(18)(150)(25)$$

$$= 270,000 + 33,750$$

$$= 303,750.00 \qquad \text{(clockwise)}$$

The calculations and the results are shown in Table 1.

Loaded Double Gable Frame

The following double bay gable frame, shown in Fig. 7, is analyzed by both the column analogy method and the moment distribution method.

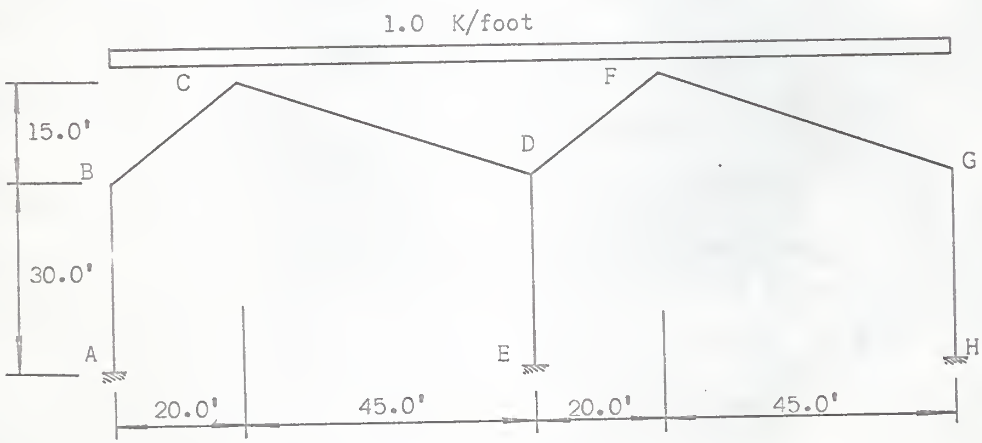


Fig. 7. Double bay gable frame

TABLE 1.- THE MOMENT AT EACH JOINT OF THE GABLE FRAME

Joint	M_d	$\frac{P}{A}$	$\frac{MxY}{Ix}$	$\frac{MyX}{Iy}$	Mi	M
A	-300	$-\frac{10350}{96} = -107.8$	$\frac{72,600}{19,780}(-24.4) = -90$	$\frac{303750}{64800}(-30) = -140.5$	-107.8 - 90 - 140.5 = 338.3	-300 - (-338.3) = +38.3
B	-300	-107.8	$\frac{72,600}{19,780}(5.6) = +20.4$	$\frac{303750}{64800}(-30) = -140.5$	-107.8 + 20.4 -140.5 = -227.9	-300 - (-227.9) = -72.1
C	0	-107.8	$\frac{72,600}{19,780}(25.6) = +94.1$	$\frac{303750}{64800}(0) = 0$	-107.8 + 94.1 + 0 = -13.7	0 - (-13.7) = 13.7
D	0	-107.8	$\frac{72,600}{19,780}(5.6) = +20.4$	$\frac{303750}{64800}(+30) = +140.5$	-107.8 + 20.4 +140.5 = +53.1	0 - (53.1) = -53.1
E	0	-107.8	$\frac{72,600}{19,780}(-24.4) = -90$	$\frac{303750}{64800}(+30) = +140.5$	-107.8 - 90 + 140.5 = -57.3	0 - (-57.3) = +57.3

Solution by the Column Analogy Method

I. Properties of the analogous column section. Take a portion ABCD of the frame as a free body with its two ends fixed as shown in Fig. 8(a). Solve this part of the frame as an independent structure by the method of column analogy. The length of the analogous column section is the same as that of the actual frame. The width of the analogous column section is a constant 1/EI, as shown in Fig. 8(b). It will be convenient to let EI = 1, so that the width at every point of the analogous column is one.

$$\text{Area of column section} = (30)(1.0) + (25)(1.0) + (47.5)(1.0)$$

$$= 102.5$$

$$\bar{Y} = \frac{(30)(15) + (25)(37.5) + (47.5)(37.5)}{102.5}$$

$$= \frac{3168.5}{102.5} = 31 \text{ feet}$$

$$X = \frac{(25)(10) + (47.5)(42.5)}{102.5}$$

$$= \frac{2268.7}{102.5} = 22.1 \text{ feet (from left)}$$

$$I_x = \frac{1}{12}(30)^{\overset{3}{\cancel{2}}} + (30)(16)^2 + \frac{1}{12}(25)^3\left(\frac{15}{25}\right)^2 + (25)(6.5)^2$$

$$+ \frac{1}{12}(47.5)^3\left(\frac{15}{47.5}\right)^2 + (47.5)(6.5)^2$$

$$= (2250 + 7680) + (469 + 1056) + (890 + 2006)$$

$$= 14{,}351.0$$

$$I_y = \frac{1}{12}(30)(1.0)^3 + (30)(22.33)^2$$

$$+ \frac{1}{12}(25)^3(\frac{20}{25})^2 + (25)(12.33)^2$$

$$+ \frac{1}{12}(47.5)^3(\frac{45}{47.5})^2 + (47.5)(20.4)^2$$

$$= 46{,}932.6$$

$$I_{xy} = (30)(-22.1)(-16.0) + (25)(-12.1)(6.5)$$

$$+ (47.5)(20.4)(6.5)$$

$$= 10{,}608.0 - 1966.3 + 6298.5$$

$$= 14{,}940.2$$

$$I_x' = I_x(1 - \frac{I_{xy}^2}{I_x I_y})$$

$$= 14351.0(1 - \frac{(14940.0)^2}{(14351.0)(46932.6)})$$

$$= 14351.0(1 - 0.333)$$

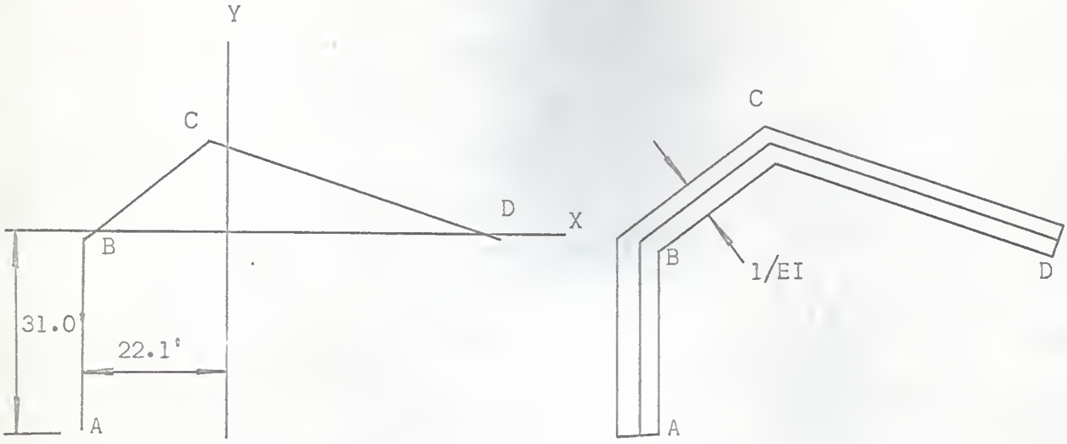
$$= 14351.0(0.667)$$

$$= 9572.1$$

$$I_y' = I_y (1 - \frac{I_{xy}^2}{I_x \cdot I_y})$$

$$= 46932.6(0.667)$$

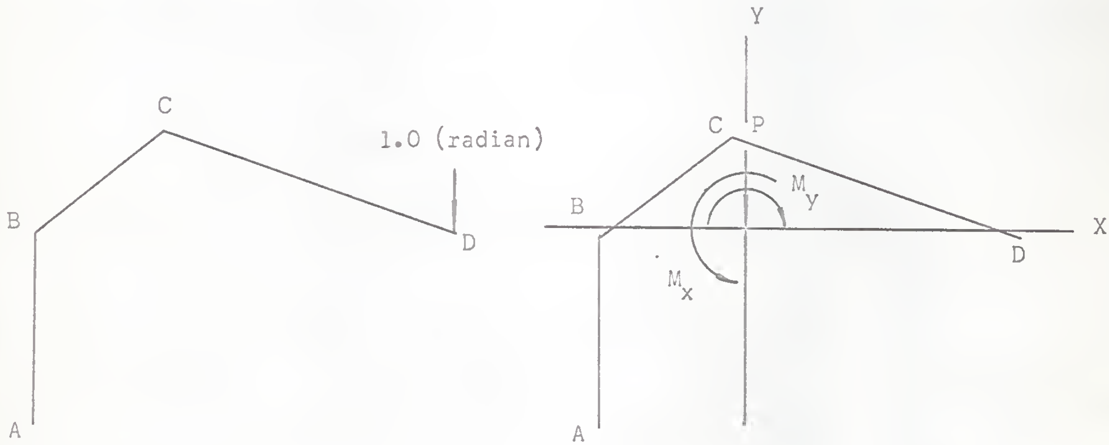
$$= 31304.0$$



(a) Free-body

(b) Analogous column section

Fig. 8



(a) Load on the analogous column

(b) Load after transfer to the centroid of the analogous column section

Fig. 9

II. Stiffness and carry-over factors. A load of 1 radian is applied at the right end, D, of the analogous column section and transferred to the centroid of the section as a concentrated load and two moments as indicated in Fig. 9(a) and (b), from which the following relationships are obtained.

$$M_x = (1.0)(-1.0) = -1.0$$

$$M_y = (1.0)(42.9) = 42.9$$

$$M_x' = M_x - M_y\left(\frac{I_{xy}}{I_y}\right)$$

$$= -(1.0) - (42.9)\left(\frac{14940.0}{46932.6}\right)$$

$$= -16.9$$

$$M_y' = M_y - M_x\left(\frac{I_{xy}}{I_x}\right)$$

$$= 42.9 - (-1.0)\left(\frac{14940.0}{14351.0}\right)$$

$$= 42.9 + 1.04$$

$$= 44.0$$

The joint moments are tabulated in Table 2.

The stiffness factor at D and the different carry-over factors are obtained as follows:

The stiffness factor at D is $S_D = -0.072$.

The carry-over factors are:

(1) From D to C.

$$C_{DC} = \frac{0.017}{0.072} = 0.236$$

TABLE 2.- THE MOMENT AT EACH JOINT OF ABCD DUE TO THE 1 RADIAN LOAD ACTING AT D

Joint	M_d	$\frac{P}{A}$	$\frac{My'x}{Iy'}$	$\frac{Mx'y}{Ix'}$	M
D	0	$\frac{1}{102.5}$ $= 0.00967$	$\frac{(44.0)(42.9)}{31304.0}$ $= 0.06$	$\frac{(-16.9)(-1.0)}{9572.1}$ $= 0.0018$	-0.072
C	0	0.00967	$\frac{(44.0)(-2.1)}{31304.0}$ $= -0.003$	$\frac{(-16.9)(14.0)}{9572.1}$ $= -0.0247$	+0.017
B	0	0.00967	$\frac{(44.0)(-22.1)}{31304.0}$ $= -0.031$	$\frac{(-16.9)(-1.0)}{9572.1}$ $= +0.0018$	+0.020
A	0	0.00967	$\frac{(44.0)(-22.1)}{31304.0}$ $= -0.031$	$\frac{(-16.9)(-31.0)}{9572.1}$ $= 0.059$	+0.038

(2) From D to B.

$$C_{DB} = \frac{0.020}{0.072} = 0.278$$

(3) From D to A.

$$C_{DA} = \frac{0.038}{0.072} = 0.527$$

III. Moments at each joint due to the applied loading. Cut the left end A of the free-body ABCD, as a redundant and make the whole free-body a determinate structure, as indicated in Fig. 10(a). The determinate moment diagram of the free-body is plotted on the compression side as shown in Fig. 10(b). Use this diagram as a load acting on the top of the analogous column.

$$P = \frac{1}{3}(25.0)(200.0) + (47.5)(200.0) - \frac{2}{3}(45.0)(252.13)$$

$$+ \frac{1}{2}(47.5)(2112.5) - 200.0)$$

$$= 49051.0$$

$$M_y = (1666.67)(7.1) - (9500.0 - 7563.75)(20.4)$$

$$- (45448.0)(45 - 15 - 2.1)$$

$$= -154262.5 \qquad \text{(counterclockwise)}$$

$$M_x = -(16666.67)(10.25) - (9500.0 - 7563.75)(6.5)$$

$$- (45448.0)(4.0)$$

$$= -211461.0 \qquad \text{(counterclockwise)}$$

$$M_x' = M_x - M_y\left(\frac{I_{xy}}{I_y}\right)$$

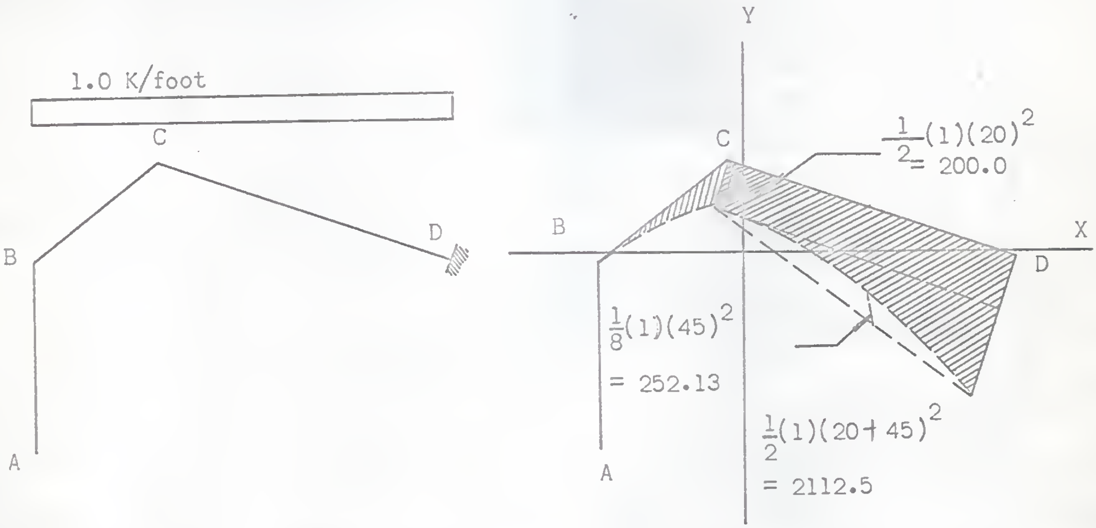
$$= -211461.0 - (-154262.5)\left(\frac{14940.0}{46932.6}\right)$$

$$M_x' = -162094.0$$

$$M_y' = M_y - M_x\left(\frac{I_{xy}}{I_x}\right)$$

$$= -154262.5 - (-211461.0)\left(\frac{14940.0}{14351.0}\right)$$

$$= 65657.0$$



(a) Loaded determinate frame

(b) M_d diagram, plotted on the compression side of the frame

Fig. 10

TABLE 3.- THE MOMENT AT EACH JOINT OF ABCD DUE TO THE APPLIED LOADS

Joint	M_d	$\frac{P}{A}$	$\frac{My'x}{Iy'}$	$\frac{Mx'y}{Ix'}$	M
D	-2112.5	$\frac{49051.0}{102.5}$ = 478.55	$\frac{(65657.0)(42.9)}{31304.0}$ = +90.1	$\frac{(-162,094.0)(-1.0)}{9572.1}$ = +16.93	-2698.08
C	-200.0	478.55	$\frac{(65657.0)(-2.1)}{31304.0}$ = -4.41	$\frac{(-162,094.0)(14)}{9572.1}$ = -237.02	-437.12
B	0	478.55	$\frac{(65657.0)(-22.1)}{31304.0}$ = -46.41	$\frac{(-162,094.0)(-1.0)}{9572.1}$ = +16.93	-449.10
A	0	478.55	$\frac{(65657.0)(-22.1)}{31304.0}$ = -46.41	$\frac{(-162,094.0)(-31.0)}{9572.1}$ = 524.83	-~~0~~9 56.97

By the same technique which has been used previously, take the right side of the frame, DFGH, as a free-body as shown in Fig. 11(a). Solve this part of the frame as an independent structure by the column analogy method. The length of the analogous column section is the same as that of the actual frame. The width of the analogous column section is a constant 1/EI, as shown in Fig. 11(b). It will be convenient to let EI = 1, so that the width at every section of the analogous column section is 1.

Area of the analogous column section = (30)(1.0) + (47.5)(1.0)

$$+ (25)(1.0) = 102.5$$

$$x = \frac{(25)(55.0) + (47.5)(22.5)}{102.5} = 23.84 \qquad \text{(from right)}$$

$$y = \frac{(30)(15.0) + (25)(37.5) + (47.5)(37.5)}{102.5}$$

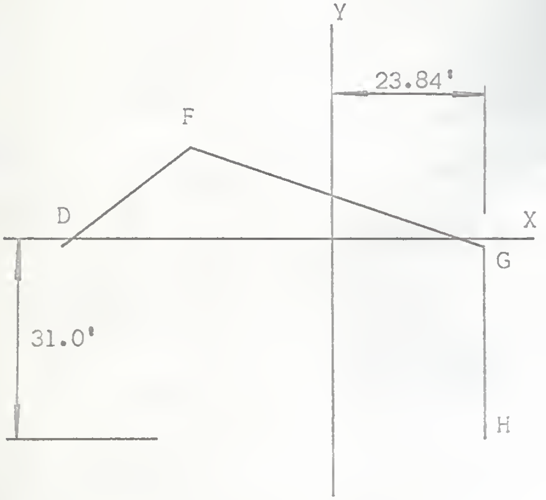
$$= 31.0 \qquad \text{(from the bottom)}$$

$$I_x = \frac{1}{12}(1.0)(30.0)^3 + (30.0)(16.0)^2$$

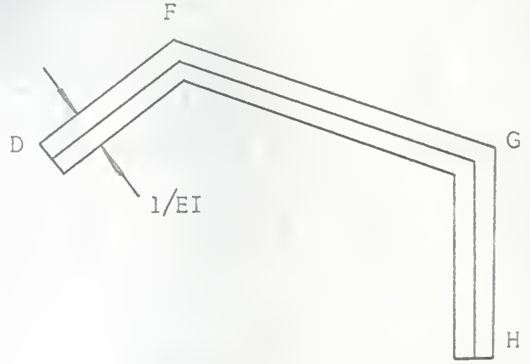
$$+ \frac{1}{12}(1.0)(47.5)^3\left(\frac{15}{47.5}\right)^2 + (47.5)(6.5)^2$$

$$+ \frac{1}{12}(25.0)^3\left(\frac{15}{25}\right)^2 + (25.0)(6.5)^2$$

$$= 14341.0$$

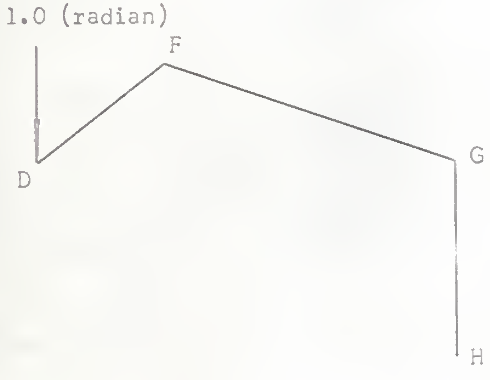


(a) Free-body

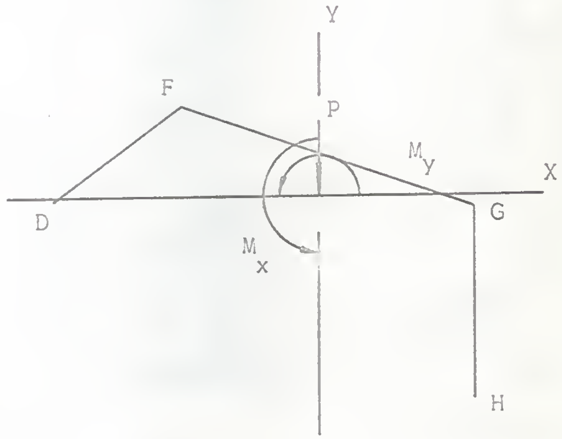


(b) Analogous column section

Fig. 11



(a) Load on the analogous column



(b) Load after transfer to the centroid of the analogous column section

Fig. 12

$$I_y = \frac{1}{12}(30.0)(1.0)^3 + (30.0)(23.84)^2$$

$$+ \frac{1}{12}(47.5)^3\left(\frac{45}{47.5}\right)^2 + (47.5)(1.34)^2$$

$$+ \frac{1}{12}(25.0)^3\left(\frac{20}{25}\right)^2 + (25.0)(31.16)^2$$

$$= 50274.4$$

$$I_{xy} = (25)(1.0)(-31.16)(6.5) + (47.5)(1.0)(1.34)(6.5)$$

$$+ (30)(1.0)(23.84)(-16.0)$$

$$= -16140.6$$

$$I_x' = I_x\left(1 - \frac{I_{xy}^2}{I_x I_y}\right)$$

$$= 14351.0 \quad 1 - \frac{(16140.6)^2}{(14351.0)(50274.4)}$$

$$= 9170.3$$

$$I_y' = I_y\left(1 - \frac{I_{xy}^2}{I_x I_y}\right)$$

$$= 50274.4 \quad 1 - \frac{(16140.6)^2}{(14351.0)(50274.4)}$$

$$= 32125.3$$

By the same technique and procedure which have been used previously, a load of 1 radian is applied at the left end, D, of the analogous column section and transferred to the centroid of the section as a concentrated load and two moments as denoted in Fig. 12(a) and (b). The moments at each point and stiffness factor at D, and carry-over factors from D to different points are obtained as follow:

$$M_x = (1.0)(-1.0) = -1.0$$

$$M_y = (1.0)(-41.17) = -41.17$$

$$M_x' = M_x - M_y\left(\frac{I_{xy}}{I_y}\right)$$

$$= -(1.0) - (41.17)\ \frac{-(16140.6)}{50274.4}$$

$$= -14.20$$

$$M_y' = M_y - M_x\left(\frac{I_{xy}}{I_x}\right)$$

$$= -(41.17) - (-1.0)\left(\frac{-16140.6}{14351.0}\right)$$

$$= -42.30$$

The stiffness factor at D, $S_D = -0.065$

The carry-over factors are:

(1) From D to F.

$$C_{DF} = \frac{0.0159}{0.0650} = 0.244$$

(2) From D to G.

$$C_{DG} = \frac{0.0202}{0.0650} = 0.311$$

(3) From D to H.

$$C_{DH} = \frac{0.0263}{0.0650} = 0.405$$

TABLE 4.- THE MOMENT AT EACH JOINT OF DFGH DUE TO 1 RADIAN LOAD ACTING AT D

Joint	M_d	$\frac{P}{A}$	$\frac{My'x}{Iy'}$	$\frac{Mx'y}{Ix'}$	M
D	0	$\frac{1}{102.5}$ = 0.00967	$\frac{(-42.30)(-41.16)}{32125.3}$ = 0.054	$\frac{(-14.20)(-1.0)}{9170.0}$ = 0.0016	-0.065
F	0	0.00967	$\frac{(-42.30)(-21.16)}{32125.3}$ = 0.0279	$\frac{(-14.20)(14.0)}{9170.0}$ = -0.0217	-0.0159
G	0	0.00967	$\frac{(-42.30)(23.84)}{32125.3}$ = -0.0314	$\frac{(-14.20)(-1.0)}{9170.0}$ = 0.0016	+0.0202
H	0	0.00967	$\frac{(-42.30)(23.84)}{32125.3}$ = -0.0314	$\frac{(-14.20)(-31.0)}{9170.0}$ = 0.048	-0.0263

IV. Moments at each joint of the free-body DFGH, due to the applied load. Cut the right end H of the free-body DFGH, and make the whole free-body a determinate structure, as indicated in Fig. 13(a). The determinate moment diagram of the free-body is plotted on the compression side as shown in Fig. 13(b). Use this diagram as a load acting on the top of the analogous column.

$$P = \frac{1}{3}(45.0)(1012.5) + (1012.5)(25) - \frac{2}{3}(20)(50)$$

$$+ \frac{1}{2}(25.0)(2112.5 - 1012.5)$$

$$= 53583.33 \qquad \text{(upward)}$$

$$M_x = -(15187.5)(11.5) - (25312.5 - 666.67)(6.5) - (13750)(4.0)$$

$$= -389845.0 \qquad \text{(counterclockwise)}$$

$$M_y = (15187.5)(9.92) + (25312.5 - 666.67)(31.16)$$

$$+ (13750.0)(34.5)$$

$$= 1,257,404.0 \qquad \text{(clockwise)}$$

$$M_x' = M_x - M_y\left(\frac{I_{xy}}{I_y}\right)$$

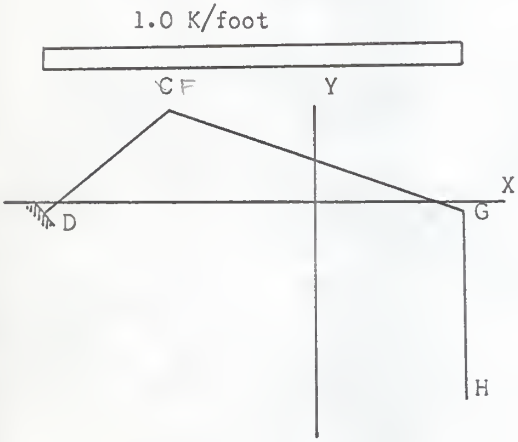
$$M_x' = -(389854.0) - (1257404.0)\left(\frac{-16140.6}{50274.4}\right)$$

$$= -27185.6$$

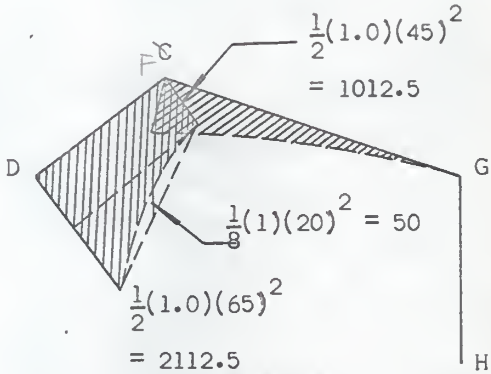
$$M_y' = M_y - M_x\left(\frac{I_{xy}}{I_x}\right)$$

$$= 1257404.0 - (-389854.0)\left(\frac{-16140.6}{14351.0}\right)$$

$$= 818818.2$$



(a) Loaded frame



(b) M_d diagram, plotted on the compression side

Fig. 13

The calculations and the resulting moments at each joint of the free-body due to the applied load are shown in Table 5.

TABLE 5.- THE MOMENT AT EACH JOINT OF DFGH DUE TO THE APPLIED LOADS

Joint	M_d	$\frac{P}{A}$	$\frac{M_{y'} x}{I_{y'}}$	$\frac{M_{x'} y}{I_{x'}}$	M
D	-2112.5	$\frac{53583.33}{102.5}$ = 522.76	$\frac{(818818.2)(-41.16)}{32125.3}$ = -1050.0	$\frac{(-27185.6)(-1.0)}{9170.0}$ = 2.96	-1588.20
F	-1012.5	522.76	$\frac{(818818.2)(-21.16)}{32125.3}$ = -539.33	$\frac{(-27185.6)(14.0)}{9170.0}$ = -41.4	-1037.33
G	0	522.76	$\frac{(818818.2)(23.84)}{32125.3}$ = 607.83	$\frac{(-27185.6)(-1.0)}{9170.0}$ = 2.96	-1133.35
H	0	522.76	$\frac{(818818.2)(23.84)}{32125.3}$ = 607.83	$\frac{(-27185.6)(-31.0)}{9170.0}$ = +91.6	-1222.0

V. Moments at each joint due to sidesway at point D.

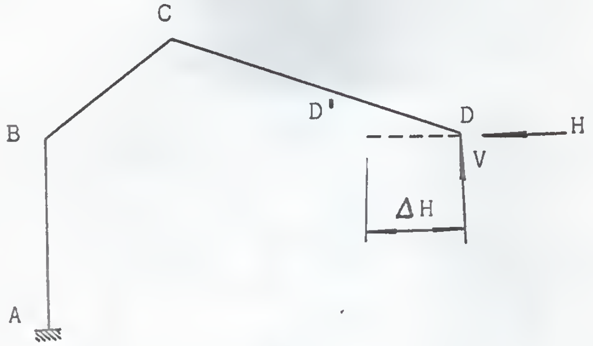


Fig. 14. Sidesway at D

$$V\int x^2 \frac{ds}{EI} + H\int xy \frac{ds}{EI} + M\int x \frac{ds}{EI} = 0$$

$$V\int xy \frac{ds}{EI} + H\int y^2\left(\frac{ds}{EI}\right) + M\int y \frac{ds}{EI} = \Delta H$$

$$V\int x \frac{ds}{EI} + H\int y \frac{ds}{EI} + M\int \frac{ds}{EI} = 0$$

Since $\int xda = 0$, $\int yda = 0$, then

$$V\int x^2 \frac{ds}{EI} + H\int xy \frac{ds}{EI} = 0$$

$$V\int xy \frac{ds}{EI} + H\int y^2 \frac{ds}{EI} = \Delta H$$

or,

$$V(I_y) + H(I_{xy}) = 0$$

$$V(I_{xy}) + H(I_x) = \Delta H$$

Solve for V and H.

$$V = \frac{-\Delta H}{I_y'}\left(\frac{I_{xy}}{I_x}\right)$$

$$H = \frac{\Delta H}{I_x'}$$

$$M = M_d - M_i$$

$$= M_d - (Vx + Hy) \qquad (6)$$

The calculations and the resulting moments at each joint of the free-body due to sidesway at D, are indicated in Table 6.

By the same technique which has been used above, when sidesway exists at the joint D of the right side free-body DFGH, the following relationships and the moments at each joint of the free-body due to this sidesway can be obtained. (7)

$$VI_y + HI_{xy} = 0$$

$$VI_{xy} + HI_x = -(\Delta H)$$

Solve for V and H from these two equations above, then

TABLE 6.- THE MOMENT AT EACH JOINT OF ABCD DUE TO SIDESWAY AT D

Joint	M_d	$V(x) = \frac{-\Delta H}{Iy'}\left(\frac{Ixy}{Ix}\right)(x)$	$H(Y) = \frac{\Delta H}{Ix'}(Y)$	M
D	0	$\left(\frac{-\Delta H}{31304}\right)\left(\frac{14940.2}{14351}\right)(42.9)$ $= -10.0$	$\left(\frac{\Delta H}{9572.1}\right)(-1.0)$ $= -123.4$	+152.4
C	0	$\left(\frac{-\Delta H}{31304}\right)\left(\frac{14940.2}{14351}\right)(-2.1)$ $= +6.0$	$\left(\frac{\Delta H}{9572.1}\right)(14.0)$ $= 145.6$	-155.1
B	0	$\left(\frac{-\Delta H}{31304}\right)\left(\frac{14940.2}{14351}\right)(-22.1)$ $= 73.15$	$\left(\frac{\Delta H}{9572.1}\right)(-1.0)$ $= 123.4$	-196.39
A	0	$\left(\frac{-\Delta H}{31304}\right)\left(\frac{14940.2}{14351}\right)(-22.1)$ $= 73.15$	$\left(\frac{\Delta H}{9572.1}\right)(-31.0)$ $= -322.4$	-249.25

$$V = \frac{H}{I_y'} \left(\frac{I_{xy}}{I_x} \right)$$

$$H = \frac{-\Delta H}{I_x'}$$

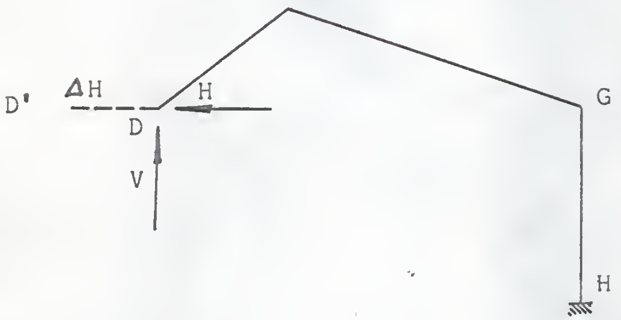


Fig. 15. Sidesway at D

$$M = M_d - M_i$$

$$= M_d - (V \cdot x + H \cdot y)$$

The calculations and the resulting moments at each joint of the free-body due to sidesway at D, are obtained as shown in Table 7.

The fixed-end moment of bar DE, due to sidesway at D, is obtained as follows:

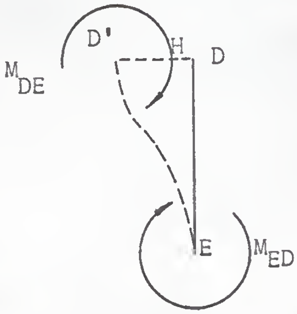


Fig. 16. Fixed-end moment of bar DE, due to sidesway at joint D

TABLE 7.- THE MOMENT AT EACH JOINT OF DFGH DUE TO SIDESWAY AT D

Joint	M_d	$V(X) = \frac{\Delta H}{I_y}\left(\frac{I_{xy}}{I_x}\right)(X)$	$H(Y) = \frac{-\Delta H}{9170}(Y)$	M
D	0	$\left(\frac{\Delta H}{32125}\right)\left(\frac{-16140.6}{14351.0}\right)(-41.16)$ $= -529.16$	$\left(\frac{-\Delta H}{9170.0}\right)(-1.0)$ $= 253.83$	+275.33
F	0	$\left(\frac{\Delta H}{32125}\right)\left(\frac{-16140.6}{14351.0}\right)(-21.16)$ $= +266.72$	$\left(\frac{-\Delta H}{9170.0}\right)(14.0)$ $= +152.6$	-419.32
G	0	$\left(\frac{\Delta H}{32125}\right)\left(\frac{-16140.6}{14351.0}\right)(-23.84)$ $= 130.02$	$\left(\frac{-\Delta H}{9170.0}\right)(-1.0)$ $= +253.83$	-383.85
H	0	$\left(\frac{\Delta H}{32125}\right)\left(\frac{-16140.6}{14351.0}\right)(-23.84)$ $= +130.02$	$\left(\frac{-\Delta H}{9170.0}\right)(-31.0)$ $= +204.66$	-334.68

$$M_{DE} = M_{ED} = -\frac{6EI\,(\Delta H)}{30^2}$$

$$= 207.27$$

Distributions of fixed-end moment at joint D, due to the applied load and sidesway at D are indicated in Tables 8 and 9. The ratio of shear stresses, due to the applied load and the sidesway at D, is

$$k = -\frac{\frac{1}{30}(1113.14 - 531.48 + 547.17 + 273.59 + 1216.9 - 1330.08)}{\frac{1}{30}(338.57 - 149.27 + 197.37 + 50.74 + 336.45 - 396.40)}$$

$$= -3.51$$

M = Moment, due to the applied load + (-3.51) (Moment, due to sidesway at D)

$M_{AB} = 1113.14 - (3.51)(338.57) = -7.50$

$M_{BA} = -M_{BC} = -531.48 - (3.51)(-149.28) = -7.47$

$M_{CB} = -M_{CD} = -507.06 - (3.51)(-107.1) = -131.08$

$M_{DC} = -2401.74 - (3.51)(-649.61) = -121.56$

$M_{DE} = 547.17 - (3.51)(107.37) = 170.00$

$M_{DF} = 1854.57 - (3.51)(542.27) = -48.8$

$M_{FD} = M_{FG} = -1102.33 - (3.51)(-382.12) = -238.16$

$M_{GF} = M_{GH} = -1050.51 - (3.51)(-289.82) = -33.20$

$M_{HG} = -1330.08 - (3.51)(-396.4) = -61.32$

$M_{ED} = 273.59 - (3.51)(50.74) = 95.49$

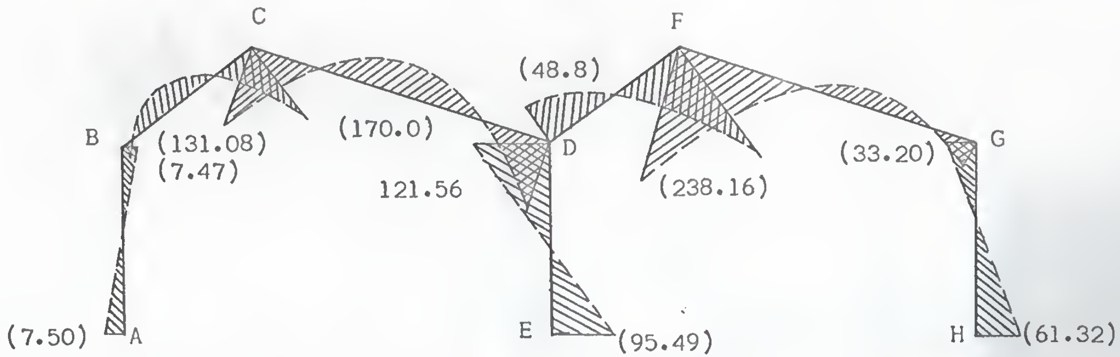


Fig. 17. The resulting moment diagram, obtained by the column analogy method, plotted on the compression side

TABLE 8 DISTRIBUTION OF F.E.M. AT D. DUE TO THE APPLIED LOAD

JOINTS	A	B		C		D			F		G		H	E
MEMBER	AB	BA	BC	CB	CD	DC	DE	DF	FD	FG	GF	GH	HG	ED
S.F.						0.072	0.133	0.065						
C.O.F.	- 0.527	- 0.278	+0.278	-0.236	+0.236				- 0.244	+0.244	- 0.311	+ 0.311	-0.405	0.5
D.F.						0.267	0.493	0.240						
F.E.M.						-2698.08		+1588.20						
BAL.						+296.34	+ 547.17	+266.37						
C.O.	+ 156.17	- 82.38	+ 82.38	- 69.94	+ 69.94				- 65.00	+ 65.00	- 82.84	+ 82.84	- 107.88	+273.59
MOMENT OBTAINED BY C.A. METHOD	+ 956.97	- 449.10	+ 449.10	- 437.12	+437.12				-1037.33	+1037.33	-1133.35	+1133.35	-1222.2	
TOTAL MOMENT	+ 1113.14	- 531.48	+ 531.48	-507.06	+ 507.06	-2401.74	+ 547.17	+1854.57	- 1102.33	+ 1102.33	- 1216.19	+ 1216.19	-1330.08	+273.59

TABLE 9 DISTRIBUTION OF F.E.M. AT D. DUE TO SIDESWAY AT D.

JOINTS	A	B		C		D			F		G		H	E
MEMBER	AB	BA	BC	CB	CD	DC	DE	DF	FD	FG	GF	GH	HG	ED
S.F.						0.072	0.133	0.065						
C.O.F.	- 0.527	- 0.278	+ 0.278	- 0.236	+ 0.236				- 0.244	+0.244	-0.311	+0.311	-0.405	0.5
D.F.						0.267	0.493	0.240						
F.E.M.						+ 152.40	+207.27	+ 275.33						+ 207.27
BAL.						- 169.50	- 313.06	- 152.4						
C.O.	+ 89.32	+ 47.12	- 47.12	+ 40.0	- 40.0				+ 37.20	- 37.20	+ 47.4	- 47.4	- 61.72	+156.53
MOMENT OBTAINED BY C.A. METHOD	+ 249.25	- 196.39	+ 196.39	- 155.10	+ 155.10				- 419.32	+ 419.32	- 383.85	+383.85	-334.68	
TOTAL MOMENT	+338.57	- 149.27	+149.27	- 107.10	+ 107.10	- 649.61	+ 107.37	+ 542.27	- 382.12	+ 382.12	- 336.45	+ 336.45	- 396.40	+ 50.74

Solution by the Moment Distribution Method

The distribution of fixed-end moment due to the applied load is indicated in Table 13. The fixed-end moment at each joint due to sidesway at B, D, and G respectively are obtained as follows:

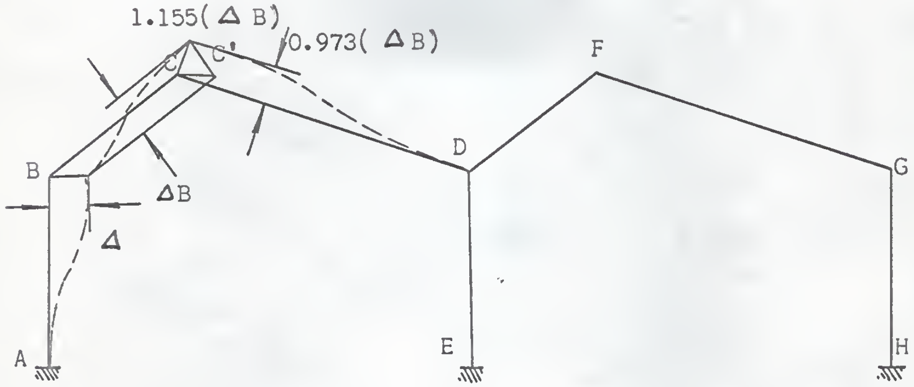


Fig. 18. Sidesway at B

TABLE 10.- FIXED-END MOMENT DUE TO SIDESWAY AT B

AB= 10

$M_{AB} = M_{BA}$	=	$\frac{6EI(\Delta B)}{30^2}$	=	66.67
$M_{BC} = M_{CB}$	=	$-\frac{6EI(1.555\ \Delta B)}{25^2}$	=	-110.88
$M_{CD} = M_{DC}$	=	$\frac{6EI(0.973\ \Delta B}{(47.5)^2}$	=	26.0

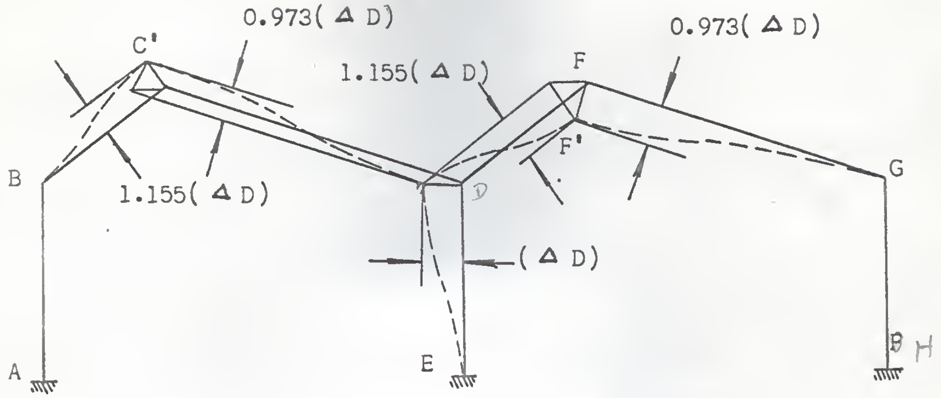


Fig. 19. Sidesway at D

TABLE 11.- FIXED-END MOMENT DUE TO SIDESWAY AT D

$M_{BC} = M_{CB}$	=	$-\dfrac{6EI(1.155\,\Delta D)}{25^2}$	=	-110.88
$M_{CD} = M_{DC}$	=	$\dfrac{6EI(0.973\,\Delta D)}{(47.5)^2}$	=	26.0
$M_{DE} = M_{ED}$	=	$\dfrac{6EI(\Delta D)}{30^2}$	=	-66.67
$M_{DF} = M_{FD}$	=	$\dfrac{6EI(1.155\,\Delta D)}{25^2}$	=	110.88
$M_{FG} = M_{GF}$	=	$-\dfrac{6EI(0.973\,\Delta D)}{(47.5)^2}$	=	-26.0

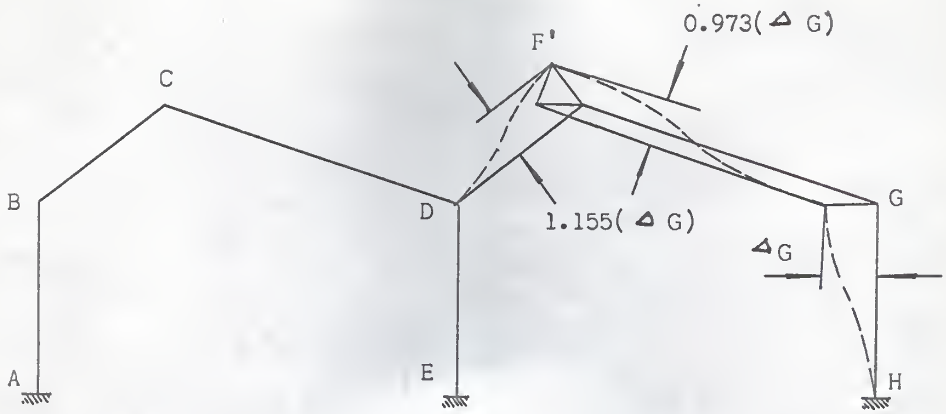


Fig. 20. Sidesway at G

TABLE 12.- FIXED-END MOMENT DUE TO SIDESWAY AT G

ΔG = 40000

$M_{DF} = M_{FD}$	=	$\dfrac{-\ 6EI(1.155\ \Delta G)}{25^2}$	=	-110.88
$M_{FG} = M_{GF}$	=	$\dfrac{6EI(0.973\ \Delta G)}{(47.5)^2}$	=	26.0
$M_{GH} = M_{HG}$	=	$\dfrac{-\ 6EI(\Delta G)}{30^2}$	=	-66.67

The distributions of the fixed-end moments due to the sidesway at point B, D, and G respectively are indicated in Tables 14, 15 and 16.

TABLE 13 DISTRIBUTION OF F.E.M. DUE TO APPIED LOAD

JOINTS	A	B		C		D			F		G		H	E
MEMBER	AB	BA	BC	CB	CD	DC	DE	DF	FD	FG	GF	GH	HG	ED
S.F.		1/30	1/25	1/25	1/47.45	1/47.45	1/30	1/25	1/25	1/47.45	1/47.45	1/30		
C.O.F.		0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5		
D.F.		0.455	0.545	0.656	0.344	0.22	0.354	0.426	0.656	0.344	0.389	0.611		
F.E.M.			+ 33.33	− 33.33	+168.75	−168.75		+ 33.33	− 33.33	+ 168.75	−168.75			
BAL.		− 15.15	−18.18	− 88.82	− 46.60	+ 29.79	+ 47.90	+ 57.63	− 88.82	− 46.60	+ 65.75	+ 103.0		
CO.	− 7.58		− 44.41	− 9.09	+14.90	− 23.30		− 44.41	+28.82	+ 32.88	− 23.30		+ 51.5	+ 23.95
BAL.		+ 20.21	+ 24.20	− 3.81	− 2.00	+ 14.90	+ 24.0	+ 28.81	− 40.5	− 21.20	+ 9.05	+ 1425		
CO.	+ 10.11		− 1.91	+ 12.10	+ 7.45	− 1.00		− 20.25	+ 14.41	+ 4.53	− 10.60		+ 7.13	+ 12.0
BAL.		+ 0.87	+ 1.04	− 12.83	− 6.72	+ 4.68	+ 7.52	+ 9.05	− 12.43	− 6.51	+ 4.12	+ 6.48		
CO.	+ 0.44		− 6.42	+ 0.52	+ 2.34	− 3.36		− 6.22	+ 4.53	+ 2.06	− 3.26		+ 3.24	+ 3.80
BAL.		+ 2.92	+ 3.50	− 1.88	− 0.98	+ 2.11	+ 3.39	+ 4.08	− 4.33	− 2.26	+ 1.27	+ 1.99		
CO.	+ 1.46		− 0.94	+ 1.75	+ 1.06	− 0.49		− 2.17	+ 2,04	+ 0.64	− 1.18		+ 0.99	+ 1.70
BAL.		+ 0.43	+ 0.51	− 1.84	− 0.97	+ 0.59	+ 0.94	+ 1.13	− 1.76	− 0.92	+ 0.46	+ 0.72		
CO.	+ 0.22		− 0.92	+ 0.26	+ 0.30	− 0.49		− 0.88	+ 0.57	+ 0.23	− 0.46		+ 0.36	+ 0.47
BAL.		+ 0.42	+ 0.50	− 0.37	− 0.19	+ 0.30	+ 0.49	+ 0.58	− 0.52	− 0.28	+ 0.18	+ 0.28		
TOTAL MOMENT	+ 4.65	+ 9.70	− 9.70	− 137.34	+ 137.34	− 145.02	+ 84.24	+ 60.68	− 131.32	+ 131.32	+ 126.72	+ 126.63	+ 63.22	+ 41.92

TABLE 14 DISTRIBUTION OF F.E.M. DUE TO SIDESWAY AT B (Δ)

JOINTS	A	B		C		D			F		G		H	E
MEMBER	AB	BA	BC	CB	CD	DC	DE	DF	FD	FG	GF	GH	HG	ED
		1/30	1/25	1/25	1/47.5	1/47.5	1/30	1/25	1/25	1/47.5	1/47.5	1/30		
C.O.	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
D.F.		0.455	0.545	0.656	0.344	0.22	0.354	0.426	0.656	0.344	0.389	0.611		
F.E.M.	+ 66.67	+ 66.67	− 110.88	− 110.88	+ 26.0	+ 26.0								
BAL.		+ 20.12	+ 24.09	+ 55.73	+ 29.15	− 5.72	− 9.20	− 11.08						
C.O.	− 6.35		− 3.02	− 7.59	− 1.61	− 1.58		+ 1.82	− 3.11		+ 0.96			− 2.58
BAL.		+ 1.37	+ 1.65	+ 6.03	+ 3.16	− 0.053	− 0.085	− 0.102	+ 2.04	+ 1.07	− 0.40	− 0.56		
C.O.	+ 10.06		+ 27.87	+ 12.05	− 2.86	+ 14.58			− 5.54					− 4.60
BAL.		− 12.69	− 15.18	− 6.03	− 3.16	− 3.21	− 5.15	− 6.22	+ 3.63	+ 1.91				
C.O.	+ 0.69		+ 3.02	+ 0.83	− 0.03	+ 1.58		+ 1.02	− 0.051	− 0.20	− 0.54		− 0.28	− 0.043
BAL.		− 1.37	− 1.65	− 0.52	− 0.28	− 0.57	− 0.92	− 1.11	+ 0.165	+ 0.086	− 0.21	− 0.33		
C.O.	− 0.69		− 0.26	− 0.83	− 0.29	− 0.14		− 0.083	− 0.56	− 0.11	+ 0.043		− 0.17	− 0.46
BAL.		+ 0.12	+ 0.14	+ 0.73	+ 0.39	+ 0.013	+ 0.02	+ 0.024	+ 0.44	+ 0.23	− 0.013	− 0.03		
	+ 70.38	+ 74.22	− 74.22	− 50.48	+ 50.47	+ 30.90	− 15.34	− 15.56	− 2.99	+ 2.99	+ 0.92	− 0.92	− 0.45	− 7.68

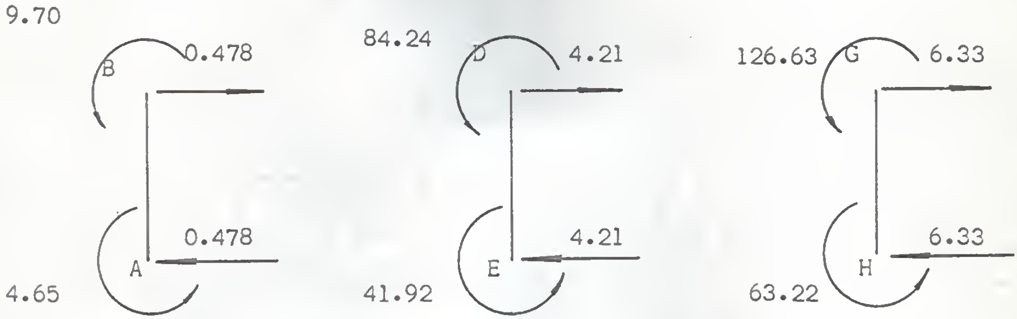
Table 15 Distributionof F.E.M., Due to sidesway at D

Joint	A	8		C		D			F		G		H	E
	A8	8A	8C	CB	CD	DC	DE	DF	FD	FG	GF	GH	HG	ED
S.F.		1/30	1/25	1/25	1/47.5	1/47.5	1/30	1/25	1/25	1/47.5	1/47.5	1/30		
C.O.F.	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	
D.F.		0.455	0.545	0.656	0.344	0.220	0.354	0.426	0.656	0.344	0.389	0.611		
F.E.M.	0	0	−110.88	−110.88	+26.00	+26.00	−66.67	+110.88	+110.88	−26.00	−26.00	0	0	−66.67
8al.		+50.40	+60.48	+55.70	+29.18	−15.45	−24.82	−29.94	−55.70	−29.18	+10.10	+15.90		
C.O.	+25.20	0	+27.85	+30.24	−7.73	+14.59	0	−27.85	−14.97	+5.05	−14.59	0	+7.95	−12.41
8al.		−12.67	−15.18	−14.76	−7.75	+2.92	+4.69	+5.65	+6.51	+3.41	+5.69	+8.90		
C.O.	−6.34	0	−7.38	−7.59	+1.46	−3.88	0	+3.26	+2.83	+2.35	+1.71	0	+4.45	+2.35
8al.		+3.36	+4.02	+4.02	+2.11	+0.14	+0.22	+0.26	−3.40	−1.78	−0.67	−1.04		
C.O.	+1.68	0	+2.01	+2.01	+0.07	+1.06	0	−1.70	+0.13	−0.34	−0.89	0	−0.52	+0.11
8al.		−0.92	−1.09	−1.36	−0.72	+0.14	+0.22	+0.26	+0.14	+0.07	+0.35	+0.54		
C.O.	−0.46	0	−0.68	−0.55	+0.07	−0.36	0	+0.07	+0.13	+0.18	+0.04	0	+0.27	+0.11
8al.		+0.31	+0.37	+0.31	+0.17	+0.064	+0.10	+0.13	−0.20	−0.11	−0.02	−0.02		
TOTAL MOMENT	+20.08	+40.48	−40.48	−42.86	+42.86	+25.22	−86.26	+61.02	+46.35	−46.35	−24.28	+24.28	+12.15	−76.51

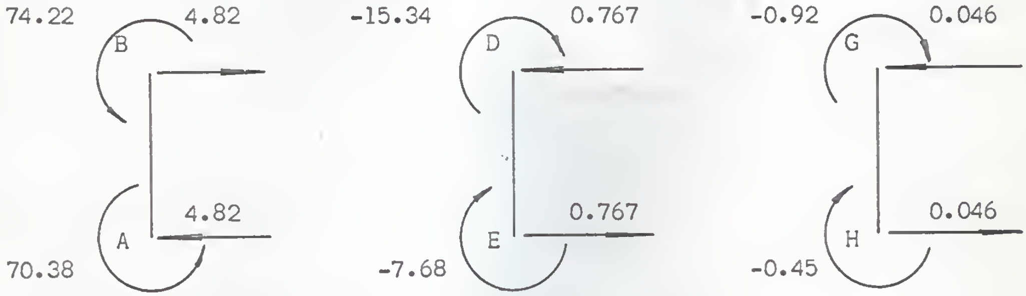
Table 16 Distribution of F.E.M. due to sidesway at G

Joint	A	B		C		D			F		G		H	E
	AB	BA	BC	CB	CD	DC	DE	DF	FD	FG	GF	GH	HG	ED
S. F.		1/30	1/25	1/25	1/47.5	1/47.5	1/30	1/25	1/25	1/47.5	1/47.5	1/30		
C.O.F.		0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5		
D. F.		0.455	0.545	0.656	0.344	0.220	0.354	0.426	0.656	0.344	0.389	0.611		
F.E.M.	0	0	0	0	0	0	0	−110.88	−110.88	+26.00	+26.00	−66.67	−6.667	0
Bal.						+24.40	+39.20	+47.28	+55.73	+29.15	+15.77	+24.90		
C.O.					+12.20	0	0	+27.87	+23.64	+7.89	+14.58	0	+12.45	+19.60
Bal.				−8.00	−4.20	−6.13	−9.85	−11.89	−20.71	−10.82	−5.68	−8.90		
C.O.			−4.00	0	−3.07	−2.10	0	−10.36	−5.95	−2.84	−5.41	0	−4.45	−4.93
Bal.		+1.82	+2.18	+2.01	+1.06	+2.74	+4.41	+5.31	+5.77	+3.02	+2.10	+3.31		
C.O.	+0.91	0	+1.01	+1.09	+1.37	+0.53	0	+2.89	+2.66	+1.05	+1.51	0	+1.66	+2.21
Bal.		−0.46	−0.55	−1.61	−0.85	−0.75	−1.21	−1.46	−2.43	−1.28	−0.59	−0.92		
C.O.	−0.23	0	−0.81	−0.28	−0.38	−0.43	0	−1.22	−0.73	−0.30	−0.64	0	−0.46	−0.61
Bal.		+0.37	+0.44	+0.07	+0.03	+0.36	+0.58	+0.71	+0.68	+0.35	+0.25	+0.39		
C.O.	+0.19	0	+0.04	+0.22	+0.18	+0.02	0	+0.34	+0.36	+0.13	+0.18	0	+0.20	+0.29
Bal.		−0.02	−0.02	−0.26	−0.14	−0.08	−0.13	−0.15	−0.32	−0.17	−0.07	−0.11		
TOTAL MOMENT	+0.87	+1.71	−1.71	−6.76	+6.76	+18.56	+33.00	−51.56	−52.08	+52.08	+48.00	−48.00	+1.57	+16.56

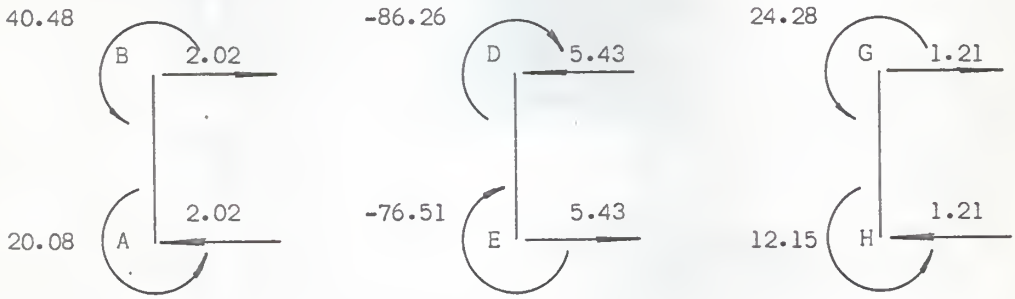
The shear condition, caused by the applied load.



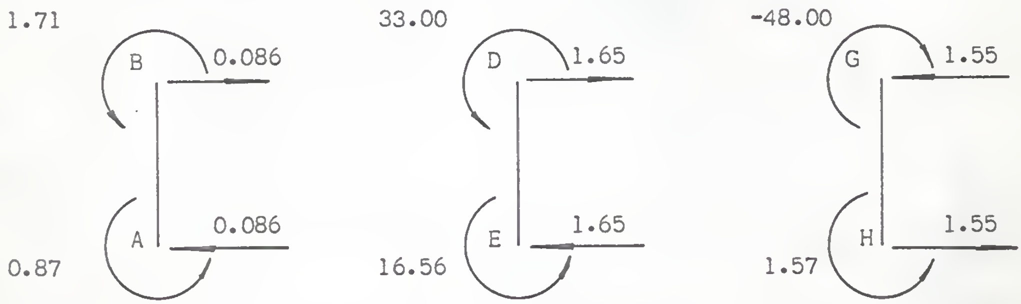
The shear condition caused by sidesway at B.



The shear condition caused by sidesway at D.



The shear condition caused by sidesway at G.



$$0.478 + 4.82k_1 - 0.767k_2 - 0.046k_3 = 0$$

$$4.210 + 2.02k_1 - 5.430k_2 + 1.210k_3 = 0$$

$$6.330 + 0.086k_1 + 1.65k_2 - 1.550k_3 = 0$$

Solve for k_1, k_2, and k_3, from these three equations.

$$k_1 = -0.11$$

$$k_2 = -0.30$$

$$k_3 = 1.80$$

$$M_{AB} = 4.64 - (0.11)(70.38) - (0.30)(20.08) + (1.8)(0.87) = -7.54$$

$$M_{BA} = -M_{BC} = 9.70 - (0.11)(74.42) - (0.30)(40.48) + (1.80)(1.71) = -7.51$$

$$M_{CB} = -M_{CD} = -137.34 - (0.11)(-50.48) - (0.30)(-42.86) + (1.80)(-6.76) = -131.13$$

$$M_{DC} = -145.02 - (0.11)(30.9) - (0.30)(25.22) + (1.80)(18.56) = -121.59$$

$$M_{DE} = 84.24 - (0.11)(-15.34) - (0.30)(-86.26) + (1.80)(33.00) = 170.21$$

$$M_{ED} = 41.92 - (0.11)(-7.68) - (0.30)(76.51) + (1.80)(16.56) = 95.49$$

$$M_{DF} = 60.68 - (0.11)(-15.52) - (0.30)(61.02) + (1.80)(-51.56) = 48.8$$

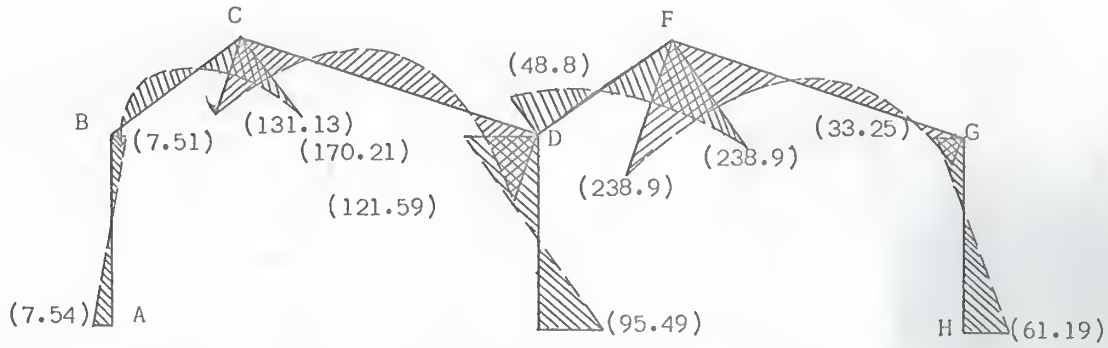


Fig. 21. The resulting moment diagram, obtained by the moment distribution method, plotted on the compression side.

$$M_{FD} = M_{FG} = 46.35 - (0.11)(-52.08) - (0.30)(46.35)$$
$$(1.80)(-52.04) = -238.9$$

$$M_{GF} = -M_{GH} = -126.72 - (0.11)(0.92) - (0.30)(-24.28)$$
$$(1.80)(48.00) = -33.25$$

$$M_{HG} = 63.22 - (0.11)(-0.45) - (0.30)(12.15)$$
$$(1.80)(0.87) = 61.19$$

CONCLUSIONS

It appears that the method of the column analogy can be applied in finding moments and stiffness and carry-over factors for statically indeterminate structures with either a symmetrical or unsymmetrical cross section, in a direct manner.

The comparison of the results of the column analogy method and the moment distribution method reveals that the results, which are obtained by these two methods, are essentially identical, as shown in Fig. 17 and Fig. 21. The column analogy method took much less time than the moment distribution method, when they were both used to solve the double bay gable frame in this report.

ACKNOWLEDGMENT

The author wishes to express his deep appreciation to his major professor, Dr. Robert R. Snell, for his advice, guidance, suggestions, and encouragement during the preparation of this report.

NOTATION

x, y = Coordinates of any point on the cross-section along any two mutually perpendicular axes X and Y through the centroid of the section.

f = Intensity of normal stress at point x, y.

A = Area of section.

$I_x = \int y^2 dA$ --- Moment of inertia about X axis (along the Y axis).

$I_y = \int x^2 dA$ --- Moment of inertia about Y axis.

$I_{xy} = \int xy dA$ --- Product of inertia about axes X, and Y

P = Normal component of external forces.

M_x = Moment of external forces about X axis.

M_y = Moment of external forces about Y axis.

$$M_x' = M_x - M_y\left(\frac{I_{xy}}{I_y}\right)$$

$$M_y' = M_y - M_x\left(\frac{I_{xy}}{I_x}\right)$$

$$I_x' = I_x - I_{xy}\left(\frac{I_{xy}}{I_x}\right)$$

$$I_y' = I_y - I_{xy}\left(\frac{I_{xy}}{I_y}\right)$$

θ = A relative rotation of the two ends of the cut.

Δ_x = Relative horizontal displacement of the two ends of the cut.

Δ_y = Relative vertical displacement of the two ends of the cut.

M_d = Determinate moment throughout the entire ring due to the loads P_1, P_2, ... after the ring is cut.

M_i = Indeterminate moment due to the forces F acting on the ends of the cut.

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ADVANCED STUDY OF THE COLUMN ANALOGY METHOD

by

DAVID K. C. CHENG

B. S., Taiwan Provincial Taipei Institute of Technology, 1957

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

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KANSAS STATE UNIVERSITY
Manhattan, Kansas

1964

The purpose of this report is to show the general derivation and application of the method of column analogy. This method is very useful in finding the moments in third degree statically indeterminate structures.

The formulas of this method are derived from two sets of equations. The first set is found by setting the rotation and displacements caused by the determinate moments equal and opposite to those caused by the indeterminate moments. In other words, by enforcing the continuity requirements of the structure.

$$\theta_d = -\theta_i$$

$$\Delta_{yd} = -\Delta_{yi}$$

$$\Delta_{xd} = -\Delta_{xi}$$

or, it can be written as:

$$\int M_d \frac{ds}{EI} + \int M_i \frac{ds}{EI} = 0$$

$$\int M_d \frac{ds}{EI}(x) + \int M_i \frac{ds}{EI}(x) = 0$$

$$\int M_d \frac{ds}{EI}(y) + \int M_i \frac{ds}{EI}(y) = 0$$

The second set of equations is obtained from the equations of static equilibrium for an imaginary column loaded with the M_d diagram of the ring. The analogous column has the same axis and the same length as the ring and has a thickness of $1/EI$.

$$\Sigma V = 0$$

$$\int M_d \frac{ds}{EI} + \int M_i \frac{ds}{EI} = 0$$

$$\Sigma M_y = 0$$

$$\int M_d \frac{ds}{EI}(x) + \int M_i \frac{ds}{EI}(x) = 0$$

$$\Sigma M_x = 0$$

$$\int M_d \frac{ds}{EI}(y) + \int M_i \frac{ds}{EI}(y) = 0$$

In comparing these two sets of equations we see that if $M_i = F$, they are identical. Therefore, we can determine the indeterminate moments in the ring by calculating the fiber stresses in the analogous column. The formula for computing the fiber stresses in a column, either concentrically loaded or eccentrically loaded, is:

$$F = \frac{P}{A} + \frac{M_x(y)}{I_x} + \frac{M_y(x)}{I_y}$$

In the application section of this report is shown the general use of the column analogy method in finding moments in a frame, fixed-end moments and stiffness and carry-over factors for members of a frame to be used in the moment distribution method.

The comparison of the results of the column analogy method and the moment distribution method reveals that the results, which are obtained by these two methods, are essentially identical. The column analogy method took much less time than the moment distribution method when they were both used to solve the double bay gable frame in this report.