ON APPLYING QUEUEING THEORY

TO A FIXED-CYCLE TRAFFIC INTERSECTION

BY

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Books! 'tis a dull and endless strife;

Come, hear the woodland linnet,

How sweet his music! on my life,

There's more of wisdom in it.

                        -William Wordsworth

My sincere thanks to...

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REVIEW OF LITERATURE

"The population of this earth has been persistently increasing throughout recorded history, but only within the twentieth century has its size become an important obstacle to orderly civilization. One of the problems created by this growth, which has proved to be of some mathematical interest, is that of congestion. On land and in the air, in vehicles and on foot, people now get in each others' way to an extent far surpassing that of any previous age. There may have been overcrowding in ancient Rome or Elizabethian London, but it can hardly have constituted the hazard, the inconvenience, or the expense it does today" Haight (10).

Although these words were written for another time, their relevance has only increased with the passing years. Fortunately, the number of ways to understand and possibly solve the problem of traffic congestion has increased also.

Consider a general intersection between two roads. The methods of controlling traffic through this intersection are many. One of the simpler designs consists of a stop sign controlling vehicles on a minor road while they wait for a gap in the traffic on a major free-flowing road. Adams (1), Gazis (8), Cleveland and Capelle (5), Tanner (21), and Wohl and Martin (23) are only a few authors that include this type of intersection in their

investigations. Most of these model the gap acceptance function. This function determines the probability of a gap being wide enough so that the driver on the minor road feels that it is safe to cross or merge with traffic.

Semi- and full-actuated traffic signals and volume density controllers are the most complicated devices of traffic control, but they also offer the greatest relief to major congestion problems, (Kennedy et al. (13)). Lin (14) models intersections with traffic actuated signals.

The fixed-cycle traffic signal is between the above extremes. Cleveland and Capelle (5) give a very good review of pre-1964 works that apply queuing theory in varying degrees to this type of intersection. Newell (19) and Haight (10) describe the arrivals by the popular Poisson distribution. Newell considers the system to be in one of 3 states: no interference between opposing directions, interference because one direction wants to turn left, and then interference because the other wants to turn left. From this approach, he derives transition probabilities and an equation for the average number of vehicles per signal cycle able to clear the intersection.

Haight derives queue length probabilities including the overflow condition. These probabilities are based on the flow of the approach, the length of red and green phases, and the constant departure headway of vehicles

during the green phase.

Allsop (2) calls the "part of the signal cycle in which one particular set of approaches has right of way ... a stage." He then determines g(i), the proportion of the cycle that is effectively green for stage i, by setting certain constraints (i.e., cycle time, minimum green time and capacity). The g(i)'s that meet these constraints and that minimize the rate of delay are the optimal values.

De Smit (7) treats the input distribution as binomial, with a constant probability, p, of a car arriving during each unit of time. This model was chosen because it "can be applied when the traffic intensity is high" and thus the transient behavior of the queue can be investigated. He then uses mathematical models from classical queuing theory to derive asymptotic equations for mean queue length, mean waiting time, and mean busy period.

Uematu (22) considers the queuing process as a one-dimensional random walk. That is the size of the queue may either increase, decrease or stay the same in a random manner. In order to determine the optimal green periods, the expected values of the time until the queues reach capacity are set equal to each other and the equation is solved for the green periods. The purpose is to give more green time to the direction that approaches its capacity faster. Conditions for the existence of the optimal green

periods are then stated.

Shock wave theory has been used by Michalolopoulos, et al. (17) Land Michalopolous and Pishardy (18) to model this type of intersection. The feeling is that conventional methods for estimating queue size which assume jam density conditions are inadequate for calculating queue sizes and delay.

By treating the green interval of the fixed cycle as one time period the cars are seen to be served in bulk. When the light turns green and service begins each car will be served up to some maximum number of cars. All other cars in the queue must wait for the next green interval. If less than the maximum number of cars are in the queue then all the cars will be served with none left at the end of the green interval. Thus, in this one time period all the cars are served as a bulk with size from 0 to some maximum. Without looking specifically at traffic intersections, Chaudhry and Templeton (4) explore queues with bulk service or bulk arrivals or both. Bailey (3) considers the embedded Markov chain generated by the length of the queue at epochs just before service is due to take place. He then derives expressions for the mean and the variance of the queue length and the mean waiting time.

This paper will treat the fixed-cycle traffic signal as a single server queue with bulk service. The

probability of the queue being of a certain length if it is currently of another length and the equation for the time until service of a car joining the queue will be derived. The intersection conditions necessary for the length of the queue to be able to return to zero at some time will also be given. Finally, some simulation results are presented.

QUEUING THEORY

Before attempting to describe a traffic intersection using queuing theory a brief discussion explaining the basics of queuing theory is in order $\left(\text{Saaty (20)}\right)$.

Customers arriving at some service point (for example a ticket window or the service counter) may find that service point busy and have to wait for some period of time. When the server becomes free, then the next customer is served for another period of time. Once finished, the customer will leave the service point on some new errand. This is a general description of a queuing process. Queues are defined by how customers arrive, how they are served and how many servers there are. The nomenclature traditionally used is

> D = deterministic (constant) arrivals or service times.
>
> M = Poisson arrivals or exponential service times.
>
> E_k= Erlangian interarrival times or service times of order k.
>
> G = General, no specific distribution is assumed.

For example, a queue with a Poisson arrival distribution, negative exponential service times, and one server would be denoted as M/M/1. Other aspects of the queue that need to be defined are the order of service, type of service and customer behavior. An example of each

is, respectively, first come, first served; bulk service; and customers may wait only a certain amount of time before they become impatient and leave.

When exploring a particular queue the items of interest may include (besides the model for the arrival and service distributions)

1) The expected number of customers in the queue,

2) The expected waiting time, or delay, for the customers,

3) The probability of transition from one length to another, and

4) The conditions necessary for the length of the queue to return to any state with probability 1.

PROBABILITY DISTRIBUTIONS

The distributions that have been used previously and will be used in this study to describe the arrival of cars at an intersection are the Poisson and the Binomial. They are briefly discussed here $\left(\text{Hogg and Craig (11)}\right)$.

POISSON

If the intersection is considered isolated in the sense that the distance to another intersection in any direction is large enough for the arrival of cars to be stochastically independent, then these arrivals can be described with the Poisson distribution.

Let X be a random variable with the rule: X = the number of cars that arrive at the intersection during some unit time. Let λ be the expected value of X or the mean number of cars arriving during that unit time. Then the probability that a particular number of cars, x, arrives at the intersection is

$$P(X=x) \;=\; \begin{cases} \dfrac{\lambda^x e^{-\lambda}}{x!} & \text{for } x = 0,1,2,\dots \\ 0 & \text{elsewhere} \end{cases}$$

Where n! = n(n-1)(n-2)... 3x2x1 and e = exp = 2.71828.... The mean and the variance for this distribution are both λ and the moment generating function is

$E(e^{-xt}) = \exp\{\lambda(e^t - 1)\}.$

BINOMIAL

Suppose the time axis is divided into equal intervals. A car will arrive at the intersection during this interval with probability, p, and no car will arrive with probability, 1-p. Again we will assume that the intersection is isolated so that the arrivals are independent and identical. Let X = the number of cars that arrive at the intersection in n time intervals. Then X can be described by the binomial probability distribution, where

$$P(X=x) = \begin{cases} \binom{n}{x}p^{x}(1-p)^{n-x} & \text{for } x=0,1,2,\ldots,n \\ 0 & \text{elsewhere} \end{cases}$$

The mean number of cars to arrive is np and the variance is np(1-p). The moment generating function of the binomial probability distribution is $E(e^{-xt}) = ((1-p) + pe^{t})^{n}$.

THE TRAFFIC SYSTEM

In this paper the approach from one of the four
directions of an isolated, fixed cycle signalized
intersection will be modeled by a single server queue with
bulk service. The following will be assumed:

1) No left or right turns,

2) A single lane,

3) The time it takes for a car to pass through
    an intersection is constant,

4) The cycle is divided into red and green
    intervals. The amber will be
    considered part of the green,

5) Cars are the only vehicles to use the
    intersection, and

6) Once a car enters the queue it does not
    leave except through the
    intersection.

Let

R = the length of the red interval,

G = the length of the green interval,

$\psi = (R + G)$ the length of one light cycle,

c = the time it takes for one car to pass
through the intersection, and

M = (integer value of $[G/c]$) the maximum number
of cars that may pass through the
intersection during one green interval. (1)

The difference $[G/c]$ - integer value of $[G/c]$ is called "lost time" and can be considered part of the red interval.

The time axis will be divided into fixed cycles, from the beginning of the red interval to the end of the green interval. Thus, cars will arrive at the intersection according to some distribution during the red interval, and wait in a first come, first served queue until the green interval begins. During this interval, cars will leave at a constant rate. Cars will still be arriving during the green interval and any that were not able to be served will wait until the next green interval.

Let

Z_n = the number of cars that arrive during the time period n-1 to n, for n = 0,1,2,3,...

Z_0 = 0.

S_n = the number of cars served during the time period n-1 to n, for n = 0,1,2,3,...

S_0 = 0

X_n = the number of cars in the queue at time n (called the state of the system), for n = 0,1,2,3,...

X_0 = 0.

Note that the number of cars that are in the queue at time n is the number of cars in the queue at time n-1 plus the number of cars that arrive minus the number that leave or

$X_n = X_{n-1} + Z_n - S_n$, for n = 0,1,2,3,...

Without making any specific assumptions about the
arrival distribution we will denote

$$a_i = Pr(Z_n = i) \text{ for } n = 0,1,2,3,\ldots, \qquad (2)$$

such that

$$a_i > 0 \text{ for all } i, \text{ and } \sum_{i=0}^{\infty} a_i = 1.0.$$

The mean of this distribution is $\sum_{k=0}^{\infty} ka_k$ and the variance is
$\sum_{k=0}^{\infty} k^2 a_k - \left(\sum_{k=0}^{\infty} ka_k\right)^2$.

The probability of transition from state i to state j
at time n will be denoted as

$$P_{ij}(n) = Pr(X_n = j|X_{n-1} = i) \text{ for } i,j = 0,1,2,3,\ldots$$

such that

$$P_{ij}(n) \geq 0 \text{ for all } i,j \text{ and } n, \text{ and}$$
$$\sum_{j=0}^{\infty} P_{ij}(n) = 1.0 \text{ for all } j,n.$$

Then

$$Pr(X_n = j|X_{n-1} = i)$$
$$= Pr(X_{n-1} + Z_n - S_n = j| X_{n-1} = i)$$
$$= Pr(Z_n - S_n = j - i|X_{n-1} = i).$$

Now, clearly, either

(i) $X_{n-1} + Z_n \leq M$. Then $S_n = X_{n-1} + Z_n \leq M$,
$X_n = j = 0$, and $Pr\left(Z_n = S_n + j - i|X_{n-1} = i\right) =$
$Pr\left(Z_n \leq M - i|X_{n-1} = i\right)$.

(ii) $X_{n-1} + Z_n > M$. Then $S_n = M$, $X_n = X_{n-1} + Z_n - M$,
and $Pr\left(Z_n = S_n + j - i|X_{n-1} = i\right) = Pr\left(Z_n = M + j - i|X_{n-1} = i\right)$.

Therefore,.

$$P_{ij}(n) = \begin{cases} P(Z_n \leq M-i) & \text{for } j=0, \quad i-j \leq M \\ P(Z_n = M+j-i) & \text{for } j>0, \quad i-j \leq M \\ 0 & i-j > M \end{cases}$$

Then from (2) and since the a_i's do not depend on n

$$P_{ij} = \begin{cases} a_{M+j-i} & \text{for } j>0, \quad i-j \leq M \\ \sum_{\ell=0}^{M-i} a_\ell & \text{for } j = 0, \quad i \leq M \\ 0 & i-j > M \end{cases}$$

The transition probability matrix is given in Table 1 on the next page.

Table 1: Probability of transition from state i at time t
to state j at time t+Δt.

Previous State (i)	Current State (j)							
	0	1	2	3	⋯	⋯	M	⋯
0	$\sum_{\ell=0}^{M} a_\ell$	a_{M+1}	a_{M+2}	a_{M+3}	⋯	⋯	a_{2M}	⋯
1	$\sum_{\ell=0}^{M-1} a_\ell$	a_M	a_{M+1}	a_{M+2}	⋯	⋯	a_{2M-1}	⋯
2	$\sum_{\ell=0}^{M-2} a_\ell$	a_{M-1}	a_M	a_{M+1}	⋯	⋯	a_{2M-2}	⋯
3	$\sum_{\ell=0}^{M-3} a_\ell$	a_{M-2}	a_{M-1}	a_M	⋯	⋯	a_{2M-3}	⋯
⋮	⋮	⋮	⋮	⋮	⋱		⋮	⋱
M−1	$\sum_{\ell=0}^{1} a_\ell$	a_2	a_3	a_4	⋯	⋯	a_{M-1}	⋯
M	a_0	a_1	a_2	a_3	⋯	⋯	a_M	⋯
M+1	0	a_0	a_1	a_2	⋯	⋯	a_{M-1}	⋯
⋮	⋮	⋮	⋮	⋮	⋱		⋮	⋱

Note however, that this is a matrix of infinite dimension. This dimension and the particular pattern of the a_i's in the matrix cause the traditional paths of showing recurrence $\big($Cohen (6), Karlin and Taylor (12)$\big)$ to be inadequate.

CONDITION FOR RECURRENCE

It is of interest to determine when the queue is transient or recurrent. If the queue is transient then the number of cars will tend towards infinity with a positive probability of never returning to state 0. On the other hand, if the queue is recurrent then the probability is one that the number of cars will return to any state after some period of time. The latter is, of course, the desired property.

Now let $\Delta_n = Z_n - S_n$ so that Δ_n is the difference between the current state and the previous state at time n. X_n can be considered as the position of a particle after n steps, where X_n can be any value from 0 to $+\infty$. Then $X_n = \Delta_1 + \Delta_2 + \Delta_3 + \ldots + \Delta_n$, where the $\{\Delta_i\}_{i=1}^{\infty}$ are, in the region where the state is greater than M, mutually independent random variables with distribution function $F_\Delta(\ )$. Let μ and σ^2 be, respectively, the finite mean and variance of $F_\Delta(\ )$.

States of the queue greater than M can be called the "danger zone". When the queue is in these states only M cars are able to pass through the intersection during one green interval and the rest must wait until a later green interval. If the queue never leaves this danger zone, but instead tends toward infinity (i.e. is transient) then the intersection becomes congested.

In this region $\Delta_n = Z_n - M$. Then $\mu = E(\Delta_n) = E(Z_n) - M = E(Z_n) - M = \sum_{k=0}^{\infty} k a_k - M$. Now, in order for the system to be recurrent the traffic intensity must be less than 1, (i.e. $\rho < 1$), which is equivalent to the drift of the particle, X_n, being less than zero (i.e. $\mu < 0$). But $\mu < 0$ implies $\sum_{k=0}^{\infty} k a_k - M < 0$, which implies $\sum_{k=0}^{\infty} k a_k / M < 1$. Hence, for the system to be recurrent the expected number of cars arriving at the intersection must be less than the maximum number of cars that can be served during one green interval.

The waiting time of a car queued at an intersection
will be defined as the length of the interval from the
instant of arrival at the queue to the instant service
begins. Let $W(k, t_k)$ be a random variable denoting the
amount of time the kth car must wait if it arrives at time
t_k, for $t_k \geq 0$, $k = 1, 2, 3, \ldots$.

If the system is in its nth period then $\frac{t_k}{\psi} = n\psi + t_k^*$,
for $n = 1, 2, 3, \ldots$, $t_k \geq 0$, $0 \leq t_k^* < c$. So far the cycle has
been considered as consisting of a red period and a green
period, in that order. However, an equivalent approach is
to reverse the order, so that the green period is
considered to be first. Now if the car arrives while the
light is red, then $R-t_k^*$, >0, is the time until the next
whole cycle begins. Similarly, if the car arrives while
the light is green, then $R-t_k^*$, ≤ 0, is the time "into" the
first whole cycle.

Let L = integer value of $(\frac{k}{M+1})$ = the number of whole
cycles the kth car must wait. Then $(R-t_k^*) + L\psi$ is the time
until the bulk of cars containing the kth car begins
service. During this time LM cars have been serviced.
This leaves $k-LM-1$ cars in front of the kth car that are
each serviced for a time c. Therefore, the waiting time of
the kth car arriving at time t_k is the sum of these or

$$W(k,t_k) = \left[(R - t_k^*) + L\psi\right] + \left[(k - LM - 1)c\right]$$
$$\text{for } k=1,2,3\ldots; \ t_k \geq 0. \tag{3}$$

MAXIMUM WAITING TIME

In a traffic intersection the item of interest is the maximum waiting time, or the maximum delay in the system. This maximum is not always associated with the last car to arrive during the cycle.

Suppose that $M < k < k+1 < aM$, $a = 2, 3, 4, \ldots$ so that the kth and the $(k+1)^{st}$ car will exit during the same green interval. Now, suppose that $0 < \tau_k \leq c$, where $\tau_k = t_{k+1} - t_k = t_{k+1}^* - t_k^*$ is the kth interarrival time. Then the difference in waiting times for the kth and $(k+1)^{st}$ car is

$$W(k+1, t_{k+1}) - W(k,t_k)$$
$$= \left[(R - t_{k+1}^*) + L\psi\right] + \left[(k+1 - LM - 1)c\right]$$
$$\quad - \left\{\left[(R - t_k^*) + L\psi\right] + \left[(k - LM - 1)c\right]\right\}$$
$$= t_k^* - t_{k+1}^* + c$$
$$= c - \tau_k$$

so that $0 \leq W(k+1, t_{k+1}) - W(k,t_k) < c$.

Suppose on the other hand, that $\tau_k > c$. Then

$$W(k+1, t_{k+1}) - W(k, t_k)$$
$$= t_k^* - t_{k+1}^* + c$$
$$= c - \tau_k < 0.$$

Thus, the waiting time of the kth car is longer than the waiting time of the $(k+1)^{st}$ car when their interarrival time is greater then the service time. Now $W(k+1, t_{k+1}) - W(k, t_k) = c - \tau_k$ holds in general so that the maximum waiting time will be associated with the ℓth car where $\ell = \min \{k : \tau_k \geq c\}$. Note that if no cars arrive after a car q, say, then $\tau_q > c$.

UPPER BOUND

Let η be the expected number of whole cycles a car must wait before it is serviced. Then

$$\eta = E(L) = \sum_{\ell=0}^{\infty} \ell \left[\sum_{j=\ell M+1}^{(\ell+1)M} a_j \right]$$

where L = integer value of $(\frac{k}{M+1})$.

Now

$$\rho = \sum_{\ell=0}^{\infty} \frac{\ell}{M} a_\ell = \sum_{\ell=0}^{\infty} \left[\sum_{j=\ell M}^{(\ell+1)M-1} \frac{j}{M} a_j \right]$$

$$\geq \sum_{\ell=0}^{\infty} \frac{\ell M}{M} \left[\sum_{j=\ell M+1}^{(\ell+1)M} a_j \right]$$

$$= \sum_{\ell=0}^{\infty} \ell \left[\sum_{j=\ell M+1}^{(\ell+1)M} a_j \right]$$

$$= \eta.$$

Therefore the expected number of cycles a car must wait is bounded above by the traffic intensity. Realizing that this traffic intensity is the ratio of the expected number of cars to arrive during one cycle divided by the maximum number of cars served, we see that if 2 times as many cars are expected to arrive as can be served during one cycle, then on the average those cars will not have to wait longer than 2 cycles. This, of course, agrees with what common sense tells us.

Finally, when the system is recurrent (i.e., $0 \leq \eta \leq \rho < 1$), on the average all cars will be served during the first cycle.

SIMULATION

In reality the arrival of vehicles at an intersection is not a constant phenomenon. The expected number of vehicles arriving is greater during morning and evening rush periods, (say, 7:00 - 9:00 am and 4:00 - 6:00 pm), than at other times, especially pre-dawn. The heterogeneity of the expected number of cars to arrive is the knife that cuts the Gordian Knot of traffic congestion.

To investigate the relationship of queue length and waiting time with the epected number of cars to arrive, a simulation of the single approach was conducted. The algorithm for this simulation is given in Appendix 1. The pattern of arrival means given in Table 1 is recurrent for a 15-minute period, transient for two 15-minute periods and finally recurrent for three periods.

Table 2: Pattern of Arrival Means

Epoch	Expected Number of Arrivals
1	14.333
2	20.000[a]
3	18.267[a]
4	17.733
5	17.267
6	19.933

a: Transient Epoch

Table 3 summarizes the results obtained with the Binomial Distribution generating inputs. As a check on the validity of the simulation the estimated probability of arrival and the actual probability of arrival were tested for equality. In all instances we could not say that the two were significantly different. (Table 3a)

Queue length and waiting time are important aspects of the traffic system. When a vehicle arrives at the intersection the waiting time is calculated using equation (3). The mean waiting time, sample standard deviation and maximum waiting time based on N arrivals are given in Table 3b.

When a vehicle either arrives at or departs from the intersection the queue length is calculated. The mean queue length, sample standard deviation, and maximum queue length are reported in Table 3c.

By inspection we see that as expected the waiting time and queue length increase and decrease with the probability of arrival. In fact, the probability of arrival explains 70.56% and 81% of the variation in mean waiting time and mean queue length, respectively.

Similar results are found (Table 4) with the Poisson distribution used to generate arrivals. Waiting time results are given in Table 4b. We see that the probability of arrival explains 81.25% of the variation in mean waiting

time.

The mean queue length, sample standard deviation, and maximum queue length are reported in Table 4c. Here the probability of arrival explains 78.72% of the variation in mean queue length.

Table 3: Binomial Distribution Results

a. Actual vs. Estimated Probability of Arrival (N = 7500)

p_0	$\hat{p}$	$\dfrac{p_0(1-p_0)}{N}$	$H_0: p=p_0$ p-value
.23889	.24320	2.424×10^{-5}	.1949
.33333	.33267	2.963×10^{-5}	.4522
.30444	.30280	2.823×10^{-5}	.3783
.29556	.29293	2.776×10^{-5}	.3085
.28778	.28747	2.733×10^{-5}	.4761
.28222	.28027	2.701×10^{-5}	.3520

b. Waiting Time (seconds)

Probability of Arrival	Mean Waiting Time	Standard Error	Maximum Waiting Time	N
.23889	3.3635	0.1155	18	1824
.33333	5.3828	0.1474	39	2102
.30444	4.6103	0.1269	27	2156
.29556	3.5307	0.1076	21	2197
.28778	3.8330	0.1245	33	2271
.28222	3.6660	0.1158	24	2495

c. Queue Length (Number of cars)

Probability of Arrival	Mean Queue Length	Standard Error	Maximum Queue Length	N
.23889	1.9348	0.0332	10	2978
.33333	4.3824	0.0526	20	4059
.30444	3.1582	0.0396	18	3737
.29556	2.3822	0.0332	9	3587
.28778	2.5265	0.0398	16	3561
.28222	2.3861	0.0364	12	4059

Table 4: Poisson Distribution Results

a. Actual vs. Estimated Expected Number of Arrival

$H_0: \lambda = \lambda_0$

λ_0	$\hat{\lambda}$	$se(\hat{\lambda})$	p-value
.23889	.23373	.00558967	0.1788
.33333	.33040	.00671397	0.3300
.30444	.30013	.00629031	0.2483
.29556	.28773	.00617285	0.1020
.28778	.29333	.00621819	0.1867
.28222	.28041	.00615302	0.3859

b. Waiting Time (seconds)

Expected Number of Arr. (Sec^{-1})	Mean Waiting Time	Standard Error	Maximum Waiting Time	N
.23889	3.3614	0.1288	23	1558
.33333	7.0758	0.1975	45	2083
.30444	4.6800	0.1489	31	1953
.29556	4.5614	0.1471	28	1881
.28778	4.6392	0.1416	30	1921
.28222	4.8494	0.1556	36	1826

c. Queue Length (Number of cars)

Expected Number of Arr. (Sec^{-1})	Mean Queue Length	Standard Error	Maximum Queue Length	N
.23889	1.9036	0.0353	9	2749
.33333	6.4383	0.0875	27	3792
.30444	3.4507	0.0492	14	3479
.29556	3.1748	0.0472	14	3376
.28778	3.0896	0.0456	15	3460
.28222	3.4367	0.0558	16	3256

"THE ROADS MUST ROLL"

Limiting the study to one single-lane approach to a fixed cycle intersection provides a strong, general foundation for expansion into more real-world applicable models. Two ways to expand the study come immediately to mind. First, relax some, or all, of the assumptions made here. Second, replicate the approach to model an entire intersection as a single system. In actuality, a combination of the two might prove the most interesting.

Some specific assumptions to relax might be

1) Allow more than one lane. This would double the capacity of the queue and the maximum number being served, thus decreasing waiting time. Allowing left turns would not adversely affect the system until it was combined with one going the opposite direction.

2) Allow the length of the red and green periods to depend on, for example, time of day, the arrival of a car, or the number of cars waiting. The system would then be traffic-actuated.

3) Allow vehicles other than cars to use the intersection. This would cause the service times to vary, thus varying the maximum number of vehicles served. If physical requirements were taken into account, then a variety of vehicles would also vary the number of vehicles the queue can hold.

Modeling the entire intersection as a single system causes interesting problems because of the interdependence of cycle length and the expected number of cars to arrive.

For ease of discussion, number the approaches to the intersection in a clockwise direction as 11, 21, 12, and 22. Let λ_{11}, λ_{21}, λ_{12}, and λ_{22} be the expected number of arrivals from each of the four approaches; T_1 and T_2 be the green times for directions 1 and 2, respectively; and M_1 and M_2 be the maximum number of cars able to be served as determined by T_1 and T_2. Direction i will be called recurrent only if both $\lambda_{11} < M_i$ and $L_{i2} < M_1$. It will be called transient otherwise. The system as a whole will be recurrent if both directions are recurrent. The λ_{ij}'s are dependent on the T_i's and thus on ψ. As the length of the cycle increases the expected number of arrivals during that cycle increases and vice versa.

Consider ψ, and therefore the λ_{ij}'s, constant. If, at this point, the system is recurrent, then no action need be taken. Suppose, however that direction 1 is transient. Then, it seems beneficial to increase T_1 enough so that M_1 is larger than λ_{11} and λ_{12} and the direction is recurrent. However, doing so will decrease T_2, possibly so much that M_2 is less than either λ_{21} or λ_{22} or both. Direction 2 then becomes transient and there is congestion.

A solution to this probelm seems to be to allow ψ to
vary. Now T_2 can be increased to cause direction 2 to be
recurrent. This solution, however, quickly becomes another
problem. If T_2 is increassed, then ψ is increased and the
λ_{ij} are increased. Again, the expected number of arrivals
may be increased so much that one or more are larger than
their corresponding maximum number able to be served. The
result is a transient system.

If the physical realities of the intersection are such
that the previous method becomes inflationary then a
solution from simultaneous equations is necessary. The
green intervals need to be chosen so as to simply minimize
rather than eliminate waiting time.

Questions that need to be answered before a solution
to this may be found are

1) How are the length of cycle and number of cars
expected to arrive related? They would seem to be highly
positively correlated, but is the slope less than, greater
than or equal to 1?

2) How is the number of cars expected to arrive
dependent on time? As mentioned before, this time
heterogeneity is what saves the system.

3) Should the system be structured to concentrate on
minimizing waiting time during the peak hours or minimizing
the waiting time for the entire day? Are these concepts

mutually exclusive?

4) How can time dependent and traffic actuated light settings be utilized to maximize the intersection's efficiency? What about real-time feedback?

APPENDIX

The algorithm for the simulation process was

LOOP 1 FOR 10 DAYS

    LOOP 2 FOR 6 EPOCHS (15 MINUTES EACH)

        LOOP 3 FOR 15 CYCLES (1 MINUTE EACH)

            LOOP 4 FOR 24 SECONDS (RED PERIOD)

                GENERATE NUMBER OF ARRIVALS

                INCREASE LENGTH BY NUMBER OF ARRIVALS

                IF NUMBER OF ARRIVALS IS GREATER THAN ZERO

                    THEN CALCULATE WAITING TIME.

                OUTPUT PER-SECOND DATA

            END LOOP 4

            LOOP 5 FOR 36 SECONDS (GREEN PERIOD)

                GENERATE NUMBER OF ARRIVALS

                INCREASE LENGTH BY NUMBER OF ARRIVALS

                IF LENGTH IS GREATER THAN ZERO AND TIME

                    IS AN EVEN NUMBER THEN ONE CAR

                    LEAVES AND DECREASE LENGTH BY ONE

                OUTPUT PER-SECOND DATA

            END LOOP 5

            OUTPUT PER-CYCLE DATA

        END LOOP 3

    END LOOP 2

END LOOP 1

GENERATING NUMBER OF ARRIVALS

Continuous Uniform random numbers were generated using
the intrinsic FORTRAN function UNI. These number were used
to generate Poisson and Binomial random numbers in the
following manner.

POISSON

```
*GENERATE TABLE OF PROBABILITIES*

    P        = EXP(-λ)
    PROBABILITY(1) = P
    LOOP 1 FOR I FROM 1 TO 98
        P = (P* λ)/I
        PROBABILITY(I+1) = PROBABILITY(I) + P
        IF PROBABILITY(I+1) IS GREATER THAN .999 THEN
            PROBABILITY(I+2) = 1.0, SIZE = I+2,
            RETURN
    END LOOP 1


*GENERATE NUMBER OF ARRIVALS*

    GENERATE RANDOM NUMBER
    LOOP 1 FOR I FROM 1 TO SIZE
        IF PROBABILITY(I) TO GREATER THAN RANDOM
            NUMBER THEN NUMBER OF ARRIVALS IS I-1,
            RETURN
    END LOOP 1
```

BINOMIAL

NUMBER OF ARRIVALS IS ZERO

GENERATE RANDOM NUMBER

IF RANDOM NUMBER IS LESS THAN OR EQUAL TO THE PROBABILITY

    OF ARRIVAL THEN NUMBER OF ARRIVALS IS ONE

RETURN

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ON APPLYING QUEUING THEORY

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ABSTRACT

     The fixed-cycle traffic signal will be treated as a
single server queue with bulk service. The transition
probabilities are derived based on this approach. Then
the condition necessary for the probability of transition
matrix to be recurrent and the equation for the waiting
time of a vehicle are given. Finally, some simulation
results are presented.