TWO RADICALS OF A RING

by

GARY DEAN GABRIELSON

B. S., Kansas State University, 1963

———————————

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1968

Approved by:

R. E. Williams

Major Professor

TABLE OF CONTENTS

INTRODUCTION

The study of radicals of a ring is helpful in the classification of rings which have properties different from the usual number-theoretic properties. Historically the radical has been based on the notion of nilpotency. This study has only yielded significant results in rings with chain conditions.

In 1945 N. Jacobson defined the first radical that gave significant results without chain conditions. Since that time about eight radicals have been defined and investigated for arbitrary rings. The purpose of this report is to investigate two of these radicals and study certain properties of these radicals.

The two radicals considered are the prime radical and the Jacobson radical. Henceforth in this paper the word radical will refer to one of these two radicals. These radicals are of interest since they are different from the classical radical which was defined as the maximal nilpotent ideal, but coincide with it in rings with the descending chain condition.

The prime radical is first defined in three ways. Then using the concept of an m-system the definitions are shown to be equivalent. In the process properties of prime and semi-prime ideals are discussed. Several properties of the prime radical are also proved including the fact that it is a nil ideal containing all nilpotent ideals. This fact is important in the relation between the prime radical and the Jacobson radical. Rings related to a ring R and the results for the prime radical are discussed. The related rings

considered are ideals in a ring R, homomorphic images of R and the ring of n by n matrices over R.

In a similar form the Jacobson radical is discussed. Again three definitions are considered. Starting with the concept of quasi-regularity these are shown to be equivalent. Several properties of quasi-regularity and the Jacobson radical are proved. Also the concepts of primitive ring and R-module are used. These are understood to mean right primitive and right R-module in the discussion of the Jacobson radical. Then the radical in the related rings are considered, that is the ideals of a ring R, homomorphic images of a ring R and the complete matrix ring of a ring R.

The final section considers rings with the descending chain condition. First the containment of the prime radical in the Jacobson radical is shown. Then an example of the distinctness of the radicals is given. Next a theorem is proved that establishes that in the rings with descending chain condition the Jacobson radical is nilpotent. This result together with others in the report yields that in rings with descending chain condition the prime radical and Jacobson radical coincide.

In this report a ring need not have an identity element unless so specified. An ideal will be two-sided unless modified by right or left. A list of conventions and notation used is found in the Appendix of this paper, roughly in the order in which they appear.

A. DEFINITIONS

The first of the two radicals considered in this report will be the prime radical. Three definitions of the prime radical will be stated and then shown to be equivalent.

1.1 Definition. The prime radical of the zero ideal in a ring R is called the prime radical of the ring.

1.2 Definition. The prime radical of a ring R is the intersection of all prime ideals of the ring R.

1.3 Definition. The prime radical of a ring R is the semi-prime ideal which is contained in every semi-prime ideal in R.

Definition 1.1 will be used as the starting point of the discussion. For this concept to be useful one needs to know some of the properties of prime ideals. First definitions of a prime ideal and a prime ring are needed.

1.4 Definition. An ideal P in a ring R is called prime if I and J are ideals in R such that $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. If the ideal (0) is prime, then R is a prime ring.

The following theorem gives several useful properties of prime ideals and will be stated without proof.

1.5 Theorem. If P is an ideal in a ring R, the following are equivalent:

(i) P is a prime ideal.

(ii) If r, $s \in R$ such that $rRs \subseteq P$ then, $r \in P$ or $s \in P$.

(iii) If (r) and (s) are principal ideals in R such that

(r)(s) $\subseteq$ P, then r ϵ P or s ϵ P.

(iv) If I and J are right (left) ideals in R such that IJ $\subseteq$ P,

then I $\subseteq$ P or J $\subseteq$ P.

From these properties one notes that the prime ideal can be characterized in terms of the elements not in the ideal. Let the complement of P, denoted $\complement$P, be the set of elements in a ring R not in the ideal P. Considering property 1.5 (ii), this leads to an interesting fact formalized in the idea of an m-system. Since rRs $\subseteq$ P implies r ϵ P or s ϵ P, if t $\notin$ P then tRt $\notin$ P. That is all the elements not in P form a multiplicative system, but moreover they form a system with the property: for all a ϵ R, ras ϵ $\complement$ P. Such a system is called an m-system.

1.6 Definition. A set M of elements of a ring R is said to be an m-system if and only if for r, s ϵ M, there exists a ϵ R such that ras ϵ M. If M=$\emptyset$, M will be considered an m-system.

Clearly $\complement$P is an m-system. In fact an ideal P in R is a prime ideal in R if and only if $\complement$P is an m-system. The convention to consider the empty set as an m-system assures one that the trivial prime ideal R is satisfied by this statement.

1.7 Definition. The prime radical of the ideal I in a ring R is the set consisting of those elements a of R with the property that every m-system in R which contains a meets I. The prime radical of I will be denoted P(I).

1.8 Theorem. If I is an ideal in the ring R, then P(I) coincides with the intersection of all prime ideals in R which contain I.

Proof: P(I) is contained in every prime ideal which contains I since if $I \subseteq Q$ and a ϵ P(I), then a $\not\in$ Q implies a in $\complement Q$. Hence $\complement Q$ is an m-system which contains a or $I \cap \complement Q \neq \emptyset$. This is a contradiction and implies a ϵ Q. Thus P(I) $\subseteq$ Q for all $I \subseteq Q$.

Choose r $\not\in$ P(I). There exists an m-system M in R such that r ϵ M and $M \cap I = \emptyset$. Let A be the set of ideals J such that $I \subseteq J$ and $M \cap J = \emptyset$. A is not empty as I is one such ideal. Now by Zorn's Lemma, A has a maximal element, say Q. Note that r $\not\in$ Q since r ϵ M and $M \cap Q = \emptyset$. Therefore $Q \subseteq P(I)$.

If Q is also prime then the assertion will be proved. Suppose a $\not\in$ Q and b $\not\in$ Q whence Q + (a) contains an element m ϵ M and Q + (b) an element n ϵ M since Q is maximal in the set A. Therefore there exists an r ϵ R such that mrn ϵ M since M is an m-system. Further mrn is an element of the ideal $(Q = (a))(Q + (b))$. If $(a)(b) \subseteq Q$ then $(Q + (a))(Q + (b)) \subseteq Q$ or mrn ϵ Q. But this contradicts $M \cap Q = \emptyset$ and therefore $(a)(b) \not\subseteq Q$ and by Theorem 1.5 (iii), Q is prime. This completes the proof since one has $P(I) \subseteq \cap Q_i$ and $\cap Q_i \subseteq P(I)$ for prime ideals in R which contain I.

From this theorem, Definition 1.2 follows from Definition 1.1 since P(R) is the intersection of all prime ideals in R which contain (0) and every prime ideal contains (0).

To show that Definition 1.3 is also equivalent to the other two definitions some properties of semi-prime ideals will be considered.

1.9 Definition. An ideal Q is a ring R is semi-prime if for I an ideal in R such that $I \subseteq Q$, then $I \subseteq Q$.

Properties analogous to those of prime ideals can also be shown. These properties are stated without proof in the following theorem.

1.10 Theorem. If Q is an ideal in a ring R the following are equivalent:

    (i)   Q is a semi-prime ideal.

    (ii)   If $s \in R$ such that $sRs \subseteq Q$, then $s \in Q$.

    (iii) If (s) is a principal ideal in R such that $(s) \subseteq Q$, then

        $s \in Q$.

    (iv) If I is a right (left) ideal in R such that $I \subseteq Q$, then

        $I \subseteq Q$.

In a manner similar to the characterization of the prime ideals one notes that semi-prime ideals can be characterized by use of n-systems.

1.11 Definition. A set N of elements of a ring R is an n-system if and only if for $n \in N$, there exists $r \in R$ such that $nrn \in N$.

One notes that an m-system is an n-system and Theorem 1.10 (i), (ii), assures that Q is a semi-prime ideal if and only if $\complement Q$ is an n-system. The following theorem, which will lead to the desired equivalences, is preceded by a lemma concerning the relation of m-systems to n-systems.

1.12 Lemma. If N is an n-system in the ring R and $a \in N$, there exists an m-system M such that $a \in M$ and $M \subseteq N$.

Proof: Define a set of elements inductively: Let $a_1 = a$, choose $a_2 \in a_1 R a_1 \cap N$ observing that $a_1 \in N$ implies $a_1 R a_1 \cap N \neq \emptyset$; in this manner define a_i as an element of $a_{i-1} R a_{i-1} \cap N$. Let this set be called $M = [a_1, a_2, a_3, \ldots]$ By definition M is contained in N. M is also an m-system for $i \leq j$ and a_i, $a_j \in M$

then $a_{j+1} \in a_j Ra_j \subseteq a_i Ra_j$ and $a_{j+1} \in M$. If $j < i$ then $a_{i+1} \in a_i Ra_i \subseteq a_j Ra_i$.
Hence M is an m-system and the proof is complete.

1.13 Theorem. An ideal Q in a ring R is a semi-prime ideal if and only
if $P(Q) = Q$.

Proof: If $P(Q) = Q$ then Q is equal to the intersection of all prime ideals
in R containing Q by Theorem 1.8. Clearly the intersection of all prime
ideals in R containing Q is a semi-prime ideal.

Conversely, let Q be a semi-prime ideal in R. From the definition of
$P(Q)$, Q must be contained in $P(Q)$. Suppose $Q \subset P(Q)$ or there exists
a $\in P(Q)$ such that a is not in Q. Since $C Q$ is an n-system and a is an
element of $C Q$, Lemma 1.12 implies there exists an m-system such that
$a \in M \subseteq C Q$. But a in $P(Q)$ together with the definition of $P(Q)$ requires
that $P(Q)$ must meet Q. This is a contradiction of a not in Q since $M \subseteq C Q$
implies $M \cap Q = \emptyset$. Therefore $Q = P(Q)$.

1.14 Corollary. If I is an ideal in the ring R, then $P(I)$ is the smallest
semi-prime ideal in R which contains I.

Proof: First note that by Theorem 1.8 it follows that $P(I)$ is the inter-
section of all prime ideals which contain I. By Theorems 1.13 and 1.8, one
knows that an ideal is semi-prime if and only if it is the intersection of all
prime ideals containing the ideal. Hence $P(I)$ is the smallest semi-prime
ideal in R which contains I.

Therefore $P(R)$ is the smallest semi-prime ideal which is contained in
every semi-prime ideal in R. Thus Definition 1.3 is equivalent to the other
two definitions.

B. PRIME RADICAL-PROPERTIES

In this section consideration will be given to the prime radical of rings related to a ring R as well as properties of the prime radical. Related rings that will be considered are the ring of n by n matrices over a ring R and homomorphic images of a ring R. At this point it is necessary to define nil and nilpotent ideals.

1.15 Definition. If I is an ideal (right, left) and every element of I is nilpotent then I is called a nil ideal (right, left).

1.16 Definition. If I is an ideal (right, left) such that there exists a positive integer n with the property that $I^n = (0)$, then I is called a nilpotent ideal (right, left).

Observe that a nilpotent ideal is always a nil ideal, but the converse is not always true, that is there are nil ideals which are not nilpotent.

Now some of the properties of the prime ideal will be considered.

1.17 Theorem. $P(R)$ is a nil ideal which contains every nilpotent right (left) ideal in R.

Proof: If $r \in R$ then the set $[\, r^i \mid i = 1, 2, 3, \ldots \,]$ forms a multiplicative system. As $r \in P(R)$ implies that every m-system in R which contains r meets (0), there exists an integer n such that $r^n = 0$. Therefore every element of $P(R)$ is nilpotent and $P(R)$ is a nil ideal.

If I is a right (left) ideal in R such that $I^n = 0$, then $I^n \subseteq P(R)$ and $I \subseteq P(R)$ since $P(R)$ is semi-prime.

1.18 Corollary. If R is a commutative ring, $P(R)$ is the ideal consisting of all nilpotent elements of R.

Proof: If r is a nilpotent element of a commutative ring R, then (r) is a nilpotent ideal. But $r \in P(R)$ implies $r^n \in (0)$ hence $P(R)$ is the set of all nilpotent elements of R.

1.19 Theorem. If I is an ideal in the ring R, the prime radical of I (considered as a ring) is $I \cap P(R)$.

Proof: Denote the prime radical of I in the usual manner $P(I)$, but remember one is now considering I as a ring and not as an ideal of R. Hence $P(I)$ is equal to the intersection of all prime ideals in I. If P is a prime ideal in R, then $P \cap I$ is a prime ideal in I by Theorem 1.5 (iii), since x, $y \in I$ such that $(x)(y) \subseteq (P \cap I)$ implies $(x)(y) \subseteq P$ and therefore $x \in P$ or $y \in P$. Thus $x \in P \cap I$ and hence $P \cap I$ is a prime ideal in I. Therefore $P(I) \subseteq I \cap P(R)$.

Conversely, if $a \in I \cap P(R)$ then $a \in I$ and $a \in P(R)$ and hence every m-system which contains a also contains zero. Moreover every m-system in I which contains a must contain zero. Therefore $a \in P(I)$ or $I \cap P(R) \subseteq P(I)$ and the proof is complete.

1.20 Theorem. If φ is a homomorphism of the ring R onto the ring S, then $P(R) \varphi \subseteq P(S)$.

Proof: Let Q be prime in S. Further let $P = Q \varphi^{-1}$ and let $aRb \subseteq P$. Then for each $x \in R$, $(axb) \varphi \in Q$ and $(axb) \varphi$ is equal to $a\varphi x \varphi b \varphi \in Q$ so that $a\varphi S b\varphi \subseteq Q$ and $a \varphi \in Q$ or $b \varphi \in Q$. But then $a R b \subseteq P$ implies $a \in P$ or $b \in P$ and P is prime in R.

Let $P(R) = \underset{i \in \alpha}{\cap} P_i$ where the P_i are all the prime ideals in R and α is an indexing set. Let $P(S) = \underset{j \in \beta}{\cap} Q_j$ where β is indexing set and the Q_j are all

prime ideals in S. Then $x \in \cap P_i$ implies $x \in \cap Q_j \varphi^{-1}$ or $P(R) = \cap P_i \subseteq \cap Q_j \varphi^{-1}$.

Thus if $x \in P(R)$, then $x\varphi \in \cap Q_j = P(S)$ so that $P(R) \varphi \subseteq P(S)$.

The prime radical of a matrix ring will now be considered. The result of interest is that $P(R_n) = (P(R))_n$. First two lemmas must be proved.

1.21 Lemma. If R is a ring with identity element, every ideal in the complete matrix ring R_n is of the form A_n, where A is an ideal in R.

Proof: Let $\mathfrak{A}$ be an ideal in R_n and let A be the set of all elements of R that occur in the first row and first column of $\mathfrak{A}$. The complete set of matrix units of R_n will be represented by E_{ij} where i=1, . . . , n and j=1, . . . , n. $\mathfrak{A}$ is closed under addition and hence so is A. By definition of A there exists an element $[a_{ij}]$ of $\mathfrak{A}$ with $a_{11} = a$. Then it follows that :

$$E_{11}[a_{ij}] \, rE_{11} = a_{11}rE_{11} = arE_{11} \in \mathfrak{A}$$

But this implies $a \, r \in A$. Similarly $r \, a \in A$ and A is an ideal in R.

Let $B = [b_{ij}]$ be an element of R_n. Then if $B \in \mathfrak{A}$ one notes that:

$$E_{1r}BE_{s1} = E_{1r}[b_{ij}]E_{s1} = b_{rs}E_{11} \in \mathfrak{A}$$

so that $b_{rs} \in A$. Since this is true for r = 1, . . . , n and s = 1, . . . , n, $B \in A_n$ or $\mathfrak{A} \subseteq A_n$.

Let C an element of A_n be $[c_{ij}]$. By definition of A there exists $B = [b_{ij}]$ of A such that $b_{11} = c_{rs}$ where r and s are any integers from 1, . . . , n. Further:

$$E_{r1}BE_{1s} = E_{r1}[b_{ij}]E_{1s} = b_{11}E_{rs} = c_{rs}E_{rs}.$$

But C is a sum of these matrices and hence is an element of $\mathfrak{A}$. Therefore $A_n \subseteq \mathfrak{A}$ and the proof is complete.

1.22 Lemma. If R is a ring with identity element, then the complete matrix ring R_n is a prime ring if and only if R is a prime ring.

Proof: By Theorem 1.5 (ii) one notes that if R is not a prime ring, there exists nonzero elements a, b ϵ R such that aRb = 0. Therefore $aE_{11}R_n bE_{11} = 0$ with aE_{11} and bE_{11} nonzero and hence R_n is not a prime ring.

Conversely, if R_n is not a prime ring, there exists nonzero matrices $[a_{ij}]$ and $[b_{ij}]$ such that $[a_{ij}]R_n[b_{ij}] = 0$. Let p, q, r, s be fixed positive integers such that a_{pq} and b_{rs} are nonzero. For any y ϵ R, it follows that:

$$[a_{ij}](yE_{qr})[b_{ij}] = [a_{iq}]y[b_{rj}] E_{ij} = 0.$$

Therefore $a_{iq}yb_{rj} = 0$ for all i, j = 1, . . . , n, $a_{pq}Rb_{rs} = 0$ and R is not a prime ring by Theorem 1.5 (ii).

One is now able to prove the desired result for R_n.

1.23 Theorem. If P(R) is the prime radical of the ring, then $P(R_n) = (P(R))_n$.

Proof: Case 1. Let R be a ring with identity element. Lemma 1.21 shows the existence of an injective mapping $\varphi\colon A \longrightarrow A_n$ (A an ideal in R) of the set all ideals in R onto the set of all ideals in R_n. Moreover since $(R/A)_n \cong R_n/A_n$, R_n/A_n is a prime ring if and only if R/A is a prime ring. Hence P_n is a prime ideal in R_n if and only if P is a prime ideal in R. Thus, if $[P_i \mid i \epsilon G]$ is the set of all prime ideals in R, $P(R_n) = \cap (P_i)_n = (P(R))_n$.

Case 2. If R does not have an identity element, R can be embedded in a ring S with identity element such that R is an ideal of S.

By Theorem 1.19 P(R) = R $\cap$ P(S). However, from the above discussion

$P(S_n) = (P(S))_n$. Since R_n is an ideal of S_n apply Theorem 1.19 again and obtain $P(R_n) = R_n \cap P(S_n) = R_n \cap (P(S))_n = (R \cap P(S))_n = (P(R))_n$.

JACOBSON RADICAL

A. DEFINITIONS

The second radical that will be considered is the Jacobson radical. Historically this radical is the first significant radical of a ring without chain conditions. The radical proposed by N. Jacobson in 1945 makes use of the notion of quasi-regularity introduced by S. Perlis. In a form similar to the discussion of the prime radical this section will start with three definitions of the Jacobson radical. These three definitions will then be shown to be equivalent.

2.1 Definition. The Jacobson radical $\mathfrak{J}(R)$ of a ring R is the set of elements a in R such that aR is right quasi-regular.

2.2 Definition. The Jacobson radical $\mathfrak{J}(R)$ of a ring R is the intersection of all modular maximal right ideals in R.

2.3 Definition. The Jacobson radical $\mathfrak{J}(R)$ is the set of all elements a in R such that a is in the kernel of every homomorphism of R onto a primitive ring.

First the definition involving quasi-regularity will be considered. The concept of quasi-regularity can best be studied by the introduction of a new binary operation called "circle" composition. "Circle" composition is defined as follows: $a \circ b = a + b - ab$ for all a, b $\in$ R. One notes the operation is associative and that 0 is an identity. Further if R is a ring with identity element, then $a \circ b = 0$ if and only if $(1-a)(1-b) = 1$.

2.4 Definition. Let a be an element of the ring R.

(i) If there exists an element b of R such that $a \circ b = 0$, a is said to be right quasi-regular (r.q.r.) and to have b as a right quasi-inverse (r.q.i.). Left quasi-regular (l.q.r.) and left quasi-inverse (l.q.i.) are defined in an analogous manner. If an element a is both r.q.r. and l.q.r. it is said to be quasi-regular.

(ii) A right ideal or left ideal in R is said to r.q.r. (l.q.r. or q.r.) if each of its elements is r.q.r. (l.q.r. or q.r.).

The usual results for inverses apply to quasi-inverses, that is if a has a r.q.i., and l.q.i. its has a unique quasi-inverse and if a has more than one r.q.i. it has no l.q.i. Some of the properties of quasi-regularity will be stated in the following lemmas. These lemmas will be needed in showing that the three definitions of the Jacobson radical are equivalent.

2.5 Lemma. If a, b ϵ R such that ab is quasi-regular, then ba is right quasi-regular.

Proof: Let c be the r.q.i. of ab or $ab \circ c = ab + c - abc = 0$, then $(ba) \circ (-ba + bca) = b[(ab) \circ c]a = 0$. Therefore ba is r.q.r. with $-ba + bca$ as its r.q.i.

2.6 Lemma. If a right ideal A in R is right quasi-regular, it is quasi-regular.

Proof: Let a be in A, then there exists an element b in R such that $a \circ b = 0$. Then b has a as its l.q.i. But from this one has $b = ab - a$ which must be in A. Therefore b has a r.q.i. which must be a. Hence a is quasi-regular.

2.7 Lemma. An element a of the ring R is right quasi-regular if and only if the right ideal $A = [ar - r \mid r \in R]$ coincides with R.

Proof: If there exists $r \in R$ such that $a \circ r = a + r - ar = 0$, then a is an element of A. Therefore for each $r \in R$, $ar \in A$. But $ar - r \in A$ and hence $r \in A$. This implies $R \subseteq A$ and therefore R = A.

Conversely if A = R, then there exists $r \in R$ such that $a = ar - r$ or $a \circ r = 0$.

Using Definition 2.1 the following property of the Jacobson radical can be proved.

2.8 Theorem. $\mathfrak{J}(R)$ is a quasi-regular ideal in R which contains every quasi-regular right ideal and every quasi-regular left ideal in R.

Proof: If $a \in \mathfrak{J}(R)$, then aR is r.q.r. Therefore for r in R one has $(ar)R \subseteq aR$ and it follows that (ar)R is r.q.r. and ar is in $\mathfrak{J}(R)$. Note then that for each r, s in R, ars is r.q.r. and by Lemma 2.5 sar is r.q.r. Therefore $\mathfrak{J}(R)$ is closed under multiplication by elements from R.

Let r be in R with b as the r.q.i. of ar where $a \in \mathfrak{J}(R)$ or $ar \circ b = 0$. Now let c be in $\mathfrak{J}(R)$, then $c(-r+rb)$ is r.q.r. Let $c(-r+rb)$ have d as its r.q.i. that is $[c(-r+rb)] \circ d = 0$. It follows that $[(a-c)r] \circ [b \circ d] = 0$. Therefore $(a-c)R$ is r.q.r. and $a-c \in \mathfrak{J}(R)$ which completes the proof that $\mathfrak{J}(R)$ is an ideal of R.

Now consider a in $\mathfrak{J}(R)$, then aR is r.q.r. and specifically $a^2 \circ b = 0$, so $a \circ [(-a) \circ b] = [a \circ (-a)] \circ b = a^2 \circ b = 0$. Therefore a has $(-a) \circ b$ as its r.q.i. Hence every element $\mathfrak{J}(R)$ is r.q.r. and by Lemma 2.6, $\mathfrak{J}(R)$ is quasi-regular.

If I is a quasi-regular right ideal in R, then a in I implies $aR \subseteq I$ and $a \in \mathfrak{J}(R)$. Hence $I \subseteq \mathfrak{J}(R)$.

Let I be a quasi-regular left ideal in R, then b ε I implies Rb ⊆ I and Rb is quasi-regular. By Lemma 2.5 bR is r.q.r. and by Lemma 2.6 bR is q.r. Hence I ⊆ $\mathfrak{J}$ (R).

2.9 Corollary. $\mathfrak{J}$ (R) contains every nil right (left) ideal in R.

Proof: This follows by noting that every nilpotent element of R is quasi-regular. For if $a^n = 0$ then $b = -\sum_{k=1}^{n-1} a^k$ is the q.i. of a.

On the basis of the above result one can prove a result similar to Theorem 1.20.

2.10 Theorem. If φ is a homomorphism of the ring R onto the ring S, then $\mathfrak{J}$ (R) $\varphi \subseteq \mathfrak{J}$ (S).

Proof: If a, b ε R then since φ is a homomorphism, φ preserves ∘. Therefore if a∘b = 0, then aφ∘bφ = 0. Hence $\mathfrak{J}$ (R)φ is a quasi-regular ideal in S and must be contained in $\mathfrak{J}$ (S) by Theorem 2.8

Before preceding to prove that Definitions 2.2 and 2.3 are also satisfactory for the Jacobson radical it is necessary to establish several properties of primitive rings.

2.11 Definition. A ring which is isomorphic to an irreducible ring of endomorphisms of some abelian group is called a primitive ring.

The concept of an irreducible ring of endomorphisms will be defined in terms of an R-subgroup.

2.12 Definition. If R is a ring of endomorphism of the abelian group G and H is a subgroup of G such that HR ⊆ H, then H is called an R-subgroup of G.

2.13 Definition. Let R be a nonzero ring of endomorphisms of the abelian group G. If the only R-subgroups H of G are H = 0 and H = G, then R is an irreducible ring of endomorphisms of G.

There will be need for the following result.

2.14 Theorem. If R is a nonzero ring of endomorphisms of an abelian group G, then R is an irreducible ring of endomorphisms of G if and only if xR = G for every nonzero element x of G.

Proof: If xR = G for every nonzero element x of G, then G is the only nonzero R-subgroup of G and hence R is irreducible.

Conversely, let R be an irreducible ring of endomorphisms of G and let x be in G. Observe that xR is a subgroup of G. Further $(xR)R \subseteq xR$ and xR is an R-subgroup of G. Therefore xR = 0 or xR = G. Let xR = 0 and define $[x] = [nx \mid n$ is an integer] which is a subgroup of G. Then $[x]R = 0$ and therefore $[x]$ is an R-subgroup of G. For $x \neq 0$ and $x \in [x]$, $[x]$ must equal G so GR = 0. But this contradicts the fact that R has nonzero elements. Hence $xR \neq 0$ which implies xR = G.

Consider next the following results for maximal and modular right ideals.

2.15 Theorem. If A is a maximal right ideal in the ring R such that $R^2 \not\subseteq A$, then the ring R/(A:R) is isomorphic to an irreducible ring of endomorphisms of the abelian group G, where $(A:R) = [r \mid r \in R, \ Rr \subseteq A]$ and $G = R^+ - A^+$ is a difference group.

Proof: The difference group G under the usual definition of addition of cosets forms an abelian group. Further if a composition is defined from

GxR into G by $(x+A)r = xr+A$, $x+A \in G$, $r \in R$, this composition makes G an R-module. From this composition and the fact that the zero of G is the coset A one has an ideal in R denoted $(A:R)$ and defined to be the set of r in R such that $Rr \subseteq A$. Further these mappings $x+A \to xr+A$ form a ring of endomorphisms, say S, of the abelian group G. And in fact the ideal $(A:R)$ is the kernel of a homomorphism of R onto S. The Fundamental Theorem of Homomorphisms assures then that $S \cong R/(A:R)$.

Since $R^2 \not\subseteq A$, then $(A:R) = R$. Hence $R/(A:R) \cong S$ has nonzero elements. S is irreducible because the right ideal $B = [b \mid b \in R, \ bR \subseteq A]$ is a right ideal such that $A \subseteq B$. Since $B \neq R$ and $R^2 \not\subseteq A$ the maximality of A implies $B = A$. If c is in R and c is not in A, then c is not in B and $cR \not\subseteq A$. The maximality of A then implies $A + cR = R$. But c not in A says that $c + A$ is a nonzero element of G. Since $(x + A)s = xs + A$ it follows that $(x + A)S = G$. Therefore by Theorem 2.14 S is an irreducible ring of endomorphisms of the group G.

The following definition and theorem will complete the discussion of maximal and modular right ideals.

2.16 Definition.   A right ideal I in a ring R is said to be a modular right ideal if there exists an element e of R such that $er-r \in I$ for every element r of R.

Note if R has an identity then every right ideal in R is modular. If A is also maximal, then A is called a modular maximal right ideal.

2.17 Theorem.   If A is a modular maximal right ideal in the ring R, then

R/(A:R) is a primitive ring. The ring R is primitive if and only if it contains a modular maximal right ideal A such that (A:R) = (0).

Proof: Let A be a modular maximal right ideal in R. Then there exists an element e in R such that er-r ϵ A for all r in R. Since A is maximal A $\neq$ R and there exists s in R such that s is not in A. Therefore es is not in A which implies $R^2 \not\subseteq$ A. Then Theorem 2.15 yields R/(A:R) isomorphic to an irreducible ring of endomorphisms so R/(A:R) is a primitive ring.

Let R be an irreducible ring of endomorphisms of the abelian group G, and let x be a fixed nonzero element of G. Define A = [a | a ϵ R, xa = 0]. A is a right ideal in R. Since x $\neq$ 0 Theorem 2.14 implies xR = G, and hence A $\neq$ R. Let c be an element in R, but not in A. Define B = A+(c)$_r$. Then since xc $\neq$ 0 and xB $\neq$ 0, (xB)R $\subseteq$ xB. The irreducibility of R implies that xB = G. Hence there exists for each element s of R, b ϵ B such that xb = xs. Therefore x(b-s) = 0 and b-s ϵ A. But A $\subseteq$ B and s must be in R so R = B = A+(c)$_r$. Since this is true for every element s in R and s not in A, A is maximal.

If xR = G then there exists an element e of R such that xe = x. Then x(er-r) = 0 since xer = xr for every r in R. Therefore er-r ϵ A and A is modular. If a ϵ (A:R), then Ra $\subseteq$ A. Then xRa = 0 and since xR = G, Ga must equal zero. But Ga = 0 implies a = 0 and (A:R) = (0).

Conversely if (A:R) = (0), then R $\cong$ R/(A:R) and from the first part of the theorem R is a primitive ring.

The above results lead to some useful properties of the Jacobson radical.

2.18 Theorem. If R is a primitive ring, then $\mathfrak{J}$ (R) = (0).

Proof: Let R be an irreducible ring of endomorphisms (a primitive ring) of an abelian group G. If $a \in \mathfrak{J}(R)$, let $xa \neq 0$ for some $x \in G$, then $xaR = G$ by Theorem 2.14. Therefore there exists $b \in R$ such that $x(ab) = x$. But ab is right quasi-regular and there exists $c \in R$ such that $ab+c-abc = 0$. Therefore $x = x-x(ab+c-abc) = 0$, which contradicts $xa = 0$. Hence $xa = 0$ for all $x \in G$ which says $a = 0$.

2.19 Theorem. If $b \in R$ such that b is not in $\mathfrak{J}(R)$, then there exists in R a modular maximal right ideal which does not contain b.

Proof: If $b \neq \mathfrak{J}(R)$, then there exists at least one element c in R such that bc does not have a right quasi-inverse. Hence the right ideal $A = [bcr-r \mid r \in R]$ does not contain bc. Zorn's Lemma shows the existence of a right ideal B which is maximal in the set of those right ideals which contain A but not the element bc. B is maximal in R for any right ideal B_1, such that $B \subset B_1$ implies $bc \in B_1$ and $bcr-r \in B_1$ for each r in R. Therefore $B_1 = R$. The modularity follows from the definition of B. Hence B does not contain bc and therefore does not contain b and is the desired right ideal.

The equivalence of Definitions 2.2 and 2.3 can now be obtained.

2.20 Theorem. Let R be a ring such that $\mathfrak{J}(R) \neq R$, then

(i) $\mathfrak{J}(R) = \cap [A_i \mid i \in G]$,

(ii) $\mathfrak{J}(R) = \cap [(A_i:R) \mid i \in G]$,

where $[A_i \mid i \in G]$ is the set of all modular maximal right ideals in R.

Proof: Theorem 2.19 shows the existence of modular maximal right ideals since $\mathfrak{J}(R) = R$. It also follows that the intersection of the A_i is

contained in $\mathfrak{J}$ (R). Now consider $R/(A_i:R)$ which is primitive by Theorem 2.17. By Theorem 2.18 $\mathfrak{J}[R/(A_i:R)] = 0$, whence the natural homomorphism of R onto $R/(A_i:R)$ and Theorem 2.10 imply that $\mathfrak{J}$ (R) $\subset \cap (A_i:R)$. Hence R $\mathfrak{J}$ (R) $\subseteq A_i$. Since the A_i are modular, $\mathfrak{J}$ (R) $\subseteq A_i$ and $\mathfrak{J}$ (R) $\subseteq \cap A_i$. To establish part (ii) of the theorem observe that $(A_i:R) \subseteq A_i$. Hence $\cap (A_i:R) \subseteq \cap A_i = \mathfrak{J}$ (R) by part (i). This completes the proof of the theorem.

2.21 Theorem. If a ϵ R, then a $\epsilon \mathfrak{J}$ (R) if and only if a is in the kernel of every homomorphism of R onto a primitive ring.

Proof: Let φ be a homomorphism from R onto a primitive ring S. Then by Theorem 2.10, $\mathfrak{J}$ (R) $\varphi \subseteq \mathfrak{J}$ (S). But by Theorem 2.18, $\mathfrak{J}$ (S) = 0 and hence $\mathfrak{J}$ (R) is in the kernel of φ.

Conversely if a is in R such that a is in the kernel of every homomorphism of R onto a primitive ring, then the result is trivial when $\mathfrak{J}$ (R) = R. If $\mathfrak{J}$ (R) $\neq$ R and I_i, i ϵ G are the modular maximal right ideals in R, then by Theorem 2.17 a is in $(I_i:R)$ for each i. Therefore by Theorem 2.20 part (ii), a is in $\mathfrak{J}$ (R).

B. JACOBSON RADICAL-PROPERTIES

Results analogous to the results for the prime radical will now be proved. The case of the radical in a homomorphic image of a ring has already been considered in Theorem 2.10. The next two theorems are similar to the results of Theorems 1.19 and 1.23.

2.21 Theorem. If I is an ideal in the ring R, then $\mathfrak{J}(I) = I \cap \mathfrak{J}(R)$.

Proof: If $a \in I \cap \mathfrak{J}(R)$, then $a \in \mathfrak{J}(R)$ and ab is right quasi-regular for each b in R. Hence there exists $c \in R$ such that $ab+c-abc = 0$ or $c = abc-ab$ and c is in I. Therefore aI is a r.q.r. ideal in the ring I and $a \in \mathfrak{J}(I)$.

Conversely, if $a \in \mathfrak{J}(I)$, then $(aR)^2 \subseteq aI$ and $(aR)^2$ is a q.r. right ideal in R. Since $\mathfrak{J}(R)$ is a semi-prime ideal in the ring R, it follows that $aR \subseteq \mathfrak{J}(R)$ and a is in $\mathfrak{J}(R)$. Hence $\mathfrak{J}(I) \subseteq I \cap \mathfrak{J}(R)$ and the proof is complete.

2.22 Theorem. If R_n is the complete matrix ring over R, then $\mathfrak{J}(R_n) = (\mathfrak{J}(R))_n$.

Proof: Consider $A = [A_{io}]$ where $A_{io} = 0$ for $i = 1$. If a_{11} is right quasi-regular, then there exists $b_{11} \in R$ such that $a_{11} \circ b_{11} = 0$. By Lemma 2.17 $[a_{11}r-r \mid r \in R] = R$. Hence there exists b_{1j} of R such that $a_{11}b_{1j} = a_{1j}$ for $j = 2, 3, \ldots, n$. Let $B = [B_{io}]$ where $B_{io} = 0$ for $i = 1$. Then $A \circ B = 0$ and A is r.q.r. in R_n.

Let Q_i be the set of all matrices of $(\mathfrak{J}(R))_n$ such that $Q_i = [Q_{io}]$ where $Q_{io} = 0$ for all but at most one i, that is a matrix with zeros in all but at most one row. If $a_{ii} \circ b_{ii} = 0$, then by the above argument every element of Q_i is r.q.r. in R_n. As Q_i is a right ideal contained in $(\mathfrak{J}(R))_n$, by Lemma 2.6 and Theorem 2.8 $Q_i \subseteq \mathfrak{J}(R_n)$ for $i = 1, \ldots, n$. Since $(\mathfrak{J}(R))_n$ can be

written as a direct sum $Q_1 \oplus Q_2 \oplus \ldots \oplus Q_n$ it follows that

$(\mathfrak{J}(R))_n \subseteq \mathfrak{J}(R_n)$.

Let $A = (a_{ij})$ be an element in $\mathfrak{J}(R_n)$. Then for arbitrary integers p and q where $1 \leq p \leq n$, $1 \leq q \leq n$, one has:

$$rE_{1p}AsE_{q1} = ra_{pq}sE_{11} \in \mathfrak{J}(R_n).$$

If $B = (b_{ij})$ is a r.q.i. of $ra_{pq}sE_{11}$ in R_n, then b_{11} is a r.q.i. of $ra_{pq}s$ in R. Hence the ideal $Ra_{pq}R$ is r.q.r. and by Theorem 2.8 it follows that $Ra_{pq}R \subseteq \mathfrak{J}(R)$. Further since $\mathfrak{J}(R/\mathfrak{J}(R)) = (0)$ and Corollary 2.9 imply $\mathfrak{J}(R)$ is a semi-prime ideal in R, then $a_{pq} \in \mathfrak{J}(R)$. Therefore $A \in (\mathfrak{J}(R))_n$ and $\mathfrak{J}(R_n) \subseteq (\mathfrak{J}(R))_n$.

THE RADICALS IN RINGS WITH DESCENDING CHAIN CONDITION

Up to this point there has been no restrictions placed on the ring R. In this section the radicals will be considered in rings with descending chain condition (DCC). This is a very strong restriction and will lead to fact that in rings with DCC the radicals coincide. First consider the following theorem which shows the relationship between the prime radical and the Jacobson radical.

3.1 Theorem. Let $P(R)$ and $\mathfrak{J}(R)$ respectively be the prime and Jacobson radicals of a ring R, then $P(R) \subseteq \mathfrak{J}(R)$.

Proof: By Corollary 2.8 $\mathfrak{J}(R)$ contains every nil right (left) ideal in R. Moreover Theorem 1.17 asserts that $P(R)$ is a nil ideal. Hence $P(R) \subseteq \mathfrak{J}(R)$.

One naturally asks now if the prime radical and Jacobson radical are ever distinct. The following example will show this to be the case.

3.2 Example. Consider K a subset of the rational numbers of the form $\frac{2p}{2q+1}$ where p and q are integers. The set K is a commutative ring under the usual operations of addition and multiplication. Since there are no nonzero nilpotent elements in K, $P(K) = (0)$ by Corollary 1.18. Now consider the following elements of K and their "circle" composition:

$$\frac{2p}{2q+1} \quad \circ \quad \frac{-2p}{2(q-p)+1} \quad = 0$$

Hence every element of K is quasi-regular and $\mathfrak{J}(K) = K$. Therefore $P(K) \subseteq \mathfrak{J}(K)$.

The next theorem will give a property of $\mathfrak{J}(R)$ in a ring with DCC.

3.3 Theorem. In a ring R with DCC, $\mathfrak{J}(R)$ is nilpotent.

Proof: First note that $\mathfrak{J}(r) \supseteq \mathfrak{J}^2(R) \supseteq \ldots$, and since R has the DCC there exists a positive integer n such that $\mathfrak{J}^n(R) = \mathfrak{J}^{n+1}(R) = \ldots$. If one sets $I = \mathfrak{J}^n(R)$, then $I \subseteq \mathfrak{J}(R)$ and $1^2 = \mathfrak{J}^{2n}(R) = \mathfrak{J}^n(R) = I$. Assume $I \neq (0)$ and consider the set of those right ideals A of R such that $A \subseteq I$ and $AI \neq (0)$. There exists at least one such ideal, namely $A = I$. Hence there exists a minimal ideal of this set since R has the DCC, say M. Let $x \in M$ such that $xI \neq (0)$. Then $(xI)I = xI \neq 0$ and $xI = M$ as $xI \subseteq M$ and M is minimal. Therefore there exists an element a of I such that $xa = x$. Since $I \subseteq \mathfrak{J}(R)$, a is right quasi-regular. Let b be the right quasi-inverse of a. Then it follows that:

$$x = x - x(a+b-ab) = x - xa - x(b-ab) = 0,$$

which contradicts $xI \neq (0)$. Then the assumption that $I \neq (0)$ is false and $\mathfrak{J}(R)$ is nilpotent.

3.4 Theorem. In a ring R with DCC, the prime radical of R and the Jacobson radical of R coincide, that is $P(R) = \mathfrak{J}(R)$.

Proof: By Theorem 3.1, $P(R) \subseteq \mathfrak{J}(R)$. Hence from Theorems 3.3 and 1.17 the result follows.

Appendix

$\subseteq$: containment,

$\subset$: proper containment,

(a)   : ideal generated by a,

$(a)_r$: right ideal generated by a,

$\mathfrak{C}A$: complement of the set A,

$P(I)$: prime radical of the ideal I,

$P(R)$: prime radical of the ring R,

E_{ij} : standard matrix unit,

R_n : complete matrix ring over the ring R,

$\cong$: isomorphic to,

$\longrightarrow$: mapped into,

R/I : difference ring,

$G-H$: difference group,

R^+ : additive group of the ring R,

A_{io} : the i th column vector of the matrix A,

DCC : descending chain condition for right ideals,

$\circ$: circle composition,

$\mathfrak{J}(R)$: Jacobson radical of the ring R,

ACKNOWLEDGMENT

The writer expresses his sincere appreciation to Dr. Robert E. Williams for his patient guidance and supervision during the preparation of this report.

REFERENCES

Baer, Reinhold. "Radical Ideals," American Journal of Mathematics, v. 65 (1943), p. 537-568.

Brown, Bailey and McCoy, Neal H. "Radicals and Subdirect Sums," American Journal of Mathematics, v. 69 (1947), p. 46-58.

Divinsky, N. J. Rings and Radicals, Mathematical Expositions No. 14. Toronto: Toronto University Press, 1965.

Fuller, Leonard E. Matrix Theory. Notes (unpublished).

Jacobson, Nathan. Lectures in Abstract Algebra, Volume I - Basic Concepts. New York: D. Van Nostrand Company, Inc. 1951.

_____. "The Radical and Semi-Simplicity for Arbitrary Rings," American Journal of Mathematics, v. 67 (1945) p. 300-320.

_____. Structure of Rings. American Mathematical Society Colloquium Publications Volume XXXVII. Providence, R. I.: American Mathematical Society, 1956.

Lambek, Joachim. Lectures on Rings and Modules. Waltham, Mass.: Blaisdell Publishing Company, 1966.

McCoy, Neal H. Rings and Ideals, The Carus Mathematical Monographs, Number 8. Buffalo, N. Y.: The Mathematical Association of America. 1948.

_____. The Theory of Rings. New York: The Macmillian Company, 1964.

Paley, Hiram and Weichsel, Paul M. A First Course in Abstract Algebra. New York: Holt, Rinehart and Winston, Inc. 1966.

Perlis, Sam. "A Characterization of the Radical of an Algebra," American Mathematical Society Bulletin. v. 48 (1942), p. 128-132.

TWO RADICALS OF A RING

by

GARY DEAN GABRIELSON

B. S., Kansas State University, 1963

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1968

The purpose of this report is to consider the properties of two radicals of a ring. The study of radicals is helpful in the classification of rings, although the classification of rings will not be considered directly in this report. For example, in rings with descending chain condition, the classical radical yields significant results. This report will consider two radicals in an arbitrary ring and then show them to coincide in rings with descending chain condition.

The two radicals considered are the prime radical and the Jacobson radical. For each radical three equivalent definitions are stated and several properties are proved. Then the radicals in a ring R, and rings related to R are considered. The related rings considered are ideals in the ring R, homomorphic images of R and the ring of n by n matrices over R.

In the final section it is established that the Jacobson radical contains the prime radical and an example is given to show that the two radicals are distinct. Also the Jacobson radical is shown to be nilpotent in a ring with descending chain condition. This result together with earlier results shows that the prime radical and the Jacobson radical coincide in rings with descending chain condition.