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PERFORMANCE OF A FREQUENCY-HOPPED NON-COHERENT QUADRATURE PHASE SHIFT KEYED MODULATOR (QPSK) IN JAMMING AND FADING ENVIRONMENTS

By

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SECTION I

INTRODUCTION

1.1 Overview

This report examines the performance of a frequency-hopped Quadrature Phase Shift Keyed (QPSK) modulator for a partially coherent communications environment. The performance is examined under different channel conditions such as noise jamming and fading. The performance criteria used is the average bit error probability.

Analytical equations for the performance are derived and the results are computed using numerical methods. The performance characteristics are plotted for these results.

1.2 Modulation System

Among the wide class of digital modulation techniques used in satellite communications, the more important techniques include Amplitude Shift Keying (ASK), Frequency Shift Keying (FSK), and Phase Shift Keying (PSK). However, among these techniques, the only band-width efficient modulation techniques are Quadrature Phase-Shift Keying (QPSK) and Quadrature Amplitude Shift Keying (QASK).

In many military communications systems, an issue which ordinarily must be addressed is the threat of jamming. The jammer intercepts a transmitted signal, performs some real-time signal processing upon it, and radiates a derived signal of his choosing in the direction of the intended receiver terminal to jam the originating communications from the transmitter. In the absence of enough protection, such "intelligent" jamming can be far more disruptive than the more commonly encountered wide-band threats. Frequency-hopped (FH) spread spectrum systems have been found as a good remedy which when super-imposed on the conventional techniques will resist the intentional interference introduced by the jammer. Although FH techniques have received a great deal of attention over the years, not many operational systems have been constructed because of the extra complex and expensive sub systems involved.

A frequency hopping modulator is no more than a code sequence generator similar to those employed in direct sequence systems, driving a frequency synthesizer that it commands to hop from frequency to frequency in a pattern that is determined by the code sequence being generated. There are two possible ways to implement the FH modulation.

o **Non-coherent FH**

The spread spectrum technique is reffered to as non-coherent FH if the individual hop pulse phases bear no relation to each other.

o **Coherent FH**

If the phase continuity is maintained from one hop pulse to another, then the spread spectrum technique is reffered to as coherent FH.

However, in either case, it is assumed in this report that the hop pulse phase is constant over a single hop interval. The performance of coherent FH modulation scheme is always superior to that of non-coherent scheme. However, it is not always possible to implement coherent schemes. Particularly, in a frequency hopped system where either the transmitter or the receiver is mobile, the doppler shift can cause phase error. Another source of phase error can be intentional or unintentional interference. Even with some phase error, it may be preferable to implement a coherent FH scheme as long as this phase error is not large enough to significantly degrade the performance. In this report, the performance of FH coherent scheme is analyzed in the presence of phase error and jamming. Since in all of the results derived in this report, phase error is assumed to be present, this scheme is called partially-coherent FH scheme. The main structure of the modulation system is shown in Figure 1.1.

The transmitted signal may undergo fading, additive noise, and jamming. The received signal is a sum of additive white Gaussian noise, the jammer, and a random phase shifted version of the transmitted signal. This signal is first frequency dehopped and then demodulated. It is assumed that the "intelligent" jammer has the knowledge of data rate, spreading bandwidth, and hop rate, but no knowledge of the frequency hopping code. The jammer will spread his energy over a fraction of the whole bandwidth to cause the maximum bit error rate. The analysis in this report assumes the fraction of the total bandwidth jammed as a parameter. The bit error rate is examined as a function of this parameter. The performance characteristics of the QPSK modulator can be examined separately under fading and non-fading channel conditions. Under the fading media it could be assumed that the signal will suffer a slow Rician fading impairment, so that the random parameters do not change appreciably during the transmission interval (i.e., over one hop interval).

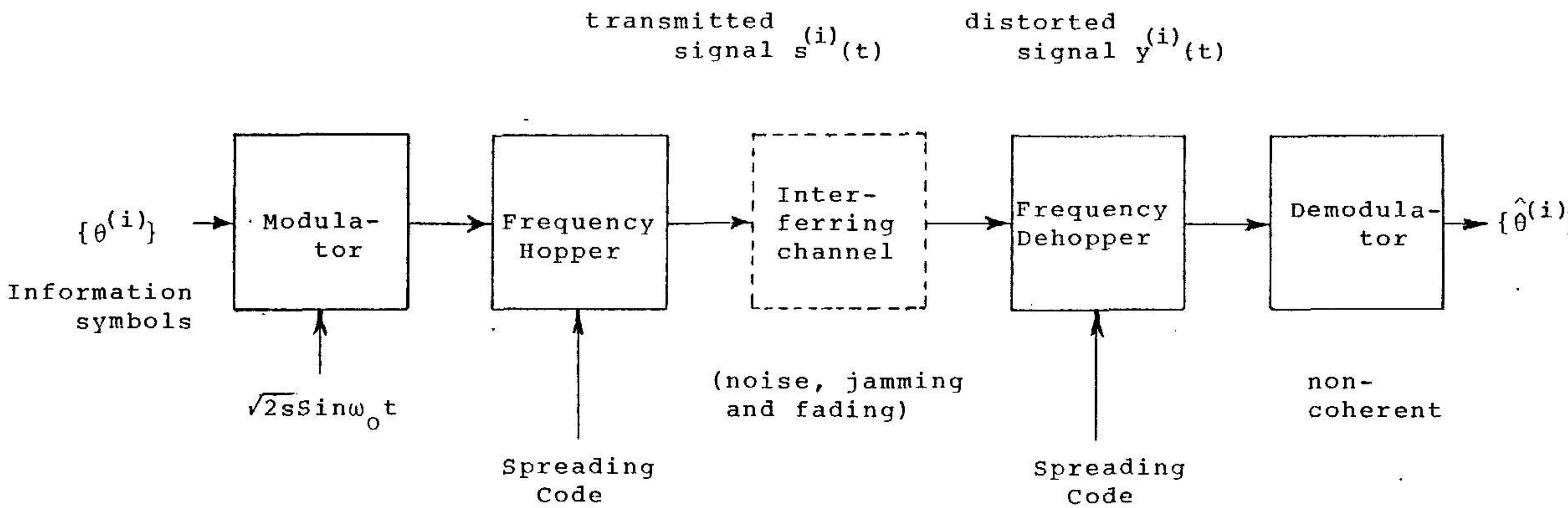


FIGURE 1.1

BLOCK DIAGRAM OF THE MODULATION SYSTEM

Therefore the first order density distribution of the random parameters will suffice to describe the fading media. Further, it can be assumed that the transmission path of the fading media is composed of a fixed component, known as specular component and a random component, known as scatter component.

The average bit error probabilities are computed and compared for different cases, as functions of signal-to-noise ratio (SNR) and signal-to-jammer ratio (SJR). The error probabilities are plotted for easier comparison.

1.3 Background Work

The analysis presented in this report is an extension of the work done by Marvin K. Simon and Polydoros [1] on coherent detection of FH-QPSK modulations in the presence of jamming. It was assumed in Simon's paper that the signals undergo ideal coherent demodulation by the frequency hopper so that the phase continuity is maintained from one hop pulse to another. However, this is not usually true in practical communications systems. There will be some phase error either due to doppler shift or due to intentional or unintentional interference. The modulation scheme employed in this analysis, called partially coherent FH scheme, assumes that some phase error is present in the received signal. The analysis technique follows the same procedure adopted by M. K. Simon. However, due to the presence of this phase error, the closed form computation of bit error rate probability turns out to be more complicated and the results can not be reduced mathematically in terms of elementary functions. Numerical methods are used as an alternative in arriving at the final results.

SECTION II

THE SYSTEM MODEL

The system model for the frequency-hopped QPSK modulator is presented in this section. Section 2.1 introduces a special density function $f_\theta(\theta;\Lambda_m)$ which is used later to examine the performance characteristics of the partially-coherent FH-QPSK modulator. Section 2.2 describes the general mathematical model for the non-fading communication media. The channel interferences are the additive white Gaussian noise n(t) and the jammer j(t). The model for the fading communication media is described in Section 2.3.

2.1 Special Density Function $f_\theta(\theta;\Lambda_m)$

In this study, instead of choosing a particular density function, a family of densities indexed by a single parameter is specified for the analysis It is important to choose a family, that will enable one to model as many cases of interest as possible. A family that will turn out to be useful is given below[2]:

$$f_\theta(\theta;\Lambda_m) = \frac{\exp(\Lambda_m \cos\theta)}{2\, I_0(\Lambda_m)}; \quad -\pi < \theta < +\pi \tag{2.1}$$

Figure 2.1 illustrates this family of density functions for different values of Λ_m. The function $I_0(\Lambda_m)$ is a modified Bessel function of the first kind which is included so that the density will integrate to unity. Λ_m is a parameter that controls the spread of the density.

From the Figure 2.1, it is seen that for $\Lambda_m = 0$;

$$f_\theta(\theta)\Big|_{\Lambda_m=0} = 1/2\pi \; ; \; -\pi < \theta < +\pi \qquad (2.2)$$

which indicates a uniform probability density function between $-\pi$ and $+\pi$. This is a logical density function for the radar problem. As Λ_m increases, the density becomes more peaked. Finally as Λ_m approaches infinity, it approaches the known signal case. Thus by varying Λ_m, it is possible to move continuously from the known signal problem to the other extreme, the uniform random variable problem.

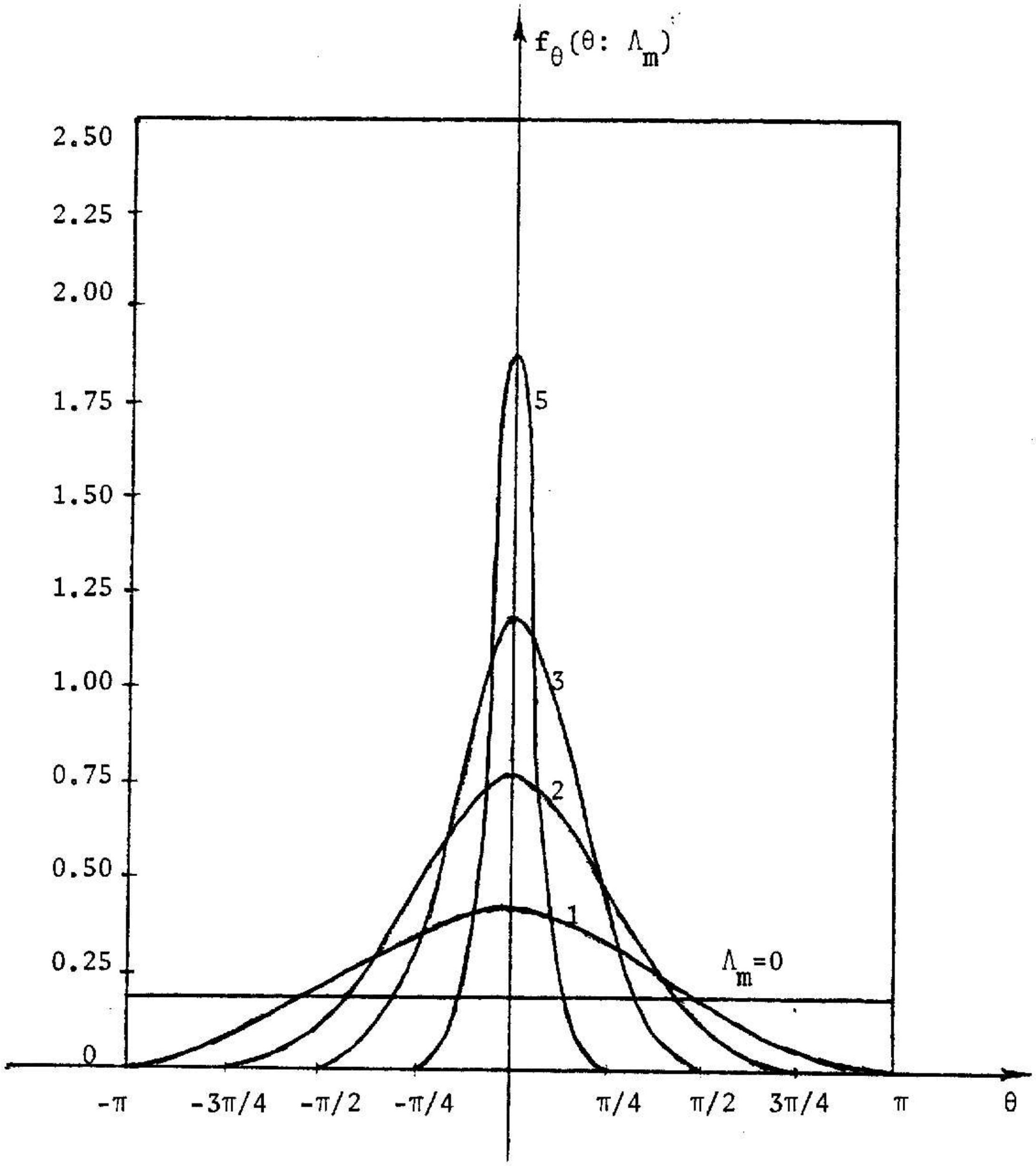


FIGURE 2-1

FAMILY OF PROBABILITY DENSITIES $f_\theta(\theta: \Lambda_m)$

2.2 Mathematical Model - Non-fading Communications Media

The block diagrams for the frequency-hopped QPSK modulator and QPSK demodulator are given in Figure 2.2.

The quadriphase modulation encodes each pair of bits into one of four phases (Figure 2.3). Alternately, one can code each pair of bits into a change in phase. One of the principle advantages of QPSK is that, under certain transmission conditions, QPSK achieves the same power efficiency as BPSK using only half the bandwidth.

The ideal QPSK signal waveform can be represented in two equivalent forms:

$$s(t) = A\mathrm{Sin}[\omega_0 t + \theta_m(t)]$$

or

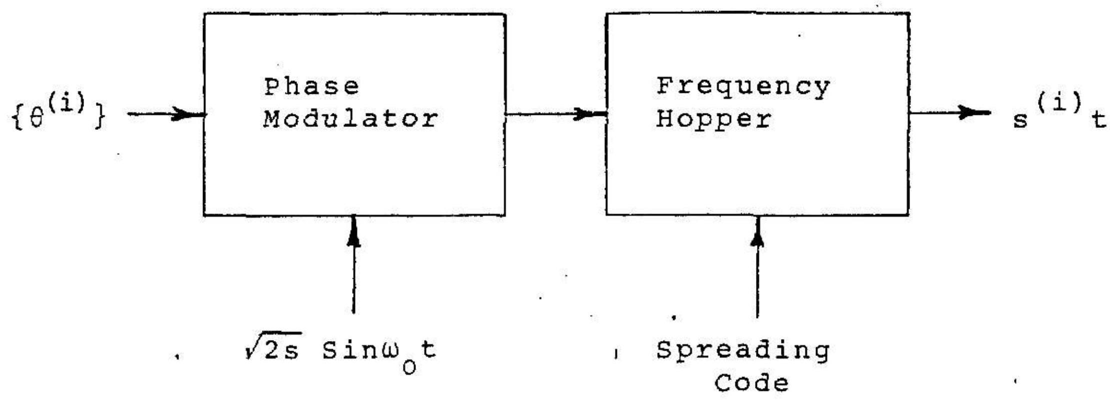
$$s(t) = \frac{A}{\sqrt{2}} U_s(t)\mathrm{Sin}[\omega_0 t + \frac{\pi}{4}] + \frac{A}{\sqrt{2}} U_c(t)\mathrm{Cos}[\omega_0 t + \frac{\pi}{4}] \tag{2.3}$$

where,

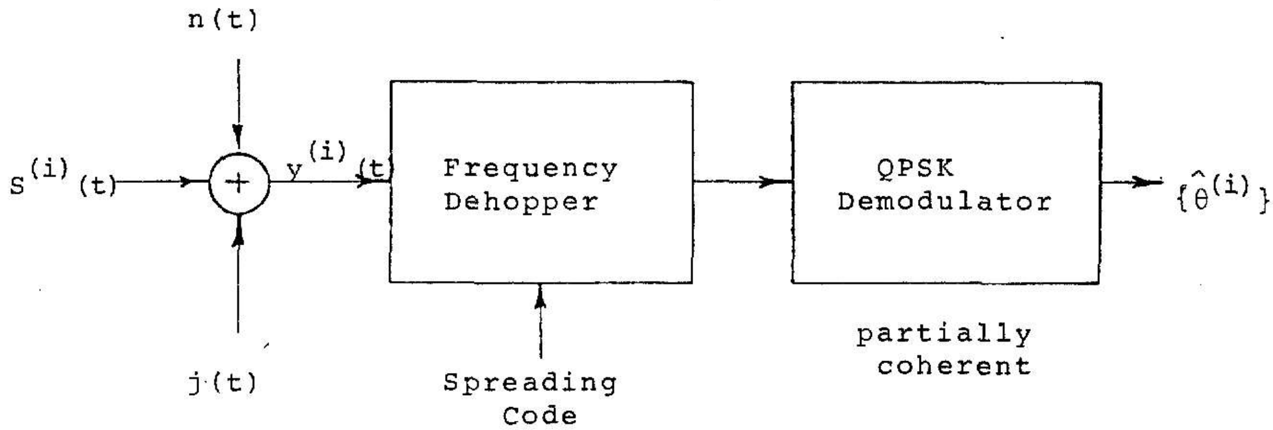
$$\theta_m = (0,\ \pi/2,\ \pi,\ 3\pi/2)$$

and U_s, $U_c = \pm 1$ represents the data modulation.
Each symbol lasts T_s seconds.

A frequency hopped QPSK signal in the ith signaling interval is represented by



(a) FH-QPSK Modulator Block Diagram



(b) FH-QPSK Demodulator Block Diagram

FIGURE 2.2

BLOCK DIAGRAM OF FH-QPSK MODULATION SYSTEM

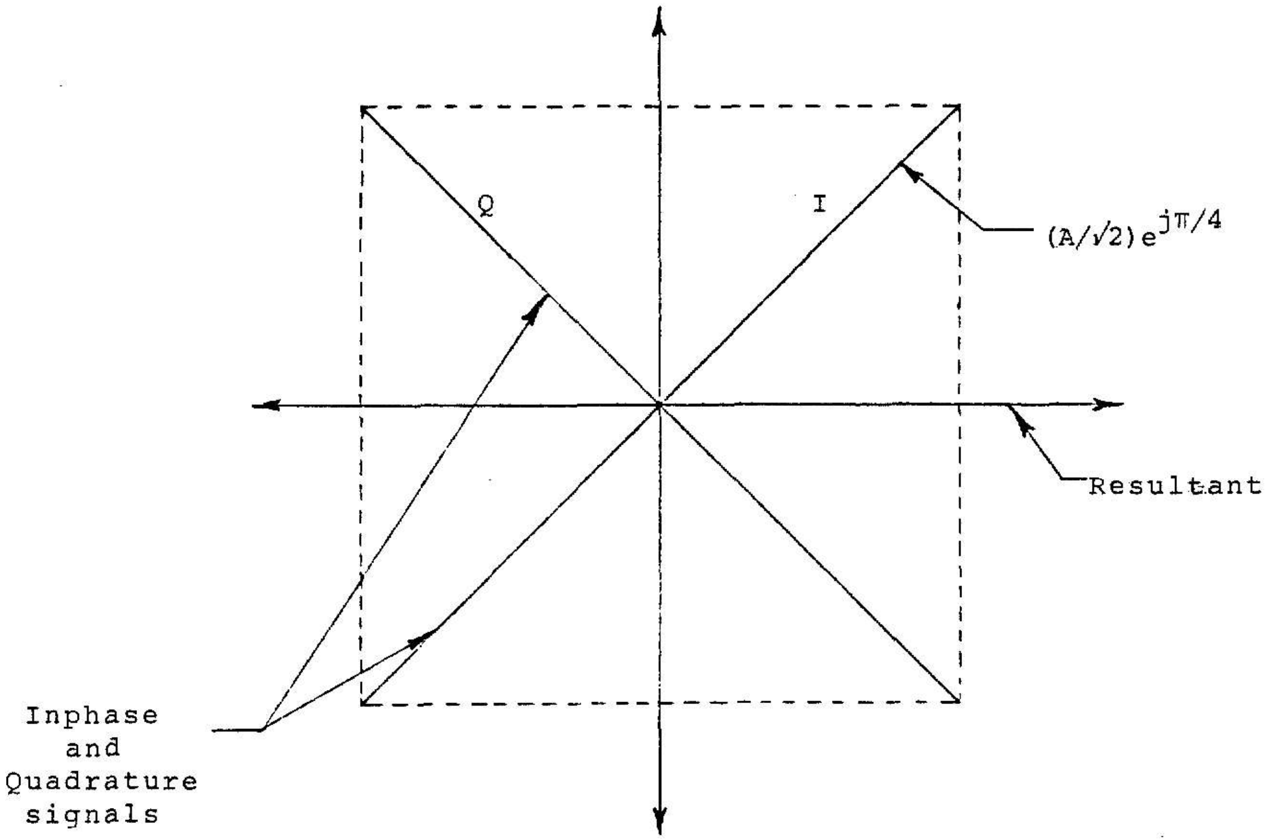


FIGURE 2.3

PHASOR REPRESENTATION OF QPSK MODULATION

$$s^{(i)}(t) = \sqrt{2S}\,\mathrm{Sin}[\omega_H^{(i)}t + \theta^{(i)}]; \quad (i-1)T_s \leq t \leq iT_s \tag{2.4}$$

where

S = transmitted average power,

$\omega_H(i)$ = particular carrier frequency selected by the frequency hopper for the given interval,

$\theta^{(i)}$ = the information symbol which is one of four possible values; $\pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$.

During the transmission, a white Gaussian noise $n(t)$ and a jammer $j(t)$ will be added. $n(t)$ is a bandpass noise and with a narrow band representation

$$n(t) = \sqrt{2}[N_c(t)\mathrm{Cos}(\omega_H^{(i)}t+\theta) - N_s(t)\mathrm{Sin}(\omega_H^{(i)}t+\theta)], \tag{2.5}$$

where $N_c(t)$ and $N_s(t)$ are statistically independent low pass white Gaussian noise processes with single sided noise spectral density N_o w/Hz, and $j(t)$ is a partial-band multitone jammer having n jammer tones.

Let J be the total jammer power evenly divided among the n tones, each tone having a power

$$J_o = J/n. \tag{2.6}$$

Since the jammer is assumed to have knowledge of hop frequencies, i.e., the exact location of the spreading bandwidth ω, and the number N of hops in this bandwidth, it is assumed that the jammer will randomly locate each of his n tones coincident with n of the N hop frequencies.

Thus the fraction of the total bandwidth which is continuously jammed with tones, each having power J_o, is given by

$$\alpha = (n/N). \tag{2.7}$$

Further, the jammer's strategy is to distribute his total power J in such a way as to cause the communicator to have maximum probability of error. At the receiver, the transmitted signal corrupted by noise and jammer will arrive as

$$y^{(i)}(t) = s^{(i)}(t;\theta) + n(t) + j(t) \tag{2.8}$$

where

$$s^{(i)}(t;\theta) = \sqrt{2s}\ \mathrm{Sin}(\omega_H^{(i)}t + \theta^{(i)} + \theta) \tag{2.9}$$

θ is the phase shift introduced by the channel,

n(t) is given by (2.5), and

$$j(t) = \sqrt{2J_o}\ \mathrm{Cos}(\omega_H^{(i)}t + \theta_j + \theta). \tag{2.10}$$

θ_j is the jammer phase and is a random variable, uniformly distributed between $(0, 2\pi)$ and independent of the information symbol phase $\theta^{(i)}$,

i.e.;

$$f_{\theta_j}(\theta_j) = \begin{cases} 1/2\pi; & 0 \leqslant \theta_j \leqslant 2\pi \\ 0; & \text{elsewhere} \end{cases} \quad (2.11)$$

Over an integral number of hop bands, the fraction α of the total number of signaling intervals will be characterized by (2.6). In the remaining fraction $(1-\alpha)$ of the signaling intervals, the received signal does not include the jammer and is simply given by

$$y^{(i)}(t) = s^{(i)}(t;\theta) + n(t). \quad (2.12)$$

The system performances and the calculation of probability of errors are described in Chapter 3.

2.3 Fading Communications Media

This section discusses the FH-QPSK model in a fading communications media. As mentioned earlier, a signal will generally undergo a slow Rician fading impairment in the communications channel.

An FH-QPSK signal in the ith signalling interval is represented by equation (2.4) and is repeated again for convenience:

$$s^{(i)}(t) = \sqrt{2s}\ \mathrm{Sin}[\omega_H^{(i)}t + \theta^{(i)}]; \quad (i-1)T_s \leqslant t \leqslant iT_s.$$

When this signal undergoes fading, it will no longer be in this form. Amplitude and carrier phase will be random variables and equation (2.4) can now be represented as

$$s_F^{(i)}(t,\theta_f) = \sqrt{2s}\ a_f e^{-j\theta_f} \mathrm{Sin}(\omega_H^{(i)}t + \theta^{(i)}) \qquad (2.13)$$

The fading media comprises of frequency non-selective path which is characterized by two quantities: a_f, the strength and θ_f, the carrier phase shift. These quantities are random and must therefore be described interms of probability density distributions. Since a slow fading environment is assumed, a_f and θ_f do not change appreciably during the transmission interval. Hence the first-order joint density distribution, $f_r(a_f, \theta_f)$ of the two path parameters will suffice to describe the fading medium. It is also assumed that a_f and θ_f are statistically independent of τ, modulation delay and hence only the distribution $f_r(a_f, \theta_f)$ is necessary to complete the description of the medium instead of $f_r(a_f, \theta_f, \tau)$.

It is further assumed that the transmission path is composed of a fixed component (specular component) and a random component (scatter component), so that equation (2.13) may be written as

$$s_F^{(i)}(t;\theta_f) = \sqrt{2S}\,(\gamma e^{-j\delta} + \beta e^{-j\varepsilon})\mathrm{Sin}(\omega_H^{(i)}t + \theta^{(i)}) \tag{2.14}$$

where, γ and δ are the strength and phase shift of the fixed component respectively, while β and ε are the strength and phase shift for the random component. The relationships among the quantities γ, δ, β, ε, a, and θ are depicted pictorially in Figure 2.4.

The joint distribution of β and ε are given by[4]

$$f_{\beta,\varepsilon}(\beta,\varepsilon) = \begin{cases} (\beta/2\pi\sigma^2)\exp(-\beta^2/2\sigma^2); & [0 < \beta < \infty;\ 0 < \varepsilon < 2\pi] \\ 0; & \text{elsewhere} \end{cases} \tag{2.15}$$

Here β and ε are independent quantities and;

β : Rayleigh distributed with mean square $2\sigma^2$ and

ε : Uniformly distributed over the interval $(0, 2\pi)$

It is easily shown [5] that the joint distribution of the strength and angle of the sum of fixed vector(γ, δ) and the random vector(β, ε) described by (2.15) is

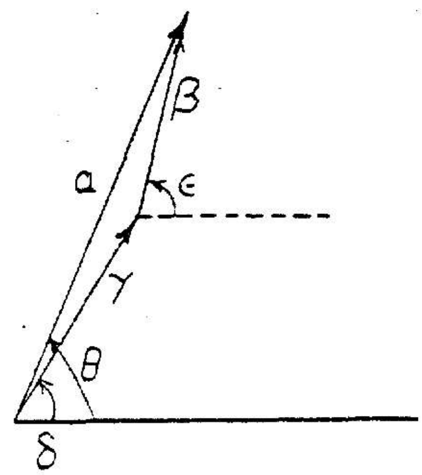


FIGURE 2-4

RELATIONSHIPS AMONG STRENGTHS AND PHASE SHIFTS
OF THE FADING MEDIA

$$f_{A_f,\theta_f}(a_f,\ \theta_f) = \begin{cases} \dfrac{a_f}{2\pi\sigma^2} \exp[-\dfrac{a_f^2+\ \gamma^2-\ 2\gamma a_f \mathrm{Cos}(\theta_f-\delta)}{2\sigma^2}] \\ \text{for } 0 \leqslant a_f \leqslant \infty \text{ and } 0 \leqslant (\theta_f-\delta) \leqslant 2\pi. \\ 0;\ \text{elsewhere} \end{cases} \tag{2.16}$$

The marginal distribution of strength a_f obtained by integrating (2.16) over θ_f, is given by

$$f_{A_f}(a_f) = \begin{cases} \dfrac{a_f}{\sigma^2} \exp\ [-\dfrac{a_f^2+\ \gamma^2}{2\sigma^2}]\ I_0(\dfrac{\gamma a_f}{\sigma^2});\quad 0 \leqslant a_f \leqslant \infty \\ 0;\quad a < 0 \end{cases} \tag{2.17}$$

where I_0 is the zeroth-order modified Bessel function of the first kind.

The assumption of Rician fading implies a strong specular or direct path as well as a scatter path which by itself would produce a Rayleigh channel.

For a short-term description of the channel, where "short" term is taken to be the period of time of sufficient duration to allow a carrier tracking loop to lockup to the specular component, and hence, permit coherent reception, the specular component is taken to be constant. However, over the long term, it is assumed that the specular component of the received signal fades accordingly to a Rayleigh density [6].

In order to find the probability of error for the fading case, it is necessary to find the probability of error, given that s is constant ($P_r(e/s)$). The result can be easily extended to the case of random variables.

If the probability density of the parameter s is known, then it can be easily shown that;

$$P_r(e) = \int_{-\infty}^{\infty} P_r(e/s) f_s(s) ds \tag{2.18}$$

where $f_s(s)$ is the probability density function of the parameter s.

SECTION III

SYSTEM PERFORMANCES

The objectives of this section are to introduce appropriate definitions of the system error probability and to obtain expressions from which this probability may be evaluated. The performance of the system can be described by several different error probabilities. The most commonly used error probability, the bit error probability criteria is used in this analysis.

3.1 System Performance Under Non-fading Channels

The system performance and the evaluation of probability of error for the non-fading channels is described in this section. The noise and jammer interferences are also taken into account.

The worst-case jammer and the worst-case performance will be determined as functions of the signal-to-noise (background noise) ratio (SNR) and signal-to-jammer power ratio (SJR). An analysis is presented which derived the average probability of bit error rate. The information rate, total bandwidth, and average transmitted power are held constant. The receiver is a partially-coherent inphase/quadrature detector employing matched filter detection. The system is stressed by the presence of an intentional jammer and the expression for the probability of error is determined. A partially-coherent QPSK detector model is shown in Figure 3-1.

The received signal will undergo partially-coherent demodulation by the frequency de-hopper. The inphase and quadrature components of the received signal, obtained from the matched filter detector are

$$E_I(t) = y^{(i)}(t)[\sqrt{2}\ \mathrm{Sin}(\omega_H^{(i)}t + \hat{\theta})] \tag{3.1a}$$

and

$$E_Q(t) = y^{(i)}(t)[\sqrt{2}\ \mathrm{Cos}(\omega_H^{(i)}t + \hat{\theta})] \tag{3.1b}$$

Unlike the coherent case, $\hat{\theta}$ will be different from θ for the partially-coherent case. Substituting the value of $y^{(i)}(t)$ from (2.8), equation (3.1) can be rewritten as

$$\begin{aligned} E_I(t) &= [s^{(i)}(t;\theta)+n(t)+j(t)][\sqrt{2}\ \mathrm{Sin}(\omega_H^{(i)}t+\hat{\theta})] \\ &= [\sqrt{2s}\ \mathrm{Sin}(\omega_H^{(i)}t+\theta^{(i)}+\theta)+n(t)+j(t)][\sqrt{2}\ \mathrm{Sin}(\omega_H^{(i)}t+\hat{\theta})]. \end{aligned}$$

Simplifying and ignoring the double harmonic terms, the above equation simplifies to

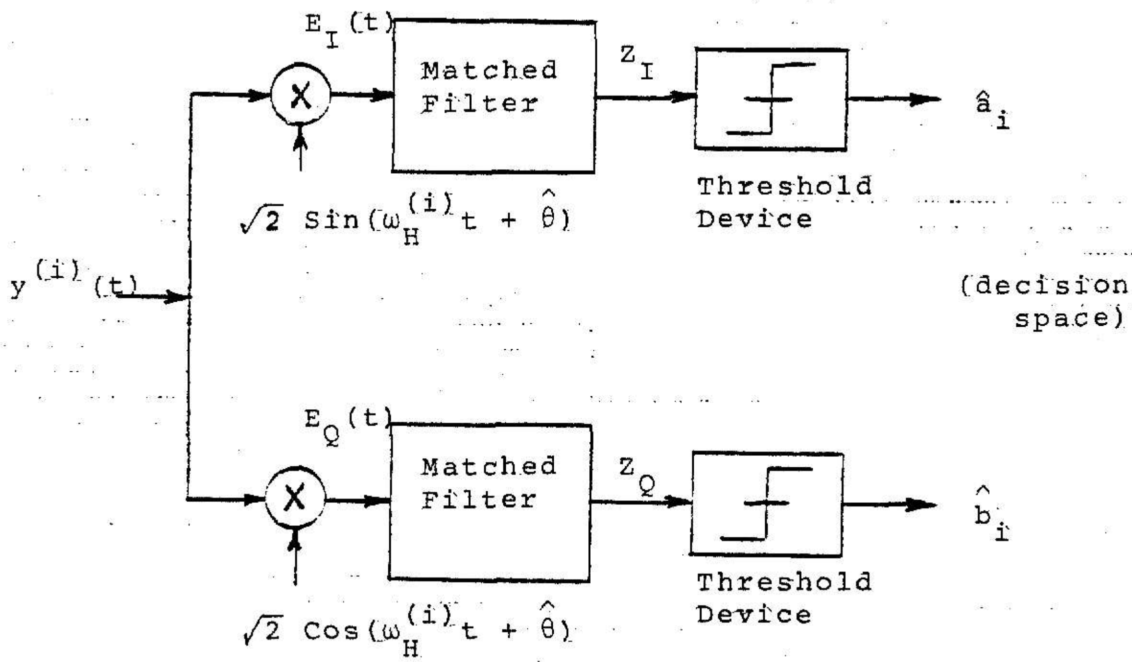


FIGURE 3.1

PARTIALLY COHERENT QPSK DETECTOR

$$E_I(t) = [\sqrt{S}\ \mathrm{Cos}(\theta^{(i)}+\theta-\hat{\theta})-N_c(t)\mathrm{Sin}(\theta-\hat{\theta})-N_s(t)\mathrm{Cos}(\theta-\hat{\theta})$$
$$-\sqrt{J}_o\mathrm{Sin}(\theta_j+\theta-\hat{\theta})]. \qquad (3.2a)$$

Similiarly,

$$E_Q(t) = [\sqrt{S}\ \mathrm{Sin}(\theta^{(i)}+\theta-\hat{\theta})+N_c(t)\mathrm{Cos}(\theta-\hat{\theta})-N_s(t)\mathrm{Sin}(\theta-\hat{\theta})$$
$$+\sqrt{J}_o\mathrm{Cos}(\theta_j+\theta-\hat{\theta})]. \qquad (3.2b)$$

These signals are then passed through matched filters of duration equal to the information symbol interval T_s to produce the in-phase and quadrature phase decision variables

$$Z_I = \int_{(i-1)T_s}^{iT_s} E_I(t)dt \quad \text{and} \qquad (3.3a)$$

$$Z_Q = \int_{(i-1)T_s}^{iT_s} E_Q(t)dt \qquad (3.3b)$$

Let $(\theta-\hat{\theta}) = \theta_1$. Then,

$$Z_I = \int_{(i-1)T_s}^{iT_s} [\ \sqrt{2}\mathrm{Cos}(\theta^{(i)}+\ \theta_1)-N_c(t)\mathrm{Sin}\theta_1-N_s(t)\mathrm{Cos}\theta_1$$
$$-\sqrt{J}_o\mathrm{Sin}(\theta_j+\theta_1)]dt \qquad (3.4a)$$

and

$$Z_Q = \int_{(i-1)T_s}^{iT_s} [\ \sqrt{2}\mathrm{Sin}(\theta^{(i)}+\ \theta_1)+N_c(t)\mathrm{Cos}\theta_1-N_s(t)\mathrm{Sin}\theta_1$$
$$+\sqrt{J}_o\mathrm{Cos}(\theta_j+\theta_1)]dt, \qquad (3.4b)$$

which inturn can be written as

$$Z_I = [\sqrt{2}T_s \text{Cos}(\theta^{(i)} + \theta_1) - N_Q \text{Sin}\theta_1 + N_I \text{Cos}\theta_1 - \sqrt{J}_o T_s \text{Sin}(\theta_j + \theta_1)] \quad (3.5a)$$

and

$$Z_Q = [\sqrt{2}T_s \text{Sin}(\theta^{(i)} + \theta_1) + N_Q \text{Cos}\theta_1 + N_I \text{Sin}\theta_1 + \sqrt{J}_o T_s \text{Cos}(\theta_j + \theta_1)]. \quad (3.5b)$$

N_I and N_Q are zero-mean Gaussian random variables with variance $N_o T_s/2$ and given by

$$N_I = -\int_{(i-1)T_s}^{iT_s} N_s(t)\,dt \quad \text{and} \quad (3.6a)$$

$$N_Q = +\int_{(i-1)T_s}^{iT_s} N_c(t)\,dt. \quad (3.6b)$$

Let $\sqrt{2}\ \text{Cos}\ \theta^{(i)} = [a_i]$ (3.7a)

$\sqrt{2}\ \text{Sin}\ \theta^{(i)} = [b_i]$, (3.7b)

where $[a_i]$ and $[b_i]$ are the equivalent independent in-phase and quadrature binary information sequences which take on values +1 or -1.

Substituting this in (3.5), the equations reduce to

$$Z_I = \left[\sqrt{S/2}\, T_s(a_i \mathrm{Cos}\theta_1 - b_i \mathrm{Sin}\theta_1) - N_Q \mathrm{Sin}\theta_1 + N_I \mathrm{Cos}\theta_1 - \sqrt{J_o}\, T_s \mathrm{Sin}(\theta_j + \theta_1)\right] \tag{3.8a}$$

and

$$Z_Q = \left[\sqrt{S/2}\, T_s(b_i \mathrm{Cos}\theta_1 - a_i \mathrm{Sin}\theta_1) + N_Q \mathrm{Cos}\theta_1 + N_I \mathrm{Sin}\theta_1 + \sqrt{J_o}\, T_s \mathrm{Cos}(\theta_j + \theta_1)\right]. \tag{3.8b}$$

The in-phase and quadrature decision variables Z_I and Z_Q are passed through hard limiters (threshold devices) to obtain the receiver estimates $\hat{a}_i$ and $\hat{b}_i$, where

$$\hat{a}_i = \mathrm{sgn}^* Z_I \text{ and}$$
$$\hat{b}_i = \mathrm{sgn}^* Z_Q. \tag{3.9}$$

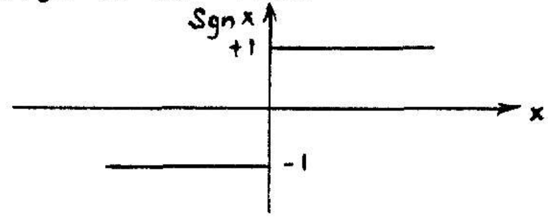
Hence, given a_i, b_i, θ_1 and θ_j, the probability that the ith symbol is in error is the probability that either a_i or b_i is in error, i.e.,

$$P_{Ei}(\theta_j) = \mathrm{Prob}\left\{\hat{a}_i \neq a_i \text{ or } \hat{b}_i \neq b_i\right\} \tag{3.10}$$

---

* The function sgn x (pronounced as signum x) is equal to +1 or -1 according to the sign of x. Thus

$$\mathrm{Sgn}\ x \begin{cases} = -1; & x < 0 \\ = +1; & x > 0 \end{cases}$$



$$P_{Ei}(\theta_j) = \text{Prob}\left\{\hat{a}_i \neq a_i\right\} + \text{Prob}\left\{\hat{b}_i \neq b_i\right\} - \text{Prob}\left\{\hat{a}_i \neq a_i\right\}\text{Prob}\left\{\hat{b}_i \neq b_i\right\}. \tag{3.11}$$

Since the signal set is not symmetric, (3.11) must be computed for four points in four different quadrants in order to obtain the average probability of symbol error conditioned on the jammer phase $P_E(\theta_j)$. Thus there are four possible combinations of $[a_i]$ and $[b_i]$.

$$\begin{aligned}
&[a_i = 1,\ b_i = 1] \\
&[a_i = 1,\ b_i = -1] \\
&[a_i = -1,\ b_i = 1] \\
&[a_i = -1,\ b_i = -1]
\end{aligned}$$

Averaging over these four cases

$$P_E(\theta_j) = 1/4[P_{E1}(\theta_j) + P_{E2}(\theta_j) + P_{E3}(\theta_j) + P_{E4}(\theta_j)], \tag{3.13}$$

where each of the terms inside brackets corresponds to the different cases of (3.12) respectively. Each of these four cases will be considered separately in the following subsections.

3.1.1 Calculation of $P_{E1}(\theta_j)$

$$P_{E1}(\theta_j) = \text{Prob}[Z_I<0;\ a_i=1,b_i=1] + \text{Prob}[Z_Q<0;\ a_i=1,b_i=1] - \text{Prob}[Z_I<0;\ a_i=1,b_i=1]\ \text{Prob}[Z_Q<0;\ a_i=1,b_i=1] \tag{3.14}$$

i.e., $P_{E1}(\theta_j) = [P_{s1}(\theta_j) + P_{c1}(\theta_j) - P_{s1}(\theta_j)\, P_{c1}(\theta_j)]$

(3.15)

where

$$P_{s1}(\theta_j) = \text{Prob}\Big\{[\sqrt{s/2}\, T_s(\text{Cos}\theta_1 - \text{Sin}\theta_1) - N_Q \text{Sin}\theta_1 + N_I \text{Cos}\theta_1 - \sqrt{J_o} T_s \text{Sin}(\theta_j + \theta_1)] \leq 0\Big\},$$

and

(3.16a)

$$P_{c1}(\theta_j) = \text{Prob}\Big\{[\sqrt{s/2}\, T_s(\text{Cos}\theta_1 + \text{Sin}\theta_1) + N_Q \text{Cos}\theta_1 + N_I \text{Sin}\theta_1 + \sqrt{J_o} T_s \text{Cos}(\theta_j + \theta_1)] < 0\Big\}.$$

(3.16b)

Rewriting (3.16) yields

$$P_{s1}(\theta_j) = \text{Prob}\Big\{[N_I \text{Cos}\theta_1 - N_Q \text{Sin}\theta_1] < [-\sqrt{s/2}\, T_s(\text{Cos}\theta_1 - \text{Sin}\theta_1) + \sqrt{J_o} T_s \text{Sin}(\theta_j + \theta_1)]\Big\},$$

and

(3.17a)

$$P_{c1}(\theta_j) = \text{Prob}\Big\{[N_I \text{Sin}\theta_1 + N_Q \text{Cos}\theta_1] < [-\sqrt{s/2}\, T_s(\text{Cos}\theta_1 + \text{Sin}\theta_1) - \sqrt{J_o} T_s \text{Cos}(\theta_j + \theta_1)]\Big\}.$$

(3.17b)

At this stage, it can be noticed that N_I, N_Q, θ_1, and θ_j are all random variables. However, if one tries to find the value of the probability, the difficult problem of finding the joint probability density function of $f(N_I, N_Q, \theta_1, \theta_j)$ will be encountered. Hence, slightly modifying the problem, at this stage only N_I and N_Q will be treated as random variables and the probability of error is conditioned on the variables θ_1 and θ_j. The unconditional probability of error can be determined by averaging the conditional probability of error over the distributions of θ_1 and θ_j. The calculations are simplified by creating two new random variables X and Y, given by

$$X = (N_I \mathrm{Cos}\theta_1 - N_Q \mathrm{Sin}\theta_1) \tag{3.18a}$$

$$Y = (N_I \mathrm{Sin}\theta_1 + N_Q \mathrm{Cos}\theta_1), \tag{3.18b}$$

whose density functions $f_X(x)$ and $f_Y(y)$ are found to be (see appendix):

$$f_X(x) = [1/\sqrt{2\ (N_o T_s/2)}]\ \exp\ [-x^2/2(N_o T_s/2)], \tag{3.19a}$$

$$f_Y(y) = [1/\sqrt{2\ (N_o T_s/2)}]\ \exp\ [-y^2/2(N_o T_s/2)]. \tag{3.19b}$$

Now equation (3.17) take the form

$$P_{s1}(\theta_j|\theta_1) = \mathrm{Prob}\ \{X < x_1\} = \int_{-\infty}^{x_1} f_X(x)\,dx \tag{3.20a}$$

and

$$P_{c1}(\theta_j|\theta_1) = \mathrm{Prob}\ \{Y < y_1\} = \int_{-\infty}^{y_1} f_Y(y)\,dy, \tag{3.20b}$$

where

$$x_1 = [-\sqrt{s/2}\ T_s(\mathrm{Cos}\theta_1 - \mathrm{Sin}\theta_1) + \sqrt{J_o} T_s \mathrm{Sin}(\theta_j + \theta_1)] \tag{3.21a}$$

and

$$y_1 = [-\sqrt{s/2}\ T_s(\mathrm{Cos}\theta_1 + \mathrm{Sin}\theta_1) - \sqrt{J_o} T_s \mathrm{Cos}(\theta_j + \theta_1)]. \tag{3.21b}$$

Substituting (3.19) in (3.20) leads to

$$P_{s1}(\theta_j|\theta_1) = 1/\sqrt{2\pi}\int_{Z_{s1}}^{\infty} \exp(-x^2/2)dx = \mathrm{erfc}(Z_{s1}) \qquad (3.22a)$$

and

$$P_{c1}(\theta_j|\theta_1) = 1/\sqrt{2\pi}\int_{Z_{c1}}^{\infty} \exp(-y^2/2)dy = \mathrm{erfc}(Z_{c1}) \qquad (3.22b)$$

where

$$\begin{aligned} Z_{s1} &= -x_1/\sqrt{(N_oT_s/2)} \\ &= \sqrt{2T_s/N_o}[\sqrt{s/2}\ (\mathrm{Cos}\theta_1 - \mathrm{Sin}\theta_1) - \sqrt{J_o}\ \mathrm{Sin}(\theta_j + \theta_1)] \end{aligned} \qquad (3.23a)$$

$$\begin{aligned} Z_{c1} &= -y_1/\sqrt{(N_oT_s/2)} \\ &= \sqrt{2T_s/N_o}[\sqrt{s/2}\ (\mathrm{Cos}\theta_1 + \mathrm{Sin}\theta_1) + \sqrt{J_c}\ \mathrm{Cos}(\theta_j + \theta_1)]. \end{aligned} \qquad (3.23b)$$

Rewriting yields

$$Z_{s1} = -\sqrt{2T_s/N_o}[\sqrt{s}\ \mathrm{Sin}(\theta_1 - \pi/4) + \sqrt{J_o}\mathrm{Sin}(\theta_j+\theta_1)] \qquad (3.24a)$$

$$Z_{c1} = +\sqrt{2T_s/N_o}[\sqrt{s}\ \mathrm{Cos}(\theta_1 - \pi/4) + \sqrt{J_o}\mathrm{Cos}(\theta_j+\theta_1)]. \qquad (3.24b).$$

Thus summarizing the expressions for $P_{E1}(\theta_j)$:

$$P_{E1}(\theta_j) = [P_{s1}(\theta_j) + P_{c1}(\theta_j) - P_{s1}(\theta_j)\ P_{c1}(\theta_j)]$$

where:

$$P_{s1}(\theta_j|\theta_1) = \mathrm{erfc}(Z_{s1})$$

$$P_{c1}(\theta_j|\theta_1) = \mathrm{erfc}(Z_{c1})$$

and

$$Z_{s1} = -\sqrt{2T_s/N_o}[\sqrt{s}\ \mathrm{Sin}(\theta_1 - \pi/4) + \sqrt{J_o}\mathrm{Sin}(\theta_j+\theta_1)]$$

$$Z_{c1} = -\sqrt{2T_s/N_o}[\sqrt{s}\ \mathrm{Cos}(\theta_1 - \pi/4) + \sqrt{J_o}\mathrm{Cos}(\theta_j+\theta_1)]$$

(3.25)

3.1.2 Calculation of $P_{E2}(\theta_j)$

$$P_{E2}(\theta_j) = \text{Prob}[Z_I<0;\ a_i=1,b_i=-1] + \text{Prob}[Z_Q<0;\ a_i=1,b_i=-1] - \text{Prob}[Z_I<0;\ a_i=1,b_i=-1]\ \text{Prob}[Z_Q<0;\ a_i=1,b_i=-1] \tag{3.26}$$

i.e.,

$$P_{E2}(\theta_j) = [P_{s2}(\theta_j) + P_{c2}(\theta_j) - P_{s2}(\theta_j)\ P_{c2}(\theta_j)] \tag{3.27}$$

where

$$P_{s2}(\theta_j) = \text{Prob}\Big\{[\sqrt{s/2}\ T_s(\text{Cos}\theta_1 + \text{Sin}\theta_1) - N_Q\text{Sin}\theta_1 + N_I\text{Cos}\theta_1 - \sqrt{J}_o T_s \text{Sin}(\theta_j + \theta_1)] < 0\Big\}, \tag{3.28a}$$

and

$$P_{c2}(\theta_j) = \text{Prob}\Big\{[\sqrt{s/2}\ T_s(-\text{Cos}\theta_1 + \text{Sin}\theta_1) + N_Q\text{Cos}\theta_1 + N_I\text{Sin}\theta_1 + \sqrt{J}_o T_s \text{Cos}(\theta_j + \theta_1)] < 0\Big\}. \tag{3.28b}$$

Rewriting (3.28) yields

$$P_{s2}(\theta_j) = \text{Prob}\Big\{[N_I\text{Cos}\theta_1 - N_Q\text{Sin}\theta_1] < [-\sqrt{s/2}\ T_s(\text{Cos}\theta_1 + \text{Sin}\theta_1) + \sqrt{J}_o T_s \text{Sin}(\theta_j + \theta_1)]\Big\}, \tag{3.29a}$$

and

$$P_{c2}(\theta_j) = \text{Prob}\Big\{[N_I\text{Sin}\theta_1 + N_Q\text{Cos}\theta_1] < [-\sqrt{s/2}\ T_s(-\text{Cos}\theta_1 + \text{Sin}\theta_1) - \sqrt{J}_o T_s \text{Cos}(\theta_j + \theta_1)]\Big\}. \tag{3.29b}$$

Using equations (3.18) and (3.19), equation (3.29) can be written as

$$P_{s2}(\theta_j|\theta_1) = \text{Prob}\left\{X < x_2\right\} = \int_{-\infty}^{x_2} f_X(x)\,dx \qquad (3.30a)$$

and

$$P_{c2}(\theta_j|\theta_1) = \text{Prob}\left\{Y < y_2\right\} = \int_{y_2}^{\infty} f_Y(y)\,dy \qquad (3.30b)$$

where

$$x_2 = [-\sqrt{s/2}\,T_s(\text{Cos}\theta_1 + \text{Sin}\theta_1) + \sqrt{J_o}T_s\text{Sin}(\theta_j + \theta_1)] \qquad (3.31a)$$

and

$$y_2 = [-\sqrt{s/2}\,T_s(-\text{Cos}\theta_1 + \text{Sin}\theta_1) - \sqrt{J_o}T_s\text{Cos}(\theta_j + \theta_1)], \qquad (3.31b)$$

which further take the form

$$P_{s2}(\theta_j|\theta_1) = 1/\sqrt{2\pi}\int_{Z_{s2}}^{\infty} \exp(-x^2/2)\,dx = \text{erfc}(Z_{s2}) \qquad (3.32a)$$

and

$$P_{c2}(\theta_j|\theta_1) = 1/\sqrt{2\pi}\int_{Z_{c2}}^{\infty} \exp(-y^2/2)\,dy = \text{erfc}(Z_{c2}) \qquad (3.32b)$$

where

$$\begin{aligned} Z_{s2} &= -\,x_2/\sqrt{(N_oT_s/2)} \\ &= -\sqrt{2T_s/N_o}[\sqrt{s}\ \text{Sin}(\theta_1 - 3\pi/4) + \sqrt{J_o}\text{Sin}(\theta_j+\theta_1)] \end{aligned} \qquad (3.33a)$$

$$\begin{aligned} Z_{c2} &= +\,y_2/\sqrt{(N_oT_s/2)} \\ &= -\sqrt{2T_s/N_o}[\sqrt{s}\ \text{Cos}(\theta_1 - 3\pi/4) + \sqrt{J_o}\text{Cos}(\theta_j+\theta_1)]. \end{aligned} \qquad (3.33b)$$

Thus summarizing the expressions for $P_{E2}(\theta_j)$

$$P_{E2}(\theta_j) = [P_{s2}(\theta_j) + P_{c2}(\theta_j) - P_{s2}(\theta_j)\, P_{c2}(\theta_j)]$$

where:

$$P_{s2}(\theta_j|\theta_1) = \operatorname{erfc}(Z_{s2})$$

$$P_{c2}(\theta_j|\theta_1) = \operatorname{erfc}(Z_{c2})$$

and

$$Z_{s2} = -\sqrt{2T_s/N_o}[\sqrt{s}\ \mathrm{Sin}(\theta_1 - 3\pi/4) + \sqrt{J_o}\mathrm{Sin}(\theta_j+\theta_1)]$$

$$Z_{c2} = -\sqrt{2T_s/N_o}[\sqrt{s}\ \mathrm{Cos}(\theta_1 - 3\pi/4) + \sqrt{J_o}\mathrm{Cos}(\theta_j+\theta_1)]$$

(3.34)

3.1.3 Calculation of $P_{E3}(\theta_j)$

$$P_{E3}(\theta_j) = \mathrm{Prob}[Z_I<0;\ a_i=-1, b_i=1] + \mathrm{Prob}[Z_Q<0;\ a_i=-1, b_i=1] - \mathrm{Prob}[Z_I<0;\ a_i=-1, b_i=1]\ \mathrm{Prob}[Z_Q<0;\ a_i=-1, b_i=1]$$

(3.35)

i.e.,

$$P_{E3}(\theta_j) = [P_{s3}(\theta_j) + P_{c3}(\theta_j) - P_{s3}(\theta_j)\, P_{c3}(\theta_j)]$$

(3.36)

where

$$P_{s3}(\theta_j) = \mathrm{Prob}\Big\{[\sqrt{s/2}\ T_s(-\mathrm{Cos}\theta_1 - \mathrm{Sin}\theta_1) - N_Q\mathrm{Sin}\theta_1 + N_I\mathrm{Cos}\theta_1 - \sqrt{J_o}T_s\mathrm{Sin}(\theta_j + \theta_1)] < 0\Big\},$$

(3.37a)

and

$$P_{c3}(\theta_j) = \mathrm{Prob}\Big\{[\sqrt{s/2}\ T_s(\mathrm{Cos}\theta_1 - \mathrm{Sin}\theta_1) + N_Q\mathrm{Cos}\theta_1 + N_I\mathrm{Sin}\theta_1 + \sqrt{J_o}T_s\mathrm{Cos}(\theta_j + \theta_1)] < 0\Big\}.$$

(3.37b)

Rewriting (3.37) yields

$$P_{s3}(\theta_j) = \text{Prob}\left\{[N_I \text{Cos}\theta_1 - N_Q \text{Sin}\theta_1] < [+\sqrt{s/2}\, T_s(\text{Cos}\theta_1 + \text{Sin}\theta_1) + \sqrt{J_o}T_s \text{Sin}(\theta_j + \theta_1)]\right\}, \tag{3.38a}$$

and

$$P_{c3}(\theta_j) = \text{Prob}\left\{[N_I \text{Sin}\theta_1 + N_Q \text{Cos}\theta_1] < [-\sqrt{s/2}\, T_s(\text{Cos}\theta_1 - \text{Sin}\theta_1) - \sqrt{J_o}T_s \text{Cos}(\theta_j + \theta_1)]\right\}. \tag{3.38b}$$

Using equations (3.18) and (3.19), equation (3.38) can be written as

$$P_{s3}(\theta_j|\theta_1) = \text{Prob}\left\{X < x_3\right\} = \int_{x_3}^{-\infty} f_X(x)dx \tag{3.39a}$$

and

$$P_{c3}(\theta_j|\theta_1) = \text{Prob}\left\{Y < y_3\right\} = \int_{-\infty}^{y_3} f_Y(y)dy \tag{3.39b}$$

where

$$x_3 = [-\sqrt{s/2}\, T_s(\text{Cos}\theta_1 + \text{Sin}\theta_1) + \sqrt{J_o}T_s \text{Sin}(\theta_j + \theta_1)] \tag{3.40a}$$

and

$$y_3 = [-\sqrt{s/2}\, T_s(\text{Cos}\theta_1 - \text{Sin}\theta_1) - \sqrt{J_o}T_s \text{Cos}(\theta_j + \theta_1)] \tag{3.40b}$$

which further take the form

$$P_{s3}(\theta_j|\theta_1) = 1/\sqrt{2\pi} \int_{Z_{s3}}^{\infty} \exp(-x^2/2)dx = \text{erfc}(Z_{s3}) \tag{3.41a}$$

and

$$P_{c3}(\theta_j|\theta_1) = 1/\sqrt{2\pi} \int_{Z_{c3}}^{\infty} \exp(-y^2/2)dy = \text{erfc}(Z_{c3}). \tag{3.41b}$$

where,

$$Z_{s3} = + x_3/\sqrt{(N_oT_s/2)}$$
$$= + \sqrt{2T_s/N_o}[-\sqrt{s}\ \mathrm{Sin}(\theta_1 + 5\pi/4) + \sqrt{J_o}\mathrm{Sin}(\theta_j + \theta_1)] \quad (3.42a)$$

and
$$Z_{c3} = - y_3/\sqrt{(N_oT_s/2)}$$
$$= + \sqrt{2T_s/N_o}[-\sqrt{s}\ \mathrm{Cos}(\theta_1 + 5\pi/4) + \sqrt{J_o}\mathrm{Cos}(\theta_j + \theta_1)]. \quad (3.42b)$$

Thus summarizing the expressions for $P_{E3}(\theta_j)$

$$P_{E3}(\theta_j) = [P_{s3}(\theta_j) + P_{c3}(\theta_j) - P_{s3}(\theta_j)\ P_{c3}(\theta_j)]$$

where:

$$P_{s3}(\theta_j|\theta_1) = \mathrm{erfc}(Z_{s3})$$
$$P_{c3}(\theta_j|\theta_1) = \mathrm{erfc}(Z_{c3})$$

and

$$Z_{s3} = \sqrt{2T_s/N_o}[-\sqrt{s}\ \mathrm{Sin}(\theta_1 + 5\pi/4) + \sqrt{J_o}\mathrm{Sin}(\theta_j + \theta_1)]$$
$$Z_{c3} = \sqrt{2T_s/N_o}[-\sqrt{s}\ \mathrm{Cos}(\theta_1 + 5\pi/4) + \sqrt{J_o}\mathrm{Cos}(\theta_j + \theta_1)]$$
$$(3.43)$$

3.1.4 Calculation of $P_{E4}(\theta_j)$

$$P_{E4}(\theta_j) = \mathrm{Prob}[Z_I<0;\ a_i=-1, b_i=-1] + \mathrm{Prob}[Z_Q<0;\ a_i=-1, b_i=-1]$$
$$- \mathrm{Prob}[Z_I<0;\ a_i=-1, b_i=-1]\ \mathrm{Prob}[Z_Q<0;\ a_i=-1, b_i=-1] \quad (3.44)$$

i.e.,

$$P_{E4}(\theta_j) = [P_{s4}(\theta_j) + P_{c4}(\theta_j) - P_{s4}(\theta_j)\ P_{c4}(\theta_j)] \quad (3.45)$$

where

$$P_{s4}(\theta_j) = \text{Prob}\Big\{[\sqrt{s/2}\ T_s(-\text{Cos}\theta_1 + \text{Sin}\theta_1) - N_Q\text{Sin}\theta_1 + N_I\text{Cos}\theta_1 - \sqrt{J_o}T_s\text{Sin}(\theta_j + \theta_1)] < 0\Big\}, \tag{3.46a}$$

and

$$P_{c4}(\theta_j) = \text{Prob}\Big\{[\sqrt{s/2}\ T_s(-\text{Cos}\theta_1 - \text{Sin}\theta_1) + N_Q\text{Cos}\theta_1 + N_I\text{Sin}\theta_1 + \sqrt{J_o}T_s\text{Cos}(\theta_j + \theta_1)] < 0\Big\}. \tag{3.46b}$$

Rewriting (3.46) yields

$$P_{s4}(\theta_j) = \text{Prob}\Big\{[N_I\text{Cos}\theta_1 - N_Q\text{Sin}\theta_1] < [+\sqrt{s/2}\ T_s(\text{Cos}\theta_1 - \text{Sin}\theta_1) + \sqrt{J_o}T_s\text{Sin}(\theta_j + \theta_1)]\Big\}, \tag{3.47a}$$

and

$$P_{c4}(\theta_j) = \text{Prob}\Big\{[N_I\text{Sin}\theta_1 + N_Q\text{Cos}\theta_1] < [+\sqrt{s/2}\ T_s(\text{Cos}\theta_1 + \text{Sin}\theta_1) - \sqrt{J_o}T_s\text{Cos}(\theta_j + \theta_1)]\Big\}. \tag{3.47b}$$

Using equations (3.18) and (3.19), equation (3.47) can be written as

$$P_{s4}(\theta_j|\theta_1) = \text{Prob}\Big\{X < x_4\Big\} = \int_{x_4}^{\infty} f_X(x)\,dx \tag{3.48a}$$

and

$$P_{c3}(\theta_j|\theta_1) = \text{Prob}\Big\{Y < y_3\Big\} = \int_{y_3}^{\infty} f_Y(y)\,dy \tag{3.48b}$$

where

$$x_4 = [+\sqrt{s/2}\ T_s(\mathrm{Cos}\theta_1 - \mathrm{Sin}\theta_1) + \sqrt{J_o}T_s\mathrm{Sin}(\theta_j + \theta_1)] \quad (3.49a)$$

and
$$y_4 = [+\sqrt{s/2}\ T_s(\mathrm{Cos}\theta_1 + \mathrm{Sin}\theta_1) - \sqrt{J_o}T_s\mathrm{Cos}(\theta_j + \theta_1)] \quad (3.49b)$$

which further take the form

$$P_{s4}(\theta_j|\theta_1) = 1/\sqrt{2\pi}\int_{Z_{s4}}^{\infty} \exp(-x^2/2)\,dx = \mathrm{erfc}(Z_{s4}) \quad (3.50a)$$

and
$$P_{c4}(\theta_j|\theta_1) = 1/\sqrt{2\pi}\int_{Z_{c4}}^{\infty} \exp(-y^2/2)\,dy = \mathrm{erfc}(Z_{c4}) \quad (3.50b)$$

where

$$\begin{aligned} Z_{s4} &= +\ x_4/\sqrt{(N_oT_s/2)} \\ &= +\sqrt{2T_s/N_o}[-\sqrt{s}\ \mathrm{Sin}(\theta_1 + 7\pi/4) + \sqrt{J_o}\mathrm{Sin}(\theta_j+\theta_1)] \end{aligned} \quad (3.51a)$$

and
$$\begin{aligned} Z_{c4} &= +\ y_4/\sqrt{(N_oT_s/2)} \\ &= +\sqrt{2T_s/N_o}[+\sqrt{s}\ \mathrm{Cos}(\theta_1 + 7\pi/4) - \sqrt{J_o}\mathrm{Cos}(\theta_j+\theta_1)] \end{aligned} \quad (3.51b)$$

Thus summarizing the expressions for $P_{E4}(\theta_j)$

$$P_{E4}(\theta_j) = [P_{s4}(\theta_j) + P_{c4}(\theta_j) - P_{s4}(\theta_j)\ P_{c4}(\theta_j)]$$

where:

$$P_{s4}(\theta_j|\theta_1) = \mathrm{erfc}(Z_{s4})$$
$$P_{c4}(\theta_j|\theta_1) = \mathrm{erfc}(Z_{c4})$$

and

$$Z_{s4} = \sqrt{2T_s/N_o}[-\sqrt{s}\ \mathrm{Sin}(\theta_1 + 7\pi/4) + \sqrt{J_o}\mathrm{Sin}(\theta_j+\theta_1)]$$
$$Z_{c4} = \sqrt{2T_s/N_o}[+\sqrt{s}\ \mathrm{Cos}(\theta_1 + 7\pi/4) - \sqrt{J_o}\mathrm{Cos}(\theta_j+\theta_1)]. \quad (3.52)$$

Using the end results of the subsections 3.1.1 through 3.1.4, the expression for the average probability of symbol error $P_E(\theta_j)$ for symbol intervals which are jammed, can be written from equation (3.13) as:

$$P_E(\theta_j) = 1/4[P_{E1}(\theta_j) + P_{E2}(\theta_j) + P_{E3}(\theta_j) + P_{E4}(\theta_j)] \tag{3.53}$$

This conditional probability value is calculated assuming that θ_1 and θ_j are not random, but known quantities. However, this is not true in practice. Proceding one step further, the probability conditioned on the jammer phase θ_j is given by averaging (3.53) over the density distribution of θ_1, $f_{\theta 1}(\theta_1)$.

Hence,

$$P_E(\theta_j) = 1/4 \int_{-\pi}^{+\pi} [P_{E1}(\theta_j)+P_{E2}(\theta_j)+P_{E3}(\theta_j)+P_{E4}(\theta_j)]f_{\theta_1}(\theta_1)d\theta_1 \tag{3.54}$$

Finally, the unconditioned average probability is obtained by averaging $P_E(\theta_j)$ of (3.54) over the uniform distribution of θ_j between $-\pi$ and $+\pi$.

This yields

$$P_{EJ} = \frac{1}{8\pi} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} [P_{E1}(\theta_j)+P_{E2}(\theta_j)+P_{E3}(\theta_j)+P_{E4}(\theta_j)]f_{\theta_1}(\theta_1)d\theta_1 d\theta_j \tag{3.55}$$

3.2 Calculation of Bit Error Rate Probability

For a QPSK signal, the symbol time T_s is twice the bit time T_b. The bit energy is given by

$$E_b = ST_b \tag{3.56}$$

Therefore

$$ST_s/N_o = 2ST_b/N_o = 2(E_b/N_o) \tag{3.57}$$

The factor (E_b/N_o) is the signal to background noise ratio and is denoted as SNR for simplicity.

Hence, $ST_s/N_o = 2(SNR)$ (3.58)

Furthermore, from equations (2.6) and (2.7)

$$J_o/S = J/\alpha NS \tag{3.59}$$

The total number of frequency hops, N is given by

$$N = W/(1/T_s) = WT_s = 2WT_b \tag{3.60}$$

where, W = total hop frequency band and

$(1/T_s)$ = width of individual frequency slot

Substituting (2.60) in (2.59)

$$J_o/S = (J/W)/2\alpha E_b = 1/2\alpha(E_b/N_J)$$

where the quantity (J/W) represents the effective jammer power spectral density in the hop band and is denoted by N_J. The factor (E_b/N_J) represents the signal to jammer power ratio and is denoted as SJR for simplicity. This yields

$$J_o/S = 1/2\alpha(SJR)$$

Further, representing Z_{s1}, Z_{c1}, Z_{s2}, Z_{c2}, Z_{s3}, Z_{c3}, Z_{s4}, and Z_{c4}, interms of SNR and SJR, their expressions take the form:

$$Z_{s1} = 2\sqrt{SNR}\,[-\mathrm{Sin}(\theta_1 - \pi/4) - \sqrt{1/2\alpha(SJR)}\,\mathrm{Sin}(\theta_j + \theta_1)]$$

$$Z_{c1} = 2\sqrt{SNR}\,[+\mathrm{Cos}(\theta_1 - \pi/4) + \sqrt{1/2\alpha(SJR)}\,\mathrm{Cos}(\theta_j + \theta_1)]$$

$$Z_{s2} = 2\sqrt{SNR}\,[-\mathrm{Sin}(\theta_1 - 3\pi/4) - \sqrt{1/2\alpha(SJR)}\,\mathrm{Sin}(\theta_j + \theta_1)]$$

$$Z_{c2} = 2\sqrt{SNR}\,[-\mathrm{Cos}(\theta_1 - 3\pi/4) - \sqrt{1/2\alpha(SJR)}\,\mathrm{Cos}(\theta_j + \theta_1)]$$

$$Z_{s3} = 2\sqrt{SNR}\,[-\mathrm{Sin}(\theta_1 + 5\pi/4) + \sqrt{1/2\alpha(SJR)}\,\mathrm{Sin}(\theta_j + \theta_1)]$$

$$Z_{c3} = 2\sqrt{SNR}\,[-\mathrm{Cos}(\theta_1 + 5\pi/4) + \sqrt{1/2\alpha(SJR)}\,\mathrm{Cos}(\theta_j + \theta_1)]$$

$$Z_{s4} = 2\sqrt{SNR}\,[-\mathrm{Sin}(\theta_1 + 7\pi/4) + \sqrt{1/2\alpha(SJR)}\,\mathrm{Sin}(\theta_j + \theta_1)]$$

$$Z_{c4} = 2\sqrt{SNR}\,[+\mathrm{Cos}(\theta_1 + 7\pi/4) - \sqrt{1/2\alpha(SJR)}\,\mathrm{Cos}(\theta_j + \theta_1)]$$

(3.61)

Using equation (3.55), the unconditioned average probability of symbol error for symbol intervals which are jammed, P_{EJ}, can be calculated for different values of SNR, SJR, and α, where;

SNR = Signal to background noise ratio

SJR = Signal to jammer power ratio and

α = the fraction of total bandwidth which is continuously jammed

For the fraction $(1-\alpha)$ of symbol intervals where the jammer is absent, the average symbol error probability P_{E0} is given by mearly substituting $\theta_j=0$ in the previous equations upto (3.54). Equation (3.55) now takes the form

$$P_{E0} = \frac{1}{4} \int_{-\pi}^{+\pi} [P_{E1_0}(\theta_j)+P_{E2_0}(\theta_j)+P_{E3_0}(\theta_j)+P_{E4_0}(\theta_j)] f_{\theta_1}(\theta_1)\,d\theta_1, \tag{3.62}$$

where the terms inside the brackets correspond to the same terms in equation (3.55), except that $\theta_j=0$ in this case.

Thus the average error probability over all symbols (jammed and unjammed) is given by

$$P_E = [\ P_{EJ} + (1-\alpha)P_{EO}], \qquad (3.62)$$

where, P_{EJ} and P_{EO} are given by equations (3.55) and (3.62) respectively.

The final step in the characterization of the performance of frequency-hopped QPSK modulator in the presence of multitone jamming is the conversion of average symbol error probability to average bit error probability.

If the information symbols are assumed to be encoded using a Gray code, the average bit error probability P_b is then approximated for large SNR by:

$$P_b = 0.5P_E \qquad (3.63)$$

where, P_E is given by (3.62).

These results can not be reduced mathematically in terms of elementary functions and their closed form evaluation requires complicated computations and approxi mations. Hence, numerical methods are used to compute the final results. The numerical results along with the computer program used to calculate these results are presented in the appendix.

These results are used to obtain the performance characteristics of the FH-QPSK modulator, which are presented graphically in the concluding section of this report.

SECTION IV

PERFORMANCE CHARACTERISTICS AND CONCLUSIONS

The performance characteristics of a partially coherent FH-QPSK modulator in the presence of jamming is presented in this section. The performance criteria is the bit error rate probability.

The probability of error is plotted for different values of α, as a function of SNR (E_b/N_o) and SJR (E_b/N_J). The factor α is the fraction of frequency band which is continuously jammed. α =1 indicates full-band jamming as the worst case. However, in the worst case jammer strategy, the worst case α for which the bit error rate probability approaches a maximum occurs for the values of α between 0 and 0.1.

The performance characteristics for SNR = 4dB and 6dB are shown in Figure 4-1 and 4-2 respectively. The probability of error, P_b, is plotted against α, for different values of SJR. It is seen that for fixed E_b/N_o and E_b/N_J, and for α greater than 0.1, the probability value decreases as a function of α. However, there exists a value of α between 0 and 0.1, which maximizes P_b and thus represents the worstcase multi-tone

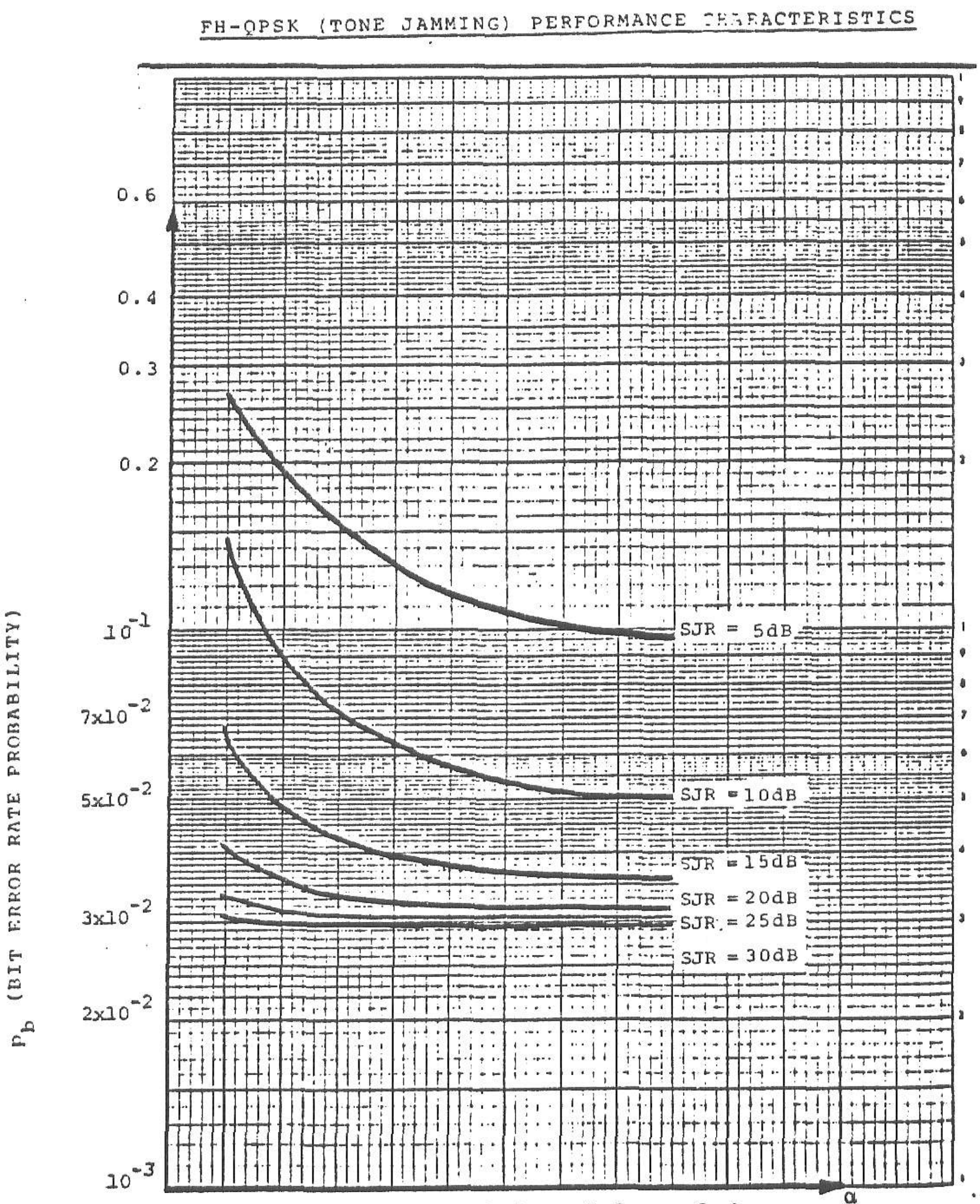


FIGURE 4-1

PLOT OF P_b VERSUS α WITH SNR = 4dB

FH-QPSK (TONE JAMMING) PERFORMANCE CHARACTERISTICS

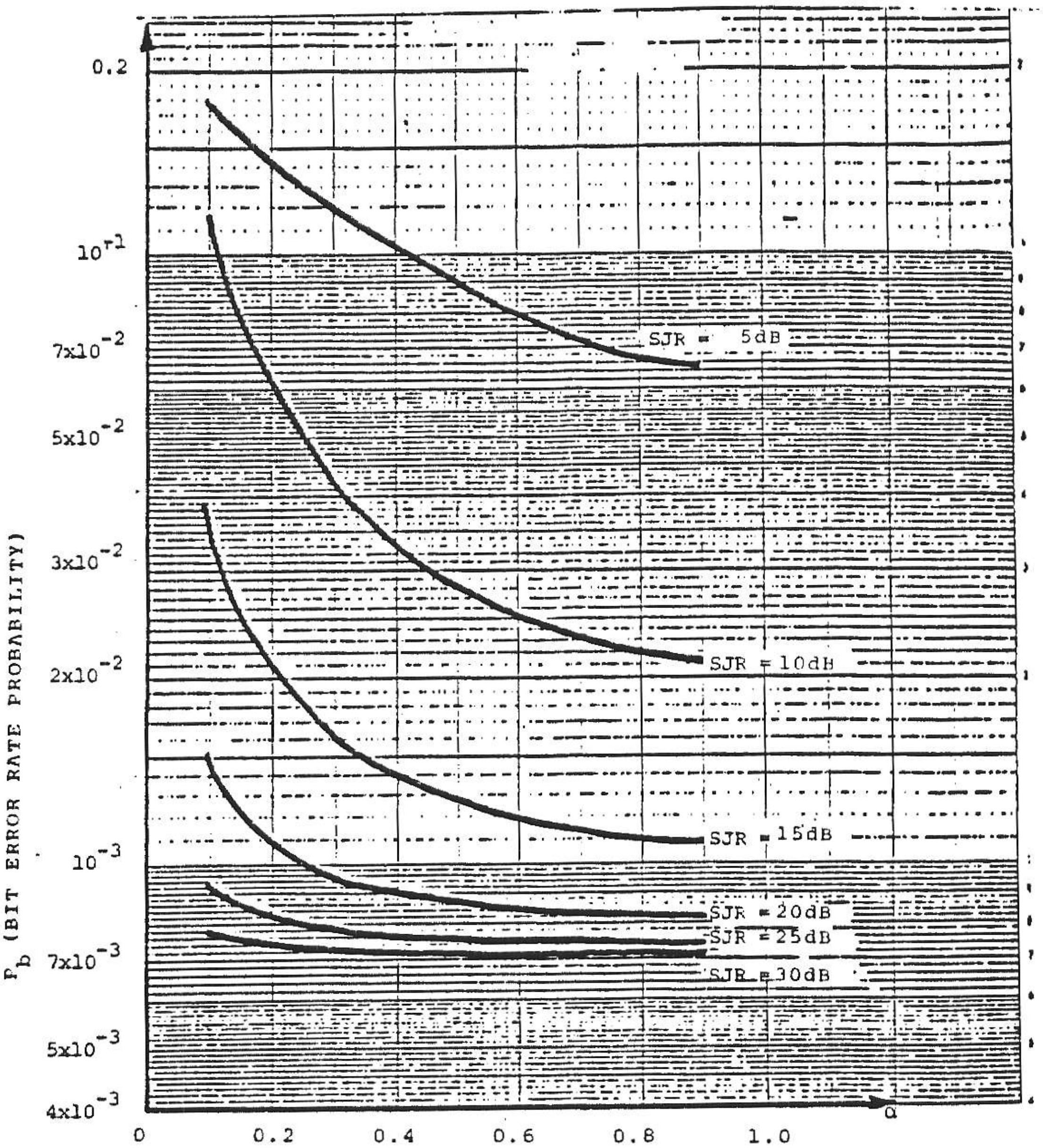


FIGURE 4-2

PLOT OF P_b VERSUS α WITH SNR = 6 dB

jammer situation. The plot is not expanded between 0 and 0.1 due to program limitations. Further, it is also seen that the probability value decreases as the signal to noise ratio increases. It can be noticed from both the plots, that for signal to jammer ratio greater than about 25dB, the curves tend to trace a horizontal line and approach each other. This indicates that the effect of jammer is significantly reduced for SJR greater than about 25dB. In other words, the jammer effect becomes insignificant for higher values of signal to jammer power ratio.

REFERENCES AND BIBLIOGRAPHY

[1] J.J. Spilker, Jr., "Digital Communications by Satellite," Information and System Sciences Series, Prentice-Hall Electrical Engineering Series, Englewood Cliffs, NJ 1977

[2] H.L. Vantrees, "Detection, Estimation and Modulation Theory," Part I

[3] M.K. Simon and A. Polydoros, "Coherent Detection of Frequency Hopped Quadrature Modulations in the Presence of Jamming - Part I," IEEE Tansactions on Communications, Vol. COM-29, November 1981

[4] G.L. Turin, "Error Probabilities for Binary Symmetric Ideal Reception Through Non-selective Slow Fading and Noise," Proc. IRE, 46, 1958

[5] S.O. Rice, "Mathematical analysis of random noise," Bell System Technical Journal, Vol. 23, July 1944

[6] L.B. Milstein, D.L. Schilling a R.L. Pickholtz, "Comparison of Performance of 16-ary QASK and MSK over a Frequency Selective Rician Fading Channel," IEEE Transactions on Communications, November 1981 (Vol. COM-29, No. 11)

[7] M. Schwartz, W.R. Bennett, and S. Stein, "Communication Systems and Techniques," N.Y. McGraw Hill 1966

[8] D.P. Taylor and D. Cheung, "The Effect of Carrier Phase Error on the Performance of a Duobinary Shaped QPSK Signal," IEEE Transactions on Communications, Volume COM-25, July 1977

APPENDIX

APPENDIX

CALCULATION OF DENSITY FUNCTIONS

This appendix describes the procedure to calculate the density functions $f_X(x)$ and $f_Y(y)$; where

$$X = (N_I \text{Cos}\theta_1 - N_Q \text{Sin}\theta_1) \text{ and}$$
$$Y = (N_I \text{Sin}\theta_1 + N_Q \text{Cos}\theta_1)$$

N_I and N_Q are zero mean Gaussian random variables with variance $N_oT_s/2$. The density functions of N_I and N_Q are given by

$$f_{N_I}(n_I) = \frac{1}{\sqrt{2\pi(N_oT_s/2)}} \exp\left[\frac{-n_I^2}{2(N_oT_s/2)}\right] \quad \text{(A-1a)}$$

$$f_{N_Q}(n_Q) = \frac{1}{\sqrt{2\pi(N_oT_s/2)}} \exp\left[\frac{-n_Q^2}{2(N_oT_s/2)}\right] \quad \text{(A-1b)}$$

Since N_I and N_Q are Gaussian random variables, X and Y, linear combinations of N_I and N_Q are also Gaussian random variables.

Thus,

$$E(x) = \text{Cos}\theta_1 E(N_I) - \text{Sin}\theta_1 E(N_Q) = 0 \quad \text{(A-2a)}$$
$$E(y) = \text{Sin}\theta_1 E(N_I) + \text{Cos}\theta_1 E(N_Q) = 0 \quad \text{(A-2b)}$$

$$Var(x) = E(x^2)$$

$$= E(N_I^2 Cos^2\theta_1 + N_Q^2 Sin^2\theta_1 - 2N_I N_Q Cos\theta_1 Sin\theta_1)$$

$$= (N_o T_s/2)Cos^2\theta_1 + (N_o T_s/2)Sin^2\theta_1 - 0$$

i.e. $$Var(x) = N_o T_s/2 \tag{A-3a}$$

Similarly,

$$Var(y) = N_o T_s/2 \tag{A-3b}$$

Using equations (A-2) and (A-3), the density functions $f_X(x)$ and $f_Y(y)$ can be written as

$$f_X(x) = \frac{1}{\sqrt{2\pi(N_o T_s/2)}} \exp\left[\frac{-x^2}{2(N_o T_s/2)}\right] \tag{A-4a}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi(N_o T_s/2)}} \exp\left[\frac{-y^2}{2(N_o T_s/2)}\right] \tag{A-4b}$$

PERFORMANCE OF A FREQUENCY-HOPPED NON-COHERENT QUADRATURE PHASE SHIFT KEYED MODULATOR (QPSK) IN JAMMING AND FADING ENVIRONMENTS

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ABSTRACT

This report examines the performance of a frequency-hopped QPSK modulator for a partially coherent communications system in the presence of jamming environment. The average bit error rate probabilities are computed and compared as functions of signal-to-noise ratio (SNR) and signal-to-jammer power ratio (SJR). Asymptotic results depict that the effect of jammer significantly reduces for SJR greater than about 25dB.