

Quantitative Imaging of Point-Like Targets

by

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Mathematics major, Statistics minor

A FINAL HONORS PROJECT

submitted in accordance with the University Honors Program requirements

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## Abstract

The inverse medium scattering problem has multiple important applications across various scientific domains, such as radar, sonar, and nondestructive testing. The goal of the project is to use scattered waves on a measured boundary to construct an imaging function capable of approximating the location and scattering strengths of point scatterers. The approach to this problem utilizes the sampling-type method developed by Dr. Nguyen and his collaborators, which is recognized for its speed and robustness against noisy data. I studied multiple equations in order to understand the construction and the proof behind the imaging function; some included the Sommerfield radiation condition and the Green function. Once given the scattered wave data and its normal derivative, the point's location and scattering strengths can be determined. The wave-based imaging method determines that the point scatterers are located on a large domain, which is discretized by using a set of sampling points  $z$ . The imaging function will approximately indicate when points  $z$  reach the scatterers. Once  $z$  reaches a scatterer, the imaging function will output a numerical result greater than zero, and when it does not reach a scatterer, it will output zero. This result occurs due to the usage of the Bessel function in the imaging function, where the output peaks at  $z$  equaling the point and it quickly decays as the absolute value of  $z$  minus  $x$  approaches infinity. Now that the imaging function is proven to find the scatterer's location, alternative equivalents are written to solve for the scattering strengths of the point scatterer.

Using the imaging function, we reconstructed point-like targets while knowing their location and scattering strength to test the viability of the function. A percentage relative error was included in the estimation of the point scatterer location and strength. The process of reconstruction was repeated with differing points within the domain  $(-2,2) \times (-2,2)$ , as well as differing scattering strengths and noise levels. All results stayed at or below a 10 percent relative error, therefore letting us conclude the imaging function was successful. Figure 1 and Figure 2 displays the successful reconstruction of the points  $(-0.5, 0.5)$  and  $(1, -1)$ .

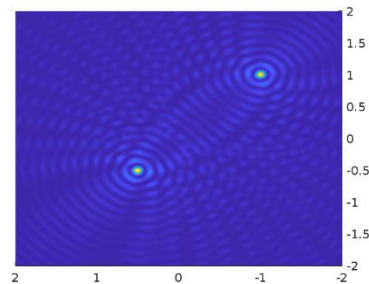


Figure 1. Top view of  $(-0.5, 0.5)$  &  $(1, -1)$ .

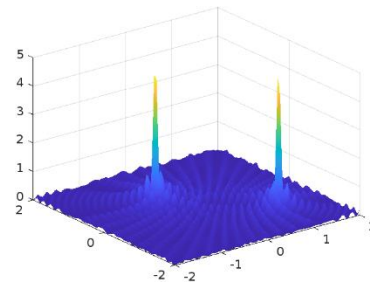


Figure 2. Side view of  $(-0.5, 0.5)$  &  $(1, -1)$ .

# Quantitative Imaging of Point-Like Targets

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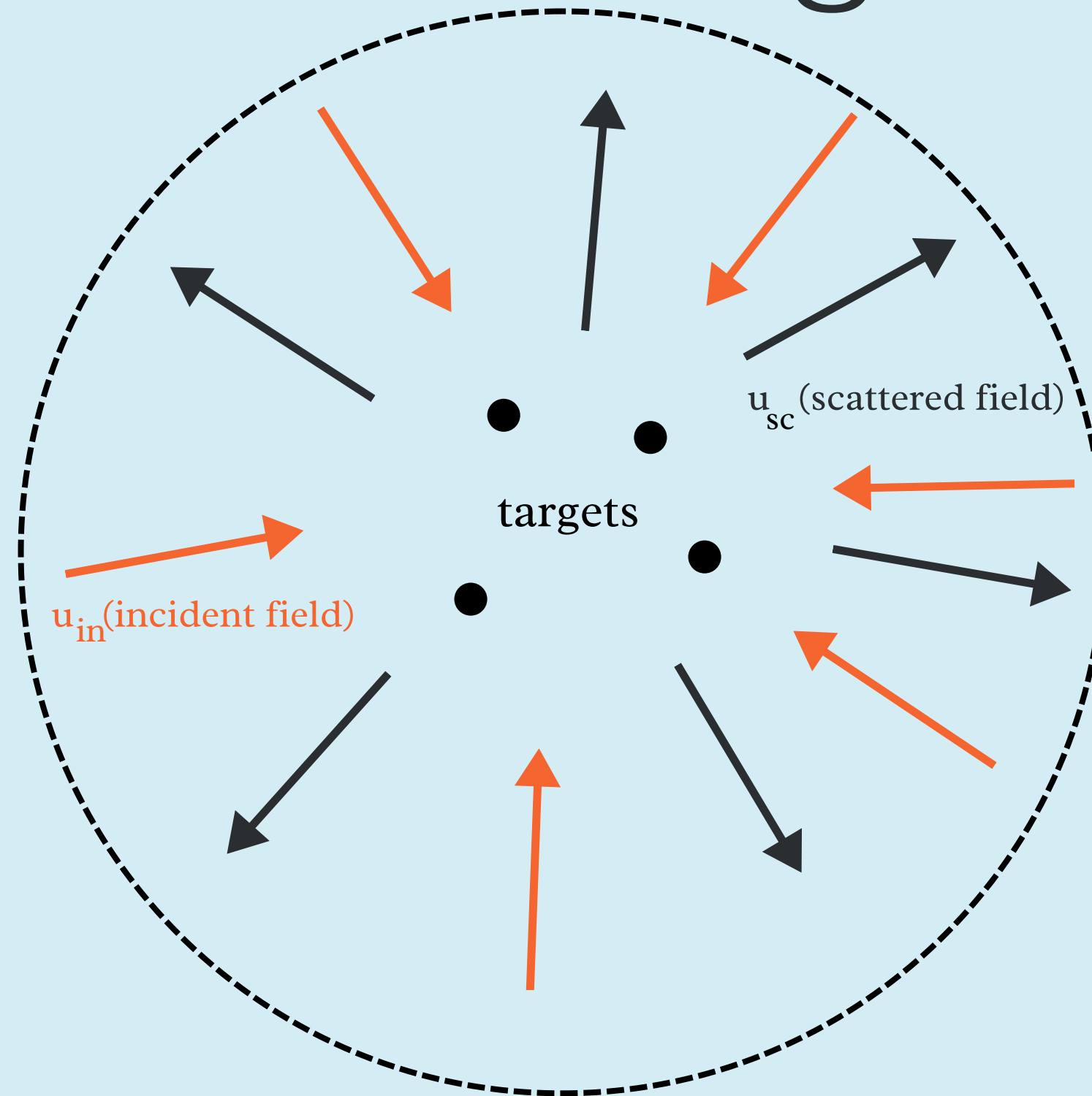
Under the guidance of Nhung Nguyen

*Directed Reading Program*

*December 4th, 2024*

# Inverse Scattering Problem

Inverse problem: Given measurement of scattered waves from incident wave(s) to determine target's location ( $x_j$ ) and scattering strengths ( $\alpha_j$ )



Challenges: non-linear, noisy data can lead to error in reconstruction results



# Applications



Medical Imaging



Non-Destructive Testing



Radar



# Mathematical Equations

$$u = u_{\text{in}} + u_{\text{sc}}$$

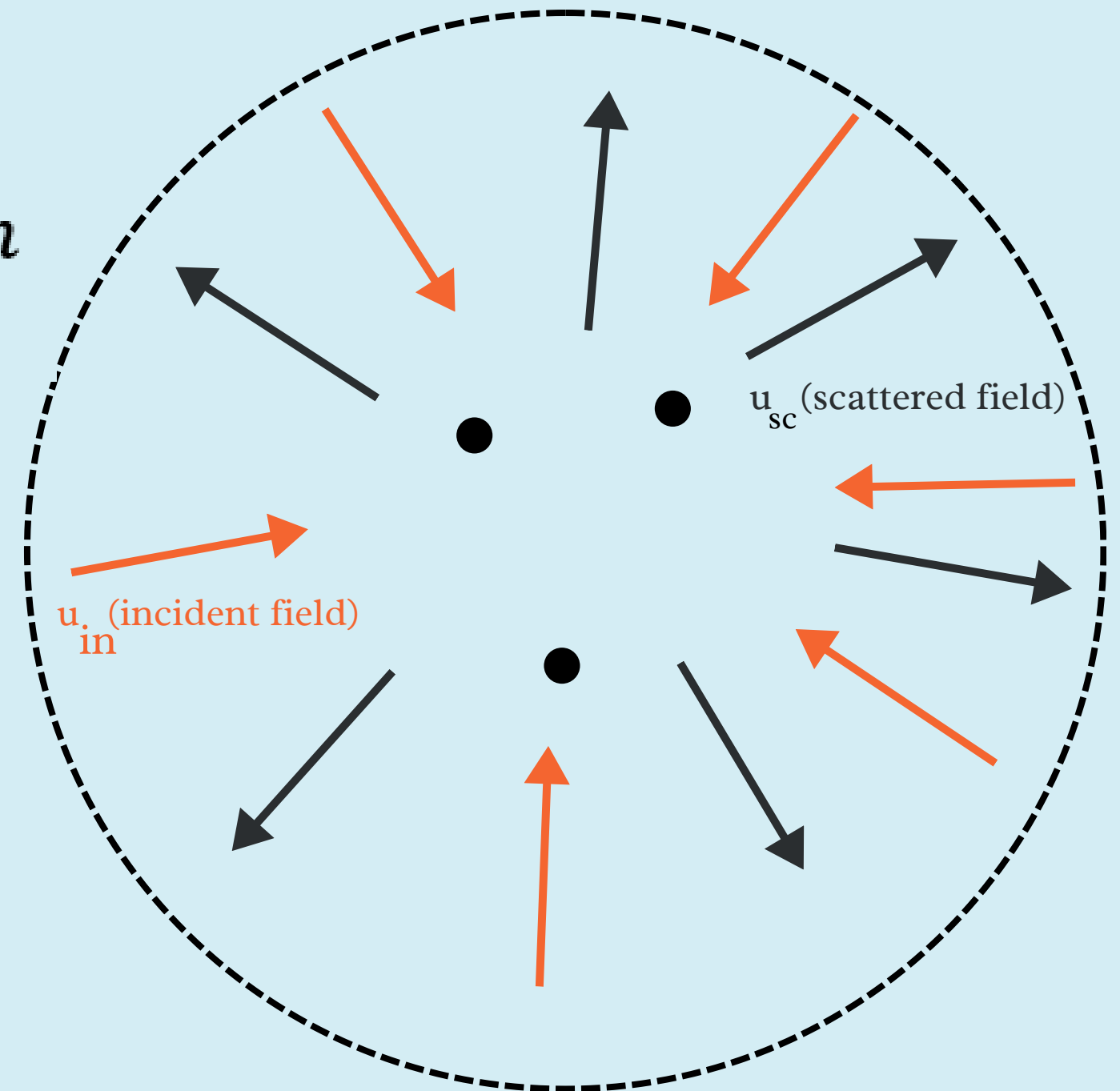
## Helmholtz Equation:

$$\Delta u + k^2(1 + \eta(x))u = 0, \quad x \in \mathbb{R}^n$$

## Sommerfeld Radiation Condition:

$$\lim_{r \rightarrow \infty} r^{\frac{n-1}{2}} \left( \frac{\partial u_{\text{sc}}}{\partial r} - iku_{\text{sc}} \right) = 0, \quad r = |x|$$

- $u$  = total field
- $u_{\text{in}}$  = incident field
- $u_{\text{sc}}$  = scattered field
- $k > 0$ : wave number (fixed)
- $\eta(x)$ : material parameter



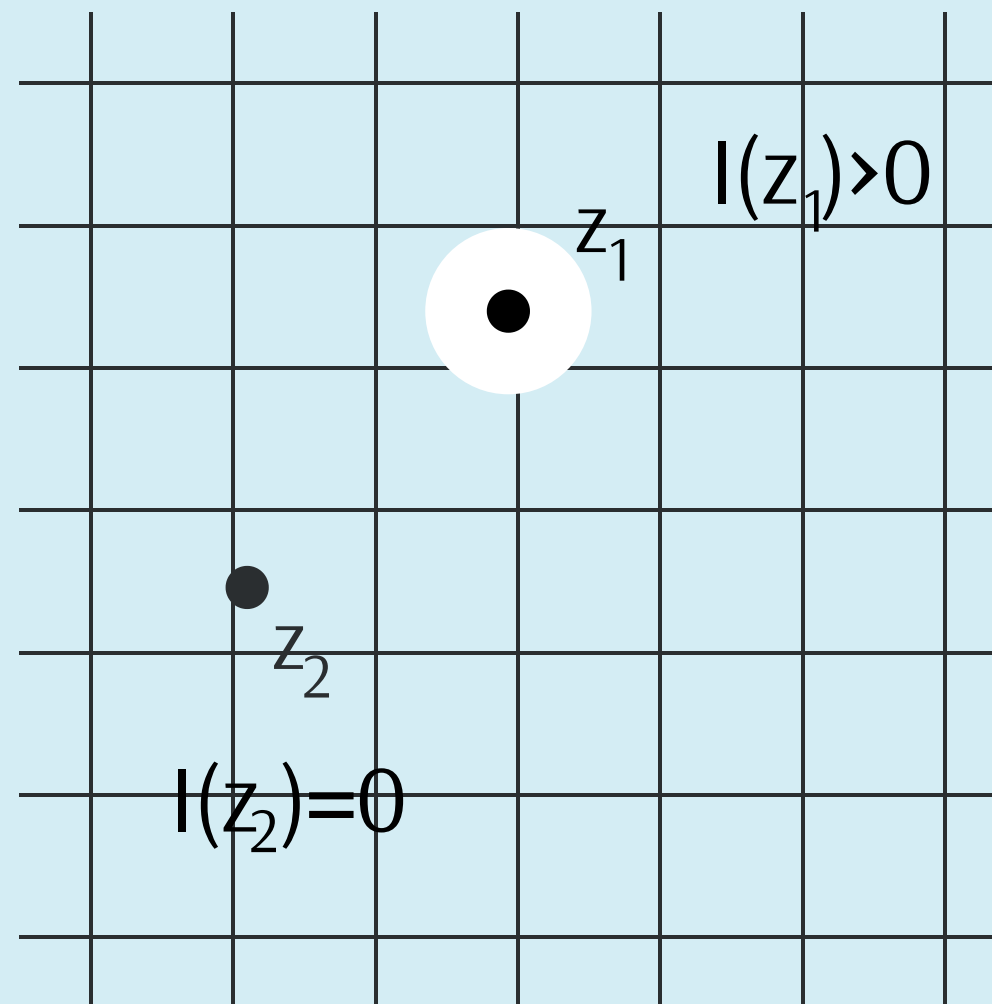
- *Given scattered wave data and its normal derivative, determine point's location and scattering strengths*

# Wave-Based Imaging Method

Solving the inverse problem:

The point scatterers are located in a large domain  $\rightarrow$  discretize it using a set of sampling points  $z$ .

Goal: Build an imaging function,  $I(z)$ , that approximately indicates when points  $z$  reach scatterers



Advantages: Cheap, quick to run, stable against noisy data,

# Imaging Function

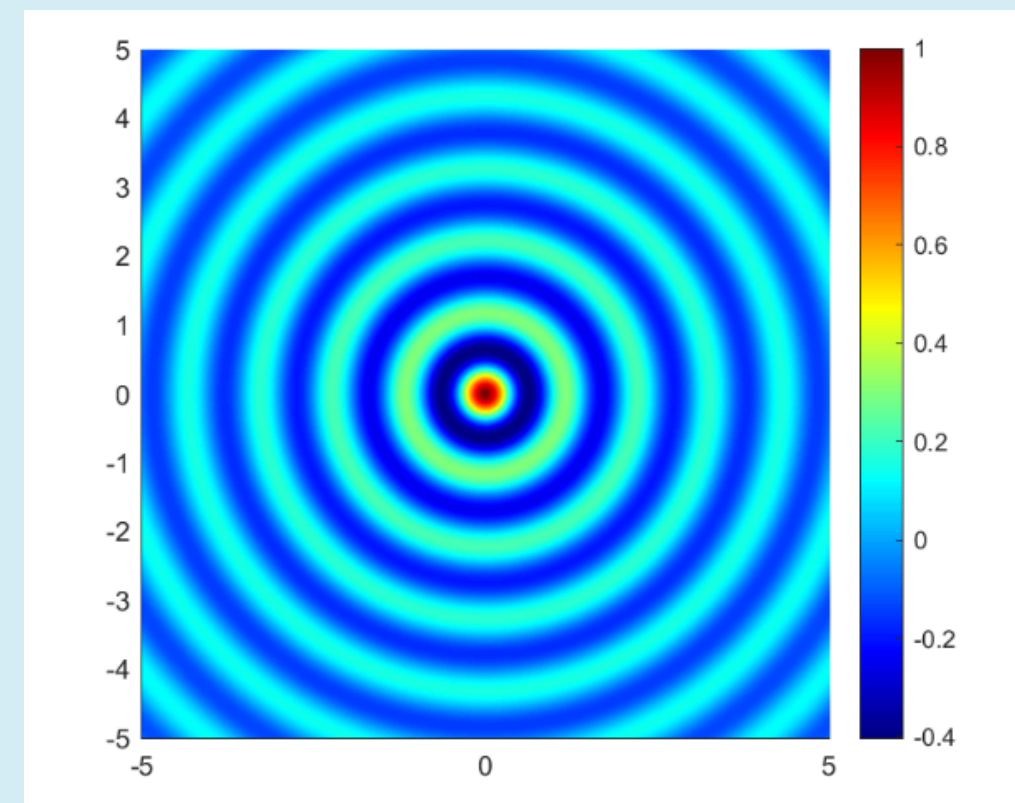
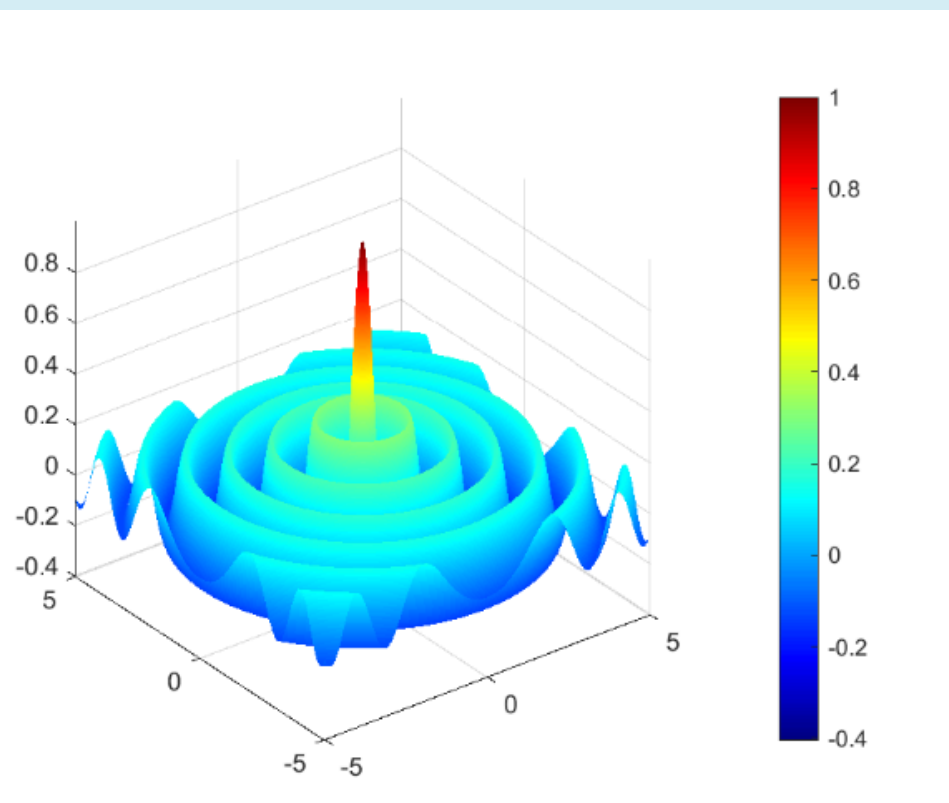
Imaging function for 1 direction: (D. Nguyen, et al.)

$$I(z) := \int_{\partial\Omega} \left( \frac{\partial \operatorname{Im}\phi(x, z)}{\partial \nu(x)} u^{sc}(x) - \operatorname{Im}\phi(x, z) \frac{\partial u^{sc}(x)}{\partial \nu} \right) ds(x)$$

Bessel Function:

$$\operatorname{Im}\Phi(x, z) = J_0(k|z - x|)$$

$\operatorname{Im}\Phi(x, z)$  peaks at  $|z = x|$  and decays quickly as  $|z - x| \rightarrow \infty$





# Imaging Function

Imaging function for 1 direction: (D. Nguyen, et al.)

$$I(z) := \int_{\partial\Omega} \left( \frac{\partial \operatorname{Im}\phi(x,z)}{\partial v(x)} u^{sc}(x) - \operatorname{Im}\phi(x,z) \frac{\partial u^{sc}(x)}{\partial v} \right) ds(x)$$

$M \geq 2$ : number of point scatterers

$\alpha_j$  = scattering strengths

Scattered Field:

$$u_{sc}(x) = \sum_{j=1}^M \alpha_j \phi(x, x_j) u_{in}(x_j)$$

# Imaging Function

Imaging function for 1 direction: (D. Nguyen, et al.)

$$I(z) := \int_{\partial\Omega} \left( \frac{\partial \operatorname{Im}\phi(x,z)}{\partial \nu(x)} u^{sc}(x) - \operatorname{Im}\phi(x,z) \frac{\partial u^{sc}(x)}{\partial \nu} \right) ds(x)$$

Imaging function Equivalent

$$= \sum_{j=1}^M \alpha_j \operatorname{Im}\Phi(z, x_j) u_{in}(x_j)$$

$M \geq 2$ : number of point scatterers

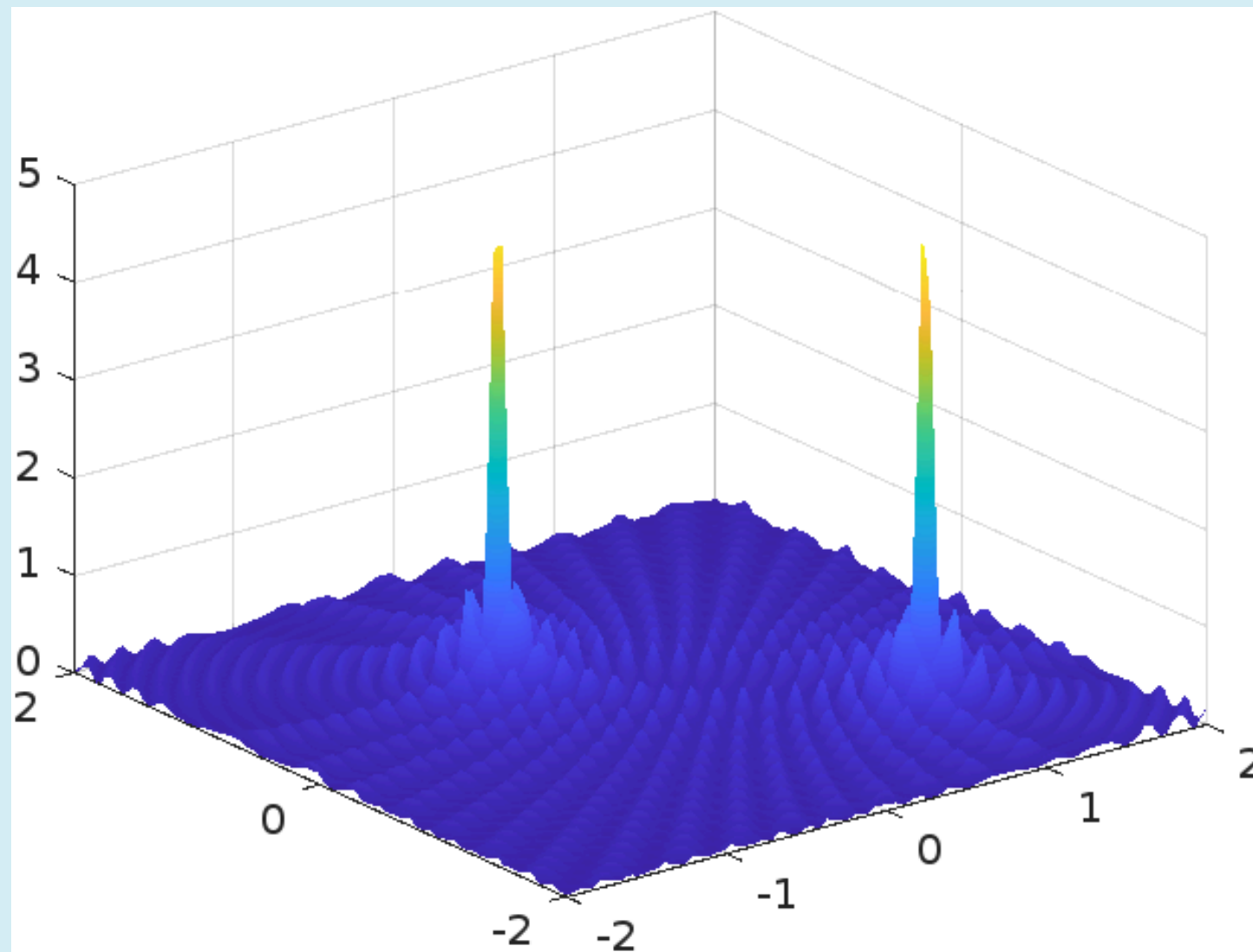
$\alpha_j$  = scattering strengths

$$\alpha_j \approx \frac{I_2(x_j)}{\operatorname{Im}\phi(x_j, x_j) u_{in}(x_j)}$$

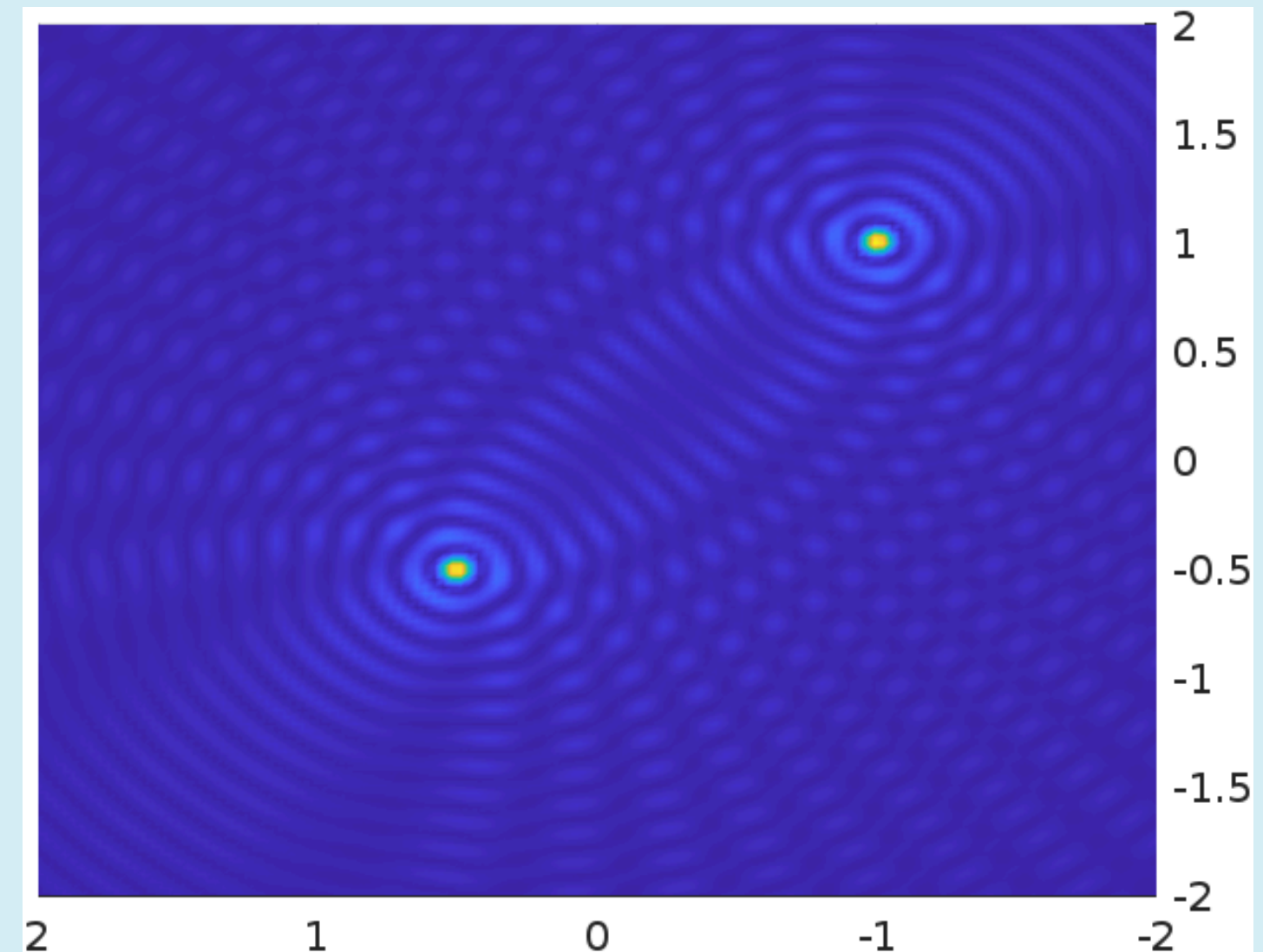
# Examples

$$k = 25$$

random noise = 0%



Side View



Top View

# Examples

$k = 25$

random noise = 0%

|              | True Location | Comp. Location  | Per. Rel. Error | True Alpha | Comp. Alpha    | Per. Rel. Error |
|--------------|---------------|-----------------|-----------------|------------|----------------|-----------------|
| First Point  | (1.0, -1.0)   | (0.992, -0.992) | 0.79            | $1 + 2i$   | 0.9961+2.0255i | 1.15            |
| Second Point | (-0.5, 0.5)   | (-0.520, 0.488) | 3.25            | $1 - 2i$   | 0.9489-2.1285i | 6.18            |

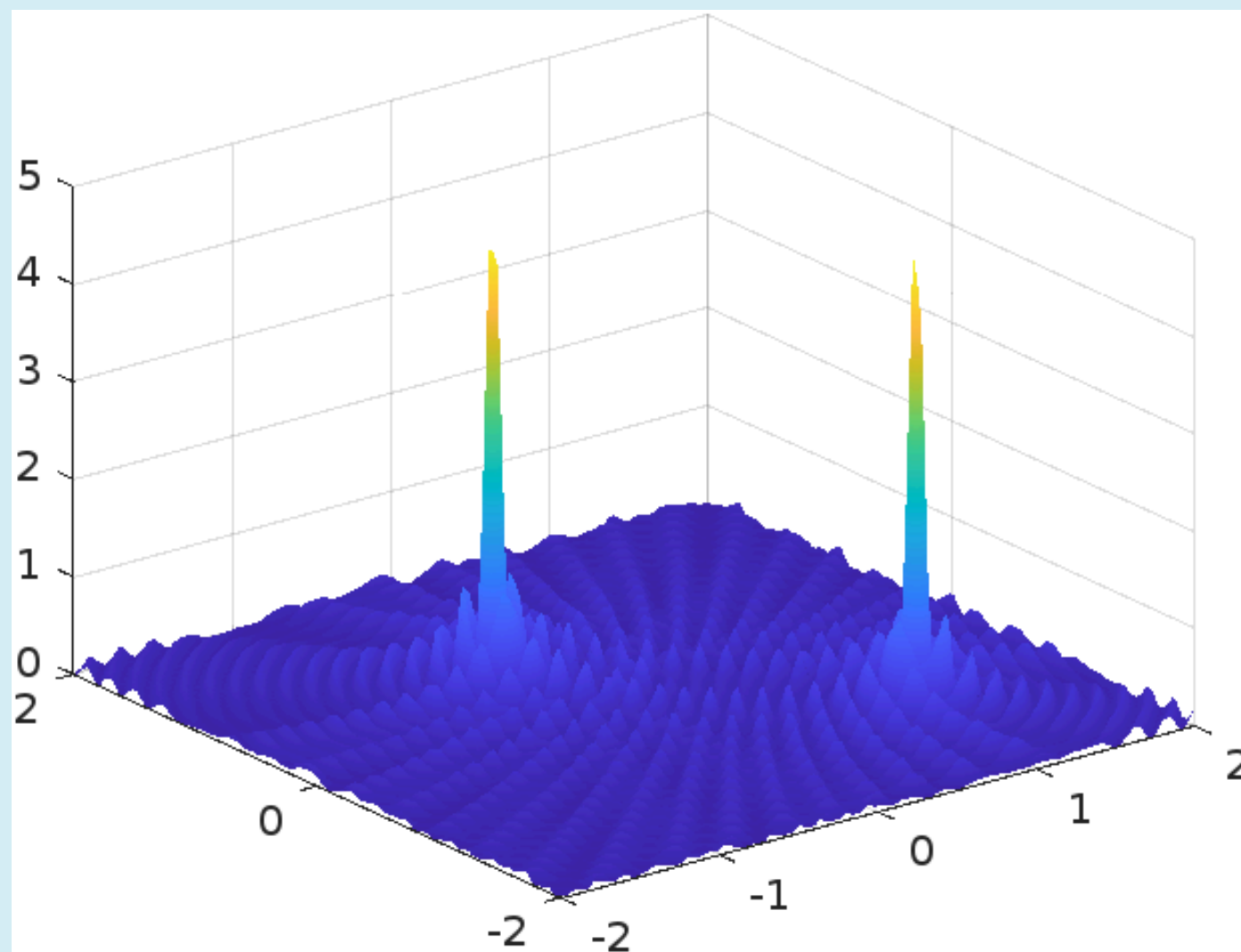


# Examples

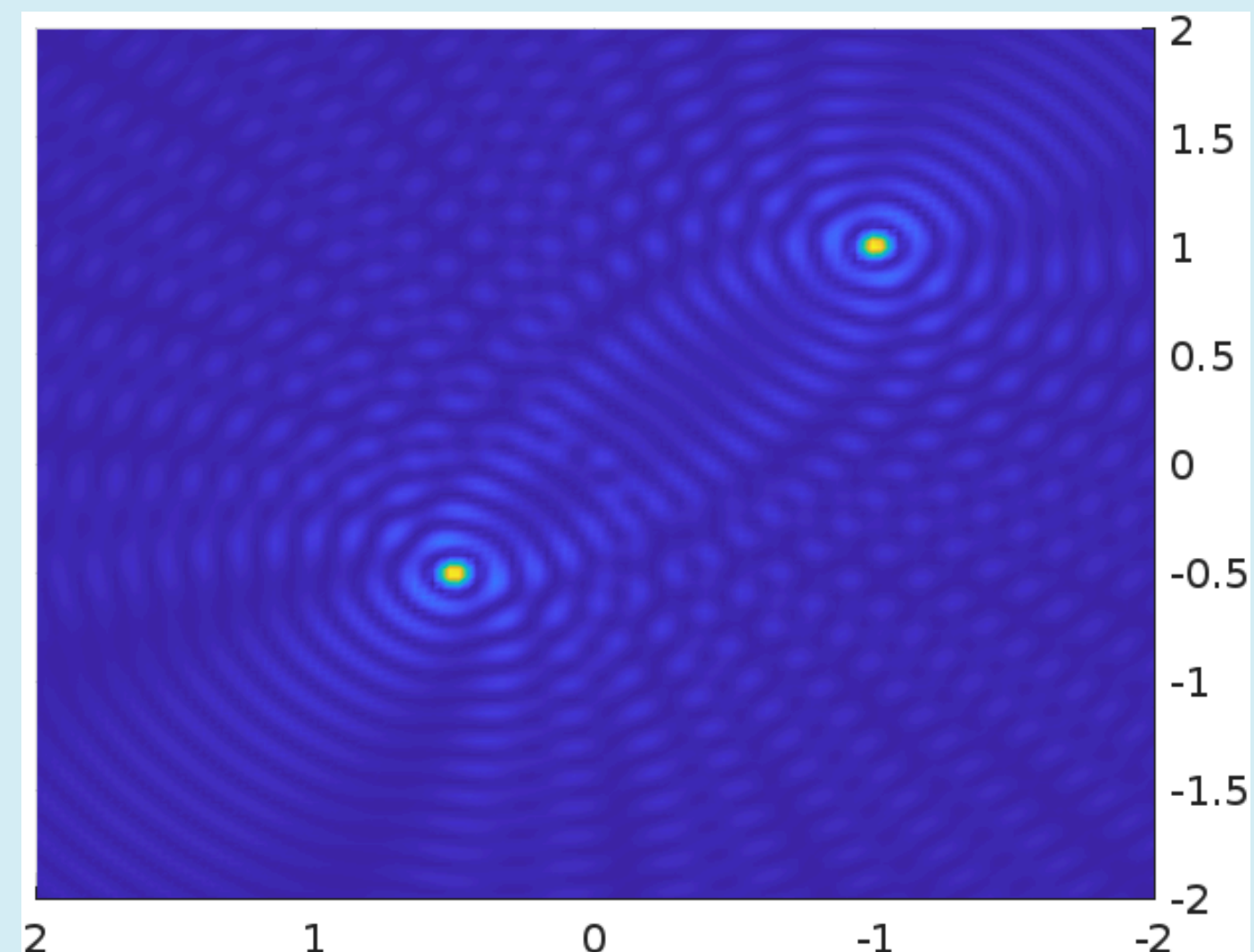
$$k = 25$$

random noise = 20%

change: noise increase



Side View



Top View

# Examples

$k = 25$

random noise = 20%

change: noise increase

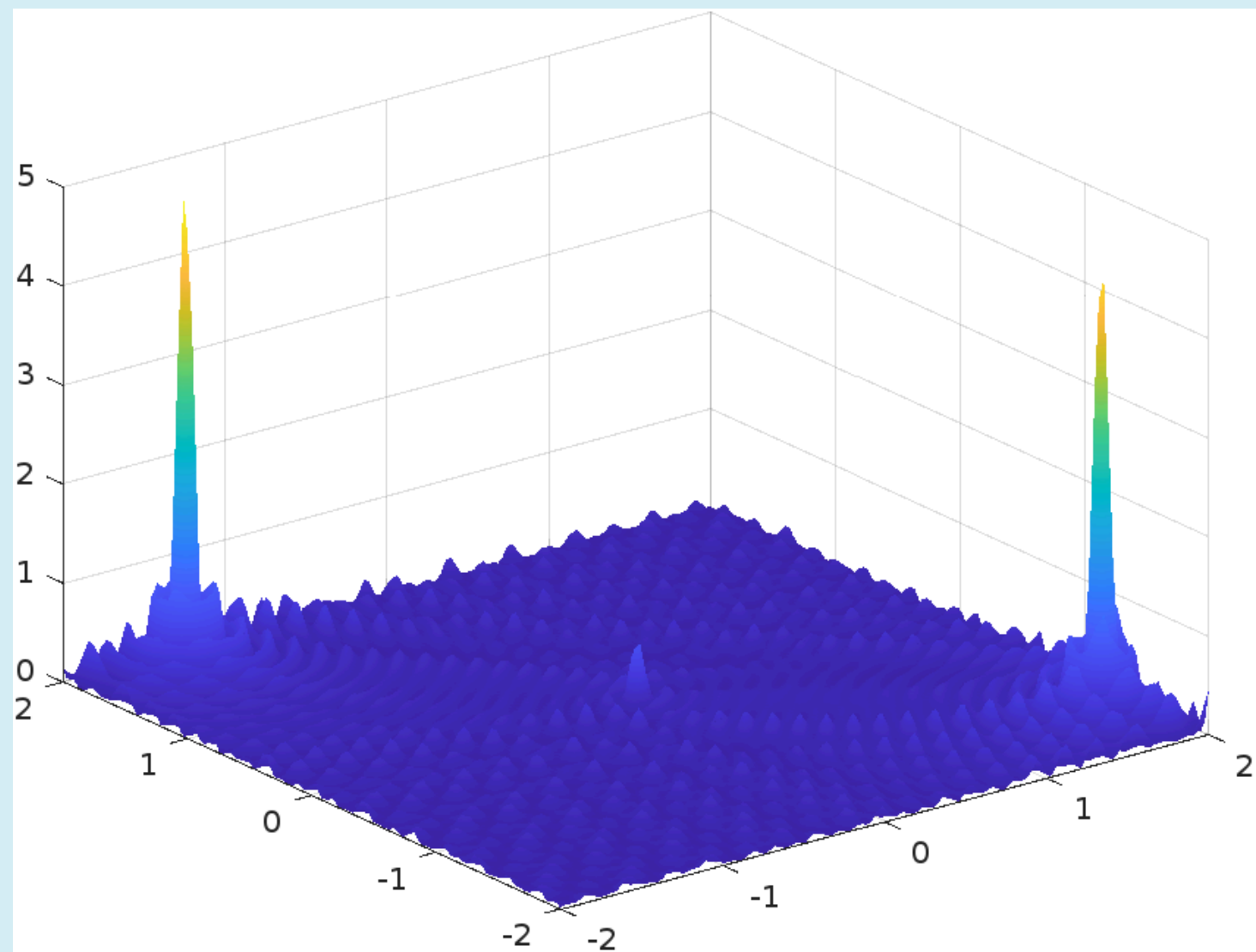
|              | True Location | Comp. Location  | Per. Rel. Error | True Alpha | Comp. Alpha    | Per. Rel. Error |
|--------------|---------------|-----------------|-----------------|------------|----------------|-----------------|
| First Point  | (1.0, -1.0)   | (0.992, -0.992) | 0.79            | $1 + 2i$   | 0.9047+2.0255i | 4.41            |
| Second Point | (-0.5, 0.5)   | (-0.520, 0.520) | 3.94            | $1 - 2i$   | 0.9480-2.2245i | 10.30           |

# Examples

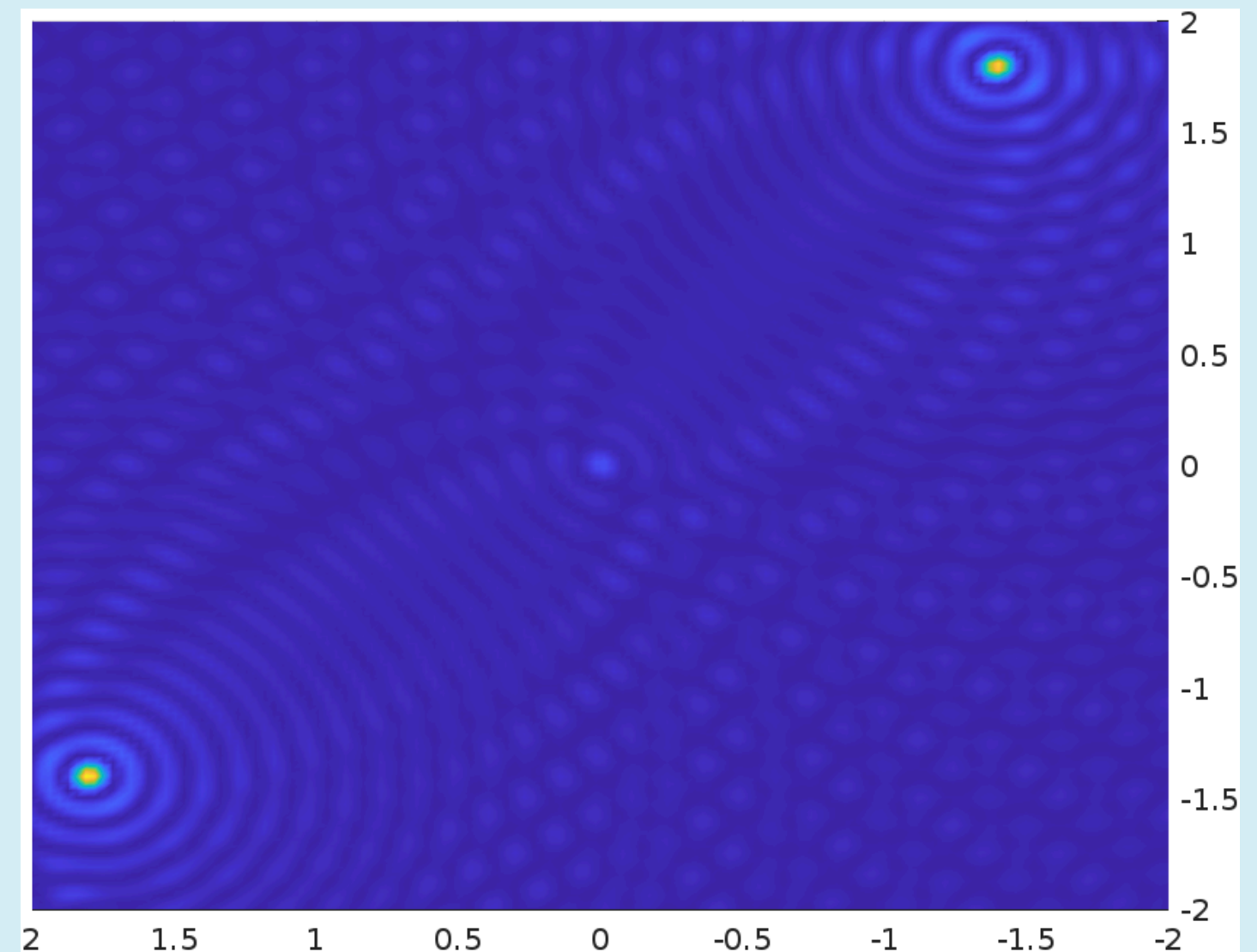
$k = 25$

random noise = 20%

change: points



Side View



Top View

# Examples

$k = 25$

random noise = 20%

change: points

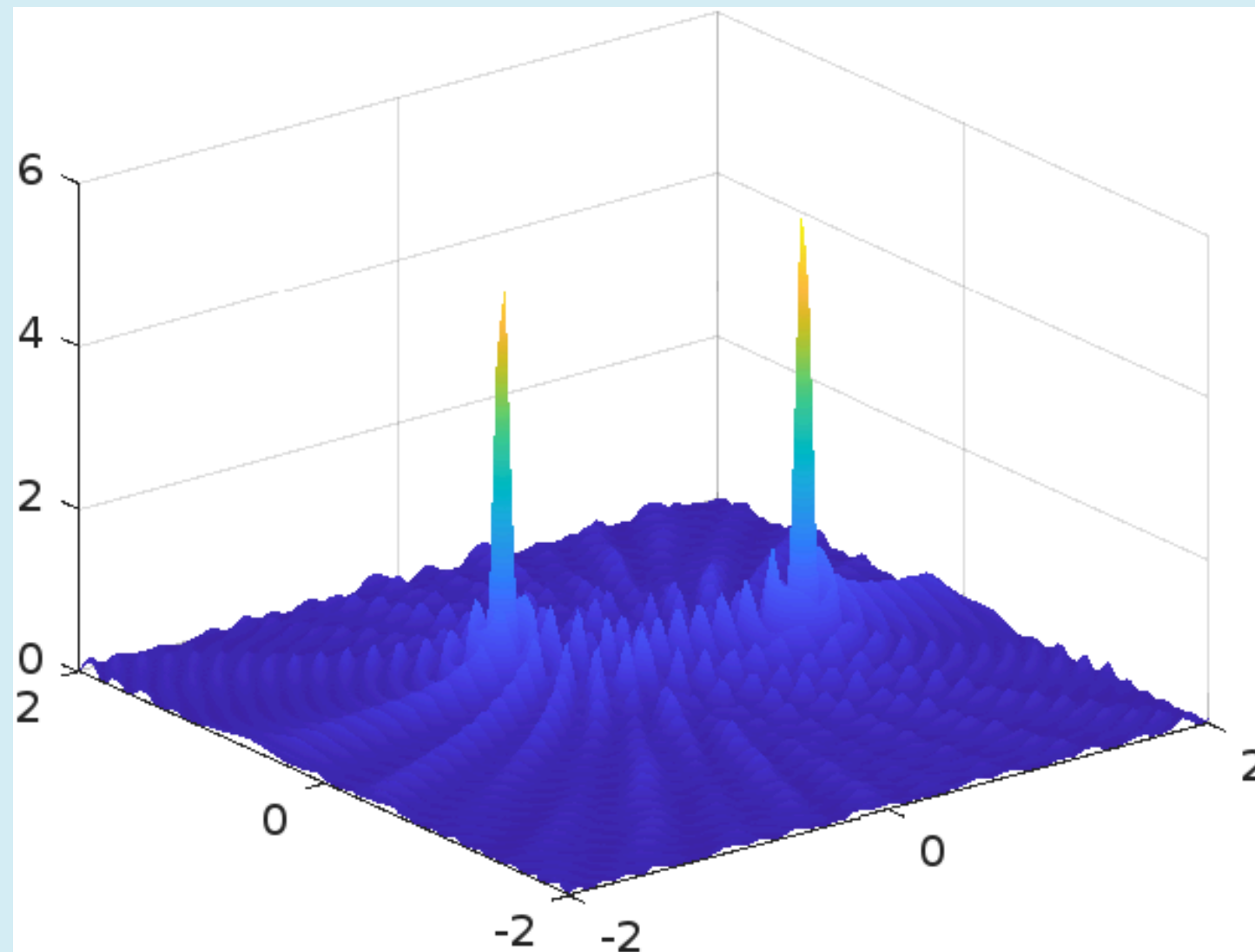
|              | True Location | Comp. Location  | Per. Rel. Error | True Alpha | Comp. Alpha        | Per. Rel. Error |
|--------------|---------------|-----------------|-----------------|------------|--------------------|-----------------|
| First Point  | (-1.4, 1.8)   | (-1.402, 1.811) | 0.49            | $1 - 2i$   | $1.1104 - 1.9277i$ | 5.90            |
| Second Point | (1.8, -1.4)   | (1.811, -1.402) | 0.49            | $1 + 2i$   | $1.1136 + 1.8136i$ | 9.76            |

# Examples

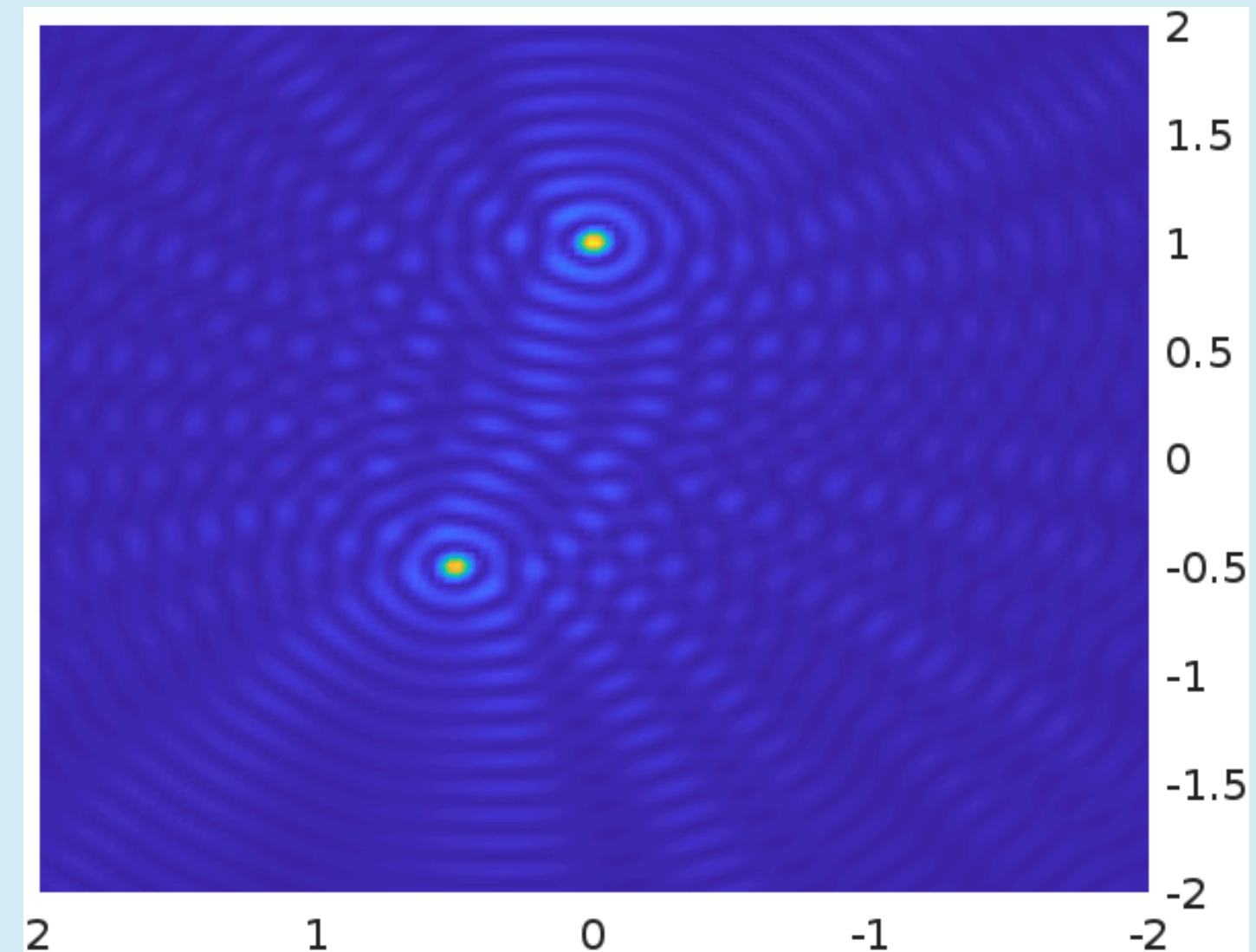
$k = 25$

random noise = 20%

change: points



Side View



Top View



# Examples

$k = 25$

random noise = 20%

change: points

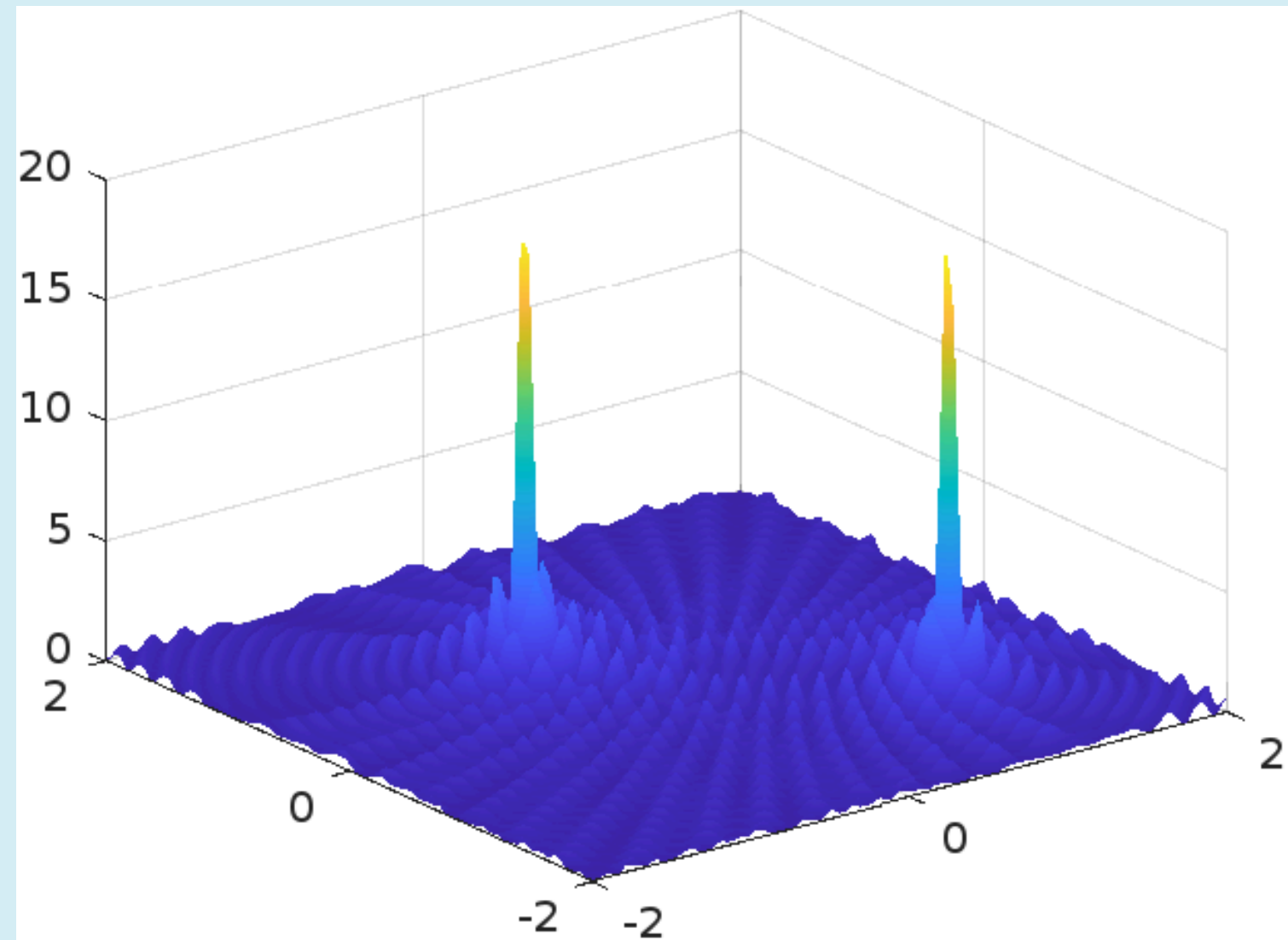
|              | True Location | Comp. Location  | Per. Rel. Error | True Alpha | Comp. Alpha      | Per. Rel. Error |
|--------------|---------------|-----------------|-----------------|------------|------------------|-----------------|
| First Point  | (1.0, 0.0)    | (0.992, 0.016)  | 1.76            | 1+2i       | 1.2732+2.0752i   | 12.67           |
| Second Point | (-0.5, 0.5)   | (-0.488, 0.488) | 2.36            | 1-2i       | 1.1671 - 2.0212i | 7.53            |

# Examples

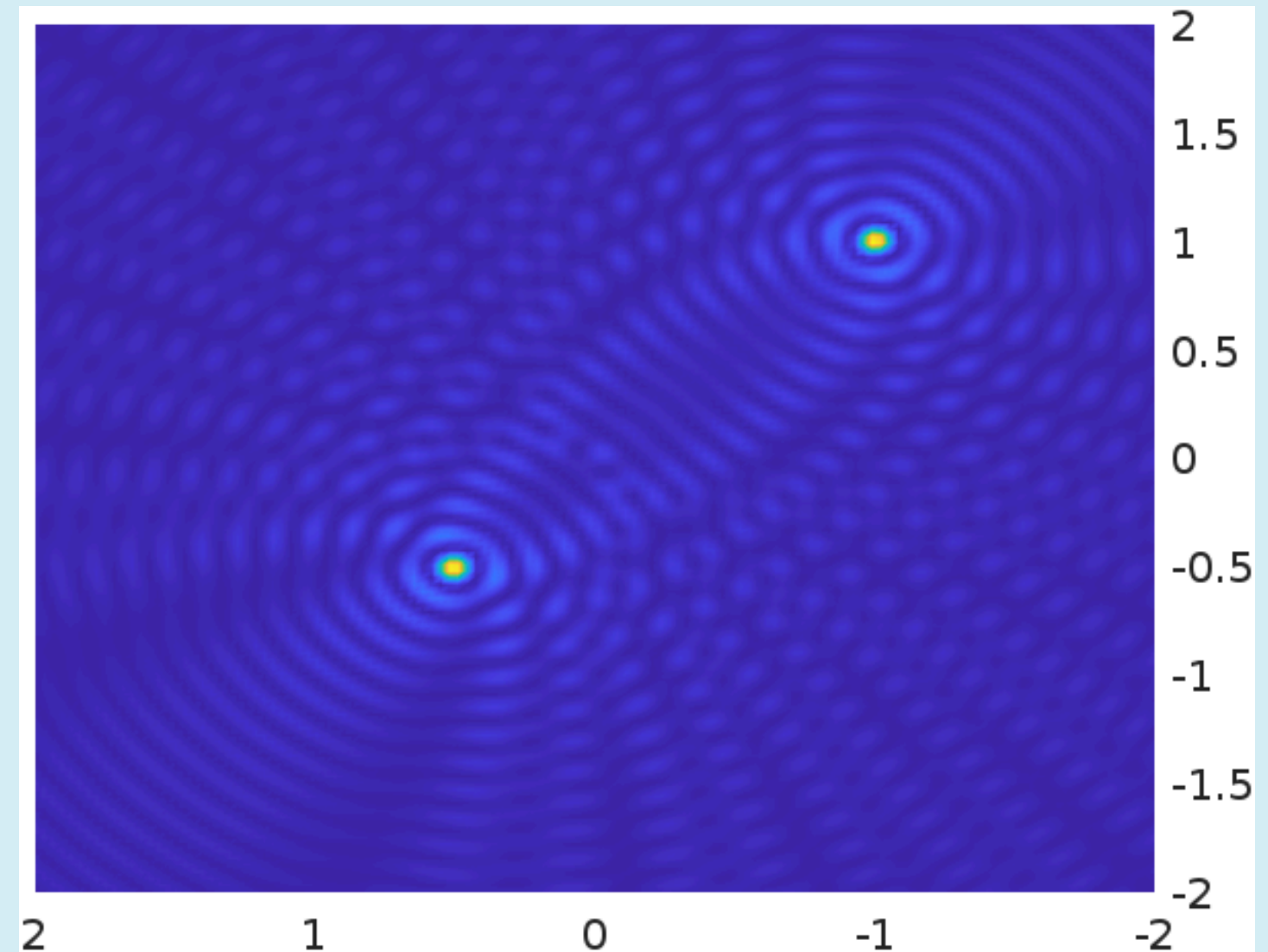
$$k = 25$$

random noise = 20%

change: intensity



Side View



Top View

# Examples

$k = 25$

random noise = 20%

change: intensity

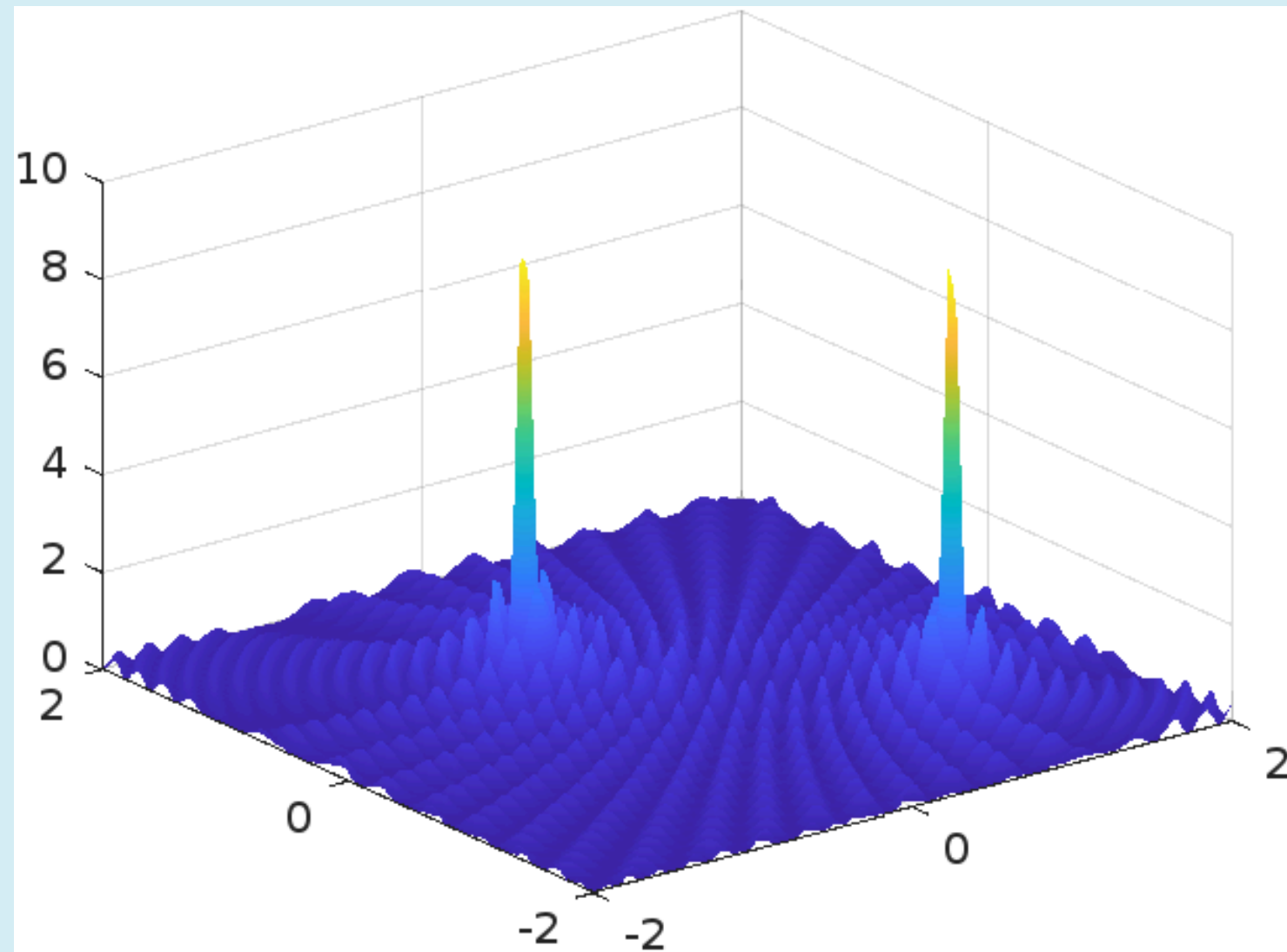
|              | True Location | Comp. Location  | Per. Rel. Error | True Alpha | Comp. Alpha    | Per. Rel. Error |
|--------------|---------------|-----------------|-----------------|------------|----------------|-----------------|
| First Point  | (1.0,-1.0)    | (0.992, -0.992) | 0.79            | 2+4i       | 1.8052+4.0510i | 4.50            |
| Second Point | (-0.5, 0.5)   | (-0.520, 0.520) | 3.94            | 2-4i       | 1.8984-4.4490i | 10.29           |

# Examples

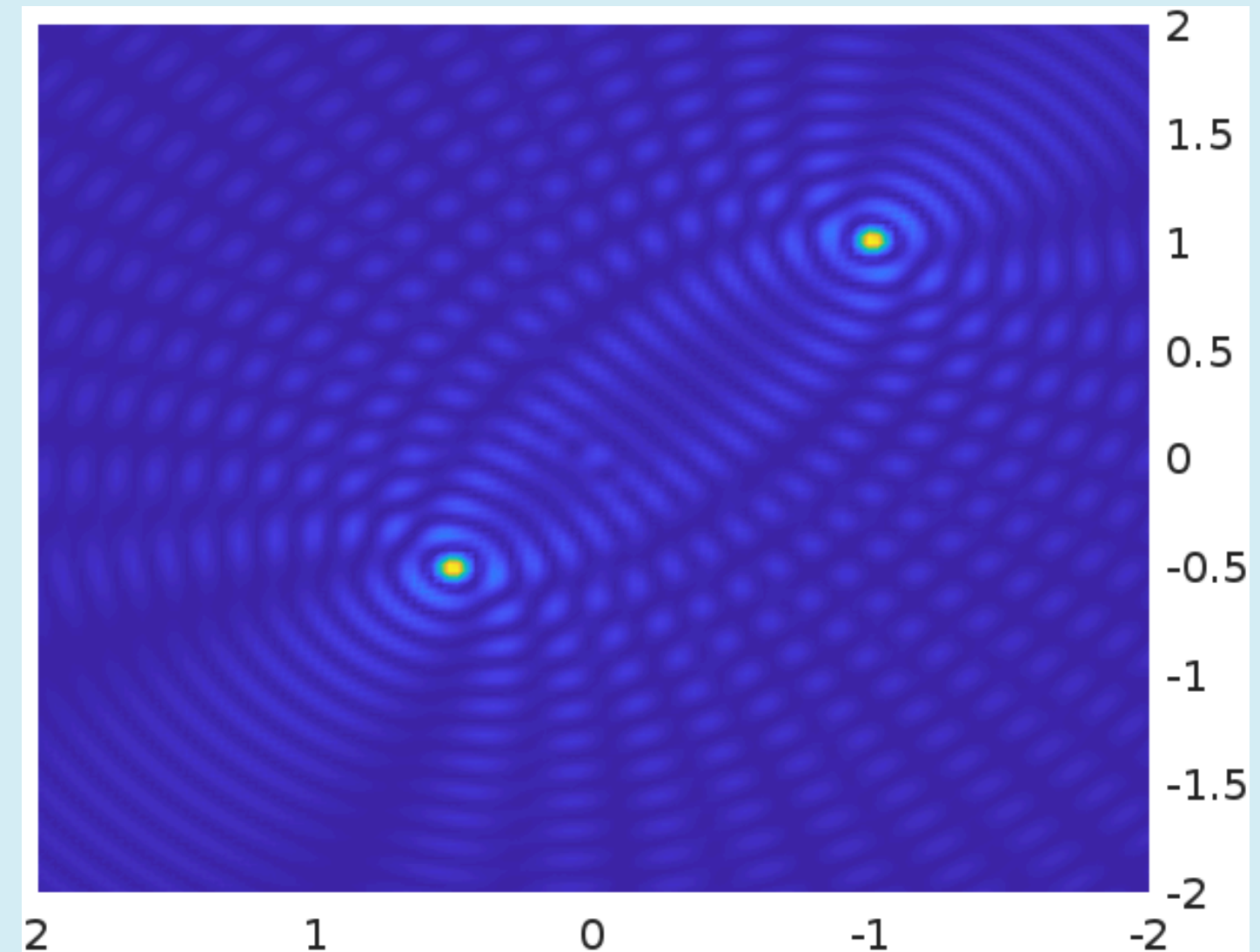
$$k = 25$$

random noise = 20%

change: intensity



Side View



Top View

# Examples

$k = 25$

random noise = 20%

change: intensity

|              | True Location | Comp. Location  | Per. Rel. Error | True Alpha | Comp. Alpha    | Per. Rel. Error |
|--------------|---------------|-----------------|-----------------|------------|----------------|-----------------|
| First Point  | (1.0,-1.0)    | (0.992, -0.992) | 0.79            | 0.5+3i     | 0.3598+3.0436i | 4.83            |
| Second Point | (-0.5, 0.5)   | (-0.520, 0.520) | 3.94            | 0.5-3i     | 0.5151-3.3307i | 10.89           |



# References

1. T. Le, D.-L. Nguyen, V. Nguyen, T. Truong, **Sampling type method combined with deep learning for inverse scattering with one incident wave**, *AMS Contemporary Mathematics* **784** (2023), 63-80.
2. I. Harris, T. Le, and D.-L. Nguyen, **On reconstruction of small sources from Cauchy data at a fixed frequency**, *Applied Mathematics Letters* 145 (2023), 108765.

Thank you!

Any questions?