ARTIFICIAL DELAY LINES

by

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1.0 INTRODUCTION

A lossless inductor-capacitor transmission line terminated in its characteristic resistance delays an input signal for T seconds. The Laplace transform of the delay operator of T seconds is e^{-Ts}.

This report considers some arbitrary artificial delay lines which approximate the delay operator. These artificial lines will be motivated by the Pade approximant of e^{-s}, $\frac{(2 - s)}{(2 + s)}$, and the time function $(1 - e^{-t})^2$.

2.0 PREVIOUS WORK

In an article entitled "Physical Theory of the Electric Wave Filter", G. A. Campbell (1) gives a generalized definition of the artificial line restricted to wave filters. It is: "An artificial line is a chain of networks connected together in sequence through two pairs of terminals, the networks being identical but, otherwise, unrestricted." This generalized artificial line possesses the well known sectional artificial line structure but it need not be an imitation of, or a substitute for, any known real transmission line connecting together distant points. The generalized artificial line consists of identical unrestricted networks which may contain resistance, self-inductance, mutual inductance, and capacitance. Networks may be either in the ladder or lattice form.

Campbell discusses the symmetric balanced lattice equivalent to the generalized artificial line. When the more specific correlation of the behavior of the generalized artificial line at different frequencies is required, it is more convenient to replace the ladder artificial line by the lattice artificial line, which avoids the necessity of considering any impedances which are not individually physically realizable. Since this is an introductory paper, the author has chosen to give only the fundamental characteristics of the artificial line, and the derivation of mathematical formulas is avoided.

The investigation followed in this report was motivated by Pade approximants. In a thesis entitled "Passive Time-delay

Networks", E. C. Bertnolli (2) investigated Pade approximants of the one-second delay operator e^{-s}. The more interesting diagonal and subdiagonal approximants were realized. Diagonal networks are Storch's (3) all-pass and have a time-delay characteristic with (2n - 1) order flatness. Both the diagonal Pade, and the cut-product approximant of the unit delay operator (developed by Warfield (4)), are ideally suited for cascading to form tapped delay lines. Poor pulse responses of diagonal networks are improved by cascading with one of the subdiagonal networks.

Bertnolli has shown that the subdiagonal networks have both time delay and magnitude responses with flatness of order (2n-1). Subdiagonal networks have time delays identical with that of the diagonal network except for $\pi/2$ radians less phase lag, but they have an inferior magnitude response. However, the subdiagonal networks could be utilized as a low-pass filter. This filter has maximally flat magnitude characteristic and maximally flat time delay.

The (1, 1) Pade approximant of e^{-s}, $(2-s)/(2+s)$, will be used in this report.

3.0 FUNCTIONS $g_n(t)$

Define $g_n(t) = (1 - e^{-t})^n$. In addition to the (1, 1) Pade approximant of e^{-s} mentioned previously, this report will extensively use the time function $(1 - e^{-t})^2$ in an exemplary manner. The general case is stated in Lemma 1.

Lemma 1. If $\bar{g}_n(s) = \dfrac{n!}{s(1+s)(2+s)\ldots(n+s)}$, n = 1, 2, 3, ...

then $g_n(t) = (1 - e^{-t})^n$.

Proof. It is true that $L(1 - e^{-t}) = \dfrac{1!}{s(1+s)}$. Verification is now required that $L(1 - e^{-t})^n = \bar{g}_n(s)$ for all integers n. If

$$g_n(t) = (1 - e^{-t})^n$$

then

$$g_{n+1}(t) = g_n(t) - g_n(t)e^{-t} .$$

Since

$$L(e^{at}g_n(t)) = \bar{g}_n(s - a)$$

one obtains

$$\bar{g}_{n+1}(s) = \bar{g}_n(s) - \bar{g}_n(1 + s) .$$

Verification is now possible. Calculation yields

$$\begin{aligned}\bar{g}_{n+1}(s) &= \left[\prod_{k=1}^{n} \left(\frac{k}{(s+k)}\right)\right]\left[\frac{1}{s} - \frac{1}{(s+n+1)}\right] \\ &= \left[\prod_{k=1}^{n} \left(\frac{k}{(s+k)}\right)\right]\left[\frac{n+1}{s(s+n+1)}\right] \\ &= \frac{1}{s} \prod_{k=1}^{(n+1)} \frac{k}{(s+k)} \qquad \text{Q.E.D.} \qquad (1)\end{aligned}$$

3.1 Time-delay Normalization of $g_n(t)$

For convenience in plotting, these functions are normalized so that $g_n(t) = 1/2$ for all integers n. We can write $(1 - e^{-1/T(n)})^n = 1/2$, for all n. Solving for T(n), we get

$$T(n) = \frac{1}{\ell n \left[2^{1/n}/(2^{1/n} - 1)\right]} \tag{2}$$

For a few values of n, T(n) are given in Table 1.

Table 1. Table of normalizing constants

n	:	T(n)
1		1.44
2		0.813
3		0.633
4		0.508
5		0.487

The normalized functions will be called $\bar{q}_n(s)$. An extremely simple form for equation (1), normalized or unnormalized, is possible. This form is given in Lemma 2.

<u>Lemma 2</u>. Delay normalized functions have a partial fraction expansion

$$\bar{q}_n(s) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{s + (k/T(n))} \tag{3}$$

3.2 Normalized Functions $q_n(t)$

From equation (3), the normalized functions are

$$q_1(t) = (1 - e^{-0.693t})$$

$$q_2(t) = (1 - e^{-1.23t})^2$$

$$q_3(t) = (1 - e^{-1.58t})^3$$

$$q_4(t) = (1 - e^{-1.835t})^4$$

$$q_5(t) = (1 - e^{-2.06t})^5$$

The normalized values of $q_n(t)$ are tabulated in Table 2.

Table 2. Normalized values of $q_n(t)$.

Time in seconds	$q_1(t)$	$q_2(t)$	$q_3(t)$	$q_4(t)$	$q_5(t)$
0	0	0	0	0	0
0.25	0.16	0.07	0.036	0.0182	0.011
0.5	0.29	0.212	0.162	0.13	0.1
0.75	0.46	0.282	0.34	0.315	0.305
1.00	0.5	0.5	0.5	0.5	0.5
1.25	0.58	0.62	0.64	0.66	0.68
1.5	0.648	0.711	0.748	0.767	0.83
1.75	0.704	0.782	0.821	0.85	0.875
2.00	0.75	0.84	0.87	0.904	0.92
2.5	0.824	0.91	0.95	0.96	0.975
4.0	0.938	0.99	0.99	0.99	0.99

Values obtained in Table 2 are plotted in Fig. 1.

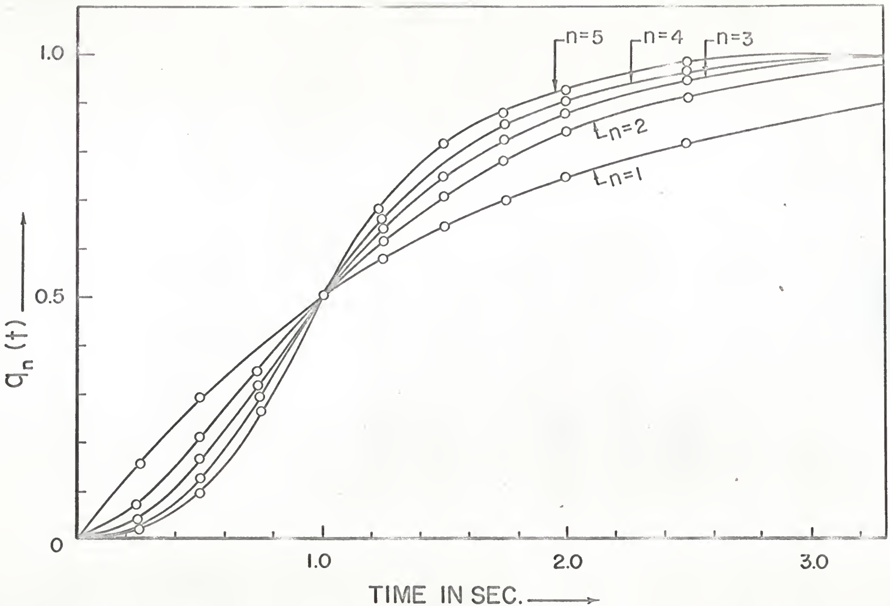

FIG. I. OUTPUT RESPONSES OF $q_n(t)$.

4.0 FUNCTIONS $f_k(t)$

In this section, we investigate the time function $f_k(t) = (1 - e^{-t})^2 p_k(t)$ which corresponds to

$$\overline{f}_k(s) = \frac{1}{s} \cdot \frac{2}{(1+s)(2+s)} \prod_{n=3}^{k} \frac{(n-s)}{(n+s)} \tag{4}$$

The general form of $f_k(t)$ is stated in Lemma 3.

Lemma 3. If

$$\overline{f}_k(s) = \frac{1}{s} \frac{2}{(1+s)(2+s)} \prod_{n=3}^{k} \frac{(n-s)}{(n+s)}$$

and $\overline{f}_2(s)$ is defined as $2/s(1+s)(2+s)$, then

$$f_k(t) = \sum_{m=0}^{k} \phi_{km} e^{-mt} .$$

Proof. By partial fraction expansion, $\overline{f}_k(s)$ can be written as

$$\overline{f}_k(s) = \frac{\phi_{k0}}{s} + \frac{\phi_{k1}}{(s+1)} + \frac{\phi_{k2}}{(s+2)} + \frac{\phi_{k3}}{(s+3)} + \cdots \frac{\phi_{km}}{(s+k)}$$

$$= \sum_{m=0}^{k} \frac{\phi_{km}}{(s+m)} , \qquad k = 2, 3, 4, \ldots \tag{5}$$

where ϕ_{km} are residues of the poles. In time domain the Laplace transform of equation (5) is written as

$$f_k(t) = \sum_{m=0}^{k} \phi_{km} e^{-mt} \tag{6}$$

4.1 Values of ϕ_{km} for $2 \leq k \leq 6$

We investigate values of ϕ_{km} for k = 1, 2, 3, 4, 5, and 6. These values are obtained by partial fraction expansion and are tabulated in Table 3.

Table 3. Values of ϕ_{km}.

k : m :	0	1	2	3	4	5	6
2	1	-2	1				
3	1	-4	5	-2			
4	1	-20/3	15	-14	14/3		
5	1	-10	35	-56	42	-12	
6	1	-14	70	-168	210	-132	33

It can be observed that for the values tabulated

$$\sum_{m=0}^{6} \phi_{km} = 0 \qquad \text{for} \quad 2 \leq k \leq 6$$

The pattern obtained suggests that (1 - z) is a factor of $f_k(z)$, for all integer values of k greater than 2.

4.2 Factors of $f_k(t)$

Further investigation is required for the multiplicity of the (1 - z) factor. From the values of ϕ_{km} tabulated in Table 3, calculations show that

$$f_2(z) = (1-z)^2$$

$$f_3(z) = (1-z)^2 \, (1-2z)$$

$$f_4(z) = (1-z)^2(1 - \frac{14}{3} z + \frac{14}{3} z^2)$$

$$f_5(z) = (1-z)^2(1 - 8z + 18z^2 - 12z^3)$$

$$f_6(z) = (1-z)^2(1 - 12z + 45z^2 - 66z^3 + 33z^4)$$

where $z = e^{-t}$.

From the factors of $f_k(z)$, it can be conjectured that $(1-z)^2$ is a factor of $f_k(z)$. The proof, in general, is given in Theorem 1.

Theorem 1. If

$$\bar{f}_k(s) = \frac{2}{s(1+s)(2+s)} \prod_{n=3}^{k} \frac{(n-s)}{(n+s)}$$

and $\bar{f}_2(s)$ is defined as $= 2/s(1+s)(2+s)$, then $f_k(z) = (1-z)^2 p_k(z)$ where $z = e^{-t}$.

Proof is given in sections 4.2a and 4.2b.

4.2a (1 - z) Factors

From relation (4), a recursive relation is conveniently written as

$$\bar{f}_{k+1}(s) = \bar{f}_k(s) \frac{(k+1-s)}{(k+1+s)}$$

$$= \bar{f}_k(s) - \frac{2s}{(k+1+s)} \bar{f}_k(s) \qquad (7)$$

Since the initial value of $L^{-1}\left[\frac{\bar{f}_k(s)}{(k+1+s)}\right] = 0$, the time-domain transform of equation (7) is

$$f_{k+1}(t) = f_k(t) - 2(d/dt)L^{-1}\left[\bar{f}_k(s)/(1+k+s)\right] .$$

By the convolution integral theorem, this is written as

$$f_{k+1}(t) = f_k(t) - 2(d/dt) \int_0^t e^{-(k+1)(t-T)} f_k(T)\,dT.$$

Substitution of $f_k(t)$ from (6) yields

$$f_{k+1}(t) = \sum_{m=0}^{k} \phi_{km} e^{-mt} - 2(d/dt)\int_0^t \sum_{m=0}^{k} \phi_{km} e^{-(k+1)t} e^{+(k+1-m)T} dT.$$

Integrating and simplifying, one obtains

$$f_{k+1}(t) = \sum_{m=0}^{k} \frac{(k+1+m)}{(k+1-m)} \phi_{km} e^{-mt} - 2 \sum_{m=0}^{k} \frac{(k+1)}{(k+1-m)} \phi_{km} e^{-(k+1)t} \tag{8}$$

For convenience, it is desirable to define $e^{-t} = z$; therefore

$$f_{k+1}(z) = \sum_{m=0}^{k} \frac{(k+1+m)}{(k+1-m)} \phi_{km} \cdot z^m - 2 \sum_{m=0}^{k} \frac{(k+1)}{(k+1-m)} \phi_{km} z^{k+1} \tag{9}$$

Evaluating equation (9) at z = 1, one obtains

$$f_{k+1}(1) = \sum_{m=0}^{k} \phi_{km} \left[\frac{(k+1+m)}{(k+1-m)} - 2 \frac{(k+1)}{(k+1-m)} \right]$$

Further simplification yields

$$f_{k+1}(1) = - \sum_{m=0}^{k} \phi_{km} = -f_k(1) \ , \qquad k \geq 2 \tag{10}$$

Relation (10) shows that (1-z) is a factor of $f_k(z)$.

4.2b $(1 - z)^2$ Factors

Taking the derivative of (8) with respect to z, one obtains

$$f'_{k+1}(z) = \sum_{m=0}^{k} \frac{(k+1+m)}{(k+1-m)} \phi_{km} m z^{m-1} - 2 \sum_{m=0}^{k} \frac{(k+1)}{(k+1-m)} \phi_{km} (k+1) z^k \tag{11}$$

At $z = 1$, the equation becomes

$$f'_{k+1}(1) = \sum_{m=0}^{k} m\phi_{km} + 2 \sum_{m=0}^{k} \phi_{km} \frac{m^2 - (k+1)^2}{(k+1-m)}$$

which can be further simplified as

$$f'_{k+1}(1) = - \sum_{m=0}^{k} m\phi_{km} - 2(k+1) \sum_{m=0}^{k} \phi_{km} .$$

In terms of $f_k(1)$, and $f_{k+1}(1)$, this can be related as

$$f'_{k+1}(1) = -f_k{}'(1) - 2(k+1)\, f_k(1) . \tag{12}$$

Since $f_k(1)$ and $f_k{}'(1)$ are equal to zero, for some integer k, equation (12) will be

$$f'_{k+1}(1) = 0 , \quad \text{for all } k \geqslant 2 . \tag{13}$$

The additional condition necessary to prove that $(1-z)^2$ is a factor of $f_k(z)$, is that $f''_{k+1}(1) \neq 0$. Differentiating (9) with respect to z and evaluating at $z = 1$, yields

$$f''_{k+1}(1) = \sum_{m=0}^{k} \phi_{km} \frac{(k+1+m)(m-1)m - 2k(1+k)^2}{(k+1-m)} . \tag{14}$$

This is not equal to zero for all $k \geqslant 2$. Relations (13) and (14) prove that $(1-z)^2$ is a factor of $f_k(z)$. Q.E.D.

4.3 Closed Form for ϕ_{km}

Lemma 4. The closed form for ϕ_{km} is

$$\phi_{km} = \frac{2(-1)^m\,(k+m)!}{m!\,(m+2)!\,(k-m)!} \quad \text{for all integers } k \geqslant 2 . \tag{15}$$

Proof. We resort to a mathematical induction proof that

ϕ_{km} is true for all k. From equation (9), which is rewritten for convenience as

$$f_{k+1}(z) = \sum_{m=0}^{k} \frac{(k+1+m)}{(k+1-m)} \phi_{km} \cdot z^m - 2 \sum_{m=0}^{k} \frac{(k+1)}{(k+1-m)} \phi_{km} z^{k+1} ,$$

one needs to prove the following:

$$\text{(i)} \quad \phi_{(k+1,m)} = \phi_{km} \cdot \frac{(k+1+m)}{(k+1-m)}$$

$$\text{and} \quad \text{(ii)} \quad \phi_{(k+1,k+1)} = -2 \sum_{m=0}^{k} \phi_{km} \frac{(k+1)}{(k+1-m)}$$

Substituting for ϕ_{km} in the right-hand side of (i), one obtains

$$\begin{aligned} \phi_{km} \cdot \frac{(k+1+m)}{(k+1-m)} &= \frac{2(-1)^m (k+m)!(k+1+m)}{m!(m+2)!(k-m)!(k+1-m)} \\ &= \frac{2(-1)^m (k+1+m)!}{m!(m+2)!(k+1-m)!} \\ &= \phi_{(k+1,m)} . \end{aligned}$$

Part (i) has been proven. Proof of part (ii) is rather difficult and has not been accomplished. It requires demonstration that

$$\begin{aligned} -2 \sum_{m=0}^{k} \phi_{km} \cdot \frac{(k+1)}{(k+1-m)} &= -4(k+1) \sum_{m=0}^{k} \frac{(-1)^m (k+m)!}{m!m!(2+m)!(k+1-m)!} \\ &= \frac{2(-1)^{k+1}(2k+2)!}{(k+1)!(k+3)!} . \end{aligned}$$

4.4 Unnormalized Functions $f_k(t)$

The values for $f_k(t)$ are tabulated for k = 2, 3, and 5 in Table 4. The functions are already calculated in section 4.2. The values obtained in Table 4 are plotted in Fig. 2.

Table 4. Unnormalized values of $f_k(t)$.

Time in seconds	$f_2(t)$	$f_3(t)$	$f_4(t)$
0	0	0	0
0.25	0.049	-0.027	0.009
0.5	0.156	-0.033	-0.019
0.75	0.28	0.016	-0.034
1.00	0.402	0.107	-0.032
1.25	0.51	0.218	0.024
1.5	0.61	0.34	0.256
1.75	0.688	0.45	0.36
2.00	0.75	0.552	0.445
2.5	0.847	0.71	0.622

4.5 Time Delay Normalization of $f_k(t)$

These functions are normalized as has been done before, so that $f_k(1) = 1/2$. Normalized $u_2(t)$ corresponds to normalized function $q_1(t)$, section 3.1. For large values of k normalization procedure becomes complex because it involves determining the stable roots or zeros of higher order polynomials. Only

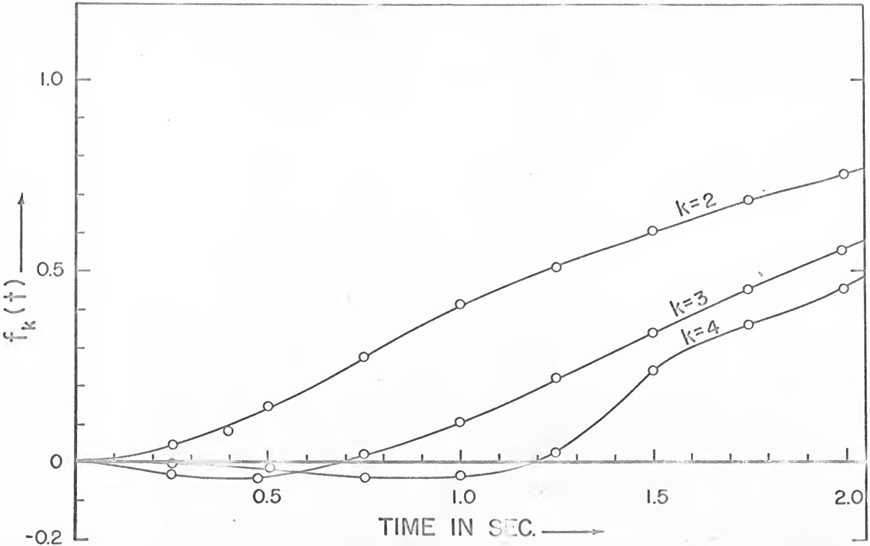

FIG. 2. OUTPUT RESPONSES OF $f_k(t)$.

two functions $f_3(t)$ and $f_4(t)$ are normalized for showing the delay characteristics of normalized functions $u_k(t)$. Sample calculations are shown below.

By normalization procedure, $f_3(1)$ can be written as

$$\frac{1}{2} = (1 - e^{-1/T})(1 - 2e^{-1/T}) \ . \tag{16}$$

Substitution for $e^{-1/T} = p$ in equation (16), one obtains

$$4p^3 - 10p^2 + 8p - 1 = 0 \ . \tag{17}$$

In order to satisfy a stability requirement of the response, p should be chosen such that it is less than 1. The value of p satisfying equation (17) is 0.149, and corresponding value of T is 0.511 as obtained from $e^{-1/T} = p$. The normalized function $u_3(t)$ obtained by substituting T = 0.511 into $f_3(t)$ is

$$u_3(t) = (1 - e^{-1.904t})^2 \ (1 - 2e^{-1.904t}) \ . \tag{18}$$

By similar procedures, $f_4(t)$ is normalized so that $f_4(1) = 1/2$. This yields

$$\frac{1}{2} = (1 - e^{-1/T})^2(1 - \frac{14}{3} e^{-1/T} + \frac{14}{3} e^{-2/T}) \ . \tag{19}$$

Substitution for $e^{-1/T} = p$ in equation (19) yields

$$28p^4 - 84p^3 + 90p^2 - 40p + 3 = 0 \ .$$

The required value of p is 0.094 and corresponding value of T is 0.422. The normalized function $u_4(t)$ obtained by substituting T = 0.422 into $f_4(t)$ is

$$u_4(t) = (1 - e^{-2.364t})^2(1 - \frac{14}{3} e^{-2.364t} + \frac{14}{3} e^{-4.73t}). \tag{20}$$

Values of normalized functions are tabulated in Table 5.

Table 5. Normalized values of $u_k(t)$.

Time in seconds	$u_2(t)$	$u_3(t)$	$u_4(t)$
0	0	0	0
0.2	0.048	-0.037	-0.038
0.25	0.07	-0.035	-0.0301
0.4	0.153	0.016	-0.067
0.5	0.212	0.089	0.01
0.6	0.274	0.166	0.106
0.75	0.282	0.301	0.245
0.8	0.395	0.344	0.346
1.0	0.5	0.5	0.5
1.2	0.6	0.64	0.7
1.25	0.62	0.673	0.697
1.4	0.68	0.75	0.702
1.5	0.71	0.787	0.83
1.6	0.741	0.861	0.876
1.8	0.84	0.905	0.99
2.0	0.842	0.932	0.995
2.4	0.9	0.96	0.997
2.5	0.915	0.983	0.99
3.0	0.95	0.99	0.99

The above normalized values are plotted in Fig. 3.

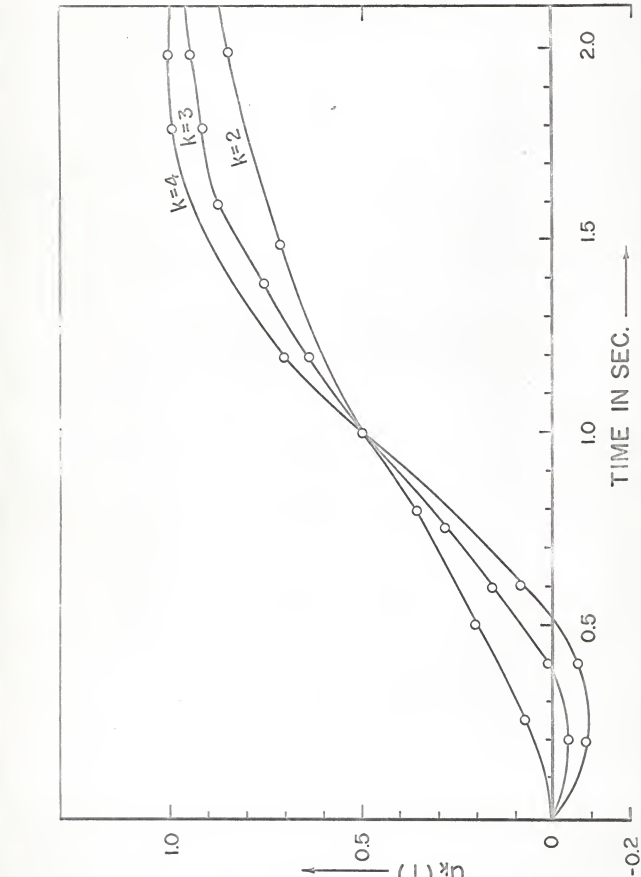

FIG. 3. OUTPUT RESPONSES OF $u_k(t)$.

5.0 TIME DOMAIN RESPONSES OF $\bar{h}_k(s)$

Define $\bar{h}_k(s) = \dfrac{2}{s(1+s)(2+s)} \left[\dfrac{(3-s)}{(3+s)}\right]^k$. Next consider the functions $\bar{h}_k(s)$. They are so defined that k cascaded networks $(3-s)/(3+s)$ are contained in them. This compares with the (1,1) diagonal Pade approximant of the delay operator e^{-s}. Responses of these functions in time domain are investigated. The general case is stated in Lemma 5.

5.1 General Form of $h_k(t)$

Lemma 5. If $\bar{h}_k(s)$ stated in section 5.0 is defined for all integers k, then a general solution in time domain can be written as

$$h_k(t) = \sum_{m=0}^{2} \phi_{km}\, e^{-mt} + \sum_{m=0}^{k-1} t^m \psi_{km}\, e^{-3t} .$$

Proof. $h_k(s)$ can be written in partial fraction expansion form

$$h_k(s) = \frac{\phi_{k0}}{s} + \frac{\phi_{k1}}{(s+1)} + \frac{\phi_{k2}}{(s+3)} + \frac{\psi_{k1}}{(s+3)^k} + \frac{\psi_{k2}}{(s+3)^{k-1}} + \cdots + \frac{\psi_{km}}{(s+3)} \tag{21}$$

The time-domain transform of equation (21) is conveniently written as

$$h_k(t) = \sum_{m=0}^{2} \phi_{km} e^{-mt} + \sum_{m=0}^{k-1} t^m \psi_{km} e^{-3t} .$$

5.2 Exact Form of $h_k(t)$

$\bar{h}_k(s)$ can be rewritten in the form

$$\bar{h}_k(s) = \frac{2}{s(1+s)(2+s)} \left[-1 + \frac{6}{(3+s)}\right]^k .$$

Binomial expansion of $\bar{h}_k(s)$ gives

$$\bar{h}_k(s) = \frac{2}{s(1+s)(2+s)} \sum_{r=0}^{k} \binom{k}{r} (-1)^{k-r} \left(\frac{6}{(3+s)}\right)^r .$$

By convolution integral, the time domain responses of $\bar{h}_k(s)$ can be written as

$$h_k(t) = \sum_{r=0}^{k} \binom{k}{r} (-1)^{k-r} 6^r \int_0^t \left[1 - e^{-(t-T)}\right]^2 T^{r-1} e^{-3T} dT . \tag{22}$$

The integral in equation (22) is written in short notation as I. I can be split into three parts.

$$I = \int_0^t e^{-3t} T^{r-1} dT - 2e^{-t} \int_0^t e^{-2T} T^{r-1} dT + e^{-2t} \int_0^t e^{-T} T^{r-1} dT .$$

The solution of this integral involves a standard form. Substitution of the integral value in equation (22) yields

$$h_k(t) = 1 - 2^{(k+1)} e^{-t} + 5^k e^{-2t} + \binom{k}{r} (-1)^{k-r} . 6^r . P(t) e^{-3t} \tag{23}$$

where

$$P(t) = \frac{(-1)^n(r-1)!}{(r-1-n)!} (3^{-(n+1)} - 2^{-n} + 1)t^{r-1-n} \qquad (24)$$

5.3 Unnormalized $h_k(t)$

By means of partial fraction expansion, time-domain responses of $h_k(t)$ for $k = 0, 1, 2,$ and 3 are

$$h_0(t) = 1 - 2e^{-t} + e^{-2t}$$
$$h_1(t) = 1 - 4e^{-t} + 5e^{-2t} - 2e^{-3t}$$
$$h_2(t) = 1 - 8e^{-t} + 25e^{-2t} - 18e^{-3t} - 12te^{-3t}$$
$$h_3(t) = 1 - 16e^{-t} + 125e^{-2t} - 110e^{-3t} - 96te^{-3t} - 36t^2e^{-3t}$$

The functions $h_k(t)$ are tabulated for different values of t in Table 6.

5.4 Time Delay Normalization of $h_k(t)$

As before, the normalization procedure is such that $h_k(1) = 1/2$ for all integers k. In particular for $k = 0, 1,$ and 2, one obtains normalization constants T_1, T_2, and T_3, respectively. From Table 1, section 3.1, $T_1 = 0.813$, and $T_2 = 0.511$ was obtained in section 4.5. T_3 is obtained from

$$1/2 = 1 - 8e^{-1/T_3} + 25e^{-2/T_3} - 12/T_3\ e^{-3/T_3} - 18e^{-3/T_3} .$$

Substitution for e^{-1/T_3} by p, yields

Table 6. Unnormalized values of $h_k(t)$.

Time in seconds	$h_0(t)$	$h_1(t)$	$h_2(t)$
0	0	0	0
0.2	0.0332	-0.036	-0.17
0.25	0.0495	-0.0273	0.04
0.4	0.11	-0.0373	0.06
0.5	0.156	-0.0331	-0.032
0.6	0.205	-0.018	0.04
0.75	0.28	0.0163	-0.015
0.8	0.305	0.03	0.075
1.0	0.402	0.107	-0.08
1.2	0.49	0.196	-0.027
1.25	0.51	0.218	-0.04
1.4	0.57	0.329	0.007
1.5	0.61	0.34	0.017
1.75	0.688	0.45	0.18
2.0	0.75	0.552	0.32
2.5	0.85	0.71	0.5

The above values are plotted in Fig. 4.

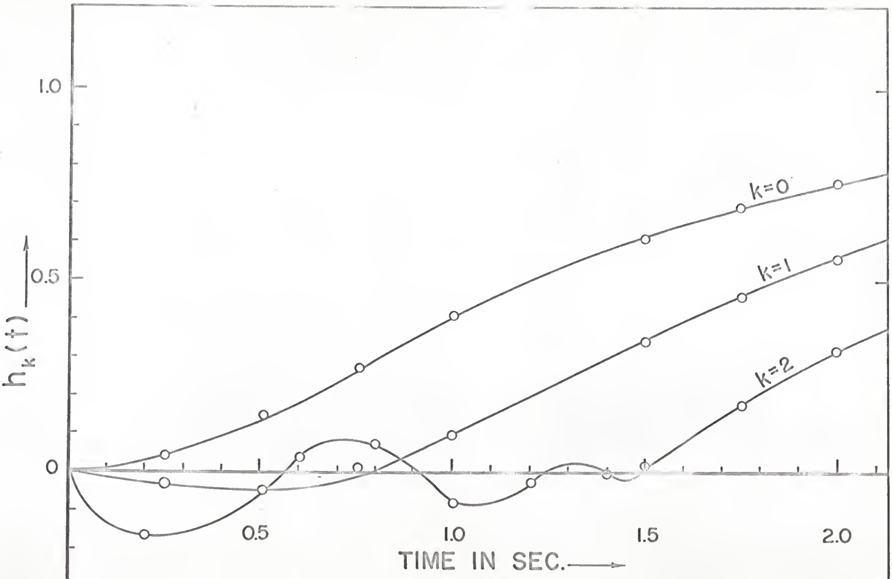

FIG. 4. OUTPUT RESPONSES OF $h_k(t)$.

$$18p^3 - 25p^2 + 8p - \frac{1}{2} - (12 \ln p)p^2 = 0 \tag{25}$$

Considering that the value of p should be positive and less than one, equation (25) yields p = 0.081. The corresponding value of $T_3 = 0.399$. The normalized function $v_2(t)$ is

$$v_2(t) = 1 - 8e^{-2.513t} + 25e^{-5.027t} - 30.1te^{-7.54t} - 18e^{-7.5t} \tag{26}$$

5.5 Normalized Functions $v_k(t)$

The functions $v_0(t)$ and $v_1(t)$ correspond to $q_2(t)$ (obtained in section 3.2) and $u_3(t)$ (normalized in section 4.5), respectively. The functional values of $v_0(t)$ and $v_1(t)$, for different values of time, have been tabulated in Table 2 and Table 5, respectively. Calculations for different values of time for the function $v_2(t)$, equation (26), are shown in Table 7.

Table 7. Values of normalized function $v_2(t)$.

Time in seconds	$v_2(t)$
0	0
0.2	0.033
0.4	0.04
0.6	0.048
0.8	0.252
1.0	0.5
1.2	0.678
1.4	0.794
2.0	0.948

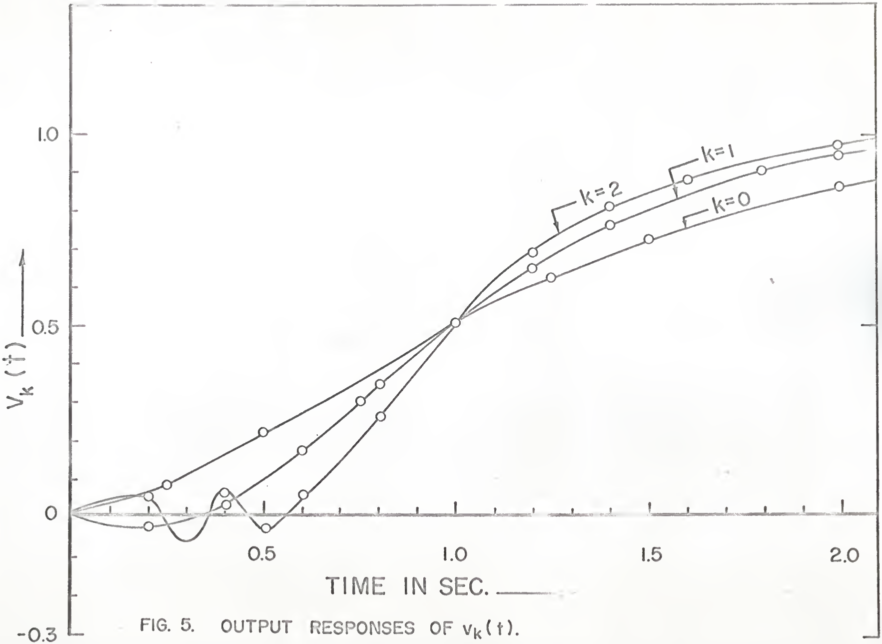

FIG. 5. OUTPUT RESPONSES OF $v_k(t)$.

The values of normalized functions $v_k(t)$ for $k = 0, 1,$ and 2 are plotted in Fig. 5.

6.0 CONCLUSION

Graphs of normalized time functions $q_n(t)$, $u_k(t)$, and $v_k(t)$ versus time, indicate that the time delay responses to a unit step input approximate a delay of one second for integers n and k. The responses of functions $q_n(t)$ are smooth and monotonically increasing. While responses to $u_k(t)$ show undershoots for initial values of time, these undershoots increase for large k. The responses of normalized $v_k(t)$, shown in Fig. 5, indicate that oscillations occur for small values of time. It can be predicted from the graphs of normalized functions that the responses of $v_k(t)$ will have faster rise time compared to $u_k(t)$ and $q_n(t)$.

Campbell's suggestion that many types of artificial delay lines are possible is specifically corroborated.

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REFERENCES

1. Campbell, G. A.
Physical Theory of the Electric Wave Filters. Bell System Technical Journal, Vol. 1, No. 2, pp. 1-32, 1922.

2. Bertnolli, E. C.
Passive Time-delay Networks. M. S. Thesis, Kansas State University, 1961.

3. Storch, L.
Synthesis of Constant-time Delay Networks. Proc. I.R.E., Vol. 42, No. 7-12, pp. 1667-1671, Nov., 1954.

4. Warfield, J. N.
Introduction to Electronic Analog Computers. Englewood Cliffs: Prentice-Hall, Inc., 1959, pp. 94-97.

5. Thomson, W. E.
Delay Networks Having Maximally Flat Frequency Characteristics. Proc. I.E.E., Part 3, Vol. 96, 1949, p. 487.

ARTIFICIAL DELAY LINES

by

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AN ABSTRACT OF A MASTER'S REPORT

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This report considers some arbitrary artificial delay lines which approximate the delay operator e^{-s}. These artificial lines will be motivated by the e^{-s} Pade approximant $\frac{2 - s}{2 + s}$, and the time function $(1 - e^{-t})^2$. The time-domain responses of functions $\overline{g}_n(s)$, $\overline{f}_k(s)$, and $\overline{h}_k(s)$ to a unit step input are considered. Delay normalized responses of these functions approximate a delay of one second. Closed forms and recursive relations of the output responses are derived as lemmas and theorems.