BEAMS ON ONE-WAY ELASTIC FOUNDATIONS

by 632

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# INTRODUCTION

Shallow foundation such as mat foundations and footing foundations are frequently designed and constructed in the form of beams on soil, and loaded by one or several concentrated column loads. One of the important steps in design of such foundations consists of analysis of bending moments and shearing forces due to the concentrated column loads. This analysis, which takes the form of the solution of a beam on an elastic foundation, requires some assumptions for properties and behavior of the soil-foundation system. E. Winkler (1) developed the first theory of beams on elastic foundations early in 1867. The theory is based on the assumption that the intensity of the continuously distributed reaction of the foundation at every point is proportional to the deflection at that point. Since then, refinements and various assumptions have been make in the solution by others notably Hetenyi(2), De Beer(3), Biot(4), and Vesic(5). Alternative mathematical techniques in solving the problem have been proposed by Levinton(6), Popov(7), Malter(8), and Bowles(9). Recently Tsai and Westmann(10) have indicated an approach based on the tensionless foundation assumption to account for the effects of beam up-lift, which satisfies the actual conditions of real soil under elastic theory. The problem of the beam subjected to a single load and supported on a one-way elastic foundation was solved by Lin(11). A matrix formulation for the numerical evaluation of the problem was developed. A study of the beam subjected to concentrated loads and moments and supported on a one-way elastic foundation will be prosented following Lin's formulation.

## PURPOSE OF THE STUDY

As mentioned previously, Lin used the matrix formulation for a beam loaded by a single concentrated load and supported on a tensionless, elastic subgrade. However, footings generally are loaded with more than one load and frequently with moments. In this report, an analysis is made for the purpose of solving problems with various loading conditions which are more likely to be encountered in the actual footing condition. The analysis is based on Lin's matrix formulation and Winkler's assumption modified by Tsai. An iterative solution of beams resting on different subgrades is presented by approximating the subgrade with equally spaced springs with a stiffness per unit length the same as the actual subgrade. The cases of finite and infinite beams under the action of concentrated loads and moments are examined. The results are compared with the results obtained by Lin(11), Levinton(6), Bowles(9) and Fraser(12).

## LITERATURE REVIEW

The theory of beams on elastic foundation was first proposed by E. Winkler(1) in 1867. By the simple assumption that the continuous reaction of the foundation is proportional to the deflection, Timoshenko(13) successively established the sulution to the differential equation which expressed the beam deflection in terms of foundation reaction. Hetenyi(2) made a comprehensive study of beam-foundation problems following the theory of elasticity and the basic mathematical relationship between the subgrade reaction and settlement. Some notable mathematic techniques in solving the problem (such as redundant method and finite difference method) have been developed by Levinton(6) and Malter(8) in 1949 and 1960 respectively.

Leonards and Harr(14) simplified the problem formulation and solution by assuming that the foundation could take tension and a further refirement was made by Kerr(15) in assuming that the subgrade properties are identical in tension and compression. The common feature to all of these works is the assumed mode of stress transfer across the beam-foundation interface. Usually the resulting analysis based on this classical solution is not acceptable, particularly in dealing with the infinite beam, because of the beam uplift. Recently Tsai and Westmann(10) indicated an approach which considered both the Winkler's assumption and the uplift effects of the beam and developed a valid problem formulation and solution by assuming that the foundation can take compression only. Lin's approach presented the digital computer program for the practical solution.

## STATEMENT OF THE PROBLEM

It is assumed in the classical solution for beams on elastic foundations that foundation properties are identical in tension and compression. The resulting analysis indicates an alternating reaction thus implying the foundation can support a tensile stress. Usually this is not an acceptable result for real soil. Therefore, the Winkler model should be modified to take into account the effect of beam uplift. This will then lead to a non-linear solution(10). As the beam is supported along its entire length by a continuous elastic medium, the problem formulation was make by Lin by assuming that the beam rests on "one-way" equal spaced, elastic springs; the more springs chosen along the length of the beam, the closer the analogy is to the continuous medium. The subgrade tensile stress in the uplift protion of beam can be relaxed simply by setting the wpring constants of those portions equal to zero. In Lin's report, two basic assumption were made;

1) The subgrade can take compression only, and
2) The compressive stress in the foundation is proportional to the deflection.

## OUTLINE OF THE STUDY

1) Present the matrix formulation of beams on elastic spring supports which are regarded as analogous to beams on elastic foundations.
2) Perform the solution process using a computer program written in Fortran IV to obtain the deflections of long and short beams under the action of several concentrated loads and moments.
3) Choose four beams on different subgrade as an illustration of the application of matrix formulation and numerical evaluation.
4) Compare results obtained with Lin's results.
5) Compare results obtained with classical solutions.
6) Compare result obtained with Fraser's computer result.

## PROBLEM FORMULATION

Elastic solutions of beam foundation problems are based on the assumption that the soil behaves as an elastic, homogeneous, infinite, and isotropic solid, defined by a modulus of elasticity, $E_s$, and a Poisson's Ratio $\nu$. It is also assumed that there are no shearing stresses at the contact between beam and soil, and in addition, possible influences of soil overburden on pressure distribution are neglected. Winkler's model can be replaced by a continuous beam resting on a set of springs with stiffness constant K. Its value is defined by

$K = K_s' a$

$K_s' = K_s B$ = modulus of subgrade reaction x width of beam.

a = cell length of beam (distance between springs equally spaced).

Once the problem is set up, it can be visualized as a continuous beam of a finite number of spans supported by a row of springs. The solution of this problem then can be expressed by a matrix formulation as follows;

Consider a beam supported by five equally spaced springs, shown in Fig. 1, where a is the cell length of beam, $\gamma$ is the uniform dead load, $Q_i$ is the concentrated load, $M_i$ is the moment load at the ith spring.

1) Load Matrix [P] and Displacement Matrix [X] (16, 11)

The load matrix [P] is defined as a column vector whose elements are the externally applied loads. Each load $P_i$, accordingly is a component of the load matrix [P] . The

Displacement Matrix [X] consists of the displacements at the prints of application of the load vector components measured in the same directions as the loads. (Fig. 2)

The load matrix [P] is expressed by

$$[P] = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \\ P_{10} \end{bmatrix} \qquad (1)$$

and the displacement matrix [X] is expressed by

$$[X] = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \\ X_8 \\ X_9 \\ X_{10} \end{bmatrix} \qquad (2)$$

Consider the beam shown in Fig. 1. The load matrix [P] can be obtained by solving the joint equilibrium equations

(Fig. 3)

$$P_1 = \frac{\gamma a^2}{12}$$

$$P_2 = \frac{\gamma a^2}{12} - \frac{\gamma a^2}{12} + M_1 = M_1$$

$$P_3 = \frac{\gamma a^2}{12} - \frac{\gamma a^2}{12} = 0$$

$$P_4 = \frac{\gamma a^2}{12} - \frac{\gamma a^2}{12} + M_2 = M_2$$

$$P_5 = -\frac{\gamma a^2}{12}$$

$$P_6 = \frac{\gamma a^2}{2}$$

$$P_7 = \gamma a + Q_1$$

$$P_8 = \gamma a + Q_2$$

$$P_9 = \gamma a + Q_3$$

$$P_{10} = \frac{\gamma a}{2}$$

Substituting into Eq. 1

$$
[P] = \begin{bmatrix} \frac{\gamma a^2}{12} \\ M_1 \\ 0 \\ M_2 \\ -\frac{\gamma a^2}{12} \\ \frac{\gamma a}{2} \\ \gamma a + Q_1 \\ \gamma a + Q_2 \\ \gamma a + Q_3 \\ \frac{\gamma a}{2} \end{bmatrix}
$$

2) Deformation Matrix [e] and Force Matrix [F]

A deformation matrix [e] consisting of member deformations $e_i$ at any joint can be defined for any structure. There will be a subset for each member. All relative movements of the end joints of the member are included in the subset of the deformation matrix for the member. (Fig. 4, Fig. 5 and Fig. 6). For matrix [e] , [F] , [A] , [S] , and their transposes, refer to Lin's Report (11).

## NUMERICAL EXAMPLES

Five numerical examples are presented. The computer program (Appendix C) is used to obtain the displacement matrix [X]. The cross section properties of beams and subgrades are shown in Table I and Table II.

Example 1:

A short beam with unit weight not included in the analysis (11, Ex. 1). The results are identical to Lin's results (Fig. 7)

Example 2:

A long beam with the unit weight not included in the analysis (Fig. 8). The results are compared with Hetenyi's Infinite Solution (2) and Bowles' solution (9), and plotted in Fig. 8. L=10 ft., Q=5 kips, q=1.2 k/ft., $K'_s$=288 K/ft$^2$., two sets of the spring numbers are compared, a=2.5 ft. and a=0.5 ft., K=720 K/ft. and K=144 K/ft. respectively.

Example 3:

A cross section of an aqueduct is shown in Fig. 9. We can consider the object as a unit width beam with uniform load on the beam, concentrated loads and moments on both ends. The results are compared with Hetenyi's Infinite Solution (2) and Levinton's Redundant Solution (6), plotted in Fig. 9.

L=14 ft., $Q_1$=$Q_2$=0.72 kips, $M_1$=$M_2$=3.57 ft.-K, r=0.54 K/ft., a=1 ft., K=285 K/ft..

Example 4:

A long beam with the unit weight not included. The results are compared with Levinton's Redundant Solution (6) and

Bowles' Infinite Solution (9), and plotted in Fig. 10.

$L=60$ ft., $Q_1=10$ kips, $Q_2=15$ kips, $M=30$ ft.-K, $K_s'=100$ K/ft$^2$.

In choosing of two sets of spring numbers, $a=5$ ft., $a=1.67$ ft. and $K=500$ K/ft., $K=167$ K/ft. respectively.

Example 5 :

A long beam with the unit weight not included. The results are compared with Hetenyi's Infinite Solution (2), Levinton's Redundant Solution and Fraser's Computer Solution (12), and plotted in Fig. 11.

$L=60$ ft., $Q_1=Q_4=75$ kips, $Q_2=Q_3=100$ kips, $M_1=120$ ft.-K, $M_2=-120$ ft.-K (sign convention, clockwise moment is positive), $a=2$ ft., $K=200$ K/ft..

Table I —— Data on beam sections

| No. of example | Width B inchs | Depth inchs | Area inch$^2$ | Moment of inertia I inch$^4$ | Modulus of Elasticity E, psi | Beam material |
|---|---|---|---|---|---|---|
| 1 | 8.0 | 8.0 | 9.12 | 109.7 | $30 \times 10^6$ | 8 WF31 |
| 2 | 10.0 | 8.0 | 80.0 | 426.7 | $1.5 \times 10^6$ | wood |
| 3 | 12.0 | 8.0 | 96.0 | 512.0 | $2.5 \times 10^6$ | concrete |
| 4 | 12.0 | 48.0 | 576.0 | 110592.0 | $3.0 \times 10^6$ | concrete |
| 5 | 12.0 | 48.0 | 576.0 | 110592.0 | $3.0 \times 10^6$ | concrete |

Table II — Properties of soil subgrade

| No. of example | Soil type | Modulus of elasticity of soil $E_s$, psi | Poisson's Ratio "$\nu$" | Modulus of subgrade reaction $K_s$, psi | Length Characteristic $\lambda L$ |
|---|---|---|---|---|---|
| 1 | Micaceous silt | 1192 | 0.25 | 454 | 0.98 |
| 2 | Silty clay | 3600 | 0.30 | 2000 | 3.57 |
| 3 | Silty clay | 3600 | 0.30 | 1980 | 3.93 |
| 4 | Sandy clay | 2100 | 0.25 | 694 | 1.98 |
| 5 | Sandy clay | 2100 | 0.25 | 694 | 1.98 |

SUMMARY OF NUMERICAL RESULTS AND COMPARISON WITH REFERENCES

Table Example 1 : Output of computer for Example 1 with comparison to (11). L=24 ft.

| Input Data | | W kips | Q kips | CL ft. | XK K/ft. | EIL K-ft.$^2$ | NC |
|---|---|---|---|---|---|---|---|
| | | 0 | 32.688 | 2.4 | 43.588 | 22896 | 10 |
| Distance from center ft. | | 0 | 2.4 | 4.8 | 7.2 | 9.6 | 12.0 |
| Deflection in. | Computer | 1.40 | 1.32 | 1.08 | 0.79 | 0.47 | 0.16 |
| | Lin | 1.40 | 1.32 | 1.08 | 0.79 | 0.47 | 0.16 |

Table Example 2 : Output of computer for Example 2 with comparison to (2),(9) and (12). L=10 ft.

| Input Data | W kips | Q kips | q K/ft. | CL ft. | XK K/ft. | EIL K-ft.$^2$ | NC |
|---|---|---|---|---|---|---|---|
| | 0 | 5 | 1.2 | 0.5<br>2.5 | 144<br>720 | 4440 | 20<br>4 |

| Distance from left end ft. | | 0 | 2.5 | 5.0 | 7.5 | 10.0 |
|---|---|---|---|---|---|---|
| Deflection in. | Computer a=0.5 ft. | 0.026 | 0.053 | 0.053 | 0.034 | 0.004 |
| | Computer a=2.5 ft. | 0.019 | 0.052 | 0.051 | 0.034 | 0.007 |
| | Hetenyi | 0.031 | 0.054 | 0.052 | 0.033 | 0.006 |
| | Bowles | 0.033 | 0.053 | 0.050 | 0.035 | 0.011 |
| | Fraser | 0.031 | 0.054 | 0.052 | 0.034 | 0.004 |

Table Example 3 : Output of computer for Example 3 with comparison to (2) and (6). L=14 ft.

| Input Data | W k/ft. | $Q_1$ kips | $Q_2$ kips | $M_1$ ft.-K | $M_2$ ft.-K | CL ft. | XK K/ft. | EIL K-ft.$^2$ | NC |
|---|---|---|---|---|---|---|---|---|---|
| | 0.54 | 0.72 | 0.72 | -3.57 | 3.57 | 1.0 | 285 | 8880 | 14 |

| Distance from center ft. | | 0 | 1 | 3 | 5 | 7 |
|---|---|---|---|---|---|---|
| Deflection in. | Computer | 0.0134 | 0.0137 | 0.0168 | 0.0273 | 0.0520 |
| | Hetenyi | 0.0110 | 0.0115 | 0.0164 | 0.0333 | 0.0695 |
| | Levinton | 0.0120 | 0.0124 | 0.0155 | 0.0277 | 0.0657 |

Table Example 4 : Output of computer for Example 4 with comparison to (6),(9). L=60 ft.

| Input Data | W k/ft. | $Q_1$ kips | $Q_2$ kips | M ft.-K | CL ft. | XK K/ft. | EIL K-ft.$^2$ | NC |
|---|---|---|---|---|---|---|---|---|
| | 0 | 10 | 15 | 30 | 1.67<br>5.00 | 167<br>500 | 2302560 | 36<br>12 |

| Distance from left end ft. | | 0 | 10 | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|---|---|---|
| Deflection in. | Computer a=1.67ft. | 0.010 | 0.044 | 0.071 | 0.079 | 0.064 | 0.032 | 0.004 |
| | Computer a=5.0 ft. | 0.010 | 0.044 | 0.070 | 0.079 | 0.064 | 0.032 | 0.004 |
| | Levinton | 0.014 | 0.046 | 0.076 | 0.080 | 0.069 | 0.036 | 0.003 |
| | Bowles | 0.014 | 0.044 | 0.069 | 0.076 | 0.060 | 0.033 | 0.012 |

Table Example 5 : Output of computer for Example 5 with comparison to (2),(6) and (12). L=60 ft.

| Input Data | W K/ft. | $Q_1$ kips | $Q_2$ kips | $Q_3$ kips | $Q_4$ kips | $M_1$ ft.-K | $M_2$ ft.-K | CL ft. | XK K/ft. | EIL K-ft.$^2$ | NC |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 75 | 100 | 100 | 75 | 120 | -120 | 2.0 | 200 | 2302560 | 30 |

| Deflection in. | Distance from center ft. | 0 | 6 | 12 | 18 | 24 | 30 |
|---|---|---|---|---|---|---|---|
| | Computer | 0.822 | 0.791 | 0.760 | 0.675 | 0.513 | 0.419 |
| | Hetenyi | 0.813 | 0.791 | 0.768 | 0.695 | 0.582 | 0.447 |
| | Fraser | 0.815 | 0.810 | 0.776 | 0.696 | 0.579 | 0.442 |
| | Levinton | 0.835 | 0.816 | 0.768 | 0.699 | 0.613 | 0.472 |

Output data of computer

1) For Example 1, a=2.4 ft

THE MATRIX X

| | | | |
|---|---|---|---|
| ROW 1 | 1.1129481D-02 | ROW 12 | 1.1999981D-02 |
| ROW 2 | 1.1063684D-02 | ROW 13 | 3.8658098D-02 |
| ROW 3 | 1.0654333D-02 | ROW 14 | 6.4830822D-02 |
| ROW 4 | 9.3339944D-03 | ROW 15 | 8.9070106D-02 |
| ROW 5 | 6.2588272D-03 | ROW 16 | 1.0823013D-01 |
| ROW 6 | 3.4703038D-04 | ROW 17 | 1.1684317D-01 |
| ROW 7 | -5.5235276D-03 | ROW 18 | 1.0992887D-01 |
| ROW 8 | -8.4842925D-03 | ROW 19 | 9.2658075D-02 |
| ROW 9 | -9.6460522D-03 | ROW 20 | 7.0643470D-02 |
| ROW 10 | -9.9041923D-03 | ROW 21 | 4.7079920D-02 |
| ROW 11 | -9.9041923D-03 | ROW 22 | 2.3309859D-02 |

2) For Example 2, a=2.5 ft

THE MATRIX X

| | | | |
|---|---|---|---|
| ROW 1 | 1.3657210D-03 | ROW 6 | 1.5643193D-03 |
| ROW 2 | 5.7299162D-04 | ROW 7 | 4.3180140D-03 |
| ROW 3 | -3.9681397D-04 | ROW 8 | 4.2817139D-03 |
| ROW 4 | -7.6371552D-04 | ROW 9 | 2.8363642D-03 |
| ROW 5 | -9.5354086D-04 | ROW 10 | 6.1069978D-04 |

3) For Example 3, a=0.5 ft

THE MATRIX X

| | | | |
|---|---|---|---|
| ROW 1 | 1.0673073D-03 | ROW 22 | 2.1491390D-03 |
| ROW 2 | 1.0585945D-03 | ROW 23 | 2.6813406D-03 |
| ROW 3 | 1.0215861D-03 | ROW 24 | 3.2030176D-03 |
| ROW 4 | 9.3242633D-04 | ROW 25 | 3.6942347D-03 |
| ROW 5 | 7.6315350D-04 | ROW 26 | 4.1220918D-03 |
| ROW 6 | 4.8207977D-04 | ROW 27 | 4.4387548D-03 |
| ROW 7 | 1.9526478D-04 | ROW 28 | 4.6032147D-03 |
| ROW 8 | 6.8176423D-06 | ROW 29 | 4.6504142D-03 |
| ROW 9 | -1.2077634D-04 | ROW 30 | 4.6201745D-03 |
| ROW 10 | -2.2425605D-04 | ROW 31 | 4.5336569D-03 |
| ROW 11 | -3.2665502D-04 | ROW 32 | 4.3960986D-03 |
| ROW 12 | -4.3405111D-04 | ROW 33 | 4.2061690D-03 |
| ROW 13 | -5.4753458D-04 | ROW 34 | 3.9610329D-03 |
| ROW 14 | -6.6643193D-04 | ROW 35 | 3.6577321D-03 |
| ROW 15 | -7.8784626D-04 | ROW 36 | 3.2941815D-03 |
| ROW 16 | -8.8928532D-04 | ROW 37 | 2.8732151D-03 |
| ROW 17 | -9.6196830D-04 | ROW 38 | 2.4096888D-03 |
| ROW 18 | -1.0112651D-03 | ROW 39 | 1.9151445D-03 |
| ROW 19 | -1.0350015D-03 | ROW 40 | 1.4026838D-03 |
| ROW 20 | -1.0429684D-03 | ROW 41 | 8.8277120D-04 |
| ROW 21 | -1.0444311D-03 | ROW 42 | 3.6079941D-04 |

4) For Example 3, a=1.0 ft

THE MATRIX X

| | |
|---|---|
| ROW 1 | -1.4063274D-03 |
| ROW 2 | -1.0232294D-03 |
| ROW 3 | -6.8759682D-04 |
| ROW 4 | -4.2527740D-04 |
| ROW 5 | -2.3965858D-04 |
| ROW 6 | -1.2006843D-04 |
| ROW 7 | -4.7790067D-05 |
| ROW 8 | 1.3484765D-18 |
| ROW 9 | 4.7790067D-05 |
| ROW 10 | 1.2006843D-04 |
| ROW 11 | 2.3965858D-04 |
| ROW 12 | 4.2527740D-04 |
| ROW 13 | 6.8759682D-04 |
| ROW 14 | 1.0232294D-03 |
| ROW 15 | 1.4063274D-03 |
| ROW 16 | 4.3374726D-03 |
| ROW 17 | 3.1250045D-03 |
| ROW 18 | 2.2751920D-03 |
| ROW 19 | 1.7253731D-03 |
| ROW 20 | 1.3990704D-03 |
| ROW 21 | 1.2240464D-03 |
| ROW 22 | 1.1431629D-03 |
| ROW 23 | 1.1203035D-03 |
| ROW 24 | 1.1431629D-03 |
| ROW 25 | 1.2240464D-03 |
| ROW 26 | 1.3990704D-03 |
| ROW 27 | 1.7253731D-03 |
| ROW 28 | 2.2751920D-03 |
| ROW 29 | 3.1250045D-03 |
| ROW 30 | 4.3374726D-03 |

5) For Example 4, a=5.0 ft

THE MATRIX X

| | |
|---|---|
| ROW 1 | 2.9106506D-04 |
| ROW 2 | 2.8891308D-04 |
| ROW 3 | 2.7636462D-04 |
| ROW 4 | 2.3738003D-04 |
| ROW 5 | 1.3768023D-04 |
| ROW 6 | 6.7254537D-05 |
| ROW 7 | -7.2095753D-06 |
| ROW 8 | -1.2083191D-04 |
| ROW 9 | -2.2692834D-04 |
| ROW 10 | -2.7556893D-04 |
| ROW 11 | -2.9235511D-04 |
| ROW 12 | -2.9555247D-04 |
| ROW 13 | -2.9555247D-04 |
| ROW 14 | 7.9280859D-04 |
| ROW 15 | 2.2445473D-03 |
| ROW 16 | 3.6646119D-03 |
| ROW 17 | 4.9641333D-03 |
| ROW 18 | 5.8829328D-03 |
| ROW 19 | 6.3897257D-03 |
| ROW 20 | 6.5487475D-03 |
| ROW 21 | 6.2523663D-03 |
| ROW 22 | 5.3529715D-03 |
| ROW 23 | 4.0788426D-03 |
| ROW 24 | 2.6503730D-03 |
| ROW 25 | 1.1779395D-03 |
| ROW 26 | -2.9982283D-04 |

6) For Example 4, a=1.67 ft

THE MATRIX X

| | |
|---|---|
| ROW 1 | 2.8516847D-04 |
| ROW 2 | 2.8508045D-04 |
| ROW 3 | 2.8468020D-04 |
| ROW 4 | 2.8364724D-04 |
| ROW 5 | 2.8156495D-04 |
| ROW 6 | 2.7792096D-04 |
| ROW 7 | 2.7210789D-04 |
| ROW 8 | 2.6342457D-04 |
| ROW 9 | 2.5107810D-04 |
| ROW 10 | 2.3418680D-04 |
| ROW 11 | 1.9608211D-04 |
| ROW 12 | 1.6353497D-04 |
| ROW 13 | 1.3543529D-04 |
| ROW 14 | 1.1061748D-04 |
| ROW 15 | 8.7870024D-05 |
| ROW 16 | 6.5943922D-05 |
| ROW 17 | 4.3560444D-05 |
| ROW 18 | 1.9418611D-05 |
| ROW 19 | -7.7971726D-06 |
| ROW 20 | -3.9408888D-05 |
| ROW 21 | -7.6735635D-05 |
| ROW 22 | -1.2108288D-04 |
| ROW 23 | -1.6464564D-04 |
| ROW 24 | -1.9957809D-04 |
| ROW 25 | -2.2706339D-04 |
| ROW 26 | -2.4821769D-04 |
| ROW 27 | -2.6408083D-04 |
| ROW 28 | -2.7560906D-04 |
| ROW 29 | -2.8366971D-04 |
| ROW 30 | -2.8903716D-04 |
| ROW 31 | -2.9239017D-04 |
| ROW 32 | -2.9430993D-04 |
| ROW 33 | -2.9527898D-04 |
| ROW 34 | -2.9568047D-04 |
| ROW 35 | -2.9579785D-04 |
| ROW 36 | -2.9581470D-04 |
| ROW 37 | -2.9581470D-04 |
| ROW 38 | 8.7033155D-04 |
| ROW 39 | 1.3465139D-03 |
| ROW 40 | 1.8223264D-03 |
| ROW 41 | 2.2969935D-03 |
| ROW 42 | 2.7691241D-03 |
| ROW 43 | 3.2365511D-03 |
| ROW 44 | 3.6961726D-03 |
| ROW 45 | 4.1437937D-03 |
| ROW 46 | 4.5739715D-03 |
| ROW 47 | 4.9798646D-03 |
| ROW 48 | 5.3382906D-03 |
| ROW 49 | 5.6378725D-03 |
| ROW 50 | 5.8869731D-03 |
| ROW 51 | 6.0920533D-03 |
| ROW 52 | 6.2575880D-03 |
| ROW 53 | 6.3859964D-03 |
| ROW 54 | 6.4775861D-03 |
| ROW 55 | 6.5305095D-03 |
| ROW 56 | 6.5407331D-03 |
| ROW 57 | 6.5020199D-03 |
| ROW 58 | 6.4059260D-03 |
| ROW 59 | 6.2418148D-03 |
| ROW 60 | 6.0019459D-03 |
| ROW 61 | 5.6967025D-03 |
| ROW 62 | 5.3395007D-03 |
| ROW 63 | 4.9418351D-03 |
| ROW 64 | 4.5133990D-03 |
| ROW 65 | 4.0622182D-03 |
| ROW 66 | 3.5947950D-03 |
| ROW 67 | 3.1162606D-03 |
| ROW 68 | 2.6305323D-03 |
| ROW 69 | 2.1404753D-03 |
| ROW 70 | 1.6480663D-03 |
| ROW 71 | 1.1545594D-03 |
| ROW 72 | 6.6065172D-04 |
| ROW 73 | 1.6665055D-04 |
| ROW 74 | -3.2736000D-04 |

7) For Example 5, a=2.0 ft

THE MATRIX X

| | | | |
|---|---|---|---|
| ROW 1 | 1.9475113D-03 | ROW 32 | 3.4903161D-02 |
| ROW 2 | 1.9414480D-03 | ROW 33 | 3.8794141D-02 |
| ROW 3 | 1.9165186D-03 | ROW 34 | 4.2656375D-02 |
| ROW 4 | 1.8585736D-03 | ROW 35 | 4.6438205D-02 |
| ROW 5 | 1.7521355D-03 | ROW 36 | 5.0058341D-02 |
| ROW 6 | 1.5413541D-03 | ROW 37 | 5.3342441D-02 |
| ROW 7 | 1.3776432D-03 | ROW 38 | 5.6255138D-02 |
| ROW 8 | 1.2419637D-03 | ROW 39 | 5.8871701D-02 |
| ROW 9 | 1.1143157D-03 | ROW 40 | 6.1228347D-02 |
| ROW 10 | 9.7383547D-04 | ROW 41 | 6.3320409D-02 |
| ROW 11 | 7.9888647D-04 | ROW 42 | 6.5100709D-02 |
| ROW 12 | 5.6715941D-04 | ROW 43 | 6.6478103D-02 |
| ROW 13 | 3.4265631D-04 | ROW 44 | 6.7374163D-02 |
| ROW 14 | 1.8898423D-04 | ROW 45 | 6.7895949D-02 |
| ROW 15 | 8.2644086D-05 | ROW 46 | 6.8161654D-02 |
| ROW 16 | 1.7269714D-15 | ROW 47 | 6.8242323D-02 |
| ROW 17 | -8.2644086D-05 | ROW 48 | 6.8161654D-02 |
| ROW 18 | -1.8898423D-04 | ROW 49 | 6.7895949D-02 |
| ROW 19 | -3.4265631D-04 | ROW 50 | 6.7374163D-02 |
| ROW 20 | -5.6715941D-04 | ROW 51 | 6.6478103D-02 |
| ROW 21 | -7.9888647D-04 | ROW 52 | 6.5100709D-02 |
| ROW 22 | -9.7383547D-04 | ROW 53 | 6.3320409D-02 |
| ROW 23 | -1.1143157D-03 | ROW 54 | 6.1228347D-02 |
| ROW 24 | -1.2419637D-03 | ROW 55 | 5.8871701D-02 |
| ROW 25 | -1.3776432D-03 | ROW 56 | 5.6255138D-02 |
| ROW 26 | -1.5413541D-03 | ROW 57 | 5.3342441D-02 |
| ROW 27 | -1.7521355D-03 | ROW 58 | 5.0058341D-02 |
| ROW 28 | -1.8585736D-03 | ROW 59 | 4.6438205D-02 |
| ROW 29 | -1.9165186D-03 | ROW 60 | 4.2656375D-02 |
| ROW 30 | -1.9414480D-03 | ROW 61 | 3.8794141D-02 |
| ROW 31 | -1.9475113D-03 | ROW 62 | 3.4903161D-02 |

# CONCLUSIONS

1) The study described herein shows that the matrix solution for beams on elastic foundations gives good agreement with the classical or recent developed method (12), for the number of springs chosen.

2) The matrix solution of the problem shows its simplicity not only in the matrix formulation but also in the numerical evaluation by computer.

3) The beam and aubgrade properties were chosen from the reference (2,6) for the convenience of investigation and results comparison, the loading conditions are both general and practical.

4) In the case of partial uniform load, increasing the number of spring will give a result showing more agreement . This is clearly illustrated in Fig. 8 and Fig. 10.

5) Physical properties of real soil are more complicated than that represented by Winkler's assumptions.

6) The modulus of subgrade $K_s'$ is mainly determined by the modulus of elasticity of soil, while the width of beam has little influence when a beam cross section is chosen.

## ACKNOWLEDGEMENTS

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# APPENDIX A —— NOTATION

The following symbols are used in this report:

$EI$ = flexural rigidity of beam

$K_s$ = subgrade modulus

$B$ = width of beam

$K_s'$ = $K_s \times B$ = subgrade modulus include the effect of beam width

$Q_i$ = magnitude of concentrated load at ith loading pt.

$M_i$ = magnitude of moment load at ith loading point

$\gamma$ = unit weight of beam

$q$ = uniform load at part of beam

$K$ = spring constant

$a$ = cell length of beam-foundation

$E_s$ = modulus of elasticity of soil

$\nu$ = Poisson's ratio

$L$ = total length of beam

$\lambda L$ = length characteristic

## APPENDIX B —— FIGURES

Fig. 1 - The given beam and loads

Fig. 2 - Force-deflection Diagram.

Fig. 3 - Load Diagram for the Given Beam

Fig. 4 - Internal Moments and Rotations

Fig. 5 - Spring Force and Deflections

Fig. 6 - Joint Equilibrium Diagram

Fig. 7 - Example 1, a short beam to compare with Lin's results

Fig. 8 - Example 2, a ong beam to compare with Hetenyi's and Bowles' results

Fig. 9 - Example 3, a long beam to compare with heteyi's and Levinton's results

Fig. 10 - Example 4, a long beam to compare with Levinton's and Bowles' results

Fig. 11 - Example 5, a long beam to compare with Levinton's Hetenyi's and Fraser's results

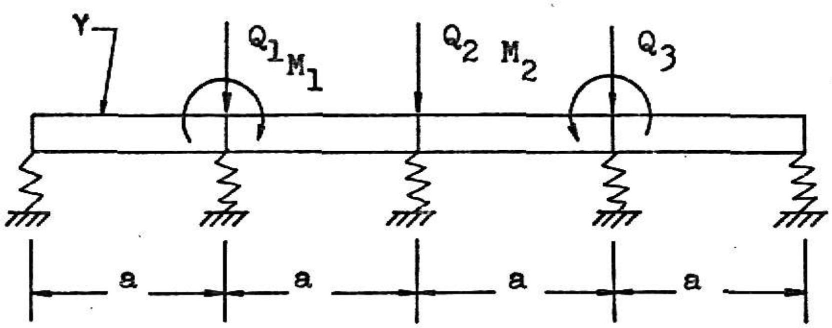



Fig. 1 Given beam and loads

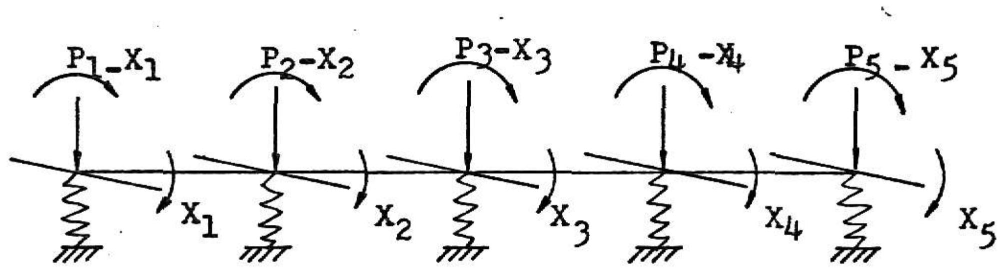



Fig. 2 Force-Deflection Diagram

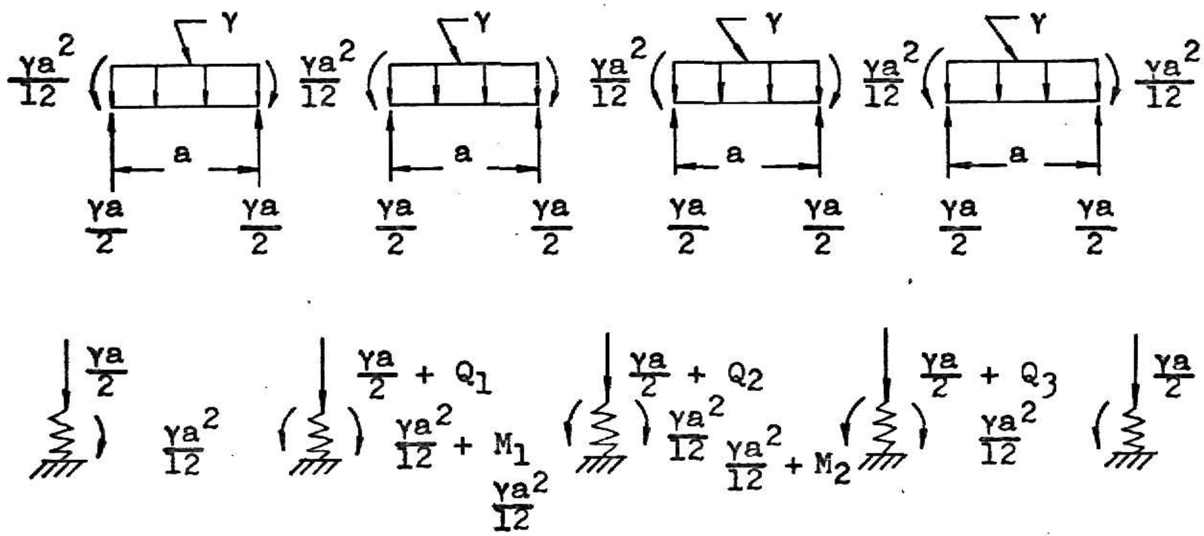



Fig. 3 Load Diagram

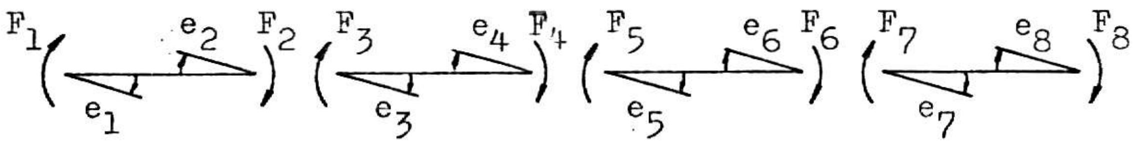



Fig. 4 Internal Moments and Rotations

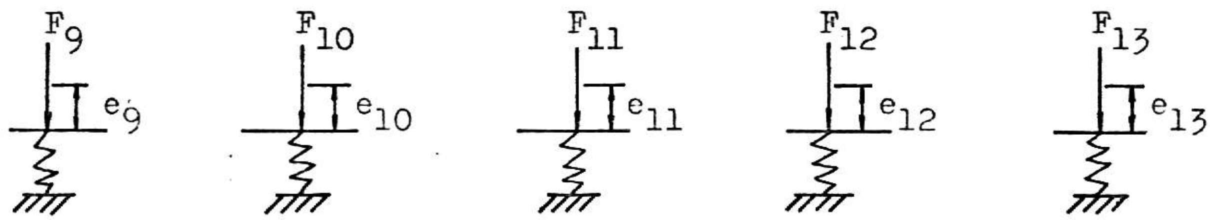



Fig. 5 Spring Forces and Deflections

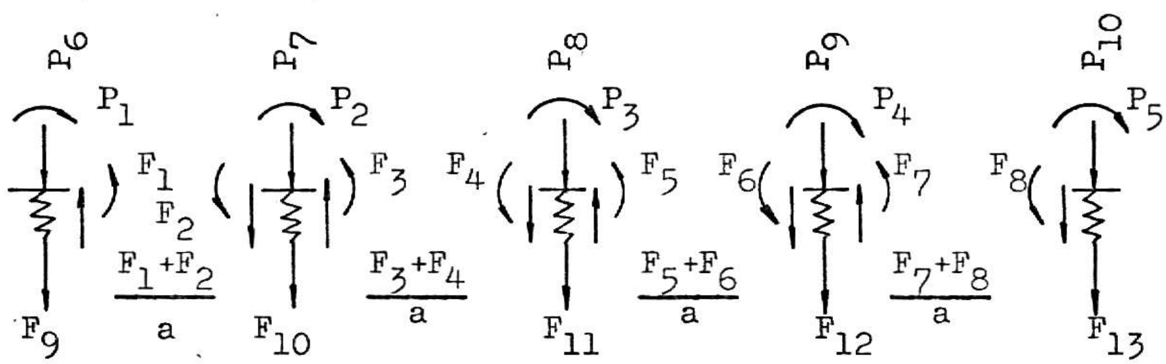



Fig. 6 Joint Equilibrium Diagram

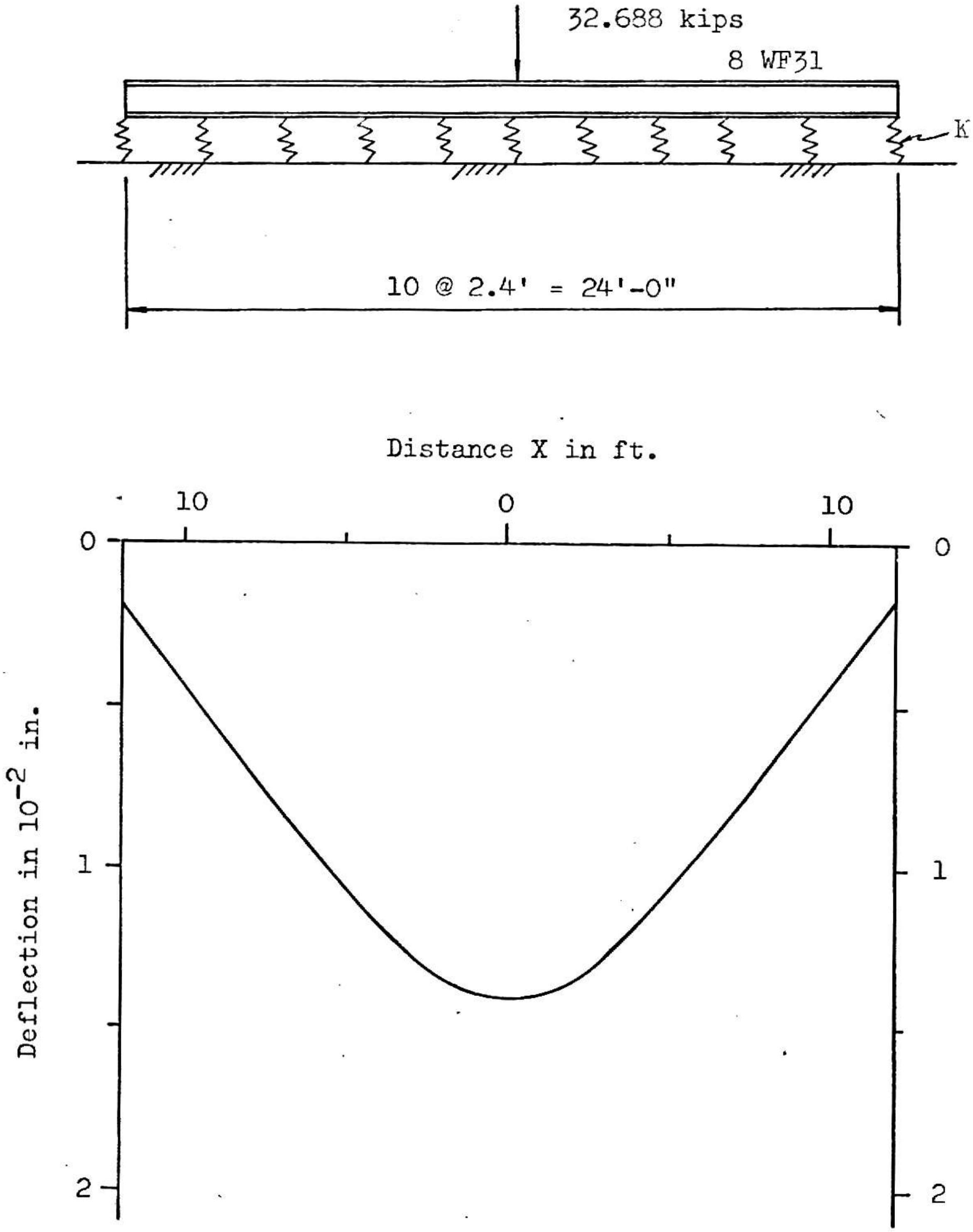



Fig. 7 Example 2, short beam to compare with Lin's results

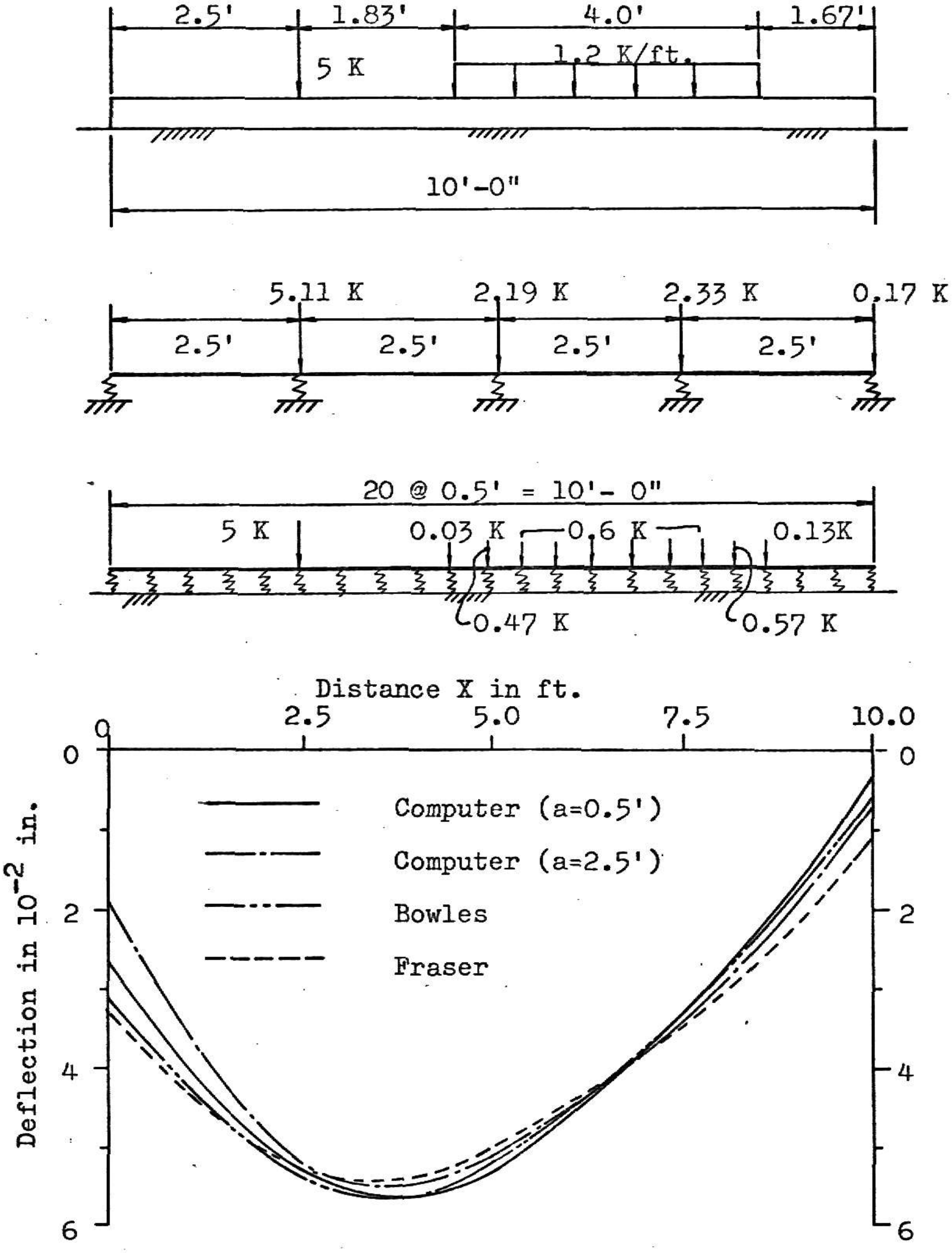



Fig. 8 Example 2, long beam to compare with Hetenyi's and Bowles' solutions

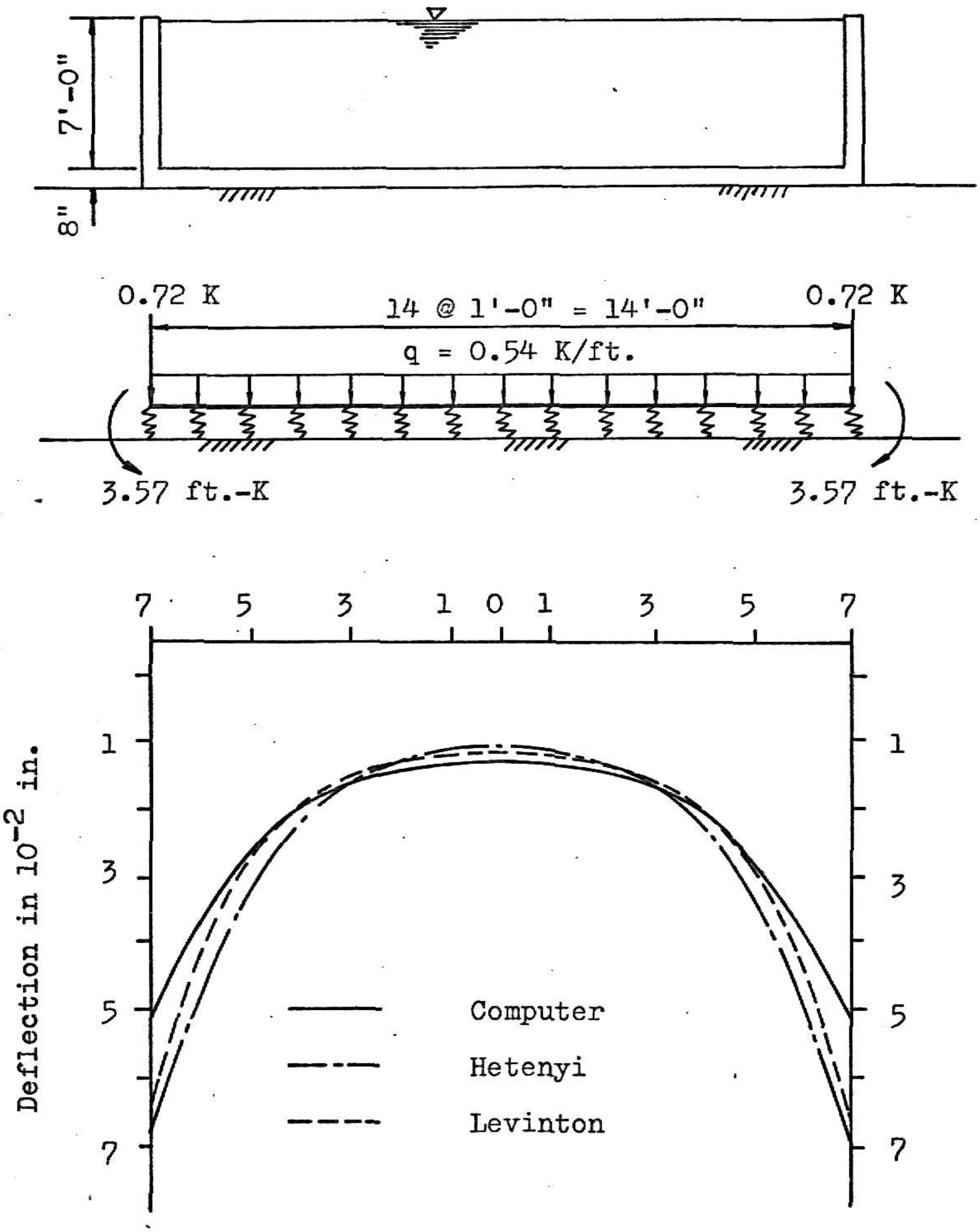
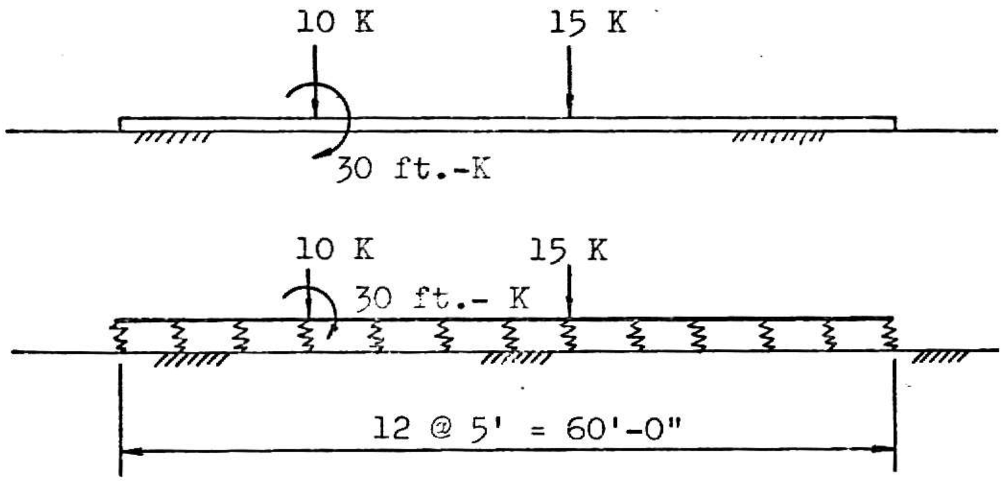



Fig. 9 Example 3, long beam to compare with Hetenyi's and Levinton's results



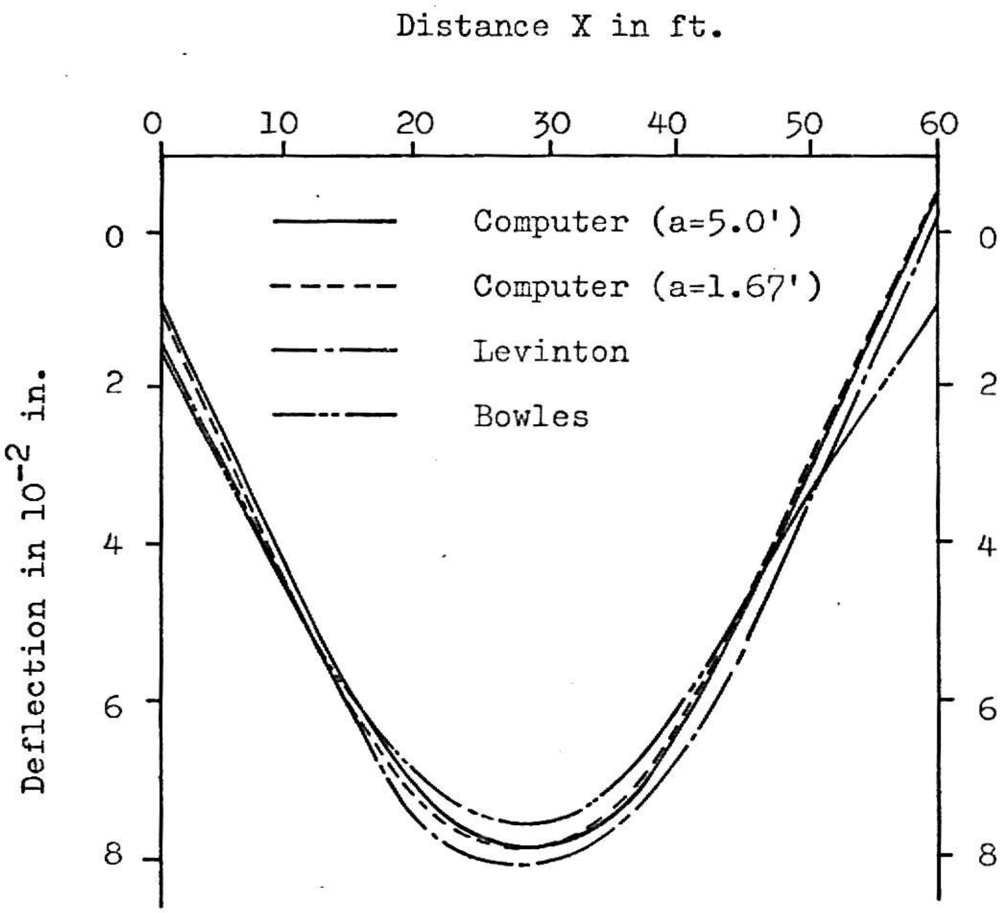
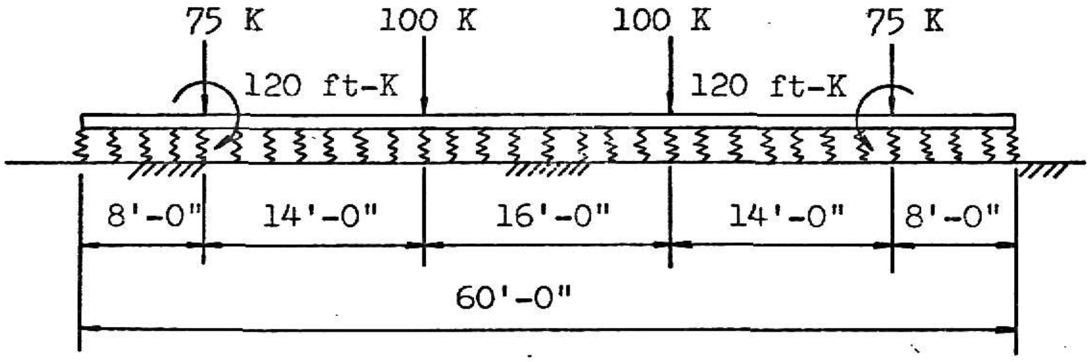




Fig. 10 Example 4, long beam to compare with Levinton's and Bowles' results



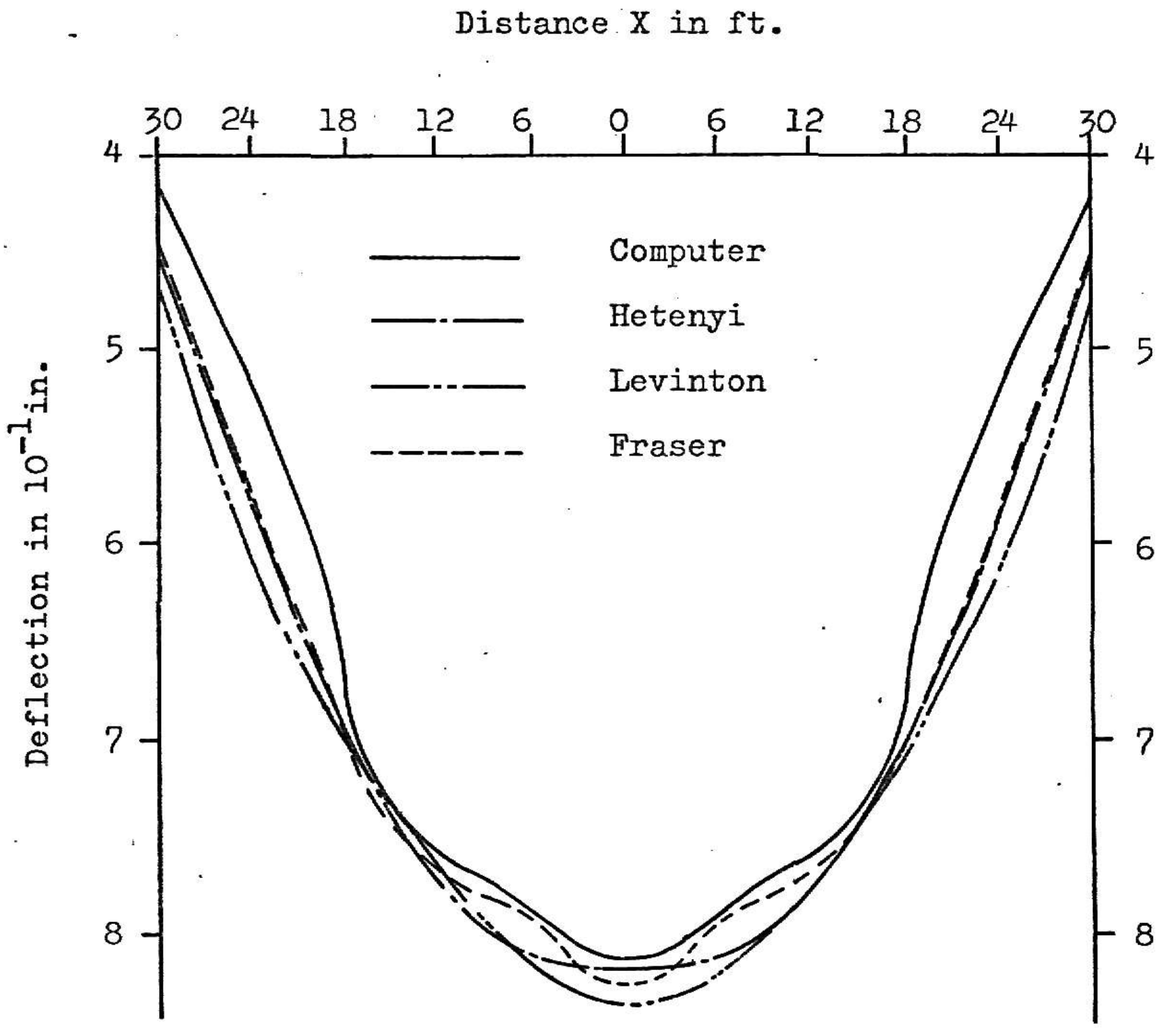




Fig. 11 Example 5, long beam to compare with Hetenyi's, Levinton's and Fraser's results

# APPENDIX C —— COMPUTER PROGRAM

Displacement method of beams on "One-way" elastic foundation analysis;

(I) Program Explanation

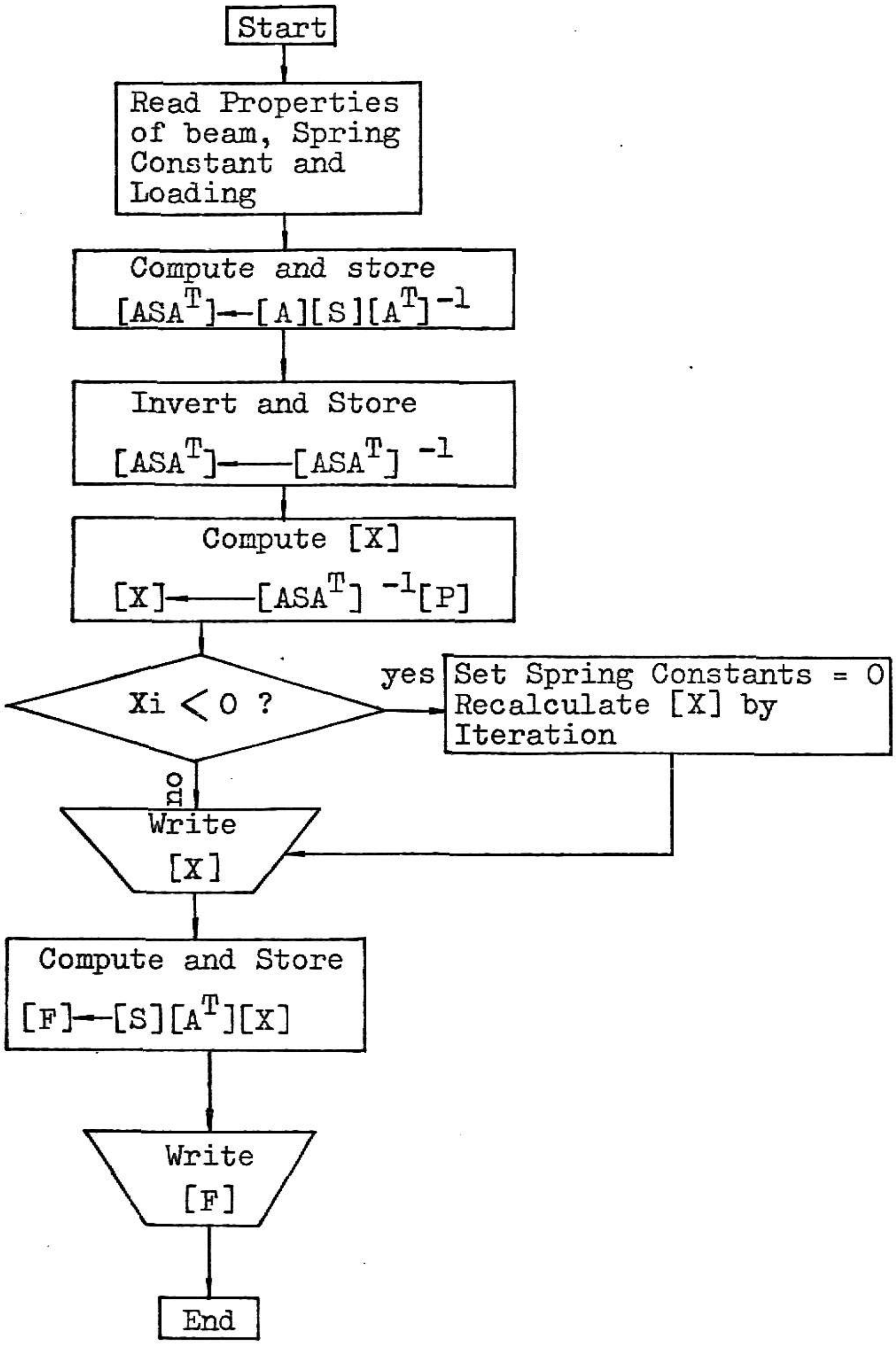
This program is a modification of the one given in Lin's report to solve the matrix equations of beams on elastic foundation by the displacement method. This was a modification of a program given by Wang (16), Where in the spring constant K, arising from upward deflections, is set equal to zero in the stiffness matrix [S] , after the first iteration. The deflections are then recalculated. If the region of upward deflections expands, iteration is continued until all upward deflections are tested and their spring constants K are set equal to zero in [S] matrix. The iteration goes on to its last cycle and the final deflections are written out. This is the essence of the program that follows;

(II) Fortran Name - The following symbols are used in this program.

| Fortran Name | Quantity |
|---|---|
| [A] | Force-load transformation matrix |
| [S] | Member stiffness matrix |
| [SAF] | Member stiffness matrix related to axial forces |
| [ASAT] | Transpose of [A] matrix |
| [P] | Load matrix |
| [X] | Displacement matrix |
| [F] | Force matrix |
| [e] | Deformation matrix |

| | |
|---|---|
| INDEX | Index of do loop taking on values from 1 to NP |
| IDONT | Index of do loop taking on values from 1 to NAF |
| NP | Degrees of freedom |
| NF | Total number of internal forces |
| NEM | Number of internal end moments |
| NAF | Number of internal axial forces |
| NLC | Load condition |
| NC | Number of cell length |
| NT | Index of tension or tensionless allowed for foundation, taking values 1 or 0, respectively |
| W | Unit weight of beam ( = $\gamma$) |
| $Q_i$ | Concentrated load at ith loading pt. |
| $M_i$ | Moment load at ith loading point |
| CL | Cell length of beam |
| XK | Spring constant |
| EIL | Flexural rigidity of beam |
| ISW | Test of upward deflection |

(III) Flow Chart of Beams on One-way Elastic Foundation Program



FORTRAN IV G LEVEL 18 MAIN DATE = 70282

```
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(80,120),S(160),SAF(40)
      DIMENSION ASAT(80,80),P(80,1) ,X(80,1),F(120,1),INDEX(80)
      DIMENSION IDONT(80)
  131 READ(1,101,END=999) NP,NF,NEM,NAF,NLC,NC,NT
      WRITE(3,101)NP,NF,NEM,NAF,NLC,NC,NT
  101 FORMAT (7I5)
      READ(1,1) LC,LP1,LP2,LP3,LP4
      WRITE(3,1) LC,LP1,LP2,LP3,LP4
    1 FORMAT(5I5)
      READ (1,2) CL,XK,EIL,W
      WRITE(3,2)CL,XK,EIL,W
    2 FORMAT (4F12.2)
      GO TO (3,4,6),LC
    3 READ (1,7) Q1,Q2,Q3,Q4
      WRITE(3,7) Q1,Q2,Q3,Q4
    7 FORMAT (4F10.2)
      GO TO 12
    4 READ (1,8) XM1,XM2,XM3,XM4
      WRITE(3,8) XM1,XM2,XM3,XM4
    8 FORMAT (4F10.2)
      GO TO 12
    6 READ (1,9) Q1,XM1,Q2,XM2,Q3,XM3,Q4,XM4
      WRITE(3,9) Q1,XM1,Q2,XM2,Q3,XM3,Q4,XM4
    9 FORMAT(8F8.2)
   12 DO 5 I=1,NP
      DO 5 J=1,NF
    5 A(I,J)=0.
      A(1,1)=1.
      A(NAF,NEM)=1.
      K=0.
      DO 10 I=2,NC
      J=I+K
      A(I,J)=1.
      A(I,J+1)=1.
```

```
10 K=K+1
   INAF=NAF+1
   A(INAF,1)=1./CL
   A(INAF,2)=1./CL
   A(INAF,NEM+1)=-1.
   A(NP,NEM-1)=-1./CL
   A(NP,NEM)=-1./CL
   A(NP,NF)=-1.
   J=1
   L=2
   DO 15 I=2,NC
   I1=I+NAF
   JJ=NEM+L
   A(I1,J)=-1./CL
   A(I1,J+1)=-1./CL
   A(I1,J+2)=1./CL
   A(I1,J+3)=1./CL
   A(I1,JJ)=-1.
   L=L+1
15 J=J+2
   NEMT2=NEM*2
   DO 20 I=1,NEMT2
20 S(I)=0.
   DO 25 I=1,NC
   I1=4*(I-1)+1
   I2=4*(I-1)+2
   I3=4*(I-1)+3
   I4=4*(I-1)+4
   S(I1)=(4.*EIL)/CL
   S(I2)=(2.*EIL)/CL
   S(I3)=(2.*EIL)/CL
25 S(I4)=(4.*EIL)/CL
   DO 30 I=1,NAF
30 SAF(I)=XK
   DO 21 I=1,NP
21 P(I,1)=0.
   P(1,1)=W*CL*CL/12.
   P(NAF,1)=-P(1,1)
   P(NAF+1,1)=W*CL/2.
```

```
      P(2*NAF,1)=P(NAF+1,1)
      DO 50 K=2,NC
      K1=K+NAF
   50 P(K1,1)=W*CL
      DO 58 I=1,NAF
      IF (I-LP1) 51,52,51
   52 P(LP1,1)=P(LP1,1)+XM1
      J=NAF+LP1
      P(J,1)=P(J,1)+Q1
      GO TO 58
   51 IF(I-LP2) 54,53,54
   53 P(LP2,1)=P(LP2,1)+XM2
      J=NAF+LP2
      P(J,1)=P(J,1)+Q2
      GO TO 58
   54 IF (I-LP3) 56,55,56
   55 P(LP3,1)=P(LP3,1)+XM3
      J=NAF+LP3
      P(J,1)=P(J,1)+Q3
      GO TO 58
   56 IF (I-LP4) 58,57,58
   57 P(LP4,1)=P(LP4,1)+XM4
      J=NAF+LP4
      P(J,1)=P(J,1)+Q4
   58 CONTINUE
      WRITE (3,103)
  103 FORMAT (52H1DISP METHOD OF BEAMS ON ELASTIC FOUNDATION ANAL
      WRITE(3,104)
  104 FORMAT (13H0THE MATRIX A)
      DO 105 I=1,NP
  105 WRITE (3,106) I,(A(I,J),J=1,NF)
  106 FORMAT (4H ROW,I3,1X,1P4E16.7/(8X,1P4E16.7))
      WRITE (3,107)
  107 FORMAT (13H0THE MATRIX S)
      DO 108 I=1,NEM
      I1=(I-1)/2*2+1
      I2=(I+1)/2*2
      I3=2*I-1
      I4=2*I
```

```
108 WRITE (3,109) I,I1, S(I3),I2,S(I4)
109 FORMAT (4H ROW,I3,5X,3HCOL,I3,1PE16.7,5X,3HCOL,I3,1PE16.7)
    DO 208 I=1,NAF
    I5=NEM+I
208 WRITE (3,209) I5,I5,SAF(I)
209 FORMAT (4H ROW,I3,5X,3HCOL,I3,1PE16.7)
    WRITE (3,110)
110 FORMAT (13HOTHE MATRIX P)
    DO 111 I=1,NP
111 WRITE (3,106) I,(P(I,J),J=1,NLC)
    DO 60 I=1,NAF
    I1=I+NAF
 60 IDONT (I1)=0
301 DO 112 I=1,NP
    DO 112 J=1,NP
    ASAT (I,J)=0.
    DO 212 K=1,NEM
    K1=(K-1)/2*2+1
    K2=(K+1)/2*2
    K3=2*K-1
    K4=2*K
212 ASAT (I,J)=ASAT(I,J)+A(I,K)*(S(K3)*A(J,K1)+S(K4)*A(J,K2))
    DO 213 K=1,NAF
213 ASAT (I,J)=ASAT(I,J)+A(I,K+NEM)*SAF(K)*A(J,K+NEM)
112 CONTINUE
    DO 113 I=1,NP
113 INDEX (I)=0
114 AMAX=-1.
    DO 115 I=1,NP
    IF (INDEX (I)) 115,116,115
116 TEMP=DABS(ASAT(I,I))
    IF (TEMP-AMAX) 115,115,117
117 ICOL=I
    AMAX=TEMP
115 CONTINUE
    IF (AMAX) 118,100,119
119 INDEX (ICOL)=1
    PIVOT=ASAT(ICOL,ICOL)
    ASAT (ICOL, ICOL)=1.0
```

```
      PIVOT=1./PIVOT
      CO 120 J=1,NP
  120 ASAT (ICOL,J)=ASAT(ICOL,J)*PIVOT
      DO 121 I=1,NP
      IF (I-ICOL) 122,121,122
  122 TEMP=ASAT(I,ICOL)
      ASAT (I,ICOL)=0.0
      DO 123 J=1,NP
  123 ASAT (I,J)=ASAT(I,J)-ASAT(ICOL,J)*TEMP
  121 CONTINUE
      GO TO 114
  118 DO 124 I=1,NP
      CO 124 J=1,NLC
      X(I,J)=0.
      DO 124 K=1,NP
  124 X(I,J)=X(I,J)+ASAT(I,K)*P(K,J)
      IF (NT) 70,70,65
   70 WRITE (3,125)
      WRITE (3,160) (I,X(I,1),I=1,NP)
  160 FORMAT (6(1X,2HX(I3,2H)=E12.5))
      ISW=0
      CO 61 I=1,NAF
      I1=I+NAF
      IF (X(I1,1)) 62,61,61
   62 IF (IDONT(I1)) 63,63,61
   63 IDONT(I1)=1
      SAF (I)=0.
      ISW=1
   61 CONTINUE
      IF (ISW) 65,65,301
   65 WRITE (3,125)
  125 FORMAT (13HOTHE MATRIX X)
      DO 126 I=1,NP
  126 WRITE (3,106) I,(X(I,J),J=1,NLC)
      CO 127 I=1,NEM
      I1=(I-1)/2*2+1
      I2=(I+1)/2*2
      I3=2*I-1
      I4=2*I
```

```
      DO 127 J=1,NLC
      F (I,J)=0.
      CO 127 K=1,NP
  127 F(I,J)=F(I,J)+X(K,J)*(S(I3)*A(K,I1)+S(I4)*A(K,I2))
      CO 228 I=1,NAF
      I5=I+NEM
      F(I5,1)=0.
      DU 228 K=1,NP
  228 F (I5,1)=F(I5,1)+SAF(I)*A(K,I5)*X(K,1)
      WRITE (3,128)
  128 FORMAT (13HOTHE MATRIX F)
      DO 129 I=1,NF
  129 WRITE (3,106) I,(F(I,J),J=1,NLC)
      GO TO 131
  100 CONTINUE
      WRITE (3,130)
  130 FORMAT (11HOZERO PIVOT)
      GO TO 131
  999 STOP
      END
```

# BEAMS ON ONE-WAY ELASTIC FOUNDATIONS

by

CHENG-MING HSU

Diploma, Taipei Institute of Technology, 1964

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AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Civil Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1970

ABSTRACT

The analysis of beams on one-way elastic foundations is based on Winkler's assumption that the continuous reaction of the foundation at every point is proportional to the deflection at that point. However, the tension property which is ordinarily assumed for a foundation is relaxed by assuming the foundation can take compression only. Under such conditions the foundation can be visualized as a set of closely spaced "one-way" springs. A matrix formulation is used to express the beam member deformations and forces in terms of spring joint displacements. Once the redundant displacements are known the elastic solution of this beam-foundation system can be obtained. Several beams on different soil subgrade with general loading conditions were chosen to illustrate the numerical evaluation of the tensionless foundation solution. The numerical process was performed using a computer program written in Fortran IV. The beam-subgrade stiffness matrix was modified to take into account beam uplift by setting appropriate spring constants equal to zero in every cycle of iteration. The final joint displacements (deflections) were calculated following the last iteration. The results are in a good agreement with previous tensionless foundation solutions.