Einstein constraints gluing and toroidal cusps

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Abstract

Isenberg, Mazzeo, and Pollack[1] construct a connected sum gluing for constant mean curvature initial data sets to the vacuum Einstein Field Equations, obtaining new initial data on the joined manifold. The new data set is arbitrarily close to the original data set outside of the gluing region, controlled by the length of the neck joining the manifolds. This approach is here summarized and then partially adapted in an attempt to create a gluing along toroidal cusps. The relevant differences and obstacles for this new gluing are discussed, restricting to the case of compact 3-manifolds and with a focus on the conformal vector Laplacian on toroidal cusps.

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Chapter 1

Introduction

The Einstein field equations are the central part of Einstein's theory of general relativity, forming a relation between the curvature of spacetime and the stress energy distribution on spacetime. They consist of a second order non-linear system of equations defined on a pseudo-Riemannian 4-manifold with metric signature (1,3) given by:

$$R_{ij} - \frac{1}{2}Rg_{ij} = T_{ij}$$

where R_{ij} is the Ricci curvature tensor and T_{ij} is the stress energy tensor. We are here concerned only with the case of $T_{ij} = 0$, known as the vacuum case.

Like most physical systems, it is desirable to rewrite it in the form of an initial value problem, allowing you to take some initial state and evolve it forward in time. Doing so for the Einstein field equations imposes constraints on the initial data, known as the Einstein constraints, defined on a 3-dimensional spacelike (Riemannian) submanifold. The constraint equations are also a system of differential equations, but an under-determined one, having both the metric and second fundamental form as variables.[2]

A number of methods of finding solutions to the Einstein constraints have been devised, including gluing techniques. We focus on the gluing by Isenberg, Mazzeo and Pollack (IMP)[1]. Their gluing has lead to a number of other results since[3–5]. The IMP gluing takes two sets of solutions to the vacuum Einstein constraints, consisting each of a smooth 3-manifold M, a Riemannian metric g, and the second fundamental form K, and produces a new family of valid initial data on the connected sum of the manifolds. This is achieved by conformally scaling the metric in a ball around the removed points by $\frac{1}{r^2}$ where r is the radius from the removed point. This creates 'cylinders,' the ends of which are cut off and then identified with each other, creating a cylindrical bridge of length T between the two manifolds. The metrics and second fundamental forms are then spliced together using cutoff functions. The result is a new manifold, metric and second fundamental form that are together only an approximate solution but have an error that tends towards zero as $T \to \infty$. This approximate solution can then be corrected into an actual solution for a sufficiently large T, creating a family of new solutions parameterized by T. The new solutions approach the original solutions outside the gluing area.

We look at some of the obstacles to adapting this to the case of gluing along the boundary of a removed S^1 embedding instead of a point. An immediate concern is that such a gluing admits no natural way of scaling the metric into the cylinder. This is sidestepped by considering instead manifolds with pre-existing hyperbolic cusps. The rest of the concerns are discussed later. Although the procedure has been generalized to gluing along sub-manifolds,[4] it only applies to co-dimension ≥ 3.

We begin in Chapter 2 with an overview of the Cauchy problem in general relativity and a derivation of the Einstein constraints. Chapter 3 introduces the conformal method for finding solutions to the Einstein constraints and chapter 4 proves existence and uniqueness of the Lichnerowicz equation, a key part of the conformal method. Chapter 5 is a brief overview of the original gluing paper as well as a discussion of some of the obstacles to adapting the gluing. Lastly, chapter 6 introduces the conformal vector Laplacian Ł and follows an analysis of the operator on $\mathbb{T}^2 \times \mathbb{R}$.

Chapter 2

Cauchy Problem in General Relativity

2.1 Background and Assumptions

One of the most natural questions one can ask about general relativity is that of the Cauchy problem: given initial conditions on a 3-manifold, how does that 3-manifold evolve under the Einstein field equations? Further, what assumptions and initial data do we need for this to be a well-posed problem?

The exact requirement on M for the Cauchy problem to be well-posed is that M is globally hyperbolic. There are several equivalent definitions of globally hyperbolic, but the most relevant one is the existence of a Cauchy surface. We first establish a few preliminary definitions.

Definition. *A curve $f : I \to M$, $I \subseteq \mathbb{R}$ is **causal** if $g(\dot{f}, \dot{f}) \leq 0$ and **timelike** if $g(\dot{f}, \dot{f}) = 0$. Note that every timelike curve is causal.*

Definition. *A causal curve f is **inextendible** if the image of f is not a strict subset of another causal curve. Similarly for timelike.*

Definition. *A subset N is **achronal** if no two points in N are joined by a timelike curve.*

Using these we can now define a Cauchy surface on M:

Definition. *A closed achronal subset N of a spacetime M is a Cauchy surface if it is intersected by every inextendible causal curve and exactly once if the curve is timelike.*[2]

A spacetime is globally hyperbolic if and only if it admits a Cauchy surface. Choquet-Bruhat proved[6] that a smooth triple (N, γ, K), where N is a smooth 3-manifold with metric γ and second fundamental form K induced by the embedding of N as a Cauchy surface of M, is a proper set of initial data if and only if that they satisfy the Einstein constraint equations, which we will derive shortly. Although this is true more generally, we will focus only on the vacuum case.

2.2 Einstein Constraints

The Einstein Constraints are a condition on the initial data that guarantees the initial data to be valid input for the evolution equation. The simplest of time evolution problems, such as ODEs or some PDEs, do not require such a condition. The 1-D wave equation, for example, can take any sufficiently regular function on the domain and evolve it. It is not true, however, that an arbitrarily chosen metric and second fundamental form are valid initial data for the Einstein Field Equations.

In the case of a vacuum spacetime the Einstein Constraints are,

$$0 = R - (trK)^2 + |K|^2 \tag{2.1}$$

$$0 = \nabla(trK) - div(K) \tag{2.2}$$

where the norms and traces are computed with respect to the metric γ on M. Similarly R is the scalar curvature of the 3 dimensional submanifold N. Since K is a rank 2 tensor, (2.1) is a scalar equation, and (2.2) is a vector equation.

The second fundamental form K is, in general, the orthogonal projection of the covariant derivative of a vector field. Note that the derivative used is the one defined on the ambient manifold, spacetime in this case, as the submanifold derivative would of course give no orthogonal component. In our case of co-dimension 1 we can write it explicitly as

$$K_{ij}X^iY^j = -\langle \nabla^g_X \hat{n}, Y\rangle = -\langle \nabla^g_Y \hat{n}, X\rangle$$

with g being the spacetime metric and $\hat{n}$ the unit vector orthogonal to T_pN.

The constraint equations can be derived using the Gauss-Codazzi equations. In vacuum, the field equations are given by

$$\hat{R}_{ij} - \frac{1}{2}\hat{R}g_{ij} = 0 \tag{2.3}$$

where $\hat{R}$ is the spacetime Ricci tensor. Taking the trace, we get

$$\begin{aligned} g^{ij}\hat{R}_{ij} &= \frac{1}{2}\hat{R}\delta^i_i \\ \hat{R} &= \frac{n}{2}\hat{R} \end{aligned}$$

showing that the Ricci tensor is zero in every dimension except $n = 2$ for the vacuum case. We may use this to simplify the Gauss-Codazzi equations into the vacuum Einstein constraints. The Gauss Codazzi equations in coordinates are:

$$\begin{aligned} \hat{R}_{ijkl}\Big|_{T_pN} &= R_{ijkl} - K_{il}K_{jk} + K_{ik}K_{jl} \\ \hat{R}_{ijk\mu}n^\mu\Big|_{T_pN} &= K_{jk;i} - K_{ik;j} \end{aligned}$$

using $\hat{R}$ as the curvature on the spacetime M and R as the curvature on the chosen spacelike 3-manifold N. Now we take the trace and set $\hat{R}_{ij} = 0$ via the vacuum field equations. It will be useful to define our coordinates as orthonormal with i starting at zero as the time coordinate and ranging to up to 3. The zeroth direction is denoted $\hat{n}$, the direction orthogonal to N.

Since R and K are restricted to T_pN,we take their trace in T_pM by first extending them to tensors on T_pM with 0 in the additional entries. Specifically $R_{ijkl} = 0$ if i, j, k or l equal 0.

$$
\begin{aligned}
g^{jk} g^{il} \hat{R}_{ijkl} &= g^{jk} g^{il} R_{ijkl} - g^{jk} g^{il} K_{il} K_{jk} + g^{jk} g^{il} K_{ik} K_{jl} \\
0 &= R - (trK)^2 + |K|^2 \\
g^{jk} \hat{R}_{ijk\mu} N^{\mu} &= g^{jk} K_{jk;i} - g^{jk} K_{ik;j} \\
0 &= \nabla(trK) - div(K)
\end{aligned}
$$

Since $g^{ij} K_{ij} = 0$ when $i, j = 0$, the trace of the extended K using the 4-metric really is the same as the trace of nonextended K using the 3-metric, denoted as (trK). The same applies to the norm of K and trace of R. The result is precisely the vacuum Einstein constraints shown earlier. Thus any solution to the vacuum Einstein Field Equations solves the Einstein constraints when restricted to a spacelike submanifold. The converse is also true but not proven here. [2;6;7]

Chapter 3

The Conformal Method

Although we know that valid initial data leads to a valid solution of the evolution problem, it is still necessary to *find* valid initial data. It is not trivial to find arbitrary solutions to the Einstein constraints. One useful method for finding valid initial data is the conformal one. First used by Lichnerowicz[8] it has since been elaborated on by York[9], Isenberg[10] and others, leading to a number of related, slightly different methods that fall under this umbrella. The one we will use works on compact manifolds and assumes $g^{ij}K_{ij} = \tau$, known as constant mean curvature (CMC).

We start by fixing a data set (N, γ, μ, τ), where μ is trace-free, symmetric, (0,2) tensor with zero divergence. By introducing conformal variance of the metric and μ, the added degrees of freedom guarantee that there exists a solution in the conformal class, under certain conditions, while also making it easier to find a solution in practice. The data set we start with does not solve the constraint equations but is instead used to construct a dataset that does, in a conformally related metric.

Theorem 1 (The Conformal Method)**.** *Given a set of data* (N, γ, μ, τ)*, consisting of a com-*

pact 3-manifold N with Reimannian metric g, constant τ, and symmetric tensor μ satisfying

$$\nabla^i \mu_{ij} = 0 \qquad\qquad \mu \not\equiv 0$$
$$\gamma^{ij}\mu_{ij} = 0 \qquad\qquad \tau \not\equiv 0$$

there exists a conformal factor ψ satisfying

$$\Delta\psi - \frac{1}{8}R\psi + \frac{1}{8}|\mu|^2\psi^{-7} - \frac{1}{12}\tau^2\psi^5 = 0 \tag{3.1}$$

such that the triple $(N, \tilde{g}, \tilde{K})$ defined by,

$$\tilde{\gamma} = \psi^4(x)\gamma \tag{3.2}$$
$$\tilde{K} = \tilde{\mu} + \frac{\tau}{3}\tilde{\gamma} = \psi^{-2}\mu + \frac{\tau}{3}\psi^4\gamma \tag{3.3}$$

solves the vacuum Einstein constraints.

The core of this method relies on finding a solution to equation 3.1, known as the Lichnerowicz equation. We are guaranteed to have a unique solution in our case, which will be proven in the next section. The Lichnerowicz equation can be derived directly from the Hamiltonian (scalar) Einstein constraint derived last chapter by considering a conformal transformation. First we write the Hamiltonian constraint in the conformally related metric and use the conformal transformation of the Ricci scalar:

$$0 = \tilde{R} - (\tilde{g}^{ij}\tilde{K_{ij}})^2 + \tilde{K^{ij}}\tilde{K_{ij}} \qquad \tilde{R} = \psi^{-4}R - 8\psi^{-5}\Delta\psi$$

Combining these, we have

$$0 = \psi^{-4}R - 8\psi^{-5}\Delta\psi - \frac{2}{3}\tau + |\mu|^2\psi^{-12}$$
$$0 = \Delta\psi - \frac{1}{8}R\psi + \frac{1}{8}|\mu|^2\psi^{-7} - \frac{1}{12}\tau^2\psi^5$$

Since ψ is a conformal factor and therefore everywhere positive there's no need to consider when $\psi = 0$.

Using the CMC assumption the momentum (vector) constraint equation becomes the requirement for μ to be divergence free

$$
\begin{aligned}
0 &= \nabla^i \tau - \nabla^i K_{ij} \\
0 &= \nabla^i \mu_{ij}
\end{aligned}
$$

This is a crucial simplification because it decouples the two constraint equations. The tensor divergence equation becomes independent of the conformal factor. With our chosen definition of $\tilde{\mu}$, $\tilde{\mathrm{div}}\tilde{\mu} = \psi^{-6}\mathrm{div}\mu$. This follows directly from computing the conformal divergence:

$$
\begin{aligned}
\tilde{\mathrm{div}}\tilde{\mu} =& \tilde{\nabla}^i(\psi^{-2}\mu_{ij}) \\
=& \psi^{-4} g^{ik}\left[(\nabla_k \psi^{-2})\mu_{ij} + \psi^{-2}\nabla_k \mu_{ij} - \psi^{-2} C^l_{ik}\mu_{lj} - \psi^{-2} C^l_{jk}\mu_{il}\right].
\end{aligned}
$$

Here C^l_{jk} is the conformal addition to the Christoffel symbol, given by

$$
\begin{aligned}
C^l_{jk} &= \frac{1}{2}\psi^{-4}\gamma^{lm}\left[\partial_j(\psi^4)\gamma_{mk} + \partial_k(\psi^4)\gamma_{mj} - \partial_m(\psi^4)\gamma_{jk}\right] \\
&= 2\left[(\partial_j \mathrm{ln}\psi)\delta^l_k + (\partial_k \mathrm{ln}\psi)\delta^l_j - (\nabla^l \mathrm{ln}\psi)\gamma_{jk}\right]
\end{aligned}
$$

Contracting with μ we have

$$
\begin{aligned}
\gamma^{ik} C^l_{ik}\mu_{lj} &= 2\left[\nabla^l \mathrm{ln}\psi + \nabla^l \mathrm{ln}\psi - 3\nabla^l \mathrm{ln}\psi\right]\mu_{lj} = -2\nabla^l \mathrm{ln}\psi \mu_{lj} \\
\gamma^{ik} C^l_{jk}\mu_{il} &= 2\left[\partial_j \mathrm{ln}\psi g^{il} + \nabla^i \mathrm{ln}\psi \delta^l_j - \nabla^l \mathrm{ln}\psi \delta^i_j\right]\mu_{il} = 0
\end{aligned}
$$

where in the final line we used the fact that μ is symmetric and trace free. Denoting $\tilde{\mathrm{div}}\tilde{\mu} = \rho$, we have

$$\rho_j = \psi^{-4}\left[(\nabla^i\psi^{-2})\mu_{ij} + \psi^{-2}\nabla^i\mu_{ij} + 2\psi^{-2}(\nabla^i\ln\psi)\mu_{ij}\right]$$
$$\psi^6\rho_j = \nabla^i\mu_{ij}$$

Thus the absence of a source term makes the source vanish in any conformally related metric. This allows us to first solve the tensor equation on the most convenient conformally related metric, and then use that result to find a suitable conformal factor.

Chapter 4

Existence and Uniqueness of Solutions to the Lichnerowicz Equation

We now proceed with a proof of existence and uniqueness of solutions to the Lichnerowicz equation introduced earlier. There are more general proofs available, however we will follow a proof by Isenberg[10], applicable to closed 3-manifolds with constant mean curvature. It relies on classifying the initial data using the Yamabe Theorem into conformal classes of constant positive, negative, or zero scalar curvature. It is slightly more general than the theorem stated above as it includes a case by case analysis for when τ or μ are identically zero.

4.1 Yamabe Classification

Yamabe's Theorem. *Let M be a smooth, compact 3-manifold with Riemannian metric g. Then g is conformally related to a metric $\tilde{g}$ with either constant positive, constant negative or zero scalar curvature. These 3 cases are mutually exclusive.*

That every compact manifold is conformally related to one of constant curvature is a proof beyond the scope of this paper[11]. However the fact that the three classes are mutually exclusive is quite simple, we need only look at the conformal relation between scalar curvatures:

$$\tilde{\nabla}^2\psi = \frac{1}{8}\tilde{R}\psi - \frac{1}{8}R\psi^5 \tag{4.1}$$

If we take $\tilde{R}$ and R to be constants of different signs then we run into problems from the maximum principle. For example in

$$\tilde{\nabla}^2\psi = \frac{1}{8}\psi + \frac{1}{8}\psi^5$$

ψ must have an everywhere positive Laplacian according to the right hand side. However as M is compact, it must somewhere have a maximum, which necessarily has a negative Laplacian. The only answer is to require ψ to be zero or -1, but these are not valid conformal factors.

4.2 Existence

4.2.1 Sub and super-solution Method

Before continuing we will want to prove two lemmas. The first is called the sub and super solution method. The idea is to bound potential solutions by an over and under estimate and then create a sequence converging to the exact solution.

Lemma. *Let $x \in M$ be a point in a closed manifold and $f : M \times \mathbb{R}^+ \to \mathbb{R}$ a differentiable function. Assume there exist strictly positive functions ϕ_-, ϕ_+ on M such that, for $p > 3$,*

i) $\phi_-, \phi_+ \in H_2^p(M)$

ii) $0 < \phi_-(x) \leq \phi_+(x)$ *at every point x in M*

iii) $\nabla^2\phi_- \geq f(x, \phi_-)$

iv) $\nabla^2\phi_+ \leq f(x, \phi_+)$

Then there exists $\phi : M \to \mathbb{R}^+$ such that $\nabla^2\phi = f(x,\phi)$. Furthermore, ϕ is an element of the Holder space $C^{2,\alpha}(M)$ for $\alpha \in (0, 1-3/p)$.

Proof. As mentioned earlier, this proof relies on the creation of a sequence $\{\phi_n\}$ which will converge to ϕ. We will first want to rearrange the equations into a more convenient operator form. To do this, we consider the function f restricted to $D := M \times [\min(\phi_-), \max(\phi_+)]$. Since M is compact, the interval is bounded in $\mathbb{R}^+$ and $f(x,y)$ has a compact domain. By assumption f is differentiable so $\frac{\partial f}{\partial y}$ has a maximum. We can therefore pick a number h such that

$$h \geq \frac{\partial f}{\partial y}$$

on all of D.

If we now define the operator

$$L := -\nabla^2 + h$$

along with a function

$$F(x,y) := -f(x,y) + hy$$

then we can rewrite the original PDE:

$$L\phi = F(x,\phi) \iff \nabla^2\phi = f(x,\phi)$$

This also applies to the inequalities in our assumptions, but we must flip the signs,

$$L\phi_- \leq F(x,\phi_-) \iff \nabla^2\phi_- \geq f(x,\phi_-)$$

$$L\phi_+ \geq F(x,\phi_+) \iff \nabla^2\phi_+ \leq f(x,\phi_+)$$

Note that by construction

$$\frac{\partial F}{\partial y} \geq 0$$

on D. Furthermore, by the maximum principle $L\phi \geq 0$ implies that $\phi \geq 0$.

We will now construct our sequence, defined by

$$
\begin{aligned}
L\phi_n &= F(x, \phi_{n-1}) \\
L\phi_1 &= F(x, \phi_+)
\end{aligned}
\tag{4.2}
$$

First it is necessary to show that the elements of this sequence even exist, as they are defined as solutions to a PDE. Note that the right hand side of (4.2) does not depend on ϕ_n, thus existence and uniqueness can be satisfied by showing that L is bijective. If there were a kernel element of L it would solve $\nabla^2\psi = h\psi$, which can be avoided by selecting an h outside the spectrum of the Laplacian on M. M is compact, so the spectrum is discrete and were are guaranteed that there is such a positive real number. Since L is self adjoint and injective it is also surjective, and our sequence is well defined.

By assumption, F is C^1, as is ϕ_+. Therefore $F(x, \phi_+)$ is C^1 and, by elliptic regularity and induction, our sequence elements ϕ_n are C^3. We now want to show that our sequence elements are bounded above by ϕ_+, below by ϕ_- and monotonically decreasing.

First, for $\phi_1 \leq \phi_+$ we have

$$
\begin{aligned}
L(\phi_+ - \phi_1) &= L\phi_+ - L\phi_1 \\
&\geq F(x, \phi_+) - F(x, \phi_+) \\
&= 0
\end{aligned}
$$

which by way of the maximum principle noted earlier for L tells us that $\phi_+ \geq \phi_1$. For the next step we must recall that we chose h and constructed F such that $\frac{\partial F}{\partial y} \geq 0$. Knowing this,

$$\begin{aligned} L(\phi_1 - \phi_2) &= L\phi_1 - L\phi_2 \\ &= F(x, \phi_+) - F(x, \phi_1) \end{aligned}$$

and that $\phi_1 \leq \phi_+$ we can conclude that $\phi_1 \geq \phi_2$. The argument for the rest of the sequence is identical,

$$\begin{aligned} L(\phi_n - \phi_{n+1}) = F(x, \phi_{n-1}) - F(x, \phi_n) \geq 0 \\ \implies \phi_n \geq \phi_{n+1}. \end{aligned}$$

We must now show that the entire sequence is bounded below by ϕ_-, which we will achieve by induction. For the base case,

$$\begin{aligned} L(\phi_1 - \phi_-) &\geq F(x, \phi_+) - F(x, \phi_-) \\ &\geq 0 \end{aligned}$$

so $\phi_1 \geq \phi_-$. For the induction step, assume $\phi_{n-1} \geq \phi_-$, then

$$\begin{aligned} L(\phi_n - \phi_-) &\geq F(x, \phi_{n-1}) - F(x, \phi_-) \\ &\geq 0 \end{aligned}$$

so $\phi_n \geq \phi_-$.

Having established that the sequence is bounded, we now endeavor to apply Arzela-Ascoli to prove that it has a convergent subsequence. Recall that Arzela-Ascoli requires a bounded, equicontinuous sequence on a compact space to produce a subsequence. It remains only to show equicontinuity.

Since M is compact, and we have already shown differentiability, we have that $\{\phi_n\} \in H_3^p$.

Having picked a norm we can look at the inequality from elliptic regularity:

$$||\phi_n||_{H_2^p} \leq \alpha_1 ||L\phi||_{H_0^p} = \alpha_1 ||F(x, \phi_{n-1})||$$

Here we can restrict F to D as by definition it contains the range of every ϕ_n. Since D is compact, F has a maximum ξ which is independent of n:

$$||\phi_n||_{H_2^p} \leq \xi$$

One of the ways to prove equicontinuity is to find an upper bound on the derivative of the sequence elements. The previous inequality is not quite what we want as it bounds the p-norm not the maximum value. Therefore we must use a Sobelev inequality to get a bound on the L^∞ norm.

For $H_k^p(M^n)$ and $C^{m,\alpha}(M^n)$ with $k - (n/p) > m + \alpha$ and $0 < \alpha < 1$, the Sobelev embedding gives $H_k^p \hookrightarrow C^{m,\alpha}$. Here $C^{m,\alpha}(M^n)$ is the Hölder space with m derivatives and α as the exponent of the Hölder condition. In our case $n = 3$, $k = 2$ and $p > 3$ by hypothesis so $H_2^p \hookrightarrow C^{1,\alpha}$ with $\alpha \in (0, 1 - 3/p)$. Furthermore since M is compact, $C^1 \hookrightarrow C^{1,\alpha}$, allowing us to write

$$||\phi_n||_{C^1} \leq ||\phi_n||_{H_2^p} \leq \xi$$

This gives a constant bound on the derivatives of the sequence elements independent of n, in turn giving us equicontinuity.

Since the sequence $\{\phi_n\}$ is equicontinuous it has a convergent subsequence by Arzela-Ascoli. Furthermore, since it is monotonically decreasing the entire sequence must converge uniformly.

Now that we know $\phi_\infty = \phi$ exists we must show that it solves the equation $\nabla^2\phi = f(x, \phi)$. First we want to prove that $\phi \in H_2^p$. We know already by Arzela-Ascoli that it must be C^0. As a sobelev space H_2^p is complete, therefore we want to show that $\{\phi_n\}$ is Cauchy in the

H_2^p norm. We will do this using elliptic regularity

$$\begin{aligned} ||\phi_n - \phi_k||_{H_2^p} &\le a_1||L(\phi_n - \phi_k)||_{H_0^p} \\ &= a_1||F(x, \phi_{n-1}) - F(x, \phi_{k-1})||_{H_0^p} \\ &\le a_2||F(x, \phi_{n-1}) - F(x, \phi_{k-1})||_{C^0}. \end{aligned}$$

The last line follows by compactness. $\{\phi_n\}$ is continuous so $F(x, \phi_n)$ is as well and converges in C^0. Since the right hand side is Cauchy, $\{\phi_n\}$ is Cauchy in H_2^p and, by completeness, convergent. Therefore $\phi \in H_2^p$. By the same Sobelev embedding used earlier, $\phi \in C^{1,\alpha}$.

We now repeat the argument to show that $\phi \in C^{2,\alpha}$

$$||\phi_n - \phi_k||_{C^{2,\alpha}} \le a_3||F(x, \phi_{n-1}) - F(x, \phi_{k-1})||_{C^{0,\alpha}}.$$

Again we know that the right hand side is Cauchy, and therefore the left hand side is Cauchy. Since $C^{2,\alpha}$ is complete, $\phi \in C^{2,\alpha}$.

Since we've now shown that ϕ is sufficiently differentiable, we now only need to show that ϕ is a weak solution to $\nabla^2\phi = f(x, \phi)$. To this end, consider the map

$$\psi \mapsto L\psi - F(x, \psi)$$

This maps H_{s+2}^p to H_s^p for $s \le 2$. By elliptic regularity L is continuous between these spaces, and by our hypothesis F is as well. Since $\{\phi_n\}$ converges in H_2^p, $\{L\phi_n - F(x, \phi_n)\}$ converges to $L\phi - F(x,\phi)$ in H^p_0. This gives us

$$\begin{aligned}
||L\phi - F(x,\phi)||_{H^p_0} &= \lim_{n\to\infty} ||L\phi_n - F(x,\phi_n)||_{H^p_0} \\
&= \lim_{n\to\infty} ||F(x,\phi_{n-1}) - F(x,\phi_n)||_{H^p_0} \\
&\le a_4 \lim_{n\to\infty} ||F(x,\phi_{n-1}) - F(x,\phi_n)||_{C^0} && \text{(By compactness)} \\
&= 0. && \text{(By continuity of } F\text{)}
\end{aligned}$$

Therefore ϕ is a weak solution. Since it it is twice differentiable it is a strong solution. Thus $\nabla^2\phi = f(x,\phi)$. □

4.2.2 Controlling the sign of scalar curvature

Our second lemma is about controlling the sign of the scalar curvature within a sufficiently smoothly bounded open region. This may not seem immediately useful, but it will allow us to split our analysis into two regions within one of which we can fix the scalar curvature to be negative.

Lemma. *Let $\mathcal{U}$ be an open set with a $\mathcal{C}^{1,1}$ boundary in a compact 3-manifold M with $\mathcal{C}^2$ Riemannian metric λ_{ab}. Assume $\bar{\mathcal{U}} \neq M$. Then there exists a positive, $\mathcal{C}^3$ function θ defined on M such that within $\mathcal{U}$ we have*

$$R(\theta^4\lambda_{ab}) < -\xi < 0$$

for some positive constant ξ.

Proof. Since λ_{ab} is $\mathcal{C}^2$, R is continuous. As M is compact, R is bounded above by some constant h. If we rescale conformally by $\tilde{\lambda} = \theta^4\lambda_{ab}$ with $\theta = \sqrt[4]{2h}$ then $R(\tilde{\lambda}) < 1/2$. This follows from the conformal relation between scalar curvatures

$$R(\tilde{\lambda}) = \theta^{-5}\Big(-8\nabla^2_\lambda\theta + R_\lambda\theta\Big).$$

Now consider the Dirichlet boundary value problem

$$8\nabla^2_{\tilde{\lambda}} u = u \qquad \text{on } \mathcal{U}$$
$$u = 1 \qquad \text{on } \partial\mathcal{U}.$$

Since $\partial\mathcal{U}$ is $\mathcal{C}^{1,1}$ there exists a unique $\mathcal{C}^2$ solution to the boundary value problem.[12] Furthermore, by the maximum principle, u is non-negative, and as M is compact, $u < k$ for some positive k.

We now choose any C^3 function ϕ on M that agrees with u on $\bar{\mathcal{U}}$. Within $\mathcal{U}$ we calculate that

$$\begin{aligned} R(\phi^4\tilde{\lambda}) &= u^{-4}\Big(R(\tilde{\lambda}) - 8u^{-5}\nabla^2_{\tilde{\lambda}} u\Big) \\ &= u^{-4}\Big(R(\tilde{\lambda}) - 1\Big) \end{aligned}$$

However we constructed $R(\tilde{\lambda})$ such that $R(\tilde{\lambda}) < 1/2$, therefore $R(\phi^4\tilde{\lambda}) < -1/2 < 0$ within $\mathcal{U}$. $\square$

4.2.3 Proof of Existence

Theorem 2. *Given a 3-manifold with Riemannian metric γ, (0,2) tensor μ and constant τ, the equation*

$$\Delta\psi - \frac{1}{8}R\psi + \frac{1}{8}|\mu|^2\psi^{-7} - \frac{1}{12}\tau^2\psi^5 = 0$$

has solutions depending on its parameters according to the table

	$\tau, \lvert\mu\rvert \equiv 0$	$\tau = 0, \lvert\mu\rvert \neq 0$	$\tau \neq 0, \mu \equiv 0$	$\tau, \mu \equiv 0$
γ^+	*No*	*Yes*	*No*	*Yes*
γ^0	*Yes*	*No*	*No*	*Yes*
γ^-	*No*	*No*	*Yes*	*Yes*

where γ^+ denotes that γ is conformally related to a metric of constant positive scalar curvature, and similarly for the other two.

Proof. We will proceed case by case. Based on the Yamabe class of each case, we will take the scalar curvature to be ± 1 or 0. Combined with the other assumptions this will simplify the equation enough to analyze it.

The simplest are the six cases in which there are no solutions, which all fail by the maximum principle.

Case 1

(γ^+ and $\tau, |\mu| \equiv 0$): No solution.

$$\Delta\psi = \frac{1}{8}\psi \tag{4.3}$$

Since ψ is a conformal factor it must be strictly positive. However on a closed manifold ψ obtains a maximum, and at this point $\Delta\psi \leq 0$, so there can be no solution that is also a valid conformal factor.

Case 2

(γ^- and $\tau, |\mu| \equiv 0$): No solution.

$$\Delta\psi = -\frac{1}{8}\psi \tag{4.4}$$

Similarly, ψ somewhere reaches a minimum at which $\Delta\psi \geq 0$.

Case 3

(γ^0 and $\tau = 0, |\mu| \neq 0$): No solution.

$$\Delta\psi = -\frac{1}{8}|\mu|^2\psi^{-7} \tag{4.5}$$

Same as case 2.

Case 4

(γ^- and $\tau = 0, |\mu| \neq 0$): No solution.

$$\Delta\psi = -\frac{1}{8}|\mu|^2\psi^{-7} - \frac{1}{8}\psi \tag{4.6}$$

Same as case 2.

Case 5

(γ^+ and $\tau \neq 0, |\mu| \equiv 0$): No solution.

$$\Delta\psi = \frac{1}{8}\psi + \frac{1}{12}\tau^2\psi^5 \tag{4.7}$$

Same as case 1.

Case 6

(γ^0 and $\tau \neq 0, |\mu| \equiv 0$): No solution.

$$\Delta\psi = \frac{1}{12}\tau^2\psi^5 \tag{4.8}$$

Same as case 1.

This concludes the cases in which no solution exists. There remains two additional immediate cases.

Case 7

(γ^0 and $\tau, |\mu| \equiv 0$): Solutions

$$\Delta\psi = 0 \tag{4.9}$$

Every positive constant solves this case.

Case 8

(γ^- and $\tau \neq 0, |\mu| \equiv 0$): Solution.

Choose a conformal factor such that $R = -1$. The equation becomes

$$\nabla^2\phi = -\frac{1}{8}\phi + \frac{1}{12}\tau^2\phi^5$$

which is solved by

$$\phi = (\frac{3}{2\tau^2})^{1/4}.$$

The remaining cases will depend on the lemma proven earlier.

Case 9

(γ^- and $\tau, |\mu| \neq 0$): Solution.

Pick a conformal factor such that $R = -1$. The Lichnerowicz equation becomes

$$\nabla^2\phi = -\frac{1}{8}\phi - \frac{1}{8}\mu^2\phi^{-7} + \frac{1}{12}\tau^2\phi^5$$

As a sub-solution we pick

$$\phi_- = (\frac{3}{2\tau^2})^{1/4}.$$

which satisfies $\nabla^2\phi \geq f(x, \phi_-)$

$$\nabla^2\phi_- = 0 \geq -\frac{1}{8}\mu^2\phi^{-7}.$$

For a super-solution we will look for a constant greater than ϕ_- such that $f(x, \phi_+)$ is positive. In other words we require of ϕ_+ that

$$\frac{1}{12}\tau^2\phi_+^5 \geq \frac{1}{8}\phi_+\frac{1}{8}\mu^2\phi_+^{-7}$$

or

$$\phi_+^4 \geq \frac{3}{2\tau^2}\left(1+\mu^2\phi_+^{-8}\right)$$

which leads us to pick

$$\phi_+ = \mathrm{Max}\left\{1, \sqrt[4]{\frac{3}{2\tau^2}\left(1+\mathrm{Max}(\mu^2)\right)}\right\}.$$

Here the maximum of μ is taken across M. This satisfies both the requirements of a valid super-solution so by our lemma a solution must exist.

Case 10

(γ^0 and $\tau, |\mu| \neq 0$): Solution.

Fix a conformal factor such that $R = 0$. The equation takes the form

$$\nabla^2\phi = -\frac{1}{8}\mu^2\phi^{-7} + \frac{1}{12}\tau^2\phi^5.$$

We will split the analysis of this case into two subclasses. Either μ is everywhere nonzero in which case the constants

$$\begin{aligned} \phi_- &= \sqrt[12]{\frac{3}{2\tau^2}\mathrm{M}in(\mu^2)} \\ \phi_+ &= \sqrt[12]{\frac{3}{2\tau^2}\mathrm{M}ax(\mu^2)} \end{aligned}$$

are valid sub and super solutions, or there is a point where $\mu^2 = 0$. In this second case we will need to pick a less simple conformal gauge and then apply Lemma 2.

Pick a point p at which μ is nonzero. Around p there exists an open metric ball B such that $|\mu|^2 > \xi$ for some positive ξ. We will denote the interior of the complement of this ball in M as $\mathcal{U}$. Note that every zero of $|\mu|$ lies within $\mathcal{U}$.

We now apply Lemma 2 to find a conformal factor ψ such that $R(\psi^4 g) < -k$ within $\mathcal{U}$ for some positive k. This changes μ^2 by ψ^{-12} throughout M. Since ψ is nonzero μ^2 will still be bounded below on B.

In summary we have $R \leq -k$ on $\bar{\mathcal{U}}$ and $|\mu|^2 \geq \tilde{\xi}$ on B. As a super solution we desire a constant ϕ_+ such that

$$\frac{2}{3}\tau^2\phi_+^5 \geq -R\phi_+ + \tilde{\mu}^2\phi^-7$$

which is satisfied if

$$\frac{2}{3}\tau^2\phi_+^5 \geq \left(\mathrm{Max}(|R|)\phi_+ + \mathrm{Max}(\tilde{\mu}^2)\phi_+^7\right).$$

We can quickly check that

$$\phi_+ = \mathrm{Max}\left\{1, \sqrt[4]{\frac{3}{2\tau^2}\left[\mathrm{Max}(|R|) + \mathrm{Max}(\tilde{\mu}^2)\right]}\right\}$$

is a valid super-solution.

As a sub-solution we require a constant ϕ_- such that

$$\frac{1}{8}R\phi_- - \frac{1}{8}\tilde{\mu}^2\phi_-^7 + \frac{1}{12}\tau^2\phi_-^5 \leq 0.$$

We will want to split our analysis of this inequality into $\bar{\mathcal{U}}$ and B. On B, $\mu^2 \geq \tilde{\xi} > 0$ so the inequality becomes

$$\tilde{\xi}\phi_-^{-7} \geq \frac{2}{3}\tau^2\phi_-^5 + \mathrm{Max}(|R|)\phi_-$$

which holds so long as

$$\phi_- \leq \mathrm{Min}\left\{1, \sqrt[8]{\frac{\tilde{\xi}}{\mathrm{Max}(|R|) + \frac{2}{3}\tau^2}}\right\}$$

However on $\bar{\mathcal{U}}$ we have $R < -k < 0$, so the inequality becomes

$$k\phi_- \geq \frac{2}{3}\tau^2\phi_-^5 - \mu^2\phi_-$$

and holds so long as

$$\phi_- \leq \sqrt[4]{\frac{3k}{2\tau^2}}.$$

Combining this with the restriction on B we arrive at

$$\phi_- \leq \mathrm{Min}\left\{1, \sqrt[4]{\frac{3k}{2\tau^2}}, \sqrt[8]{\frac{\tilde{\xi}}{\mathrm{Max}(|R|) + \frac{2}{3}\tau^2}}\right\}$$

Clearly $\phi_+ \geq \phi_- \geq 0$ so there must exist a solution between.

Case 11

(γ^+ and $\tau, |\mu| \neq 0$): Solution.

In this case we will again separate our analysis into a case where μ^2 has zeros and one where it has none. In the case where μ^2 has zeroes, the analysis is identical to that of case 10 and we will not repeat it.

In the case that μ^2 has no zeros, we will take the conformal transform such that $R = 1$ giving the equation

$$\nabla^2\phi = \frac{1}{8}\phi - \frac{1}{8}\mu^2\phi^{-7} + \frac{1}{12}\tau^2\phi^5.$$

We now choose the super and sub solutions

$$\begin{aligned}\phi_+ &= \sqrt[8]{\mathrm{Max}(\mu^2)}\\ \phi_- &= \mathrm{Min}\left\{1, \sqrt[8]{\mathrm{Min}(\mu^2)\frac{1}{1 + \frac{2}{3}\tau^2}}\right\}\end{aligned}$$

Case 12

(γ^+ and $\tau = 0, |\mu| \neq 0$): Solution.

This final class is the only one for which we cannot pick a constant sub and super solution. As such we will instead find a function as the lower bound. We choose the conformal gauge such that $R = 8$ giving the equation

$$\nabla^2\phi = \phi - \frac{1}{8}\mu^2\phi^{-7} \tag{4.10}$$

If there are no zeros of μ^2 then we take $\phi_- = \sqrt[8]{\frac{1}{8}\mathrm{Min}(\mu^2)}$ and $\phi_+ = \sqrt[8]{\frac{1}{8}\mathrm{Max}(\mu^2)}$.

Otherwise, let

$$A := \mathrm{Max}\left\{1, \frac{1}{8}\mathrm{Max}(\mu^2)\right\}$$

and consider the PDE

$$-\nabla^2\phi_- + \phi_- = \frac{\mu^2}{8}A^{-7}$$

Since kernel of the operator $\nabla^2 + 1$ is zero, the metric is 3 times differentiable and μ^2 is twice differentiable, there must be a unique C^4 solution ϕ_- to this equation on M. Since the right hand side is positive and not identically zero, we have $\phi_- > 0$ by the maximum principle.

We will now show that ϕ_- is a valid sub-solution. Consider the function

$$G(x, s) := \frac{\mu^2}{8}s^{-7}.$$

Since $\frac{\partial G}{\partial s} = -\frac{7}{8}\mu^2 s^{-8} \leq 0$, G is monotonically decreasing. By definition $A \geq 1$, so

$$G(x, 1) \geq G(x, A)$$

Since $G(x, 1) = \frac{\mu^2}{8} \leq A$ we have

$$G(x, A) \leq A$$

giving

$$-\nabla^2\phi_- + \phi_- \leq A.$$

Note that at a maximum of ϕ_-, $-\nabla^2\phi_- \geq 0$ so $A - \phi_- \geq 0$. So by the monotonicity of G

we have $g(x, A) \leq G(x, \phi_-)$ giving

$$\begin{aligned} -\nabla^2\phi_- + \phi_- &\leq G(x, \phi_-) \\ &= \frac{\mu^2}{8}\phi_-^{-7} \end{aligned}$$

so ϕ_- is a valid sub solution.

For a super solution we take $\phi_+ = A$. As we've already shown $\phi_+ \geq \phi_-$. plugging ϕ_+ into 4.10 we get

$$\begin{aligned} -\nabla^2\phi_+ + \phi_+ &= \phi_+ \\ &\geq G(x, \phi_+) \\ &= \frac{\mu^2}{8}\phi_+^{-7} \end{aligned}$$

showing that $\phi_+ = A$ is a valid supersolution. □

4.3 Uniqueness

We will now show that the solutions found in the previous section are unique, with the obvious exception of the case corresponding to the equation $\nabla^2\phi = 0$. First, however, we require another lemma. This one gives uniqueness of solutions to the inhomogenous Laplacian provided monotonicity of the right hand side.

4.3.1 Uniqueness from monotonic functions

Lemma. *Let $f : M \times \mathbb{R}^+ \to \mathbb{R}$ be C^1 and not identically zero, with*

$$\frac{\partial f}{\partial s}(x, s) \geq 0$$

for $s \in I$ an interval (potentially infinite). If ψ_1 and ψ_2 are solutions to $\nabla^2\psi = f(x,\psi)$ that both take values only in I then $\psi_1 = \psi_2$.

Proof. Consider the difference

$$\mathcal{I}(\psi_1,\psi_2)(x) = \big[\nabla^2\psi_1 - f(x,\psi_1)\big] - \big[\nabla^2\psi_2 - f(x,\psi_2)\big].$$

Clearly if both functions satisfy the equation then $\mathcal{I} = 0$. Now let us rewrite this quantity using the fundamental theorem of calculus.

$$\int_0^1 \left\{ \frac{d}{dv}\big[\nabla^2(v\psi_1 + (1-v)\psi_2) - f(x, v\psi_1 + (1-v)\psi_2)\big] \right\} dv \tag{4.11}$$

If we evaluate using the chain rule and pull out the terms independent of v we get

$$\mathcal{I}(\psi_1,\psi_2)(x) = \nabla^2(\psi_1 - \psi_2) - \left[\int_0^1 \partial_s f(x,s)dv\right](\psi_1 - \psi_2)$$

where $s = v\psi_1 + (1-v)\psi_2$. Now we assume that both ψ_1 and ψ_2 solve the original PDE. This means that $\mathcal{I} = 0$ as previously established, leaving us with

$$\nabla^2(\psi_1 - \psi_2) = \left[\int_0^1 \partial_s f(x,s)dv\right](\psi_1 - \psi_2)$$

In the next lines we will denote the integral above as $\mathcal{F}$ for brevity. Now we multiply both sides by $\psi_1 - \psi_2$

$$(\psi_1 - \psi_2)\nabla^2(\psi_1 - \psi_2) = \mathcal{F}(\psi_1 - \psi_2)^2$$

integrate over M

$$\int_M \Big[(\psi_1 - \psi_2)\nabla^2(\psi_1 - \psi_2)\Big] = \int_M (\psi_1 - \psi_2)^2\mathcal{F}$$

and integrate by parts to arrive at (potentially) only positive terms

$$0 = \int_M (\psi_1 - \psi_2)^2 \mathcal{F} + \int_M \left[\nabla(\psi_1 - \psi_2)\right]^2.$$

Since M is compact the boundary term is zero leaving us with the equation above. If $\mathcal{F}$ is positive, then the only way to satisfy this equation is by having $\psi_1 = \psi_2$. $\mathcal{F}$ is defined as

$$\left[\int_0^1 \partial_s f(x,s)dv\right]$$

which will be positive by our assumptions if s is within the interval I required by the lemma. The only other restriction on I is that ψ_1 and ψ_2 take values only in I. However $s = v\psi_1 + (1-v)\psi_2$ is just a homotopy formula from ψ_2 to ψ_1. Any interval that contains ψ_1 and ψ_2 must also contain every value of s. Therefore $\mathcal{F} \geq 0$ (and must be somewhere non-zero by hypothesis), leaving us with $\psi_1 = \psi_2$. □

Having established this lemma we can proceed with the 5 cases for which unique solutions exist. The conformal guages used previously for each case to establish existence will again be used for uniqueness unless otherwise mentioned.

4.3.2 Proof of Uniqueness

Case 1

(γ^- and $\tau \neq 0, |\mu| \equiv 0$)

$$\nabla^2\phi = -\frac{1}{8}\phi + \frac{1}{12}\tau^2\phi^5$$

which is solved by

$$\phi = (\frac{3}{2\tau^2})^{1/4}.$$

This is the only solution, which can be seen by applying our lemma.

$$\partial_\phi f(x,\phi) = -\frac{1}{8} + \frac{5}{12}\tau^2\phi^4$$

is not necessarily positive. However using the maximum principle, we can consider a minimum of ϕ

$$0 \leq \nabla^2\phi(x_{min}) = -\frac{1}{8}\phi + \frac{1}{12}\tau^2\phi^5$$

implying that

$$\phi(x_{min}) \geq \sqrt[4]{\frac{3}{2\tau^2}}$$

This is the minimum value of any possible solution, and as such we only need $\partial_\phi f \geq 0$ on values greater than it. Luckily, this is true,

$$\begin{aligned}\partial_\phi f &\geq -\frac{1}{8} + \frac{5}{12}\tau^2\left(\frac{3}{2\tau^2}\right)\\ &\geq \frac{1}{2}\end{aligned}$$

guaranteeing uniqueness.

Case 2

(γ^+ and $\tau = 0, |\mu| \neq 0$)

$$\nabla^2\phi = \phi - \frac{1}{8}\mu^2\phi^{-7}$$

Taking the derivative of the right hand side we see that it is positive,

$$\partial_\phi f(x,\phi) = 1 + \frac{7}{8}\mu^2\phi^{-8}$$

guaranteeing uniqueness.

Case 3

(γ^+ and $\tau, |\mu| \neq 0$)

Here we will not use the non-constant scalar curvature gauge used earlier and will instead elect to fix $R = 1$, giving

$$\nabla^2\phi = \frac{1}{8}\phi - \frac{1}{8}\mu^2\phi^{-7} + \frac{1}{12}\tau^2\phi^5.$$

Taking the derivative, we see that it is positive,

$$\partial_\phi f(x,\phi) = \frac{1}{8} + \frac{7}{8}\mu^2\phi^{-8} + \frac{5}{12}\tau^2\phi^4.$$

guaranteeing uniqueness.

Case 4

(γ^0 and $\tau, |\mu| \neq 0$)

Here we will fix $R = 0$. Having done so, this case proceeds as the previous two:

$$\nabla^2\phi = -\frac{1}{8}\mu^2\phi^{-7} + \frac{1}{12}\tau^2\phi^5.$$
$$\partial_\phi f(x,\phi) = \frac{7}{8}\mu^2\phi^{-8} + \frac{5}{12}\tau^2\phi^4.$$

The derivative is positive for all values of ϕ so uniqueness is guaranteed.

Case 5: (γ^- and $\tau, |\mu| \neq 0$) Fix $R = -1$. This gives

$$\nabla^2\phi = -\frac{1}{8}\phi - \frac{1}{8}\mu^2\phi^{-7} + \frac{1}{12}\tau^2\phi^5.$$

Taking the derivative,

$$\partial_\phi f(x,\phi) = -\frac{1}{8} + \frac{7}{8}\mu^2\phi^{-8} + \frac{5}{12}\tau^2\phi^4$$

we see that it is not necessarily positive. We will again apply the maximum principle at the minimum of phi:

$$\begin{aligned} 0 \leq \nabla^2\phi_{min}) &= -\frac{1}{8}\phi_{min} - \frac{1}{8}\mu^2\phi_{min}^{-7} + \frac{1}{12}\tau^2\phi_{min}^5. \\ \phi_{min}^4 &\geq \frac{3}{2\tau^2}(1+\mu^2\phi_{min}^{-8}) \\ \phi_{min} &\geq \sqrt[4]{\frac{3}{2\tau^2}}. \end{aligned}$$

Now we check that $\partial_\phi f$ is positive for ϕ greater than this lower bound:

$$\begin{aligned} \partial_\phi f(x,\phi) &= -\frac{1}{8} + \frac{7}{8}\mu^2\phi^{-8} + \frac{5}{12}\tau^2\phi^4 \\ &\geq -\frac{1}{8} + \frac{7}{8}\mu^2\left(\frac{4}{9}\tau^4\right) + \frac{5}{12}\frac{3}{2} \qquad \geq -\frac{1}{2}. \end{aligned}$$

By our lemma the solution must be unique.

Case 5

(γ^- and $\tau, |\mu| \neq 0$)

$$\nabla^2\phi = -\frac{1}{8}\phi - \frac{1}{8}\mu^2\phi^{-7} + \frac{1}{12}\tau^2\phi^5$$

Taking the derivative we get

$$\partial_\phi f = -\frac{1}{8} + \frac{7}{8}\mu^2\phi^{-8} + \frac{5}{12}\tau^2\phi^4$$

which again is not necessarily positive. However we can again apply the maxmimum principle giving us

$$0 \leq \nabla^2 \phi_{min} = -\frac{1}{8}\phi_{min} - \frac{1}{8}\mu^2 \phi_{min}^{-7} + \frac{1}{12}\tau^2 \phi_{min}^5.$$

Thus

$$\phi_{min}^4 \geq \frac{3}{2\tau^2}\left[1 + \mu^2(x_{min})\phi_{min}^{-8}\right].$$

The second term on the right hand side is necessarily positive so we have

$$\phi_{min} \geq \sqrt[4]{\frac{3}{2\tau^2}}.$$

Now we can restrict $\partial_\phi f$ to values of ϕ greater than or equal to this lower bound. Having done so we get

$$\partial_\phi f \geq \frac{1}{2} + \frac{7}{8}\mu^2\left(\frac{4}{9}\tau^2\right)$$

which is positive. Uniqueness follows. □

Chapter 5

Gluing

5.1 The IMP Gluing

We will now outline the goal and methodology of the gluing by Isenberg, Mazzeo and Pollack [1]. Although their paper includes results for gluing in the asymptotically flat and hyperbolic cases, we will confine ourselves to the compact case.

Starting with a manifold equipped with valid solutions to the Einstein constraints, the goal of their procedure is to glue together two separate points on the manifold creating an explicit, topologically distinct, new solution to the Einstein constraints that asymptotically approaches the original solution as we stretch the neck between the two gluing points. It's worth noting that the starting manifold need not be connected; we can glue together two separate components or a single component to itself.

The gluing is done by removing two different points P_1 and P_2. In a ball around each point we then conformally scale by $\psi = r^{-1/2}$ where r is the radius from the removed point. The result is a 'cylinder' which resembles something like $\mathbb{R}^+ \times \mathbb{S}^2$. The original paper includes useful diagrams of this part of the procedure which can be seen in figure 5.1.

Here Σ is the starting manifold and γ the starting metric. We will continue with this notation for the remainder of this section. After creating the cylinders we then truncate

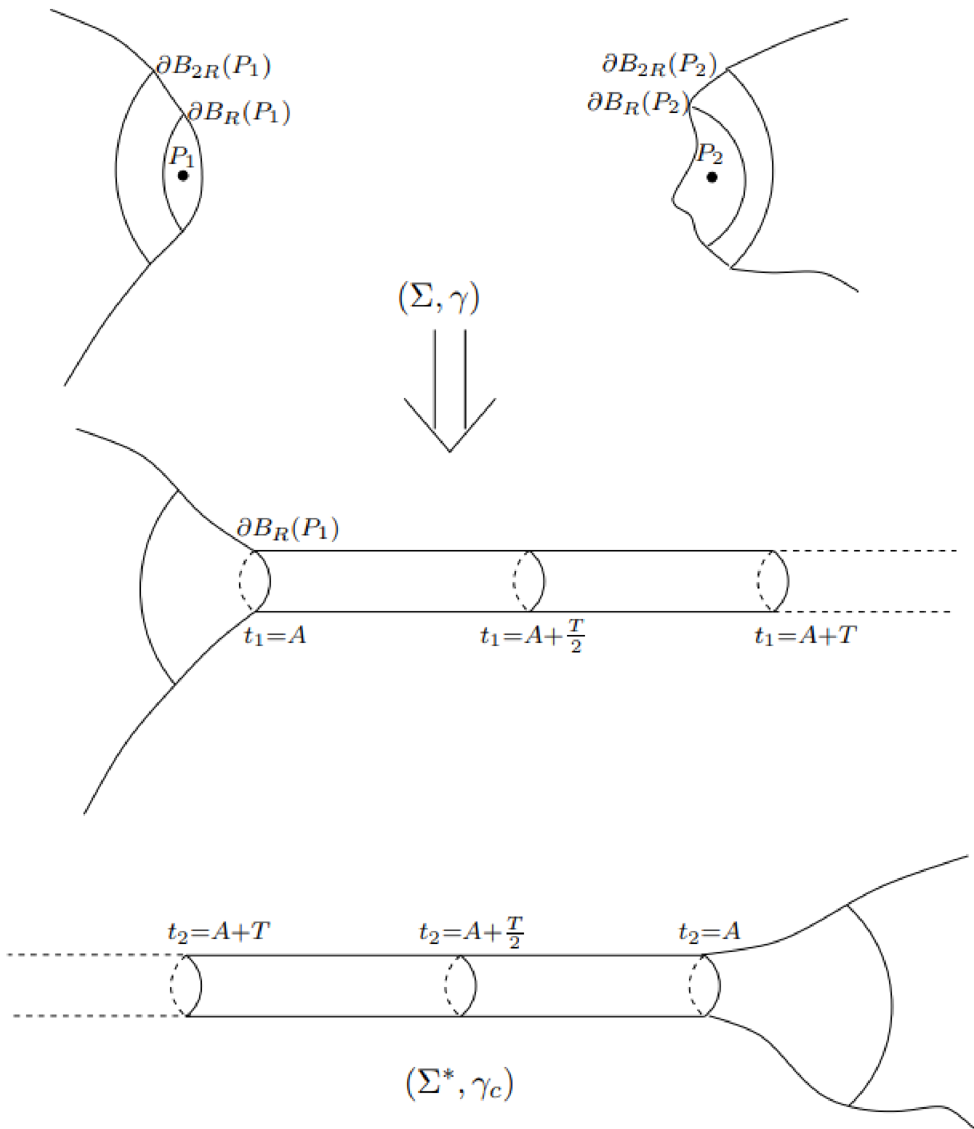

Figure 5.1: *The conformal scaling of the ball around each point. (Reproduced with permission from Springer Nature.)*[1]

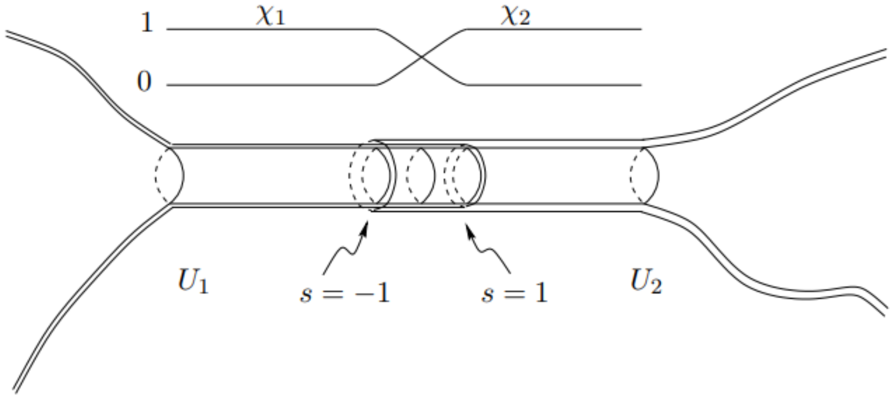

Figure 5.2: *The truncation and identification of the ends of the cylinders. The relevant structures are then smoothed together with cutoff functions* χ_1 *and* χ_2*. (Reproduced with permission from Springer Nature.)*[1]

them at a sufficient distance (the control of which allows us to control the accuracy of the approximation) and then identify sections at the end of each cylinder. This can be seen in figure 5.2. Also notable in 5.2 are the cutoff functions χ_1 and χ_2. These are used to smooth together the 3 relevant structures on the manifold: the metric γ, the traceless part μ of the second fundamental form and the conformal factor used to scale the gluing region ψ. Although the cutoff functions shown are not smooth this is just for visual convenience. The functions actually used would be smooth cutoffs. The result of this is a new manifold Σ_c, new metric γ_c, new second fundamental form μ_c and new conformal factor ψ_c. These functions are smooth on all of Σ_c, but do *not* satisfy the constraint equations. The chief problem is that by modifying μ and γ we are no longer guaranteed that $\nabla^k \mu_{ki} = 0$ or that the modified conformal factor ψ solves the Lichnerowicz equation. We will need to construct correction terms.

Having produced our approximate solutions, denoted $(\gamma_T, \mu_T, \psi_T)$ for a cylinder length

T we now want to add small correction terms

$$\tilde{\mu_T} = \mu_T + \sigma_T$$

$$\tilde{\psi_T} = \psi_T + \eta_T$$

so that the modified results are actual solutions. For μ we construct the correction term using a vector Laplacian Ł which is treated in depth later. This corrected μ is then used in the Lichnerowicz equation $\mathcal{N}_T$ allowing us to see the error term $\mathcal{N}_T(\psi_T) = E_T$. $\mathcal{N}_T$ is linearized and and the correction produced using the contraction mapping argument. These correction terms are then bounded to show that they tend towards zero outside the neck as $T \to \infty$.

5.2 Gluing along Hyperbolic Cusps

We now consider gluing in the toroidal case. As mentioned earlier, we sidestep the complications of surgery by considering manifolds with pre-existing cusps. These manifolds are obviously non-compact which seems to contradict our restriction to compact manifolds. We solve this by considering only complete manifolds with a single cusp. That is, the only end of the manifold is the that of the cusp being glued. Since compactness is only used after the gluing, this resolves the issue. The rest of the construction precedes nearly exactly as the original gluing, until the correction terms are needed.

Constructing the correction terms differs in several ways between the connected sum and S^1 case. First are topological differences between $\mathbb{T}^2$ and S^2 which effect the analysis of Ł but are ultimately non-issues. Second is the non-degeneracy condition. The original paper requires that compact manifolds have no non-trivial conformal killing vectors vanishing at the removed points. This cannot be directly translated to the case of gluing along hyperbolic cusps so must be modified or replaced.

The 2-sphere has only a single conformal class by the uniformization theorem, while the

2-torus has many, parametrized by the upper half-plane. This complicates the invertability argument for Ł. We assume here that the conformal class of the torus remains constant across the cylinder.

The difference in conformal scaling between the two considered cylinders affects the bounds we can put on the errors of the new solution. In the IMP gluing[1], the metric around a point is scaled by $\psi_c = r^{1/2}$ inside the cylinder:

$$\gamma = dr^2 + r^2 h(r)$$
$$\gamma_c = \psi_c^4 \gamma = r^{-2} dr^2 + h(r)$$

and then re-parameterized with $t = -\log r$

$$\gamma_c = dt^2 + h(e^{-t})$$

which gives us $|\psi_c| = e^{-t/2}$. So in the cylinder $|\psi_c| \approx e^{-T/2}$.

For the hyperbolic case, we scale the metric by $\psi_c = z^{1/2}$:

$$\gamma = \frac{1}{z^2}\left(dx^2 + dy^2 + dz^2\right)$$
$$\gamma_c = dx^2 + dy^2 + dz^2$$

So $\psi_c \approx T^{1/2}$. However almost all of the bounds come from how small ψ_c is. For example, equation 15 in the original gluing paper[1] bounds the divergence correction term σ by

$$||\sigma|| \leq CT^3 |\psi^6|$$

As such the gluing procedure would need to be drastically altered to work on hyperbolic cusps.

Chapter 6

Conformal Vector Laplacian

6.1 Derivation and Application

The conformal vector laplacian Ł(x) is a useful tool for creating correction terms necessary when gluing. It is constructed by the composition of the conformal killing operator and its adjoint. The conformal killing operator is given by:

$$\begin{aligned} \mathcal{D}(X) &= \frac{1}{2}\mathcal{L}_x g - \frac{1}{3}div(X)g \\ (\mathcal{D}(X))_{ij} &= \frac{1}{2}(X_{i;j} + X_{j:i}) - \frac{1}{3}X^k_{;k}g_{ij} \end{aligned} \tag{6.1}$$

This operator is constructed such that it maps vectors fields to the space of symmetric, trace free (0,2) tensors, and has kernel equal to the conformal killing fields of the underlying manifold. It is clearly symmetric by construction, and has zero trace in three dimensions. We are interested in the dual of this operator on this space, which is $-div(A)$. It is crucial to restrict to symmetric and trace free tensors for this calculation. Neglecting this, one arrives at a different adjoint for $-div(A)$:

$$\begin{aligned}\langle X, -div(A)\rangle &= \int_M -X^\lambda \nabla^\sigma A_{\sigma\lambda} \\ &= \int_M -\nabla^\sigma (X^\lambda A_{\sigma\lambda}) + \nabla^\sigma (X^\lambda) A_{\sigma\lambda} \\ &= \langle \nabla X, A\rangle \end{aligned} \tag{6.2}$$

However on symmetric trace free tensors, we also have:

$$\begin{aligned}\langle \mathcal{D}(X), A\rangle &= \int_M \frac{1}{2}(\nabla_\nu X_\mu + \nabla_\mu X_\nu)A^{\nu\mu} - \int_M \frac{1}{3} div(X) g_{\mu\nu} A^{\nu\mu} \\ &= \int_M \nabla_\nu X_\mu A^{\nu\mu} - \int_M \frac{1}{3} div(X) tr(A) \\ &= \langle \nabla X, A\rangle \\ &= \langle X, -div(A)\rangle \end{aligned} \tag{6.3}$$

Where we use the symmetry of A in the second line and the fact that A is trace free in the third. Thus the operator we are interested in, $-\mathcal{D}^* \circ \mathcal{D}$, can be written

$$Ł(X) = div(\frac{1}{2}\mathcal{L}_x g - \frac{1}{3} div(X) g) \tag{6.4}$$
$$(Ł(X))^i = \frac{1}{2}\nabla_j \nabla^j X^i + \frac{1}{2}\nabla_j \nabla^i X^j - \frac{1}{3}\nabla^i \nabla_l X^l$$

6.1.1 Kernel

The kernel of $\mathcal{D}$ is the space of conformal killing fields. This comes by construction, since conformal killing fields are defined as

$$\mathcal{L}_x g = \lambda(x) g$$

Taking the trace,

$$\nabla_\mu X^\mu = 3\lambda(x)$$

we see that a vector field is in the kernel of $\mathcal{D}$ if and only if it is a conformal killing field

$$\mathcal{L}_x g = \frac{1}{3} div(X) g$$

Since Ł is constructed by composing $\mathcal{D}$ with its adjoint, clearly $\mathrm{Ker}(\mathcal{D}) \subset \mathrm{Ker}(Ł)$. We also have the converse, since

$$\langle Ł(x), X \rangle = \langle \mathcal{D}(X), \mathcal{D}(X) \rangle = |\mathcal{D}(X)|^2$$

6.1.2 Weitzenbock identity

We can relate this operator to the rough Laplacian by a Weitzenbock identity. Defining the Riemann curvature as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{6.5}$$
$$\langle R(X,Y)Z, W \rangle = R_{lijk} W^l Z^i X^j Y^k$$

allows us to write

$$\nabla_i \nabla_j X^k - \nabla_j \nabla_i X^k = R^k_{lij} X^l \tag{6.6}$$

We can then apply this to the final term of Ł above:

$$\begin{aligned}\nabla^i\nabla_l X^l &= g_{ij}\delta^l_m(R^l_{kjm}X^k - \nabla_m\nabla_j X^l) \\ &= (R^{m\ i}_{\ k\ m}X^k - \nabla_l\nabla^i X^l) \\ &= -R^i_k X^k + \nabla_l\nabla^i X^l\end{aligned} \tag{6.7}$$

Where we defined the Ricci tensor as $R_{jk} = \Sigma_i\langle R(e_j, e_i)e_i, e_k\rangle$. Substituting this into Ł gives

$$(Ł(X))^i = \frac{1}{2}\nabla_j\nabla^j X^i + \frac{1}{2}R^i_{\ j}X^j + \frac{1}{6}\nabla^i\nabla_l X^l \tag{6.8}$$

This is potentially a more attractive form of the equation for the purpose of gluing.

6.1.3 Correction Term

Having explored the properties of Ł we can now use it to construct the correction term for μ as mentioned earlier. We do this by exploiting the fact that $ŁX = div(\mathcal{D}X)$ and inverting Ł. Explicitly, define W as the vector associated to the 1-form divμ, then we want to solve the equation $ŁX = W$. Let $\mathcal{D}X = \sigma$ and we have

$$div(\mu - \sigma) = W - div\mathcal{D}X = W - ŁX = 0$$

Thus by finding X we can then apply $\mathcal{D}$, giving us our correction term. This relies on inverting Ł on the cylinder.

6.2 Ł(X) on $\mathbb{T}^2 \times \mathbb{R}$

The IMP gluing procedure[1] applies the conformal vector Laplacian to $\mathbb{S}^2 \times \mathbb{R}$. We will take the similar case of $\mathbb{T}^2 \times \mathbb{R}$ and fix the conformal class of the torus along the cylinder, allowing us to apply the separation of variables used in the IMP gluing. On this domain, our metric looks like $ds^2 + h$, where h is the metric on the torus. Starting with this, every vector field can be decomposed as

$$X = f\partial_s + Y(s) \tag{6.9}$$

Where f is a function on $\mathbb{T}^2 \times \mathbb{R}$ and Y is a vector field on the torus for each s. Then $\mathcal{D}(X)$ can be written:

$$\mathcal{D}(X) = \mathcal{D}(f\partial_s) + \mathcal{D}(Y)$$

$$\mathcal{D}(X) = \begin{bmatrix} \frac{2}{3}\partial_s f & \frac{1}{2}\nabla_{\mathbb{T}} f \\ \\ \frac{1}{2}\nabla_{\mathbb{T}} f & (-\frac{1}{3}\partial_s f)h \end{bmatrix} + \begin{bmatrix} -\frac{1}{3}\mathrm{div}_{\mathbb{T}}(Y) & \frac{1}{2}\partial_s Y \\ \\ \frac{1}{2}\partial_s Y & \mathcal{D}_{\mathbb{T}}(Y) + \frac{1}{6}\mathrm{div}_{\mathbb{T}}(Y)h \end{bmatrix} \tag{6.10}$$

where $\mathrm{div}_{\mathbb{T}}$ and $\nabla_{\mathbb{T}}$ are the corresponding operators on torus. Similarly, $\mathcal{D}_{\mathbb{T}}$ is the 2 dimensional conformal killing operator $\mathcal{D}(X)$ on the torus. In coordinates:

$$\nabla_i Y_j + \nabla_j Y_i - \frac{1}{2}\mathrm{div}_{\mathbb{T}}(Y)h_{ij} = \mathcal{D}_{\mathbb{T}}(Y)$$

Applying -div to $\mathcal{D}(X)$ we get

$$Ł(X) = \begin{bmatrix} -\frac{2}{3}\partial_s^2 f + \frac{1}{2}\Delta_{\mathbb{T}} f + \frac{1}{6}\partial_s \mathrm{div}_{\mathbb{T}}(Y) \\ \\ -\frac{1}{6}\partial_s \nabla_{\mathbb{T}} f - \frac{1}{2}\partial_s^2 Y + Ł_{\mathbb{T}}(Y) - \frac{1}{6}\nabla_{\mathbb{T}}\mathrm{div}_{\mathbb{T}}(Y) \end{bmatrix} \tag{6.11}$$

Recall the Weitzenbock identity derived earlier in 6.8, and note that two dimensional operator $Ł_{\mathbb{T}}$ lacks the final $\frac{1}{6}\nabla\mathrm{div}$ term. Additionally, on the flat torus the Ricci curvature vanishes, giving us

$$Ł_{\mathbb{T}} = -\frac{1}{2}\nabla^* \circ \nabla$$

On a flat space it is further true that $\nabla^* \circ \nabla = \delta \mathrm{d} + \mathrm{d}\delta$, when acting on forms. With this identity, it is useful to rewrite 6.11 by swapping Y with its associated 1-form ω and replacing $\nabla_{\mathbb{T}}$ and $\mathrm{div}_{\mathbb{T}}$ with d and $-\delta$ respectively. Having done all this, we now write Ł as

$$L \begin{pmatrix} f \\ \\ \omega \end{pmatrix} = \begin{pmatrix} -\frac{2}{3}\partial_s^2 + \frac{1}{2}\Delta_{\mathbb{T}} & \frac{1}{6}\partial_s \delta \\ \\ -\frac{1}{6}\partial_s \mathrm{d} & -\frac{1}{2}\partial_s^2 + \frac{2}{3}\mathrm{d}\delta + \frac{1}{2}\delta\mathrm{d} \end{pmatrix} \begin{pmatrix} f \\ \\ \omega \end{pmatrix} \tag{6.12}$$

This differs from the operator on $\mathbb{S}^2 \times \mathbb{R}$ used in the original gluing[1] only slightly, due to the fact that the torus is flat.

6.2.1 Scalar Laplacian on $\mathbb{T}^2$

It will be relevant to know the eigenvalues and eigenfunctions of the scalar Laplacian and the Hodge Laplacian on the torus. Consider a torus defined as

$$\mathbb{T}^2 := \mathbb{R}^2 / \langle \hat{x}, a\hat{x} + b\hat{y} \rangle$$

Here a and b are constant in s which is equivalent to our assumption of fixed conformal class through the cylinder. The eigenfunctions of the scalar Laplacian on this space can be found by first considering the eigenfunctions on $\mathbb{R}^2$:

$$u_{pq} = e^{2\pi i(px+qy)}$$
$$\lambda_{pq} = 4\pi^2(p^2 + q^2)$$

Requiring periodicity under translation by $\hat{x}$ and $a\hat{x} + b\hat{y}$ gives constraints on what p and q can be. Specifically, we require that

$$\begin{aligned}
e^{2\pi p} &= 1 \\
e^{2\pi(pa+qb)} &= 1
\end{aligned}$$

which is true if and only if p and $pa + qb$ are integers. Knowing this, we may write the eigenfunctions and eigenvalues as as

$$\begin{aligned}
u_{nm} &= e^{2\pi i(mx - ma/by + n/by)} \qquad n, m \in \mathbb{Z} \\
\lambda_{nm} &= 4\pi^2 m^2 + 4\pi^2(\frac{n}{b} - \frac{ma}{b})^2
\end{aligned} \tag{6.13}$$

6.2.2 Hodge Laplacian on $\mathbb{T}^2$

Knowing the eigenfunctions of the scalar Laplacian, we can now examine the Hodge Laplacian:

$$\Delta\omega = (\delta \mathrm{d} + \mathrm{d}\delta)\omega$$

Using the Hodge decomposition, eigenfunctions of Δ are

$$\begin{aligned}
\Delta\alpha &= \xi\alpha \\
(\delta\mathrm{d} + \mathrm{d}\delta)(\mathrm{d}u + \delta\beta) &= \xi(\mathrm{d}u + \delta\beta) \\
\mathrm{d}\delta\mathrm{d}u + \delta\mathrm{d}\delta\beta &= \xi(\mathrm{d}u + \delta\beta) \\
\mathrm{d} \star \mathrm{d} \star \mathrm{d}u + \star\mathrm{d} \star \mathrm{d}v &= \xi(\mathrm{d}u + \star\mathrm{d}v)
\end{aligned} \tag{6.14}$$

where $\star\beta = v$. Splitting this equation into u and v parts by orthogonality of d and δ, and noting that the scalar Laplacian is δd, we get:

$$\mathrm{d}(\Delta u - \xi u) = 0$$
$$\mathrm{d}(\Delta v - \xi v) = 0$$

Therefore u differs from an eigenfunction of Δ by a constant C. However, if we define u' as $u - \frac{\xi}{C}$, then u' will be an eigenfunction. Similarly for v. Knowing this, we can write the eigenvalues of the Hodge laplacian in terms of the scalar laplacian:

$$\begin{aligned} \Delta\omega &= \xi\omega & \xi_{pq} &= \lambda_{pq} \\ \omega &= \mathrm{d}u & \omega &= \star\mathrm{d}u \end{aligned} \tag{6.15}$$

In contrast with the case on the sphere, the torus has a two dimensional space of zero modes, corresponding to constant forms along the two axes

$$\{a\mathrm{d}x + b\mathrm{d}y | a, b \in \mathbb{R}\}$$

6.2.3 Separation of Variables

We now proceed by separation of variables in order to study the eigenspaces of Ł. First, we normalize the eigenfunctions of Δ,

$$|\mathrm{d}u_{nm}|^2 = (2\pi n u_{nm})^2 + (\frac{n}{b} - \frac{ma}{b})^2(2\pi u_{nm})^2 = \lambda_{nm}$$

and similarly for $\star\mathrm{d}u$. Thus the normalized eigenbasis is $\{\lambda_\alpha^{-1/2}\mathrm{d}u_\alpha, \lambda_\alpha^{-1/2} \star \mathrm{d}u_\alpha\}$ where we use the multi-index $\alpha = (n, m)$ with $|\alpha| = n + m$ from now on when necessary. Expanding a vector field X on $\mathbb{T}^2 \times \mathbb{R}$ in terms of these normalized eigenfunctions as

$$X = f(s)u\mathrm{d}s + \frac{g(s)}{\sqrt{\lambda}}\mathrm{d}u + \frac{h(s)}{\sqrt{\lambda}} \star \mathrm{d}u$$

and acting on such a vector field by Ł as

$$\begin{aligned}ŁX = \Big(\frac{-2}{3}f'' + \frac{\lambda}{2}f - \frac{\sqrt{\lambda}}{6}g'\Big)u\mathrm{d}s + \Big(\frac{-1}{2}g'' - \frac{\sqrt{\lambda}}{6}f' + \frac{2\sqrt{\lambda}}{3}g\Big)\frac{\mathrm{d}u}{\sqrt{\lambda}} \\ +\Big(\frac{-1}{2}w'' + \frac{\lambda}{2}w\Big)\frac{\star\mathrm{d}u}{\sqrt{\lambda}}\end{aligned}$$

gives us a much more tractable form of the operator. Alternatively,

$$ŁX = \begin{pmatrix} -\frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \partial_s^2 + \begin{pmatrix} 0 & \frac{\sqrt{\lambda}}{6} & 0 \\ -\frac{\sqrt{\lambda}}{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \partial_s + \begin{pmatrix} \frac{\lambda}{2} & 0 & 0 \\ 0 & \frac{2\lambda}{3} & 0 \\ 0 & 0 & \frac{\lambda}{2} \end{pmatrix}$$

acting on the column vector $(f, g, h)^t$. Using this form, it's easier to see how to further decompose Ł. For $|\alpha| \geq 1$, the operator can be broken apart into

$$Ł'_\alpha = \begin{pmatrix} -\frac{2}{3} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \partial_s^2 + \begin{pmatrix} 0 & \frac{\sqrt{\lambda}}{6} \\ -\frac{\sqrt{\lambda}}{6} & 0 \end{pmatrix} \partial_s + \begin{pmatrix} \frac{\lambda}{2} & 0 \\ 0 & \frac{2}{3}\lambda \end{pmatrix} \qquad \text{on } (f, g)^t$$

$$Ł''_\alpha = -\frac{1}{2}\partial_s^2 + \frac{\lambda}{2} \qquad \text{on } h$$

However, we must also consider the zero modes of Δ on $\mathbb{T}^2$ discussed earlier. For $|\alpha| = 0$, $\lambda = 0$, and we have $X = f\mathrm{d}s + g\mathrm{d}x + h\mathrm{d}y$. Applying Ł,

$$Ł_0 = \begin{pmatrix} -\frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \partial_s^2$$

and writing as the direct sum of these operators,

$$Ł = Ł_0 \oplus \bigoplus_{|\alpha| \geq 1} Ł'_\alpha \oplus Ł''_\alpha$$

which closely mirrors the decomposition on $\mathbb{S}^2 \times \mathbb{R}$. However, it differs in two relevant ways; $Ł_0$ acts on $(f, g, h)^t$ rather than just f, and the lack of curvature on $\mathbb{T}^2$ means that for all $|\alpha| \geq 1$, h grows exponentially, which limits the space of temperate solutions.

6.2.4 Temperate Solutions

We are interested in the solutions to $ŁX = 0$ that grow less than exponentially along the cylinder. The space of these temperate solutions is 6 dimensional, all corresponding to $Ł_0$. $Ł_0$ is proportional to ∂_s^2 on every component of X, so the nullspace is linear functions:

$$(a + bs)\mathrm{d}s$$

$$(e + fs)\mathrm{d}y$$

$$(g + hs)\mathrm{d}x$$

These are the only temperate solutions. $Ł'' = 0$ clearly has no non-exponential solutions since $|\alpha| \geq 1$. $Ł' = 0$ similarly fails to have non-exponential solutions. The proof is nearly identical to that in the original gluing.[1]

6.2.5 Lowest Eigenvalue on C_T

Theorem 3. *The lowest eigenvalue on $\mathbb{T}^2 \times [-T/2, T/2]$ for $X = 0$ at $\pm T/2$ satisfies*

$$\lambda_0(T) := \langle ŁX, X\rangle \geq \frac{C}{T^2}$$

where C is constant independent of T and $|X| = 1$.

This follows immediately for $Ł_0$ and $Ł''$. Eigenvalues of $Ł_0$ are of the form $j^2\pi/T^2$ for integer j, and thus satisfy the inequality. Eigenvalues of $Ł''$ are bounded below by the lowest eigenvalue of Δ on the torus, which is constant in T, and therefore also satisfy the inequality.

For $Ł'$ we have,

$$\begin{aligned}\langle Ł'_\alpha \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}\rangle &= \int_{-T/2}^{T/2} \frac{2}{3}(f')^2 + \frac{1}{2}(g')^2 + \frac{\sqrt{\lambda}}{6}(fg' - f'g) + \frac{\lambda}{2}f^2 + \frac{2}{3}\lambda g^2 ds \\ &= \int_{-T/2}^{T/2} \left(f' - \frac{\sqrt{\lambda}}{8}g\right)^2 + \frac{1}{2}\left(g' + \frac{\sqrt{\lambda}}{6}f\right)^2 + \frac{35}{72}\lambda f^2 + \frac{63}{96}\lambda g^2 ds\end{aligned}$$

Since $|\alpha| > 0$ for $Ł'$, λ is strictly positive, and the above integral is bounded below by

$$C\int_{-T/2}^{T/2} f^2 + g^2 ds$$

which is a constant, the spectrum of $Ł'$ is bounded from zero.

6.2.6 Growth Estimates

There are two rather specific estimates we would need to show invertability of Ł. The first is a way of controlling the size of an eigenfunction with sufficiently small eigenvalue on the interior by its values on the ends. The second is an estimate on the inhomogeneous eigenvalue problem.

Theorem 4. *Given $ŁX = \mu X$, where $Ł$ is defined on $\mathbb{T}^2 \times [-T/2, T/2]$ with the product* *metric, and* $\mu < \frac{1}{8}\lambda_0(T)$*, and* X *satisfying*

$$\int_{\mathbb{T}^2} |X(-T/2,\theta)|^2 + |X(T/2,\theta)|^2 d\theta \le C_1$$

then for any $a \in [-T/2+1, T/2-1]$*,*

$$\int_{a-1}^{a+1} \int_{\mathbb{T}^2} |X(s,\theta)|^2 d\theta ds \le C_2$$

where C_2 *depends on* C_1 *but not* a *or* T *and* $C_2 \to 0$ *as* $C_1 \to 0$*.*

We will check this estimate on each of component of the decomposition of Ł as in previous sections. For $Ł_0$ and L'' we proceed directly using solutions of the eigenvalue problem.

An arbitrary eigenfunction of $Ł_0$ has norm $|exp(i\mu s + \alpha)|^2$ for complex α. Setting the volume of the torus equal to 1, the first integral gives $2|e^\alpha|^2$ since the norm is constant in θ. Similarly, the second integral gives $(|e^\alpha|^2)\big|_{a-1}^{a+1} = 2|e^\alpha|$. So in the case of $Ł_0$, $C_1 = C_2$. The same argument applies for $Ł''$. Note that for these components of Ł, the condition that μ be small is not required.

$Ł'$ requires slightly more finesse, but the argument in the end is essentially the same. We rewrite $Ł'$ as

$$Ł'_\alpha = -A\partial_s^2 + B\partial_s + C$$

where A, B and C are $2x2$ the matrices of coefficients given earlier, and $|\alpha| > 0$.

Let $X = (f,g)^t$, and define $\rho(s) = \langle AX, X\rangle$.

$$\frac{1}{2}\partial_s^2\rho(s) = \langle AX'', X\rangle + \langle AX', X'\rangle.$$

Using $Ł_\alpha = \mu X$ and the skew symmetry of B, we rewrite this as,

$$\frac{1}{2}\partial_s^2 \rho(s) = -\langle X', BX\rangle + \langle CX, X\rangle + \langle AX', X'\rangle - \mu\langle X, X\rangle$$

which then becomes

$$\frac{1}{2}\partial_s^2 \rho(s) = |A^{1/2}X' - \frac{1}{2}A^{-1/2}BX|^2 + \langle DX, X\rangle$$

with

$$D = \frac{1}{4}BA^{-1}B + C - \mu I = \begin{pmatrix} \frac{35}{72}\lambda - \mu & 0 \\ 0 & \frac{63}{96} - \mu \end{pmatrix}.$$

D is positive definite for $\mu < \frac{1}{8}\lambda_0$ and sufficiently large T.

This gives us that $\rho''(s) \geq 0$, and we can therefore apply the same argument to $\rho(s)$ as used on solutions of $Ł_0$.

Inhomogeneous Eigenvalue Estimate

The second estimate is based on the Holder norm of X on segments of the cylinder. Denote $||\cdot||_{0,\alpha,\sigma}$ for the norm $\mathcal{C}^{0,\alpha}$ on $[\sigma, \sigma+1]$.

Theorem 5. *With $Ł$ computed with respect to the product metric, suppose that $ŁX = \mu X + F$ with $X = 0$ on the boundaries of the cylinder and $\mu T^2 < \nu_0$ a sufficiently small constant independent of T. Further assume that for all $|s| \leq T/2$,*

$$||F(s,\theta)||_{0,\alpha,\sigma} \leq \rho(\sigma)e^{-T/2}\cosh\sigma$$

with $|\rho(\sigma)| \leq 1$ for all σ Since F implicitly depends on T, we also assume $\forall\epsilon > 0, A > 0$, $\exists T_0$ such that $T \geq T_0$ and $T/2 - A \leq |\sigma| \leq T/2$ implies $|\rho(\sigma)| \leq \epsilon$. Then $\forall\eta > 0$ we have $|X| \leq \eta$ on the cylinder for large enough T.

We will again proceed by separation of variables. For $Ł'_\alpha$ and $Ł''_\alpha$ with $|\alpha| > 1$ we have shown that the eigenvalues are bounded below by a constant. If the theorem were false, then we could produce a non-trivial bounded function such that $Ł''X = 0$ of $Ł'X = 0$ which contradicts the strictly positive spectrum. It then remains to prove the the theorem for $Ł_0$. This is identical to the proof in the original paper[1].

6.2.7 Invertability

Isenberg, Mazzeo, and Pollack[1] proceed from here using the analogous results on $\mathbb{S}^2 \times \mathbb{R}$ to prove that the lowest eigenvalue λ_0 of Ł is bounded below by CT^{-2} for some constant C independent of T. Their proof comes by contradiction, assuming that there exists a sequence $T_i \to \infty$ such that $\lambda_0(T_i)T_i^2 \to 0$ and using a convergent subsequence to construct a non trivial X such that $ŁX = 0$. They then show that X is a bounded conformal killing field in the conformally scaled metric and corresponds to a non-trivial conformal killing field that vanishes at at the removed points on the original manifolds. But the nonexistence of such a vector field was exactly the assumed non-degeneracy condition.

This assumption works well for the connected sum, but doesn't make much sense in the case of pre-existing toroidal cusps. There are no removed points at which we prohibit vanishing. A stronger assumption such as assuming there are no non-trivial conformal killing fields is too limiting for the case of pre-existing hyperbolic cusps. So a different method or assumption would be necessary. The assumption is not necessary for non-compact asymptotically euclidean or asymptotically hyperbolic manifolds, but they are not the focus of this paper. While manifolds with toroidal cusps are obviously non-compact before the gluing, complete manifolds with only a single cusp and no other ends are compact once glued.

Chapter 7

Conclusion

Isenberg, Mazzeo and Pollack[1] construct a gluing of solutions to the Einstein Constraints along a connected sum. Directly adapting this procedure to a gluing along pre-existing toroidal cusps runs into a few differences and several obstacles. The lack of curvature on the torus slightly alters the form of Ł, changing the nullspace and cutting down the number of temperate solutions. The eigenvalues and eigenfunctions of Ł also change, inheriting the differences of the scalar and Hodge Laplacians on $\mathbb{T}^2$. Both of these changes are minor variations that don't affect much of the overall gluing.

The inapplicability of the non-degeneracy condition is, however, a meaningful difference. It is the key component of the IMP invertability argument and needs to either be replaced with another assumption or the entire argument needs to be altered. Although not necessary for the non-compact cases, a substitution would be needed for manifolds that are compact once joined. Generalizations and adaptations of the IMP gluing have used other non-degeneracy conditions such as the assumption that there are no conformal killing fields at all on the pre-gluing manifolds[4] or the 'no KIDs' (Killing Initial Datasets) assumption[5], requiring that there are no non-trivial solutions to the L^2 adjoint of the linearization of the constraints in a region around the gluing.

The conformal class of the torus varying along the cylinder is not considered here, but would be necessary for a fuller treatment of the problem. The torus defining lattice constants a and b become functions of s which prevents us from applying separation of variables using Laplacian eigenfunctions, as they become s-dependent.

The biggest obstacle is the difference in conformal scaling along the cylinders. In the case of the connected sum, the conformal factor decays exponentially: $|\psi_c| \approx e^{-T/2}$. In our case it grows: $|\psi_c| \approx T^{1/2}$. The bounds ensuring that the gluing has arbitrarily small deviation outside the neck all depend on this value. Correction terms for connected sum gluings have since been completely localized within the gluing region.[5;13] Adapting this method, considering other geometries to represent a toroidal cusp or seeking other ways to get bounds on the correction terms are potential paths forward in that regard.

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