A COMPUTER SIMULATION STUDY FOR COMPARING THREE METHODS OF ESTIMATING VARIANCE COMPONENTS

by

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CHAPTER # 1

Introduction and Notation

1.1 Objective

The objective of this paper is to investigate three methods of estimating variance components for both the balanced two-way mixed effects and random effects models with interaction. These methods of estimation are the analysis of variance, restricted maximum likelihood, and pretest methods.

This paper consist of a statistical derivation and/or explanation for each method of estimating variance components as well as a brief discussion of their properties and shortcomings. Lastly, all three methods are compared by a computer simulation study. The results of the simulation form the basis for a discussion of empirical conclusions associated with each method of estimation.

Before examining these three methods of estimation, the two-way mixed and random effects models are described.

1.2 Fixed and Random Effects

The two-way model considered involves a set of treatments (or populations) generated by a cross classified system involving the levels of two factors. The experimental design used to generate the data is a completely randomized design where each treatment is observed on an equal number of experimental units.

The purpose of the analysis of variance is to make inferences concerning the levels of the two factors and the interaction between the two factors. If the levels of a factor are predetermined and the inferences apply to those treatments alone, then the factor is defined as a fixed effect. To visualize sampling for factors with fixed effects consider "... data as being one possible set of data involving these same treatments that could be derived in repetitions of the experiment, repetitions in which the differences in observations on each occasion would be a random sample from a population of differences having a mean of zero and a variance σ^2_e ...", Searle (1971). From this point on σ^2_e will represent the variability in observations due to random sampling error.

On the other hand, if the inferences concerning the levels of a factor are to apply to a population of levels corresponding to that factor, then the factor is defined as a random effect. Here we select a random sample of treatments from a population of possible treatments and our primary interest is in the population as a whole.

In selecting observations for a random effects model there are two sources of variation present. First, by selecting a random sample of similar treatments we do not restrict ourselves to a specific group thereby allowing a degree of variability from treatment to treatment. Secondly, as with the fixed effects model, for whatever treatments selected there is variation observed within these treatments. This type of variability was defined previously as random sampling error.

The variability associated with random sampling for both the random and mixed effects models will have mean zero and variance σ^2_e . For the random effects model, the variability associated with the random effect will constitute a sample from a population with mean zero and variance σ^2_T.

In order to construct a model, we use the rule that if an interaction involves a random factor, then the interaction is also a random effect.

For both the mixed and random effects models, the interaction is a random effect with mean zero and variance $\sigma^2{}_{AT}$. All the random effects and the sampling error are assumed to be uncorrelated.

The parameter of interest for a random effect is the corresponding variance of the population of levels while the parameters of interest for a fixed effect are the means of the corresponding levels. It is often assumed that the distribution associated with each random effect is the normal distribution with mean zero and the effects variance. The AOV method of estimation discussed in Chapter 2 does not require this assumption, but for distributional properities of the estimators to be examined the assumption of normality is a convenience.

1.3 Models

This section describes the two-way mixed and random effects models which are used in the simulation study.

A two-way mixed effects model consists of a two-way factorial with interaction where the levels of one factor are fixed and the levels of one factor are random. A linear model describing the observations from this type of design is

$$Y_{ijk} = \mu + \alpha_i + T_j + G_{ij} + \varepsilon_{ijk} \qquad (1.3.1)$$

$$i = 1,\ldots,a \qquad j = 1,\ldots,t \qquad k = 1,\ldots,n \quad ,$$

where μ is the common mean for all observations,

α_i is the i^{th} fixed effect associated with factor A ,

T_j is the j^{th} random effect associated with factor T, $T_j \sim N(0, \sigma^2{}_T)$,

	G_{ij}	is the random interaction effect between i^{th} level of factor A and j^{th} level of factor T, $G_{ij} \sim N(0, \sigma^2_{AT})$,
	ε_{ijk}	is the random sampling error $\varepsilon_{ijk} \sim N(0, \sigma^2_e)$,
and	Y_{ijk}	is the k^{th} observation for the i^{th} level of factor A and j^{th} level of factor T.

Throughout this paper model (1.3.1) will be referred to as the mixed effects model. The variance of an observation from this model is

$$VAR\ (Y_{ijk}) = \sigma^2_T + \sigma^2_{AT} + \sigma^2_e$$

where σ^2_T, σ^2_{AT}, and σ^2_e are the components of the variance of an observation Y_{ijk}.

A two-way random effects model consists of a two-way factorial where the levels of both factors are random. A linear model describing observations from this type of design is

$$Y_{ijk} = \mu + A_i + T_j + G_{ij} + \varepsilon_{ijk} \qquad (1.3.2)$$

$$i = 1,\ldots,a \qquad j = 1,\ldots,t \qquad k = 1,\ldots,n\ ,$$

where	μ	is the common mean of all observations,
	A_i	is the i^{th} level of random factor A $A_i \sim N(0, \sigma^2_A)$,
	T_j	is the j^{th} level of random factor T $T_j \sim N(0, \sigma^2_T)$,
	G_{ij}	is the interaction between i^{th} level of A and j^{th} level of T $G_{ij} \sim N(0, \sigma^2_{AT})$,

ε_{ijk} is the random sampling error

$\varepsilon_{ijk} \sim N(0, \sigma^2_e)$,

and Y_{ijk} is the k^{th} observation of the i^{th} level of A and j^{th} level of T.

This model will be referred to as the random effects model throughout this paper. The variance of an observation from this model is

$$VAR\ (Y_{ijk}) = \sigma^2_A + \sigma^2_T + \sigma^2_{AT} + \sigma^2_e$$

where σ^2_A, σ^2_T, σ^2_{AT}, and σ^2_e are the components of the variance of the observation Y_{ijk}.

This paper is concerned with obtaining estimates of the variance components for models (1.3.1) and (1.3.2). There are several methods for estimating variance components, three of which are discussed in Chapter 2.

CHAPTER # 2

Methods of Estimating Variance Components

2.1 Introduction

The purpose of this chapter is to describe in detail the three methods of estimating variance components that are employed in the simulation study. These methods are the analysis of variance, restricted maximum likelihood, and pretest method. Various properties and shortcomings of each method are presented. Also, similarities between methods are discussed.

2.2 Analysis of Variance (AOV) Estimators

The first method presented is the method of analysis of variance (AOV). This method is explained with respect to models(1.3.1) and (1.3.2). First, the mean squares associated with each source of variation are computed as if the model were two-way fixed effect. Second, the expected value of the mean squares are computed with respect to the conditions of the correct model, i.e., either (1.3.1) or (1.3.2). The next step is to generate a system of equations by equating these expectations to their corresponding mean squares. The solutions of the resulting system of equations yield the AOV estimators of the variance components.

The two-way fixed effect model is defined as

$$Y_{ijk} = \mu + \alpha_i + \tau_j + \delta_{ij} + \varepsilon_{ijk}$$

$$i = 1,\ldots,a \qquad j = 1,\ldots,t \qquad k = 1,\ldots,n\ ,$$

where μ is the common mean for all observations,

	α_i	is the i^{th} fixed effect of factor A,
	T_j	is the j^{th} fixed effect of factor T,
	δ_{ij}	is the interaction between i^{th} level of A, and j^{th} level of T,
	ε_{ijk}	is the random sampling error $\varepsilon_{ijk} \sim N(0, \sigma^2_e)$,
and	Y_{ijk}	is the k^{th} observation of the i^{th} level of A and j^{th} level of T.

The mean squares from the analysis of variance of the two-way fixed effects model are

$$MSA = nt \sum_{i=1}^{a} (\overline{Y}_{i..} - \overline{Y}_{...})^2/(a-1),$$

$$MST = na \sum_{j=1}^{t} (\overline{Y}_{.j.} - \overline{Y}_{...})^2/(t-1),$$

$$MSAT = n \sum_{i=1}^{a} \sum_{j=1}^{t} (\overline{Y}_{ij.} - \overline{Y}_{i..} - \overline{Y}_{.j.} + \overline{Y}_{...})^2/(a-1)(t-1), \text{ and} \tag{2.2.1}$$

$$MSE = \sum_{i=1}^{a} \sum_{j=1}^{t} \sum_{k=1}^{n} (Y_{ijk} - \overline{Y}_{ij.})^2/at(n-1).$$

By taking the expected value of the mean squares (2.2.1) with respect to the conditions of the mixed effects model (1.3.1), it can be shown that

$$E[MSA] = \sigma^2_e + n\sigma^2_{AT} + \frac{nt}{a-1} \sum_{i=1}^{a} (\alpha_i - \overline{\alpha}_.)^2$$

$$E[MST] = \sigma^2_e + n\sigma^2_{AT} + na\sigma^2_T \tag{2.2.2}$$

$E[MSAT] = \sigma^2_e + n\sigma^2_{AT}$ and

$E[MSE] = \sigma^2_e$.

A system of equations is generated by equating equations (2.2.2) to their corresponding mean squares. The AOV estimates are the solution to the resulting system of equations, which are presented in Table 2.2.1.

Table 2.2.1: AOV estimators for the mixed effects model

$$\tilde{\sigma}^2_{AT} = (MSAT - MSE)/n$$
$$\tilde{\sigma}^2_T = (MST - MSAT)/an$$
$$\tilde{\sigma}^2_e = MSE$$

By taking the expectation of the mean squares (2.2.1) with respect to the conditions of the random model (1.3.2), it can be shown that

$$\begin{aligned} E[MSA] &= \sigma^2_e + n\sigma^2_{AT} + nt\sigma^2_A \\ E[MST] &= \sigma^2_e + n\sigma^2_{AT} + na\sigma^2_T \\ E[MSAT] &= \sigma^2_e + n\sigma^2_{AT} \text{ and} \\ E[MSE] &= \sigma^2_e. \end{aligned} \tag{2.2.3}$$

For the random model, a similar set of equations is generated by equating equations (2.2.3) to their corresponding mean squares. The AOV estimates are the solution to the resulting system of equations, which are presented in Table 2.2.2.

Table 2.2.2: AOV Estimators for Random Model

$$\tilde{\sigma}^2_A = (MSA - MSAT)/nt$$

$$\tilde{\sigma}^2_T = (MST - MSAT)/na$$

$$\tilde{\sigma}^2_{AT} = (MSAT - MSE)/n$$

$$\tilde{\sigma}^2_e = MSE$$

The AOV estimators in tables (2.2.1) and (2.2.2) can be easily shown to be unbiased, although this unbiasedness is restricted to the balanced design case, see Searle (1971).

The AOV estimates can be shown to have minimum variance for the set of all unbiased estimates which are quadratic functions of the observations see Graybill and Hulquist (1961).

Up to this point everything discussed so far concerning the AOV estimates holds with or without the assumption of normality as defined in equations (1.3.1) and (1.3.2). The assumption of normality allows us to say that the AOV estimates have minimum variance for all unbiased estimators, regardless of whether or not they are quadratic functions of the observations, see Graybill (1954) and Graybill and Wortham (1956).

With AOV estimates in table 2.2.1 and 2.2.2 there exists the possibility of obtaining negative estimates. This is a distinct disadvantage as a variance component is defined to be non-negative. Searle (1971) suggests several courses of action when negative estimates are observed. One is to set the value of the variance component to zero. The next two methods presented, consider two different approaches to eliminating the possibility of negative estimates.

2.3 Restricted Maximum Likelihood Estimators (REML)

The REML estimates employ a method of estimating variance components which eliminates the possibility of observing a negative estimate. The idea is to maximize the likelihood function of equations (1.3.1) and (1.3.2) over a restricted positive parameter space, one that is free of the fixed effects. This results in a system of equations which can be solved for the variance component estimates. The resulting estimates are free of the fixed effects, invariant under translation similar to the maximum likelihood estimates, and also reduce to the AOV estimators in certain cases for the balanced design, see Corbeil and Searle (1976).

The following is a general derivation of the Restricted Maximum Likelihood Estimators. Consider the model

$$\underline{Y} = \underline{X}\,\underline{\mu} + \underline{U}_1\,\underline{b}_1 + \underline{U}_2\underline{b}_2 + \ldots + \underline{U}_c\underline{b}_c + \underline{e} \qquad (2.3.1)$$

where $\underline{Y}$ is an $N \times 1$ vector of observations,

$\underline{\mu}$ is a $K \times 1$ vector of fixed effects,

$\underline{X}$ is an $N \times K$ design matrix of rank K,

$\underline{U}_i$ is an $N \times M_i$ design matrix for the i^{th} random factor

$i = 1,\ldots,c$, $\sum_{i=1}^{c} M_i + K < N$

$\underline{b}_i$ is an $M_i \times 1$ vector of random effects associated with factor i , $\underline{b}_i \sim N(\underline{0}, \sigma^2{}_i\underline{I})$, and

$\underline{e}$ is an $N \times 1$ vector of unobservable independent random variables $\underline{e} \sim N(\underline{0}, \sigma^2{}_e\, \underline{I}_n)$.

This model has $E[\underline{Y}] = \underline{X}\,\underline{\mu}$ and $VAR[\underline{Y}] = \sigma^2{}_e\underline{H}$ where $\underline{H} = \sum_{i=1}^{c} \frac{\sigma^2{}_i}{\sigma^2{}_e} \underline{U}_i\,\underline{U}'_i + \underline{I}_N$ and $\sigma^2{}_i$ is the variance of the i^{th} random factor. $\underline{H}$ is a positive definite matrix of rank N.

Assume $\underline{Y} \sim N[\underline{Y}; \underline{X\mu}, \sigma^2{}_e\, \underline{H}]$.

The log of the likelihood function is

$$-\tfrac{1}{2}N \log 2\pi - \tfrac{1}{2}N \log \sigma^2{}_e - \tfrac{1}{2}\log |\underline{H}| - \tfrac{1}{2}(\underline{Y} - \underline{X\mu})'\underline{H}^{-1} (\underline{Y} - \underline{X\mu})/\sigma^2{}_e. \qquad (2.3.2)$$

Patterson and Thompson (1971) suggest the singular transformation of $\underline{Y}'[\underline{S},\underline{H}^{-1}\underline{X}]$ to partition the above log likelihood function into two parts, one of which is free of the fixed effects. Let

$$\underset{N\times N}{\underline{S}} = \underline{I}_N - \underline{X}(\underline{X}'\underline{X})^{-1}\underline{X}' = \sum_{t=1}^{K}{}^{+}(\underline{I}_{n_t} - \frac{1}{n_t}\underline{J}_{n_t})$$

where the notation $\sum_{t=1}^{K}{}^{+}\underline{B}$ denotes an NxN blocked designed matrix with $\underline{B}$ as the diagonal matrices. Since a balance design is assumed $n_1 = n_2 = \ldots = n_K = n$, where n is the number of observations associated with each fixed effect. The matrix $\underline{S}$ is NxN symmetric and idempotent of rank N-K. The distribution of $\underline{SY}$ is

$$\underline{S}\,\underline{Y} \sim N[\underline{S}\,\underline{Y}; \underline{0}, \underline{S}\,\underline{H}\,\underline{S}\,\sigma^2{}_e]$$

where $\underline{S}\,\underline{Y}$ and $\underline{X}'\underline{H}^{-1}\underline{Y}$ are independent.

The distribution of $\underline{S}\,\underline{Y}$ is free of fixed effects in the original model. The covariance matrix of $\underline{S}\,\underline{Y}$, $\underline{S}\,\underline{H}\,\underline{S}\,\sigma^2{}_e$, is singular which results in the need for another transformation. To modify for singularity, the n_1^{th}, $(n_1 + n_2)^{th}$, ... , $(n_1 + n_2 + \ldots + n_K)^{th}$ rows of $\underline{S}$ are deleted. The resulting matrix, denoted by $\underline{T}$, has order (N - K) x N of rank N-K, where

$$\underline{T} = \sum_{t=1}^{K}{}^{+} (\underline{I}_{n-1} - \frac{1}{n}\underline{J}_{n-1}, -\frac{1}{n}\underline{J}_{n-1}), \qquad (2.3.3)$$

see Corbeil and Searle (1976).

It can be shown that $\underline{T}\,\underline{X} = \underline{0}$ and $\underline{T}'(\underline{T}\,\underline{T}')^{-1}\underline{T} = \underline{S}$.

To modify for the singularity of $\underline{S}\,\underline{H}\,\underline{S}$, Corbeil and Searle suggest the following transformation to break up the likelihood function into two parts, again one being free of the fixed effects.

Let $\underline{Z} = \begin{bmatrix} \underline{T} \\ \underline{X}'\underline{H}^{-1} \end{bmatrix} \underline{Y}$, which results in $\underline{Z}$ being distributed as

$$\underline{Z} \sim N \left[\begin{bmatrix} \underline{T}\,\underline{Y} \\ \underline{X}'\underline{H}^{-1}\underline{Y} \end{bmatrix} ; \begin{bmatrix} \underline{0} \\ \underline{X}'\underline{H}^{-1}\underline{X}\underline{\mu} \end{bmatrix} , \begin{bmatrix} \underline{T}\,\underline{H}\,\underline{T}'\sigma^2{}_e \\ \underline{0} \end{bmatrix} \begin{bmatrix} \underline{0} \\ \underline{X}'\underline{H}^{-1}\underline{X}\sigma^2{}_e \end{bmatrix} \right] \quad (2.3.4)$$

where $\underline{T}\,\underline{Y}$ and $\underline{X}'\underline{H}^{-1}\underline{Y}$ are independent.

The log of the likelihood function of $\underline{T}\,\underline{Y}$ is free of the fixed effects and is

$$\lambda = -\tfrac{1}{2}(N-K)\log 2\pi - \tfrac{1}{2}(N-K)\log \sigma^2{}_e - \tfrac{1}{2}\log |\underline{T}\,\underline{H}\,\underline{T}'| - \tfrac{1}{2}\,\underline{Y}'\underline{T}'(\underline{T}\,\underline{H}\,\underline{T}')^{-1}\underline{T}\,\underline{Y}/\sigma^2{}_e.$$

By differentiating λ with respect to the variance components and equating them to zero, the REML estimators can be obtained.

The derivative of λ with respect to $\sigma^2{}_e$ is

$$\frac{\delta\lambda}{\delta\sigma^2{}_e} = -\tfrac{1}{2}(N-K)/\sigma^2{}_e + \tfrac{1}{2}\underline{Y}'\underline{T}'(\underline{T}\,\underline{H}\,\underline{T}')^{-1}\underline{T}\,\underline{Y}/\sigma^4{}_e$$

which results in the following estimate of $\sigma^2{}_e$,

$$\tilde{\sigma}^2{}_e = \frac{1}{N-K}\,\underline{Y}'\underline{T}'(\underline{T}\,\underline{H}\,\underline{T}')^{-1}\underline{T}\,\underline{Y}. \quad (2.3.5)$$

Similarly for the other variance components,

$$\frac{\delta\lambda}{\delta\sigma^2{}_i} = -\tfrac{1}{2}\mathrm{tr}[\underline{U}'{}_i\underline{T}'(\underline{T}\,\underline{H}\,\underline{T}')^{-1}\underline{T}\,\underline{U}_i]$$

$$+ \tfrac{1}{2}\,\underline{Y}'\underline{T}'(\underline{T}\,\underline{H}\,\underline{T}')^{-1}\underline{T}\,\underline{U}_i\underline{U}_i{}'\underline{T}'(\underline{T}\,\underline{H}\,\underline{T}')^{-1}\underline{T}\,\underline{Y}/\sigma^2{}_e\,. \quad (2.3.6)$$

Consider the two-way mixed effects model (1.3.1). By expressing equation

(1.3.1) in terms of equations (2.3.1), we obtain

$$\underline{X\mu} = (\underline{j}_n \otimes \underline{j}_t \otimes \underline{I}_a) \begin{bmatrix} \mu + \alpha_1 \\ \mu + \alpha_2 \\ \vdots \\ \mu + \alpha_a \end{bmatrix},$$

where

$\underline{X}$ is an N x a design matrix of rank a, and

$\underline{\mu}$ is an a x 1 vector of fixed effects,

$$\underline{U_1 b_1} = (\underline{j}_n \otimes \underline{I}_t \otimes \underline{j}_a) \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_t \end{bmatrix},$$

where

$\underline{U}_1$ is an N x t design matrix of rank t, and

$\underline{b}_1$ is a t x 1 vector of random effects,

and

$$\underline{U_2 b_2} = (\underline{j}_n \otimes \underline{I}_t \otimes \underline{I}_a) \begin{bmatrix} G_{11} \\ \vdots \\ G_{1t} \\ \vdots \\ G_{a1} \\ \vdots \\ G_{at} \end{bmatrix},$$

where

$\underline{U}_2$ is an N x ta design matrix of rank ta, and

$\underline{b}_2$ is a ta x 1 vector of random effects.

The notation $\underset{a \times s}{\underline{A}} \otimes \underset{b \times t}{\underline{B}}$ represents the abxst matrix

$$\begin{bmatrix} \underline{A}b_{11} & \underline{A}b_{12} & \cdots & \underline{A}b_{1t} \\ \underline{A}b_{21} & \underline{A}b_{22} & \cdots & \underline{A}b_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{A}b_{b1} & \underline{A}b_{b2} & \cdots & \underline{A}b_{bt} \end{bmatrix}$$

where b_{ij} is the element in the i^{th} row and j^{th} column of the matrix $\underline{B}$.

The two-way Mixed effects model has

$$E[\underline{Y}] = \underline{X\mu} \quad \text{and } VAR[\underline{Y}] = \underline{H}\, \sigma^2{}_e \ ,$$

$$\text{where } \underline{H} = \frac{\sigma^2{}_T}{\sigma^2{}_e} \underline{U_1U'}_1 + \frac{\sigma^2{}_{AT}}{\sigma^2{}_e} \underline{U_2U'}_2 + \underline{I}_N \qquad (2.3.7.a)$$

$$= \frac{\sigma_T{}^2}{\sigma^2{}_e} (\underline{J}_n \otimes \underline{I}_t \otimes \underline{J}_a) + \frac{\sigma^2{}_{AT}}{\sigma^2{}_e} (\underline{J}_n \otimes \underline{I}_t \otimes \underline{I}_a) + \underline{I}_N \ ,$$

$\underline{J}_n$ is an n x n matrix of ones,

$\underline{j}_t$ is a t x 1 vector of ones,

$\underline{I}_s$ is an s x s identity matrix, and

$$T = \sum_{i=1}^{a}{}^{+} \left(\underline{I}_{n-1} - \frac{1}{n} \underline{J}_{n-1} \ \vdots \ -\frac{1}{n} \underline{J}_{n-1}\right) \quad n_1 = \ldots = n_t = n \qquad (2.3.7.b)$$

By solving equations (2.3.5) and (2.3.6) with respect to (2.3.7 a & b), the REML estimators for the variance components are obtained as shown in Table 2.3.1.

Table (2.3.1) Reml estimators for the mixed model

$$\tilde{\sigma}^2_{AT} = \begin{cases} (MSAT - MSE)/n & \text{if } MSAT \geq MSE \\ 0 & \text{if } MSAT < MSE \end{cases} \qquad (2.3.8)$$

$$\tilde{\sigma}^2_{T} = \begin{cases} (MST - MSAT)/na & \text{if } MST \geq MSAT \\ 0 & \text{if } MST < MSAT \end{cases}$$

$$\tilde{\sigma}^2_{e} = \underline{Y}'\underline{T}'(\underline{THT}')^{-1}\underline{TY}/(N-K) = MSE$$

Consider the two-way random model with interaction as defined by (1.3.2). Expressing equation (1.3.2) in terms of equation (2.3.1) it follows that

$$\underline{Y} = \underline{j}_N\mu + \underline{U}_1\underline{b}_1 + \underline{U}_2\underline{b}_2 + \underline{U}_3\underline{b}_3 + \underline{e}$$

where $\underline{j}_N\mu$ is an N x 1 vector containing the constant μ,

$$\underline{U}_1\underline{b}_1 = (\underline{j}_n \otimes \underline{j}_t \otimes \underline{I}_a) \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_a \end{bmatrix}$$

where

$\underline{U}_1$ is an N x a design matrix of rank a, and

$\underline{b}_1$ is an a x 1 vector of random effects,

$$\underline{U}_2\underline{b}_2 = (\underline{j}_n \otimes \underline{I}_t \otimes \underline{j}_a) \begin{bmatrix} T_1 \\ T_2 \\ \cdot \\ \cdot \\ \cdot \\ T_t \end{bmatrix}$$

where

$\underline{U}_2$ is an N x t design matrix of rank t, and

$\underline{b}_2$ is a t x 1 vector of random effects, and $\underline{U}_3\underline{b}_3 = (\underline{j}_n \otimes \underline{I}_t \otimes \underline{I}_a) \begin{bmatrix} G_{11} \\ \cdot \\ \cdot \\ \cdot \\ G_{1t} \\ \cdot \\ \cdot \\ \cdot \\ G_{a1} \\ \cdot \\ \cdot \\ \cdot \\ G_{at} \end{bmatrix}$

where

$\underline{U}_3$ is an N x ta design matrix of rank ta, and

$\underline{b}_3$ is an at x 1 vector of random effects.

The two-way random effects model has

$$E[\underline{Y}] = \underline{X}\mu, \text{ and } VAR[\underline{Y}] = \underline{H}\ \sigma^2{}_e\ ,$$

where $\underline{H} = \frac{\sigma^2{}_A}{\sigma^2{}_e}\ \underline{U}_1\underline{U}'_1 + \frac{\sigma^2{}_T}{\sigma^2{}_e}\ \underline{U}_2\underline{U}'_2 + \frac{\sigma^2{}_{AT}}{\sigma^2{}_e}\ \underline{U}_3\underline{U}'_3 + \underline{I}_N$

$$= \frac{\sigma^2{}_A}{\sigma^2{}_e} (\underline{J}_n \otimes \underline{J}_t \otimes \underline{I}_a) + \frac{\sigma^2{}_T}{\sigma^2{}_e} (\underline{J}_n \otimes \underline{I}_t \otimes \underline{J}_a) \qquad (2.3.9.a)$$

$$+ \frac{\sigma^2{}_{AT}}{\sigma^2{}_e} (\underline{J}_n \otimes \underline{I}_t \otimes \underline{I}_a) + \underline{I}_n.$$

and $\underline{T} = \sum_{i=1}^{a} {}^{+} \left(\underline{I}_{n_t-1} - \frac{1}{n_t}\underline{J}_{n_t-1} \ \vdots\ - \frac{1}{n_t}\underline{J}_{n_t-1}\right).$ (2.3.9.b)

By solving equations (2.3.5) amd (2.3.6) with respect to equations (2.3.9.a) and (2.3.9.b), the REML estimators are obtained as shown in table 2.3.2.

Table (2.3.2): Reml estimators for the random model

$$\tilde{\sigma}^2_{AT} = \begin{cases} (MSAT - MSE)/n & MSAT \geq MSE \\ 0 & MSAT > MSE \end{cases}$$

$$\tilde{\sigma}^2_{T} = \begin{cases} (MST - MSAT)/na & MST \geq MSAT \\ 0 & MST > MSAT \end{cases} \qquad (2.3.10)$$

$$\tilde{\sigma}^2_{A} = \begin{cases} (MSA - MSAT)/nt & MSA \geq MSAT \\ 0 & MSA < MSAT \end{cases}$$

$$\tilde{\sigma}^2_{e} = MSE$$

The Reml estimators have a unique characteristic in that they reduce to the AOV estimators when there is "no chance" of observing a negative estimate. This results in the Reml estimators possessing propertie similar to those associated with the AOV estimators.

In the event of a chance for a negative AOV estimate, the REML estimators are no longer identical to the AOV estimates and obviously no longer unbiased.

2.4 Pretest Method

The pretest method employs a model building approach to estimating variance components. Each variance component is examined individually prior to being estimated. If there is evidence that a particular source of variability is present then it is estimated. If the source of variabilit is found to be negligible, then it is eliminated from the model and a new model is defined. This process is repeated until all variance components have been examined.

To determine if a source of variation is present the following hypothesis is tested,

H_0: $\sigma^2_i = 0$ $\qquad$ H_a: $\sigma^2_i > 0$ (2.4.1)

where σ^2_i is the variance of the i^{th} random factor.

If at a specified type I error rate, the variance component is found to be negligible (Fail to Reject H_0) then the sum of squares due to that effect is pooled with the appropriate effect based on expected mean squares. Then zero is chosen as the estimate of the variance component and a new model is constructed where that variance component is set equal to zero. This will result in developing a new ANOVA table where certain mean squares have been modified.

On the other hand, if the variance component was found to be significant,no pooling would take place and it would then be estimated by the AOV method.

The hypothesis stated in (2.4.1) is then tested for the next variance component. The test statistic for hypothesis (2.4.1) is based on the most recently defined ANOVA table. Similarly, if this component is significant it is estimated by the AOV method. If the variance component is found to be negligible then it is set equal to zero and the appropriate sums of squares are pooled. This process is continued until all variance components have been estimated or equated to zero.

The first variance component examined will be the one involving the most factors. This will be the interaction variance component for models (1.3.1) and (1.3.2).

Consider the model defined by (1.3.1). The sources of variation corresponding to the random effects and their expected mean squares are given in Table 2.4.1.

Table 2.4.1. Random sources of variation for model (1.3.1)

SOURCE	EMS
T	$\sigma^2_e + n\sigma^2_{AT} + an\sigma^2_T$
AT	$\sigma^2_e + n\sigma^2_{AT}$
ERROR	σ^2_e

First, the interaction effect is tested, i.e.,

$$\begin{array}{ll} H_0: \sigma^2_{AT} = 0 & \text{no interaction} \\ H_a: \sigma^2_{AT} > 0 & \text{interaction present.} \end{array} \qquad (2.4.2)$$

If this effect is found to be negligible then $\tilde{\sigma}^2_{AT}$ would be set equal to zero. The sums of square due to interaction would be pooled with sampling error as directed by the expected mean squares. The pooling would result in new values for certain mean squares in the ANOVA table.

Otherwise, if the interaction effect was significant, the interaction variance component would be estimated by the AOV method in Table (2.2.1), from the original ANOVA table with no pooling taking place.

The T effect is tested next, i.e.,

$$\begin{array}{ll} H_0: \sigma^2_T = 0 & \text{no T effect} \\ H_a: \sigma^2_T > 0 & \text{T effect present.} \end{array} \qquad (2.4.4)$$

If the T effect is found to be negligible then $\tilde{\sigma}^2_T$ would be set equal to zero. The sums of square due to the T effect would be pooled with either the sums of square error or interaction, depending upon the outcome of the first test of hypothesis (2.4.2). Again, this results in modifying values in the previous ANOVA table.

If the T effect was found to be significant, then it would be estimated by the AOV method using values from most recently updated ANOVA table.

After the variances of both the T and interaction effects have been estimated, the estimate of the sampling error is the final value of MSE after all pooling has taken place.

The pretest method does not allow for 'retesting upward'. 'Retesting upward' means, if the interaction effect was found to be significant and the T effect was negligible, interaction would not be retested based on the sums of squares obtained by pooling SST with the SSAT. A hypothesis is tested only once.

However, this method does allow for 'reestimation upward'. For example, if the interaction effect was found to be significant it would be estimated by the AOV method. Then, assume the T effect was found to be negliglble. This results in a modification to the values of the ANOVA table. This method will reestimate the interaction variance component based on a new pooled mean square interaction due to the negligible T effect.

For the mixed effect model (1.3.1), the pretest method allows four different possible estimates of each variance component. Tables 2.4.2.a and 2.4.2.b define all possible estimates of the variance components when interaction is both negligible and significant, respectively. The tables also show a detailed path of the hypotheses tested and their conclusions that yield the respective estimates. There is only one occurrence where reestimation upward is used, i.e., in Table 2.4.2.b, path M3.

The notation in Tables 2.4.2.a and 2.4.2.b with the prefixes SS, MS, and I represent a particular effects sums of squares, mean square, and degrees of freedom, respectively.

Table 2.4.2.a Pretest Estimators for mixed model, no interaction

H_0: $\sigma^2_{AT} = 0$

F_{AT} = MSAT/MSE

"Fail to Reject H_0"

$\tilde{\sigma}^2_{AT} = 0$

SSE1 = SSE + SSAT

IDF1 = IEDF + IATDF

MSE1 = SSE1/IDF1

H_0: $\sigma^2_T = 0$

F_T = MST/MSE1

"Fail to Reject H_0"

$\tilde{\sigma}^2_T = 0$

SSE2 = SSE1 + SST

IDF2 = IDF1 + ITDF

MSE2 = SSE2/IDF2

$\tilde{\sigma}^2_{AT} = 0$

$\tilde{\sigma}^2_T = 0$

$\tilde{\sigma}^2_e$ = MS2

M1

"Reject H_0"

$\tilde{\sigma}^2_T$ = (MST - MSE1)/na

$\tilde{\sigma}^2_{AT} = 0$

$\tilde{\sigma}^2_T$ = (MST - MSE1)/na

$\tilde{\sigma}^2_e$ = MSE1

M2

Table 2.4.2.b Pretest Estimators for Mixed Model, with interaction

H_o: $\sigma^2_{AT} = 0$

F_{AT} = MSAT/MSE

"Reject H_o"

$\tilde{\sigma}^2_{AT}$ = (MSAT - MSE)/n

H_o: $\sigma^2_T = 0$

F_T = MST/MSAT

"Fail to Reject H_o"

$\tilde{\sigma}^2_T = 0$

SSAT3 = SSAT + SST

IDF3 = IATDF + ITDF

MSAT3 = SSAT3/IDF3

$\tilde{\sigma}^2_{AT}$ = (MSAT3 - MSE)/n

$\tilde{\sigma}^2_T = 0$

$\tilde{\sigma}^2_e$ = MSE

M3

"Reestimation upward"

"Reject H_o"

$\tilde{\sigma}^2_T$ = (MST - MSAT)/na

$\tilde{\sigma}^2_{AT}$ = (MSAT - MSE)/n

$\tilde{\sigma}^2_T$ = (MST - MSAT)/na

$\tilde{\sigma}^2_e$ = MSE

M4

'AOV estimators'

Consider the random effects model (1.3.2). The random sources of variation and their expected mean squares are in table 2.4.3.

Table 2.4.3 Sources of Variation for Model (1.3.2)

SOURCE	EMS
A	$\sigma^2_e + nt\sigma^2_A + n\sigma^2_{AT}$
T	$\sigma^2_e + na\sigma^2_T + n\sigma^2_{AT}$
AT	$\sigma^2_e + n\sigma^2_{AT}$
ERROR	σ^2_e

For the random effects model there are two cases to consider depending upon the outcome of the first test.

First, if the interaction effect is found to be negligible, the sum of squares due to that effect are pooled with the sum of squares due to sampling error as directed by the expected mean squares in Table 2.4.3. Then for each subsequent main effect that is found to be negligible, their sums of squares are also pooled with sum of square error.

Secondly, if the interaction effect is significant then no initial pooling occurs. For subsequent main effects found to be negligible, their sum of squares are pooled with the interaction sum of squares, not the sum of squares error.

Similar to the mixed effects model, there is no retesting upward.

For the random effects model there are eight different ways to estimate the variance components. Tables 2.4.4.a and 2.4.4.b give a detailed description for all estimates when interaction is negligible and significant, respectively. Reestimation upward occurs in Table 2.4.4.a on path R3 and in table 2.4.4.b in paths R5, R6, and R7. Finally, there are different estimates of the sampling error depending on which sums of squares are pooled.

Table 2.4.4.a Pretest estimators for Random Model, no interaction

H_0: $\sigma^2_{AT} = 0$
F_{AT} = MSAT/MSE
"Fail to Reject H_0"
$\sigma^2_{AT} = 0$
SSE1 = SSE + SSAT
IEDF1 = IEDF + IATDF
MSE1 = SSE1/IEDF1

↓

H_0: $\sigma^2_T = 0$
F_T = MST/MS1

"Fail to Reject H_0"	"Reject H_0"
$\sigma^2_T = 0$	σ^2_T = (MST-MSE1)/na
SSE2 = SSE1 + SST	
IEDF2 = IEDF1 + ITDF	
MSE2 = SSE2/IEDF2	
↓	↓
H_0: $\sigma^2_A = 0$	H_0: $\sigma^2_A = 0$
F_A = MSA/MSE2	F_A = MSA/MSE1

"Fail to Reject H_0"	"Reject H_0"	"Fail to Reject H_0"	"Reject H_0"
$\sigma^2_A = 0$	σ^2_A = (MSA-MSE2)/nt	$\sigma^2_A = 0$	σ^2_A =(MSA-MSE1)/nt
SSE3 = SSE2 + SSA		SSE4 = SS1 + SSA	
IEDF3 = IEDF2 + IADF		IEDF4 = IEDF1 + IADF	
MSE3 = SSE3/IEDF3		MSE4 = SSE4/IEDF4	
↓	↓	↓	↓
$\sigma^2_A = 0$	σ^2_A = (MSA-MSE2)/nt	$\sigma^2_A = 0$	σ^2_A = (MSA-MSE1)/nt
$\sigma^2_T = 0$	$\sigma^2_T = 0$	* σ^2_T = (MST-MSE4)/na	σ^2_T=(MST-MSE1)/na
$\sigma^2_{AT} = 0$	$\sigma^2_{AT} = 0$	$\sigma^2_{AT} = 0$	$\sigma^2_{AT} = 0$
σ^2_e = MSE3	σ^2_e = MSE2	σ^2_e = MSE4	σ^2_e = MSE1
R1	R2	R3	R4

Table 2.4.4.b Pretest Estimators for Random Model, with interaction

H_0: $\sigma^2_{AT} = 0$

F_{AT} = MSAT/MSE

"Reject H_0"

σ^2_{AT} = (MSAT - MSE)/n

H_0: $\sigma^2_T = 0$

F_T = MST/MSAT

Fail to Reject H_0	Reject H_0
$\sigma^2_T = 0$	σ^2_T = (MST-MSAT)/na
SSAT1 = SSAT + SST	
IATDF1 = IATDF + ITDF	
MSAT1 = SSAT1/IATDF1	
H_0 $\sigma^2_A = 0$	H_0 $\sigma^2_A = 0$
F_A = MSA/MSAT1	F_A = MSA/MSAT

Fail to Reject H_0	Reject H_0	Fail to Reject H_0	Reject H_0
$\sigma^2_A = 0$	σ^2_A = (MSA-MSAT1)/nt	$\sigma^2_A = 0$	σ^2_A = (MSA-MSAT)/nt
SSAT2 = SSAT1 + SSA		SSAT3 = SSAT + SSA	
IATDF2 = IATDF1 + IADF		IATDF3 = IATDF + IADF	
MSAT2 = SSAT2/IATDF2		MSAT3 = SSAT3/IATDF3	
$\sigma^2_A = 0$	σ^2_A = (MSA-MSAT1)/nt	$\sigma^2_A = 0$	AOV
$\sigma^2_T = 0$	$\sigma^2_T = 0$	* σ^2_T = (MST-MSAT3)/na	estimators
* σ^2_{AT} = (MSAT2-MSE)/n	* σ^2_{AT}=(MSAT1-MSE)/n	* σ^2_{AT}=(MSAT3-MSE)/n	
σ^2_e = MSE	σ^2_e = MSE	σ^2_e = MSE	
R5	R6	R7	R8

*cases where reestimation upward is employed.

Negative estimates with the pretest method are very unlikely, although the possibility does exist. If the presence of a variance component was tested with a very large type I error rate, it may be found to be significant. This can result in the AOV method yielding a negative estimate when it should be set to zero. When the AOV method is used to estimate negligible variance components, negative estimates are likely to occur. Assuming the pretest method is employed at a reasonably small type I error rate, say .01 to .10, no negative estimates should be obtained.

In the case where all sources of variation are very large, the pretest method is identical to the AOV method (Table 2.2.1 and 2.2.2) as is the REML method. This is shown by Table 2.4.2.b path M4 and Table 2.4.4.b path R8.

CHAPTER # 3

The Simulation

3.1 Introduction

This chapter presents a computer simulation study that compares the three methods of variance component estimation presented in the previous chapter. The general layout of the computer program is also presented as well as the reasoning behind the determination of initial parameter values. Three different parameter specifications are considered for the mixed effects model and four different parameter specifications are considered for the random effects model. Statistics associated with each method of estimation are examined as well as any pertinent graphs for both the mixed and random effects model.

3.2 The Computer Program

The computer program in the simulation study consists of both a Fortran and SAS™ job step.

The Fortran step first generates observations for a two-way completely randomized design with interaction. The data are generated according to models (1.3.1) and (1.3.2) where each effect is added in at its appropriate level. Each effect must be defined at all the levels in which it occurs in the design.

Random effects are defined by multiplying the standard deviation of that effect by a standard normal random variable generated by the FORTRAN function RNOR(Ø), i.e. $\sigma_i * Z$. This is repeated for all levels wher the random effect is defined. The standard deviation of the random effect is held constant where as the standard normal random variable is generated at all appropriate levels.

For example, the random effect interaction is defined for all combinations of rows and columns, therefore a unique interaction effect is defined for all combinations as,

```
DO OVER ROWS
    DO OVER COLUMNS
        .
        .
        .
        INTER(ROW, COL) = σAT * RNOR(Ø)
        .
        .
        .
    END COLUMNS
END ROWS.
```

Fixed effects are defined by simply adding in a predetermined constant at the appropriate levels, where the constant represents the value of the fixed effect.

Once all the effects for both fixed and random effects have been initialized at their appropriate levels, the observations are generated by adding all the effects in at once. For example, the observations for the mixed effects model as defined by equation (1.3.1) are generated by

```
DO OVER ROWS
    DO OVER COLUMNS
        DO OVER REPLICATIONS
        .
        .
        .
        OBS  =  μ  + A(ROW) + SIGT(COL)
```

```
                    + INTER(ROW,COL) +  σe *RNOR(Ø)
                .
                .
                .
                END REPLICATIONS
            END COLUMNS
        END ROWS.
```

The observations for the random model (1.3.2) are generated in a similar manner. The subroutines used to generate observations for the mixed and random effects model are MIXGEN* and RANGEN*, respectively.

Once the data is generated, an analysis of variance table is computed under the assumption of having a two-way fixed effects model with interaction, (see 2.2.1). The AOV estimates of variance components are then computed by the equations in table (2.2.1) and (2.2.2) and stored on a disk. Estimates of variance components based on the pretest method are computed by subroutines PRETST* and RANTST*. The pretest method is employed at alpha levels of .01, .05, .10, .25, and .50. These estimates are also stored on a disk. Hypotheses tested by this method are deemed significant or not by comparing the predefined alpha values to a computed alpha $\hat{\alpha}$, or PR>F. The PR>F is computed by a convergent series expansion of positive terms for an F-distribution with the corresponding degrees of freedom within the function subroutine FPROB*, (see American Statistician, 55, pg. 944).

The SAS™ step reads in the AOV and pretest estimates from disk storage. A data set based on REML estimates of equation (2.3.8) and (2.3.10) is obtained by restricting estimates in the AOV data set over a positive estimation space, i.e. negative estimates are set equal to zero. The

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*see Appendix I for actual source code.

pretest estimates are then separated into 5 data sets which correspond to the level of significance employed when the estimate was obtained.

Once all data sets of estimates are defined with respect to the method of estimation, then basic statistics such as the mean, mean square error and bias are computed.

The estimates generated in the Fortran job steps are stored on a permanent disk in catalogued data sets for further accessing and analyzing if initial results deem it appropriate.

3.3 Initial Parameter Values

This section defines all parameter initializations that are examined in the simulation for both the mixed and random effects models.

The parameters that need to be initialized for the mixed effects model (1.3.1) are μ, α_i, $i=1,\ldots,a$, $\sigma^2{}_T$, $\sigma^2{}_{AT}$, and $\sigma^2{}_e$. For the mixed effects model μ, α_i, and $\sigma^2{}_e$ will remain constant throughout the simulation study. The variance components $\sigma^2{}_T$ and $\sigma^2{}_{AT}$ are varied. The initial values of $\sigma^2{}_T$ and $\sigma^2{}_{AT}$ are expressed in terms of the ratio $\sigma^2{}_i/\sigma^2{}_e$, where $\sigma^2{}_i$ is the variance of the i^{th} random effect. A ratio larger than one implies the presence of a particular random effect. Similarly a ratio much smaller then one implies a negligible random effect relative to the random sampling error.

First consider the case when both $\sigma^2{}_T$ and $\sigma^2{}_{AT}$ are very large relative to sampling error, i.e.,

$$\sigma^2{}_T/\sigma^2{}_e > 1 \text{ and } \sigma^2{}_{AT}/\sigma^2{}_e > 1.$$

This type of parameter initialization implies a large MST and MSAT relative to MSE, which would cause the AOV and REML estimates of equations

(2.2.23) and (2.3.8) respectively, to yield identical results. When both σ^2_T and σ^2_{AT} are large relative to σ^2_e the pretest method reduces to the AOV method. Therefore estimates from all three methods should be identical (and possess the same distributional properties) assuming the pretest method is employed at a reasonably small Type I error rate, say less than 10%.

Consider the other extreme, when both σ^2_T and σ^2_{AT} are negligible relative to σ^2_e, i.e.,

$$\sigma^2_T/\sigma^2_e \ll 1 \quad \text{and} \quad \sigma^2_{AT}/\sigma^2_e \ll 1. \tag{3.3.1}$$

As σ^2_T and σ^2_{AT} approach zero, then the expected mean squares for T, AT, and ERROR approach equality. This implies values of MST, MSAT, and MSE that are of about the same magnitude. This presents the possibility of obtaining negative estimates by the AOV method, zero estimates by the REML method, and estimates based on pooling by the pretest method, as well as zero estimates. Expression (3.3.1) will be the basis for the first parameter initialization in the simulation study for the mixed effects model, as each method is likely to obtain different estimates.

The next consideration is when one effect is large and one effect is negligible.

The second parameter initialization is when the interaction effect is negligible and the T effect is large relative to sampling error, expressed as

$$\sigma^2_T/\sigma^2_e > 1 \quad \text{and} \quad \sigma^2_{AT}/\sigma^2_e \ll 1. \tag{3.3.2}$$

The third parameter initialization is when the T effect is negligible and the interaction effect is large relative to sampling error, expressed as

$$\sigma^2_T/\sigma^2_e << 1 \quad \text{and} \quad \sigma^2_{AT}/\sigma^2_e > 1 \qquad (3.3.3)$$

The results of the simulation for cases 3.3.1-3 are presented in section 3.4.

The parameters that need to be initialized for the random effects model (1.3.2) are μ, σ^2_A, σ^2_T, σ^2_{AT}, and σ^2_e. For this model μ and σ^2_e will remain constant for all four parameter initializations throughout the simulation study. Similar to the mixed effects model, parameter initializations for σ^2_a, σ^2_T, and σ^2_{AT} are expressed as a ratio, with the same interpretation as defined earlier.

As with the mixed effects model, when all components are large relative to σ^2_e the REML and pretest methods are identical to the AOV method.

The other extreme is when all components are negligible relative to σ^2_e, expressed as

$$\sigma^2_A/\sigma^2_e << 1, \ \sigma^2_T/\sigma^2_e << 1, \text{ and } \sigma^2_{AT}/\sigma^2_e << 1. \qquad (3.3.4)$$

This is the first case considered in the simulation study for the random effects model.

The remaining three parameter initializations consists of a combination of negligible and large variance components relative to σ^2_e.

The second case is when both the A and T effect are large and the interaction effect is small relative to sampling error, expressed as

$$\sigma^2_A/\sigma^2_e > 1, \ \sigma^2_T/\sigma^2_e > 1, \text{ and } \sigma^2_{AT}/\sigma^2_e << 1. \qquad (3.3.5)$$

The third case is when both the A and T effects are negligible and the interaction effect is large relative to sampling error, expressed as

$$\sigma^2_A/\sigma^2_e << 1,\ \sigma^2_T/\sigma^2_e << 1, \text{ and } \sigma^2_{AT}/\sigma^2_e > 1. \tag{3.3.6}$$

The last case is when the A effect is negligible, T effect is large, and the interaction effect is moderate, expressed as

$$\sigma^2_A/\sigma^2_e <<,\ .\ \sigma^2_T/\sigma^2_e > 1, \text{ and } \sigma^2_{AT}/\sigma^2_e = 1 \tag{3.3.7}$$

The results of the simulation for cases 3.3.4-7 are presented in section 3.5.

3.4 Mixed Model Results

This section presents the results of the simulation study for the mixed model parameter initializations defined by expressions (3.3.1-3). The parameters μ, σ^2_e, and α_i for i=1,2, and 3 remain constant throughout the simulation at the following values 10, 100, -25, 0, and 25, respectively.

A completely randomized design with three levels of the fixed effects and four levels of the random effect is examined. Four observations are generated for each of the twelve possible treatment combinations.

The simulation generates the completely randomized design and estimates the variance components by each method one hundred times. Basic statistics of the estimates are then computed and presented in tabular form. These tables consist of the mean, mean square error (MSE), and average bias (BIAS) of the estimates for all methods. The MSE and (BIAS) are defined as

$$\frac{\sum_{i=1}^{100} (\sigma^2_0 - \tilde{\sigma}^2_i)^2}{100} \quad \text{and} \quad \frac{\sum_{i=1}^{100} (\sigma^2_0 - \tilde{\sigma}^2_i)}{100}$$

respectively, where σ^2_0 is the parameter value and $\tilde{\sigma}^2_i$ is the estimate of the appropriate variance component.

The first case considered is expression (3.3.1), where both the T and interaction effects are negligible relative to the random sampling error. The initial values of $\sigma^2{}_T$ and $\sigma^2{}_{AT}$ are 2.25 and 4.0 respectively. Table 3.4.M1 presents basic statistics computed from the actual estimates of the variance components obtained in the simulation.

The mean of the estimates based on the AOV method for $\sigma^2{}_T$, $\sigma^2{}_{AT}$, and $\sigma^2{}_e$ are very accurate relative to their MSE at 1.32, 4.97 and 100.12. The mean square errors for the AOV estimators are larger than those for the other two methods. This implies a very wide range for possible estimates. This could be attributed to the fact that negative estimates are possible and also more likely in this case where both $\sigma^2{}_T$ and $\sigma^2{}_{AT}$ are small relative to $\sigma^2{}_e$. The average bias for the estimators based on the AOV method are negligible which is consistent with the statement of unbiasedness in chapter two. As there is no chance of ever obtaining a negative estimate for $\sigma^2{}_e$, the AOV and Reml method yield the same estimates

The mean of the REML estimators for $\sigma^2{}_T$ and $\sigma^2{}_{AT}$ are greater than the means based on the AOV method, implying that negative estimates were observed and zero was taken as the estimate. The AOV method is shown to be both analytically and empirically unbiased, which implies the REML estimates yield a negative bias, i.e. overestimate, as the REML estimates are greater than or equal to the AOV estimates when $\sigma^2{}_T$ and $\sigma^2{}_{AT}$ are small relative $\sigma^2{}_e$. The case of a negative bias for the REML estimates was observed for this parameter configuration. The REML estimates also yielded a smaller mean square error then the AOV estimates when negative estimates were observed.

Table 3.4.M1

	σ^2_T = 2.25			σ^2_{AT} = 4.0			σ^2_e = 100.0		
	Mean	MSE	BIAS	Mean	MSE	BIAS	Mean	MSE	BIAS
AOV	1.32	123.18	0.93	4.97	400.59	-0.97	100.12	591.28	-0.12
REML	5.06	67.84	-2.81	10.29	275.50	-6.29	100.12	591.28	-0.12
Pretest									
.01	1.26	41.21	0.98	1.24	58.64	2.76	103.15	658.04	-3.15
.05	1.62	32.99	0.63	5.04	197.84	-1.04	99.83	615.54	0.17
.10	2.36	40.93	-0.11	5.94	217.64	-1.94	98.51	556.92	1.49
.25	3.81	51.24	-1.56	7.10	176.96	-3.10	96.43	536.80	3.57
.50	4.00	50.69	-1.75	6.79	177.28	-2.79	96.52	541.90	3.48

The pretest method was employed for parameter initialization (3.3.1) at alpha levels of .01, .05, .10, .25, and .50. The results for this method are also presented in table 3.4.M1.

The mean of the estimates for both $\sigma^2{}_T$ and $\sigma^2{}_{AT}$ increase as alpha increases. This seems reasonable for as alpha increases an effect is more likely to be deemed significant and consequently estimated. Similarly, as alpha decreases an effect is more likely to be found to be non-significant, resulting in an estimate of zero. The average estimates for $\sigma^2{}_e$ increase as alpha decreases. This results as the T and interaction effects are more likely to be negligible and their sums of squares pooled with sums of square error at smaller alpha values.

The MSE gives a feel for the range of the occurences of estimates and the BIAS indicates at approximately what level of alpha the pretest method yields nearly unbiased estimators.

By visual inspection of the MSE and BIAS, optimal alpha values can be determined for estimating each variance component. Optimal alpha values for estimating $\sigma^2{}_T$, $\sigma^2{}_{AT}$, and $\sigma^2{}_e$ where the Pretest estimators are nearly unbiased (i.e. BIAS $\rightarrow$ 0) are approximately .09, .04, and .05. Optimal alpha values for estimating $\sigma^2{}_T$, $\sigma^2{}_{AT}$, and $\sigma^2{}_e$ where the Pretest estimators yielded a minimal MSE are .05, .01, and .25. From this point an overall optimal alpha value for estimating all of the variance components in the model will be based on the average of the individual optimal alpha values. For this parameter initialization, the overall optimal alpha values for the Pretest method being both nearly unbiased and having minimal MSE are approximately .06 and .09 respectively.

As alpha increases one would expect all the random effects to be deemed significant. This results in the pretest method reducing to the AOV method. This is obviously not the case as indicated by table 3.4.M1.

The reason being is that $\sigma^2{}_T$ and $\sigma^2{}_{AT}$ are still found to non-significant at an alpha value as high as .50, resulting in estimates that are based on pooling or equated to zero which differ from estimates based on the AOV method.

Assuming an optimal alpha level is determined the pretest method is seemingly better then both the AOV and the REML method, for this parameter initialization.

The next case considered is expression (3.3.2), where the interaction effect is negligible and the T effect is large relative to the random sampling error. The initial values of $\sigma^2{}_T$ and $\sigma^2{}_{AT}$ are 400.0 and 9.0 respectively. Table 3.4.M2 presents basic statistics computed from the estimates obtained in the simulation.

The mean of the estimates obtained by the AOV method are very accurate with a small average bias relative to the estimates mean square error. The T effect is large relative to the random sampling error which implies all three methods of estimation are the same, or equally good in terms of estimators.

For this particular parameter initialization the main interest is in estimating $\sigma^2{}_{AT}$ and $\sigma^2{}_e$.

Due to differences in the mean of the estimates between the AOV and REML method for $\sigma^2{}_{AT}$, it follows that negative estimates were observed. Simular to the previous case, the REML method yields biased estimators, although they do posess a smaller mean square then the AOV method. From a standpoint of wanting an unbiased estimator, the AOV method would seem preferable, whereas the REML method would be better from a minimal MSE viewpoint.

For the pretest method there is no change when estimating $\sigma^2{}_T$ at different alpha levels. This results as $\sigma^2{}_T$ is very large relative to

Table 3.4.M2

	$\sigma^2_T = 400.0$			$\sigma^2_{AT} = 9.0$			$\sigma^2_e = 100.0$		
	Mean	MSE	BIAS	Mean	MSE	BIAS	Mean	MSE	BIAS
AOV	414.76	128226.24	-14.76	7.43	511.62	1.57	100.93	632.74	-0.93
REML	414.82	128179.48	-14.82	12.25	347.87	-3.25	100.93	632.74	-0.93
Pretest									
.01	413.56	129798.16	-13.56	5.76	710.02	3.24	103.23	588.66	-3.23
.05	414.85	128424.11	-14.85	6.82	326.47	2.18	101.34	619.60	-1.34
.10	413.78	128609.74	-13.78	10.16	373.08	-1.16	99.54	601.41	0.46
.25	413.65	128259.89	-13.65	11.47	353.84	-2.47	98.59	572.10	1.41
.50	413.45	128287.36	-13.65	12.17	345.64	-3.17	98.19	572.79	1.81

σ^2_e. The average estimates of σ^2_{AT} increase and σ^2_e decrease as alpha increases by the same reasoning explained for the first parameter configuration.

Individual optimal alpha values for estimating σ^2_{AT} and σ^2_e, by the Pretest method, that yield estimators which are nearly unbiased and have a minimal MSE are approximately .09, .10, and .05, .25, respectively. Overall optimal alpha value for this parameter initialization are .095 and .15 for obtaining both nearly unbiased and minimal MSE estimators.

A particular point of interest for this parameter configuration is the value of the MSE for the estimates of σ^2_{AT} by the Pretest Method at α = .01, which is 710.02. This value is inconsistent with the corresponding MSE's obtained by the other methods of estimation. This unusually large value resulted as σ^2_T was found to be nonsignificant at α = .01, and σ^2_{AT} was found to be significant. Consequently, the SST were pooled with SSAT and σ^2_{AT} was estimated. The estimate was extremely large at $\tilde{\sigma}^2_{AT}$ = 227.45. This is an obvious flaw in the Pretest method strategy. This situation implies the possibility of bad estimation if the method employs too small of an optimal alpha value. Assuming this erroneous estimate is deleted the MSE would be 232.95, which is more consistent with the other methods.

The last case examined for the mixed effects model is parameter initialization (3.3.3), where the T effect is negligible and the interaction effect is large relative to the random sampling error. The initial values used for this case are σ^2_T = 9.0 and σ^2_{AT} = 324.0. Table 3.4.M3 presents basic statistics computed from the estimates obtained in the simulation study for this parameter initialization.

All three methods of estimation yielded almost identical results when estimating σ^2_{AT}. This is due to the magnitude of σ^2_{AT} relative to

$\sigma^2{}_e$. Therefore, the main interest in comparing the three methods is in the estimation of $\sigma^2{}_T$.

Both the AOV and REML methods do not estimate $\sigma^2{}_T$ with a good deal of accuracy, based on the MSE and BIAS. The AOV method does posess a smaller BIAS than does the REML method. Although, as seen before, the REML method has a smaller MSE than the AOV method.

For the Pretest Method, the estimates of $\sigma^2{}_{AT}$ and $\sigma^2{}_e$ are similar if not identical across all levels of alpha. The little change in $\sigma^2{}_e$ is caused by sums of squares from negligible T effects which are usually being pooled with sums of squares interaction, not sums of square error. This results in estimates of $\sigma^2{}_T$ increasing, and estimates of $\sigma^2{}_{AT}$ decreasing as α increases. At small values of alpha, zero is taken as the estimate of $\sigma^2{}_T$ and SST are pooled into the interaction effect. Simarly for larger values of alpha, $\sigma^2{}_T$ is estimated more often and equated to zero less frequently.

Optimal alpha values for estimating $\sigma^2{}_T$, based on both a zero bias and minimal MSE are .03, and .01. As $\sigma^2{}_T$ was the only variance component of interest for this parameter initialization, the individual optimal alpha values will also serve as overall optimal alpha values.

Table 3.4.M3

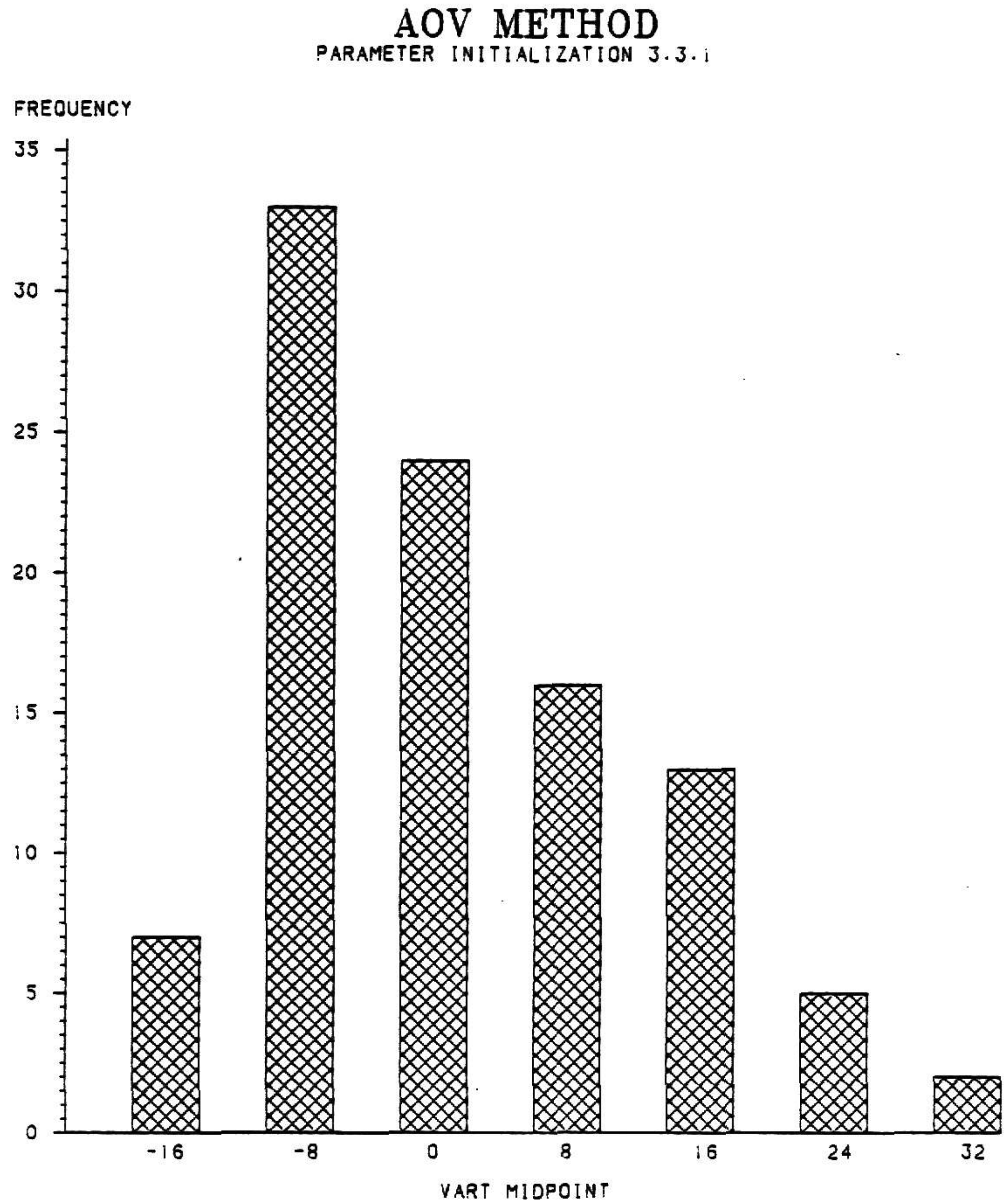
	$\sigma^2_T = 9.0$			$\sigma^2_{AT} = 324.0$			$\sigma^2_e = 100.0$		
	Mean	MSE	BIAS	Mean	MSE	BIAS	Mean	MSE	BIAS
AOV	-8.35	15225.73	17.35	349.62	49378.61	-25.62	100.63	532.16	-0.63
REML	42.07	6471.41	-33.07	349.62	49378.61	-25.62	100.63	532.16	-0.63
Pretest									
.01	5.71	1260.24	3.29	333.89	30310.66	-9.89	101.97	673.51	-1.97
.05	13.42	2992.53	-4.42	327.78	30424.27	-3.78	100.69	531.40	-0.69
.10	20.22	4130.84	-11.22	320.97	31371.23	+3.03	100.69	531.40	-0.69
.25	36.41	6270.29	-27.41	304.86	30400.65	19.14	100.63	532.16	-0.63
.50	41.69	6481.80	-32.69	299.58	29556.80	24.42	100.63	532.16	-0.63

By visual inspection of the statistics in tables 3.4.1.M1-M3, the following overall optimal alpha values for obtaining both nearly unbiased and minimal mean square error estimators when estimating σ^2_T, σ^2_{AT}, and σ^2_e are observed to be .06, .095, .03, and .09, .15, .01, respectively.

Using the overall optimal alpha values based on estimators that are nearly unbiased, graphs of the estimates obtained by the Pretest method are presented. Graphs of estimates obtaied by the AOV method are also presented. Estimates based on the REML method are not presented as these graphs can be intuitively visualized by examining the AOV graphs.

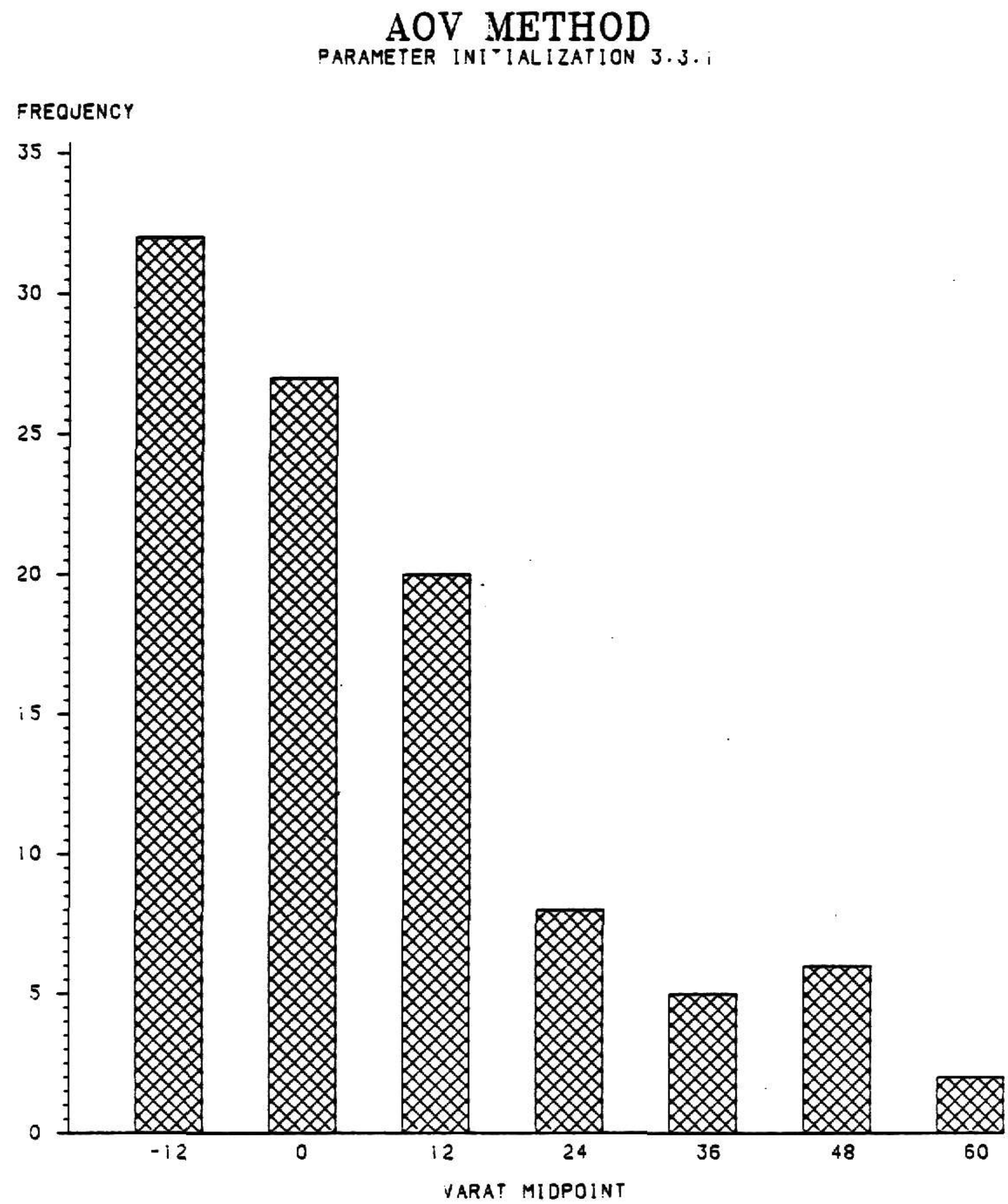
The estimates of the variance components σ^2_T and σ^2_{AT} are examined. The best estimate of σ^2_e was MSE, hence eliminating the need to consider pretest plots of these estimates. The graphs of the estimates for σ^2_T and σ^2_{AT} are displayed for parameter initialization 3.3.1, 3.3.2, and 3.3.3.

An interesting point displayed by the graphs is the large number of negative estimates observed by the AOV method for certain parameter initializations. For extremely large values of σ^2_T and σ^2_{AT} the Pretest method yields estimates just as good as the AOV method, i.e., they are identical. Graph 3.4.12 shows that for a large value of σ^2_{AT} relative to σ^2_e, the Pretest method appears to be empirically unbaised with a much smaller range of estimates than the estimates based on the AOV method, graph 3.4.10.



GRAPH 3.4.1: ESTIMATES OF THE VARIANCE OF T
FOR THE MIXED EFFECTS MODEL (1.3.1)

VARIANCE OF T = 2.25

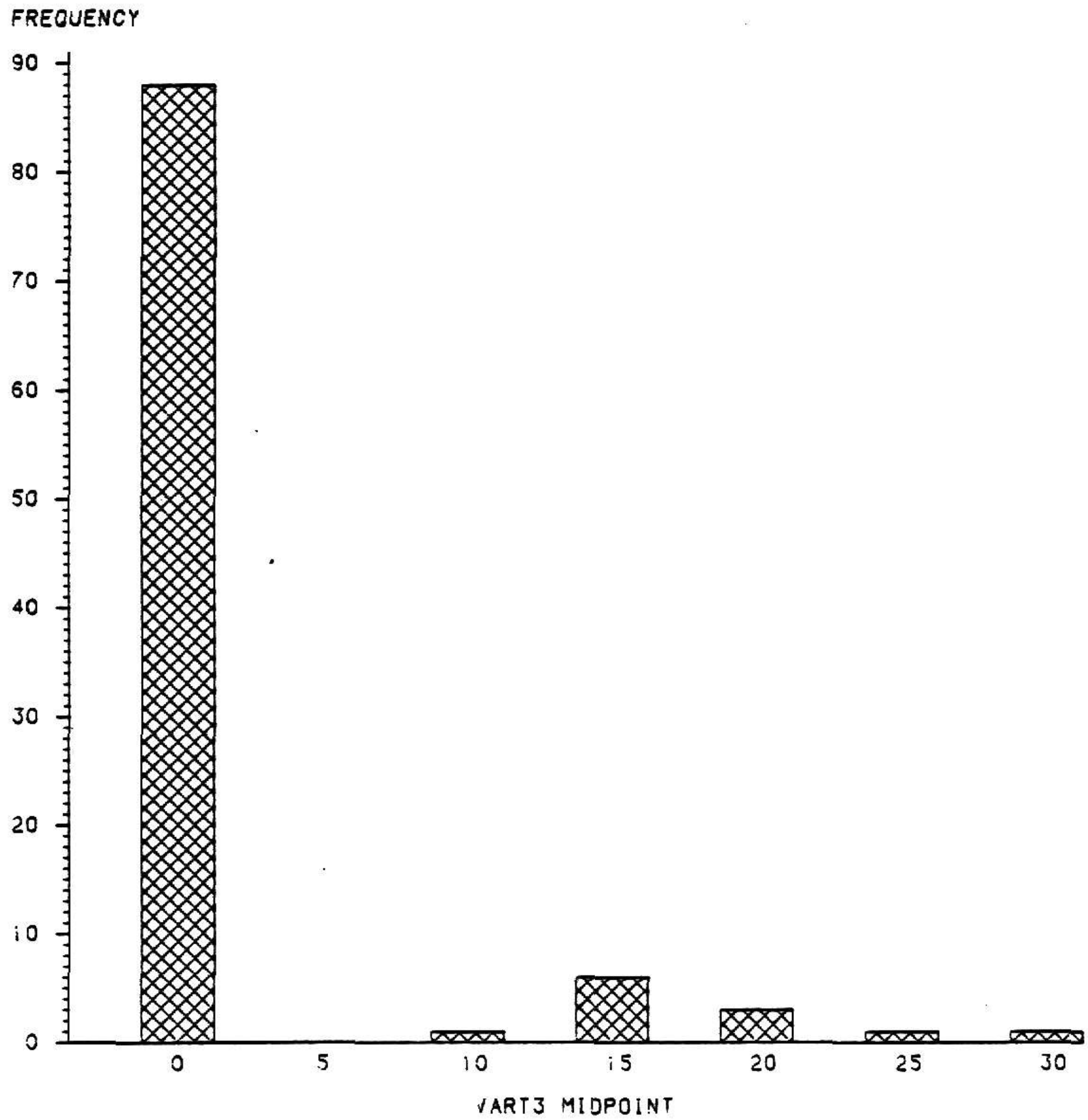


GRAPH 3.4.2: ESTIMATES OF THE VARIANCE OF AT
FOR THE MIXED EFFECTS MODEL (1.3.1)

VARIANCE OF AT = 4.00

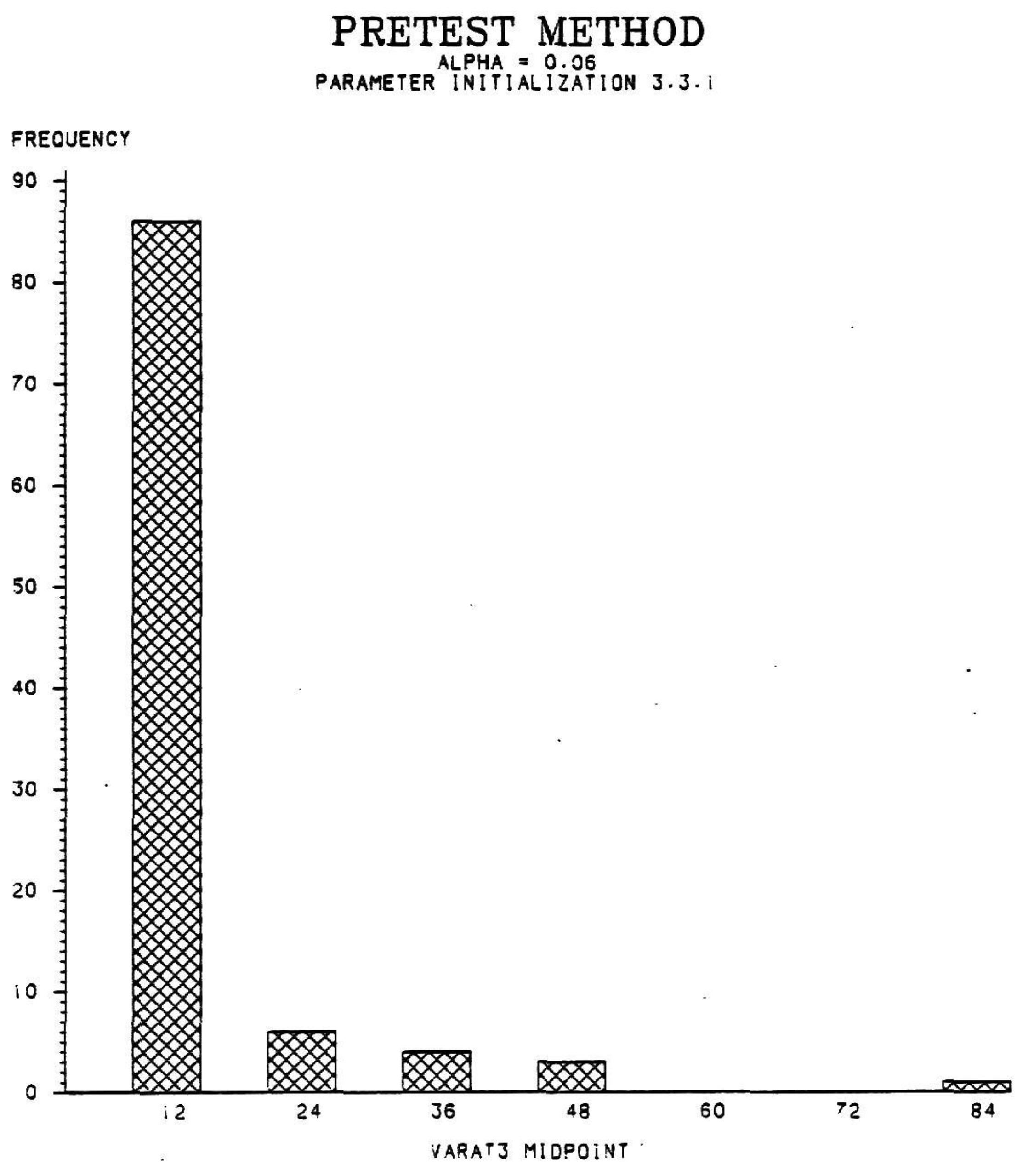
PRETEST METHOD

ALPHA = 0.06
PARAMETER INITIALIZATION 3.3.1



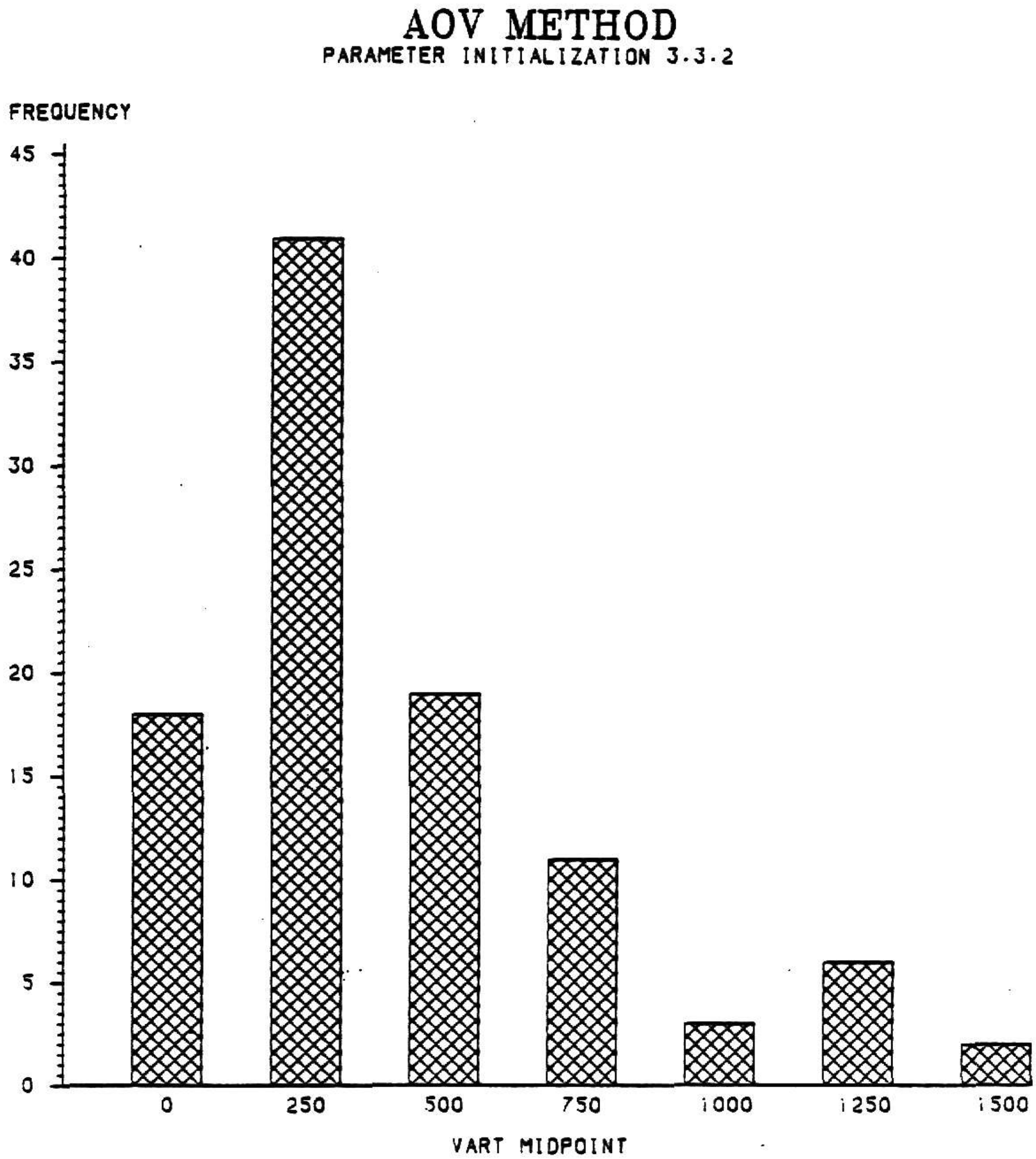
GRAPH 3.4.3: ESTIMATES OF THE VARIANCE OF T
FOR THE MIXED EFFECTS MODEL (1.3.1)

VARIANCE OF T = 2.25



GRAPH 3.4.4: ESTIMATES OF THE VARIANCE OF AT
FOR THE MIXED EFFECTS MODEL (1.3.1)

VARIANCE OF AT = 4.00

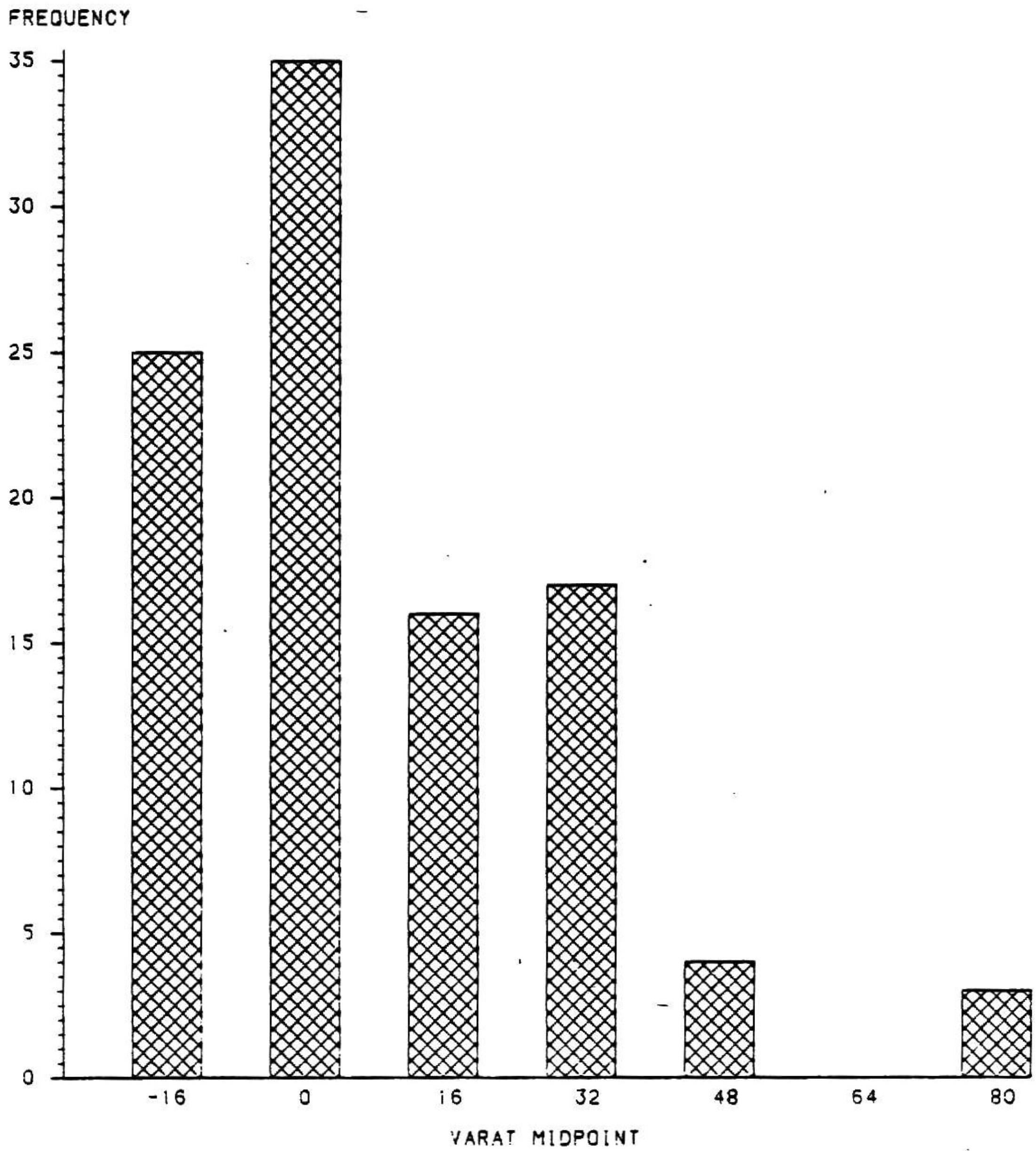


GRAPH 3.4.5: ESTIMATES OF THE VARIANCE OF T
FOR THE MIXED EFFECTS MODEL (1.3.1)

VARIANCE OF T = 400.0

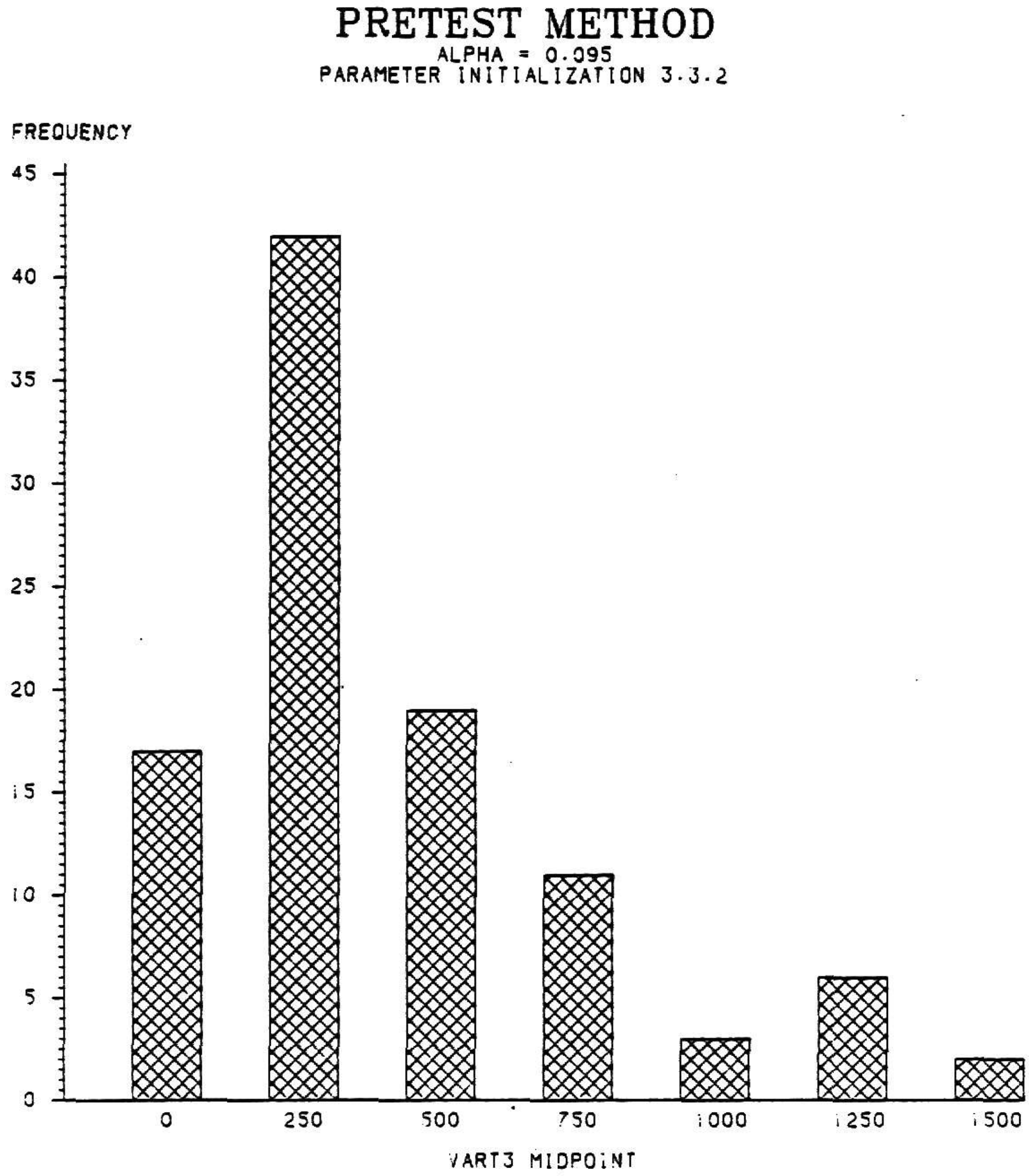
AOV METHOD

PARAMETER INITIALIZATION 3.3.2



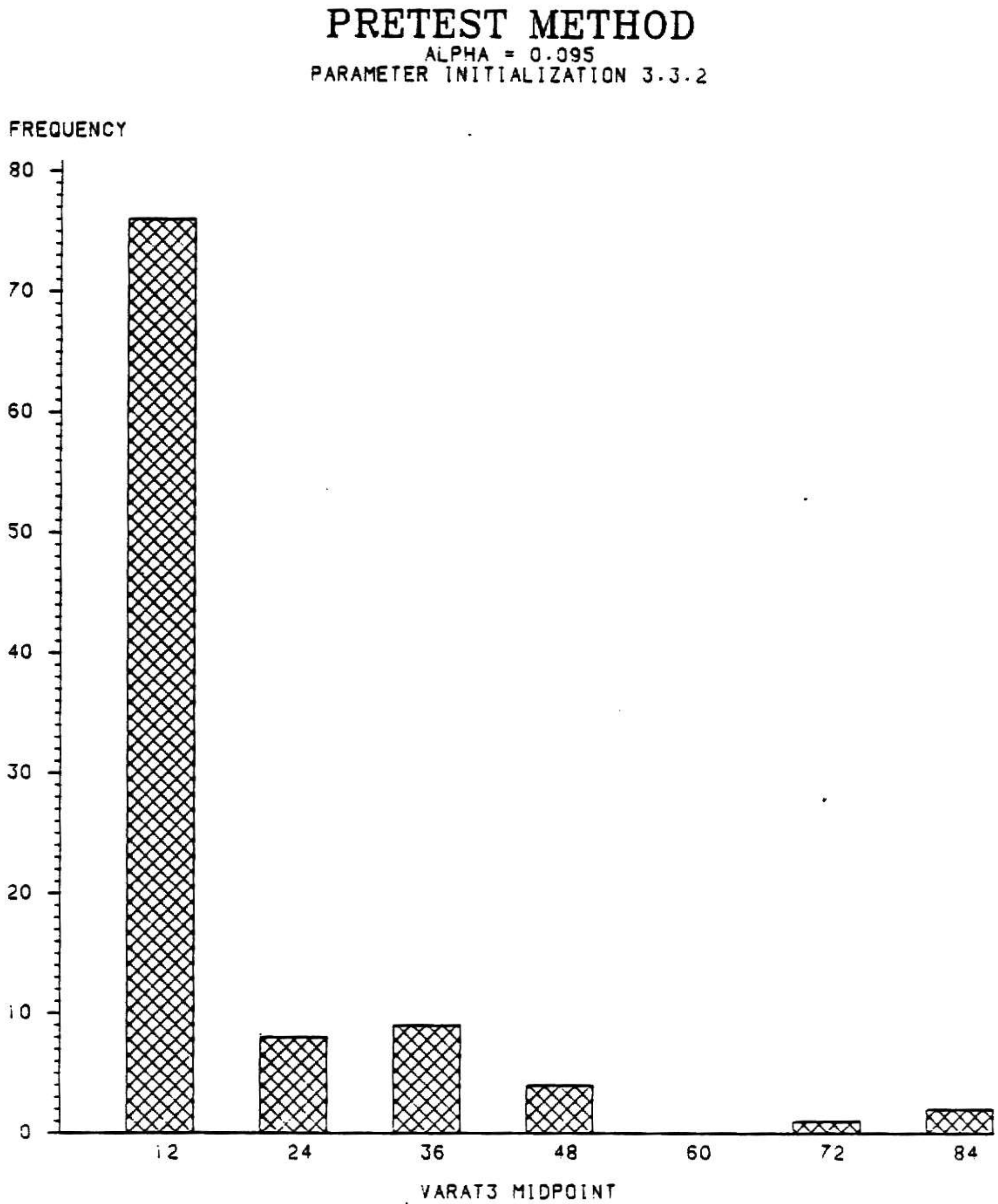
GRAPH 3.4.6: ESTIMATES OF THE VARIANCE OF AT
FOR THE MIXED EFFECTS MODEL (1.3.1)

VARIANCE OF AT = 9.0



GRAPH 3.4.7: ESTIMATES OF THE VARIANCE OF T
FOR THE MIXED EFFECTS MODEL (1.3.1)

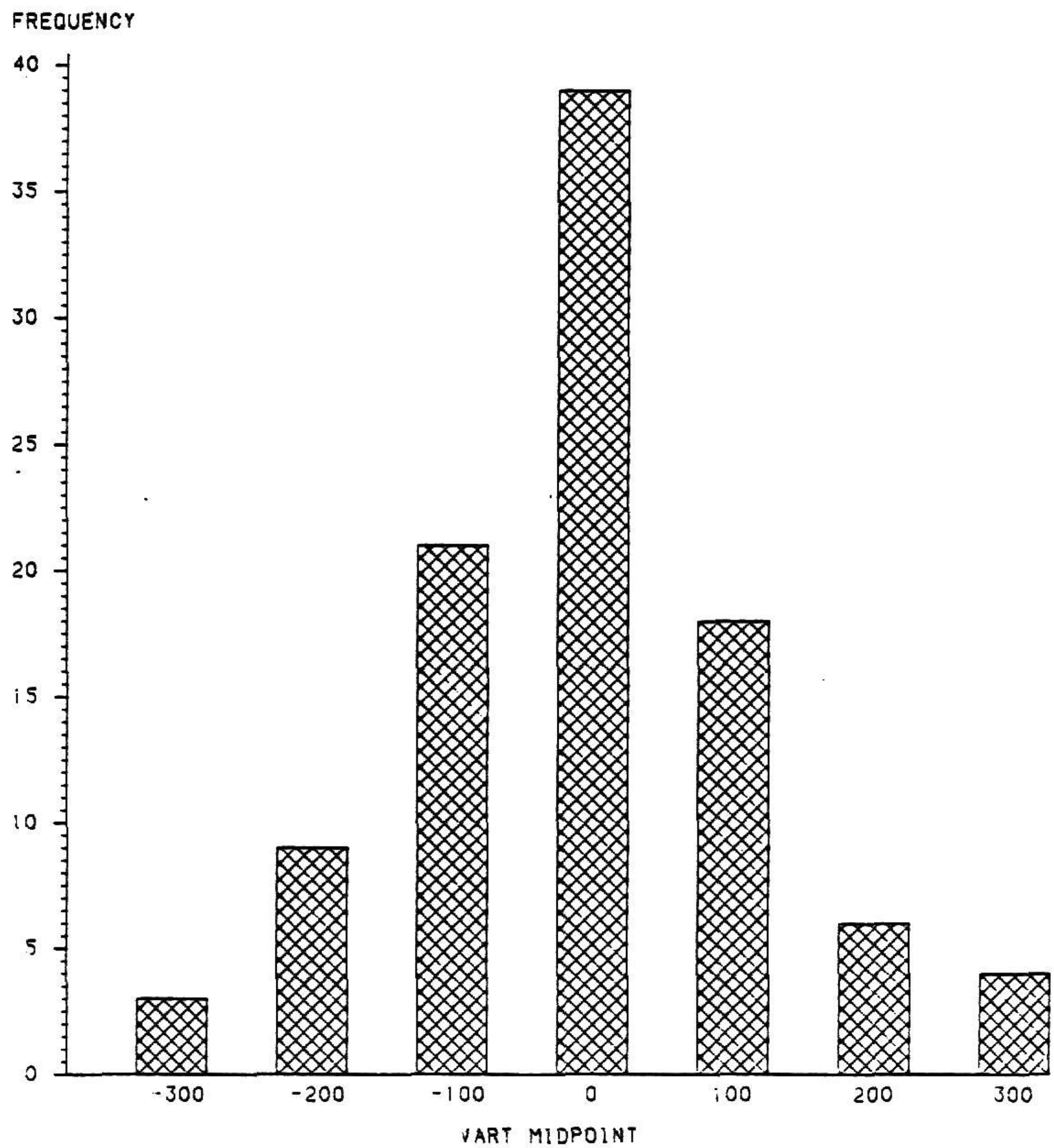
VARIANCE OF T = 400.0



GRAPH 3.4.8: ESTIMATES OF THE VARIANCE OF AT FOR THE MIXED EFFECTS MODEL (1.3.1)

VARIANCE OF AT = 9.00

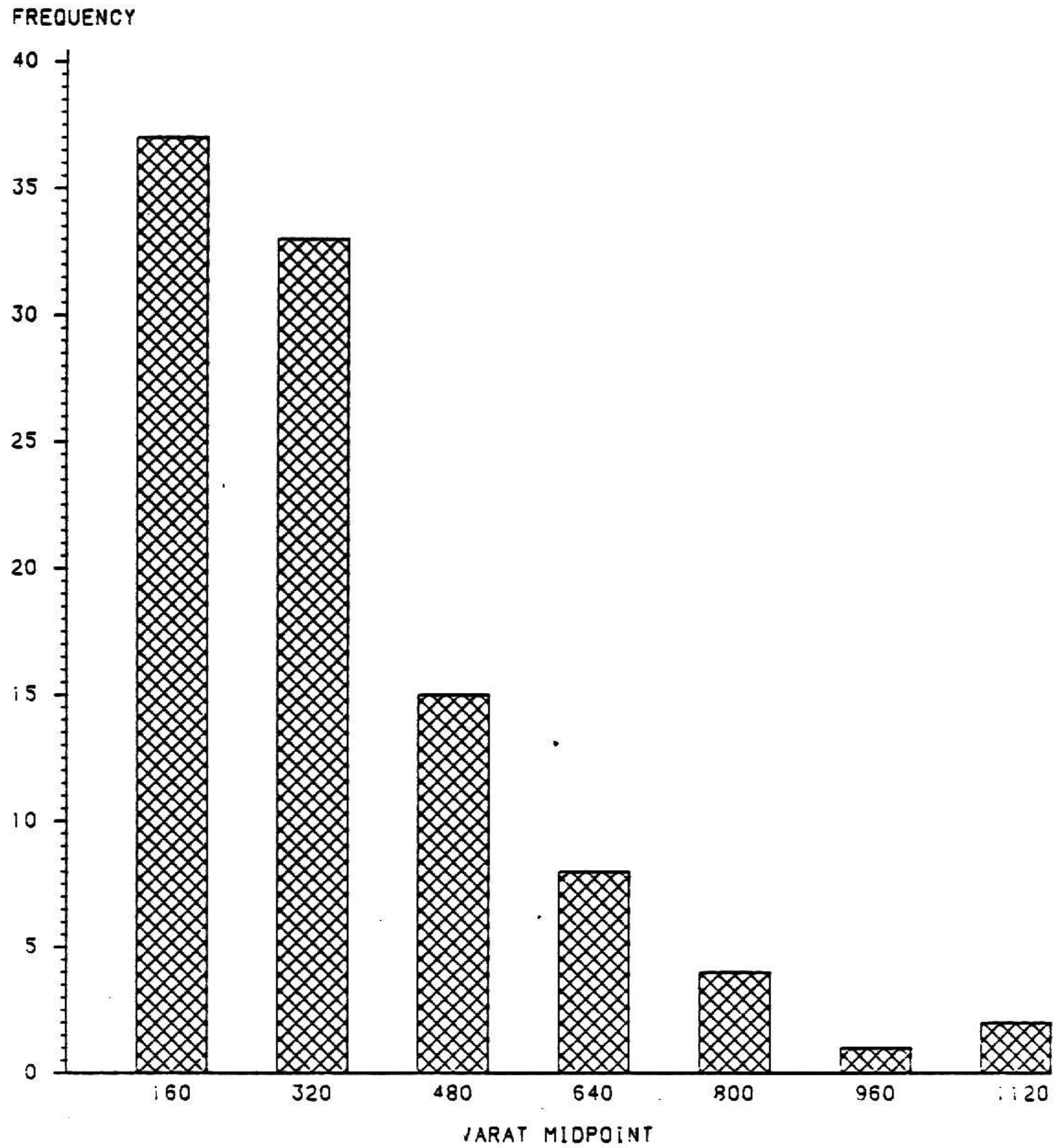
AOV METHOD
PARAMETER INITIALIZATION 3.3.3



GRAPH 3.4.9: ESTIMATES OF THE VARIANCE OF T
FOR THE MIXED EFFECTS MODEL (1.3.1)

VARIANCE OF T = 9.0

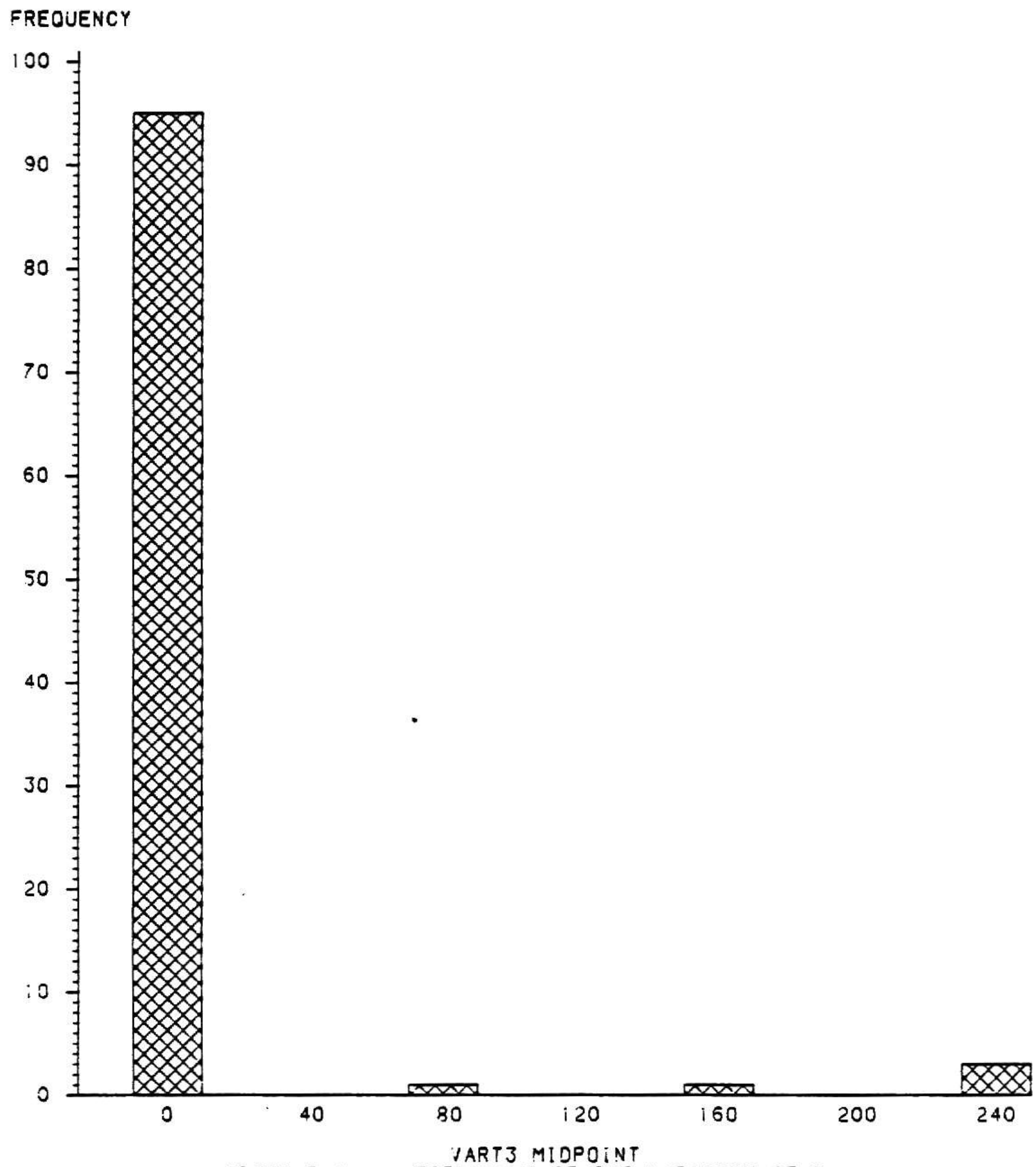
AOV METHOD
PARAMETER INITIALIZATION 3.3.3



GRAPH 3.4.10: ESTIMATES OF THE VARIANCE OF AT
FOR THE MIXED EFFECTS MODEL (1.3.1)

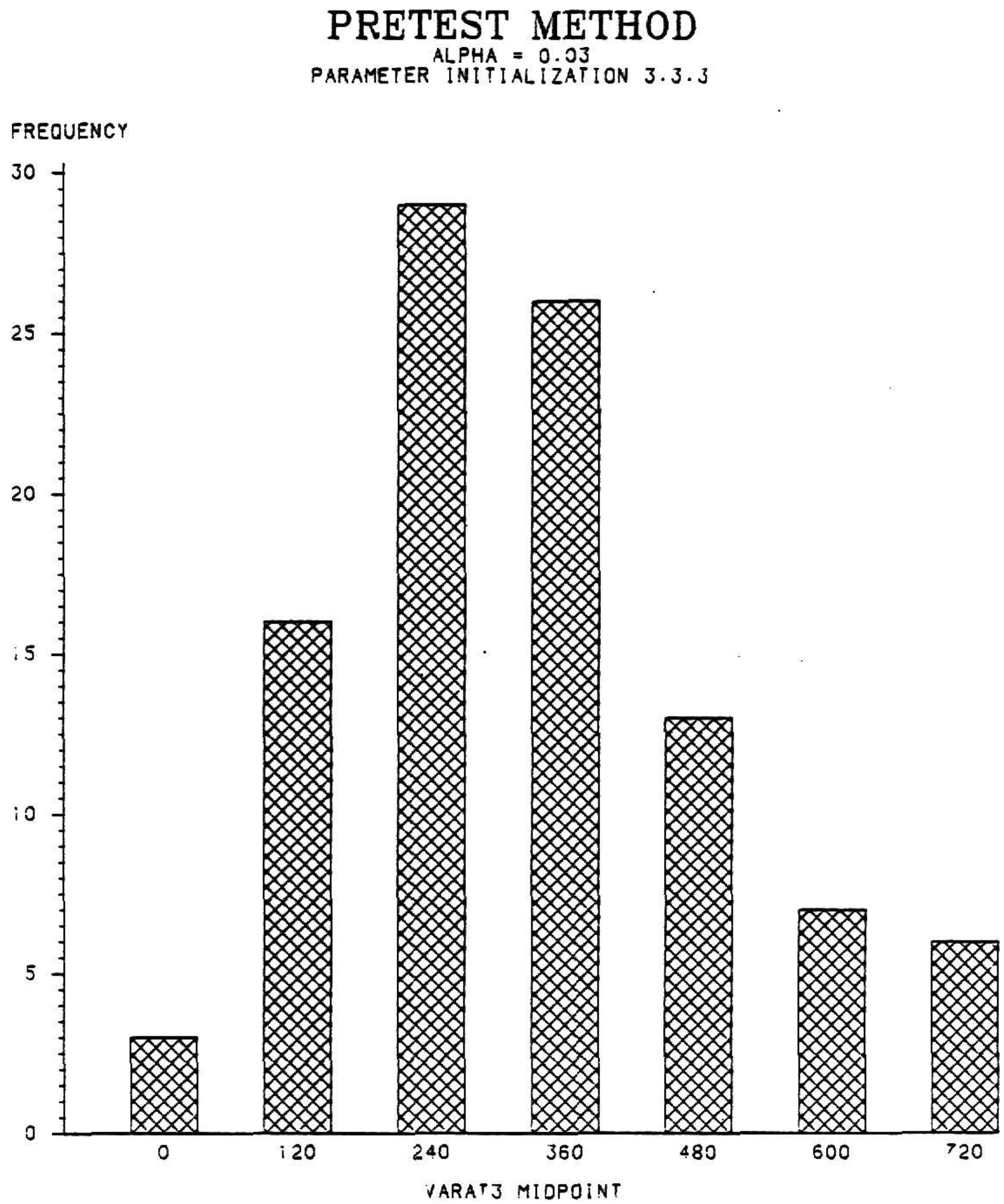
VARIANCE OF AT = 324.0

PRETEST METHOD
ALPHA = 0.03
PARAMETER INITIALIZATION 3.3.3



GRAPH 3.4.11: ESTIMATES OF THE VARIANCE OF T
FOR THE MIXED EFFECTS MODEL (1.3.1)

VARIANCE OF T = 9.0



GRAPH 3.4.12: ESTIMATES OF THE VARIANCE OF AT
FOR THE MIXED EFFECTS MODEL (1.3.1)

VARIANCE OF AT = 324.0

3.5 Random Model Results

This section presents the results of the simulation study for the random model parameter initializations defined in expressions (3.3.4-7). For all initializations considered, the parameters μ and σ^2_e remain constant throughout the simulation at the values 10.0 and 100.0, respectively.

A completely randomized design with four levels of both random factors is considered. There are sixteen possible treatment combinations where each contains a sample of four observations.

Similar to the mixed effects model, basic statistics which are presented in Tables (3.5.R1-R4) form the basis for a discussion of empirical conclusions associated with each method of estimation. There were one hundred generations of estimates for all four parameter initializations considered.

The first case considered is expression (3.3.4), where all the effects A, T, and interaction are negligible relative to the sampling error. The initial values for σ^2_A, σ^2_T, and σ^2_{AT} are 4, 9, and 16, respectively. Table 3.5.R1 presents statistics computed from the estimates of the variance components.

Table 3.5.R1

	$\sigma^2_T = 9.0$			$\sigma^2_A = 4.0$			$\sigma^2_{AT} = 16.0$			$\sigma^2_e = 100.0$		
	Mean	MSE	BIAS	Mean	MSE	BIAS	Mean	MSE	BIAS	Mean	MSE	BIAS
AOV	10.19	256.27	-1.19	6.11	184.18	-2.11	13.17	374.95	2.83	99.43	310.73	0.57
REML	11.85	212.59	-2.85	8.62	139.50	-4.62	14.91	301.41	1.09	99.43	310.73	0.57
Pretest												
.01	7.37	240.42	1.63	4.42	120.41	-0.42	4.97	415.34	11.03	110.67	512.67	-10.67
.05	9.21	231.71	-0.21	6.79	134.42	-2.79	8.62	424.00	7.38	103.99	378.18	-3.99
.10	9.82	222.63	-0.82	7.41	139.53	-3.41	10.76	375.30	5.24	101.01	326.09	-1.01
.25	11.21	212.80	-2.21	8.33	140.51	-4.33	11.56	247.82	4.44	98.50	284.10	1.50
.50	11.84	207.65	-2.84	8.42	137.99	-4.42	11.21	223.22	4.79	98.27	288.34	1.73

Based on Table 3.5.R1, there appears to be very little difference between the REML and AOV method of estimating variance components. This implies that few negative estimates were observed for this particular parameter initialization. The REML method would seem to be better as it restricts the possibility of negative estimates without severly affectin the average bias. It also possesses a smaller mean square error then the AOV method.

The pretest method will again be employed at the following alpha values of .01, .05, .10, .25, and .50. The mean of the estimates of $\sigma^2{}_A$, $\sigma^2{}_T$, and $\sigma^2{}_{AT}$ increase and $\sigma^2{}_e$ decrease as alpha increases. This is because of zero estimates for $\sigma^2{}_A$, $\sigma^2{}_T$, and $\sigma^2{}_{AT}$ and pooling of negligible affects with the sampling error at smaller values of alpha.

Optimal alpha values, obtained by visual inspection of statistics in table 3.5.R1 for both nearly unbiased and minimal mean square error estimators can be observed. For estimating the variance components $\sigma^2{}_A$, $\sigma^2{}_T$, and $\sigma^2{}_{AT}$, they are .01, .04, and .25, and .01, .50, and .50 respectively. Overall optimal alpha values, which again are based on the average of the individual optimal alpha values, are .10 and .33 for this parameter initializations.

The second case considered is expression (3.3.5), where the interaction effect is negligible and the main effects are large relative to the random sampling error. The initial values used are $\sigma^2{}_A = 256.0$, $\sigma^2{}_T = 324.0$, and $\sigma^2{}_{AT} = 9.0$.

For this parameter initialization, all methods of estimation are identical for large main effects, i.e., $\sigma^2{}_A$ and $\sigma^2{}_T$. Therefore the estimation of the variance of the random interaction effect is the main interest for this case.

The AOV method yields estimates with a smaller average bias then the REML method when estimating σ^2_{AT}. Similar to previous cases the REML method yields a larger average bias but a smaller mean square error. The smaller mean square error implies the occurance of negative estimates by the AOV method.

For the pretest method σ^2_A and σ^2_T are almost always found to be significant, which results in σ^2_A, σ^2_T, σ^2_{AT}, and σ^2_e being estimated by estimators given in path R4 and R8 as defined in Table 2.4.4. If the interaction effect is found to be nonsignificant then path R4 is used, otherwise path R8 is used. At smaller levels of alpha, σ^2_{AT} will be deemed nonsignificant more often and consequently path R4 employed. This yields smaller estimates of σ^2_{AT} and larger estimates of σ^2_e, as SSAT will be pooled with SSE and the estimate of σ^2_{AT} will be taken as zero. At larger values of alpha less pooling of SSAT with SSE takes place, and consequently σ^2_{AT} is estimated.

Optimal alpha values for estimating σ^2_{AT} and σ^2_e for both minimal bias and mean square error are .10, .09 and .50, .10, respectively. The overall optimal alpha values for both considerations are approximately .095 and .30.

Similar to the second case for the mixed effects model, the MSE for estimating σ^2_{AT} by the pretest method at α = .01 was observed to be unusually large at 791.34. This is due to σ^2_{AT} being found significant and the main effects A and T being found nonsignificant. This will result in estimates of σ^2_A and σ^2_T being zero, and the sum of squares from A and T being pooled with the sum of squares interaction. The estimate of σ^2_{AT} is then grossly inflated. This situation was observed three times in the simulation for this parameter configuration. If these unusually large estimates of σ^2_{AT} were deleted then the MSE is observed to be 171.88 which is smaller than the MSE for both the AOV and REML methods.

Table 3.5.R2

	$\sigma^2_T = 324.0$			$\sigma^2_A = 256.0$			$\sigma^2_{AT} = 9.0$			$\sigma^2_e = 100.0$		
	Mean	MSE	BIAS	Mean	MSE	BIAS	Mean	MSE	BIAS	Mean	MSE	BIAS
AOV	292.19	60727.32	31.81	258.86	48821.60	-2.86	8.32	304.65	0.62	99.65	361.59	0.35
REML	292.19	60727.32	31.81	258.93	48785.42	-2.93	11.47	207.02	-2.47	99.65	361.59	0.35
Pretest												
.01	290.96	61943.09	33.04	257.67	49431.82	-1.67	5.66	791.24	3.34	104.03	373.70	-4.03
.05	292.32	60860.52	31.68	258.92	48890.13	-2.92	7.03	240.53	1.97	100.74	356.09	-0.74
.10	291.87	60906.76	32.13	258.35	48957.54	-2.35	9.14	264.47	-0.14	99.5	316.87	0.50
.25	291.69	60784.75	32.31	258.37	48814.48	-2.37	10.74	217.01	-1.74	98.10	319.79	1.90
.50	291.55	60807.03	32.45	258.27	48777.95	-2.27	11.39	208.57	-2.39	97.67	316.62	2.33

The third parameter initialization is expression (3.3.6), where the main effects A and T are negligible and the interaction effect is large relative to the random sampling error. The initial parameter values are $\sigma^2_A = 4.0$, $\sigma^2_T = 9.0$, and $\sigma^2_{AT} = 400.0$.

Similar to the previous cases the AOV method yields a smaller average bias whereas the REML method yields a smaller mean square error. Both the AOV and REML method yielded identical results for estimating σ^2_{AT} which indicates no negative AOV estimates were observed. Although negative AOV estimates were observed when estimating σ^2_A and σ^2_T.

The estimates of σ^2_e are identical for all three methods. The large magnitude of σ^2_{AT} causes subsequent pooling of main effects to occur with SSAT and not with SSE. Therefore the MSE is the estimate of σ^2_e at all levels of alpha.

At small alpha values SST and SSA are being pooled with SSAT. The table shows underestimation of σ^2_A and σ^2_T and overestimation of σ^2_{AT} at $\alpha = .01$. At $\alpha = .50$ there is less pooling of SSA and SST with SSAT and now there is overestimation of σ^2_A and σ^2_T and underestimation of σ^2_{AT}. This implies the presence of some optimal alpha between .01 and .50 where just enough pooling takes place in order to obtain good estimates of σ^2_A, σ^2_T, and σ^2_{AT}.

Table 3.5.R3

	$\sigma^2_T = 9.0$			$\sigma^2_A = 4.0$			$\sigma^2_{AT} = 400.0$			$\sigma^2_e = 100.0$		
	Mean	MSE	BIAS	Mean	MSE	BIAS	Mean	MSE	BIAS	Mean	MSE	BIAS
AOV	15.46	12003.17	-6.46	2.67	8775.14	1.33	408.84	51071.49	-8.84	103.56	358.60	-3.56
REML	47.81	7639.73	-38.81	37.23	4471.09	-33.23	408.84	51071.49	-8.84	103.56	358.60	-3.56
Pretest												
.01	3.33	1128.90	5.67	0.0	16.00	4.00	420.68	26063.55	-20.68	103.56	358.60	-3.56
.05	15.77	4316.40	-6.77	11.89	2536.69	-7.89	401.22	29153.29	-1.22	103.56	358.60	-3.56
.10	22.33	5247.93	-13.33	17.84	3253.47	-13.84	391.20	31452.55	8.80	103.56	358.60	-3.56
.25	43.01	8089.19	-34.01	32.54	4598.05	-28.54	362.91	30508.35	37.09	103.56	358.60	-3.56
.50	51.03	8379.64	-42.03	40.20	4959.76	-36.20	350.36	30305.06	49.64	103.56	358.60	-3.56

The optimal alpha values to be employed when estimating σ^2_A, σ^2_T, and σ^2_{AT} for both nearly unbiased and minimal mean square error estimators, are approximately .02, .03, .06 and .01, .01, .01. Overall optimal alpha values for both considerations are .037 and .01.

The last parameter initialization considered for the random effects model is expression (3.3.7), where the T effect is large, the A effect is negligible, and the interaction effect is moderate relative to the random sampling error. The initial values for σ^2_A, σ^2_T, and σ^2_{AT} are 4.0, 361.0, and 100.00, respectively.

Similar to previous cases, the estimates of σ^2_T are very similar for all methods (due to the large magnitude of σ^2_T). The AOV and REML methods also yield identical results when estimating σ^2_{AT}, where the interaction effect was defined to be comparable to the sampling error. For estimating σ^2_A the AOV method yields a negligible average bias, but continues to observe negative estimates. This results in the REML method posessing a smaller mean square error when estimating σ^2_A.

For the pretest method, estimates of σ^2_T are identical for all levels of alpha as it was defined to be large. The estimates of σ^2_e also vary little over all alpha values. This is due to presence of an interaction effect. With interaction present, pooling of sum of squares will occur between the A and the interaction effect. This results in estimates of σ^2_A increasing, and σ^2_{AT} decreasing as alpha increases.

Table 3.5.R4

	σ^2_T = 361.0			σ^2_A = 4.0			σ^2_{AT} = 100.0			σ^2_e = 100.0		
	Mean	MSE	BIAS	Mean	MSE	BIAS	Mean	MSE	BIAS	Mean	MSE	BIAS
AOV	384.85	131121.19	-23.85	3.21	1298.87	0.79	98.62	4143.86	1.38	100.43	441.62	-0.43
REML	386.01	130246.73	-25.01	14.74	823.98	-10.74	98.62	4143.86	1.38	100.43	441.62	-0.43
Pretest												
.01	360.99	145079.77	0.01	6.41	462.30	-2.41	109.71	8720.90	-9.71	105.60	795.12	-5.60
.05	383.77	132305.23	-22.77	7.62	556.24	-3.62	94.85	4157.77	5.15	101.47	507.44	-1.47
.10	385.21	131290.99	-24.21	9.01	671.02	-5.01	93.10	3763.28	6.90	100.99	480.21	-0.99
.25	387.83	130943.13	-26.83	14.19	824.17	-10.19	87.45	3128.08	12.55	100.45	441.61	-0.45
.50	388.55	130504.48	-27.55	14.63	825.10	-10.63	86.52	3045.86	13.48	100.42	441.62	-0.42

The optimal alpha values for both unbiased and minimal mean square error estimators when estimating $\sigma^2{}_A$ are both approximately .007. This value will also serve as the overall optimal alpha as the other variance component are large.

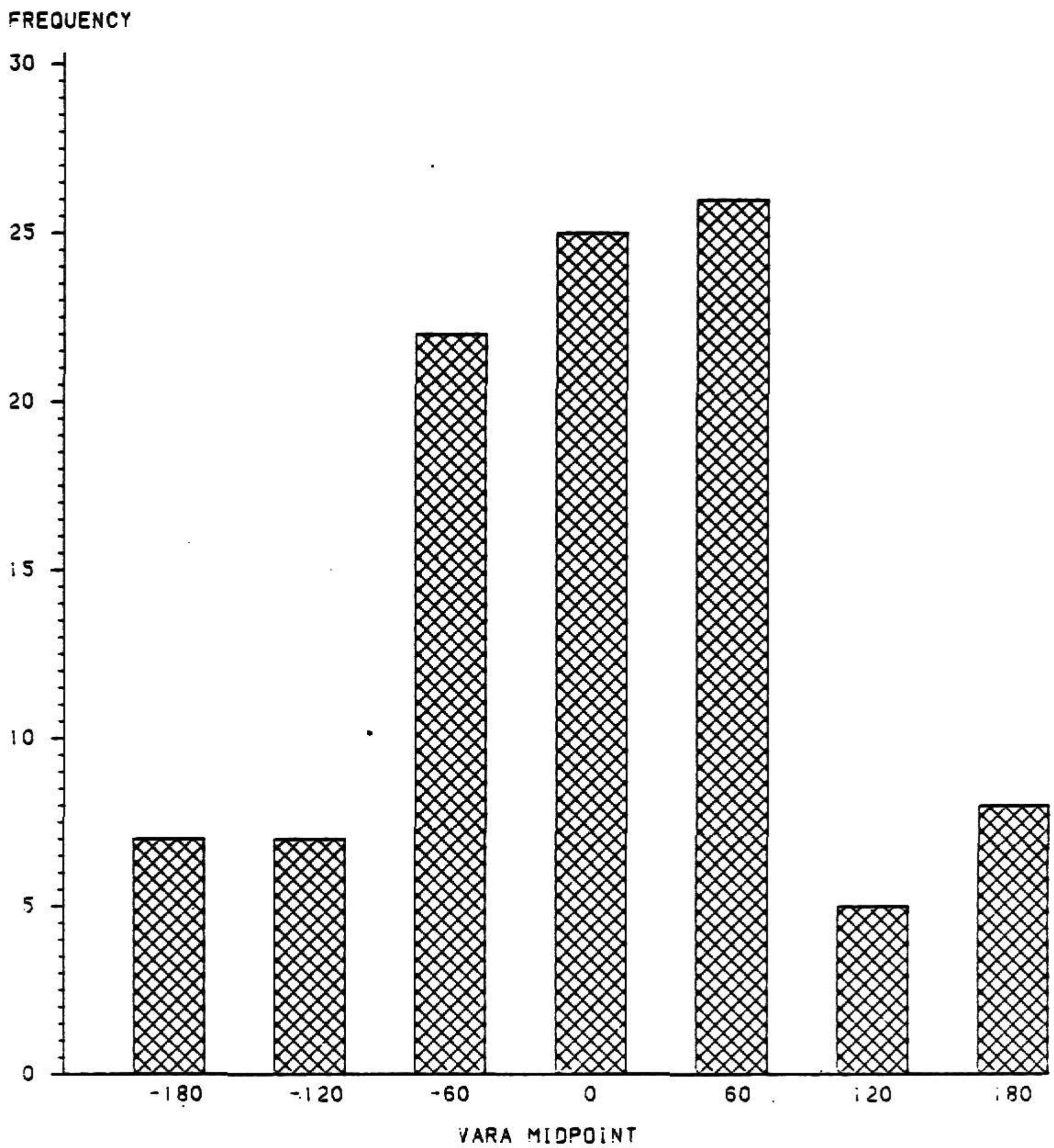
The following pages are graphs of the estimates of $\sigma^2{}_A$, $\sigma^2{}_T$, and $\sigma^2{}_{AT}$ for both the AOV and pretest method. The pretest method is employed at optimal alpha values associated with nearly unbiased estimators. Graphs are presented for all four parameters initializations considered for the random effects model.

Similar to the mixed effects model, there are a large number of occurences of negative estimates for small values of $\sigma^2{}_T$, $\sigma^2{}_A$, and $\sigma^2{}_{AT}$ relative to $\sigma^2{}_e$, see graphs 3.5.1, 3.5.2, 3.5.7, etc.

For certain cases, the pretest method yields estimates with a smaller range of estimation than the AOV estimators as implied by graphs 3.5.3 and 3.5.6.

AOV METHOD

PARAMETER INITIALIZATION 3.3.6

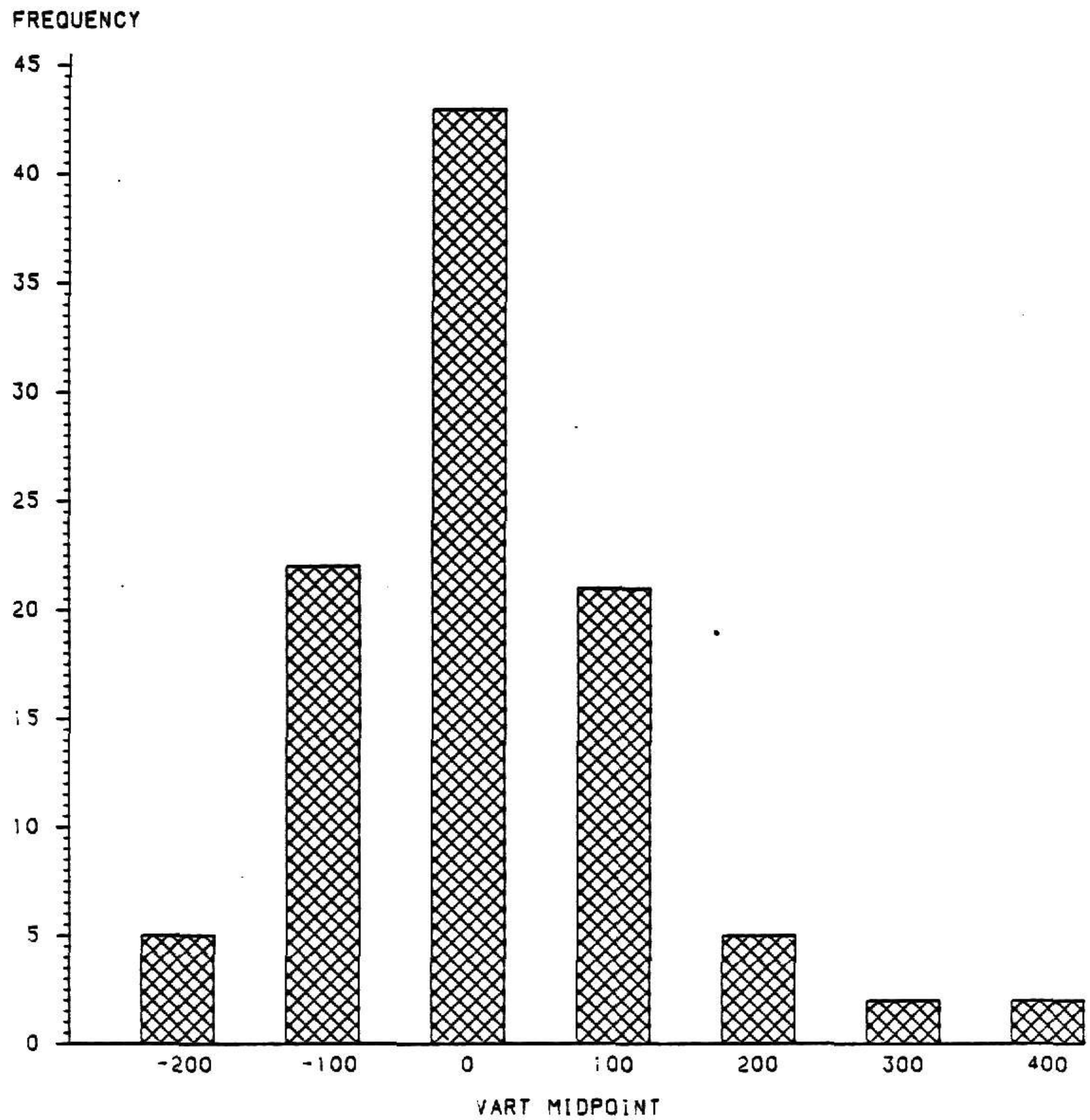


GRAPH 3.5.1: ESTIMATES OF THE VARIANCE OF A
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF A = 4.00

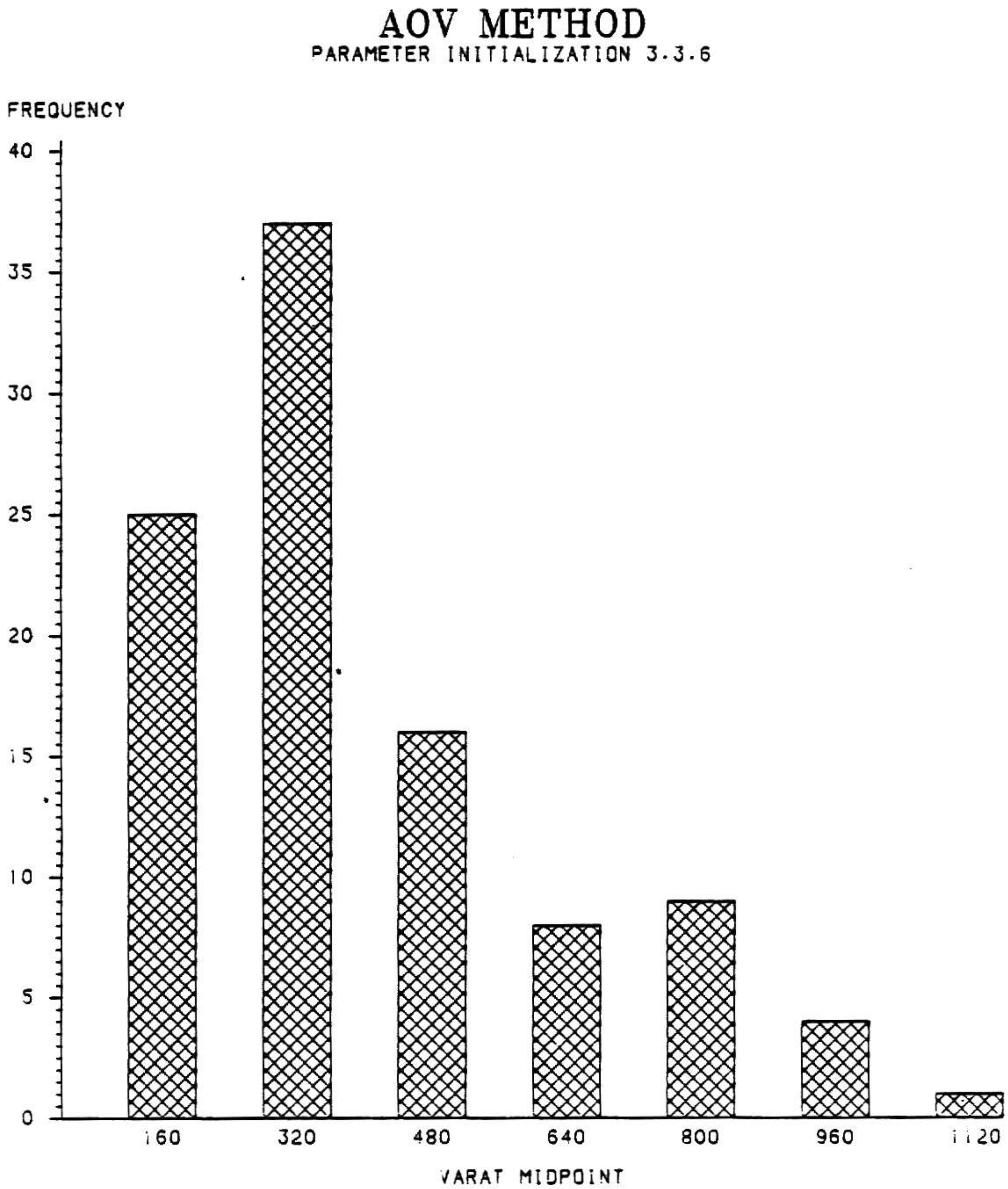
AOV METHOD

PARAMETER INITIALIZATION 3.3.6



GRAPH 3.5.2: ESTIMATES OF THE VARIANCE OF T
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF T = 9.00

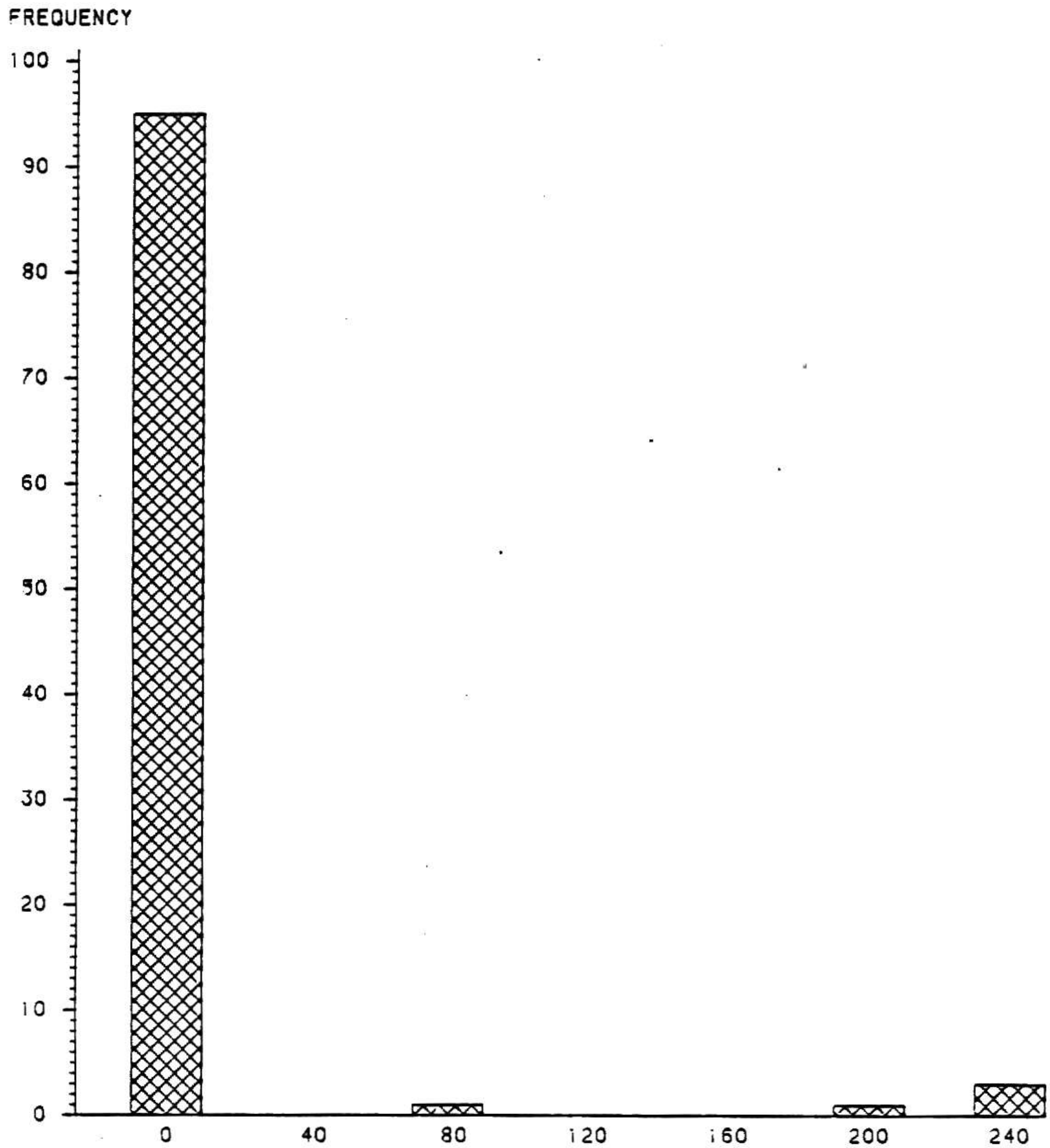


GRAPH 3.5.3: ESTIMATES OF THE VARIANCE OF AT
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF AT = 400.0

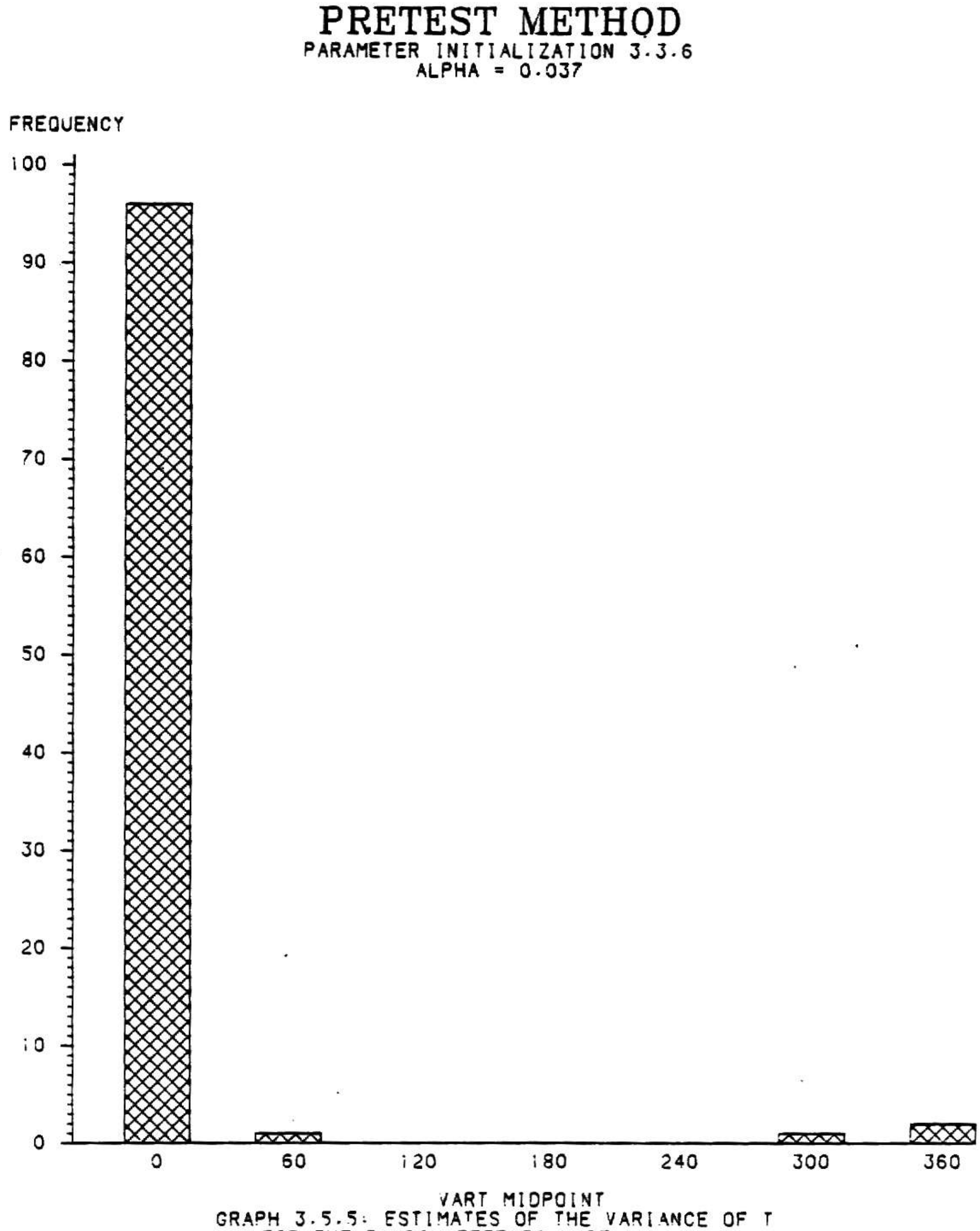
PRETEST METHOD

PARAMETER INITIALIZATION 3.3.6
ALPHA = 0.037



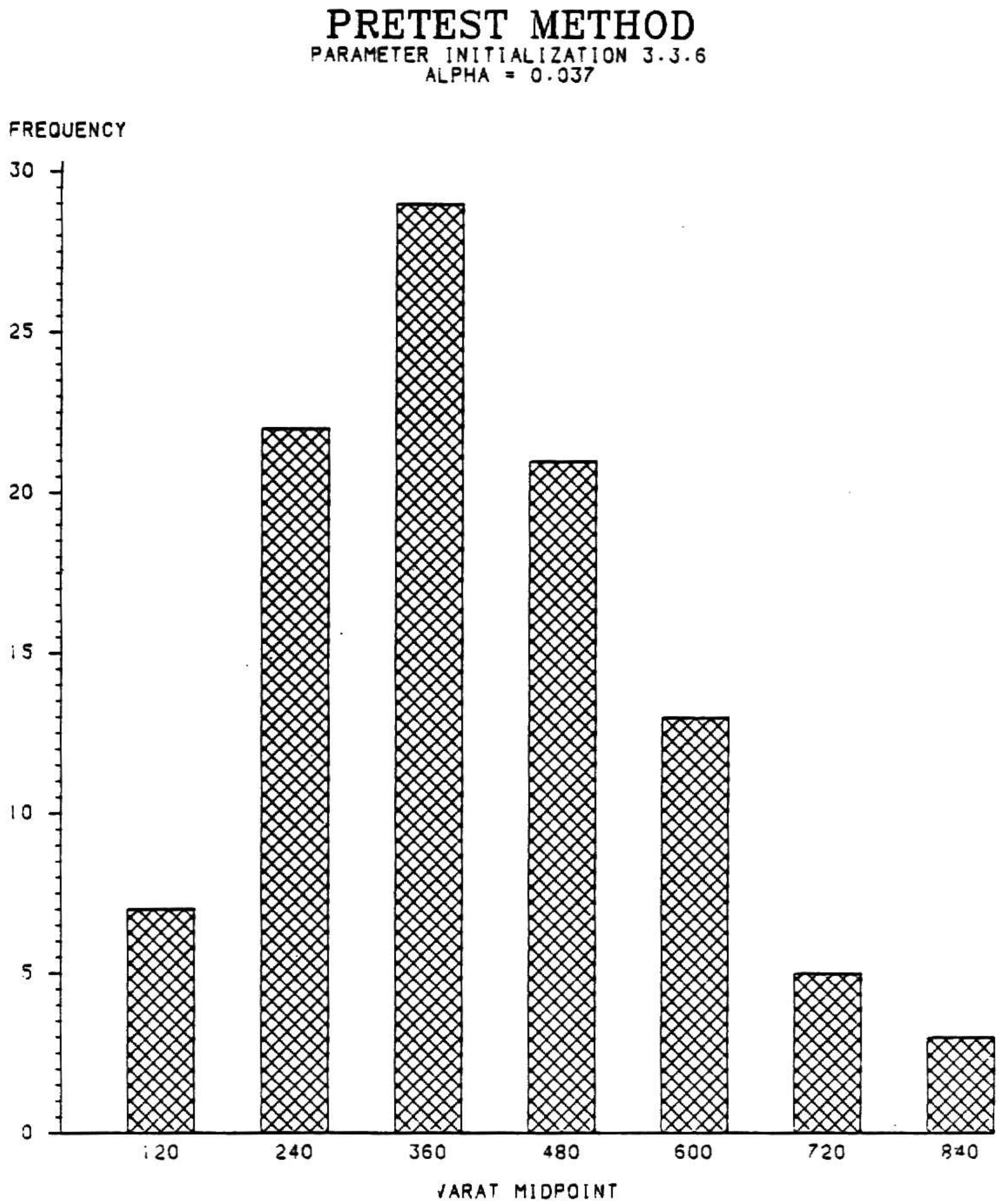
GRAPH 3.5.4: ESTIMATES OF THE VARIANCE OF A
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF A = 4.0



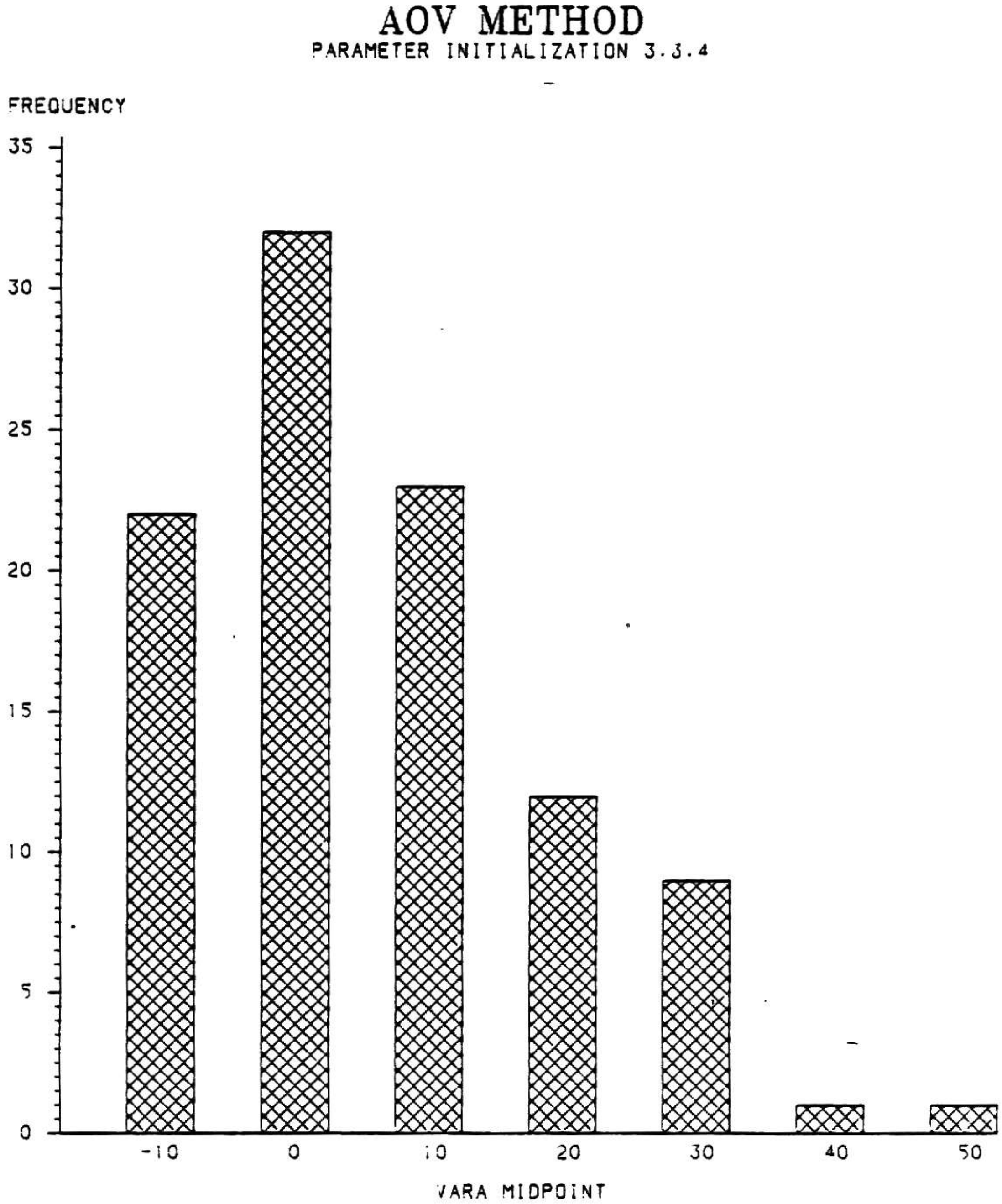
GRAPH 3.5.5: ESTIMATES OF THE VARIANCE OF T
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF T = 9.0



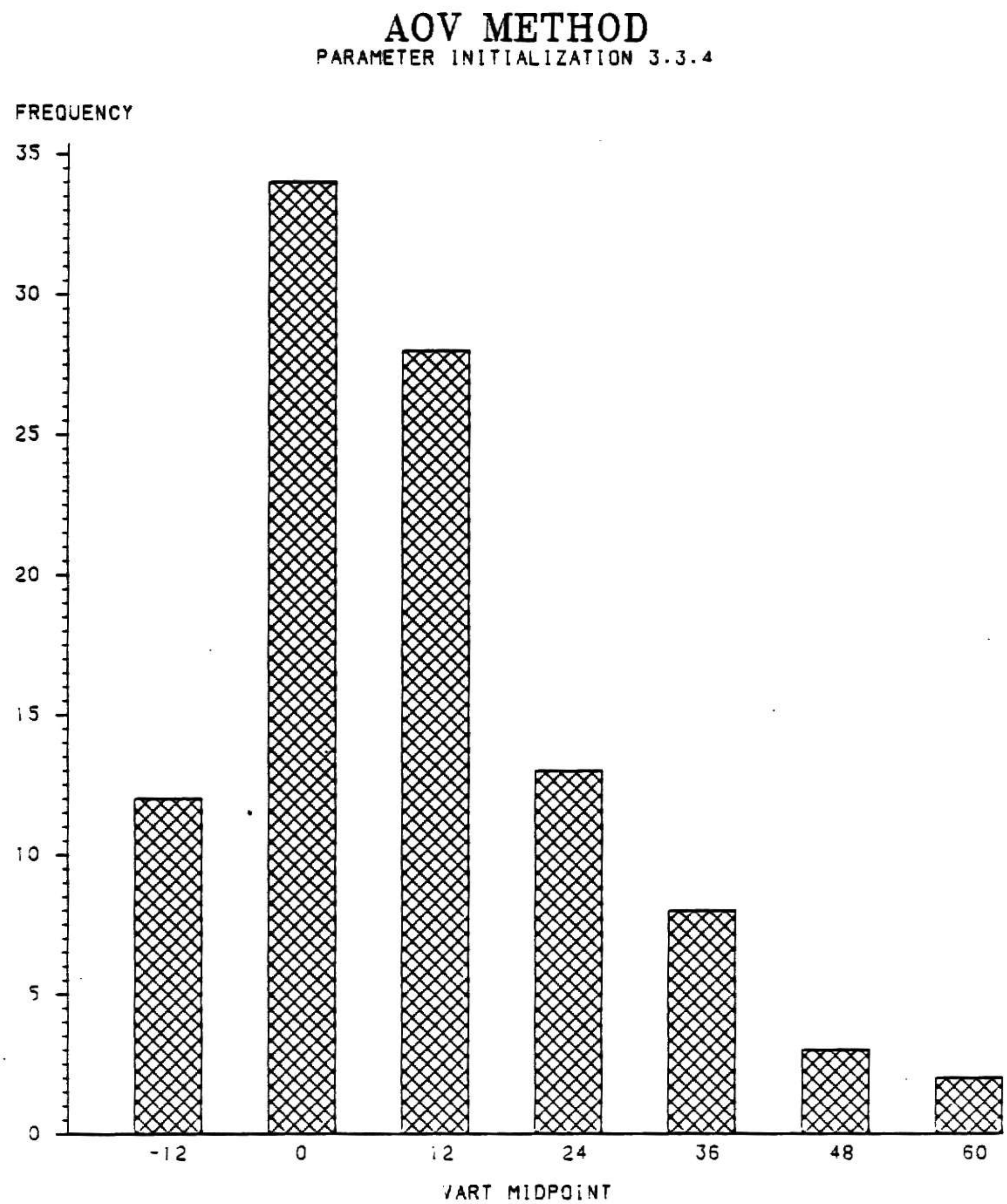
GRAPH 3.5.6: ESTIMATES OF THE VARIANCE OF AT
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF AT = 400.0



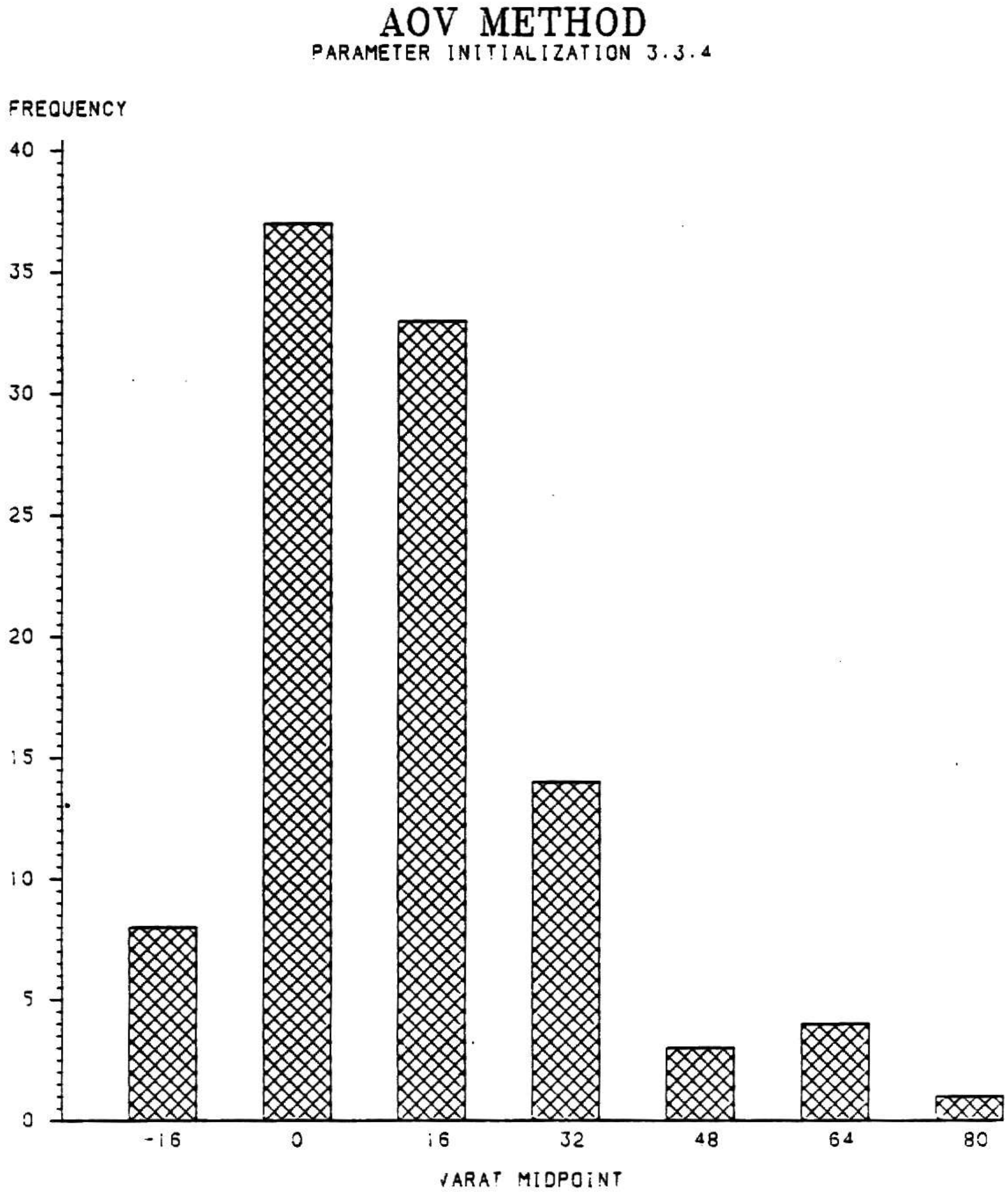
GRAPH 3.5.7: ESTIMATES OF THE VARIANCE OF A
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF A = 4.00



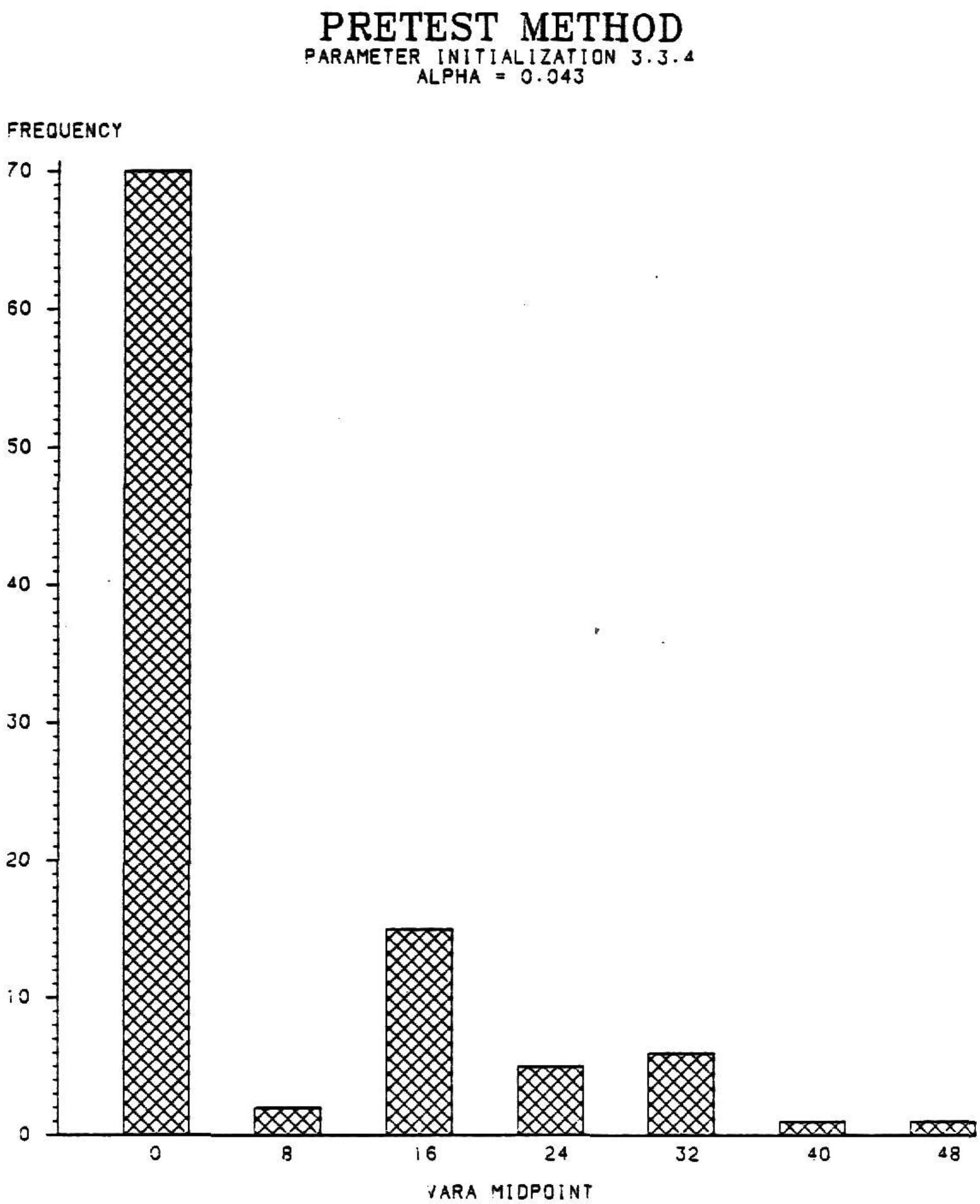
GRAPH 3.5.8: ESTIMATES OF THE VARIANCE OF T
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF T = 9.00



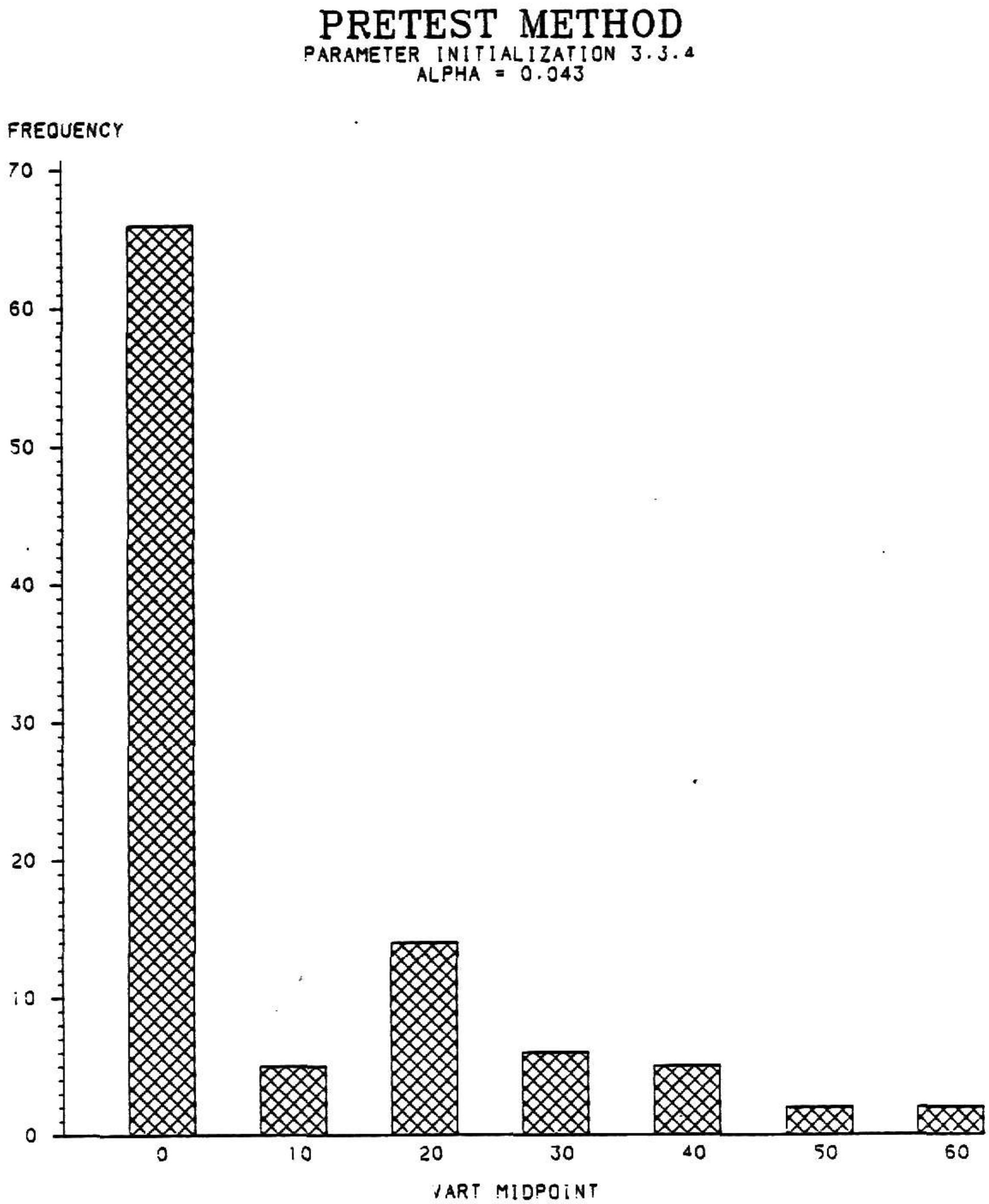
GRAPH 3.5.9. ESTIMATES OF THE VARIANCE OF AT
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF AT = 16.0



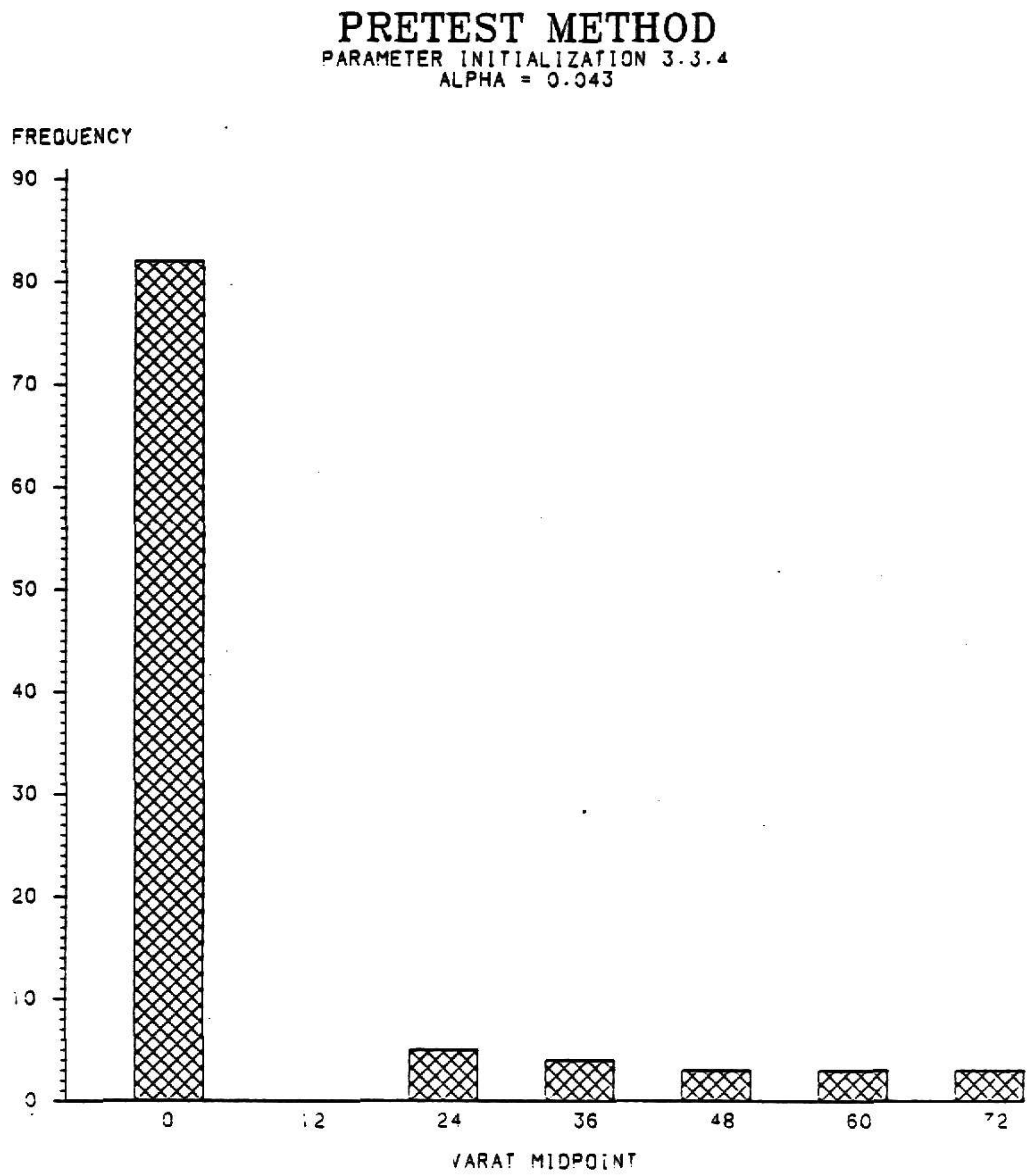
GRAPH 3.5.10: ESTIMATES OF THE VARIANCE OF A
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF A = 4.0



GRAPH 3.5.11: ESTIMATES OF THE VARIANCE OF T
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF T = 9.0

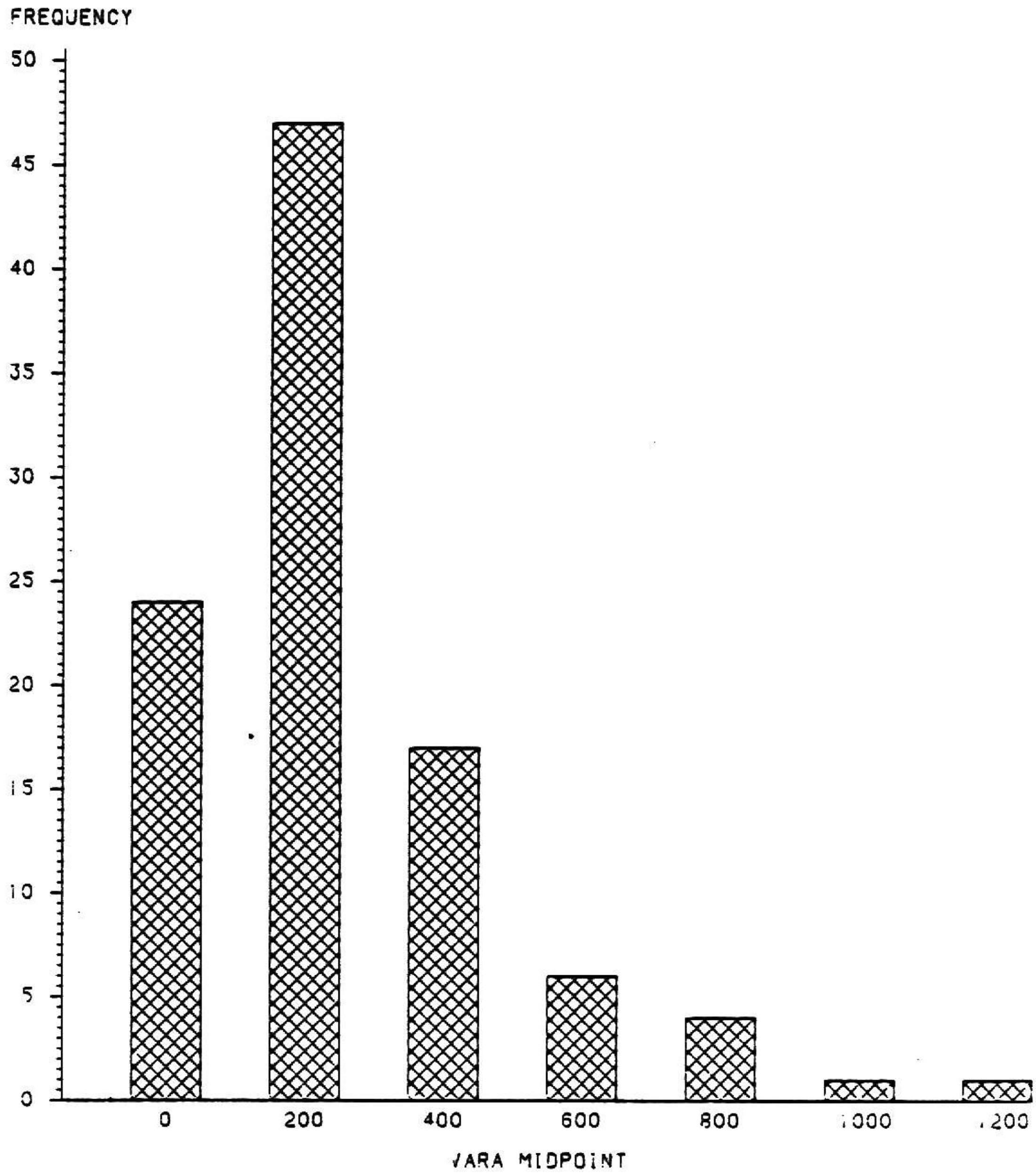


GRAPH 3.5.12. ESTIMATES OF THE VARIANCE OF AT
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF AT = 16.0

AOV METHOD

PARAMETER INITIALIZATION 3.3.5

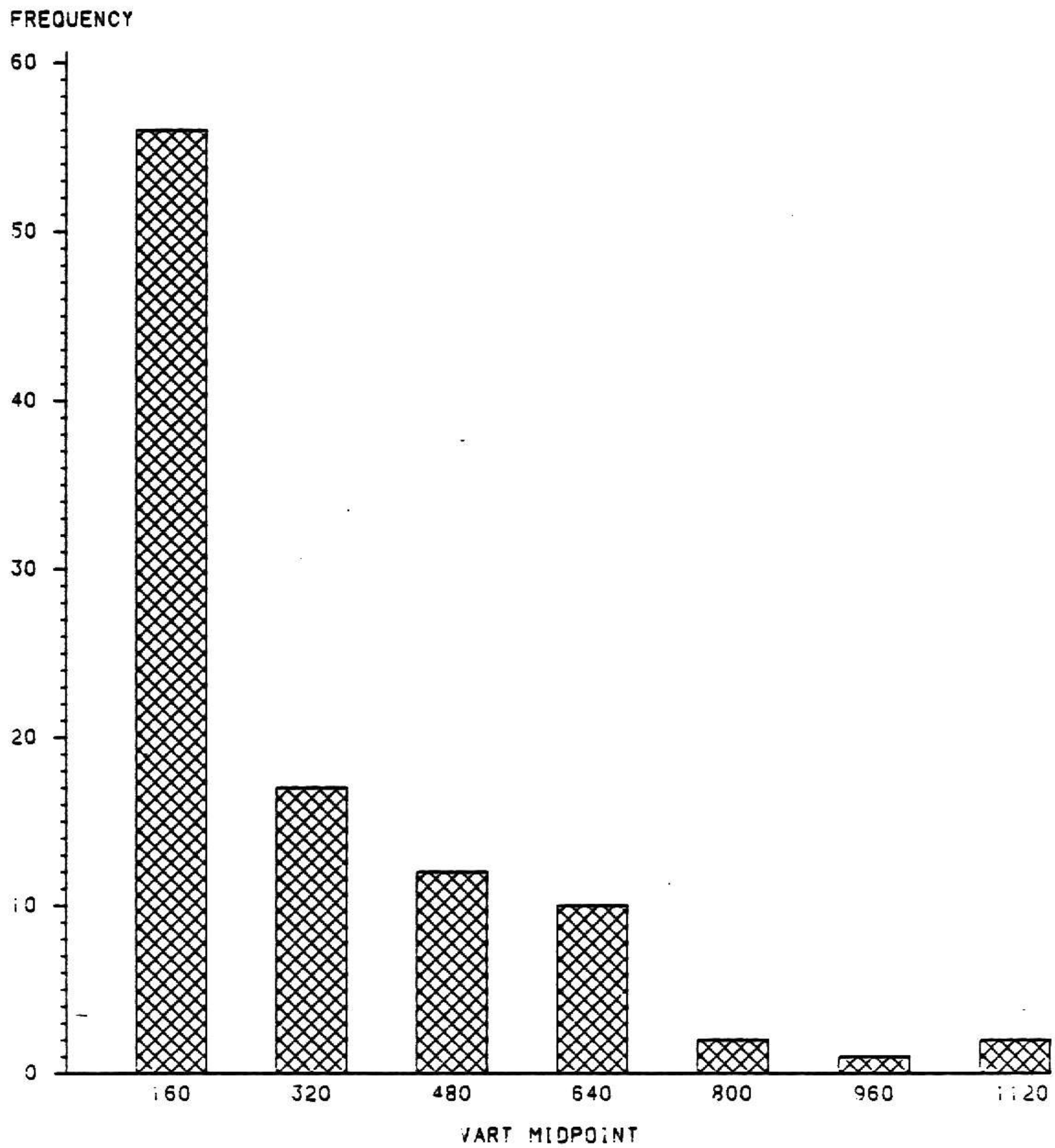


GRAPH 3.5.13: ESTIMATES OF THE VARIANCE OF A
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF A = 256.0

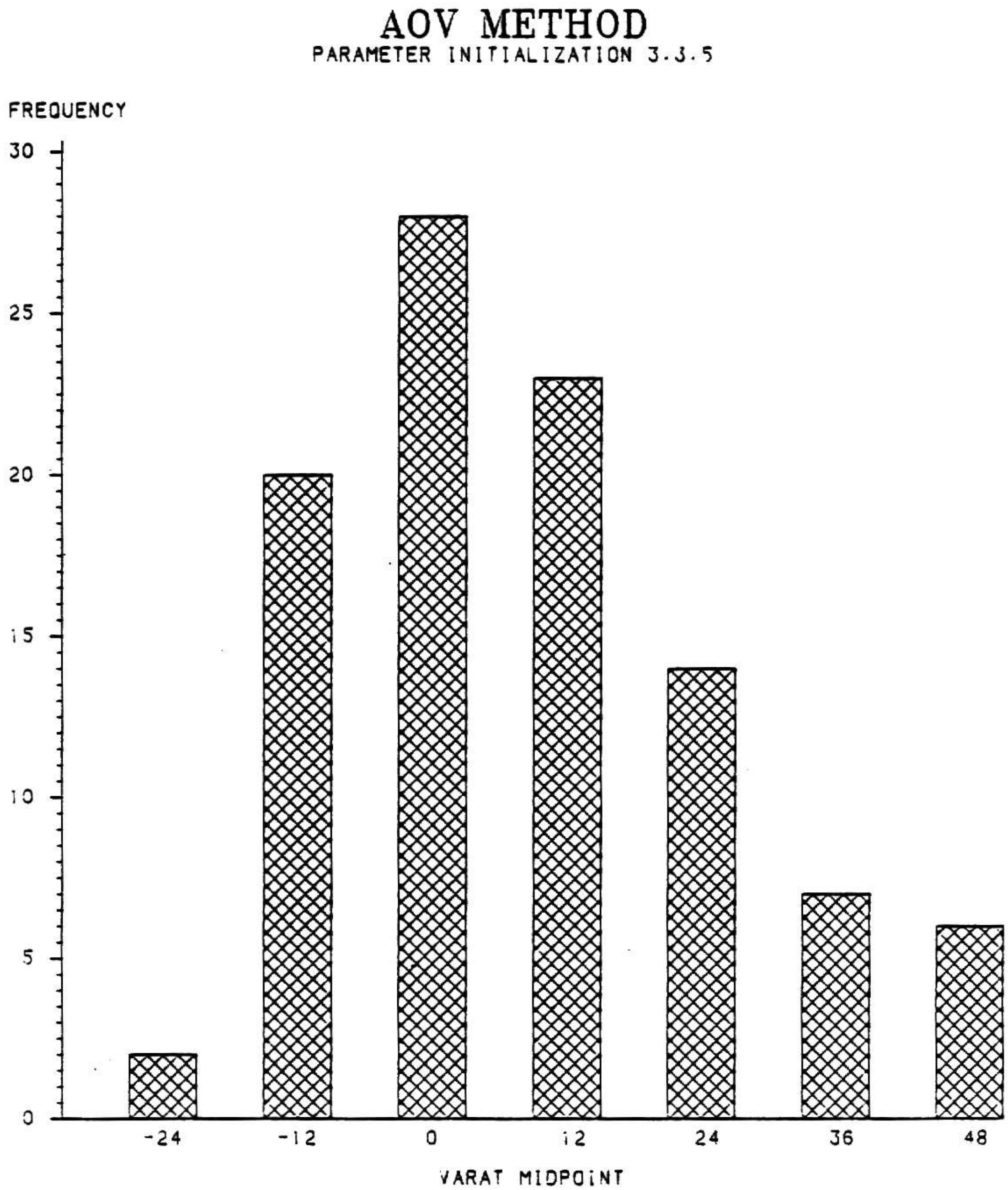
AOV METHOD

PARAMETER INITIALIZATION 3.3.5



GRAPH 3.5.14: ESTIMATES OF THE VARIANCE OF T
FOR THE RANDOM EFFECTS MODEL (1.3.2)

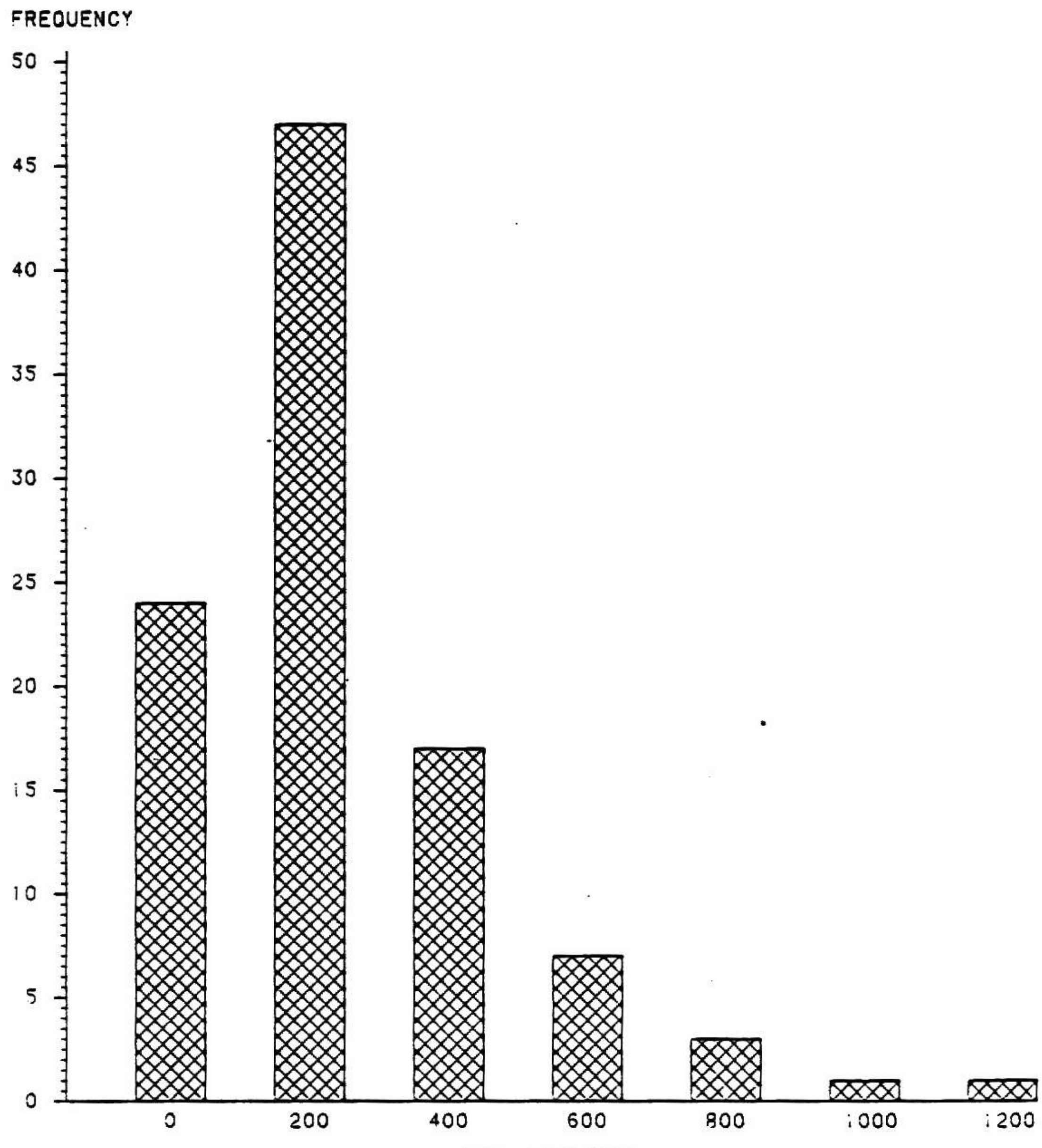
VARIANCE OF T = 324.0



GRAPH 3.5.15: ESTIMATES OF THE VARIANCE OF AT
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF AT = 9.0

PRETEST METHOD
PARAMETER INITIALIZATION 3.3.5
ALPHA = 0.095

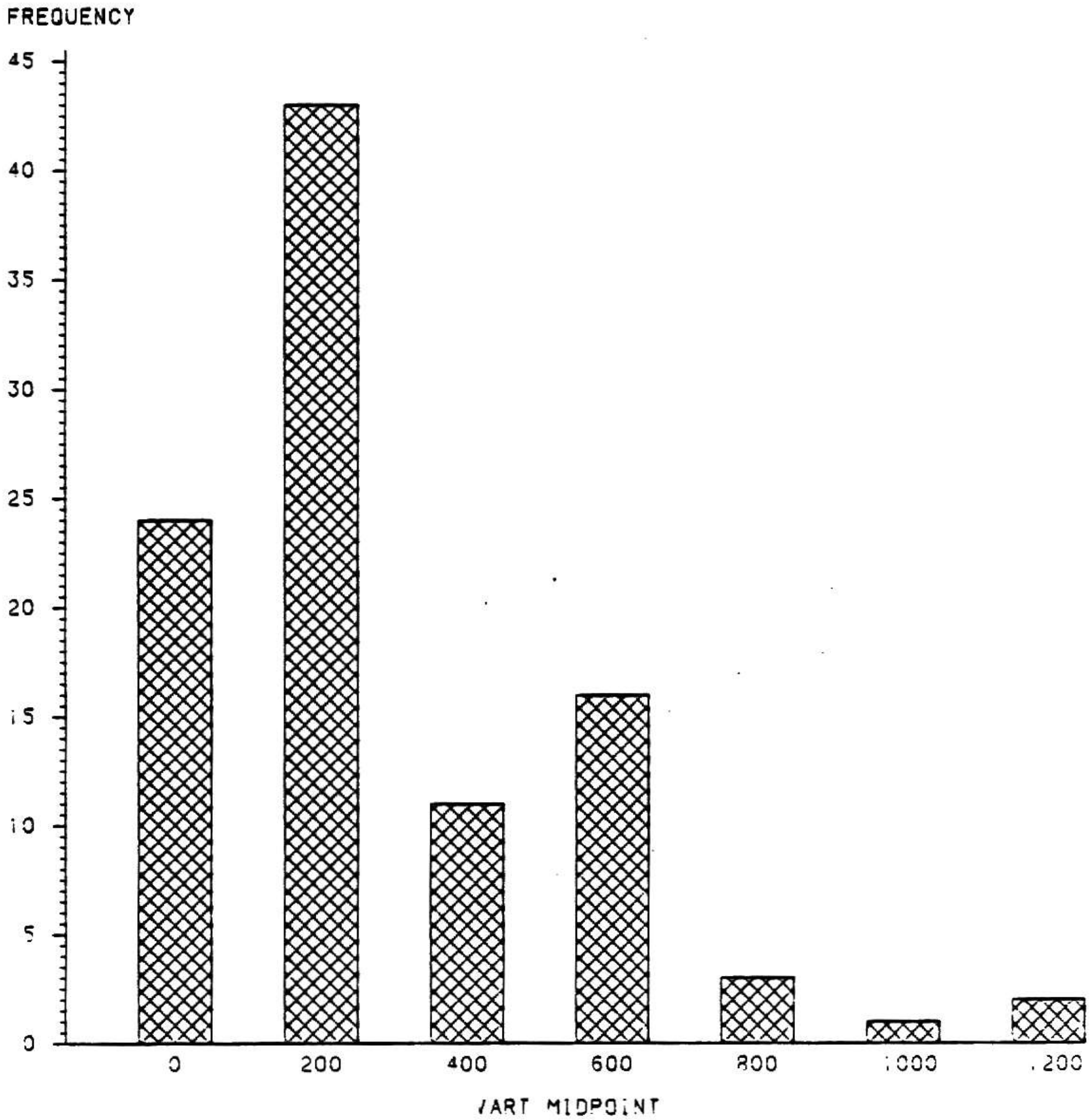


GRAPH 3.5.16: ESTIMATES OF THE VARIANCE OF A
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF A = 256.0

PRETEST METHOD

PARAMETER INITIALIZATION 3.3.5
ALPHA = 0.095

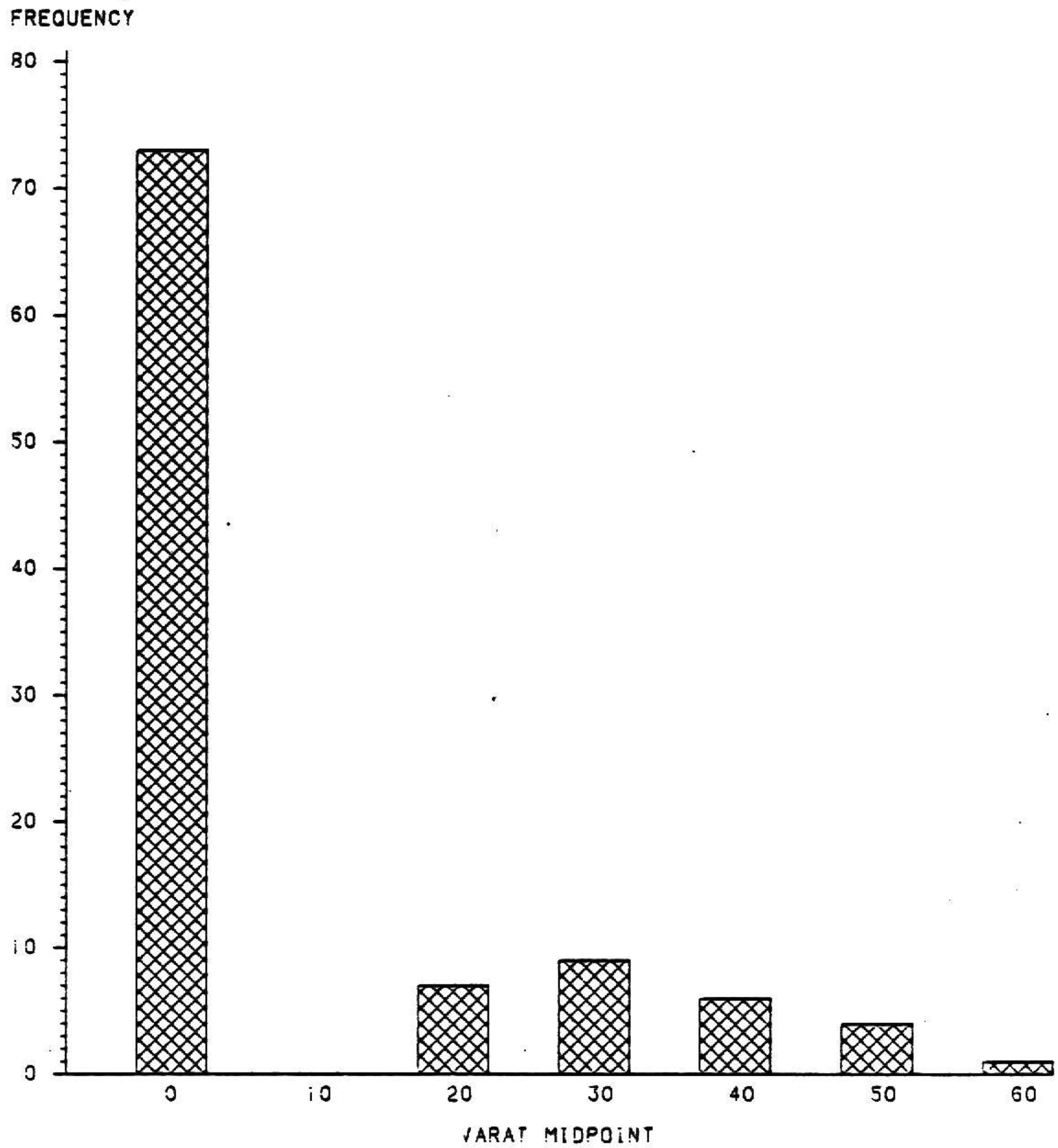


GRAPH 3.5.17: ESTIMATES OF THE VARIANCE OF T
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF T = 324.0

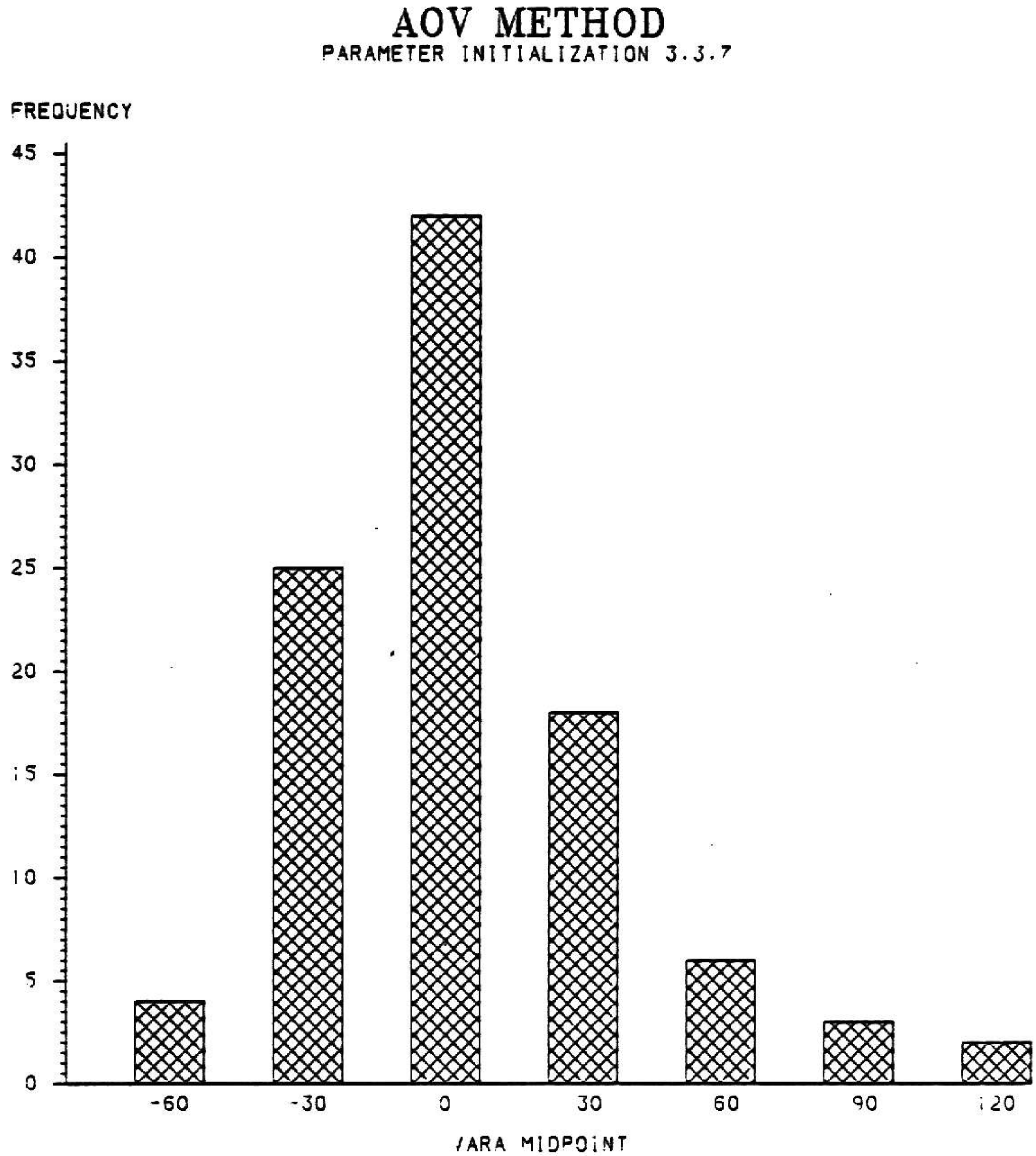
PRETEST METHOD

PARAMETER INITIALIZATION 3.3.5
ALPHA = 0.095



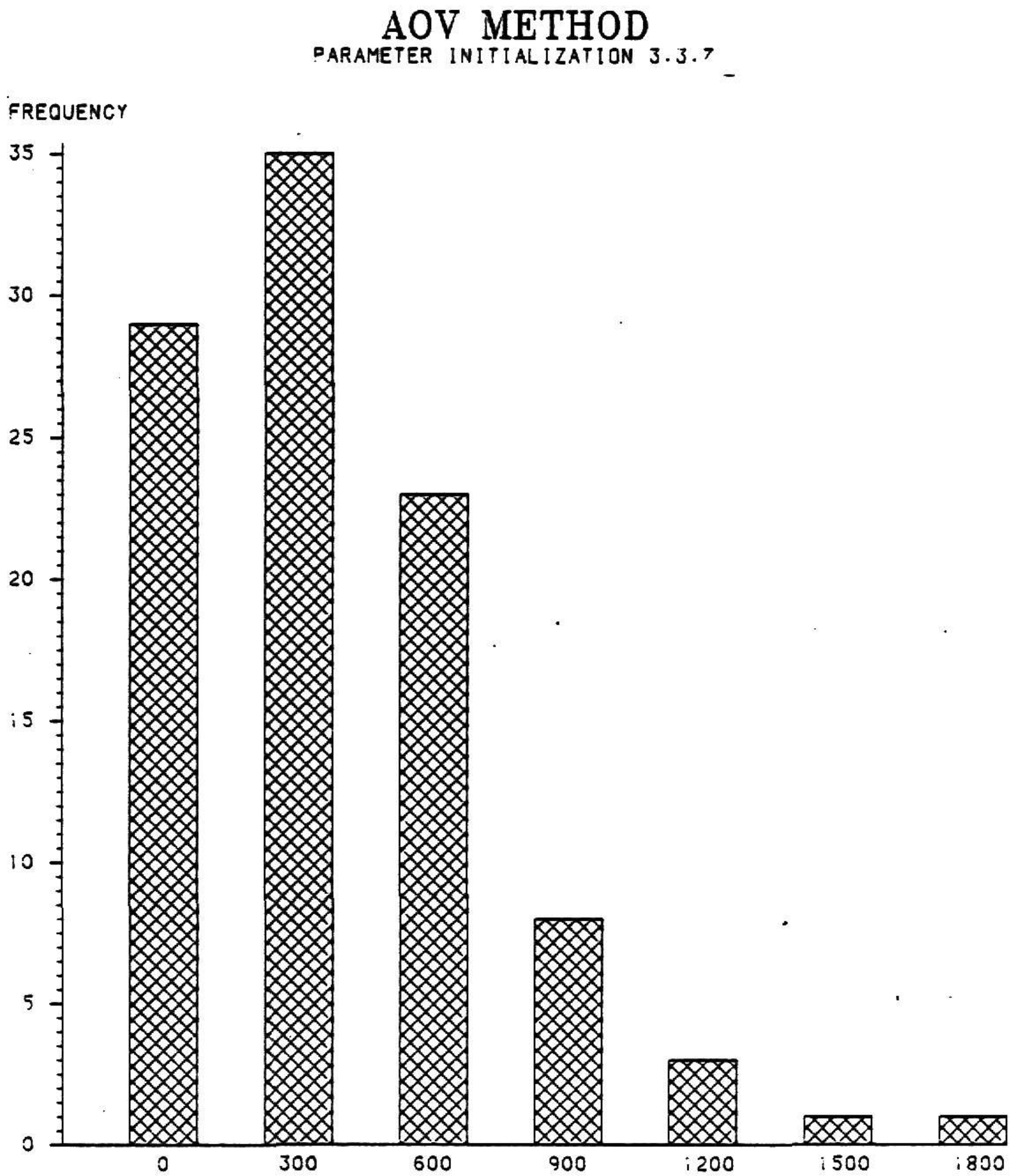
GRAPH 3.5..8: ESTIMATES OF THE VARIANCE OF AT
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF AT = 9.0



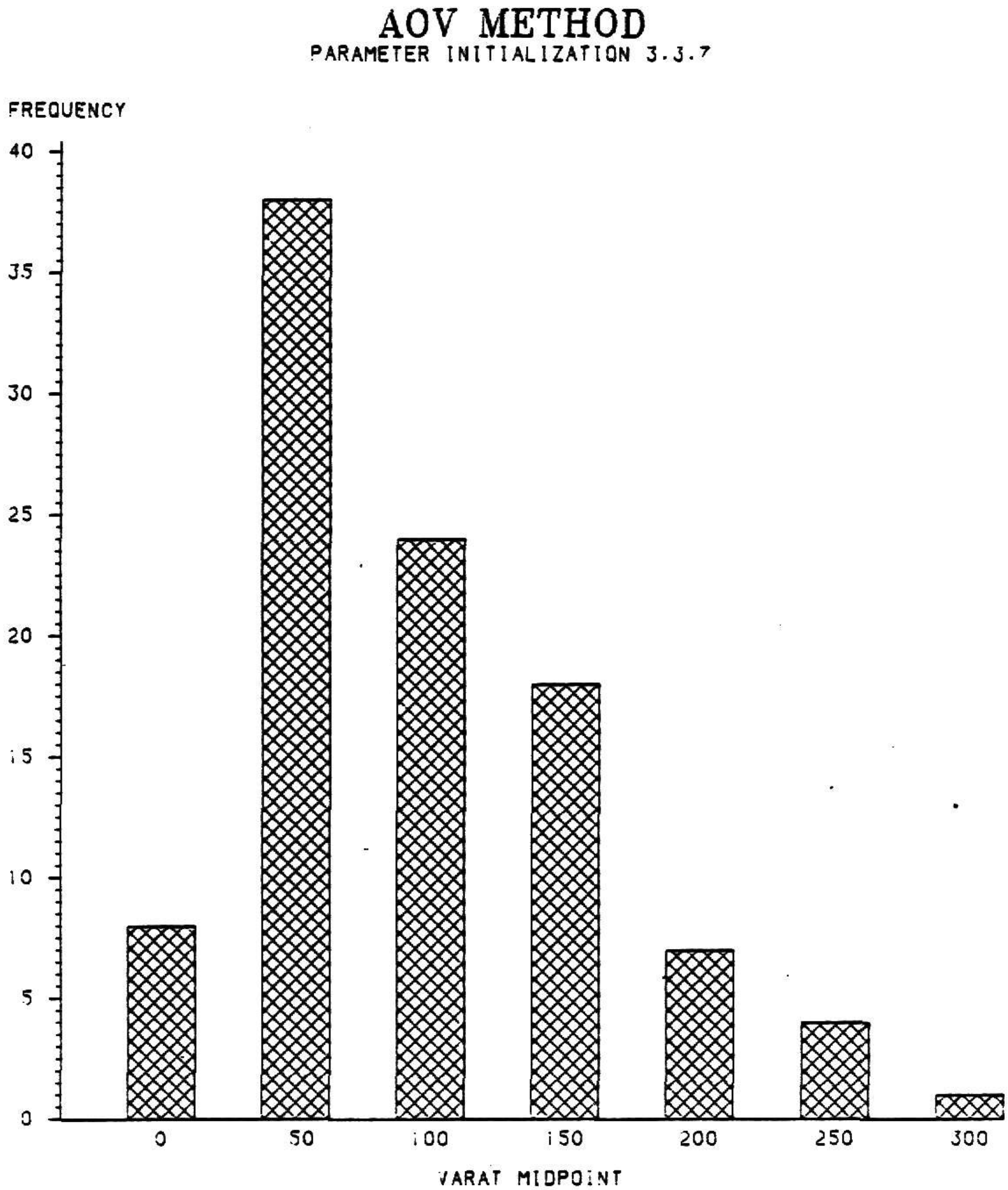
GRAPH 3.5.19: ESTIMATES OF THE VARIANCE OF A
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF A = 4.0



GRAPH 3.5.20: ESTIMATES OF THE VARIANCE OF T
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF T = 361.0

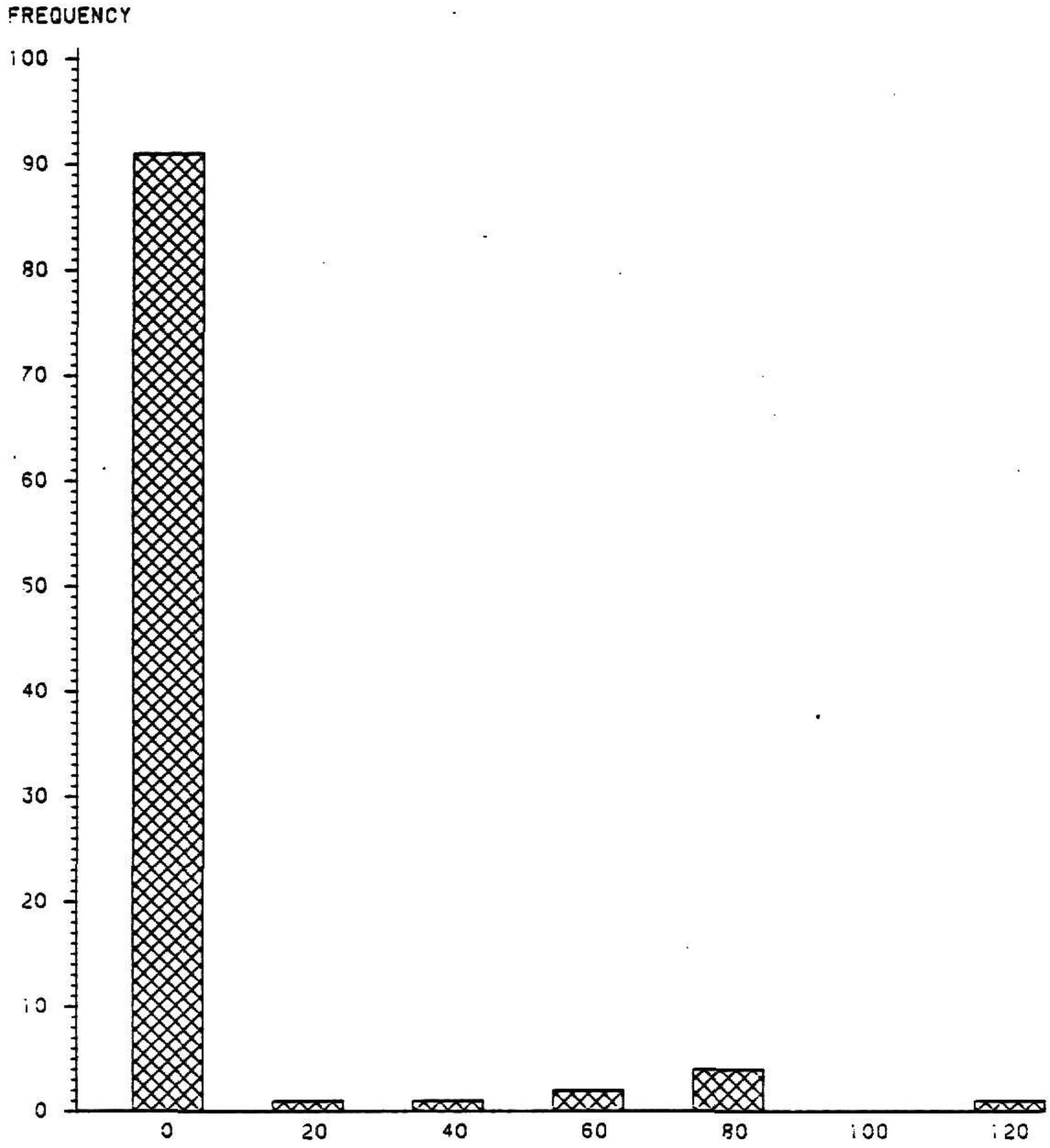


GRAPH 3.5.21: ESTIMATES OF THE VARIANCE OF AT
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF AT = 100.0

PRETEST METHOD

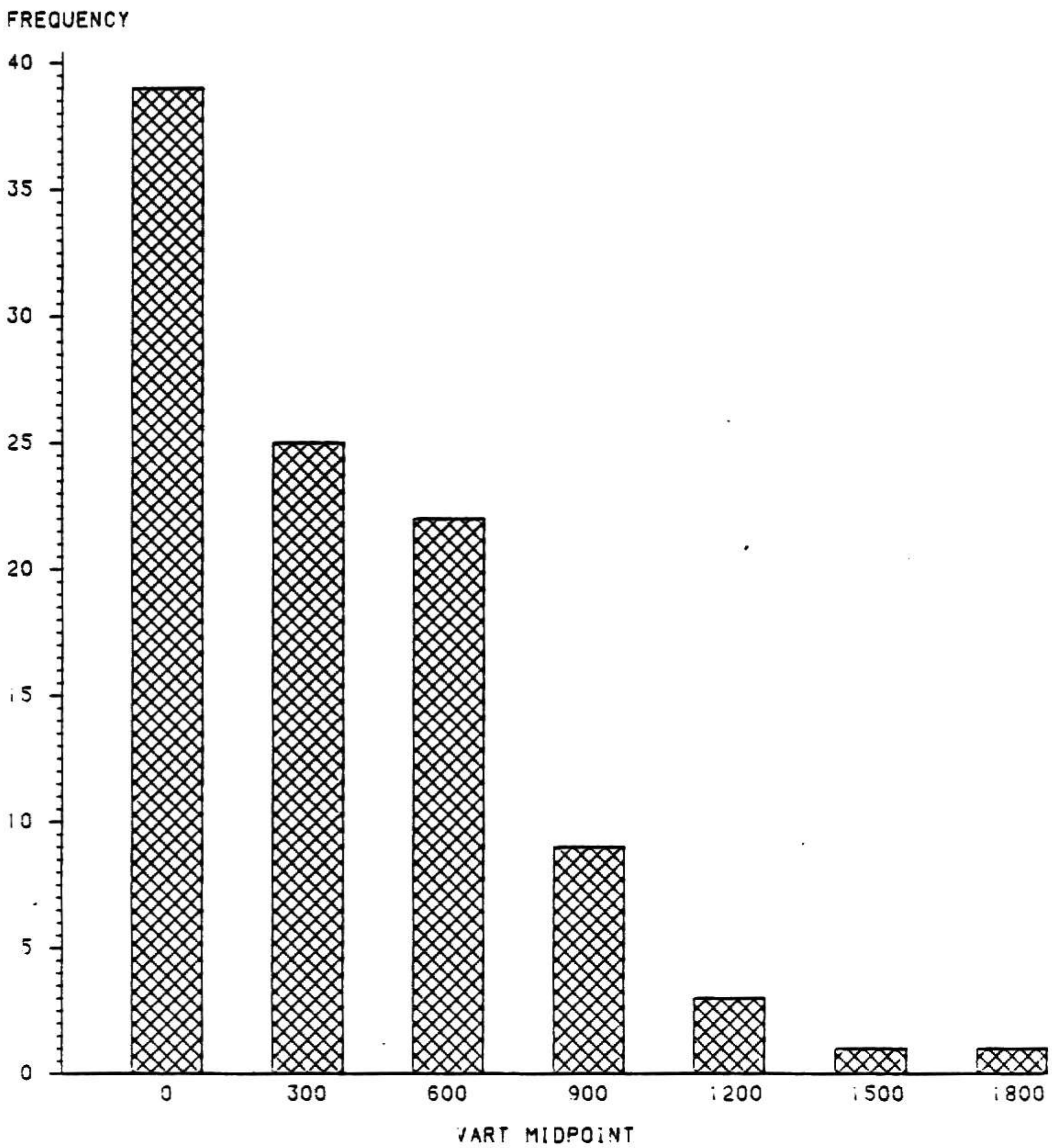
PARAMETER INITIALIZATION 3.3.7
ALPHA = 0.007



GRAPH 3.5.22: ESTIMATES OF THE VARIANCE OF A
FOR THE RANDOM EFFECTS MODEL (1.3.2)

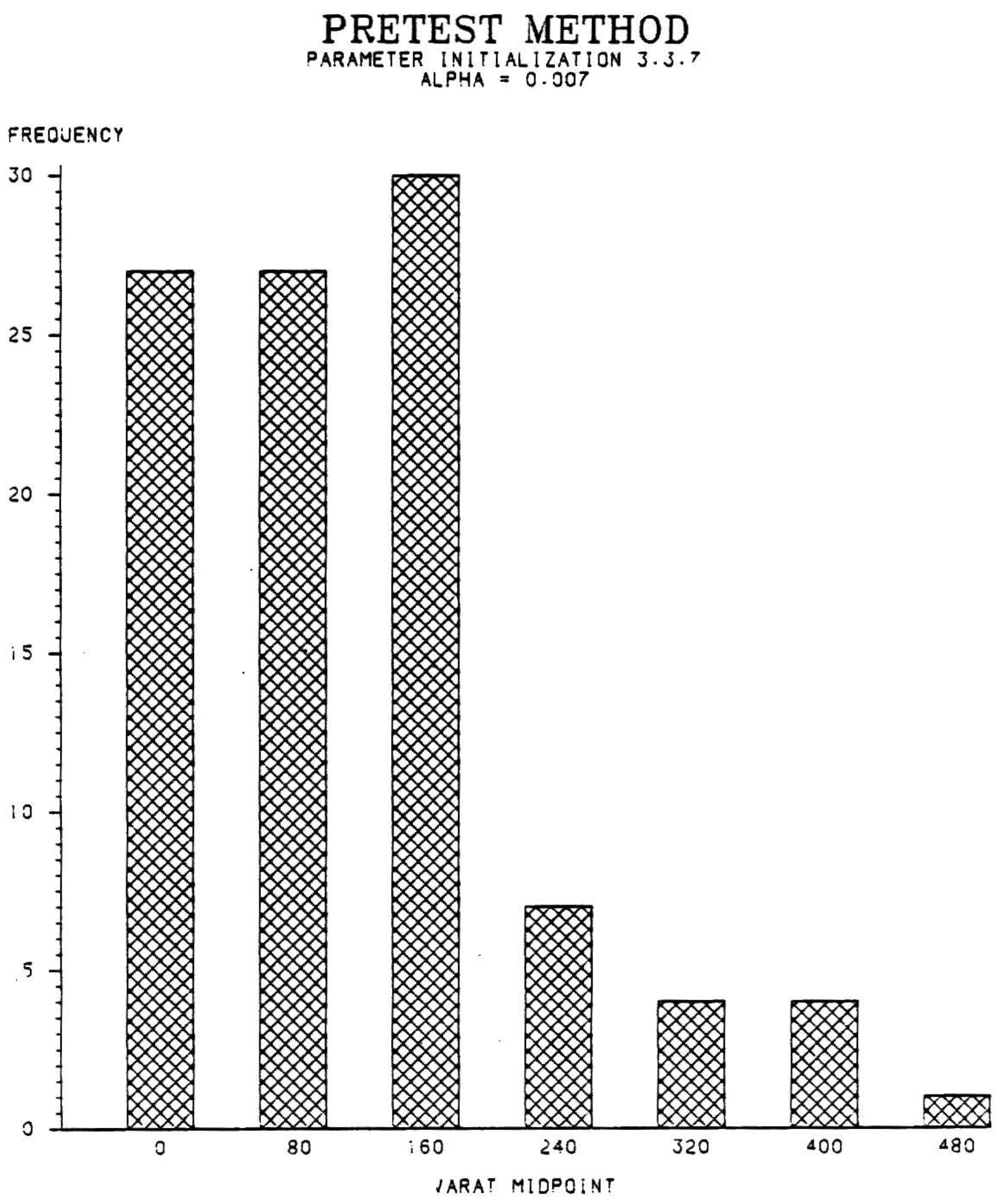
VARIANCE OF A = 4.0

PRETEST METHOD
PARAMETER INITIALIZATION 3.3.7
ALPHA = 0.007



GRAPH 3.5.23: ESTIMATES OF THE VARIANCE OF T
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF T = 361.0



GRAPH 3.5.24: ESTIMATES OF THE VARIANCE OF AT
FOR THE RANDOM EFFECTS MODEL (1.3.2)

VARIANCE OF AT = 100.0

CHAPTER 4

CONCLUSIONS

4.1 Summary

This chapter first reviews and restates the objective of the simulation study. Then, based on the simulation study, the advantages and disadvantages associated with each method of estimation are discussed. A concluding statement concerning the best method of estimating variance components is presented. Lastly, there is a section suggesting further investigations related to the topics of estimating variance components.

As was stated previously in chapter one, the objective of this paper is to compare three methods of estimating variance components for models (1.3.1) and 1.3.2).

The first method considered is the AOV method. This method provides estimators which are minimum variance and unbiased. The AOV method has one severe problem which is the possibility of observing a negative estimate. Negative estimates are more likely to be observed when the variance component being estimated is small relative to σ^2_e. When the variance component is large relative to σ^2_e, a negative estimate is less likely but still possible. Therefore two other methods were considered to see if the properties of the AOV estimates could be retained while at the same time eliminating the possibility of obtaining a negative estimate. The other two methods are the REML and Pretest method, which are both based on the AOV method with slight modification.

The REML method yields estimates that have a smaller mean square error than the AOV method, resulting from negative estimates being set equal to zero. These estimates also tend toward a negative average bias, i.e., they overestimate the respective parameters. The magnitude of the average bias associated with this method is directly related to the number of negative AOV estimates observed.

The simulation showed that the estimators based on the Pretest method have minimum mean square error when compared to the other estimates. The Pretest method was also shown to be nearly unbiased at some optional alpha value. The only obvious disadvantage of the Pretest method is determining an optimal alpha value to be employed.

It is suggested that the optimal alpha value to be employed in the Pretest method, when estimating variance components for both the two-way mixed and random effects models, be choosen in the interval .10 to .25.

To obtain a better feel for estimating a variance component the Pretest method should be employed at several alpha levels, using as a starting point the previously defined optimal alpha value.

It is the recommendation of this paper, based on both study and simulation, that the Pretest method yields better estimates of variance components with more accuracy than do the REML and AOV methods. Based on the simulation study, it is the conclusion of this paper that Pretest method can approach the properties of unbiasedness and minimum variance associated with the AOV method, while eliminating the possibility of observing negative estimates.

All conclusions from this report strictly apply to only those parameter configurations explored.

4.2 Additional Comments

Throughout the initial stages of the simulation study for the two-way mixed effects model an inconsistency was observed. A very small PR > F was observed when testing the hypothesis

$$H_0: \sigma^2_T = 0 \qquad \text{no T effect present}$$
$$H_a: \sigma^2_T > 0 \qquad \text{T effect present} \qquad (4.2.1)$$

for the following parameter configuration,

$$\sigma^2_T << \sigma^2_e << \sigma^2_{AT}.$$

The F statistic that yielded these erroneous results was computed as follows

$$F_T = MST/MSE,$$

based on the notes from ANALYSIS OF VARIANCE AND COVARIANCE, Statistics 704, and DESIGN OF EXPERIMENTS, Statistics 720.

Searle (1971) points out there are two different expected mean squares for the two-way mixed effects model, assuming a balanced design. These expected mean squares are presented in tables 4.2.1 and 4.2.2, where table 4.4.2 makes the assumption that

$$\sum_{i=1}^{a} \delta_{ij} = \delta_{.j} = 0 \quad \forall j \ ,$$

where a is the number of fixed effects

δ_{ij} is the interaction effect of i^{th} level of factor A and j^{th} level of factor T.

Table 4.2.1: EMS for two-way mixed model, balance data

Source	EMS
A	$\sigma^2_e + n\sigma^2_{AT} + \phi$
T	$\sigma^2_e + n\sigma^2_{AT} + an\sigma^2_T$
AT	$\sigma^2_e + n\sigma^2_{AT}$
ERROR	σ^2_e

Table 4.2.2: EMS for two-way mixed model, balance data with restriction on interaction effect

Source	EMS
A	$\sigma^2_e + n(\frac{a}{a-1})\ \sigma^2_{AT} + \phi$
T	$\sigma^2_e + na\sigma^2_T$
AT	$\sigma^2_e + n(\frac{a}{a-1})\sigma^2_{AT}$
ERROR	σ^2_e

The F statistics used to test hypothesis (4.2.1) are F_1 = MST/MSAT and F_2 = MST/MSE corresponding to tables 2.4.1 and 2.4.2, respectively.

Based on the simulation study, table 4.2.3 presents the approximate average value of the PR > F for both F_1 and F_2 for all parameter configurations defined in chapter three that were employed in the simulation study.

Table 4.2.3: Average PR > F values for F_1 and F_2 for hypothesis 4.2.1

	Parameter Configuration	PR > F_1	PR > F_2
1.	σ^2_T, $\sigma^2_{AT} << \sigma^2_e$	.3733	.3120
2.	$\sigma^2_{AT} << \sigma^2_e << \sigma^2_T$	.0306	.0001
3.	$\sigma^2_T << \sigma^2_e << \sigma^2_{AT}$	.4148	.0047

For the first two parameter configurations there appears to be no differences in conclusions based on testing hypothesis 4.2.1 by both F statistics. Although for the third configuration, the F statistics yield extremely different conclusions. F_1, yields a more realistic interpretation than does F_2 for the third parameter configuration.

Based on this simulation, for the parameter initializations defined in chapter three, F_1 is recommended as the F statistic to be used when testing hypothesis (4.2.1). For a further investigation of how realistic the assumption is and where it is applicable the reader is referred to chapter nine of Searle (1971) and also Hartley and Searle (1969).

References

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APPENDIX A

```
C
C
C  A COMPUTER SIMULATION STUDY FOR THE TWO-WAY MIXED EFFECTS MODEL
C
C
      IMPLICIT  REAL*8(A-H,O-Z)
      DIMENSION CELL(5,5),ROW(5),COL(5),A(5)
      READ(5,501) SIGT,SIGAT,SIGE,DMU,IR,IC,IK
  501 FORMAT(4F10.3,3I2)
      WRITE(6,601) SIGT,SIGAT,SIGE,DMU,IR,IC,IK
  601 FORMAT('1',4F10.3,3I4)
      READ(5,502) (A(I),I=1,IR)
  502 FORMAT(5F10.3)
      WRITE(6,602) (A(I), I=1,IR)
  602 FORMAT(' ',5F10.3)
C
C    THIS PROGRAM WILL GENERATE DATA FOR A TWO-WAY ANOVA MIXED MODEL
C    WITH INTERACTION.  IT WILL WORK FOR UP TO FIVE LEVELS OF THE
C    RANDOM EFFECT, FIVE LEVELS OF THE FIXED EFFECT AND FIVE
C    REPLICATIONS PER TREATMENT COMBINATIONS.  TO EXPAND THE SIZE OF
C    THE DESIGN YOU WILL NEED TO MODIFY JCL PARAMETERS AND INITIAL
C    VALUES OF MAXIMUM ROW(IR), COL(IC), AND REPLICATION(IK).
C
C   INITIAL SEED VALUES FOR NORMAL NUMBER GENERATOR
C
      IX=467584331
      IY=222658302
      CALL RSTART(IX,IY)
      DO 93 M = 1,10
      DO 80 I = 1,5
            ROW(I)=0.0D0
            COL(I)=0.0D0
            DO 81  J = 1, 5
                  CELL(I,J)=0.0D0
 81         CONTINUE
 80   CONTINUE
      TOTAL=0.0D0
      SSQS=0.0D0
      SSQCEL=0.0D0
      SSQROW=0.0D0
      SSQCOL=0.0D0
C
C  GENERATE DATA ACCORDING TO MODEL (1.3.1)
C
      CALL MIXGEN(IC,IR,IK,SIGT,DMU,A,SIGAT,SIGE,TOTAL,SSQS,CELL,ROW,COL
     +,SSQCEL,SSQCOL,SSQROW,M)
C
C  COMPUTE COMPONENTS FOR AOV TABLE
C
      AA1=SSQROW/(IK*IC)
      BB2=TOTAL**2/(IR*IC*IK)
      CC3=SSQCOL/(IK*IR)
      DD4=SSQCEL/IK
      SSA=AA1-BB2
      SST=CC3-BB2
      SSAT=DD4-AA1-CC3+BB2
      SSERR=SSQS-DD4
      SSTOT=SSQS-BB2
      IADF=(IR-1)
      ITDF=(IC-1)
```

```
      IATDF=(IR-1)*(IC-1)
      IEDF=(IR*IC*(IK-1))
      ITOTDF=IR*IC*IK - 1
      DMSA=SSA/IADF
      DMST=SST/ITDF
      DMSAT=SSAT/IATDF
C
C  COMPUTE THE AOV ESTIMATES OF THE VARIANCE COMPUNENTS
C
      DMSE=SSERR/IEDF
      VART =(DMST-DMSAT)/(IK*IR)
      VARAT=(DMSAT-DMSE)/IK
C
      FA=DMSA/DMSAT
      FT=DMST/DMSAT
      FAT=DMSAT/DMSE
      ALPA=FPROB(IADF,IATDF,FA)
      ALPT=FPROB(ITDF,IATDF,FT)
      ALPAT=FPROB(IATDF,IEDF,FAT)
C
C  PUT AOV ESTIMATES OUT ON DISK
C
      WRITE(12) DMSE, VART, VARAT
C
C  COMPUTE ESTIMATES BASED ON THE PRETEST METHOD
C
      CALL PRETST(ALPT,ALPAT,SSERR,SSAT,SST,ITDF,IATDF,IEDF,DMST,DMSAT,D
     +MSE,IC,IR,IK,M)
 93   CONTINUE
      STOP
      END
      FUNCTION FPROB(U,V,F)                                             FPR0001
C
C          THIS ROUTINE WILL GIVE THE PROBABILITY > FCOMP
C
CFPROC      F-PROBABILITY VIA CONVERGENT SERIES OF POSITIVE TERMS       FPR0002
C                                                                       FPR0003
      REAL*8 F
      INTEGER U,V
      A=.5*U                                                            FPR0004
      B=.5*V                                                            FPR0005
      TEMP=B+A*F                                                        FPR0006
      X=A*F/TEMP                                                        FPR0007
      FPROB=1.0                                                         FPR0008
      IF(F.LE.0.0.OR.X.LE.0.0)RETURN                                    FPR0009
      XC=B/TEMP                                                         FPR0010
      AB=A+B                                                            FPR0011
      CON=0.                                                            FPR0012
      SGN=+1.                                                           FPR0013
      IF(F-1.) 110,120,120                                              FPR0014
  110 TEMP=A                                                            FPR0015
      A=B                                                               FPR0016
      B=TEMP                                                            FPR0017
      TEMP=XC                                                           FPR0018
      XC=X                                                              FPR0019
      X=TEMP                                                            FPR0020
      CON=1.                                                            FPR0021
      SGN=-1.                                                           FPR0022
C     CONVERGENT SERIES EXPANSION - SEE AMS-55 PG. 944                  FPR0023
  120 TOP=AB                                                            FPR0024
```

```
      BOT=B+1.
      SUM=1.
      TERM=1.
  130 TEMP=SUM
      TERM=TERM*(TOP/BOT)*XC
      SUM=SUM+TERM
      TOP=TOP+1.
      BOT=BOT+1.
      IF(SUM-TEMP) 140,140,130
  140 FPROB=CON+SGN*EXP(A*ALOG(X)+B*ALOG(XC)+ALGAMA(AB)-ALGAMA(A)
     1 -ALGAMA(B))*SUM/B
  150 RETURN
      END
      SUBROUTINE MIXGEN(IC,IR,IK,SIGT,MU,A,SIGAT,SIGE,TOTAL,SSQS,CELL,RO
     +W,COL,SSQCEL,SSQCOL,SSQROW,M)
      IMPLICIT  REAL*8(A-H,O-Z)
      DIMENSION A(IR),CELL(IR,IC),ROW(IR),COL(IC),SIGTT(5),SIGAT1(5,5)
      REAL*8  MU
C
C THIS SUBROUTINE WILL GENERATE DATA FOR A TWO-WAY DESIGN, MIXED
C MODEL WITH AT MOST 5 LEVELS OF RANDOM EFFECT,FIXED EFFECT, AND
C REPLICATION.
C
      DO 4 J = 1,IC
      SIGTT(J) = SIGT*RNOR(0)
           DO 5 I = 1,IR
           SIGAT1(I,J) = SIGAT*RNOR(0)
  5        CONTINUE
  4   CONTINUE
      DO 1 J=1,IC
          DO 2 I=1,IR
              DO 3 N=1,IK
              OBS = MU + A(I) + SIGTT(J) + SIGAT1(I,J) + SIGE*RNOR(0)
              TOTAL=TOTAL + OBS
              SSQS=SSQS + OBS**2
              CELL(I,J)=CELL(I,J) + OBS
              ROW(I)= ROW(I) + OBS
              COL(J)= COL(J) + OBS
  3           CONTINUE
          SSQCEL=SSQCEL + CELL(I,J)**2
  2       CONTINUE
      SSQCOL=SSQCOL + COL(J)**2
  1   CONTINUE
      DO 82 L=1,IR
            SSQROW=SSQROW + ROW(L)**2
 82   CONTINUE
      RETURN
      END
C
C
      SUBROUTINE PRETST(ALPT,ALPAT,SSERR,SSAT,SST,ITDF,IATDF,IEDF,DMST,D
     +MSAT,DMSE,IC,IR,IK,M)
C
C  THE PRETEST METHOD IS EMPLOYED AT THE FOLLOWING ALPHA VALUES
C
      IMPLICIT REAL*8(A-H,O-Z)
      DO 10 I=1,5
      IF(I.EQ.1) ALPHA=.01D0
      IF(I.EQ.2) ALPHA=.05D0
      IF(I.EQ.3) ALPHA=.10D0
```

```
      IF(I.EQ.4) ALPHA=.25D0
      IF(I.EQ.5) ALPHA=.50D0
      ALPT1=0.0D0
      IPATH=0
      ERR=DMSE
C
C  IS INTERACTION PRESENT?
C
      IF(ALPHA.GT.ALPAT)                          GOTO 100
      VARAT=0.0D0
      SSE1=SSERR + SSAT
      IDF1=IEDF + IATDF
      DMSE1=SSE1/IDF1
      FTT=DMST/DMSE1
      ALPT1=FPROB(ITDF,IDF1,FTT)
C
C  NO INTERACTION, IS T EFFECT PRESENT?
C
            IF(ALPHA.GT.ALPT1)                    GOTO 200
            VART=0.0D0
            SSE2=SSERR + SST + SSAT
            IDF2=IEDF + ITDF + IATDF
            DMSE2=SSE2/IDF2
            ERR=DMSE2
            IPATH=1
            GOTO 999
  200       VART=(DMST-DMSE1)/(IK*IR)
            ERR=DMSE1
            IPATH=2
            GOTO 999
  100 VARAT=(DMSAT-DMSE)/IK
C
C  INTERACTION PRESENT, IS T EFFECT PRESENT?
C
            IF(ALPHA.GT.ALPT)                     GOTO 500
            VART=0.0D0
            SSAT3=SSAT+SST
            IDF3=IATDF + ITDF
            DMSAT3=SSAT3/IDF3
            VARAT=(DMSAT3 -DMSE)/IK
            ERR=DMSE
            IPATH=3
            GOTO 999
  500       VART=(DMST-DMSAT)/(IK*IR)
            IPATH=4
C
C  PLACE THE PRETEST ESTIMATES OUT ON DISK
C
  999       WRITE(14) I,VART,VARAT,ERR,IPATH
   10 CONTINUE
      RETURN
      END
//LKED.SYSPRINT DD DUMMY
//GO.FT12F001  DD DSN=&&TEMP1,DISP=(NEW,PASS),UNIT=SYSDA,
//     SPACE=(TRK,(30,30),RLSE),DCB=(RECFM=VBS,LRECL=28,BLKSIZE=3200)
//GO.FT14F001  DD DSN=&&TEMP2,DISP=(NEW,PASS),UNIT=SYSDA,
//     SPACE=(TRK,(30,30),RLSE),DCB=(RECFM=VBS,LRECL=36,BLKSIZE=3200)
//GO.SYSIN DD *
    3.0        18.0        10.0        10.0    3 4 4
  -25.0         0.0        25.0
```

```
//   EXEC SAS,REGION=650K
//DISK1  DD DSN=&&TEMP1,DISP=(OLD,DELETE),UNIT=SYSDA
//DISK3  DD DSN=&&TEMP2,DISP=(OLD,DELETE),UNIT=SYSDA
//SYSIN DD *
DATA ESTIMATE;
INFILE DISK1;
INPUT (DMSE VART VARAT ) ( 3*RB8.);
BT = (  9  - VART )**2;
BAT = (324  - VARAT )**2;
BE = ( 100 - DMSE )**2;
DATA PRETST;
INFILE DISK3;
INPUT ( INDEX VART3 VARAT3 MSE3 IPATH ) ( IB4. 3*RB8. IB4. );
T = (  9  - VART3)**2;
AT = (324  - VARAT3)**2;
E = ( 100 - MSE3 )**2;
DATA REML;
   SET ESTIMATE;
   DROP BT BAT BE;
   IF VART < 0
   THEN
       VART = 0;
   IF VARAT < 0
   THEN
       VARAT = 0;
T1 = (  9  - VART )**2;
AT1 = ( 324 - VARAT )**2;
E1 = ( 100 - DMSE )**2;
DATA PRETST01 PRETST05 PRETST10 PRETST25 PRETST50;
    SET PRETST;
    IF INDEX =   1
    THEN
        OUTPUT PRETST01;
    IF INDEX =   2
    THEN
        OUTPUT PRETST05;
    IF INDEX =   3
    THEN
        OUTPUT PRETST10;
    IF INDEX =   4
    THEN
        OUTPUT PRETST25;
    IF INDEX =  5
    THEN
        OUTPUT PRETST50;
PROC PRINT DATA=ESTIMATE;
TITLE USUAL ANOVA ESTIMATES OF VARIANCE COMPONENTS;
PROC PRINT DATA=REML;
TITLE RESTRICTED MAXIMUM LIKELYHOOD ESTIMATES OF VARIANCE COMPONENTS;
PROC PRINT DATA=PRETST01;
TITLE ESTIMATES BASED ON PRETEST WITH ALPHA=.01;
PROC PRINT DATA=PRETST05;
TITLE ESTIMATES BASED ON PRETEST WITH ALPHA=.05;
PROC PRINT DATA=PRETST10;
TITLE ESTIMATES BASED ON PRETEST WITH ALPHA=.10;
PROC PRINT DATA=PRETST25;
TITLE ESTIMATES BASED ON PRETEST WITH ALPHA=.25;
PROC PRINT DATA=PRETST50;
TITLE ESTIMATES BASED ON PRETEST WITH ALPHA=.50;
PROC MEANS DATA=ESTIMATE;
```

```
TITLE STATISTICS OF USUAL ESTIMATES;
PROC MEANS DATA=REML;
TITLE STATISTICS OF REML ESTIMATORS;
PROC MEANS DATA=PRETST01;
TITLE STATISTICS OF PRETEST WITH ALPHA=.01;
PROC MEANS DATA=PRETST05;
TITLE STATISTICS OF PRETEST WITH ALPHA=.05;
PROC MEANS DATA=PRETST10;
TITLE STATISTICS OF PRETEST WITH ALPHA=.10;
PROC MEANS DATA=PRETST25;
TITLE STATISTICS OF PRETEST WITH ALPHA=.25;
PROC MEANS DATA=PRETST50;
TITLE STATISTICS OF PRETEST WITH ALPHA=.50;
/*
```

```
C
C
C  A COMPUTER SIMULATION STUDY FOR THE TWO-WAY RANDOM EFFECTS MODEL
C
C
      IMPLICIT  REAL*8(A-H,O-Z)
      DIMENSION CELL(5,5),ROW(5),COL(5),A(5)
      READ(5,501) SIGT,SIGA,SIGAT,SIGE,DMU,IR,IC,IK
  501 FORMAT(5F10.3,3I2)
      Z=SIGA**2
      T=SIGT**2
      AT=SIGAT**2
      E=SIGE**2
      WRITE(6,601) T,Z,AT,E,DMU,IR,IC,IK
  601 FORMAT('1','VARIANCE OF T',F9.3/' VARIANCE OF A',F9.3/' VARIANCE O
     +F A*T',F9.3/' SAMPLING ERROR',F10.3//' ','THE MEAN IS',F9.3/' ROW=
     +',I2/' COLUMN=',I2/' REPLICATION PER CELL IS',I3)
C
C     THIS PROGRAM WILL GENERATE DATA FOR A TWO-WAY ANOVA RANDOM MODEL
C     WITH INTERACTION.  IT WILL WORK FOR UP TO FIVE LEVELS OF THE
C     RANDOM A EFFECT, FIVE LEVELS OF THE RANDOM T EFFECT AND FIVE
C     REPLICATIONS PER TREATMENT COMBINATIONS.  TO EXPAND THE SIZE OF
C     THE DESIGN YOU WILL NEED TO MODIFY JCL PARAMETERS AND INITIAL
C     VALUES OF MAXIMUM ROW(IR), COL(IC), AND REPLICATION(IK).
C
C     INITIALIZE THE SEED FOR RANDOM NUMBER GENERATOR
C
      IX=614250061
      IY=110036553
      CALL RSTART(IX,IY)
      DO 93 M = 1,10
      DO 80 I = 1,5
            ROW(I)=0.0D0
            COL(I)=0.0D0
            DO 81 J = 1, 5
                  CELL(I,J)=0.0D0
 81         CONTINUE
 80   CONTINUE
      TOTAL=0.0D0
      SSQS=0.0D0
      SSQCEL=0.0D0
      SSQROW=0.0D0
      SSQCOL=0.0D0
C
C  GENERATE DATA ACCORDING TO MODEL (1.3.2)
C
      CALL RANGEN(IC,IR,IK,SIGT,DMU,A,SIGAT,SIGE,TOTAL,SSQS,CELL,ROW,COL
     +,SSQCEL,SSQCOL,SSQROW,M,SIGA)
C
C  COMPUTE THE COMPONENNTS OF THE AOV TABLE
C
      AA1=SSQROW/(IK*IC)
      BB2=TOTAL**2/(IR*IC*IK)
      CC3=SSQCOL/(IK*IR)
      DD4=SSQCEL/IK
      SSA=AA1-BB2
      SST=CC3-BB2
      SSAT=DD4-AA1-CC3+BB2
      SSERR=SSQS-DD4
      SSTOT=SSQS-BB2
```

```
      IADF=(IR-1)
      ITDF=(IC-1)
      IATDF=(IR-1)*(IC-1)
      IEDF=(IR*IC*(IK-1))
      ITOTDF=IR*IC*IK - 1
      DMSA=SSA/IADF
      DMST=SST/ITDF
      DMSAT=SSAT/IATDF
C
C  COMPUTE THE AOV ESTIMATES
C
      DMSE=SSERR/IEDF
      VARA=(DMSA-DMSAT)/(IK*IC)
      VART=(DMST-DMSAT)/(IK*IR)
      VARAT=(DMSAT-DMSE)/IK
C
      FA=DMSA/DMSAT
      FT=DMST/DMSAT
      FAT=DMSAT/DMSE
      ALPA=FPROB(IADF,IATDF,FA)
      ALPT=FPROB(ITDF,IATDF,FT)
      ALPAT=FPROB(IATDF,IEDF,FAT)
C
C  STORE THE AOV ESTIMATES OUT ON DISK
C
      WRITE(12) DMSE, VARA, VART, VARAT
C
C  COMPUTE ESTIMATES BY THE PRETEST METHOD
C
      CALL RANTST(ALPA,ALPT,ALPAT,SSERR,SST,SSA,SSAT,IADF,ITDF,IATDF,
     +IEDF,DMSA,DMST,DMSAT,DMSE,IC,IR,IK,M)
 93   CONTINUE
      STOP
      END
C
      FUNCTION FPROB(U,V,F)
C
C         THIS ROUTINE WILL GIVE THE PROBABILITY > FCOMP
C
CFPROC       F-PROBABILITY VIA CONVERGENT SERIES OF POSITIVE TERMS
C
      REAL*8 F
      INTEGER U,V
      A=.5*U
      B=.5*V
      TEMP=B+A*F
      X=A*F/TEMP
      FPROB=1.0
      IF(F.LE.0.0.OR.X.LE.0.0)RETURN
      XC=B/TEMP
      AB=A+B
      CON=0.
      SGN=+1.
      IF(F-1.) 110,120,120
  110 TEMP=A
      A=B
      B=TEMP
      TEMP=XC
      XC=X
      X=TEMP
```

```
      CON=1.
      SGN=-1.
C     CONVERGENT SERIES EXPANSION - SEE AMS-55 PG. 944
  120 TOP=AB
      BOT=B+1.
      SUM=1.
      TERM=1.
  130 TEMP=SUM
      TERM=TERM*(TOP/BOT)*XC
      SUM=SUM+TERM
      TOP=TOP+1.
      BOT=BOT+1.
      IF(SUM-TEMP) 140,140,130
  140 FPROB=CON+SGN*EXP(A*ALOG(X)+B*ALOG(XC)+ALGAMA(AB)-ALGAMA(A)
     1 -ALGAMA(B))*SUM/B
  150 RETURN
      END
C
      SUBROUTINE RANGEN(IC,IR,IK,SIGT,DMU,A,SIGAT,SIGE,TOTAL,SSQS,CELL,R
     +OW,COL,SSQCEL,SSQCOL,SSQROW,M,SIGA)
      IMPLICIT  REAL*8(A-H,O-Z)
      DIMENSION A(IR),CELL(IR,IC),ROW(IR),COL(IC),SIGAA(5),SIGTT(5),
     +SIGAT1(5,5)
C
C THIS SUBROUTINE WILL GENERATE DATA FOR A TWO-WAY DESIGN, RANDOM
C MODEL WITH AT MOST 5 LEVELS OF RANDOM EFFECT AND
C REPLICATION.
C
      DO 4 I=1,IR
            SIGAA(I)= RNOR(0)*SIGA
 4    CONTINUE
      DO 5 J=1,IC
            SIGTT(J)= RNOR(0)*SIGT
 5    CONTINUE
      DO 6 K = 1,IR
            DO 7 L = 1,IC
            SIGAT1(K,L) = RNOR(0)*SIGAT
 7          CONTINUE
 6    CONTINUE
      DO 1  J = 1,IC
             DO 2 I = 1,IR
                   DO 3 N=1,IK
                   OBS=DMU+SIGAA(I)+SIGTT(J)+SIGAT1(I,J)+RNOR(0)*SIGE
                   TOTAL=TOTAL + OBS
                   SSQS=SSQS + OBS**2
                   CELL(I,J)=CELL(I,J) + OBS
                   ROW(I)= ROW(I) + OBS
                   COL(J)= COL(J) + OBS
  3                CONTINUE
             SSQCEL=SSQCEL + CELL(I,J)**2
  2          CONTINUE
      SSQCOL=SSQCOL + COL(J)**2
  1   CONTINUE
      DO 82 L=1,IR
             SSQROW=SSQROW + ROW(L)**2
 82   CONTINUE
      RETURN
      END
C
      SUBROUTINE RANTST(ALPA,ALPT,ALPAT,SSE,SST,SSA,SSAT,IADF,ITDF,IATDF
```

```
     +,IEDF,DMSA,DMST,DMSAT,DMSE,IC,IR,IK,M)
      IMPLICIT REAL*8(A-H,O-Z)
C
C  THE PRETEST METHOD IS EMPLOYED AT THE FOLLOWING ALPHA VALUES
C
      DO 10 I=1,5
      IF(I.EQ.1) ALPHA=0.01D0
      IF(I.EQ.2) ALPHA=0.05D0
      IF(I.EQ.3) ALPHA=0.10D0
      IF(I.EQ.4) ALPHA=0.25D0
      IF(I.EQ.5) ALPHA=0.50D0
      IPATH=0
      ALPT1=0.0D0
      ALPA1=0.0D0
      ALPA2=0.0D0
      ALPA3=0.0D0
      ERR=DMSE
C
C  IS INTERACTION PRESENT?
C
      IF(ALPHA.GT.ALPAT)                  GOTO 100
      VARAT=0.0D0
      SSE1=SSE+SSAT
      IEDF1=IEDF+IATDF
      DMSE1=SSE1/IEDF1
      ERR=DMSE1
      FTT=DMST/DMSE1
      ALPT1=FPROB(ITDF,IEDF1,FTT)
C
C  NO INTERACTION, IS T PRESENT?
C
          IF(ALPHA.GT.ALPT1)              GOTO 200
          VART=0.0D0
          SSE2=SSE+SSAT+SST
          IEDF2=IEDF+IATDF+ITDF
          DMSE2=SSE2/IEDF2
          ERR=DMSE2
          FAA=DMSA/DMSE2
          ALPA1=FPROB(IADF,IEDF2,FAA)
C
C  NO INTERACTION OR T,IS A PRESENT?
C
               IF(ALPHA.GT.ALPA1)         GOTO 300
               VARA=0.0D0
               SSE3=SSE+SSAT+SST+SSA
               IEDF3=IEDF+IATDF+ITDF+IADF
               DMSE3=SSE3/IEDF3
               ERR=DMSE3
               IPATH=1
               GOTO 999
  300          VARA=(DMSA-DMSE2)/(IK*IC)
               ERR=DMSE2
               IPATH=2
               GOTO 999
  200     VART=(DMST-DMSE1)/(IK*IR)
          FAA=DMSA/DMSE1
          ALPA2=FPROB(IADF,IEDF1,FAA)
C
C  T PRESENT AND NO INTERACTION, IS A PRESENT?
C
```

```
              IF(ALPHA.GT.ALPA2)        GOTO 400
              VARA=0.0D0
              SSE4=SSE+SSAT+SSA
              IEDF4=IEDF+IATDF+IADF
              DMSE4=SSE4/IEDF4
              ERR=DMSE4
              VART=(DMST-DMSE4)/(IK*IR)
              IPATH=3
              GOTO 999
 400          VARA=(DMSA-DMSE1)/(IK*IC)
              ERR=DMSE1
              IPATH=4
              GOTO 999
 100 VARAT=(DMSAT-DMSE)/IK
C
C  INTERACTICN PRESENT,IS T PRESENT?
C
         IF(ALPHA.GT.ALPT)              GOTO 500
         VART=0.0D0
         SSAT1=SSAT + SST
         IATDF1=IATDF+ITDF
         DMSAT1=SSAT1/IATDF1
         FAA=DMSA/DMSAT1
         ALPA3=FPROB(IADF,IATDF1,FAA)
C
C  INTERACTION BUT NO T, IS A PRESENT?
C
              IF(ALPHA.GT.ALPA3)        GOTO 600
              VARA=0.0D0
              SSAT2=SSAT+SST+SSA
              IATDF2=IATDF+ITDF+IADF
              DMSAT2=SSAT2/IATDF2
              VARAT=(DMSAT2-DMSE)/IK
              IPATH=5
              GOTO 999
 600          VARA=(DMSA-DMSAT1)/(IK*IC)
              VARAT=(DMSAT1-CMSE)/IK
              IPATH=6
              GOTO 999
 500     VART=(DMST-DMSAT)/(IK*IR)
C
C  INTERACTION AND T , IS A PRESENT?
C
              IF(ALPHA.GT.ALPA)        GOTO 700
              VARA=0.0D0
              SSAT3=SSAT+SSA
              IATDF3=IATDF+IADF
              DMSAT3=SSAT3/IATCF3
              VART=(DMST-DMSAT3)/(IK*IR)
              VARAT=(DMSAT3-DMSE)/IK
              IPATH=7
              GOTO 999
 700          VARA=(DMSA-DMSAT)/(IK*IC)
              IPATH=8
C
C  PUT PRETEST ESTIMATES OUT CN DISK
C
 999 WRITE(14)  I,VARA,VART,VARAT,ERR,IPATH
 10  CONTINUE
     RETURN
```

```
      END
//LKED.SYSPRINT DD DUMMY
//GO.FT12F001  DD DSN=&&TEMP1,DISP=(NEW,KEEP),UNIT=SYSDA,
//       SPACE=(TRK,(30,30),RLSE),DCB=(RECFM=VBS,LRECL=36,BLKSIZE=4000)
//GO.FT14F001  DD DSN=&&TEMP2,DISP=(NEW,KEEP),UNIT=SYSDA,
//       SPACE=(TRK,(30,30),RLSE),DCB=(RECFM=VBS,LRECL=44,BLKSIZE=4000)
//GO.SYSIN DD *
    19.0        2.0       10.0       10.0       10.0   4 4 4
//  EXEC SAS,REGION=650K
//DISK1  DD DSN=&&TEMP1,DISP=(OLD,DELETE),UNIT=SYSDA
//DISK2 DD  DSN=&&TEMP2,DISP=(OLD,DELETE),UNIT=SYSDA
//SYSIN DD *
DATA ESTIMATE;
INFILE DISK1;
INPUT (DMSE VARA VART VARAT ) ( 4*RB8.);
BT = (  9   -  VART )**2;
BA = ( 4    -  VARA )**2;
BAT= (400   -  VARAT )**2;
BE = ( 100 -   DMSE   )**2;
DATA PRETST;
INFILE DISK2;
INPUT ( I VARA VART VARAT MSE IPATH ) ( IB4. 4*RB8. IB4. );
T = (  9    -  VART )**2;
AT =( 400   -  VARAT)**2;
A = (   4   -  VARA )**2;
E = ( 100   -  MSE   )**2;
DATA REML;
   SET ESTIMATE;
   DROP BT BA BAT BE;
   IF VART < 0
   THEN
       VART = 0;
   IF VARAT < 0
   THEN
       VARAT = 0;
   IF VARA < 0
   THEN VARA = 0;
T1 = (  9    -  VART )**2;
AT1= ( 400   -  VARAT)**2;
A1 = (   4   -  VARA )**2;
E1 = ( 100   -  DMSE )**2;
DATA PRE01 PRE05 PRE10 PRE25 PRE50;
      SET PRETST;
      IF I = 1
      THEN
          OUTPUT PRE01;
      IF I = 2
      THEN
          OUTPUT PRE05;
      IF I = 3
      THEN
          OUTPUT PRE10;
      IF I = 4
      THEN
          OUTPUT PRE25;
      IF I = 5
      THEN
          OUTPUT PRE50;
PROC PRINT DATA=ESTIMATE;
TITLE USUAL ANOVA ESTIMATES OF VARIANCE COMPONENTS;
```

```
PROC PRINT DATA=REML;
TITLE RESTRICTED MAXIMUM LIKELYHOOD ESTIMATES OF VARIANCE COMPONENTS;
PROC MEANS DATA=ESTIMATE;
TITLE STATISTICS OF USUAL ESTIMATES;
PROC MEANS DATA=REML;
TITLE STATISTICS OF REML ESTIMATORS;
PROC PRINT DATA=PRE01;
TITLE ESTIMATES BASED ON ALPHA = .01;
PROC PRINT DATA=PRE05;
TITLE ESTIMATES BASED ON ALPHA = .05;
PROC PRINT DATA=PRE10;
TITLE ESTIMATES BASED ON ALPHA = .10;
PROC PRINT DATA=PRE25;
TITLE ESTIMATES BASED ON ALPHA = .25;
PROC PRINT DATA=PRE50;
TITLE ESTIMATES BASED ON ALPHA = .50;
PROC MEANS DATA=PRE01;
TITLE STATS OF LEVEL .01 ESTIMATES;
PROC MEANS DATA=PRE05;
TITLE STATS OF LEVEL .05 ESTIMATES;
PROC MEANS DATA=PRE10;
TITLE STATS OF LEVEL .10 ESTIMATES;
PROC MEANS DATA=PRE25;
TITLE STATS OF LEVEL .25 ESTIMATES;
PROC MEANS DATA=PRE50;
TITLE STATS OF LEVEL .05 ESTIMATES;
```

A COMPUTER SIMULATION STUDY FOR COMPARING THREE METHODS OF ESTIMATING VARIANCE COMPONENTS

by

THOMAS RICHARD WALSH

B.S., Kansas State University, 1981

AN ABSTRACT OF A MASTER'S REPORT

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Department of Statistics

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Manhattan, Kansas

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ABSTRACT

This report investigates three different methods of estimating variance components for both the two-way mixed and random effects model. These methods of estimation are the Analysis of Variance, Restricted maximum likelihood, and pretest methods.

A statistical derivation and/or detailed explanation for each method of estimation is presented. Various statistical properties and short comings are also discussed.

The three methods of estimating variance components are compared by a computer simulation study. Empirical conclusions associated with each method of estimation are based on statistics computed from each methods estimates observed in the simulation. Graphs of the estimates are presented for further investigation.

An overall 'best' method of estimating variance components for the two-way mixed and random effects model is selected. This report then concludes with a discussion of topics related to variance components for further investigation.