An introduction to uniform distributions and Weyl's Criterion

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Abstract

This report is an exploration into the basics of the uniform distribution of sequences and a proof of Weyl's Criterion. After describing what it means for a sequence to be uniformly distributed, we develop the tools to prove Weyl's Criterion. In order to do this, we split Weyl's Criterion into two theorems and prove each of them. Finally, we will show an example which applies Weyl's Criterion to prove that a certain sequence of irrational numbers is uniformly distributed.

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Chapter 1

Uniform Distribution

The concept of uniform distribution generally can be described as a constant probability that exists within a set. Each element of the set possesses an equal share of the distribution where, if one were to be chosen at random, then all elements would have equal probability of being chosen. In addition, because each element is equal within the distribution, this conveys the idea that a uniform distribution creates an ideal situation.

Suppose that $(s_i)_{i\in\mathbb{N}}$ is a sequence of real numbers in the interval $I = [0, 1)$. For every natural number n and any subset E of I, we write

$$Z(E, n) = \sum_{i=1}^{n} \chi_E(s_i),$$

where χ_E is the characteristic function for the set E. $Z(E, n)$ counts the number of terms among $s_1, s_2, ..., s_n$ that lie in the set E.

Definition 1. The sequence of real numbers $(s_i)_{i\in\mathbb{N}}$ in I is said to be **uniformly distributed in** I if for every $\alpha, \beta \in \mathbb{R}$ satisfying $0 \leq \alpha < \beta \leq 1$, we have

$$\lim_{n\to\infty} \frac{Z([\alpha, \beta), n)}{n} = \beta - \alpha. \tag{1.1}$$

In order for a sequence to be uniformly distributed, the number of terms falling in a subinterval must be proportional to the length of that interval.

Suppose that the real sequence $(s_i)_{i\in\mathbb{N}}$ in I is uniformly distributed in I. Let $f:[0,1]\to\mathbb{R}$ be a continuous function. Suppose that the natural number n is large. Then one can expect that the **discrete average**

$$\frac{1}{n}\sum_{i=1}^{n} f(s_i)$$

of the function f over the first n terms of the sequence $(s_i)_{i\in\mathbb{N}}$ may not differ significantly from the **continuous average**

$$\int_0^1 f(x)dx$$

of the function f over the interval $[0,1]$.

For a real number x, $\lfloor x\rfloor$ denotes the largest integer less than or equal to x and $\{x\} = x - \lfloor x\rfloor$ is the fractional part of x. A sequence of real numbers (x_n) is called **uniformly distributed mod 1** provided that $(\{x_n\})$ is uniformly distributed in $I = [0,1]$.

Let $\chi_{[\alpha,\beta)}$ be the characteristic function of the interval $[\alpha,\beta)\subseteq I$. Then the definition of uniformly distributed mod 1 can be written in the form

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi_{[\alpha,\beta)}(\{x_n\}) = \int_0^1 \chi_{[\alpha,\beta)}(x)dx. \tag{1.2}$$

This observation, along with an approximation technique, gives us the following theorem.[2]

Theorem 1.1. *The sequence (x_n), with $n = 1,2,\ldots,$ of real numbers is uniformly distributed mod 1 if and only if for every real-valued continuous function f defined on the closed unit interval $I = [0,1]$ we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)dx. \tag{1.3}$$

Proof. Let (x_n) be uniformly distributed mod 1, and let $f(x) = \sum_{i=0}^{k-1} d_i \chi_{[\alpha_i, \alpha_{i+1})}(x)$ be a step function on I, where $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ and $d_i \in \mathbb{R}$. Since (x_n) is uniformly distributed mod 1, then we know (1.2) holds for the characteristic function of an interval in I. Therefore we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{i=0}^{k-1} d_i \chi_{[\alpha_i, \alpha_{i+1})}(\{x_n\})$$

$$= \sum_{i=0}^{k-1} d_i \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[\alpha_i, \alpha_{i+1})}(\{x_n\}) = \sum_{i=0}^{k-1} d_i \int_0^1 \chi_{[\alpha_i, \alpha_{i+1})}(x)dx$$

$$= \int_0^1 \sum_{i=0}^{k-1} d_i \chi_{[\alpha_i, \alpha_{i+1})}(x)dx = \int_0^1 f(x)dx.$$

Thus for every such f, equation (1.3) holds.

Now assume that f is a real-valued continuous function defined on I. Given any $\epsilon > 0$, there exist, by the definition of the Riemann integral, two step functions, say f_1 and f_2, such that $f_1(x) \leq f(x) \leq f_2(x)$ for all $x \in I$ and $\int_0^1 (f_2(x) - f_1(x))dx \leq \epsilon$. We are sandwiching our function f in between these two step functions, f_1 and f_2, which have a very small area between them. Then we have the following:

$$\int_0^1 f(x)dx - \epsilon \leq \int_0^1 f_1(x)dx = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_1(\{x_n\}).$$

Since $f_1(x) \leq f(x) \leq f_2(x)$ and the area between f_1 and f_2 is at most ϵ, then $\int_0^1 f(x)dx - \epsilon$ must be smaller than $\int_0^1 f_1(x)dx$. The equality above comes from the fact that (1.3) holds for step functions, which we can apply because we are assuming (x_n) is uniformly distributed mod 1. Because $f_1(x) \leq f(x) \leq f_2(x)$ on $[0, 1)$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_1(\{x_n\}) \le \liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) \le \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) \le \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_2(\{x_n\}).$$

We use the lim inf and lim sup here, because we do not know for sure if the limit exists for this function f. Whether $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\})$ exists or not, this chain of inequalities still holds. Also,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_2(\{x_n\}) = \int_0^1 f_2(x)dx \le \int_0^1 f(x)dx + \epsilon.$$

Once again, the equality above comes from the fact that (1.3) holds for step functions; and since $f_1(x) \le f(x) \le f_2(x)$ and the area between f_1 and f_2 is at most ϵ, then adding ϵ to $\int_0^1 f(x)dx$ must give us a quantity that is greater than $\int_0^1 f_2(x)dx$.

Looking at the big picture of the chain of inequalities above, we get the result that $\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\})$ and $\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\})$ are at most 2ϵ apart. If we shrink ϵ small enough, then these limits become equivalent and our chain of inequalities becomes a chain of equalities. Thus in the case of a continuous function f, the relation

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)dx$$

holds, and we have completed the first half of our proof.

Conversely, let a sequence (x_n) be given, and suppose that (1.3) holds for every real-valued continuous function f on I. Let $[\alpha,\beta)$ be an arbitrary subinterval of I. Given any $\epsilon > 0$, there exist two continuous functions, say g_1 and g_2, such that $g_1(x) \le \chi_{[\alpha,\beta)}(x) \le g_2(x)$ for $x \in I$ and at the same time $\int_0^1 (g_2(x) - g_1(x))dx \le \epsilon$. Then we have

$$\beta - \alpha - \epsilon \le \int_0^1 g_2(x)dx - \epsilon \le \int_0^1 g_1(x)dx = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} g_1(\{x_n\}).$$

The function $g_2(x) = 1$ on $[\alpha, \beta)$. Since $[\alpha, \beta)$ has a length of $\beta - \alpha$, then $\beta - \alpha \leq \int_0^1 g_2(x)dx$. Since $g_1(x) \leq g_2(x)$ and the area between g_1 and g_2 is at most ϵ, then subtracting ϵ from $\int_0^1 g_2(x)dx$ must result in a quantity smaller than $\int_0^1 g_1(x)dx$. The equality above comes from our assumption that (1.3) holds for continuous functions. Because $g_1(x) \leq \chi_{[\alpha,\beta)}(x) \leq g_2(x)$ on $[0, 1)$ and

$$\frac{Z([\alpha, \beta), N)}{N} = \frac{1}{N} \sum_{n=1}^{N} \chi_{[\alpha,\beta)}(\{x_n\}),$$

then we get

$$\beta - \alpha - \epsilon \leq \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g_1(\{x_n\}) \leq \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[\alpha,\beta)}(\{x_n\}) = \liminf_{N\to\infty} \frac{Z([\alpha, \beta), N)}{N}$$

$$\leq \limsup_{N\to\infty} \frac{Z([\alpha, \beta), N)}{N} = \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[\alpha,\beta)}(\{x_n\}) \leq \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g_2(\{x_n\})$$

$$= \int_0^1 g_2(x)dx \leq \int_0^1 g_1(x)dx + \epsilon \leq \beta - \alpha + \epsilon.$$

The last few steps in the chain of inequalities follow by similar arguments to those made earlier. As we make ϵ arbitrarily small, the above $\liminf$ and $\limsup$ become equivalent and the chain of inequalities turns into a chain of equalities. Thus we have that (1.2) holds, and the sequence is uniformly distributed mod 1. Therefore, our proof is complete.

$\square$

Corollary 1. *The sequence (x_n) is uniformly distributed mod 1 if and only if for every complex-valued continuous function f on $\mathbb{R}$ with period 1 we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)dx. \tag{1.4}$$

Proof. Assume the sequence (x_n) is uniformly distributed mod 1, and assume $f(x)$ is a complex-valued function with period 1. Then we can split it up into the sum of its real and imaginary parts. Say $f(x) = f_r(x) + if_i(x)$ where $f_r(x)$ is the real part of $f(x)$ and $f_i(x)$ is the imaginary part, then we have

$$\int_0^1 f(x)dx = \int_0^1 f_r(x) + if_i(x)dx = \int_0^1 f_r(x)dx + i\int_0^1 f_i(x)dx.$$

Then we can apply Theorem 1.1 to the real and imaginary parts of f, since both f_r and f_i are real-valued continuous functions.

$$\int_0^1 f_r(x)dx + i\int_0^1 f_i(x)dx = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f_r(\{x_n\}) + i\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f_i(\{x_n\})$$

$$= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}(f_r(\{x_n\}) + if_i(\{x_n\})) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}).$$

The periodicity condition implies that $f(\{x_n\}) = f(x_n)$. Therefore,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n).$$

Thus we obtain (1.4).

Conversely, we can note that in the second part of the proof of Theorem 1.1, the real-valued continuous functions g_1 and g_2 can be chosen in such a way that they satisfy the additional requirements $g_1(0) = g_1(1)$ and $g_2(0) = g_2(1)$, so that (1.4) can be applied to the periodic extensions of g_1 and g_2 to $\mathbb{R}$. For $i \in \{1, 2\}$, the periodicity condition implies that $g_i(\{x_n\}) = g_i(x_n)$, so we get

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} g_i(\{x_n\}) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} g_i(x_n) = \int_0^1 g_i(x)dx.$$

Then, following the argument in the second half of the proof of Theorem 1.1,

$$\lim_{N\to\infty} \frac{Z([\alpha,\beta),N)}{N} = \beta - \alpha,$$

and the sequence (x_n) is uniformly distributed mod 1.

$\square$

Chapter 2

Weyl's Criterion

Hermann Weyl was a German mathematician and physicist, active in the early 1900's. He has made many contributions to multiple fields in both mathematics and physics, such as the distribution of eigenvalues, geometric foundations of manifolds, topological groups, Lie groups, representation theory, and quantum mechanics. The theorem of his that is the focus of this paper, Weyl's Criterion, was a fundamental step in analytic number theory. His original publication [in German] of this theory is cited here[3]. In this chapter, we will prove Weyl's Criterion using the tools we have developed in Chapter 1 and see an application of the criterion.

Theorem 2.1 (Weyl's Criterion)**.** *A real sequence $(s_i)_{i\in\mathbb{N}}$ in the interval $I = [0, 1]$ is uniformly distributed in I if and only if for every non-zero integer h, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} e(hs_i) = 0.$$

Note that $e(x)$ denotes $e^{2\pi ix} = \cos(2\pi x) + i\sin(2\pi x)$. We will prove Weyl's Criterion loosely following Chen's 2012 lecture notes[4] and L. Kuipers and H. Niederreiter[2] by dividing the criterion into two parts, Corollary 1 and Theorem 2.3. As pointed out by Kuipers and Niederreiter, the necessity in Weyl's Criterion follows directly from Corollary 1 (which we will justify below), and so the remaining challenge is to prove the other direction.

For the necessity direction, the collection of exponential functions we are considering is a sub-collection of all continuous functions $f : [0, 1] \to \mathbb{C}$ satisfying $f(0) = f(1)$, so we are able to apply Corollary 1. Note that for every non-zero integer h, we have

$$\int_0^1 e(hx)dx = 0. \tag{2.1}$$

We will quickly justify this fact by using the Fundamental Theorem of Calculus:

$$\int_0^1 e(hx)dx = \int_0^1 e^{2\pi ihx}dx = \frac{e^{2\pi ihx}}{2\pi ih}\bigg|_0^1 = \frac{e^{2\pi ih(1)}}{2\pi ih} - \frac{e^{2\pi ih(0)}}{2\pi ih} = \frac{1}{2\pi ih}(e^{2\pi ih} - 1).$$

Since h is a non-zero integer, $2\pi ih$ will always be a multiple of $2\pi i$, so $e^{2\pi ih}$ will always land on the point $(1, 0)$ on the unit circle. Thus $e^{2\pi ih} = 1$, and

$$\frac{1}{2\pi ih}(e^{2\pi ih} - 1) = \frac{1}{2\pi ih}(1 - 1) = \frac{1}{2\pi ih}(0) = 0.$$

When we assume (s_i) is uniformly distributed in I, Corollary 1 implies

$$\lim_{N\to\infty} \frac{1}{n}\sum_{i=1}^{n} e(hs_i) = \int_0^1 e(hx)dx = 0.$$

Therefore, we have the first direction of the proof of Weyl's Criterion.

We will simply state the Weierstrass Approximation Theorem for trigonometric polynomials since it supports our proof below for Theorem 2.3. The full proof can be found in Chaudhury's notes[5].

Definition 2. A **trigonometric polynomial**, $p(x)$, is a finite linear combination of functions $\sin(nx)$ and $\cos(nx)$ with $n \in \mathbb{N}$. The function p may be written in the form

$$p(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + i\sum_{n=1}^{N} b_n \sin(nx) = a_0 + \sum_{n=1}^{N} c_n e(nx).$$

Theorem 2.2 (Weierstrass Approximation Theorem). [5] *If f is a continuous complex-valued function in $\mathbb{R}$ that is 1-periodic and if any $\epsilon > 0$ is given, then there exists a trigonometric polynomial $p(x)$ such that*

$$|f(x) - p(x)| < \epsilon$$

for all $x \in \mathbb{R}$. (I.e. any continuous, 1-periodic function can be uniformly approximated by trigonometric polynomials to any degree of accuracy.)

In the proof below, we will be approximating a continuous 1-periodic function $f : [0, 1) \to \mathbb{C}$ with a series of trigonometric polynomials, namely $p(x) = c_0 + \sum_{j=1}^{k} c_j e(h_j x)$, to show (1.4) holds.

Theorem 2.3. *Suppose that $(s_i)_{i \in \mathbb{N}}$ is a real sequence in $I = [0, 1]$. Suppose further that for every non-zero integer h, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} e(hs_i) = 0.$$

Then for every continuous function $f : [0, 1] \to \mathbb{C}$ satisfying $f(0) = f(1)$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(s_i) = \int_0^1 f(x)dx.$$

Proof. Suppose that $f : [0, 1] \to \mathbb{C}$ is continuous and satisfies $f(0) = f(1)$. Let $\epsilon > 0$. Then there exists a trigonometric polynomial $p : [0, 1] \to \mathbb{C}$, i.e. a linear combination of functions of the type $e(hx)$ where $h \in \mathbb{Z}$, such that

$$\sup_{x\in[0,1]} \Big| f(x) - p(x) \Big| < \frac{\epsilon}{3}. \tag{2.2}$$

Note that

$$\left| \frac{1}{n} \sum_{i=1}^{n} f(s_i) - \int_0^1 f(x)dx \right| \leq \tag{2.3}$$

$$\left|\frac{1}{n}\sum_{i=1}^{n} f(s_i) - \frac{1}{n}\sum_{i=1}^{n} p(s_i)\right| + \left|\frac{1}{n}\sum_{i=1}^{n} p(s_i) - \int_0^1 p(x)dx\right| + \left|\int_0^1 p(x)dx - \int_0^1 f(x)dx\right|.$$

Since we know (2.2), we see that the first and last terms on the second line of (2.3) are each less than $\frac{\epsilon}{3}$. We just need to show that the second term is also less than $\frac{\epsilon}{3}$ for all sufficiently large n:

$$\left|\frac{1}{n}\sum_{i=1}^{n} p(s_i) - \int_0^1 p(x)dx\right| < \frac{\epsilon}{3}.$$

Let's suppose that

$$p(x) = c_0 + \sum_{j=1}^{k} c_j e(h_j x), \tag{2.4}$$

where $c_0 \in \mathbb{C}, c_1, \dots, c_k \in \mathbb{C}\backslash\{0\}$ and $h_1, \dots, h_k \in \mathbb{Z}\backslash\{0\}$. Then we have

$$\frac{1}{n}\sum_{i=1}^{n} p(s_i) - \int_0^1 p(x)dx = \frac{1}{n}\sum_{i=1}^{n}\Big(c_0 + \sum_{j=1}^{k} c_j e(h_j s_i)\Big) - \int_0^1 \Big(c_0 + \sum_{j=1}^{k} c_j e(h_j x)\Big)dx$$

$$= \frac{1}{n}\sum_{i=1}^{n} c_0 + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{k} c_j e(h_j s_i) - \int_0^1 c_0 dx - \int_0^1 \sum_{j=1}^{k} c_j e(h_j x)dx$$

$$= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{k} c_j e(h_j s_i) - \int_0^1 \sum_{j=1}^{k} c_j e(h_j x)dx$$

$$= \sum_{j=1}^{k} c_j \left(\frac{1}{n}\sum_{i=1}^{n} e(h_j s_i) - \int_0^1 e(h_j x)dx\right).$$

By (2.1), we know $\int_0^1 e(h_j x)dx = 0$ for every $j = 1, \dots, k$. Then using the triangle inequality, we get

$$\left|\frac{1}{n}\sum_{i=1}^{n}p(s_i)-\int_0^1 p(x)dx\right| \le \sum_{j=1}^{k}|c_j|\left|\frac{1}{n}\sum_{i=1}^{n}e(h_js_i)\right|. \tag{2.5}$$

For every $j=1,\dots,k$, it follows from the hypotheses that there exists $n_j\in\mathbb{N}$ such that for every $n>n_j$, we have

$$\left|\frac{1}{n}\sum_{i=1}^{n}e(h_js_i)\right| < \frac{\epsilon}{3k|c_j|}. \tag{2.6}$$

Let $n_0=\max\{n_1,\dots,n_k\}$. Then for every $n>n_0$, taking (2.5) and (2.6) into consideration, we have

$$\left|\frac{1}{n}\sum_{i=1}^{n}p(s_i)-\int_0^1 p(x)dx\right| \le \sum_{j=1}^{k}|c_j|\left|\frac{1}{n}\sum_{i=1}^{n}e(h_js_i)\right| < \sum_{j=1}^{k}|c_j|\frac{\epsilon}{3k|c_j|} = \sum_{j=1}^{k}\frac{\epsilon}{3k} = \frac{\epsilon}{3}$$

which is what we wanted. Therefore,

$$\left|\frac{1}{n}\sum_{i=1}^{n}f(s_i)-\int_0^1 f(x)dx\right| \le$$

$$\left|\frac{1}{n}\sum_{i=1}^{n}f(s_i)-\frac{1}{n}\sum_{i=1}^{n}p(s_i)\right| + \left|\frac{1}{n}\sum_{i=1}^{n}p(s_i)-\int_0^1 p(x)dx\right| + \left|\int_0^1 p(x)dx-\int_0^1 f(x)dx\right|$$

$$< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon,$$

implying that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f(s_i)=\int_0^1 f(x)dx.$$

□

Combining our conclusions from Theorem 2.3 and Corollary 1, we get that if $(s_i)_{i\in\mathbb{N}}$ is a sequence of real numbers in I satisfying

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} e(hs_i) = 0$$

$\forall h \in \mathbb{Z}\setminus\{0\}$, then $(s_i)_{i\in\mathbb{N}}$ is uniformly distributed in I. This completes our proof of Weyl's Criterion.

2.4 Application of Weyl's Criterion

The following describes an example of a certain irrational number sequence which is uniformly distributed:

Theorem 2.5. *Let θ be a fixed real number. Then the sequence $(\{i\theta\})_{i\in\mathbb{N}}$ is uniformly distributed in the interval $[0,1)$ if θ is irrational.*

Proof. To show this, we can use Weyl's Criterion. For every non-zero integer h, we have

$$\left|\frac{1}{n}\sum_{i=1}^{n} e(h\{i\theta\})\right| = \left|\frac{1}{n}\sum_{i=1}^{n} e(hi\theta)\right| = \left|\frac{1}{n}\frac{e(h(n+1)\theta)-e(h\theta)}{1-e(h\theta)}\right| \le \frac{2}{n|1-e(h\theta)|} = \frac{1}{n|\sin(\pi h\theta)|}.$$

The last rational expression in the above equation clearly approaches zero as n approaches infinity, since $\sin(\pi h\theta)$ cannot equal zero with θ being an irrational number and h a non-zero integer. Thus

$$\frac{1}{n}\sum_{i=1}^{n} e(h\{i\theta\}) \to 0$$

as $n\to\infty$. Then by Weyl's Criterion, the sequence $(\{i\theta\})_{i\in\mathbb{N}}$ is uniformly distributed in the interval $[0,1)$ with the condition of θ being irrational.

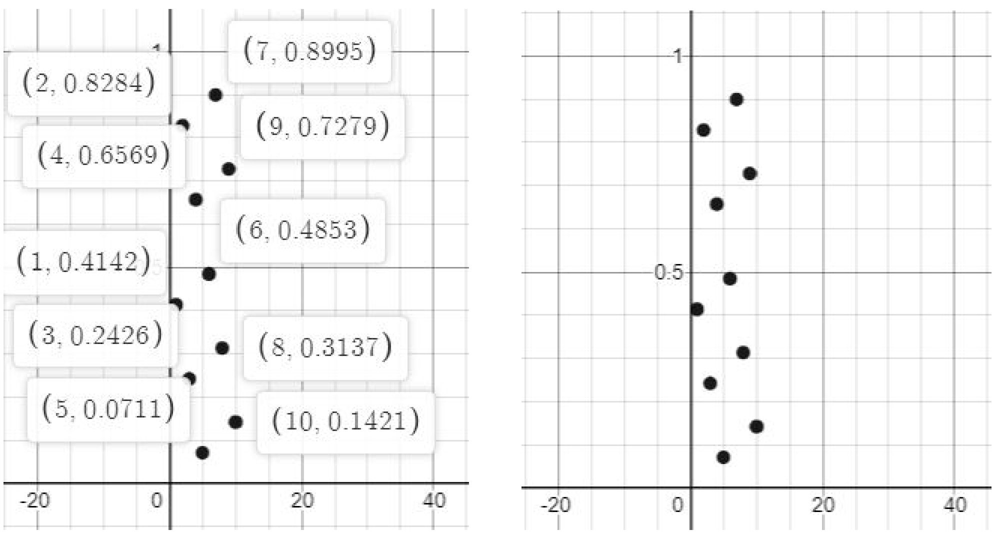
□

This theorem tells us that concrete examples like

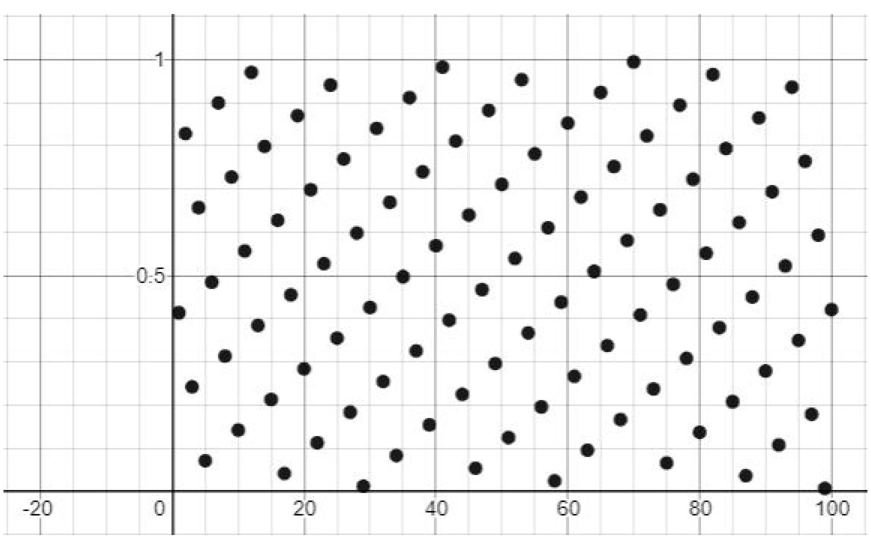
$$(\{n \cdot \sqrt{2}\})_{n=1}^{\infty} = (\{\sqrt{2}\}, \{2\sqrt{2}\}, \{3\sqrt{2}\}, \{4\sqrt{2}\}, \dots)$$

are uniformly distributed in $[0, 1)$. We can pictorially show what this distribution looks like. The following graphs plot the first ten points in the sequence $(\{n \cdot \sqrt{2}\})_{n=1}^{\infty}$, then the first hundred points. Note that the horizontal axis is the value of n, starting at $n = 1$, and the vertical axis shows the value of the point in the sequence which corresponds to that n.

First ten points:



First hundred points:



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