A COMPARISON OF METHODS FOR OBTAINING THE PROBABILITY DENSITY OF FUNCTIONS RELATED TO RADAR BACKSCATTER

by

ABDUL LATEEF ABDUSSATTAR

B.E., University of Madras, 1957

M.E., Indian Institute of Science, 1959

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Department of Electrical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

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Approved by:

D H Lenhert

Major Professor

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INTRODUCTION

1.1 Historical Sketch

In 1821, the French scientist Fresnel, established the formulae for determining the intensities and directions of reflected and refracted rays of light incident on the plane surface of a transparent body. These formulae were based on the "Elastic Theory" of light which assumes the existence of infinitely elastic medium ether, now of only historical interest. However, the formulae themselves had brilliant experimental justification and were used for the verification of every theory of light which was proposed there-after including the Electromagnetic Theory of light developed by Maxwell in 1865.

For electromagnetic waves, the Fresnel reflection laws can be deduced from Maxwell's equations and the appropriate boundary conditions. If a wave, travelling in free space, is incident on the plane surface of a medium of relative dielectric constant K, then, for the horizontal and the vertical polarizations, the respective reflection coefficients R_h & R_v are given by (Harrington, 1961):

$$R_h = \frac{\mathrm{Cos}\theta_1 - \sqrt{K^2 - \mathrm{Sin}^2\theta_1}}{\mathrm{Cos}\theta_1 + \sqrt{K^2 - \mathrm{Sin}^2\theta_1}} \qquad (1.1.1)$$

$$R_v = \frac{K^2 \cos\theta_1 - \sqrt{K^2 - \sin^2\theta_1}}{K^2 \cos\theta_1 + \sqrt{K^2 - \sin^2\theta_1}} \qquad (1.1.2)$$

where θ_1 is the angle of reflection (equal to the angle of incidence).

1.2 Reflection From A Rough Surface

If the reflecting surface becomes rough, the electromagnetic energy is scattered in various directions, though certain privileged directions may receive more energy than others (Beckmann & Spizzichino, 1963). It becomes impossible to apply the boundary conditions imposed by the rough surface for determining the scattered field. With the advent of radar, the study of scattering of electromagnetic waves from rough surfaces, particularly from terrains, has achieved special importance. Also the back scattered field is required to calculate the radar cross section of targets of general shapes.

A number of simplifying assumptions are required to determine rough surface scatter. Some of the important ones are (Beckmann & Spizzichino, 1963):

1. The dimensions of the scattering elements of the rough surface are either much smaller or much greater than the wave length of the incident radiation.

2. The radius of curvature of the scattering elements is much greater than the wave length of the incident radiation.

3. Shadowing and multiple scattering effects of the scattering elements are negligible.

4. The rough surface is assumed to have specific types of roughness e.g. sinusoidal, sawtooth, random variations in height, with respect to a "mean" surface, describable by their statistical distributions.

5. Only the far field is calculated.

In spite of all these and several other assumptions, it is still not possible to obtain a general solution for a rough surface. By far the largest number of rough surface scatter theories is based on the Kirchhoff approximations of the boundary conditions which are required to evaluate the Helmholtz integral for calculating the scattered field. This evaluation requires that the total field distribution on the scattering surface be known. Based on the assumption number 2 above, the field at any point of the surface is approximated by the one that would be present on the "tangent plane" at that point. This field is then the sum of the incident and reflected fields at that point. The reflected field is assumed to be given by the Fresnel coefficients.

The angle θ_1, now, is no longer constant but, as shown in Fig. 1, is a function of the angle θ which is called the generalized angle of reflection (or angle of incidence).

1.3. Rough Sphere

Obviously, the functional relationship between θ_1, and θ

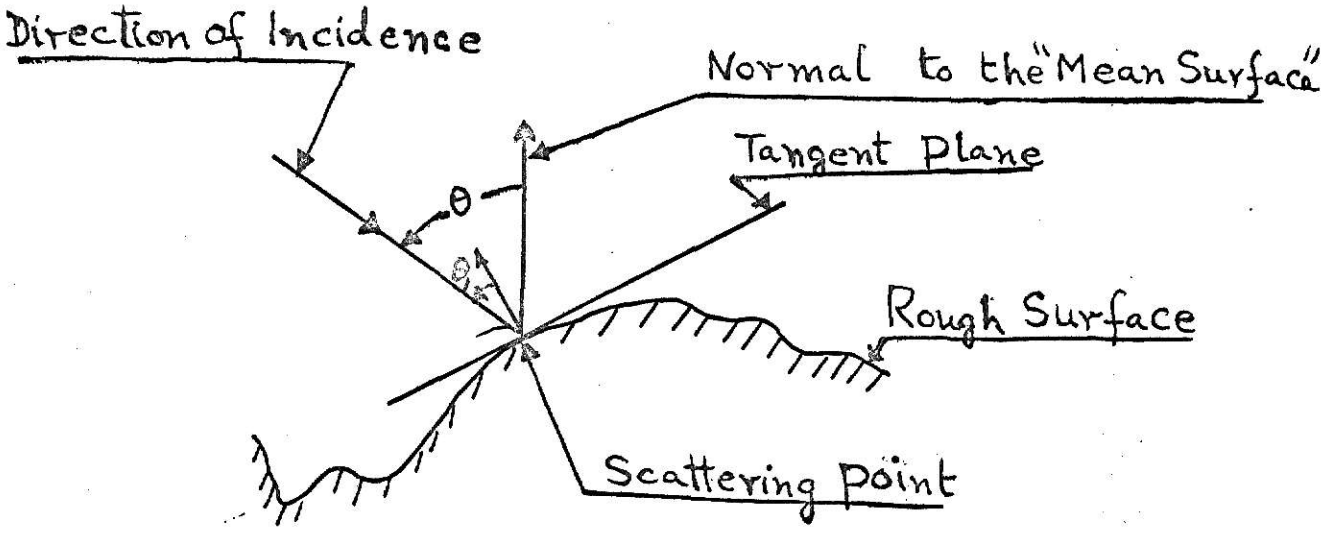

Fig. 1. Scattering geometry of a rough surface.

depends on the type of "mean" surface chosen. One particular surface of both theoretical and practical interest and of great importance is the statistically rough sphere with gaussian roughness. In this section the coordinate system necessary for describing reflection from a rough sphere is described and the functional relationship between θ_1, and θ is derived (Lenhert, 1966).

Let the rough sphere be of average radius a. and the variation in the radius at a point (θ, ϕ) be $H(\theta, \phi)$. If $\vec{r} = r\ \vec{a}_r$, is the vector from the origin (Fig. 2) to a point on the reflecting surface, then the equation for the rough surface is:

$$\psi = r - \left[a + H(\theta,\phi)\right] = 0 \qquad (1.3.1)$$

The unit outward surface normal $\vec{a}_n$ is then given by:

$$\vec{a}_n = \frac{\nabla \psi}{|\nabla \psi|} \qquad (1.3.2)$$

From (1.3.1),

$$\nabla\psi = \vec{a}_r - \frac{1}{r}\frac{\partial H(\theta,\phi)}{\partial \theta}\ \vec{a}_\theta - \frac{1}{r\sin\theta}\ \frac{\partial H(\theta,\phi)}{\partial \phi}\ \vec{a}_\phi \qquad (1.3.3)$$

If the sphere has gaussian roughness, $H(\theta,\phi)$ can be generated by a normal random process and its partial derivatives are therefore also normal (Middleton, 1960).

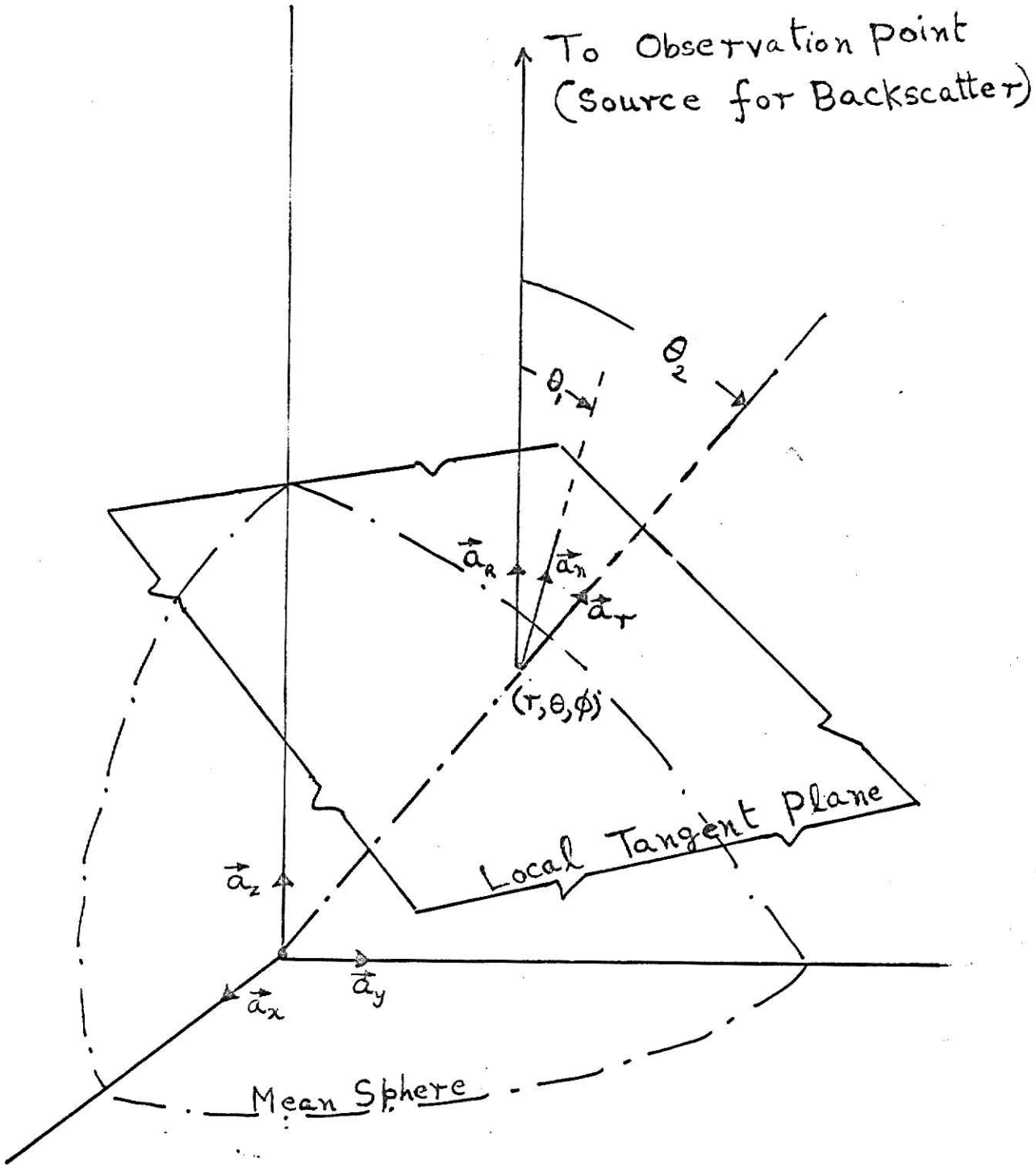

Fig. 2. Coordinate system for rough sphere scatter.

Let $X = \frac{1}{r \sin\theta} \frac{\partial H(\theta,\phi)}{\partial \phi}$ (1.3.4)

and $Y = \frac{1}{r} \frac{\partial H(\theta,\phi)}{\partial \theta}$ (1.3.5)

X and Y are therefore normal random variables.

Let $\vec{a}_R$ denote the direction of the receiver at the point (r, θ, ϕ) on the rough sphere and the angle between $\vec{a}_R$ and $\vec{a}_n$ be θ_1 (local angle of reflection). Assuming that only the far field is of interest, we have:

$$\vec{a}_R = \vec{a}_Z = \vec{a}_r \cos\theta - \vec{a}_\theta \sin\theta \quad (1.3.6)$$

Taking the dot product, $\vec{a}_R \cdot \vec{a}_n$ to get $\cos\theta_1$ and substituting from the above equations, we get:

$$\cos\theta_1 = \frac{\cos\theta + Y \sin\theta}{\sqrt{1 + X^2 + Y^2}} \quad (1.3.7)$$

1.4 Problem Defined

In this report an attempt is made to obtain the probability density function of $\cos\theta_1$ and from this to estimate the distributions of R_h and R_v. Because of the complexity of $\cos\theta_1$, its distribution cannot be determined analytically. The "Monte Carlo" and "numerical integration" techniques are therefore used with the IBM 360/50 computer and the results are then compared.

1.5 Summary of Chapter Development

Chapter I contains the statement of the problem and the necessary derivations. Chapter II presents the analytical technique for determining the distribution of $\text{Cos}\theta_1$. In Chapter III, two methods of evaluating the distribution of $\text{Cos}\theta_1$ by numerical techniques are discussed and the results are compared in the following chapter. The distribution of R_h and R_v are determined in Chapter V and the report is concluded with recommendations for further investigations in the last chapter.

CHAPTER II

PROBABILITY DENSITY OF FUNCTIONS OF RANDOM VARIABLES

2.1 Introduction

The distribution of a random variable which is a function of other random variables can be calculated only if the joint density of the constituent random variables is known (Papoulis, 1965). The joint density of normally distributed random variables is assumed to be also normal. Based on this, the density of a function which is the sum of several normal variables can easily be calculated. If the function, however, is not just the sum of random variables but incorporates other functional relationships, the problem becomes very difficult and an analytical solution is often impossible.

2.2 Density Function of $\text{Cos}\theta_1$

$$\text{Let} \quad Z = \text{Cos}\theta_1 \tag{2.2.1}$$

$$\therefore \; Z = \frac{\text{Cos}\theta \; + Y \; \text{Sin}\theta}{\sqrt{1 + X^2 + Y^2}} \tag{2.2.2}$$

If X and Y are assumed to be normally distributed with zero mean and equal standard deviation σ, then their joint probability density $f(U)$ is given by:

$$f(U) = \frac{1}{2\pi\sqrt{1 - \rho^2}\,\sigma^2} \, e^{-(x^2+y^2-2\rho xy)/2\sigma^2(1-\rho^2)} \tag{2.2.3}$$

where ρ is the correlation coefficient (Papoulis, 1965).

$$\text{Let } A = \frac{1}{2\pi\sqrt{1-\rho^2}\,\sigma^2} \tag{2.2.4}$$

$$\text{and } B = \frac{1}{2\sigma^2(1-\rho^2)} \tag{2.2.5}$$

$$\therefore f(U) = A\, e^{-B(x^2+y^2-2\rho xy)} \tag{2.2.6}$$

The problem therefore reduces to the determination of the density function f(Z), knowing equation (2.2.3). The most general method is to determine the distribution function F(Z) and then evaluate f(Z) using the relationship:

$$f(Z) = \frac{\partial F(Z)}{\partial z} \tag{2.2.7}$$

Solving (2.2.2) for x, (X is the random variable and x is the value it takes)

$$x = \pm\left[\left(\frac{\text{Cos}\theta + y\ \text{Sin}\theta}{z}\right)^2 - (1 + y^2)\right]^{\frac{1}{2}} \tag{2.2.8}$$

$$\text{Let } \alpha = \left[\left(\frac{\text{Cos}\theta + y\ \text{Sin}\theta}{z}\right)^2 - (1 + y^2)\right]^{\frac{1}{2}} \tag{2.2.9}$$

$$\therefore x = \pm\,\alpha \tag{2.2.10}$$

$$\therefore F(Z) = \int_{R_y}\int_{R_x} \left[A\, e^{-B(x^2 + y^2 - \rho 2\, xy)}\right] d\,x\,d\,y \tag{2.2.11}$$

where R_x and R_y are regions of integration for x and y respectively. From (2.2.2), it is clear that:

$Z \leqslant 0$ for $-\infty < Y \leqslant -\cot\theta$

and $Z \geqslant 0$ for $-\cot\theta \leqslant Y < \infty$

$\therefore$ For Z negative,

$$F(Z) = A\int_{-\infty}^{-\cot\theta}\left\{\int_{-\alpha}^{0} e^{-B\,(x^2 + y^2 - 2\rho x y)}\,dx + \int_{0}^{\alpha} e^{-B(x^2 + y^2 - 2\rho x y)}\,dx\right\} d y \qquad (2.2.12)$$

and for Z positive,

$$F(Z) = A\int_{-\cot\theta}^{\infty}\left\{\int_{-\infty}^{-\alpha} e^{-B\,(x^2 + y^2 - 2\rho x y)}\,dx + \int_{\alpha}^{\infty} e^{-B\,(x^2 + y^2 - 2\rho x y)}\,dx\right\} d y \qquad (2.2.13)$$

From (2.2.9)

$$\frac{\partial \alpha}{\partial z} = -\frac{(\cos\theta + y \sin\theta)^2}{\alpha z^3} \qquad (2.2.14)$$

Using equations (2.2.7), (2.2.14) and Leibnitz rule for differentiation under the integral, we get:

$$f(Z) = A\int_{-\infty}^{-\cot\theta} \left\{ \frac{-(\cos\theta + y\,\sin\theta)^2}{\alpha z^3} \left[e^{-B(\alpha^2+y^2+2\beta\alpha y)} + e^{-B(\alpha^2+y^2-2\beta\alpha y)} \right] \right\} dy \qquad (2.2.15)$$

for Z negative, and:

$$f(Z) = A\int_{-\cot\theta}^{\infty} \left\{ \frac{(\cos\theta + y\,\sin\theta)^2}{\alpha z^3} \left[e^{-B(\alpha^2+y^2+2\beta\alpha y)} + e^{-B(\alpha^2+y^2-2\beta\alpha y)} \right] \right\} dy \qquad (2.2.16)$$

for Z positive.

The integration indicated in the last two equations cannot be performed analytically and therefore an exact expression for f(Z) cannot be obtained.

CHAPTER III

METHODS OF EVALUATING f(Z)

3.1 Introduction

As f(Z) cannot be determined exactly, numerical integration or Monte Carlo technique is used to get approximate density function of $\text{Cos}\theta_1$. Both utilise the IBM 360/50 computer and the accuracy of the results depends upon the computer time that can be economically used.

3.2 Monte Carlo Technique

In the Monte Carlo method (IBM Manual; Shrieder, 1966; Cheng, 1970) two sets of uniformly distributed random numbers in the interval (0,1) are first generated. These numbers are then transformed into normally distributed random numbers with zero mean and unit standard deviation using the following relationships:

$$Z_1 = (- 2 \log_e R_1)^{\frac{1}{2}} \text{ Cos } 2\pi R_2 \qquad (3.2.1)$$

$$Z_2 = (- 2 \log_e R_2)^{\frac{1}{2}} \text{ Sin } 2\pi R_2 \qquad (3.2.2)$$

where Z_1, Z_2 give the normally distributed set corresponding to the uniformly distributed set R_1 and R_2.

These normal "random pairs" are then converted into X and Y with the appropriate correlation coefficient ρ, standard deviation σ and mean zero. The relationships used are:

$$X = \sigma Z_1 \tag{3.2.3}$$

$$Y = \sigma (Z_1 \rho + Z_2 \sqrt{1 - \rho^2}) \tag{3.2.4}$$

For each pair of values of X and Y and given θ, Z is evaluated using

$$Z = \frac{\text{Cos}\theta + Y\ \text{Sin}\theta}{\sqrt{1 + X^2 + Y^2}} \tag{3.2.5}$$

Since Z (i.e. $\text{Cos}\theta_1$) can take values between -1 and +1, the interval (-1,1) is divided into subintervals and the number of values of Z which lie in each subinterval is calculated. After normalizing this number with the total number of generated random numbers, we get the density function f(Z).

The accuracy of the Monte Carlo method depends mainly upon the sample size M, that is, the total number of random numbers generated and the validity of randomness. The sample size used for this work was 10000 (R_1 and R_2 each) numbers because of economic considerations of computer running time and the limitations of the memory of IBM 360/50. As for the validity of randomness, a chi-square test (Wadsworth and Bryan, 1960) was used (with all its limitations). Obviously, the random numbers generated by the computer are not purely random but are completely determined by the starting data.

3.3 Numerical Integration

In this method (Conte, 1965) f(Z) is evaluated using the

equations (2.2.15) and (2.2.16). The integration has to be carried out in several steps because of the factor $\{(\text{Cos}\theta + y\ \text{Sin}\theta)^2 - (1 + y^2)\}$ which has to be greater than zero for real values of x in equation (2.2.8). This implies:

$$\left(y + \frac{\text{Sin}\theta\ \text{Cos}\theta}{\text{Sin}^2\theta - z^2}\right)^2 > \frac{z^2\ (1 - z^2)}{(\text{Sin}^2\theta - z^2)^2} \qquad (3.3.1)$$

Step 1.

For the region

$$-1 \leqslant z \leqslant -\text{Sin}\theta$$

$$f(Z) = 0 \qquad (3.3.2)$$

as the minimum value Z can take is $-\text{Sin}\theta$ when Y becomes $-\infty$ (this has zero probability by definition).

Step 2.

In the region

$$-\text{Sin}\theta < z \leqslant 0$$

$$\text{Sin}^2\theta - z^2 > 0$$

Therefore (3.3.1) is satisfied if y does not take values between $(-m - n)$ and $(-m + n)$ where

$$m = \frac{\text{Sin}\theta\ \text{Cos}\theta}{\text{Sin}^2\theta - z^2} \qquad (3.3.3)$$

and $$n = \frac{z\ (1 - z^2)^{\frac{1}{2}}}{|\text{Sin}^2\theta - z^2|} \qquad (3.3.4)$$

Subject to these restrictions, from (2.2.15),

$$f(Z) = \frac{A}{z^3} \int_{-\infty}^{-\mathrm{Cot}\theta} \left\{ \frac{(\mathrm{Cos}\theta + y\ \mathrm{Sin}\theta)^2}{\alpha} \left[e^{-B(\alpha^2+y^2+2\beta\alpha y)} + e^{-B(\alpha^2+y^2-2\beta\alpha y)} \right] \right\} dy \qquad (3.3.5)$$

Step 3.

For

$$0 \leqslant z < \mathrm{Sin}\theta,$$

all arguments under Step 2 hold except that Z is now positive.

$$\therefore\ f(Z) = \frac{A}{z^3} \int_{-\mathrm{Cot}\theta}^{\infty} \left\{ \frac{(\mathrm{Cos}\theta + y\ \mathrm{Sin}\theta)^2}{\alpha} \left[e^{-B(\alpha^2+y^2+2\beta\alpha y)} + e^{-B(\alpha^2+y^2-2\beta\alpha y)} \right] \right\} dy \qquad (3.3.6)$$

Step 4.

If $z = \mathrm{Sin}\theta$, to make x real in equation (2.2.8),

$$y > -\mathrm{Cot}2\theta.$$

With this restriction on y, f(Z) is given by (3.3.6).

Step 5.

Finally, in the region

$$\mathrm{Sin}\theta < z \leqslant 1,$$

$$\mathrm{Sin}^2\theta - z^2 < 0.$$

Therefore, to satisfy (3.3.1), y should take values only between (-m -n) and (-m +n) and subject to this, f(Z) is given by (3.3.6).

From the above discussion it is clear that determination of f(Z) finally boils down to the evaluation of the integral of the general form:

$$\frac{A}{z^3}\int\left\{\frac{(\cos\theta + y\sin\theta)^2}{\alpha}\left[e^{-B(\alpha^2+y^2+2\beta\alpha y)} + e^{-B(\alpha^2+y^2-2\beta\alpha y)}\right]\right\} dy.$$

Simpson's rule was used to evaluate the above integral. Additional simplifications of the integration limits result from comparing $\frac{m}{\sin^2\theta}$ with $-\cot\theta$ in different regions. The accuracy of this method depends mainly on the computer running time. Because of the presence of factors like $\frac{1}{z^3}$, $\frac{1}{\alpha}$ and the exponentials in the integral, the contribution of these integrals to f(Z) gets restricted to only a small range of values of y. It is very hard to determine this range for each value of Z. It was found that for satisfactory results, the number of steps used for integration should be at least 1000. The actual number used was 2000.

CHAPTER IV

COMPARISON OF THE TWO METHODS

4.1 Introduction

The Monte Carlo method is a very general one and the same program can be adapted to get the density of any function with minor modifications (Cheng, 1970). The numerical integration method, though equally general, requires a different program for each function. However, as the results show, this method has better accuracy and can be made as exact as desired at the expense of computing time.

4.2 Results Compared

Fig. 3 shows the density function f(Z) for $\sigma = 100$, $\theta = 60°$ and $\rho = 0.4$ obtained by integration. Fig. 4 gives same f(Z) by Monte Carlo method. Table-1 gives a comparison between the values obtained for f(Z) by the two methods and also when different values of random numbers are used in the Monte Carlo method.

When the chi-square test is applied for determining the randomness of a set of 10000 random numbers and the first 1000 of these numbers, the two sets will differ as regards to their acceptability. Therefore to get reasonably "acceptable" chi-square values for the two sets, M = 10000 and M = 1000, they must be generated by changing the initial data. For example, the M = 10000 set used in Fig. 4 was generated using the initial numbers 129140163 and 787972333

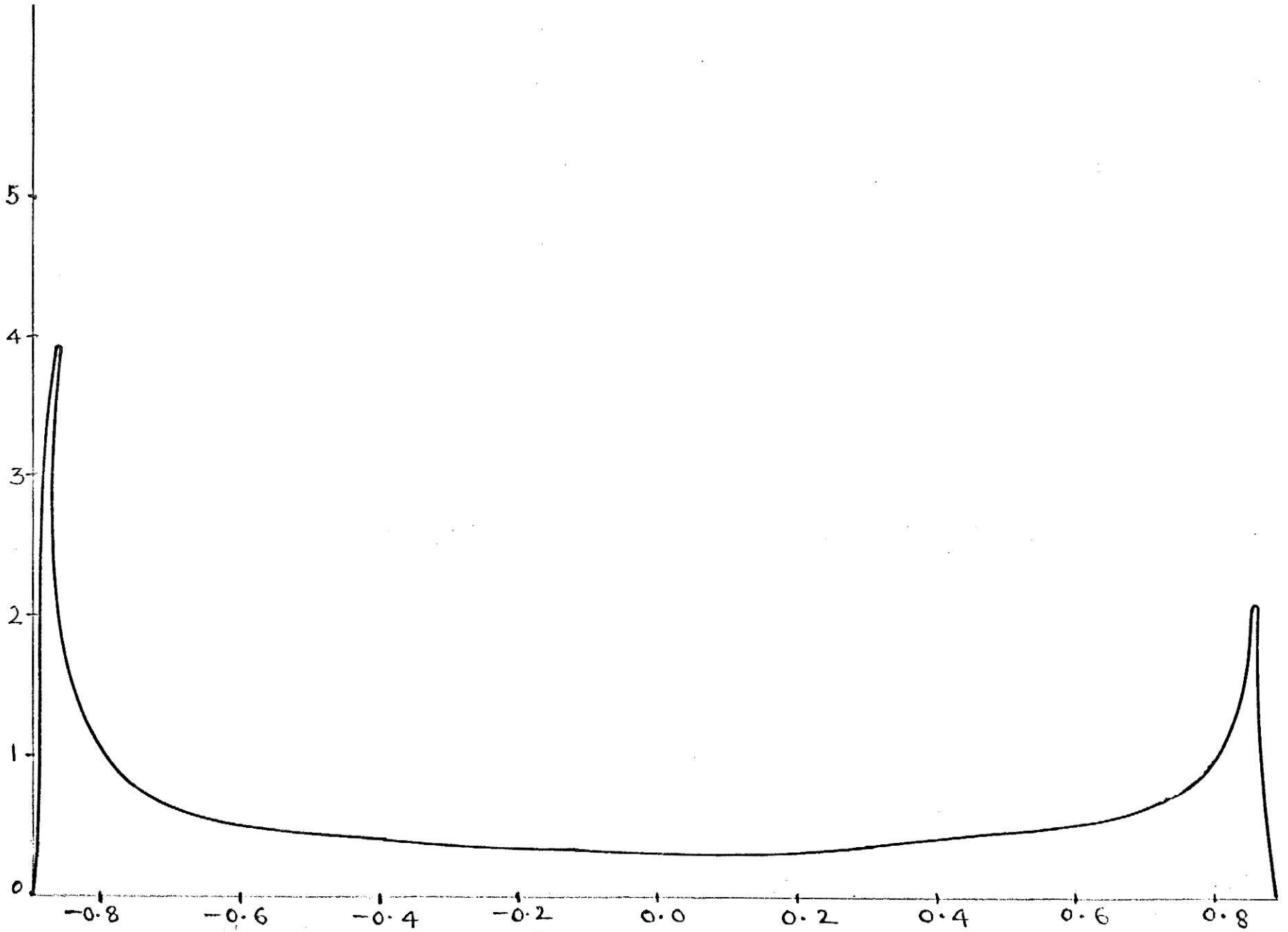

Fig. 3. Probability density of $\mathrm{Cos}\theta_1$ ($\rho = 0.4$, $\sigma = 100$, $\theta = 60°$) by numerical integration.

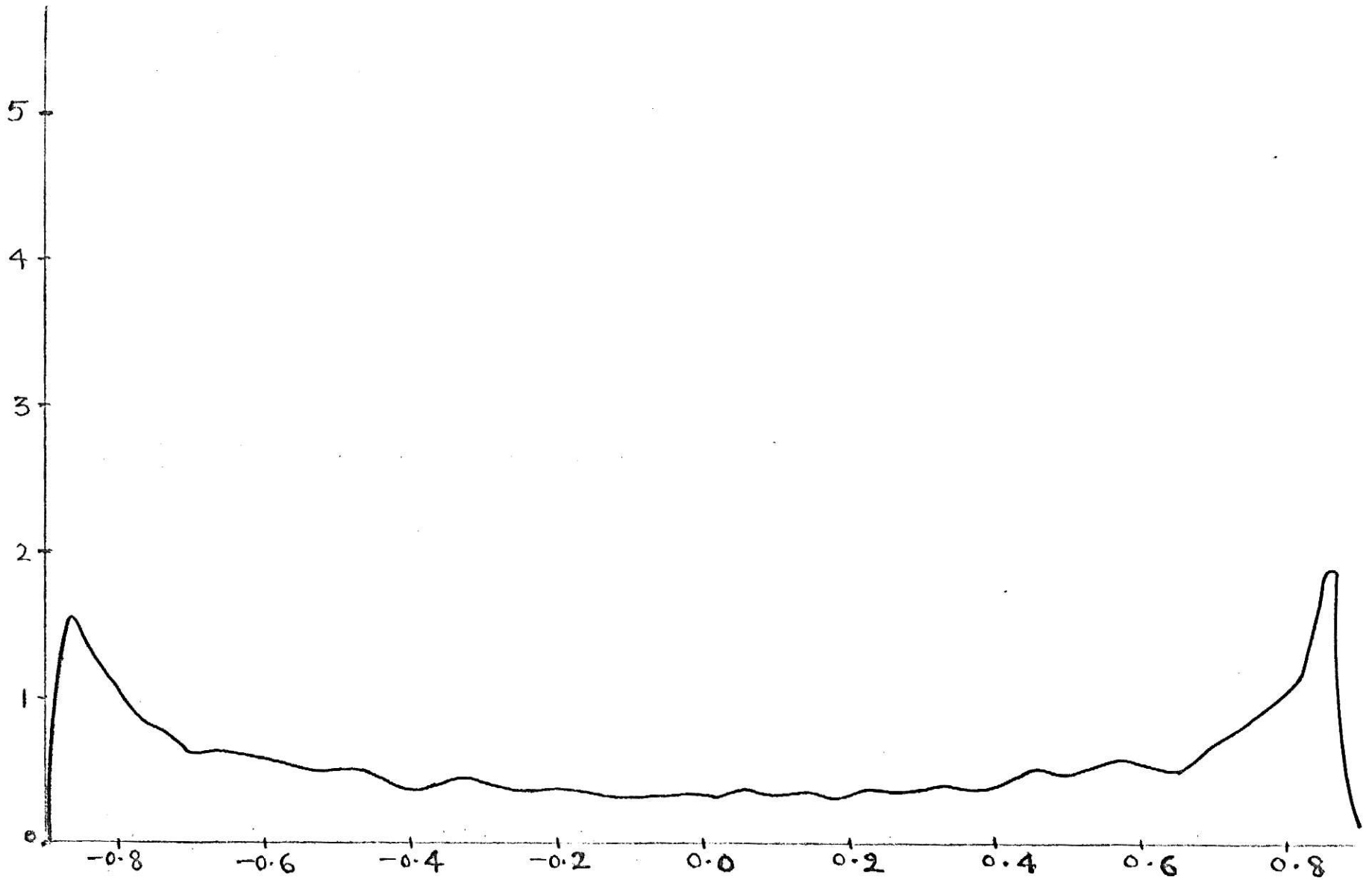

Fig. 4. Probability density of $\mathrm{Cos}\theta_1$ ($\rho = 0.4, \sigma = 100, \theta = 60°$) by Monte Carlo method.

TABLE I

Density of $\cos\theta_1$

Z	f(Z) ($\rho = 0.4$, $\sigma = 100$, $\theta = 60°$)		
	By Integration	By Monte Carlo Method	
		M=10000	M=1000
-0.98	0.0000	0.0000	0.0000
-0.94	0.0000	0.0000	0.0000
-0.90	0.0000	0.0000	0.0000
-0.86	3.9181	1.5375	1.8500
-0.82	1.2002	1.2275	1.1750
-0.78	0.8918	0.9200	0.9500
-0.74	0.7597	0.7700	0.6500
-0.70	0.6807	0.6325	0.6500
-0.66	0.6247	0.6450	0.6000
-0.62	0.5808	0.6175	0.7250
-0.58	0.5442	0.5800	0.5250
-0.54	0.5125	0.5125	0.5000
-0.50	0.4846	0.5075	0.3500
-0.46	0.4599	0.5050	0.4750
-0.42	0.4380	0.4050	0.3500
-0.38	0.4186	0.3650	0.3750
-0.34	0.4016	0.4350	0.5000
-0.30	0.3869	0.4100	0.3500
-0.26	0.3742	0.3600	0.3750
-0.22	0.3635	0.3875	0.4250

TABLE I CONTINUED

Z	Density of $\cos\theta_1$ f(Z) ($\rho = 0.4$, $\sigma = 100$, $\theta = 60°$)		
	By Integration	By Monte Carlo Method M=10000	M=1000
-0.18	0.3548	0.3550	0.3000
-0.14	0.3478	0.3325	0.5000
-0.10	0.3426	0.3275	0.2250
-0.06	0.3391	0.3450	0.5000
-0.02	0.3373	0.3325	0.2500
0.02	0.3371	0.3025	0.3000
0.06	0.3386	0.3750	0.3000
0.10	0.3417	0.3400	0.4500
0.14	0.3465	0.3625	0.3500
0.18	0.3530	0.3050	0.4500
0.22	0.3614	0.3775	0.2250
0.26	0.3716	0.3500	0.2750
0.30	0.3838	0.3850	0.3000
0.34	0.3980	0.4275	0.4500
0.38	0.4144	0.3800	0.2000
0.42	0.4331	0.4125	0.4500
0.46	0.4542	0.5250	0.5000
0.50	0.4779	0.4800	0.4000

TABLE I CONTINUED

Density of $\cos\theta_1$			
	f(Z)		
Z	($\rho = 0.4$, $\sigma = 100$, $\theta = 60°$		
	By Integration	By Monte Carlo Method	
		M=10000	M=1000
0.54	0.5046	0.5075	0.6250
0.58	0.5346	0.5750	0.5000
0.62	0.5690	0.5150	0.4500
0.66	0.6095	0.4875	0.5250
0.70	0.6598	0.6500	0.5000
0.74	0.7282	0.7550	0.6250
0.78	0.8353	0.9025	0.7500
0.82	1.5400	1.1075	1.1250
0.86	2.1185	1.8825	2.5250
0.90	0.0306	0.0750	0.1250
0.94	0.0042	0.0075	0.0000
0.98	0.0013	0.0025	0.0000

and showed a chi-square value of 53.02 and 35.15 (for the two normal number pair sets) for the whole sample and 72.15 and 48.40 respectively for the first 1000 numbers of this sample. The M = 1000 set used in Table-1 were therefore obtained using the initial numbers 21435881 and 776179721 and had a chi-square value of 53.34 and 44.94 respectively.

The disadvantages of the Monte Carlo method is obvious from a comparison of Figs. 3 and 4. The density function shows abrupt variations (Cheng, 1970) when Z changes by small amount and does not show a smooth variation as given by the integration method which is short of being exact only due to the limitations set by the economic considerations of the computer time. Moreover, different values are obtained as shown in Table-1 for different number of random numbers used. There is no way of getting consistant results because of this. A detailed study based on Monte Carlo techniques is given elsewhere (Cheng, 1970).

A close study of Figs. 3 and 4 reveals a very striking difference between the peak values obtained by the two methods. This is because of the essential difference in evaluating f(Z) in the two methods. In Monte Carlo techniques, we actually get the probability of Z taking a value z_1 between $z_1 - \frac{\Delta z_1}{2}$ and $z_1 + \frac{\Delta z_1}{2}$ and hence corresponds to the area $f(z_1)\Delta z_1$ under the density curve. We then calculate $f(z_1)$ and consider this to be the value of f(Z) at $Z = z_1$. In the integration method, we calculate directly f(Z) at $Z = z_1$.

When sharp peaks occur in f(Z), these two methods will therefore differ in the values obtained.

The results of integration performed in the close neighborhood of the two peaks is shown in Figs. 5 and 6. The approximate areas evaluated with these figures for Z = -.86 and Z = +.86 are respectively .065 and .065. The corresponding areas by the Monte Carlo method with M = 10000 are 0.053 and .073 and with M = 1000 are .073 and .101. These compare favorably well. It should however be noted, that the areas obtained with Monte Carlo method are bound to differ for different sets of random numbers employed with the larger number sets giving better accuracy.

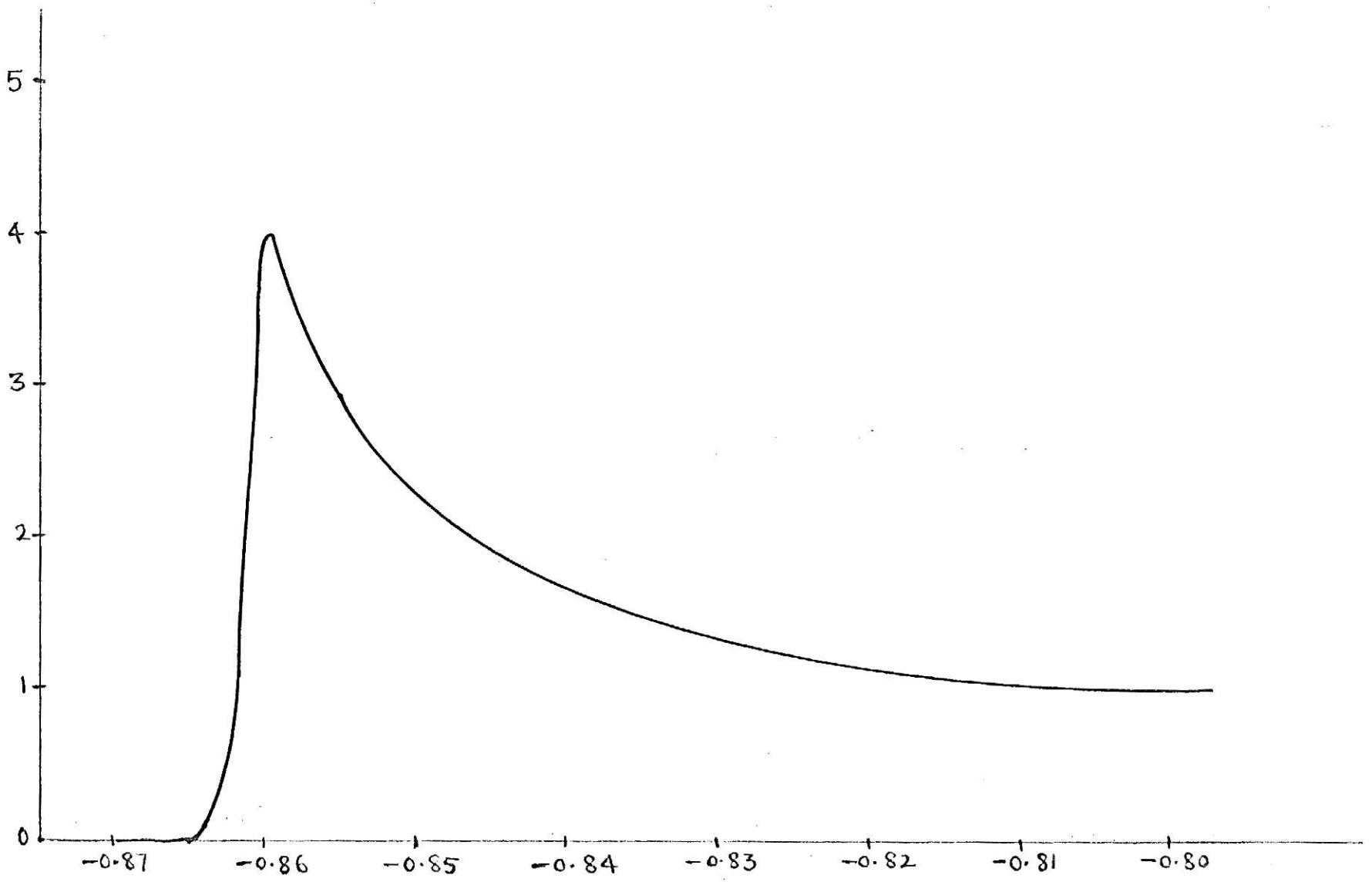

Fig. 5. Probability density in the neighborhood of $\text{Cos}\theta_1 = -0.86$.

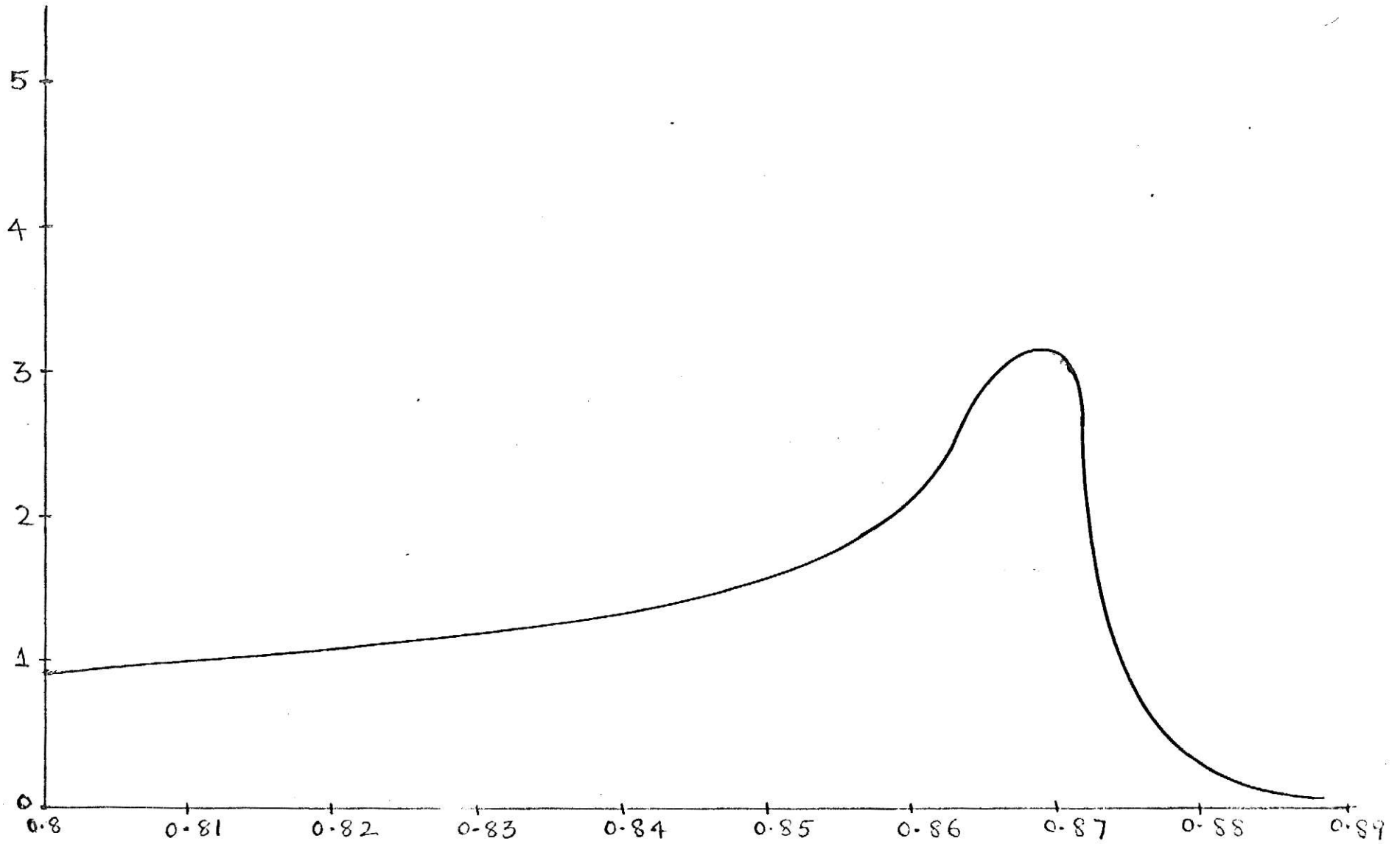

Fig. 6. Probability density in the neighborhood of $\text{Cos}\theta_1 = 0.86$.

CHAPTER V

DENSITY FUNCTIONS OF FRESNEL COEFFICIENTS

5.1 Introduction

Both the Monte Carlo and the numerical integration methods can be applied to determine the distribution of R_h and R_v. The first one is simpler and the computer program requires only minor modifications. The latter method, as mentioned earlier, requires elaborate study of the functions and entirely new programs. Therefore a histogram technique (Wadsworth & Bryan, 1960) has been used to determine the density of R_h and R_v after obtaining f(Z).

5.2 By Monte Carlo Method

In calculating the Fresnel coefficients, the angle θ_1 must be restricted to the range $0 \leqslant \theta_1 \leqslant \pi/2$. Angles greater than $\pi/2$ will imply the propagation of energy away from receiver which may give rise to multiple scattering and hence the tangent plane approximations discussed in sections 1.2 cannot be justified. Therefore the density of Z for $-1 \leqslant z \leqslant 0$ has to be discarded and this factor must be taken into account in calculating the densities $f(R_h)$ and $f(R_v)$ for $0 \leqslant z \leqslant 1$. In Monte Carlo method, the number N of random numbers which give rise to f(Z) in $-1 \leqslant z \leqslant 0$ is counted and the $f(R_h)$ and $f(R_v)$ curves corresponding to $0 \leqslant z \leqslant 1$ are normalized by dividing the ordinates with $(1 - \frac{N}{M})$.

The program used is given in Appendix A. The results are plotted in Fig. 7 and 8.

5.3 By Histogram Method

Here again, the negative values of Z are not acceptable. Therefore the area A under the distribution curve f(Z) for negative Z is evaluated (using trapezoidal rule).

We have:

$$R_h = \frac{\cos\theta_1 - \sqrt{K^2 - \sin^2\theta_1}}{\cos\theta_1 + \sqrt{K^2 - \sin^2\theta_1}} \qquad (5.2.1)$$

$$\therefore \quad \frac{\partial R_h}{\partial(\cos\theta_1)} = \frac{2(K^2-1)}{(K^2-\sin^2\theta_1)^{\frac{1}{2}}(\cos\theta_1+\sqrt{K^2-\sin^2\theta_1})^2} \qquad (5.2.2)$$

Therefore for $0 \leqslant z \leqslant 1$, $\frac{\partial R_h}{\partial Z}$ remains positive and hence R_h increases monotonically over this range.

For this reason, the probability of Z taking values between z_1 and z_2 with $z_2 > z_1$, is the same as R_h taking values between r_{h1} and r_{h2} with $r_{h2} > r_{h1}$. Therefore the area under the curve f(Z) between z_1 and z_2 and the corresponding r_{h1} and r_{h2} are calculated. This area is divided by $r_{h2} - r_{h1}$ to get the average height of the density curve $f(R_h)$ at the point $(r_{h2} + r_{h1})/2.0$. To account for the negative values of Z, this height is further divided by

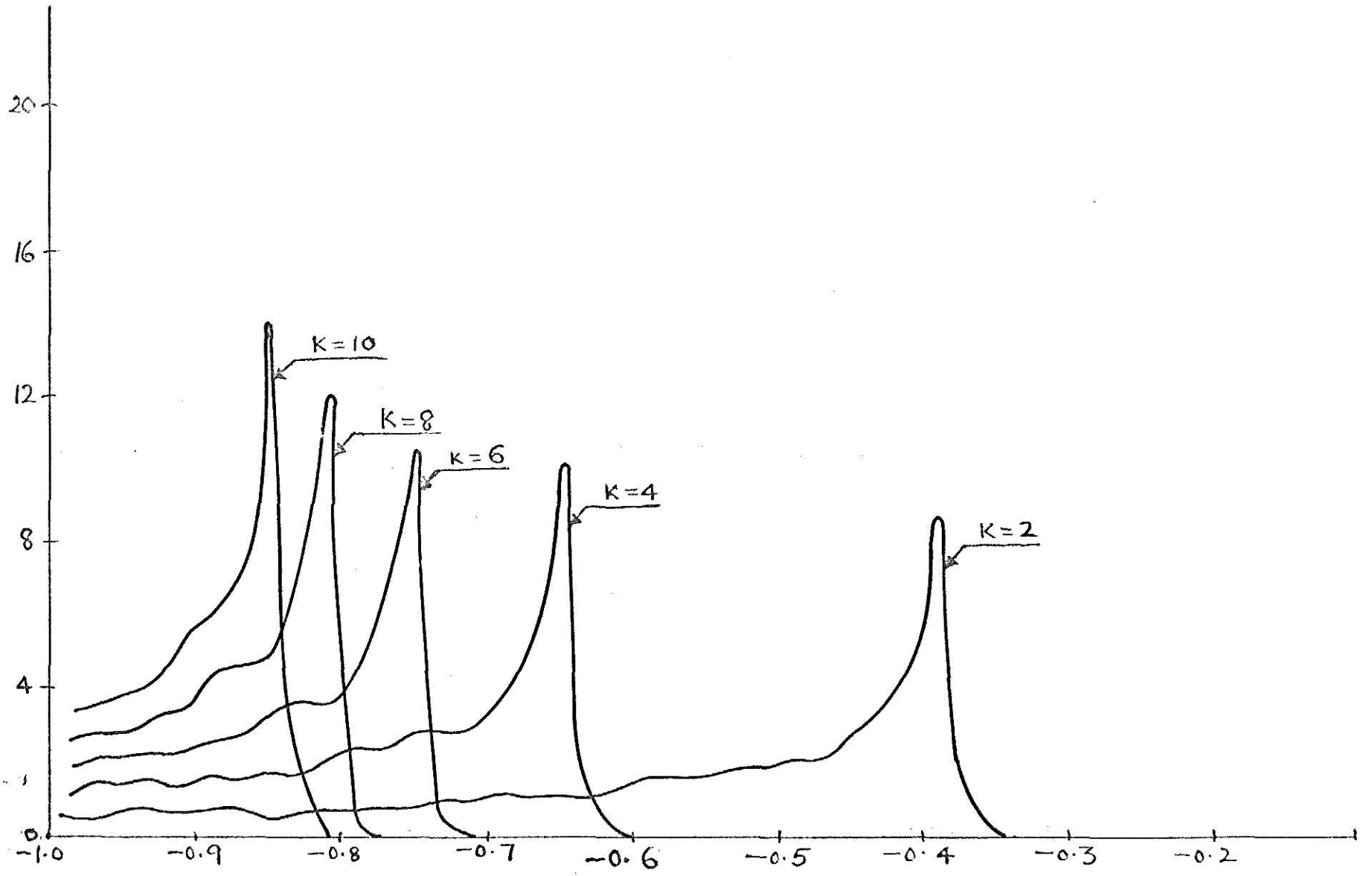

Fig. 7. Probability density of reflection coefficient (horizontal polarisation) by the Monte Carlo method.

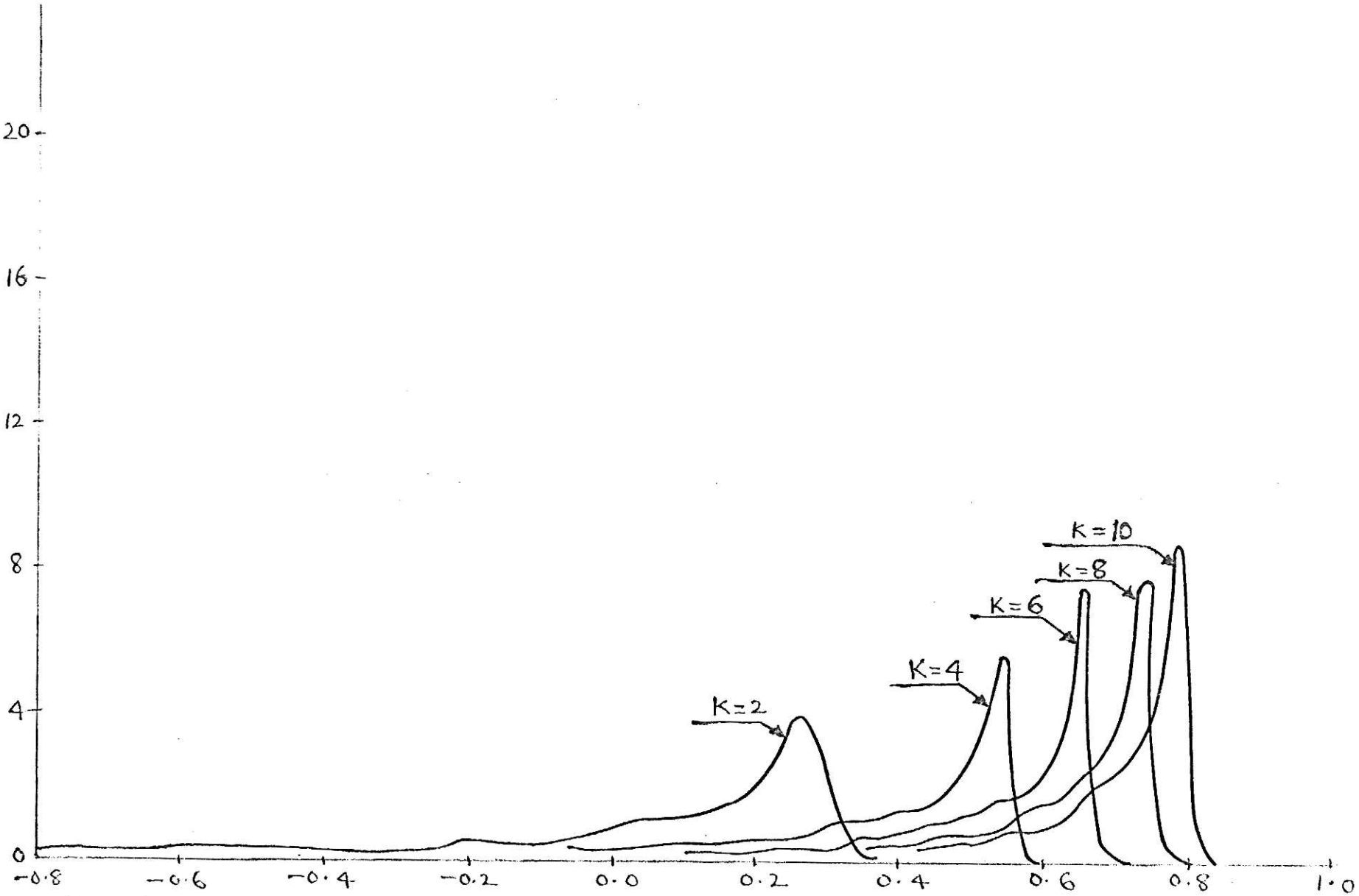

Fig. 8. Probability density of reflection coefficient (vertical polarisation) by the Monte Carlo method.

(1 - A) (A is the area as discussed above).

In case of R_v, we have:

$$R_v = \frac{K^2\text{Cos}\theta_1 - \sqrt{K^2 - \text{Sin}^2\theta_1}}{K^2\text{Cos}\theta_1 + \sqrt{K^2 - \text{Sin}^2\theta_1}} \qquad (5.2.3)$$

and

$$\frac{\partial R_v}{\partial(\text{Cos}\theta_1)} = \frac{2K^2\,(K^2-1)}{(K^2-\text{Sin}^2\theta_1)^{\frac{1}{2}}(K^2\text{Cos}\theta_1+\sqrt{K^2-\text{Sin}^2\theta_1})^2} \qquad (5.2.4)$$

$\frac{\partial R_v}{\partial Z}$ remains positive over $0 \leqslant z \leqslant 1$ and hence R_v also increases monotonically over this range. The same procedure as in the case of R_h is therefore applied to R_v also.

The density curves are plotted in Figs. 9 and 10, and the computer program is given in Appendix B.

A set of density curves for selected values of ρ, σ and θ are given in Appendix C for comparison purposes. These curves were obtained by numerical integration. The dotted portion in some of these curves shows the expected variation of f(Z) over the interval $0.98 \leqslant Z \leqslant 1.0$ since $Z = 0.98$ is the last point where the computer will evaluate f(Z) as per the method used to increment Z in the computer program given in Appendix B.

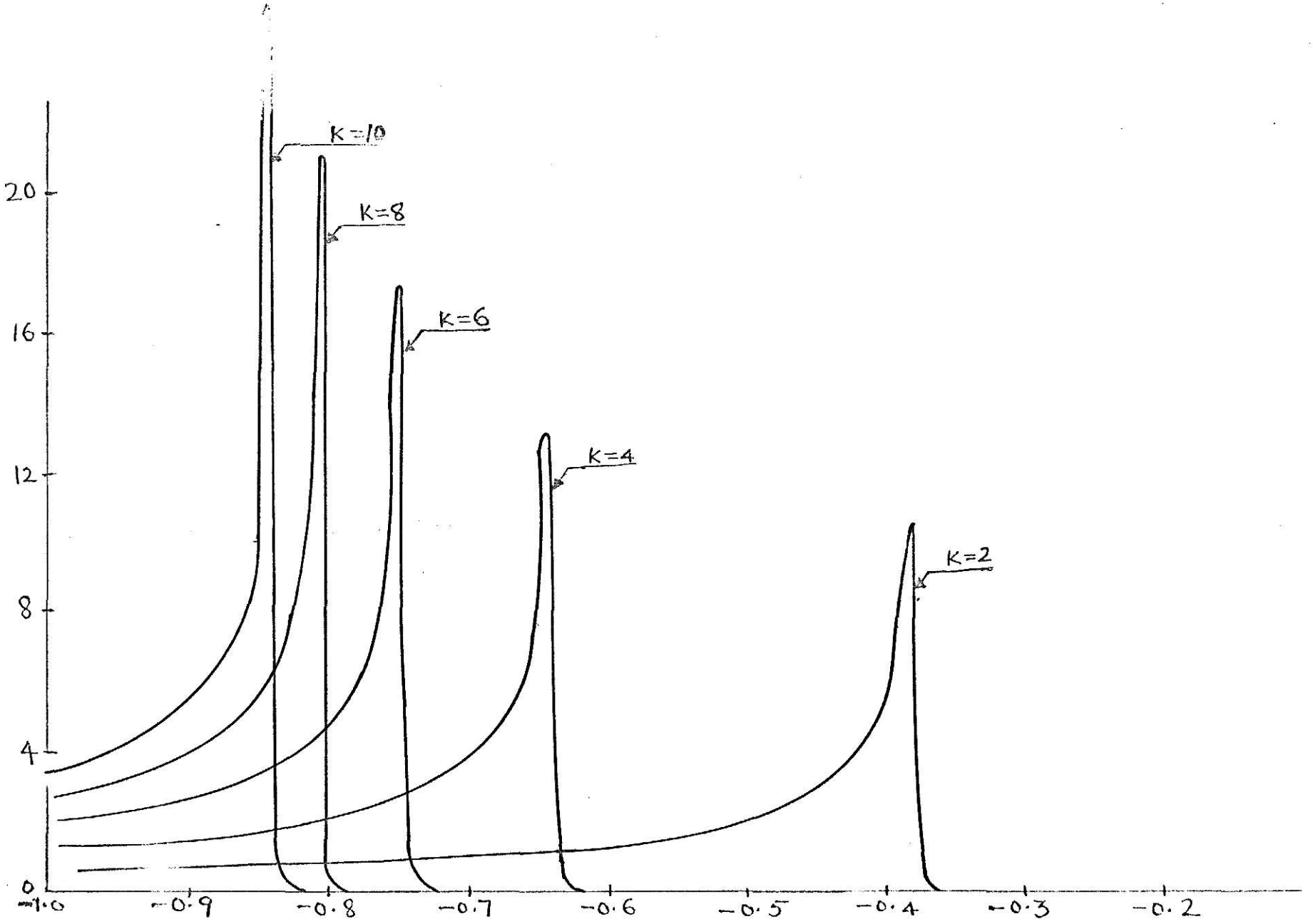

Fig. 9 Probability density of reflection coefficient (horizontal polarisation) by numerical integration.

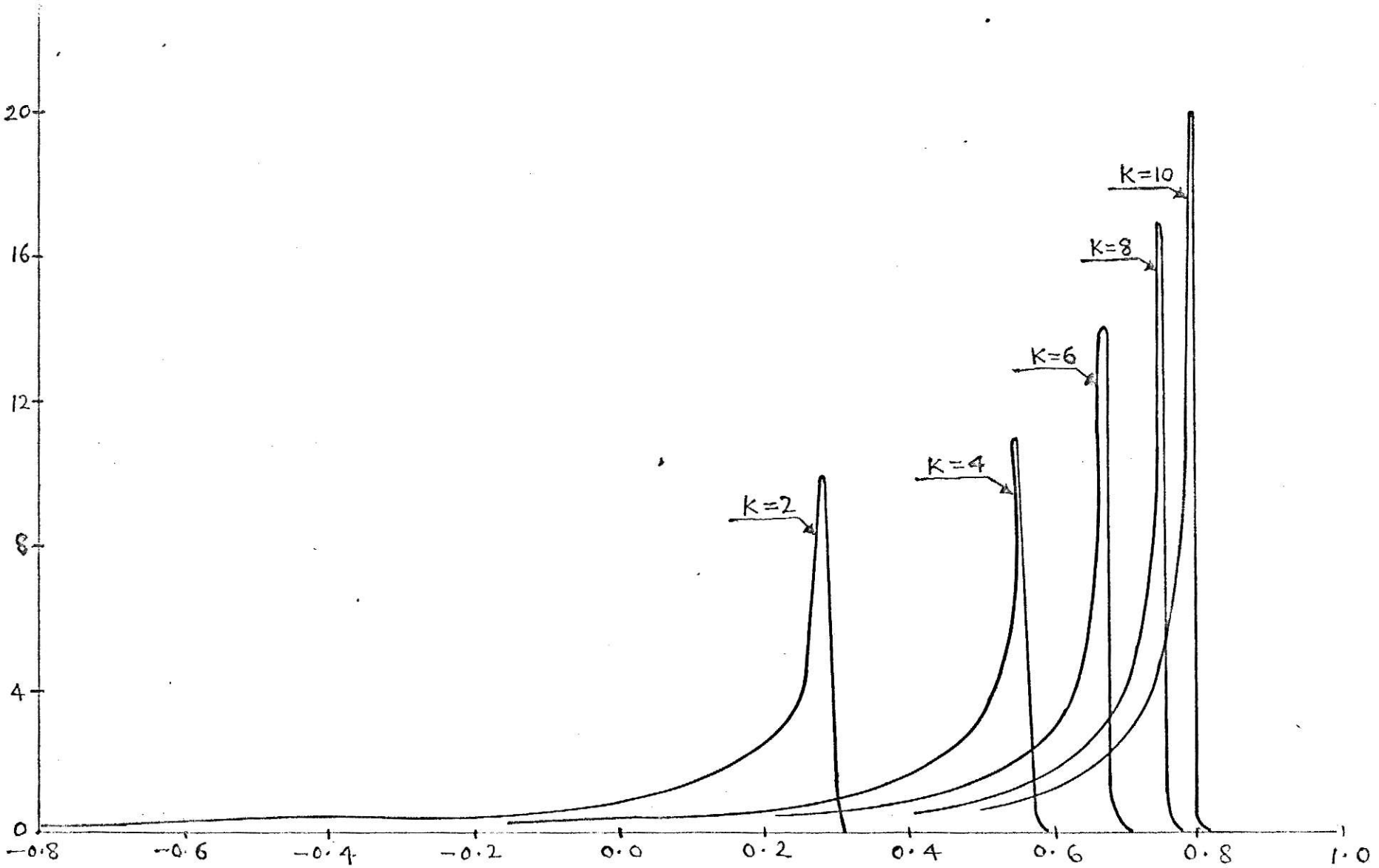

Fig. 10. Probability density of reflection coefficient (vertical polarisation) by numerical integration.

5.4 Computing Time

The computing time required to evaluate f(Z) is 0.061 hours by Monte Carlo method (with M = 10000) and 0.143 hours by numerical integration. However, a comparison of Figs. 3 and 4 shows that the curve obtained with Monte Carlo method is not sufficiently smooth and is in very large error near the sharp peaks of f(Z). One way to get better accuracy near the peaks is to evaluate f(Z) at smaller increments of Z. In this work increments of Z used were 0.04 as indicated in Table-1. If the increment is made 0.02, the time for Monte Carlo method will be considerably increased.

For evaluating Fresnel coefficients, for five values of dielectric constant numerical integration requires 0.145 hours whereas Monte Carlo method requires 0.149 hours. Again, a comparison of Figs. 7 and 8 with 9 and 10 shows the large error near sharp peaks when Monte Carlo method is used (incidentally, $f(R_h)$ and $f(R_v)$ in Figs. 7 and 8 were evaluated by incrementing R_h and R_v by 0.02 and 0.04 respectively. The marked difference in peak values of these curves compared to their numerical integration counterparts in Figs. 9 and 10 clearly indicate the effect of evaluating the density functions at smaller increments in the respective random variables.), though both methods need almost the same computing time.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

6.1 Conclusions

Exact analytical expression cannot be obtained for density of $\text{Cos}\theta_1$. Monte Carlo or numerical integration method has therefore to be used for determining the densities of $\text{Cos}\theta_1$, R_h and R_v. From the set of curves obtained using various ρ, σ, θ and K, the following conclusions are made:

1. $\text{Cos}\theta_1$ takes values only in the range ($-\text{Sin}\theta$, 1).

2. If σ is such that the 3σ points cannot make Y exceed $\text{Cot}\theta$, then the area under the density curve of $\text{Cos}\theta_1$ in region $-1 \leqslant \text{Cos}\theta_1 \leqslant 0$ is almost zero. Otherwise $\text{Cos}\theta_1$ is distributed with two peaks, one in the positive region and the other in negative region. It is to be noted that if $\theta = 90°$, $\text{Cos}\theta_1$ is symmetrically distributed about the f(Z) axis irrespective of the value of σ, since:

$$\text{Cos}\theta_1 = \frac{Y}{\sqrt{1 + X^2 + Y^2}} \quad .$$

3. The correlation coefficient ρ has less effect on the density of $\text{Cos}\theta_1$ compared to that of σ and θ as shown by the density curves given in Appendix C.

4. Monte Carlo techniques are satisfactory only if a rough estimate of the density is needed. For high accuracy the numerical integration method must be used.

5. As the dielectric constant becomes large the range over which R_h and R_v have significant spread decreases.

6.2 Recommendations

Once the densities of $\text{Cos}\theta_1$, R_h and R_v are known, it will be interesting to estimate the density of radar cross-section of the rough sphere. It is hoped that this will not get tied up with very complex integrals.

The Monte Carlo method must be studied in greater details to find out how to overcome its inaccuracies. The larger the number of random numbers used, the better will be the accuracy. This will increase the required computational time proportionately. If this method can be made to give as high an accuracy as that obtained by numerical integration, and the random numbers used can be kept less than 10000, the computer time needed will be much less.

When σ' is small compared to unity, f(Z) has significant spread over only a small range of values of Z. For example, if $\sigma' = 0.1$, f(Z) is spread over $0.4 \leqslant Z \leqslant 0.9$ as per Figs. 11 and 12. When f(Z) is evaluated by numerical integration, the integration has to be performed over the complete range $-\text{Sin}\theta \leqslant Z \leqslant 1$ which is more than 3 times the range $0.4 \leqslant Z \leqslant 0.9$. This suggests that a large saving in computer time can be achieved if the range over which f(Z) has significant spread is determined before actual integration. A hybrid method incorporating both Monte Carlo and numerical integration

techniques is therefore worth studying. This method may use 1000 random numbers to predetermine roughly, the significant spread of f(Z). Numerical integration is then performed only over this range. This will give better accuracy at lesser compution time.

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The financial assistance provided by the Department of Electrical Engineering is gratefully acknowledged.

APPENDIX A

```
      IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
      DIMENSION B(50,2),C(50,10),HH(2,2),R(10000,2),RHV(10)
      M=10000
      DO 15 K=1,2
      READ (1,5) IX
 5    FORMAT(I10)
      WRITE (3,6)K,IX
 6    FORMAT(10X,'VALUE OF IX',I1,'=',I10//)
      DO 15 J=1,M
      IY=IX*65539
      IF(IY)10,11,11
10    IY=IY+2147483647+1
11    YFL=IY
      YFL=YFL*0.4656613E-9
      IX=IY
15    R(J,K)=YFL
      P=2.0*3.1415927
      DO 16 J=1,M
      X=(-2.0*DLOG(R(J,1)))**0.5
      Y=P*R(J,2)
      R(J,1)=X*DCOS(Y)
16    R(J,2)=X*DSIN(Y)
      N=50
      DO 25 K=1,2
      READ (1,24) HH(1,K),HH(2,K),B(1,K)
24    FORMAT(3F10.3)
      DO 25 I=2,N
25    B(I,K)=B(I-1,K)+HH(2,K)
      DO 30 K=1,10
      DO 30 I=1,N
30    C(I,K)=0.0
      READ(1,31)RHO,SIGMA,THETA
31    FORMAT(3F10.2)
      Z=DSQRT(1.0-RHO**2)
      T=P*THETA/360.0
      CT=DCOS(T)
      ST=DSIN(T)
      L=0
      DO 41 J=1,M
      X=SIGMA*R(J,1)
      Y=SIGMA*(R(J,1)*RHO+R(J,2)*Z)
      F1=DSQRT(1.0+X**2+Y**2)
      F=(CT+Y*ST)/F1
      IF(F.GE.0.0)GO TO 32
      L=L+1
```

```
      GO TO 41
   32 Z1=F
      K=1
      DC=2.0
   33 DC2=DC**2
      Z2=(DC2-(1.0-Z1**2))**0.5
      Z3=DC2*Z1
      RHV(K)=(Z1-Z2)/(Z1+Z2)
      RHV(K+5)=(Z3-Z2)/(Z3+Z2)
      DC=DC+2.0
      K=K+1
      IF(DC.LE.10.0)GO TO 33
      KK=1
      K=1
   34 H=HH(1,KK)
      DO 35 I=1,N
      IF(RHV(K).LE.H)GO TO 40
   35 H=H+HH(2,KK)
   40 C(I,K)=C(I,K)+1.0
      K=K+1
      IF(K.LE.5)GO TO 34
      KK=2
      IF(K.LE.10)GO TO 34
   41 CONTINUE
      BN=M-L
      BX1=HH(2,1)*BN
      BX2=HH(2,2)*BN
      DO 50 K=1,5
      DO 50 I=1,N
      C(I,K)=C(I,K)/BX1
   50 C(I,K+5)=C(I,K+5)/BX2
      WRITE(3,60)RHO,SIGMA,THETA
   60 FORMAT(10X,'RHO=',F6.2,10X,'SIGMA=',F6.2,10X,'THETA=',F6.2////)
      WRITE(3,65)
   65 FORMAT(10X,'DENSITY OF R-HORIZONTAL'///)
      WRITE(3,70)
   70 FORMAT(12X,'DIELECTRIC CONSTANT',12X,'2.0',12X,'4.0',12X,'6.0',
     112X,'8.0',12X,'10.0'////)
      WRITE(3,75)(B(I,1),(C(I,K),K=1,5),I=1,N)
   75 FORMAT(20X,6F15.4//)
      WRITE(3,80)
   80 FORMAT(///10X,'DENSITY OF R-VERTICAL'////)
      WRITE(3,70)
      WRITE(3,75)(B(I,2),(C(I,K),K=6,10),I=1,N)
      STOP
      END
```

APPENDIX B

```
      IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
      DIMENSION FC(100),RH(26),RV(26)
      P=3.14159625
      P2=P/2.0
      P1=-P2
      N=1000
    6 READ(1,10)R,S,T
   10 FORMAT(3F10.3)
      WRITE(3,15)R,S,T
   15 FORMAT(10X,'RHO=',F5.3,10X,'SIGMA=',F6.2,10X,'THETA=',F5.1////)
      A=1.0/(2.0*P*(S**2)*DSQRT(1.0-R**2))
      B=1.0/(2.0*(S**2)*(1.0-R**2))
      I=T*P/180.0
      CT=DCOS(T)
      ST=DSIN(T)
      SS=ST**2
      Q=P1+T
      W=Q+T
      WRITE (3,20)
   20 FORMAT(16X,'Z',19X,'F(Z)'////)
      K=1
      Z=-0.98
   22 IF(Z.GT.-ST)GO TO 25
      FZ=0.0
      GO TO 42
   25 ZZ=Z**2
      SZ=SS-ZZ
      IF(SZ)26,28,31
   26 XM=ST*CT/SZ
      XN=(ZZ*(1.0-ZZ)/SZ**2)**0.5
      X1=DATAN(-XM-XN)
      X2=DATAN(-XM+XN)
      IF(Q.GE.X1)GO TO 27
      CALL INTEG(X1,X2,N,CT,ST,Z,ZZ,A,B,R,FZ1)
      FZ=FZ1
      GO TO 42
   27 CALL INTEG(Q,X2,N,CT,ST,Z,ZZ,A,B,R,FZ1)
      FZ=FZ1
      GO TO 42
   28 CALL INTEG(W,P2,N,CT,ST,Z,ZZ,A,B,R,FZ2)
      FZ=FZ2
      GO TO 42
   31 XM=ST*CT/SZ
      XN=(ZZ*(1.0-ZZ)/SZ**2)**0.5
      X1=DATAN(-XM-XN)
```

```
      X2=DATAN(-XM+XN)
      FZ3=0.0
      IF(Z.GT.0.0)GO TO 36
      IF(X2.GE.Q)GO TO 33
      CALL INTEG(X2,Q,N,CT,ST,Z,ZZ,A,B,R,FZ3)
      FZ=FZ3
   33 CALL INTEG(P1,X1,N,CT,ST,Z,ZZ,A,B,R,FZ4)
      FZ=FZ3+FZ4
      GO TO 42
   36 IF(X2.GE.Q)GO TO 37
      CALL INTEG(Q,P2,N,CT,ST,Z,ZZ,A,B,R,FZ5)
      FZ=FZ5
      GO TO 42
   37 CALL INTEG(X2,P2,N,CT,ST,Z,ZZ,A,B,R,FZ6)
      FZ=FZ6
   42 WRITE(3,50)Z,FZ
   50 FORMAT(10X,F10.4,10X,F10.4//)
      FC(K)=FZ
      K=K+1
      Z=Z+0.04
      IF(Z.LE.1.0)GO TO 22
      K=0
   55 K=K+1
      IF((FC(K).LE.0.0).AND.(K.LT.23))GO TO 55
      FZS=(FC(K)+FC(25))/2.0
      DO 60 I=1,24
      IF(I.LE.K)GO TO 60
      FZS=FZS+FC(I)
   60 CONTINUE
      FZS=FZS*0.04
      DC=2.0
   75 WRITE(3,77)DC
   77 FORMAT(10X,'DIELECTRIC CONSTANT=',F5.2,10X,'RH',10X,'PRH
     1',10X,'RV',10X,'PRV'////)
      DC2=DC**2
      Z1=0.0
      DO 80 I=1,26
      Z2=(DC2-(1.0-Z1**2))**0.5
      Z3=DC2*Z1
      RH(I)=(Z1-Z2)/(Z1+Z2)
      RV(I)=(Z3-Z2)/(Z3+Z2)
   80 Z1=Z1+0.04
      DO 85 I=1,25
      PAREA=FC(I+25)*0.04
      PAREAN=PAREA/(1.0-FZS)
      R1=(RH(I+1)+RH(I))/2.0
      PRH=PAREAN/(RH(I+1)-RH(I))
      R2=(RV(I+1)+RV(I))/2.0
      PRV=PAREAN/(RV(I+1)-RV(I))
   85 WRITE(3,90)R1,PRH,R2,PRV
```

```
90   FORMAT(40X,F10.5,2X,F10.5,3X,F10.5,2X,F10.5//)
     DC=DC+2.0
     IF(DC.LE.10.0)GO TO 75
95   STOP
     END


     SUBROUTINE INTEG(Q1,Q2,N,CT,ST,Z,ZZ,A,B,R,FZ)

     IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
     DIMENSION X(2001),Y(2001)
     N2=2*N
     NN=N2+1
     AN=NN
     DX=(Q2-Q1)/AN
     X(1)=Q1+DX/2.0
     DO 5 I=1,N2
5    X(I+1)=X(I)+DX
     DO 34 I=1,NN
     U=DTAN(X(I))
     UU=U**2
     D=CT+ST*U
     DD=D**2
     V=1.0+UU
     CC=DD/ZZ-V
     C=CC**0.5
     CU=B*(CC+UU)
     RCU=2.0*B*R*C*U
     ARG1=CU+RCU
     ARG2=CU-RCU
     Y(I)=0.0
     IF(ARG1.GE.174.63)GO TO 32
     Y(I)=DEXP(-ARG1)
32   IF(ARG2.GE.174.63)GO TO 34
     Y(I)=DEXP(-ARG2)+Y(I)
34   Y(I)=(V*DD/C)*Y(I)
35   FZ=Y(1)+Y(NN)+4.0*Y(2)
     DO 40 I=3,N2,2
40   FZ=FZ+2.0*Y(I)+4.0*Y(I+1)
     FZ=(A/DABS(Z**3))*FZ*DX/3.0
     RETURN
     END
```

APPENDIX C

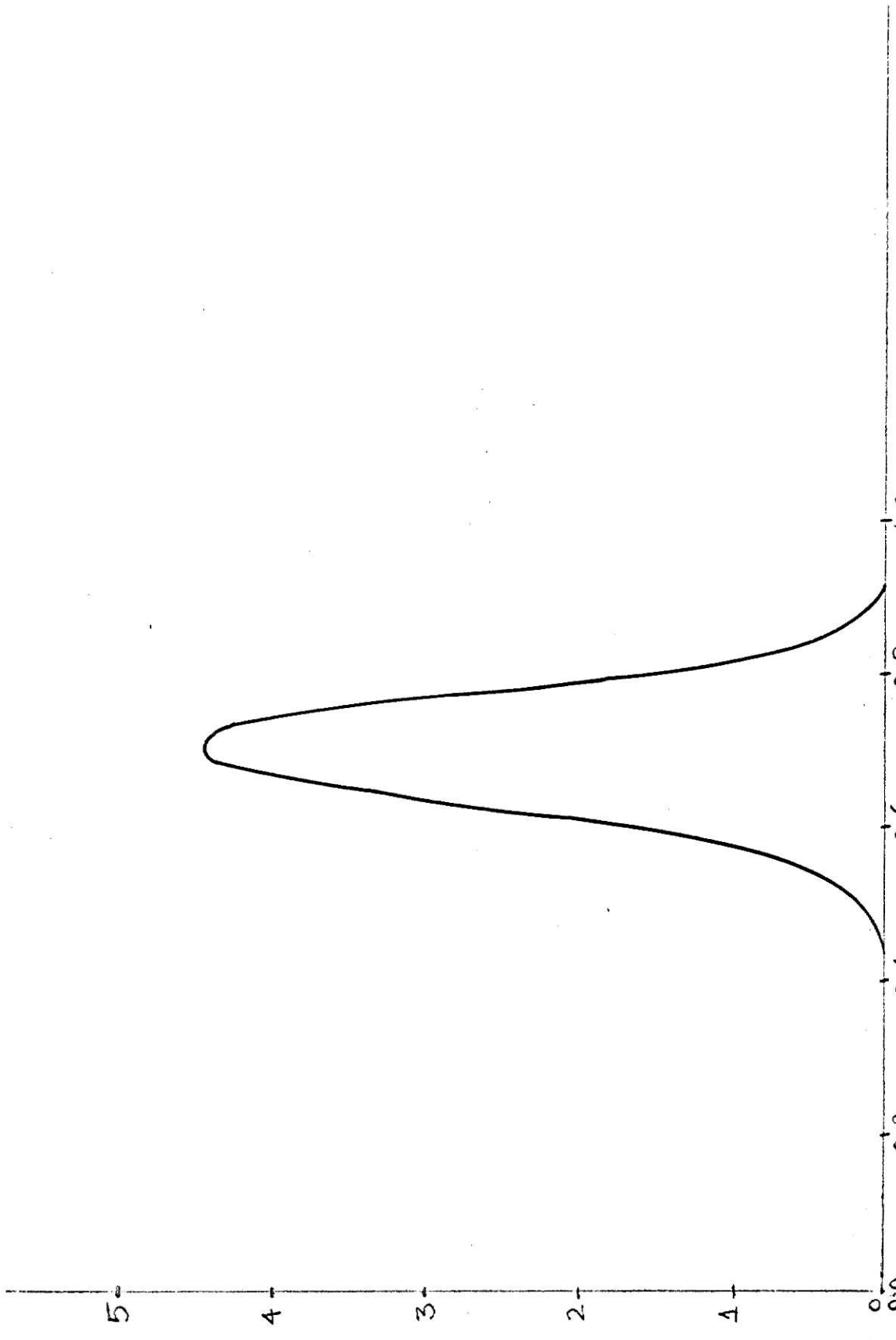

Fig. 11. Probability density of $\cos\theta_1$ (δ = 0.01, σ = 0.1, θ = 45°).

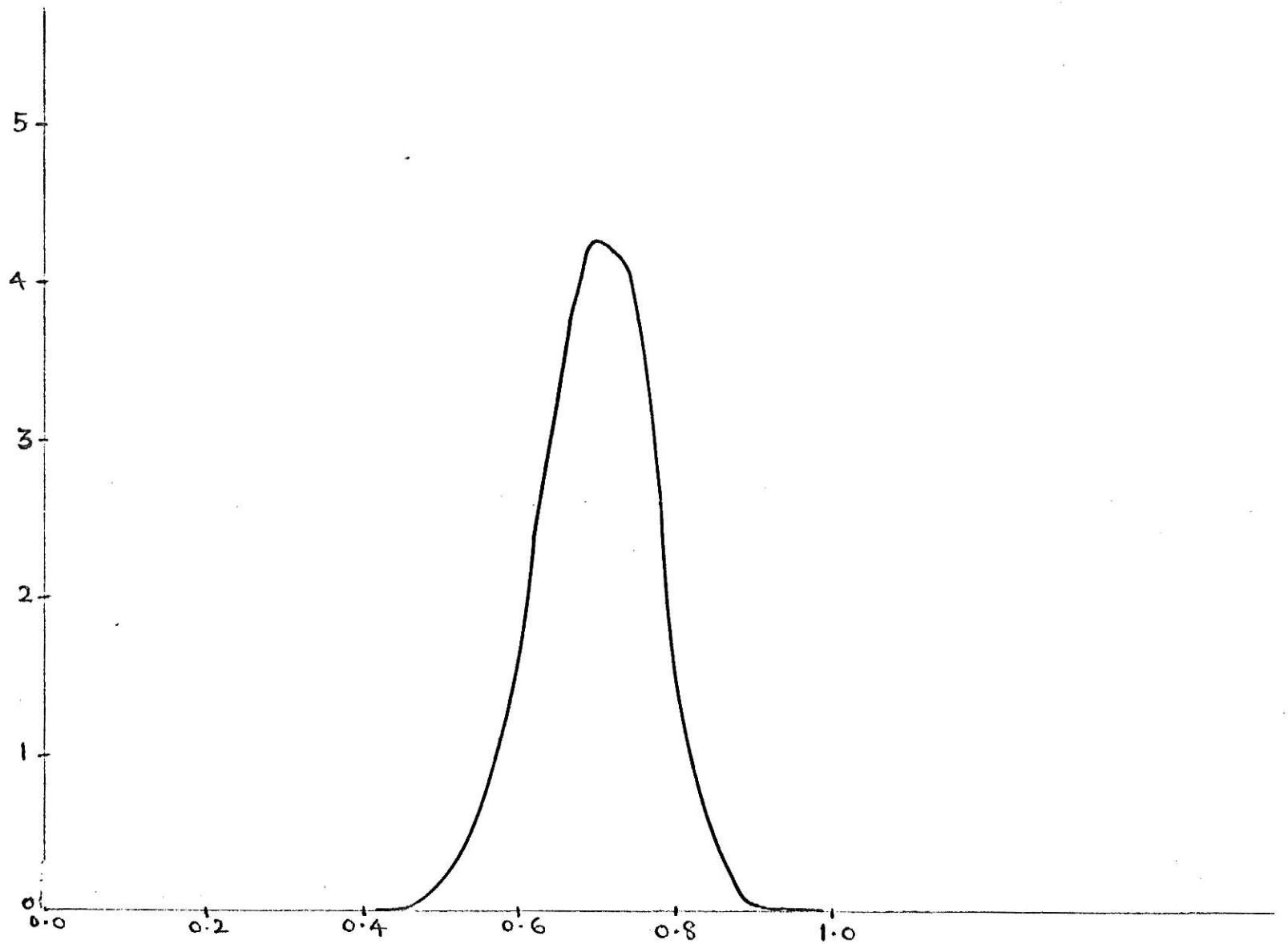

Fig. 12. Probability density of $\mathrm{Cos}\theta_1$ ($\rho = 0.4, \sigma = 0.1, \theta = 45°$).

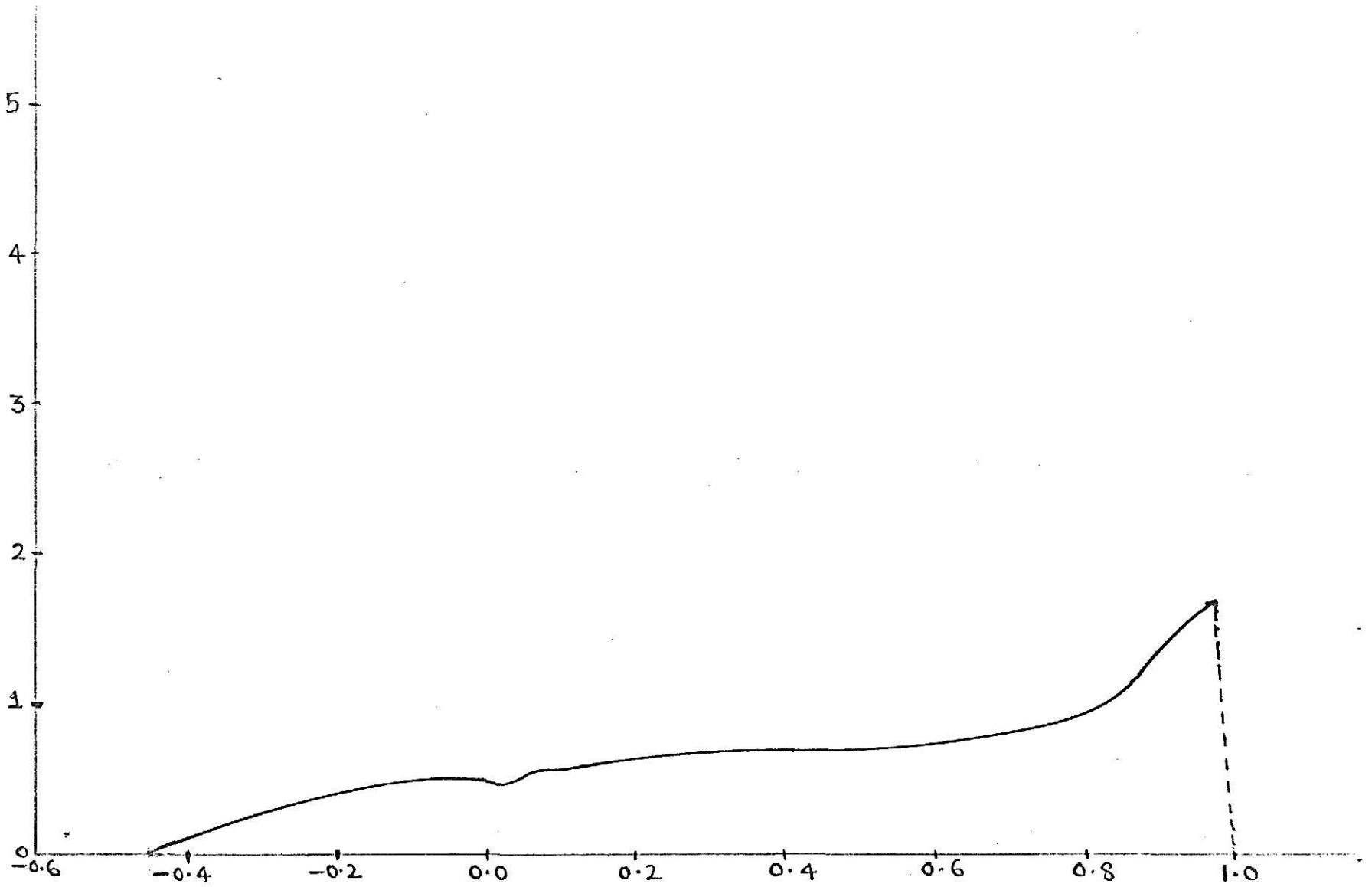

Fig. 13. Probability density of $\cos\theta_1$ ($\rho = 0.01, \sigma = 1.0, \theta = 45°$).

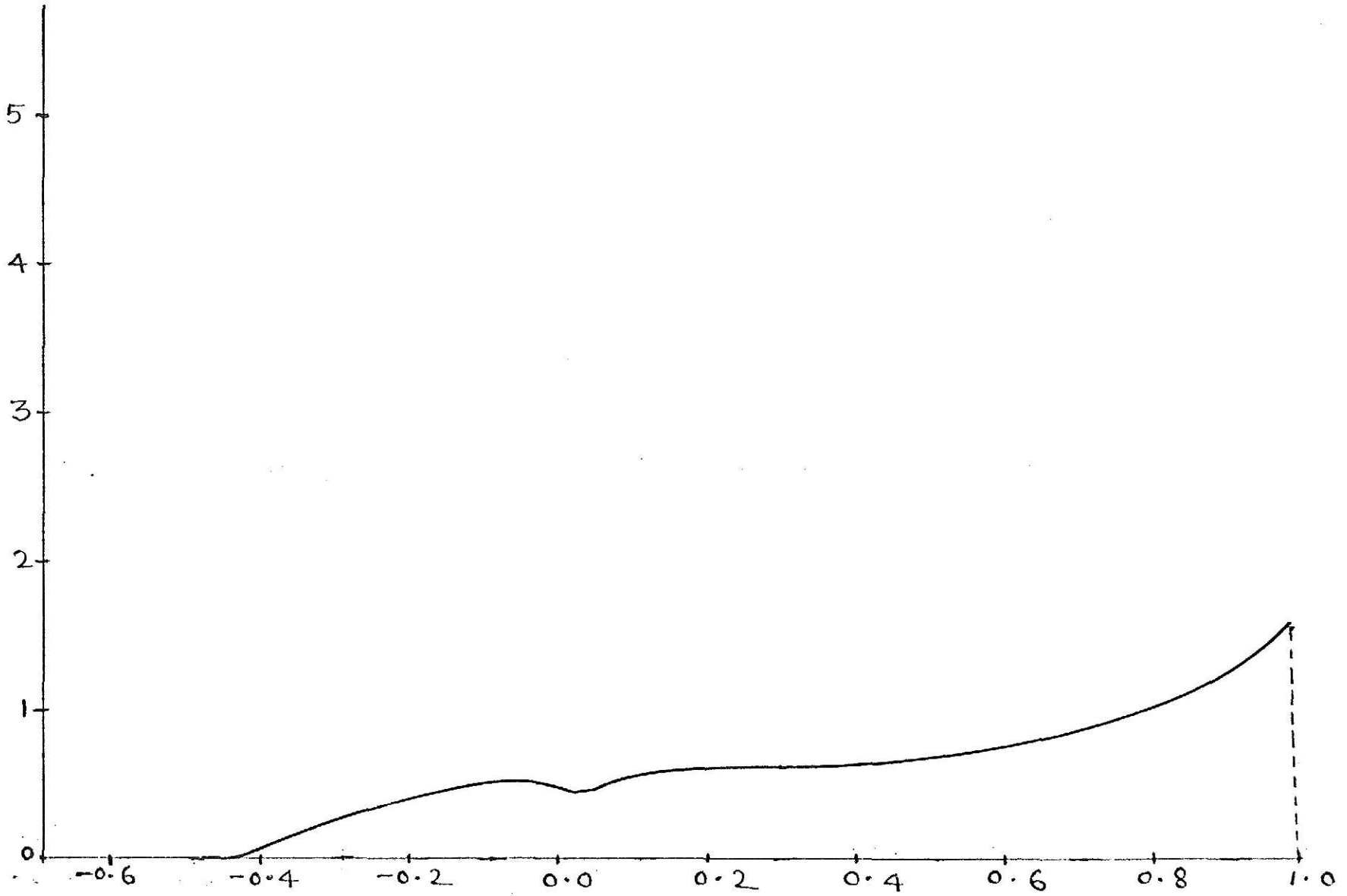

Fig. 14. Probability density of $\mathrm{Cos}\theta_1$ ($\rho = 0.4, \sigma = 1.0, \theta = 45°$).

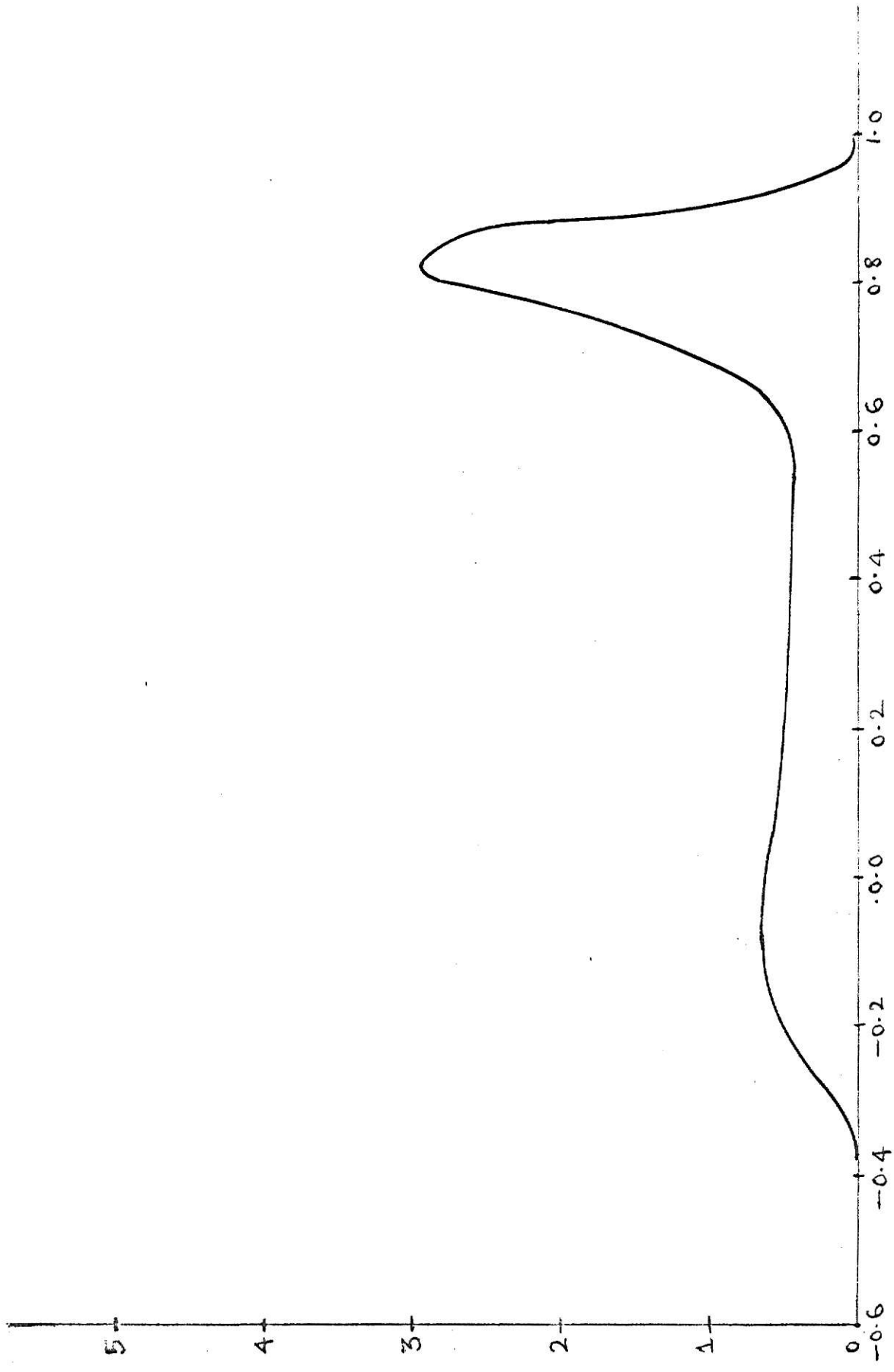

Fig. 15. Probability density of $\cos\theta_1$ ($\rho = 0.99, \sigma = 1.0, \theta = 45°$).

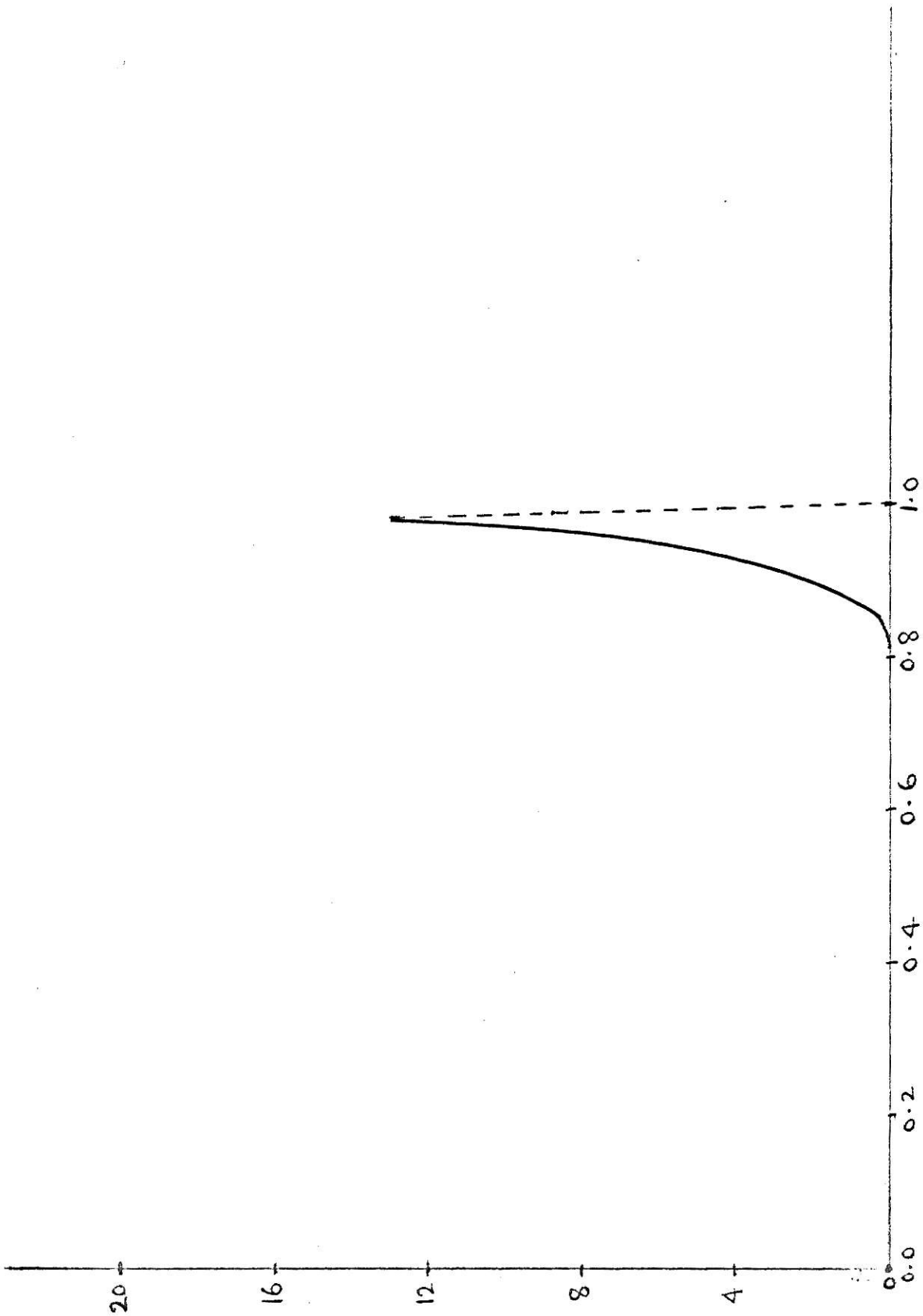

Fig. 16. Probability density of $\cos\theta_1$ ($\rho = 0.4$, $\sigma = 0.1$, $\theta = 15°$).

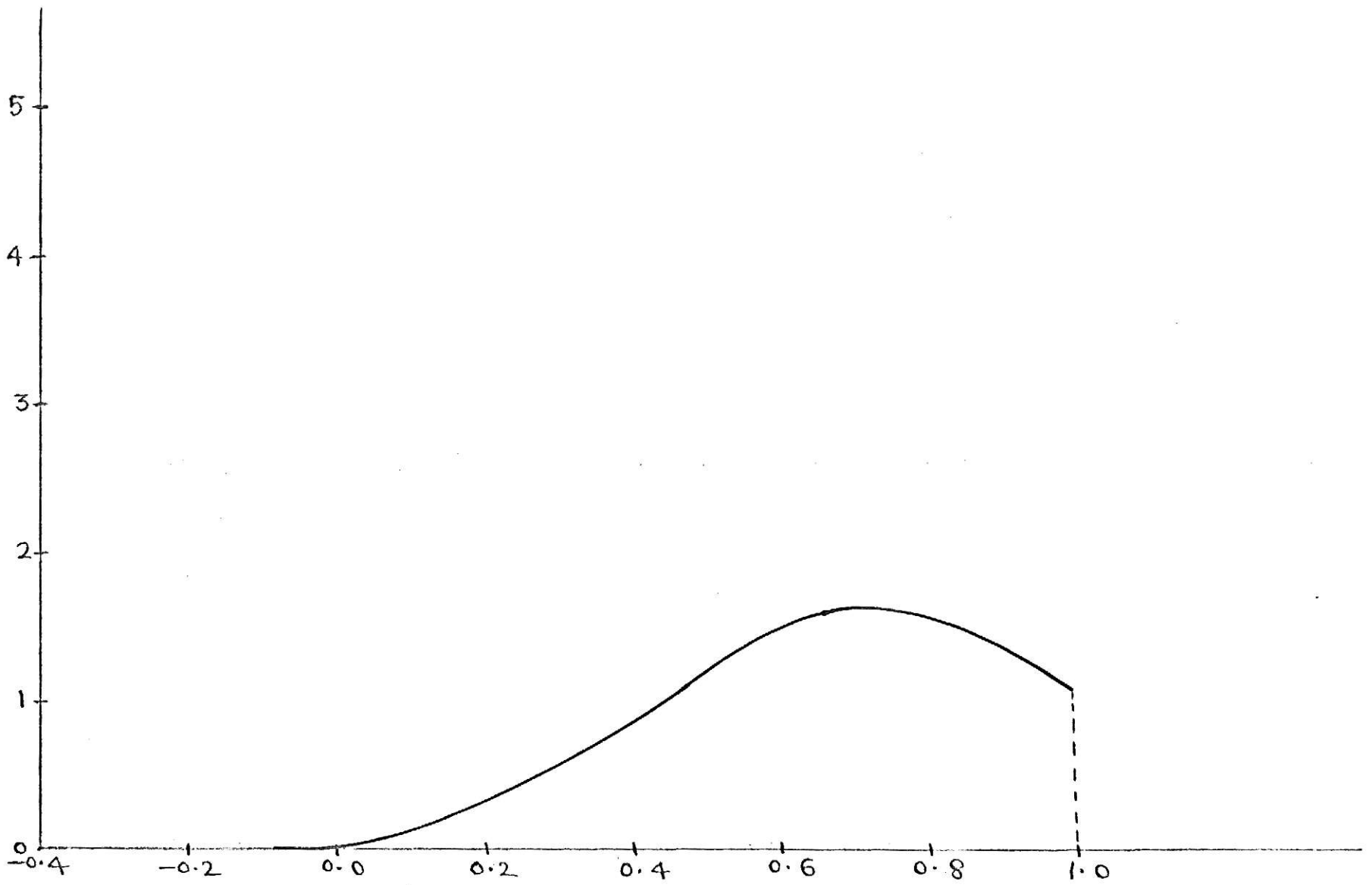

Fig. 17. Probability density of $\mathrm{Cos}\theta_1$ ($\rho = 0.4, \sigma = 1.0, \theta = 15°$).

A COMPARISON OF METHODS FOR OBTAINING THE PROBABILITY DENSITY OF FUNCTIONS RELATED TO RADAR BACKSCATTER

by

ABDUL LATEEF ABDUSSATTAR

B.E., University of Madras, 1957

M.E., Indian Institute of Science, 1959

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Electrical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1970

ABSTRACT

The Fresnel reflection coefficients for a smooth surface are given by:

$$R_h = \frac{\mathrm{Cos}\theta_1 - \sqrt{K^2 - \mathrm{Sin}^2\theta_1}}{\mathrm{Cos}\theta_1 + \sqrt{K^2 - \mathrm{Sin}^2\theta_1}}$$

$$R_v = \frac{K^2\,\mathrm{Cos}\theta_1 - \sqrt{K^2 - \mathrm{Sin}^2\theta_1}}{K^2\,\mathrm{Cos}\theta_1 + \sqrt{K^2 - \mathrm{Sin}^2\theta_1}}$$

where θ_1 is the angle between the normal to the surface and the direction of the receiver. For a rough spherical surface, this angle is a function of θ, the angle between the direction of the receiver and a "generalized" normal to the surface (equal to the angle of incidence for a plane surface) and is given by:

$$\mathrm{Cos}\theta_1 = \frac{\mathrm{Cos}\theta + Y\,\mathrm{Sin}\theta}{\sqrt{1 + X^2 + Y^2}} \quad .$$

The determination of the probability density function of $\mathrm{Cos}\theta_1$ with normally distributed X and Y is the main subject matter of this report. Because of the complex relationship between $\mathrm{Cos}\theta_1$ and X and Y, analytical expressions cannot be obtained for the density function. Among the available numerical

methods, the Monte Carlo technique is the easiest to use. The accuracy achievable with this method is poor. Numerical integration techniques were therefore tried with the IBM 360/50 computer. The density functions of R_h and R_v were determined using a "histogram" method after obtaining the distribution of $Cos\theta_1$.

A comparison of results obtained by the two methods shows that the numerical integration technique is to be preferred where high accuracy is desired.