On the fourth moment of Dirichlet L-functions: revisiting Soundararajan's approach

by

Palash Khanra

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Major Professor,
Prof. Xiannan Li

Abstract

Moments of L-functions are central objects in analytic number theory, providing insight into the size, distribution, and arithmetic behavior of L-functions. This report presents a detailed and self-contained exposition of Soundararajan's asymptotic formula for the fourth moment of Dirichlet L-functions associated with primitive Dirichlet characters modulo q at the central point $s = \frac{1}{2}$, valid uniformly for all moduli, including large highly composite ones. The proof relies on a splitting technique applied to the approximate functional equation. The diagonal terms of the short range sum yield the leading asymptotic of order $q(\log q)^4$, while the remaining contributions are shown to be of lower order using analytic estimates. The report develops the necessary background, establishes key lemmas, and carefully analyzes both main and error terms. This work provides a clear and accessible treatment of a significant result in the theory of L-functions, serving both as a reference for Soundararajan's method and as an introduction to the analytic techniques underlying higher moments of Dirichlet L-functions.

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Chapter 1

Introduction

The study of L-functions occupies a central position in modern number theory, connecting arithmetic and geometry. Their special values, particularly at the central point $s = \frac{1}{2}$, encode deep arithmetic information and are closely related to problems such as non-vanishing and equidistribution. This naturally leads to the study of moments of L-functions.

1.1 Moments of Dirichlet L-functions

In this report, we focus on moments of Dirichlet L-functions associated with primitive Dirichlet characters modulo q. For a positive integer k, the k^{th} moment of Dirichlet L-functions associated with primitive Dirichlet characters $\chi \pmod q$ at $s = \frac{1}{2}$ is defined by

$$M_k(q) = \frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} \left| L\left(\frac{1}{2}, \chi\right) \right|^k,$$

where $\sideset{}{^*}\sum_{\chi \pmod q}$ is the sum over primitive characters and $\varphi^*(q)$ is the number of such characters. Individual values of $\left|L\left(\frac{1}{2}, \chi\right)\right|$ can be large or small, and are difficult to estimate. However, their averages over χ often admits precise asymptotic formulas.

The first moment measures the average size of $\left|L\left(\frac{1}{2}, \chi\right)\right|$, indicates whether these values tend to be biased toward being large or small, and in particular can be used to detect non-vanishing. However, the first moment can be deceptive; since one or two massive values could pull the average up even if most values are tiny. Thus, from the first moment alone, one cannot tell whether large values are typical or instead rare but extreme. In contrast, very large values of $\left|L\left(\frac{1}{2}, \chi\right)\right|$ are much more visible in the second moment than in the first, because squaring amplifies their contribution relative to typical values. Higher moments continue this process; by raising the values to higher powers, they increasingly emphasize the contribution of extreme values. As the order of the moment increases, the number of summation variables arising from the Dirichlet series expansion also increases, leading to arithmetic relations among these variables. After applying the approximate functional equation and expanding the $2k^{th}$ moment, one is led to the expression, roughly of the form

$$\sideset{}{^*}\sum_{\chi \pmod q} \sum_{\substack{n_1, n_2, \cdots, n_k \\ m_1, m_2, \cdots, m_k}} \frac{\chi(n_1 n_2 \cdots n_k)\overline{\chi}(m_1 m_2 \cdots m_k)}{\sqrt{n_1 n_2 \cdots n_k \; m_1 m_2 \cdots m_k}} W,$$

where W is a smooth rapid decay function. Orthogonality yields congruence conditions of the form

$$n_1 n_2 \cdots n_k \equiv m_1 m_2 \cdots m_k \pmod q.$$

The main asymptotic contribution arises from the diagonal terms. The off-diagonal terms become substantially more difficult to handle for higher moments due to the emergence of shifted convolution sums

$$\sum_{a \leq x} d_k(a) d_k(a \pm l),$$

where $d_k(a)$ is the divisor function when $a = a_1 a_2 \ldots a_k$.

Modern perspectives on moments are guided by conjectures from random matrix theory. Following Katz and Sarnak, families of L-functions are classified into three principal symmetry types: unitary, symplectic, and orthogonal, each leading to different growth rates for moments at the central point. The family of Dirichlet L-functions associated with primitive Dirichlet characters modulo q is unitary. In this setting, random matrix theory predicts that the $2k^{th}$ moment grows on the order of $(\log q)^{k^2}$. In particular for the fourth moment,

$$M_4(q) = \frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \bmod q} \left| L\left(\frac{1}{2}, \chi\right) \right|^4 \asymp (\log q)^4.$$

1.2 Historical Background and the Fourth Moment

Historically, the problem of computing moments dates back to Hardy and Littlewood[1]. They studied the integrals of the type

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt, \quad k \geq 1$$

in order to study the Lindelöf hypothesis[1]. They mentioned the Lindelöf hypothesis is equivalent[2] to obtaining sharp bounds on all the positive, even integral moments. Since then, the moments of the Riemann zeta-function and of more general L-functions have become natural objects of study. In 1916, Hardy and Littlewood[1] computed the second moment of $\zeta(s)$ as follows

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt \sim T \log T.$$

Later in 1926, Ingham[3] obtained an asymptotic formula for the fourth moment of $\zeta(s)$ as

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 dt \sim \frac{T}{2\pi^2} (\log T)^4.$$

[1] It states that as $t \to \infty$, $\quad \zeta(\frac{1}{2} + it) \ll t^{\epsilon}, \quad \forall \epsilon > 0$. For the family of Dirichlet L-functions with Dirichlet character modulo q, it states that for any $\epsilon < 0$, $\quad L\left(\frac{1}{2}, f\right) \ll_{\epsilon} q^{\epsilon}$.

[2] Let $\zeta\left(\frac{1}{2} + it\right) \ll_{\epsilon} t^{\epsilon}$, $\forall \epsilon > 0$. If this holds then for any fixed k,

$$\left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} \ll t^{2k\epsilon} \ll t^{\epsilon}.$$

Therefore,

$$\int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt \ll \int_0^T t^{\epsilon} dt = T^{1+\epsilon}.$$

The converse part of this equivalence is straight-forward. One can also prove the equivalence for the family of Dirichlet L-functions. In 1918, Backlund[2] mentioned a necessary and sufficient condition for the Lindelöf hypothesis. It states that LH is equivalent to the statement that the number of non-trivial zeros of the Riemann zeta function with $\Re(s) > \frac{1}{2}$ and imaginary part between T and $T+1$ is $o(\log T)$ as $T \to \infty$.

More precisely

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt = \frac{T}{2\pi^2}(\log T)^4 + O(T(\log T)^3).$$

In 1979, Heath-Brown[4] improved this result as follows

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt = \sum_{i=0}^{4} a_i T(\log T)^i + O\left(T^{\frac{7}{8}+\epsilon}\right),$$

where $a_4 = \frac{1}{2\pi^2}$, and $a_3 = \frac{2}{\pi^2}\left(4\gamma - 1 - \log(2\pi) - \frac{12\zeta'(2)}{\pi^2}\right)$. In the Theorem 3 of the 1980 paper[5], Iwaniec mentioned that

$$\int_T^{T+T^{\frac{2}{3}+\epsilon}} \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \ll_\epsilon T^{\frac{2}{3}+\epsilon}.$$

In 1978, Heath-Brown[6] proved the t-aspect bound for the twelfth moment of the Riemann zeta function as follows

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{12} dt \ll_\epsilon T^{2+\epsilon}.$$

One can also obtain the Weyl-strength sub-convexity bound as

$$\zeta\left(\frac{1}{2}+it\right) \ll t^{\frac{1}{6}+\epsilon}.$$

In 1993, Motohashi[7,8] proved an exact spectral formula for a smoothed fourth moment of the Riemann zeta function: after applying a smooth weight, one can express the integral of $\left|\zeta\left(\frac{1}{2}+it\right)\right|^4$ in terms of a sum over automorphic forms, specifically involving the cubic central values $L\left(\frac{1}{2}, f\right)^3$ of Maass forms f for $SL_2(\mathbb{Z})$. No other asymptotic formulas for higher moment of $\zeta(s)$ has been computed so far.

The family of Dirichlet L-functions with primitive character χ modulo q is analogous in some ways to the Riemann zeta function in t-aspect. But for the Riemann zeta function, moments arise from continuous averaging over the parameter t, while for Dirichlet L-functions one averages discretely over characters. In 1979, Ramachandra[9] conjectured that, for Dirichlet characters modulo some prime p,

$$\sum_{\chi \pmod p} \left| L\left(\frac{1}{2}, \chi\right) \right|^4 \sim \frac{p}{2\pi^2} (\log p)^4.$$

In 1981, for a general q Heath-Brown[10] showed that

$$\sum_{\chi \pmod q}^{*} \left| L\left(\frac{1}{2}, \chi\right) \right|^4 = \frac{\varphi^*(q)}{2\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})} (\log q)^4 + O\left(2^{w(q)} q (\log q)^3\right), \tag{1.1}$$

where $w(q)$ denotes the number of distinct prime divisors of q. Since there are no primitive characters with $q \equiv 2 \pmod 4$, see Lemma 3.2, we can assume that $q \not\equiv 2 \pmod 4$. Given

$$\varphi^*(q) \asymp q \prod_{p|q} \left(1 - \frac{1}{p}\right)^2, \quad \prod_{p \le p_{w(q)}} \left(1 - \frac{1}{p}\right) \asymp \frac{1}{\log\log q}, \quad \text{and} \quad \prod_{p \le p_{w(q)}} \left(1 + \frac{1}{p}\right) \asymp \log\log q,$$

we obtain

$$\begin{aligned} g = \frac{\text{Main terms growth}}{\text{Error terms growth}} \asymp \frac{\varphi^*(q)(\log q)^4 \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})}}{2^{w(q)} q (\log q)^3} &\asymp \frac{\log q}{2^{w(q)}} \prod_{p|q} \frac{(1-p^{-1})^5}{(1+p^{-1})} \\ &\asymp \frac{\log q}{2^{w(q)} (\log\log q)^6}. \end{aligned}$$

If $w(q) \le \left(\frac{1}{\log 2} - \epsilon\right) \log\log q$ then as $q \to \infty$,

$$g \ge \frac{\log q}{2^{\frac{\log\log q}{\log 2} - \epsilon \log\log q} (\log\log q)^6} = \frac{2^{\epsilon \log\log q}}{(\log\log q)^6} \longrightarrow \infty.$$

Thus, when $w(q) \le \left(\frac{1}{\log 2} - \epsilon\right) \log\log q$ (which holds for almost all q), the main term dominates the error term, and (1.1) gives a q-analog of Heath-Brown's result for $\zeta(s)$. But if q is a large and highly composite[3] (or product of small primes) with $w(q) \ge \frac{\log\log q}{\log 2}$, then as $q \to \infty$,

$$g \le \frac{1}{(\log\log q)^6} \longrightarrow 0.$$

Therefore, the error term dominates the main term in (1.1). To improve the Heath-Brown's[10]

[3] We call a number x highly composite if $\forall\, n \le x,\ d(n) \le d(x)$.

result, in 2007 Soundararajan[11] provided the following theorem suggesting an asymptotic formula for the fourth moment valid for all q including the large highly composite ones.

Theorem. For all positive integer q we have

$$\sideset{}{^*}\sum_{\chi \pmod q} \left| L\left(\frac{1}{2}, \chi\right) \right|^4 = \frac{\varphi^*(q)}{2\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})} (\log q)^4 \left(1 + O\left(\frac{w(q)}{\log q}\sqrt{\frac{q}{\varphi(q)}}\right)\right) + O\left(q\,(\log q)^{\frac{7}{2}}\right). \tag{1.2}$$

Here $O\left(\frac{w(q)}{\log q}\sqrt{\frac{q}{\varphi(q)}}\right)$ is the primary error and $O\left(q(\log q)^{\frac{7}{2}}\right)$ is the secondary error term. Now one can observe that, as $q \to \infty$,

$$\frac{q(\log q)^{\frac{7}{2}}}{\varphi^*(q)\prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})}(\log q)^4} \asymp \frac{1}{(\log q)^{\frac{1}{2}}\prod_{p|q}\frac{(1-p^{-1})^5}{(1+p^{-1})}} \asymp \frac{(\log\log q)^6}{(\log q)^{\frac{1}{2}}} \longrightarrow 0.$$

Using the classical bounds, see Lemma 3.7 and Lemma 3.8,

$$w(q) \ll \frac{\log q}{\log\log q} \quad \text{and} \quad \varphi(q) \gg \frac{q}{\log\log q},$$

one can see that as $q \to \infty$,

$$\frac{w(q)}{\log q}\sqrt{\frac{q}{\varphi(q)}} \ll \frac{1}{\sqrt{\log\log q}} \longrightarrow 0.$$

Therefore indeed Soundararajan's theorem provides an asymptotic formula for the fourth moment, valid for all q.

1.3 Outline of the Report

The purpose of this report is to present a detailed and self-contained account of Soundararajan's proof of the fourth moment of Dirichlet L-functions associated with primitive characters, with particular emphasis on the uniformity of the argument in the modulus q. The report develops the necessary analytic tools, establishes the required background lemmas, and carefully carries out the estimation of both the main and error terms. Soundararajan's proof relies on a delicate splitting technique, in which the approximate functional equation for $\left|L\left(\frac{1}{2},\chi\right)\right|^2$ is decomposed into a short-range sum $B(\chi)$ and a long-range sum $C(\chi)$, with a cut-off parameter $Z = \frac{q}{2^{w(q)}}$. One can observe that it is relatively easy to extract an asymptotic from the diagonal structure of $B(\chi)$, while the contribution of $C(\chi)$ can be bounded effectively.

Chapter 2 introduces the splitting method and presents heuristics for extracting the asymptotic formula for the fourth moment. It proves Soundararajan's theorem using some important background lemmas and propositions from Chapter 4, 5 and 6 respectively.

Chapter 3 collects preliminary material, including Dirichlet characters, Dirichlet L-functions, the Dirichlet hyperbola method, the Mellin transform, and several other related things. Section 3.5 contains a collection of important lemmas required for proving the main background lemmas and propositions used in Soundararajan's work. This chapter also discusses the proof of the classical bounds $w(q) \ll \frac{\log q}{\log\log q}$, $\frac{q}{\varphi(q)} \ll \log\log q$, and $d(n) \ll_\epsilon n^\epsilon$. In addition, it includes the proofs of several properties of the divisor function, including the shifted divisor sum result by Heath-Brown (Lemma 3.11) and another result (Lemma 3.10) due to Linnik and Vinogradov.

Chapter 4 establishes the key background lemmas required for the analysis. Lemma 4.1 recalls the orthogonality relations for primitive Dirichlet characters. Lemma 4.2 provides a sharp uniform bound for off–diagonal solutions to certain multiplicative congruences, ensuring that off-diagonal terms contribute only lower order errors. Lemma 4.3 gives a precise asymptotic formula for harmonic sums restricted to integers coprime to a modulus, allowing for the clean extraction of main terms. Lemma 4.4 establishes sharp bounds and asymptotics for divisor-weighted sums under coprimality conditions, enabling the precise evaluation of diagonal contributions.

Chapters 5 and 6 are devoted to the proofs of the main propositions leading to the final result. Proposition 1 facilitates the extraction of the asymptotic formula for the fourth moment of Dirichlet L-functions associated with the primitive Dirichlet characters. The expansion of $|B(\chi)|^2$ leads to quadruple sums whose averages over characters impose multiplicative congruence conditions. The diagonal contribution of the $B(\chi)$ part produces a main term of size $(\log q)^4$, while all off–diagonal terms are shown to be of lower order. Proposition 2 shows that both the diagonal and off-diagonal contributions arising from $C(\chi)$ are also of lower order.

Chapter 7 looks at the bigger picture of L-function research and explains where this report fits into the field. It compares the progress made on the fourth moment from prime to general moduli, with recent breakthroughs on the sixth, eighth moment, and twelfth moments. As a further direction, the author has noticed and mentioned a few research questions in these works.

Chapter 2

The Splitting Technique and the Fourth Moment of Dirichlet L-Functions

Assuming the work in the Chapters 4, 5 and 6, this chapter establishes an asymptotic formula for the fourth moment of Dirichlet L-functions,

$$\frac{1}{\varphi^*(q)} \sum_{\chi \bmod q}^{*} \left| L(\frac{1}{2}, \chi) \right|^4 .$$

Using the approximate functional equation, we outline the main strategy of the proof: decomposing the character sum $A(\chi)$ (a Dirichlet polynomial of length q, stated explicitly later.) into a short-range part, $B(\chi)$, and a long-range part, $C(\chi)$. We justify the specific choice of the cut-off parameter, $Z = \frac{q}{2^{\omega(q)}}$, and subsequently show how the asymptotic result is derived primarily from the diagonal contribution of $B(\chi)$, while the off-diagonal terms of $B(\chi)$ and the whole of $C(\chi)$ only contribute to the error.

2.1 The splitting, $A(\chi) = B(\chi) + C(\chi)$

Let $\chi \pmod q$ be a primitive Dirichlet character. The approximate functional equation for the fourth moment of $L(s, \chi)$ on the critical line $s = \frac{1}{2}$ is given by

$$\left| L\left(\frac{1}{2}, \chi\right) \right|^4 = 4|A(\chi)|^2, \tag{2.1}$$

where

$$A(\chi) = \sum_{a,b\geq 1} \frac{\chi(a)\overline{\chi}(b)}{\sqrt{ab}} W_\alpha\left(\frac{\pi ab}{q}\right) \tag{2.2}$$

is a Dirichlet polynomial of length q and $W_\alpha(x)$ is defined as in (3.4).

The orthogonality for the primitive Dirichlet characters is as follows, see Lemma 4.1. If $(r,q)=1$, then

$$\sideset{}{^*}\sum_{\chi \pmod q} \chi(r) = \sum_{k|(q,r-1)} \mu\left(\frac{q}{k}\right)\varphi(k).$$

If q is prime, the character sum is easy to handle, since k can be just 1 and q. However, when q is highly composite or product of small primes, we must deal with $2^{w(q)}$ separate congruence conditions, because of the $2^{w(q)}$ many square-free divisors of q. Let us consider

$$A(\chi) = B(\chi) + C(\chi),$$

where $Z = \frac{q}{2^{w(q)}}$, and

$$B(\chi) = \sum_{\substack{a,b\geq 1,\\ ab\leq Z}} \frac{\chi(a)\bar{\chi}(b)}{\sqrt{ab}} W_\alpha\left(\frac{\pi ab}{q}\right), \tag{2.3}$$

$$C(\chi) = \sum_{\substack{a,b\geq 1,\\ ab> Z}} \frac{\chi(a)\bar{\chi}(b)}{\sqrt{ab}} W_\alpha\left(\frac{\pi ab}{q}\right). \tag{2.4}$$

An important question is why we split the range and why we choose the specific cut-off $Z = \frac{q}{2^{w(q)}}$. Let k be a small divisor of q and consider integers $1 \leq a,b,c,d \leq Z$ with $ab \asymp Z$, and $cd \asymp Z$. Heuristically, the number of off-diagonal solutions (a,b,c,d) to the congruence $ac \equiv \pm bd \pmod k$ is of the same order of magnitude as $\frac{Z^2}{k}$; that is,

$$\sum_{\substack{ab,cd\asymp Z,\\ ac\neq bd,\\ ac\equiv \pm bd \pmod k}} 1 \asymp \frac{Z^2}{k}.$$

While dealing with off-diagonal part of the short range sum $B(\chi)$, one come across a sum like the below one and heuristically,

$$\sum_{k|q} \mu^2\left(\frac{q}{k}\right)\varphi(k) \sum_{\substack{ab,cd\asymp Z,\\ ac\neq bd,\\ ac\equiv \pm bd \pmod k,\\ (abcd,k)=1}} \frac{1}{\sqrt{abcd}} \asymp \sum_{k|q} \mu^2\left(\frac{q}{k}\right)\frac{\varphi(k)}{k} Z \asymp 2^{w(q)}Z.$$

If $Z \asymp q$ (in particular, if $Z = q$), then the term $2^{w(q)}q$ would dominate the main term $q(\log q)^4$. Indeed, when q is large and highly composite (for example, a product of many small primes), the factor $2^{w(q)}$ can exceed $(\log q)^4$, thereby preventing us from obtaining the desired moment structure. In contrast, choosing Z such that $2^{w(q)}Z \asymp q$ preserves the main term. This motivates our choice of the cut-off $Z = \frac{q}{2^{w(q)}}$. With this choice, it is easier to derive an asymptotic formula for $B(\chi)$ and to obtain a satisfactory bound for $C(\chi)$. The idea is explained heuristically through the lemmas below.

Lemma 2.1. For positive integers $a, b, c, d, q \geq 1$, and $Z = \frac{q}{2^{w(q)}}$,

$$\sum_{\substack{ab\leq Z, cd\leq Z,\\ ac=bd,\\ (abcd,q)=1}} \frac{1}{\sqrt{abcd}} \asymp (\log q)^4.$$

Proof. Let $(r, s) = 1$. With the change of variables $a = gr, b = gs, c = hs, d = hr$, we have $ac = ghrs = bd$, and $ab = g^2rs$, $cd = h^2rs$. Since $(r, s) = 1$, $n = rs$ is squarefree and the number of ways of writing $n = rs$ is $d(n) = 2^{w(n)}$. Therefore,

$$S = \sum_{\substack{ab\leq Z, cd\leq Z,\\ ac=bd,\\ (abcd,q)=1}} \frac{1}{\sqrt{abcd}} = \sum_{\substack{n\leq Z,\\ (n,q)=1,\\ n \text{ is square-free}}} \frac{2^{w(n)}}{n} \left(\sum_{\substack{g\leq\sqrt{\frac{Z}{n}},\\ (g,q)=1}} \frac{1}{g} \right)^2.$$

Using Lemma 4.4, and Lemma 4.3 respectively,

$$S \asymp \sum_{\substack{n\leq Z,\\ (n,q)=1,\\ n \text{ is square-free}}} \frac{2^{w(n)}}{n} \left(\frac{\varphi(q)}{q} \log\frac{Z}{n}\right)^2 \asymp \left(\frac{\varphi(q)}{q}\right)^2 \sum_{\substack{n\leq Z,\\ (n,q)=1,\\ n \text{ is square-free}}} \frac{2^{w(n)}}{n}\left(\log\frac{Z}{n}\right)^2 \asymp (\log q)^4.$$

□

Lemma 2.1 shows that the diagonal contribution to the weighted sum in the short range is asymptotic to $(\log q)^4$. Thus, the short range determines the asymptotic structure of the fourth moment. Moreover, to bound the off-diagonal in the short-range, we get $2^{w(q)}Z$, which is natural since it does not exceed q. This is why it is easy to obtain an asymptotic formula for $B(\chi)$.

Lemma 2.2. Let m, n, q be positive integers and $Z = \frac{q}{2^{w(q)}}$. For sufficiently large q, let

$$S_1 = \sum_{\substack{m,n \le Z, \\ m=n}} \frac{1}{\sqrt{mn}}, \quad \text{and} \quad S_2 = \sum_{\substack{Z<m,n \le q, \\ m=n}} \frac{1}{\sqrt{mn}}.$$

Then, $S_2 = o(S_1)$.

Proof. **Case 1: Short-range, $m, n \le Z$.**

$$S_1 = \sum_{m \le Z} \frac{1}{m} \asymp \log Z = \log q - w(q) \log 2 \asymp \log q \left(1 - \frac{\log 2}{\log\log q}\right) \asymp \log q.$$

Case 2: Mid-range, $Z < m, n \le q$.

$$S_2 \asymp \log \frac{q}{Z} = \log 2^{w(q)} \asymp \frac{\log q}{\log\log q}, \quad \text{using} \quad w(q) \asymp \frac{\log q}{\log\log q}.$$

Therefore,

$$\lim_{q\to\infty} \frac{S_2}{S_1} \asymp \lim_{q\to\infty} \frac{1}{\log\log q} = 0.$$

$\square$

Thus, Lemma 2.2 implies that for sufficiently large q, the diagonal contribution from the interval $[1, Z]$ is significantly larger than the one from $(Z, q]$.

Lemma 2.3. Let m, q be positive integers. For all $c > 0$,

$$\sum_{m>q} \frac{1}{m} \left(W_\alpha\left(\frac{\pi m}{q}\right)\right)^2 \ll_c 1.$$

Proof. Let us consider a dyadic interval $2^k q \le m < 2^{k+1}q$, then

$$S = \sum_{m>q} \frac{1}{m}\left(W_\alpha\left(\frac{\pi m}{q}\right)\right)^2 = \sum_{k}^{d} \sum_{2^k q \le m < 2^{k+1} q} \frac{1}{m}\left(W_\alpha\left(\frac{\pi m}{q}\right)\right)^2.$$

Since $\frac{1}{m} \asymp \frac{1}{2^k q}$, $\frac{m}{q} \asymp 2^k$, $W_\alpha(x) \ll_c x^{-c}$, we have

$$S \ll_c \sum_{k}^{d} \frac{1}{2^k q}\, 2^{-2kc}\, 2^k q \ll_c \sum_{k}^{d} \left(\frac{1}{2^{2c}}\right)^k \ll_c 1.$$

□

Therefore all the other digonal terms has smaller contribution than the diagonal contribution of the range $[1, Z]$.

Lemma 2.4. Let q be a large positive integer, $Z = \frac{q}{2^{w(q)}}$, and k be a divisor of q. Then

$$\sum_{\substack{k|q,\\ k \text{ is square-free}}} \sum_{\substack{a,b,c,d \ge 1,\\ ab,cd \le Z,\\ ac \ne bd,\ k|(ac \pm bd),\ (abcd,k)=1}} \frac{k}{\sqrt{abcd}} \ll q(\log q)^3.$$

Proof. Let us break the range $ab, cd \le Z$ into dyadic blocks and consider one such dyadic block as $Z_1 \le ab < 2Z_1$ and $Z_2 \le cd < 2Z_2$. Using Lemma 4.2,

$$\sum_{\substack{Z_1 \le ab < 2Z_1,\\ Z_2 \le cd < 2Z_2,\\ ac \ne bd,\ k|(ac \pm bd),\ (abcd,k)=1}} \frac{k}{\sqrt{abcd}} \ll \sqrt{Z_1 Z_2}(\log Z_1 Z_2)^3 \ll \sqrt{Z_1 Z_2}(\log q)^3.$$

The total dyadic contribution is

$$\sum_{Z_1 \le Z}^{d} \sum_{Z_2 \le Z}^{d} \sqrt{Z_1 Z_2}(\log q)^3 \ll (\sqrt{Z})(\sqrt{Z})(\log q)^3 = Z(\log q)^3.$$

Therefore,

$$\sum_{\substack{k|q,\\ k \text{ is square-free}}} Z(\log q)^3 = 2^{w(q)} Z(\log q)^3 = q(\log q)^3.$$

□

Lemma 2.5. Let q be a large positive integer, $Z = \frac{q}{2^{w(q)}}$. Then

$$\sum_{\substack{a,b,c,d \geq 1, \\ Z < ab, cd \leq q, \\ (abcd,q)=1,\ ac \neq bd,\ q|(ac \pm bd)}} \frac{1}{\sqrt{abcd}} \ll q(\log q)^3.$$

Proof. Considering the dyadic block with $Z_1, Z_2 \leq q$, and applying Lemma 4.2, we have,

$$\sum_{\substack{Z_1 \leq ab < 2Z_1, \\ Z_2 \leq cd < 2Z_2, \\ ac \neq bd,\ (abcd,q)=1, \\ q|(ac \pm bd)}} \frac{1}{\sqrt{abcd}} \ll \sqrt{Z_1 Z_2}(\log Z_1 Z_2)^3.$$

Total dyadic contribution is

$$\sum_{Z_1 \leq q}^{d} \sum_{Z_2 \leq q}^{d} \sqrt{Z_1 Z_2}(\log q)^3 \ll (\sqrt{q})(\sqrt{q})(\log q)^3 = q(\log q)^3.$$

□

Thus the contribution to off-diagonal terms of short-range ($ab, cd \leq Z$) and mid-range ($Z < ab, cd \leq q$) is $O(q \log q)^3$.

Lemma 2.6. Let m, n, q be positive integers. Then $\forall c > \frac{1}{2}$,

$$\sum_{\substack{m,n > q, \\ m \neq n}} \frac{1}{\sqrt{mn}} W_\alpha\left(\frac{\pi m}{q}\right) W_\alpha\left(\frac{\pi n}{q}\right) \ll q.$$

Proof. Consider two dyadic intervals $2^k q \leq m < 2^{k+1} q$, and $2^l q \leq n < 2^{l+1} q$. Then $\forall c > \frac{1}{2}$,

$$\begin{aligned} \sum_{\substack{m,n > q, \\ m \neq n}} \frac{1}{\sqrt{mn}} W_\alpha\left(\frac{\pi m}{q}\right) W_\alpha\left(\frac{\pi n}{q}\right) &= \sum_{k,l}^{d} \sum_{\substack{2^k q \leq m < 2^{k+1} q, \\ 2^l q \leq n < 2^{l+1} q, \\ m \neq n}} \frac{1}{\sqrt{mn}} W_\alpha\left(\frac{\pi m}{q}\right) W_\alpha\left(\frac{\pi n}{q}\right) \\ &\ll_c \sum_{k,l}^{d} (2^k q)\ (2^l q)\ \frac{1}{2^{l/2}\ 2^{k/2}\ q}\ 2^{-lc}\ 2^{-kc} \\ &\ll_c q \sum_{k}^{d} \left(\frac{1}{2^{c-\frac{1}{2}}}\right)^k \sum_{l}^{d} \left(\frac{1}{2^{c-\frac{1}{2}}}\right)^l \ll_c q. \end{aligned}$$

□

Thus, all off-diagonal terms are of smaller order, while the diagonal of the short-range sum $B(\chi)$ provides the main contribution to the asymptotic of the fourth moment.

2.2 Proof of the main theorem

Let us the recall the Soundararajan's theorem.

Theorem. For all positive integer q we have

$$\sideset{}{^*}\sum_{\chi \pmod q} \left| L\left(\frac{1}{2}, \chi\right) \right|^4 = \frac{\varphi^*(q)}{2\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})} (\log q)^4 \left(1 + O\left(\frac{w(q)}{\log q}\sqrt{\frac{q}{\varphi(q)}}\right)\right) + O\left(q\,(\log q)^{\frac{7}{2}}\right).$$

In Soundararajan's paper[11], there are two important propositions. The above main theorem follows from these two key propositions.

- Proposition 1:

$$\sideset{}{^*}\sum_{\chi \pmod q} |B(\chi)|^2 = \frac{\varphi^*(q)}{8\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})} (\log q)^4 \left(1 + O\left(\frac{w(q)}{\log q}\right)\right). \tag{2.5}$$

- Proposition 2:

$$\sum_{\chi \pmod q} |C(\chi)|^2 \ll q \left(\frac{\varphi(q)}{q}\right)^5 (w(q) \log q)^2 + q(\log q)^3. \tag{2.6}$$

In Proposition 2, we are using $\sum_{\chi \pmod q}$ instead of $\sideset{}{^*}\sum_{\chi \pmod q}$ because of the easier orthogonality of Dirichlet characters than the primitive ones (see Lemma 4.1).

Proof. Since $A(\chi) = B(\chi) + C(\chi)$, we have

$$\sideset{}{^*}\sum_{\chi \pmod q} \left| L\left(\frac{1}{2}, \chi\right) \right|^4 = 4 \sideset{}{^*}\sum_{\chi \pmod q} \left(|B(\chi)|^2 + 2\Re\left[B(\chi)\bar{C}(\chi)\right] + |C(\chi)|^2\right).$$

Since $\Re\left[B(\chi)\bar{C}(\chi)\right] \le |B(\chi)C(\chi)|$, by Cauchy-Schwarz inequality,

$$\sideset{}{^*}\sum_{\chi \pmod q} \left|B(\chi)C(\chi)\right| \le \left(\sideset{}{^*}\sum_{\chi \pmod q} |B(\chi)|^2 \right)^{\frac{1}{2}} \left(\sideset{}{^*}\sum_{\chi \pmod q} |C(\chi)|^2 \right)^{\frac{1}{2}}$$
$$\le \left(\sideset{}{^*}\sum_{\chi \pmod q} |B(\chi)|^2 \right)^{\frac{1}{2}} \left(\sum_{\chi \pmod q} |C(\chi)|^2 \right)^{\frac{1}{2}}.$$

Thus,

$$\begin{aligned}
\sideset{}{^*}\sum_{\chi \pmod q} \left|B(\chi)C(\chi)\right| &= O\left(\left(\sideset{}{^*}\sum_{\chi \pmod q} |B(\chi)|^2\right)^{\frac{1}{2}} \left(\sum_{\chi \pmod q} |C(\chi)|^2\right)^{\frac{1}{2}}\right) \\
&= O\left(\sqrt{\varphi^*(q)}(\log q)^2 \sqrt{q\left(\frac{\varphi(q)}{q}\right)^5 (w(q)\log q)^2 + q(\log q)^3}\right) \\
&= O\left(\sqrt{\varphi^*(q)}(\log q)^2 \sqrt{q}\left(w(q)\log q\left(\frac{\varphi(q)}{q}\right)^{\frac{5}{2}} + (\log q)^{\frac{3}{2}}\right)\right) \\
&= O\left(\sqrt{\varphi^*(q)}\sqrt{q}\left(w(q)(\log q)^3\left(\frac{\varphi(q)}{q}\right)^{\frac{5}{2}} + (\log q)^{\frac{7}{2}}\right)\right).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\sideset{}{^*}\sum_{\chi \pmod q} \left|L\left(\frac{1}{2},\chi\right)\right|^4 &= \frac{\varphi^*(q)}{2\pi^2}\prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})}(\log q)^4\left(1+O\left(\frac{w(q)}{\log q}\right)\right) \\
&+ O\left(\sqrt{\varphi^*(q)}\sqrt{q}\left(w(q)(\log q)^3\left(\frac{\varphi(q)}{q}\right)^{\frac{5}{2}} + (\log q)^{\frac{7}{2}}\right)\right) \\
&+ O\left(q\left(\frac{\varphi(q)}{q}\right)^5 (w(q)\log q)^2 + q(\log q)^3\right) \\
&= \frac{\varphi^*(q)}{2\pi^2}\prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})}(\log q)^4\left(1+O\left(\frac{w(q)}{\log q}\right)\right) + O\left(q\left(\frac{\varphi(q)}{q}\right)^5 (w(q)\log q)^2\right) \\
&+ O\left(\sqrt{\varphi^*(q)}\sqrt{q}\left(w(q)(\log q)^3\left(\frac{\varphi(q)}{q}\right)^{\frac{5}{2}}\right)\right) + O\left(q(\log q)^{\frac{7}{2}}\right) \\
&= \frac{\varphi^*(q)}{2\pi^2}\prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})}(\log q)^4\left(1+O\left(\frac{w(q)}{\log q}\right)\sqrt{\frac{q}{\phi(q)}}\right) + O\left(q(\log q)^{\frac{7}{2}}\right).
\end{aligned}$$

The last line normalization is clear, since,

$$\frac{\sqrt{\varphi^*(q)}\sqrt{q}(\log q)^3 w(q)}{\varphi^*(q)(\log q)^4} = \sqrt{\frac{q}{\varphi^*(q)}}\frac{w(q)}{\log q} = \frac{w(q)}{\log q}\sqrt{\frac{q}{\varphi(q)}}\sqrt{\frac{\varphi(q)}{\varphi^*(q)}} \leq \frac{w(q)}{\log q}\sqrt{\frac{q}{\varphi(q)}}$$

and

$$\frac{qw(q)(\log q)^2}{\varphi^*(q)(\log q)^4} \leq \frac{w(q)}{(\log q)^2}.$$

$\square$

Thus, we obtain

$$\frac{1}{\varphi^*(q)} \sideset{}{^*}\sum_{\chi \pmod q} \left| L\left(\frac{1}{2}, \chi\right)\right|^4 = \frac{1}{2\pi^2}\prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})}(\log q)^4\left(1 + O\left(\frac{w(q)}{\log q}\right)\sqrt{\frac{q}{\varphi(q)}}\right) + O\left(\frac{q(\log q)^{\frac{7}{2}}}{\varphi^*(q)}\right).$$

To summarize, the asymptotic structure of the fourth moment comes from the diagonal contribution of the short-range terms. All off-diagonals have smaller contribution. Moreover, $C(\chi)$ is hard to estimate but easier to bound. Therefore, the decomposition $A(\chi) = B(\chi) + C(\chi)$ makes the extraction of the asymptotic structure transparent.

Chapter 3

Preliminaries

This chapter covers some preliminaries including Dirichlet characters, Dirichlet L-functions, the Dirichlet hyperbola method, the Mellin transform and its inversion, and explains how these tools are used to derive the functional, and approximate functional equations, along with several related concepts. The chapter also introduce a few lemmas which are immensely helpful to prove the background lemmas of Soundararajan's work.

3.1 Dirichlet Characters

Definition 1 (Dirichlet Character)**.** A function $\chi : (\mathbb{Z}/q\mathbb{Z})^{\times} \to \mathbb{C}^{\times}$ is called a Dirichlet character $\pmod q$ if χ is a completely multiplicative group homomorphism with a period q i.e. $\forall\, m, n \in \mathbb{Z}$, the function χ satisfies $\chi(mn) = \chi(m)\chi(n)$, $\chi(m+q) = \chi(m)$, and $\chi(1) = 1$. If χ is a Dirichlet character with moduli q then it can be extended to an arithmetic function $\chi : \mathbb{Z} \to \mathbb{C}$ via

$$\chi(m) = \begin{cases} 0 & \text{if } (m, q) > 1, \\ \chi(m \bmod q) \neq 0 & \text{if } (m, q) = 1. \end{cases}$$

Hence a Dirichlet character $\pmod q$ is a completely multiplicative arithmetic function with period q such that $\chi(m) = 0$ whenever $(m, q) > 1$. Conversely, any such function is a Dirichlet character $\pmod q$.

Here we mention some properties of Dirichlet characters.

1. $\forall m \in (\mathbb{Z}/q\mathbb{Z})^{\times}$, $\chi(m)$ is a $\varphi(q)^{th}$ root of unity.

Proof. Since $\left|(\mathbb{Z}/q\mathbb{Z})^\times\right| = \varphi(q)$, we have $m^{\varphi(q)} = 1$, $\forall m \in (\mathbb{Z}/q\mathbb{Z})^\times$. Therefore, $\chi(m)^{\varphi(q)} = \chi(m^{\varphi(q)}) = \chi(1) = 1$. □

2. Let G be the set of all Dirichlet characters $\chi \pmod q$. Then G is a finite abelian group of order $\varphi(q)$ under point-wise multiplication, $(\chi_1\chi_2)(m) = \chi_1(m)\chi_2(m)$.

Proof. It is straightforward to verify that G is a group. Indeed, the complete multiplicativity of χ ensures closure and associativity, and the identity element is the principal character

$$\chi_0(n) = \begin{cases} 0 & \text{if } (n,q) > 1, \\ 1 & \text{if } (n,q) = 1. \end{cases}$$

The inverse of the character χ is $\bar{\chi}$ defined by

$$\overline{\chi}(n) = \begin{cases} 0 & \text{if } (n,q) > 1, \\ \overline{\chi(n)} & \text{if } (n,q) = 1. \end{cases}$$

Thus G really forms a group and it is exactly $Hom\left((\mathbb{Z}/q\mathbb{Z})^\times, \mathbb{C}^\times\right)$.

Let $q = \prod_{i=1}^r p_i^{\alpha_i}$. Then by Chinese remainder theorem, we have

$$\mathbb{Z}_q^\times \cong \mathbb{Z}_{p_1^{\alpha_1}}^\times \times \mathbb{Z}_{p_2^{\alpha_2}}^\times \times \ldots \times \mathbb{Z}_{p_r^{\alpha_r}}^\times.$$

If p is odd then for any positive integer α, $\mathbb{Z}_{p^\alpha}^\times$ is cyclic group of order $\varphi(p^\alpha)$. For $p = 2$, $\mathbb{Z}_2^\times \cong \{e\}$, $\mathbb{Z}_4^\times \cong \mathbb{Z}_2$, and for $\alpha \geq 3$, $\mathbb{Z}_{2^\alpha}^\times \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{\alpha-2}}$. In particular $|\mathbb{Z}_{2^\alpha}^\times| = 2^{\alpha-1} = \varphi(2^\alpha)$. Now using the Chinese remainder theorem

$$|G| = \prod_{i=1}^r \varphi(p_i^{\alpha_i}) = \varphi\left(\prod_{i=1}^r p_i^{\alpha_i}\right) = \varphi(q).$$

Moreover,

$$(\chi_1\chi_2)(m) = \chi_1(m)\chi_2(m) = \chi_2(m)\chi_1(m) = (\chi_2\chi_1)(m).$$

Thus G forms an abelian group. □

3. If $a \neq 1$ in $X = (\mathbb{Z}/q\mathbb{Z})^\times$ then $\exists$ a character χ of X in G such that $\chi(a) \neq 1$.

Proof. Since X is a finite abelian group, by the fundamental theorem of finite abelian group, we can express

$$X \cong C_{n_1} \times C_{n_2} \times \cdots \times C_{n_k},$$

where each C_{n_j} is a cyclic group of order n_j. Let $a = (a_1, a_2, ..., a_k) \in C_{n_1} \times \cdots \times C_{n_k}$. Since $a \neq 1$, there exists at least one j such that $a_j \neq 1$. Let $C_{n_j} =< \gamma >$, for some γ. Thus for some $k \not\equiv 0 \pmod{n_j}$, we have $a_j = \gamma^k$. Define a mapping

$$\beta_j : C_{n_j} \to \mathbb{C}^\times \text{ by } \beta_j(\gamma^m) = e^{\frac{2\pi i m}{n_j}}.$$

Take two elements $\gamma^{m_1}, \gamma^{m_2} \in C_{n_j}$. Then

$$\beta_j(\gamma^{m_1}\gamma^{m_2}) = e^{\frac{2\pi i(m_1+m_2)}{n_j}} = \beta_j(\gamma^{m_1})\beta_j(\gamma^{m_2}),$$

and

$$\beta_j(1) = \beta_j(\gamma^0) = 1 = e^{2\pi i} = \beta(\gamma^{n_j}).$$

Hence, β_j is a well-defined character on C_{n_j}.

As a consequence,

$$\beta_j(a_j) = e^{\frac{2\pi i k}{n_j}} \neq 1.$$

Consider the projection map $\pi_j : X \to C_{n_j}$. The projection π_j is a group homomorphism. Then $\chi = \beta_j \circ \pi_j$ is a character. Explicitly,

$$\chi(gg') = \beta_j(\pi_j(gg')) = \beta_j(\pi_j(g)\pi_j(g')) = \beta_j(\pi_j(g))\beta_j(\pi_j(g')) = \chi(g)\chi(g').$$

Subsequently

$$\chi(a) = \beta_j(a_j) \neq 1.$$

$\square$

4. Let $q \in \mathbb{N}$, and $n \in \mathbb{Z}$ such that $(n, q) = 1$. Then $\forall n \in \mathbb{Z}$ we have

$$\sum_{\chi \pmod q} \chi(m)\overline{\chi(n)} = \begin{cases} \varphi(q) & \text{if } m \equiv n \pmod q, \\ 0 & \text{otherwise .} \end{cases}$$

where $\sum_{\chi \pmod q} = \sum_{\chi \in G}$.

Proof. If $m = n$, then the sum consists of $\varphi(q)$ terms, each equal to 1. Hence the result follows. Let $m \neq n$. Using the property 3, $\exists\ \chi_1 \in G$ such that $\chi_1(m) \neq \chi_1(n)$. Define

$$T = \sum_{\chi \pmod q} \chi(m)\overline{\chi(n)}.$$

Therefore

$$\chi_1(m)\overline{\chi_1(n)}T = \sum_{\chi \pmod q} \chi_1(m)\chi(m)\overline{\chi_1(n)\chi(n)} = \sum_{\chi \pmod q} \chi_1\chi(m)\overline{\chi_1\chi(n)}.$$

Now since $\chi_1\chi \in G$, we have

$$\sum_{\chi \pmod q} \chi_1\chi(m)\overline{\chi_1\chi(n)} = \sum_{\chi \pmod q} \chi(m)\overline{\chi(n)} = T.$$

Thus we have

$$\chi_1(m)\overline{\chi_1(n)}T = T \implies \left(\chi_1(m)\overline{\chi_1(n)} - 1\right) T = 0.$$

But $\chi_1(m)\overline{\chi_1(n)} \neq 1$. Therefore, $T = 0$. Hence,

$$\sum_{\chi \pmod q} \chi(m)\overline{\chi(n)} = 0.$$

□

There are two types of Dirichlet characters modulo q : **"primitive"** and **"non-primitive"**. Let $d|q$ and let χ^* be a character $\pmod d$. Define

$$\chi(n) = \begin{cases} \chi^*(n) & \text{if } (n, q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then χ is a Dirichlet character $\pmod q$ induced by the character χ^*. Note that if a prime $p|q$ but $p \nmid d$ then χ does not have a period d. If all the prime divisors of q and d are same then $\chi(n) = \chi^*(n), \quad \forall n \in \mathbb{Z}$.

Definition 2 (Conductor)**.** Let $\chi \pmod q$ be a Dirichlet character. A divisor d of q is called a **quasi-period** of χ if $\chi(m) = \chi(n)$ whenever $m \equiv n \pmod d$ and $(mn, q) = 1$. The least of all such quasi-periods is called the conductor of χ.

Definition 3 (Primitive Character)**.** A Dirichlet character $\chi \pmod q$ is called a primitive character if q is the minimum quasi-period of χ or, it has the conductor q.

Therefore using the primitive character idea, we can redefine the conductor of χ as $d_\chi = \min \big\{ d : d \mid q$ such that $\exists$ a primitive character $\chi^* \pmod d$ and $\chi = \chi^* \circ \pi$, where π is the projection map from $\mathbb{Z}_q^\times \to \mathbb{Z}_d^\times \big\}$.

Definition 4 (Non-primitive Character)**.** A Dirichlet character $\chi \pmod q$ is called non-primitive if and only if the smallest positive divisor d of q, for which $\chi(n + d) = \chi(n), \forall n$, is strictly smaller than q that is χ has minimum period $< q$.

Lemma 3.1. Let χ be a Dirichlet character modulo q. The following are equivalent:

1. χ is primitive.
2. If $d \mid q$ with $d < q$, then there exists an integer $c \equiv 1 \pmod d$ which is coprime to q such that $\chi(c) \neq 1$.
3. If $d \mid q$ with $d < q$, then for any $a \in \mathbb{Z}$ we have $\sum_{\substack{1 \le n \le q \\ n \equiv a \pmod d}} \chi(n) = 0$.

Proof. (1) $\Rightarrow$ (2). Let χ be a primitive character and let $d|q, \ d < q$. Since χ does not factor modulo d, there exists n such that $\chi(n + d) \neq \chi(n)$. Since $\chi(n)$ and $\chi(n + d)$ are defined, we have $(n, q) = 1, \ (n + d, q) = 1$. Let $c \in \mathbb{Z}$ such that $c \equiv n(n + d)^{-1} \pmod q$. Then $(c, q) = 1$ and $c \equiv 1 \pmod d$, but $\chi(c)\chi(n + d) = \chi(n) \implies \chi(c) \neq 1$.

(2) $\Rightarrow$ (3). Define

$$S = \sum_{\substack{1 \le n \le q \\ n \equiv a \pmod d}} \chi(n).$$

Then consider the map $c \mapsto cn$. Since $(c, q) = 1$, this map is a permutation of the residue class modulo q. Therefore

$$S = \sum_{\substack{1 \le n \le q \\ n \equiv a \pmod d}} \chi(cn) = \sum_{\substack{1 \le n \le q \\ n \equiv a \pmod d}} \chi(c)\chi(n) = \chi(c) \sum_{\substack{1 \le n \le q \\ n \equiv a \pmod d}} \chi(n) = \chi(c) S.$$

Thus we have $(\chi(c) - 1)\, S = 0$. Since $\chi(c) \neq 1$, $S = 0$.

(3) $\Rightarrow$ (1). If let possible χ is not primitive. So χ has a period d such that $d|q, d < q$. Then for any $n \equiv 1 \pmod d$ with $(n, q) = 1$,

$$\chi(n) = \chi(1) = 1.$$

Hence we obtain

$$S \neq 0,$$

which is a contradiction. Hence χ is a primitive character. $\square$

Lemma 3.2. If $q \equiv 2 \pmod 4$ then there exists no primitive characters $\chi \pmod q$.

Proof. Assume $q \equiv 2 \pmod 4$ then $q = 2r$ with odd r, with $r \geq 1$. Therefore by Chinese Remainder Theorem, we have

$$\mathbb{Z}_q^\times = \mathbb{Z}_{2r}^\times \cong \mathbb{Z}_2^\times \times \mathbb{Z}_r^\times \cong \{e\} \times \mathbb{Z}_r^\times \cong \mathbb{Z}_r^\times.$$

That implies every character modulo $q = 2r$ is induced by a character modulo r. And since $r < 2r = q$, there exists no primitive character modulo q. $\square$

Lemma 3.3. Let $\varphi^*(q)$ denote the number of primitive Dirichlet characters with modulo q. Then φ^* is a multiplicative function, and for a prime p we have

$$\varphi^*(p^k) = \begin{cases} p - 2 & \text{if } k = 1, \\ p^k \left(1 - \frac{1}{p}\right)^2 & \text{if } k \geq 2 \end{cases}.$$

Proof. Using the orthogonality for the primitive Dirichlet characters (discussed in the Lemma **??**), for $r = 1$ we have

$$\varphi^*(q) = \sum_{\chi \pmod q}^{*} \chi(1) = \sum_{d|q} \mu(d)\, \varphi\Big(\frac{q}{d}\Big)\,,$$

where μ is the Möbius function. Since both μ and Euler's totient function φ are multiplicative, their Dirichlet convolution $\mu * \varphi = \varphi^*$ is also multiplicative.

Let $q = p^k$. The only divisors d of p^k with $\mu(d) \neq 0$ are $d = 1$ and $d = p$. Therefore

$$\varphi^*(p^k) = \mu(1)\varphi(p^k) + \mu(p)\varphi(p^{k-1}) = \varphi(p^k) - \varphi(p^{k-1}).$$

- If $k = 1$, then $\varphi(p) = p - 1$ and $\varphi(1) = 1$. So

$$\varphi^*(p) = \varphi(p) - \varphi(1) = (p-1) - 1 = p - 2.$$

- If $k \geq 2$, then $\varphi(p^k) = p^{k-1}(p-1)$ and $\varphi(p^{k-1}) = p^{k-2}(p-1)$, hence

$$\varphi^*(p^k) = p^{k-1}(p-1) - p^{k-2}(p-1) = p^{k-2}(p-1)^2 = p^k\Big(1 - \frac{1}{p}\Big)^2.$$

□

3.2 Dirichlet L-functions

For a Dirichlet character χ mod q, the associated Dirichlet L-function $L(s,\chi)$ is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

which converges absolutely only for $\Re(s) > 1$.

For the trivial Dirichlet character $\mathbb{1}$ (it is the unique Dirichlet character of period 1), $L(s,\mathbb{1}) = \zeta(s)$. For the principal character $\mathbb{1}_m$ of modulus m induced by $\mathbb{1}$ we have

$$L(s,\mathbb{1}_m) = \zeta(s)\prod_{p|m}(1 - p^{-s}) \sim \frac{1}{s-1}\prod_{p|m}(1 - p^{-s}).$$

The product on the right is finite as $s \to 1^+$. The L-function $L(s, 1_m)$ has a simple pole at $s = 1$ of residue

$$\mathbf{Res}_{s=1}\ L(s, 1_m) = \prod_{p|m}(1 - p^{-1}) = \frac{\varphi(m)}{m}.$$

Lemma 3.4. Let $\chi \pmod q$ be a Dirichlet character.

1. If χ is non-principal then $L(s, \chi)$ extends to a holomorphic function on $\Re(s) > 0$.
2. If $\chi = \chi_0$ is the principal character, then

$$L(s, \chi_0) = \zeta(s) \prod_{p|q}(1 - p^{-s})$$

extends to a meromorphic function on $\Re(s) > 0$ with a simple pole at $s = 1$ and no other singularities.

Proof.

- Case 1: Let $\chi \pmod q$ be a non-principal character. Define $T : \mathbb{R}_{\geq 0} \to \mathbb{C}$ by

$$T(x) = \sum_{0<n\leq x} \chi(n).$$

Therefore, since χ is non-principal

$$T(x + q) - T(x) = \sum_{x<n\leq x+q} \chi(n) = \sum_{n\in\mathbb{Z}/q\mathbb{Z}} \chi(n) = 0.$$

Thus $T(x)$ is periodic modulo q and is therefore bounded. For $\Re(s) > 1$, writing $L(s, \chi)$ as a Stieltjes integral and integrating by parts yields

$$\begin{aligned} L(s, \chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s} = \int_1^\infty x^{-s}\, dT(x) &= \Big[x^{-s}T(x)\Big]_1^\infty - \int_1^\infty T(x)\, d(x^{-s}) \\ &= 0 - \int_1^\infty T(x)\,(-sx^{-s-1})\, dx \\ &= \int_1^\infty s\ T(x)\, x^{-s-1}\, dx. \end{aligned}$$

Since T is bounded, the right-hand side integration converges for $\Re(s) > 0$, and the integrand defines a holomorphic function on $\Re(s) > 0$.

- Case 2: Let $\chi = \chi_0$ be the principal Dirichlet character modulo q. Then

$$T(x) = \sum_{n \le x} \chi_0(n) = \sum_{\substack{n \le x, \\ (n,q)=1}} 1.$$

Let $x = kq + m$ such that $0 \le m < q$. The first kq integers contains exactly $k\varphi(q)$ many coprime integers. Thus

$$T(x) = k\varphi(q) + O(1) = \frac{\varphi(q)}{q}(x - m) + O(1) = \frac{\varphi(q)}{q}x + O(1).$$

Using the Stieltjes form, for $\Re(s) > 1$ we obtain

$$L(s, \chi_0) = s\int_1^\infty T(x)x^{-s-1}dx = s\frac{\varphi(q)}{q}\int_1^\infty x^{-s} \ + \ O(s) = \frac{\varphi(q)}{q}\frac{s}{1-s} \ + \ O(s).$$

Thus the function $L(s, \chi_0)$ defined for $\Re(s) > 1$ extends to a meromorphic function on $\Re(s) > 0$ with a pole at $s = 1$.

$\square$

The completed L-function $\Lambda(s, \chi)$ is given by

$$\Lambda(s, \chi) = \left(\frac{q}{\pi}\right)^{(\frac{s+\alpha}{2})}\Gamma\left(\frac{s+\alpha}{2}\right)L(s, \chi), \tag{3.1}$$

and satisfies the functional equation

$$\Lambda(s, \chi) = \frac{i^{-\alpha}\tau(\chi)}{\sqrt{q}}\Lambda(1 - s, \overline{\chi}) \quad := \epsilon(\chi)\ \Lambda(1 - s, \overline{\chi}) \tag{3.2}$$

where $\tau(\chi)$ is the Gauss sum given by

$$\tau(\chi) = \sum_{n \bmod q} \chi(n)e^{\frac{2\pi i n}{q}},$$

and

$$\alpha = \begin{cases} 0, & \text{if } \chi(-1) = 1, \\ 1, & \text{if } \chi(-1) = -1 \end{cases}. \tag{3.3}$$

The functional equation expresses a symmetry of the Dirichlet L-function. It relates values in the right half-plane $\Re(s) > 1$, where the Dirichlet series converges, to values in the left half-plane $\Re(s) < 0$, where the series no longer well defined. This also ensures the meromorphic continuation of L-function to $\mathbb{C}$, except the possible simple pole at $s = 1$.

3.2.1 Approximate Functional Equation

For $x,\ c > 0$, and α as in (3.3) we define the following smooth function,

$$W_\alpha(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{\Gamma\left(\frac{s+\frac{1}{2}+\alpha}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\alpha}{2}\right)}\right)^2 x^{-s}\frac{ds}{s}. \tag{3.4}$$

Let $s = \sigma + it$ with t large and σ fixed. Then $\log|\Gamma(s)| = \Re(\log\Gamma(s))$. By Stirling's formula, we have

$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + O\left(\frac{1}{|s|}\right).$$

Therefore

$$\log|\Gamma(s)| \ll \Re\left(\left(\sigma - \frac{1}{2} + it\right)\log(\sigma + it) - (\sigma + it)\right) \ll \left(\sigma - \frac{1}{2}\right)\log|t| - \frac{\pi}{2}|t|,$$

since

$$\log(\sigma + it) = \log(it) + \log\left(1 + \frac{\sigma}{it}\right) = \log(it) + O\left(\frac{1}{|t|}\right) = \log|t| + i\frac{\pi}{2}sgn(t) + O\left(\frac{1}{|t|}\right).$$

Hence, for fixed σ in a bounded strip,

$$\Gamma(s) \ll e^{-\frac{\pi}{2}|t|}\ |t|^{(\sigma-\frac{1}{2})}. \tag{3.5}$$

Thus, $\Gamma(s)$ decays rapidly along vertical lines, ensuring the convergence of the integral defining $W_\alpha(x)$ in (3.4).

Lemma 3.5. For any $c > 0$ and some $\epsilon > 0$ we have

$$W_\alpha(x) = \begin{cases} 1 + O\big(x^{\frac{1}{2}-\epsilon}\big), & \text{if } x \le 1, \\ O(x^{-c}), & \text{if } x > 1. \end{cases}$$

Proof. Case 1: Let $x > 1$. If we shift the contour to the right, σ increases and $|x^{-s}|$ decreases, so the integrand decays. Hence using (3.5), we have

$$W_\alpha(x) = O_c(x^{-c}).$$

Case 2: Let $x \leq 1$. If we shift the contour to the right, $|x^{-s}|$ does not decrease, and the integrand can grow. To ensure a better bound, we should move it to the left- but how far? For $\alpha = 0$, $\Gamma\left(\frac{s+\frac{1}{2}+\alpha}{2}\right)$ has poles at $s = -\frac{1}{2}$, and for $\alpha = 1$, the poles occur at $s = -\frac{3}{2}$. Since we do not want to cross any poles of $\Gamma\left(\frac{s+\frac{1}{2}+\alpha}{2}\right)$, we can shift the contour to $\Re(s) = -\frac{1}{2} + \epsilon$ for some $\epsilon > 0$. Shifting contour to $c = -\frac{1}{2} + \epsilon$ and applying Cauchy's residue theorem,

$$W_\alpha(x) = \mathbf{Res}_{s=0}\left[\left(\frac{\Gamma\left(\frac{s+\frac{1}{2}+\alpha}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\alpha}{2}\right)}\right)^2 \frac{x^{-s}}{s}\right] + O\big(x^{\frac{1}{2}-\epsilon}\big) = 1 + O\big(x^{\frac{1}{2}-\epsilon}\big).$$

$\square$

With the change of variable $s \mapsto \frac{1}{2} + s$, the functional equation 3.2 becomes

$$\Lambda\left(\frac{1}{2} + s, \chi\right) = \epsilon(\chi)\ \Lambda\left(\frac{1}{2} - s, \overline{\chi}\right), \quad |\epsilon(\chi)| = 1.$$

Thus we have,

$$\Lambda\left(\frac{1}{2} + s, \chi\right)\Lambda\left(\frac{1}{2} + s, \overline{\chi}\right) = \Lambda\left(\frac{1}{2} - s, \chi\right)\Lambda\left(\frac{1}{2} - s, \overline{\chi}\right).$$

For $c > \frac{1}{2}$, consider

$$I = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Lambda\left(\frac{1}{2} + s, \chi\right)\Lambda\left(\frac{1}{2} + s, \overline{\chi}\right)}{\left(\Gamma\left(\frac{\frac{1}{2}+\alpha}{2}\right)\right)^2}\frac{ds}{s}. \tag{3.6}$$

Since $L(\frac{1}{2}+s, \chi) = \sum_{a\geq 1}\frac{\chi(a)}{a^{\frac{1}{2}+s}}$ and $L(\frac{1}{2}+s, \overline{\chi}) = \sum_{b\geq 1}\frac{\overline{\chi}(b)}{b^{\frac{1}{2}+s}}$, using the completed L-function,

$$\Lambda\left(\frac{1}{2} + s, \chi\right)\Lambda\left(\frac{1}{2} + s, \overline{\chi}\right) = \left(\frac{q}{\pi}\right)^{\frac{1}{2}+s+\alpha}\left(\Gamma\left(\frac{s + \frac{1}{2} + \alpha}{2}\right)\right)^2 \sum_{a,b\geq 1}\frac{\chi(a)\overline{\chi}(b)}{(ab)^{\frac{1}{2}+s}}.$$

Thus,

$$I=\sum_{a,b\geq 1}\frac{\chi(a)\overline{\chi}(b)}{\sqrt{ab}}\left(\frac{q}{\pi}\right)^{\frac{1}{2}+\alpha}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{\Gamma\left(\frac{s+\frac{1}{2}+\alpha}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\alpha}{2}\right)}\right)^{2}\left(\frac{\pi ab}{q}\right)^{-s}\frac{ds}{s}.$$

Therefore,

$$I=\sum_{a,b\geq 1}\frac{\chi(a)\overline{\chi}(b)}{\sqrt{ab}}\left(\frac{q}{\pi}\right)^{\frac{1}{2}+\alpha}W_\alpha\left(\frac{\pi ab}{q}\right)=A(\chi)\left(\frac{q}{\pi}\right)^{\frac{1}{2}+\alpha},$$

where $A(\chi)$ is as defined in (2.2). By shifting the contour to $\Re(s)=c<0$ in (3.6), we encounter a simple pole at $s=0$. Applying Cauchy's residue theorem, we obtain,

$$I=\frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty}\frac{\Lambda\Big(\frac{1}{2}+s,\chi\Big)\Lambda\Big(\frac{1}{2}+s,\overline{\chi}\Big)}{\left(\Gamma\left(\frac{\frac{1}{2}+\alpha}{2}\right)\right)^{2}}\frac{ds}{s}+\mathrm{Res}_{s=0}\left(\frac{\Lambda\left(\frac{1}{2}+s,\chi\right)\Lambda\left(\frac{1}{2}+s,\overline{\chi}\right)}{\left(\Gamma\left(\frac{\frac{1}{2}+\alpha}{2}\right)\right)^{2}s}\right).$$

With a change of variable and using the functional equation (3.2), we obtain,

$$\begin{aligned}I&=\frac{1}{2\pi i}\int_{c+i\infty}^{c-i\infty}\frac{\Lambda\Big(\frac{1}{2}-s,\chi\Big)\Lambda\Big(\frac{1}{2}-s,\bar{\chi}\Big)}{\left(\Gamma\left(\frac{\frac{1}{2}+\alpha}{2}\right)\right)^{2}}\frac{d(-s)}{-s}+\frac{\Lambda\Big(\frac{1}{2},\chi\Big)\Lambda\Big(\frac{1}{2},\overline{\chi}\Big)}{\left(\Gamma\left(\frac{\frac{1}{2}+\alpha}{2}\right)\right)^{2}}\\&=-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Lambda\Big(\frac{1}{2}+s,\chi\Big)\Lambda\Big(\frac{1}{2}+s,\overline{\chi}\Big)}{\left(\Gamma\left(\frac{\frac{1}{2}+\alpha}{2}\right)\right)^{2}}\frac{ds}{s}+\frac{\left|\Lambda\Big(\frac{1}{2},\chi\Big)\right|^{2}}{\left(\Gamma\left(\frac{\frac{1}{2}+\alpha}{2}\right)\right)^{2}}=-I+\frac{\left|\Lambda\Big(\frac{1}{2},\chi\Big)\right|^{2}}{\left(\Gamma\left(\frac{\frac{1}{2}+\alpha}{2}\right)\right)^{2}}.\end{aligned}$$

Thus we have,

$$2I=\left(\frac{q}{\pi}\right)^{\frac{1}{2}+\alpha}\left|L\left(\frac{1}{2},\chi\right)\right|^{2}.$$

Hence the approximate functional equation becomes

$$\left|L\left(\frac{1}{2},\chi\right)\right|^{4}=4|A(\chi)|^{2}.$$

3.3 Dirichlet Hyperbola Method

The hyperbola method, first introduced by Dirichlet, is a technique for estimating sums of the form

$$\sum_{n\leq x} f(n)$$

where $f = g * h$ is the Dirichlet convolution of arithmetic functions g and h. A classical application is to the divisor function $d(n) = (1 * 1)(n)$, which counts the number of divisors of n. This method is useful to find an asymptotic formula for

$$\sum_{n\leq x} d(n).$$

First we will start with a naive approach.

$$\sum_{n\leq x} d(n) = \sum_{n\leq x}\sum_{d|n} 1 = \sum_{d\leq x}\sum_{\substack{n\leq x,\\ d|n}} 1 = \sum_{d\leq x}\sum_{kd\leq x} 1 = \sum_{d\leq x}\sum_{k\leq \frac{x}{d}} 1 = \sum_{d\leq x}\left\lfloor \frac{x}{d}\right\rfloor = \sum_{d\leq x}\left(\frac{x}{d} + O(1)\right).$$

The sum then becomes $x\log x + O(x)$. This works well when d is small. In our computation, we have used the approximation $\left\lfloor \frac{x}{d}\right\rfloor = \frac{x}{d} + O(1)$. However the error term $O(1)$ can be problematic in some cases.

- When d is small, or more precisely, for any fixed $n \in \mathbb{N}$, $d \leq x^{\frac{1}{n}}$, we have $\frac{x}{d} \geq x^{1-\frac{1}{n}}$. Therefore, for any fixed n, as $x \to \infty$, $x^{1-\frac{1}{n}} \to \infty$. Hence, in this range, the error term $O(1)$ is small.

- If d is large and close to x, say $x^{\frac{1}{n}} < d \leq x$, then for each $\frac{1}{n}, \exists\, 0 < \delta < 1$ such that $\frac{x}{1+\delta} < d \leq x$. In this case $1 \leq \frac{x}{d} < 1 + \delta$. So we can write $\frac{x}{d} = 1 + \epsilon$, with $0 \leq \epsilon < 1$. It follows that the main term $\left\lfloor \frac{x}{d}\right\rfloor = 1$ and the fractional part (error) $\{\frac{x}{d}\} < 1$. Hence, both the error and main term are of size $O(1)$.

So we need to treat these two cases separately, which is the central idea behind the hyperbola method. Since $f = g * h$, we can write,

$$\sum_{n\le x} f(n) = \sum_{n\le x}\sum_{d|n} g(d)h\left(\frac{n}{d}\right) = \sum_{de\le x} g(d)h(e), \quad n = de.$$

Consequently, we have

$$\sum_{n\le x} d(n) = \sum_{de\le x} 1.$$

Thus the sum counts the number of lattice points on or below the hyperbola $de = x$. This geometric interpretation explains why the method is called the Dirichlet hyperbola method. We split our analysis into two parts with a parameter A such that $1 < A \le x$.

- If $d \le A$. Then

$$\begin{aligned}\sum_{\substack{de\le x,\\ d\le A}} 1 = \sum_{d\le A}\sum_{e\le \frac{x}{d}} 1 = \sum_{d\le A}\left(\frac{x}{d} + O(1)\right) &= x\left(\log A + \gamma + O\left(\frac{1}{A}\right)\right) + O(A)\\ &= x\log A + \gamma x + O\left(A + \frac{x}{A}\right).\end{aligned}$$

  Here, γ is the Euler-Mascheroni constant, defined as

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log(n+1)\right).$$

- If $d > A$. Then we have

$$\begin{aligned}\sum_{\substack{de\le x,\\ d>A}} 1 = \sum_{e<\frac{x}{A}}\sum_{A<d\le\frac{x}{e}} 1 = \sum_{e<\frac{x}{A}}\left(\frac{x}{e} - A + O(1)\right) &= x\sum_{e<\frac{x}{A}}\frac{1}{e} - A\sum_{e<\frac{x}{A}} 1 + O(1)\sum_{e<\frac{x}{A}} 1\\ &= x\left(\log x - \log A + \gamma + O\left(\frac{A}{x}\right)\right) - A\left(\frac{x}{A} + O(1)\right) + O\left(\frac{x}{A}\right)\\ &= x\log x - x\log A + \gamma x + O(A) - x - O(A) + O\left(\frac{x}{A}\right)\\ &= x\log x - x\log A + (\gamma - 1)x + O\left(\frac{x}{A}\right).\end{aligned}$$

Now combining the two cases together, we have

$$\sum_{de\le x} 1 = x\log x + (2\gamma-1)x + O\left(A+\frac{x}{A}\right).$$

Choosing an optimal split $A=\sqrt{x}$, we have

$$\sum_{n\le x} d(n) = x\log x + (2\gamma-1)x + O(\sqrt{x}).$$

Here the error $O(\sqrt{x})$ beats the error $O(x)$ obtained from our naive approach.

Theorem. (Dirichlet Hyperbola Method)

Let f,g,h be some arithmetic functions such that

$$f(n) = (g*h)(n) = \sum_{de=n} g(d)h(e).$$

Define,

$$G(x) = \sum_{n\le x} g(n), \quad \text{and} \quad H(x) = \sum_{n\le x} h(n).$$

Then for all real number $1<A<x$ we have

$$\sum_{n\le x} f(n) = \sum_{de\le x} g(d)h(e) = \sum_{d\le A} g(d)\,H\left(\frac{x}{d}\right) + \sum_{e\le \frac{x}{A}} h(d)\,G\left(\frac{x}{d}\right) - G(A)\,H\left(\frac{x}{A}\right).$$

Proof. Let us do a similar approach to before

$$\begin{aligned}
\sum_{n\le x} f(n) = \sum_{\substack{de\le x\\ d\le A}} g(d)h(e) + \sum_{\substack{de\le x\\ d>A}} g(d)h(e) &= \sum_{d\le A} g(d) \sum_{e\le \frac{x}{d}} h(e) + \sum_{e\le \frac{x}{A}} h(e) \sum_{A<d\le \frac{x}{e}} g(d)\\
&= \sum_{d\le A} g(d)\,H\left(\frac{x}{d}\right) + \sum_{e\le \frac{x}{A}} h(e)\left(G\left(\frac{x}{e}\right) - G(A)\right)\\
&= \sum_{d\le A} g(d)\,H\left(\frac{x}{d}\right) + \sum_{e\le \frac{x}{A}} h(e)G\left(\frac{x}{e}\right) - G(A)\sum_{e\le \frac{x}{A}} h(e)\\
&= \sum_{d\le A} g(d)\,H\left(\frac{x}{d}\right) + \sum_{e\le \frac{x}{A}} h(d)\,G\left(\frac{x}{d}\right) - G(A)\,H\left(\frac{x}{A}\right).
\end{aligned}$$

$\square$

3.4 The Mellin Transform and Its Inversion

The Mellin transform plays an important role in analytic number theory, particularly in the study of the Riemann zeta function or general L-functions. It can be viewed as a Fourier transform on the multiplicative group $G = (\mathbb{R}^+, \times)$, whose Haar measure is $d\mu(x) = \frac{dx}{x}$, and whose continuous characters are given by $\chi_s(x) = x^s$ such that $s \in \mathbb{C}$.

Definition 5 (Mellin Transform)**.** Let $f : \mathbb{R}^+ \to \mathbb{C}$ be a measurable and locally integrable function, satisfying suitable decay and growth conditions as $x \to \infty$ and $x \to 0^+$, respectively. For $s = \sigma + it \in \mathbb{C}$, the Mellin transform of f is defined by

$$\mathcal{M}\{f\}(s) = \tilde{f}(s) = \int_0^\infty f(x)\ x^s\ \frac{dx}{x}.$$

For the Mellin integral to converge, the function $f(x)$ must have suitable behavior as $x \to 0^+$ and $x \to \infty$. For large $x \geq 1$, suppose that as $x \to \infty$, with $\alpha > 1$ (requires for the local integrability of f) the function satisfies $f(x) = O(x^{-\alpha})$. Then

$$|f(x)x^{s-1}| \ll x^{\sigma-1-\alpha},$$

and the convergence of $\int_1^\infty f(x)\, x^{s-1} dx$ requires $\sigma - 1 - \alpha < -1 \Leftrightarrow \sigma < \alpha$. For small x, with $0 < x < 1$, suppose that as $x \to 0^+$ with $\beta > -1$ (requires for the local integrability of f) the function satisfies $f(x) = O(x^{\beta})$. Then

$$|f(x)\, x^{s-1}| \ll x^{\sigma-1+\beta},$$

and the convergence of $\int_0^1 f(x)\, x^{s-1} dx$ requires $\sigma - 1 + \beta > -1 \Leftrightarrow \sigma > -\beta$. Thus the Mellin transform exists in the fundamental strip $-\beta < \sigma < \alpha$, and defines an analytic function.

The change of variable $x = e^u, \quad u \in \mathbb{R}$, maps the the additive group $(\mathbb{R}, +)$ onto multiplicative group $(\mathbb{R}^+, \times)$. Under this substitution,

$$\widetilde{f}(s) = \int_0^\infty f(x)\, x^{s-1}\, dx = \int_{-\infty}^\infty f(e^u)\, e^{(s-1)u}\, e^u\, du = \int_{-\infty}^\infty f(e^u)\, e^{su}\, du.$$

In other words, the Mellin transform on $(\mathbb{R}^+, \times)$ is the Fourier transform (up to normalization) of the function $u \mapsto f(e^u)$.

Definition 6 (Mellin Inversion)**.** Suppose $\tilde{f}(s)$ converge in a vertical strip $a < \sigma < b$. We define the inverse Mellin transform as

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \widetilde{f}(s)\, x^{-s}\, ds, \quad c \in (a, b).$$

Note that for $\Re(s) > 0$, the Gamma function $\Gamma(s) = \int_0^\infty e^{-x}\, x^{s-1}\, dx$ is the Mellin transform of the function $f(x) = e^{-x}$. With the change of variable $x = \pi n^2 t$, we obtain $\Gamma(s) = \pi^s\, n^{2s} \int_0^\infty e^{-\pi n^2 t}\, t^{s-1}\, dt$. Now consider the function $w(t) = \sum_{n=1}^\infty e^{-\pi n^2 t}$, then

$$\pi^{-s}\Gamma(s)\zeta(2s) = \int_0^\infty w(t)\, t^s\, \frac{dt}{t},$$

where the interchange of summation and integration is justified by the rapid convergence of the integral. For $c > \frac{1}{2}$, the Mellin inversion yields

$$w(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \pi^{-s}\Gamma(s)\, \zeta(2s)\, x^{-s}\, ds.$$

Finally, after a change of variables $s \mapsto \frac{s}{2}$, we obtain

$$w(t) = \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)\, x^{-\frac{s}{2}}\, ds.$$

Lemma 3.6. For all $s \in \mathbb{C} \setminus \{0, 1\}$, we have

$$\zeta(s)\, \Gamma\left(\frac{s}{2}\right)\, \pi^{-\frac{s}{2}} = \frac{1}{s(s-1)} + \int_1^\infty \frac{t^{\frac{s}{2}-1} + t^{-\frac{s+1}{2}}}{2}\, w(t)\, dt.$$

Proof. We know that,

$$\zeta(s)\, \Gamma\left(\frac{s}{2}\right)\, \pi^{-\frac{s}{2}} = \int_0^\infty w(t)\, t^{\frac{s}{2}-1}\, dt = \int_0^1 w(t)\, t^{\frac{s}{2}-1}\, dt + \int_1^\infty w(t)\, t^{\frac{s}{2}-1}\, dt = I_1 + I_2.$$

Using the functional equation[1] of $w(t)$, we have

$$I_1 = \frac{1}{2}\int_0^1 t^{\frac{s}{2}-1-\frac{1}{2}}\left(2\,w\left(\frac{1}{t}\right)+1\right)dt\,-\,\frac{1}{2}\int_0^1 t^{\frac{s}{2}-1}\,dt$$
$$= \int_0^1 t^{\frac{s}{2}-\frac{3}{2}}w\left(\frac{1}{t}\right)dt\,+\,\frac{1}{2}\left(\int_0^1 t^{\frac{s}{2}-\frac{3}{2}}\,dt\,-\,\int_0^1 x^{\frac{s}{2}-1}\,dt\right) = \int_1^\infty t^{-\frac{s+1}{2}}w(t)\,dt\,+\,\frac{1}{s(s-1)}.$$

□

The right-hand side of the above lemma is an analytic function $\forall s \in \mathbb{C}$, except the simple poles at $s = 0, 1$. The integral converges absolutely for $\Re(s) > 1$ and, in fact, defines an entire function of s. Consequently, the right-hand provides an meromorphic continuation of $\zeta(s)\,\Gamma\left(\frac{s}{2}\right)\,\pi^{-\frac{s}{2}}$ to the whole $\mathbb{C}$ with the possible simple poles at $s = 0,\,1$. Therefore,

$$\begin{aligned}\zeta(1-s)\,\Gamma\left(\frac{1-s}{2}\right)\,\pi^{-\frac{1-s}{2}} &= \frac{1}{s(s-1)} + \int_1^\infty \frac{t^{\frac{1-s}{2}-1}+t^{-\frac{1-s}{2}-1}}{2}\,w(t)\,dt,\\ &= \frac{1}{s(s-1)} + \int_1^\infty \frac{t^{\frac{s}{2}-1}+t^{-\frac{s+1}{2}}}{2}\,w(t)\,dt,\\ &= \zeta(s)\,\Gamma\left(\frac{s}{2}\right)\,\pi^{-\frac{s}{2}}.\end{aligned}$$

Define the completed ζ function using the Riemann Xi-function as

$$\Xi(s) := \zeta(s)\,\Gamma\left(\frac{s}{2}\right)\,\pi^{-\frac{s}{2}}.$$

Then we have the functional equation of Riemann Zeta function as

$$\Xi(s) = \Xi(1-s).$$

We now illustrate how Mellin transform provides a natural framework for deriving the functional equation and the approximate functional equation associated with an Dirichlet L-function. Let $\chi \pmod q$ be a primitive Dirichlet character. For $\Re(s) > \frac{1}{2}$, $\Gamma(s)\,\pi^{-s}L(2s,\chi)$

[1] $2\,w(t)+1 = x^{-\frac{1}{2}}\left(2\,w\left(\frac{1}{t}\right)+1\right)$.

is the Mellin transform of the function

$$w_\chi(t) = \sum_{n\geq 1} \chi(n)\, e^{-\pi n^2 t}, \quad t > 0.$$

For $\Re(s) > 1$, using the functional equation[2] satisfied by $w_\chi(t)$, we obtain

$$I(s) = \int_0^\infty w_\chi(t)\, t^{\frac{s}{2}-1} dt = \frac{i^{-\alpha}\tau(\chi)}{q} \int_0^\infty w_{\overline{\chi}}\left(\frac{1}{q^2 t}\right) t^{\frac{s-1}{2}} dt.$$

With the change of variable $u = \frac{1}{q^2 t}$,

$$I(s) = i^{-\alpha}\, \tau(\chi)\, q^{-s}\, \Gamma\left(\frac{1-s}{2}\right) \pi^{-\frac{1-s}{2}}\, L(1-s, \overline{\chi}).$$

Therefore,

$$\begin{aligned}
\Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} L(s,\chi) = I(s) &= i^{-\alpha}\, \tau(\chi)\, q^{-s}\, \Gamma\left(\frac{1-s}{2}\right) \pi^{-\frac{1-s}{2}}\, L(1-s, \overline{\chi}), \\
&= \frac{i^{-\alpha}\, \tau(\chi)}{\sqrt{q}}\, q^{\frac{1}{2}-s}\, \Gamma\left(\frac{1-s}{2}\right) \pi^{-\frac{1-s}{2}}\, L(1-s, \overline{\chi}), \\
&= \epsilon(\chi)\, q^{\frac{1}{2}-s}\; \Gamma\left(\frac{1-s}{2}\right) \pi^{-\frac{1-s}{2}}\, L(1-s, \overline{\chi}).
\end{aligned}$$

Multiplying both sides by

$$q^{\frac{s}{2}}\, p^{-\frac{s+\alpha}{2}}$$

and using the parity of characters, we obtain

$$q^{\frac{s}{2}}\, \pi^{-\frac{s}{2}} \Gamma\left(\frac{s+\alpha}{2}\right) L(s,\chi) = \epsilon(\chi)\, q^{\frac{1-s}{2}}\, \pi^{-\frac{1-s+\alpha}{2}}\, \Gamma\left(\frac{1-s+\alpha}{2}\right) L(1-s, \overline{\chi}).$$

Hence the functional equation follows as

$$\Lambda(s,\chi) = \epsilon(\chi)\Lambda(1-s,\chi),$$

[2] For $t > 0$ and primitive characters, $w_\chi(t) = \frac{i^{-\alpha}\tau(\chi) t^{-\frac{1}{2}}}{q} w_{\overline{\chi}}\left(\frac{1}{q^2 t}\right)$.

where the completed L-function is

$$\Lambda(s,\chi) = q^{\frac{s}{2}}\,\pi^{-\frac{s+\alpha}{2}}\Gamma\left(\frac{s+\alpha}{2}\right)\,L(s,\chi).$$

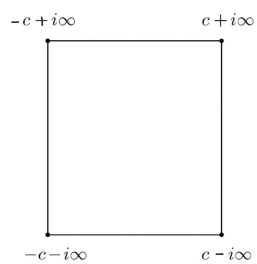
Let $G(u)$ be a an entire function with rapid decay on vertical lines and satisfying $G(0) = 1$. Fix $c > 0$, we define its inverse Mellin transform as

$$V(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(u)\,t^{-u}\,du.$$

For $t = \frac{n}{X}$, we then have

$$V\left(\frac{n}{X}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(u)\,\left(\frac{n}{X}\right)^{-u}\,du.$$

Since G is entire and decays rapidly on vertical lines, we can consider a closed rectangle contour with vertical sides $\Re(u) = +c$ and $\Re(u) = -c$. Using the Cauchy's residue theorem,



the integral around the rectangle is

$$2\pi i\,\mathbf{Res}_{u=0}\,G(u)\,\left(\frac{n}{X}\right)^{-u} = 2\pi i.$$

Due to the decay of G, the integral along the horizontal segments vanish. Hence

$$\int_{c-i\infty}^{c+i\infty} G(u)\,\left(\frac{n}{X}\right)^{-u}\,du + \int_{-c-i\infty}^{-c+i\infty} G(u)\,\left(\frac{n}{X}\right)^{-u}\,du = 2\pi i,$$

or equivalently,

$$V\left(\frac{n}{X}\right) + \frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty} G(u)\,\left(\frac{n}{X}\right)^{-u}\,du = 1.$$

Using this identity, we can express the L-function as

$$L(s,\chi)=\sum_{n\geq 1}\frac{\chi(n)}{n^s}.1=\sum_{n\geq 1}\frac{\chi(n)}{n^s}V\left(\frac{n}{X}\right)\;+\;\frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty}G(u)\,X^u\,L(s+u,\chi)\,du,$$
$$=\sum_{n\geq 1}\frac{\chi(n)}{n^s}V\left(\frac{n}{X}\right)\;+\;\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}G(-u)\,X^{-u}\,L(s-u,\chi)\,du.$$

Using the completed L-function and functional equation, we can express

$$L(s-u,\chi)=q^{-\frac{s-u}{2}}\,\pi^{\frac{s-u+\alpha}{2}}\left(\Gamma\left(\frac{s-u+\alpha}{2}\right)\right)^{-1}\epsilon(\chi)\,\Lambda(1-s+u,\overline{\chi}),$$
$$=\epsilon(\chi)\,\left(\frac{q}{\pi}\right)^{\frac{1}{2}-s}\frac{\Gamma\left(\frac{1-s+u+\alpha}{2}\right)}{\Gamma\left(\frac{s-u+\alpha}{2}\right)}\,L(1-s+u,\overline{\chi}).$$

Thus we have the approximate functional equation as

$$L(s,\chi)=\sum_{n\geq 1}\frac{\chi(n)}{n^s}V\left(\frac{n}{X}\right)\;+\;\epsilon(\chi)\,\left(\frac{q}{\pi}\right)^{\frac{1}{2}-s}\sum_{n\geq 1}\frac{\overline{\chi(n)}}{n^{1-s}}\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}G(-u)\frac{\Gamma\left(\frac{1-s+u+\alpha}{2}\right)}{\Gamma\left(\frac{s-u+\alpha}{2}\right)}\left(\frac{n}{X}\right)^{-u}du,$$
$$=\sum_{n\geq 1}\frac{\chi(n)}{n^s}V\left(\frac{n}{X}\right)\;+\epsilon(\chi,s)\,\sum_{n\geq 1}\frac{\overline{\chi(n)}}{n^{1-s}}=I_1+I_2.$$

I_1 converges absolutely for $\Re(s)>0$, since $V\left(\frac{n}{X}\right)$ decays rapidly. In addition, I_2 converges absolutely for $\Re(s)<1$. The approximate functional equation helps us to represent an L-function near the critical line as a finite length sum. It allows analysis where the Dirichlet series fails to converge.

3.5 A Few Other Important Things

Lemma 3.7. Let q be a sufficiently large highly composite number or it is product of many small primes. Let $w(q)$ be the number of distinct prime divisors of q. Then

$$w(q)\ll\frac{\log q}{\log\log q}.$$

Proof. Let, $\pi(x)$ be the prime counting function. Therefore by prime number theorem,

$$w(q)=\pi(p_{w(q)})\sim\frac{p_{w(q)}}{\log p_{w(q)}}.$$

Let,

$$q = \prod_{i=1}^{w(q)} r_i^{\alpha_i} \quad \text{such that} \quad r_1 < r_2 < ... < r_{w(q)}.$$

Let $p_1, p_2, \ldots, p_{w(q)}$ be the first $w(q)$ known primes. Therefore $\prod_{i=1}^{w(q)} p_i^{\alpha_i} \leq \prod_{i=1}^{w(q)} r_i^{\alpha_i} = q$. Consequently

$$\begin{aligned}\log q &\geq \alpha_1 \log p_1 + \alpha_2 \log p_2 + ... + \alpha_{w(q)} \log p_{w(q)} \\ &\geq \min(\alpha_1, \alpha_2, ..., \alpha_{w(q)}) \sum_{p \leq p_{w(q)}} \log p \\ &\geq \sum_{p \leq p_{w(q)}} \log p = \theta(p_{w(q)}),\end{aligned}$$

where θ is the Chebyshev function. By prime number theorem,

$$\theta(p_{w(q)}) \sim p_{w(q)} \Rightarrow \log q \gg p_{w(q)}.$$

□

Lemma 3.8. Let q be a sufficiently large highly composite number or it is product of many small primes. Then,

$$\varphi(q) \gg \frac{q}{\log \log q}.$$

Proof. We know that,

$$\varphi(q) = q \prod_{p|q} \left(1 - \frac{1}{p}\right).$$

Thus it suffices to show that

$$\prod_{p|q} \left(1 - \frac{1}{p}\right) \gg \frac{1}{\log \log q}.$$

Let $q = \prod_{i=1}^{w(q)} r_i^{\alpha_i}$ so that $r_1 < r_2 < ... < r_{w(q)}$, and let $p_1, p_2, \ldots, p_{w(q)}$ be the first $w(q)$ known primes. Thus $p_i \leq r_i$ for each i. Consequently

$$\prod_{i=1}^{w(q)} \left(1 - \frac{1}{r_i}\right) \geq \prod_{i=1}^{w(q)} \left(1 - \frac{1}{p_i}\right) = \prod_{p \leq p_{w(q)}} \left(1 - \frac{1}{p}\right).$$

By Mertens' theorem, $\prod_{p\le p_{w(q)}}\left(1-\frac{1}{p}\right)\asymp\frac{1}{\log p_{w(q)}}$. Then, using the prime number theorem,

$$\prod_{p\le p_{w(q)}}\left(1-\frac{1}{p}\right)\gg\frac{1}{\log\log q}.$$

$\square$

Lemma 3.9. Let $d(n)$ be the divisor function and $n\in\mathbb{N}$. Then for any $\epsilon>0$, we have

$$d(n)\ll n^{\epsilon},\quad\text{and}\quad\sum_{n\le x}\frac{d(n)}{n}\ll(\log x)^2.$$

Proof. Let $n\in\mathbb{N}$, and $n=\prod_{i=1}^{w(n)}p_i^{\alpha_i}$. Then $d(n)=\prod_{i=1}^{w(n)}(\alpha_i+1)$.

For a fixed $\epsilon>0$, define $f(n)=\frac{d(n)}{n^{\epsilon}}$. Then we have

$$f(n)=\prod_{i=1}^{w(n)}\frac{(\alpha_i+1)}{p_i^{\alpha_i\epsilon}}=\prod_{p\ge 2^{\frac{1}{\epsilon}}}\frac{(\alpha+1)}{p^{\alpha\epsilon}}\prod_{p<2^{\frac{1}{\epsilon}}}\frac{(\alpha+1)}{p^{\alpha\epsilon}}.$$

If $p_i^{\epsilon}\ge 2$, that is $p_i\ge 2^{\frac{1}{\epsilon}}$, then $p_i^{\alpha_i\epsilon}\ge 2^{\alpha_i}\ge(\alpha_i+1)$. This implies $\frac{(\alpha_i+1)}{p_i^{\alpha_i\epsilon}}\le 1$, and subsequently $\prod_{p\ge 2^{\frac{1}{\epsilon}}}\frac{(\alpha+1)}{p^{\alpha\epsilon}}\le 1$.

Define $\mathbb{P}=\{p\text{ is prime}:p<2^{\frac{1}{\epsilon}}\}$. Then $\mathbb{P}$ is a finite set.

For $p\in\mathbb{P}$, let $f_p(\alpha)=\frac{\alpha+1}{p^{\alpha\epsilon}}$. It is easy to see that $\lim_{\alpha\to\infty}f_p(\alpha)=0$. The critical point $\alpha_{\text{critical}}=\frac{1}{\epsilon\log p}-1$ is obtained by solving $f_p'(\alpha)=0$.

Hence, the local maxima of $f_p(\alpha)$ is $f_p^{\max}=\max\{f_p(1),\,f_p(\alpha_{\text{critical}})\}<+\infty$, since both values are finite. Therefore

$$\prod_{p<2^{\frac{1}{\epsilon}}}\frac{(\alpha+1)}{p^{\alpha\epsilon}}<+\infty.$$

Thus we have

$$\frac{d(n)}{n^{\epsilon}}=\prod_{p\ge 2^{\frac{1}{\epsilon}}}\frac{(\alpha+1)}{p^{\alpha\epsilon}}\prod_{p<2^{\frac{1}{\epsilon}}}\frac{(\alpha+1)}{p^{\alpha\epsilon}}<+\infty.$$

Hence

$$d(n)\ll n^{\epsilon}.$$

Since ϵ was chosen arbitrarily, this bound holds $\forall\epsilon>0$.

Recall that, for any $n \in \mathbb{N}$, the divisor function satisfies $d(n) = \sum_{d|n} 1$. Therefore,

$$\sum_{n\le x} \frac{d(n)}{n} = \sum_{n\le x} \frac{1}{n} \sum_{d|n} 1 = \sum_{d\le x} 1 \sum_{\substack{n\le x,\\ d|n}} \frac{1}{n} = \sum_{d\le x} 1 \sum_{m\le \frac{x}{d}} \frac{1}{dm} = \left(\log\left(\frac{x}{d}\right) + \gamma + O\left(\frac{d}{x}\right)\right) \sum_{d\le x} \frac{1}{d}$$

$$\ll \log x \sum_{d\le x} \frac{1}{d} \ll (\log x)^2.$$

□

Lemma 3.10 (Vinogradov, Linnik[12]). Let $k \geq 1,\ (r,k) = 1$ and $x \geq k^{1+\delta}$ for a fixed $\delta > 0$. Let $n \in \mathbb{N}$, then

$$\sum_{\substack{n\le x,\\ n\equiv r \pmod k}} d(n) \ll \frac{\varphi(k)}{k^2} x \log x.$$

Proof. Since $\sum_{ab\equiv r \pmod k} 1 = 0$ whenever $(a,k) \neq 1$, and since $d(n) = \sum_{ab=n} 1$, it follows that

$$\sum_{\substack{n\le x,\\ n\equiv r \pmod k}} d(n) = \sum_{a\le x} \sum_{\substack{b\le \frac{x}{a},\\ ab\equiv r \pmod k}} 1 = \sum_{\substack{a\le x,\\ (a,k)=1}} \left(\frac{x}{ak} + O(1)\right) = \frac{x}{k} \sum_{\substack{a\le x,\\ (a,k)=1}} \frac{1}{a} + O(x) = I_1 + I_2.$$

Here $I_2 = O(x)$. Since $x \geq k^{1+\delta}$, we have $\frac{\log x}{k} \geq \frac{(1+\delta)\log k}{k}$. Therefore for some $c > 0$,

$$\frac{\varphi(k)}{k^2} x \log x \geq \frac{c}{k} x \log x \geq c(1+\delta) \frac{\log k}{k} x \gg x.$$

Now,

$$\begin{aligned} I_1 = \frac{x}{k}\, \delta_{(a,k)=1} \sum_{a\le x} \frac{1}{a} = \frac{x}{k} \sum_{d|(a,k)} \mu(d) \sum_{a\le x} \frac{1}{a} &= \frac{x}{k} \sum_{d|k} \mu(d) \sum_{\substack{a\le x,\\ d|a}} \frac{1}{a} \\ &= \frac{x}{k} \sum_{d|k} \frac{\mu(d)}{d} \sum_{m\le \frac{x}{d}} \frac{1}{m} \\ &= \frac{x\,\varphi(k)}{k^2} \left(\log\left(\frac{x}{d}\right) + \gamma + O\left(\frac{d}{x}\right)\right) \\ &\ll \frac{\varphi(k)}{k^2} x \log x. \end{aligned}$$

□

Remark: In 1980, P. Shiu[13] extended the Lemma 3.10 for any positive, multiplicative, arithmetic function.

Lemma 3.11 (Heath-Brown[10])**.** If k, l are positive integers with $lk \ll x^{\frac{5}{4}}$ for the plus case and $x \leq lk$ for the minus case. Then

$$\sum_{n \leq x,\ (n,k)=1} d(n)d(kl \pm n) \ll x(\log x)^2 \sum_{d|l} d^{-1}.$$

Proof. Since

$$d(kl \pm n) = \sum_{a|(kl \pm n)} 1,$$

we have

$$S = \sum_{n \leq x,\ (n,k)=1} d(n)d(kl \pm n) = \sum_{1 \leq a \leq kl \pm n} \sum_{\substack{n \leq x,\ (n,k)=1, \\ n \equiv \pm kl \pmod{a}}} d(n).$$

Now $(a, k) = 1$, otherwise it contradicts the fact that $(n, k) = 1$. Then

$$S \leq \sum_{\substack{1 \leq a \leq kl \pm x, \\ (a,k)=1}} \sum_{\substack{n \leq x,\ (n,k)=1, \\ n \equiv \pm kl \pmod{a}}} d(n).$$

Let $d = (a, l)$. Then $\exists\, a_1 \in \mathbb{Z}$ such that $a = da_1$. Consequently $\left(a_1, \frac{l}{d}\right) = 1$. Since $(a, k) = 1$, it follows that $(a_1, k) = 1$, and hence $(a_1, \pm k\frac{l}{d}) = 1$. Fix an $\epsilon > 0$, and suppose that $a_1 \leq x^{1-\epsilon}$. Then $\exists\, \delta > 0$ such that $1 + \delta < \frac{1}{1+\epsilon}$ and, $a_1^{1+\delta} < a_1^{\frac{1}{1+\epsilon}} \leq x$. Having satisfied all the conditions of Lemma 3.10 we have

$$\sum_{\substack{n \leq x, \\ n \equiv \pm k\frac{l}{d} \pmod{a_1}}} d(n) \ll \frac{\varphi(a_1)}{(a_1)^2} x \log x.$$

Therefore

$$S \ll x \log x \sum_{d|l} \sum_{a_1 \leq x^{1-\epsilon}} \frac{\varphi(a_1)}{(a_1)^2} \ll x(\log x)^2 \sum_{d|l} 1 \leq x(\log x)^2 \sum_{d|l} \frac{l}{d} \ll x(\log x)^2 \sum_{d|l} \frac{1}{d}.$$

Let $a_1 > x^{1-\epsilon}$. Since $kl \ll x^{\frac{5}{4}}$ and $a_1|a$, we have

$$a_1 \leq kl + x \leq x^{\frac{5}{4}} + x \ll x^{\frac{5}{4}}.$$

Therefore

$$x^{1-\epsilon} < a_1 \ll x^{\frac{5}{4}}.$$

Now since $\forall \beta > 0$,

$$d(n) \ll n^{\beta} \ll x^{\beta},$$

we have

$$\sum_{\substack{n \leq x, \\ n \equiv \pm k\frac{l}{d} \pmod{a_1}}} d(n) \ll \quad x^{\beta} \sum_{\substack{n \leq x, \\ n \equiv \pm k\frac{l}{d} \pmod{a_1}}} 1 \ll x^{\beta} \left(\frac{x}{a_1} + O(1) \right) \ll x^{\epsilon+\beta}.$$

Thus

$$S \ll \sum_{x^{1-\epsilon} \leq a_1 \ll x^{\frac{5}{4}}} x^{\epsilon+\beta} \ll x^{\frac{5}{4}+\epsilon+\beta}.$$

Since $x(\log x)^2 \ll_{\epsilon} x^{1+2\epsilon}$, for all $\beta > \frac{1}{4}$ we have $1 + 2\epsilon > \frac{5}{4} + \beta + \epsilon$. Hence, for all $\beta > \frac{1}{4}$, the contribution from the case $a_1 \leq x^{1-\epsilon}$ dominates the contribution from the case $a_1 > x^{1-\epsilon}$. $\square$

Lemma 3.12. For every positive integer q, we have

$$\sum_{d|q} \frac{\mu(d) \log d}{d} = -\frac{\varphi(q)}{q} \sum_{p|q} \frac{\log p}{p-1}.$$

Proof. For $\Re(s) > 1$, define,

$$F(s) = \sum_{d|q} \frac{\mu(d)}{d^s} = \prod_{p|q} \left(1 - \frac{1}{p^s} \right).$$

Therefore

$$\log F(s) = \sum_{p|q} \log \left(1 - \frac{1}{p^s} \right).$$

Taking derivative for both sides

$$\frac{F'(s)}{F(s)} = \sum_{p|q} \frac{p^{-s}\log p}{1-\frac{1}{p^s}} = \sum_{p|q} \frac{\log p}{p^s-1}, \quad \text{where}$$

$$F'(s) = \frac{d}{ds}\left(\sum_{d|q} \mu(d)\, d^{-s}\right) = -\sum_{d|q} \mu(d)\, d^{-s}\, \log d.$$

Hence for $s=1$, we have

$$\sum_{d|q} \frac{\mu(d)\,\log d}{d} = -F'(1) = -F(1)\sum_{p|q}\frac{\log p}{p-1} = -\sum_{d|q}\frac{\mu(d)}{d}\sum_{p|q}\frac{\log p}{p-1} = -\frac{\varphi(q)}{q}\sum_{p|q}\frac{\log p}{p-1}.$$

□

Lemma 3.13. Let $q \geq 2$ and $\Re(s) > 1$. Then we have,

$$\sum_{\substack{n\geq 1,\\ (n,q)=1}} \frac{2^{w(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)} \prod_{p|q} \frac{1-p^{-s}}{1+p^{-s}}.$$

Proof. Let $m, n \in \mathbb{N}$ with $(m,n) = 1$. The arithmetic function $2^{w(n)}$ is multiplicative. Define,

$$\delta_{\text{Condition}} = \begin{cases} 1 & \text{if condition holds,} \\ 0 & \text{otherwise.} \end{cases}$$

Since $\delta_{(n,q)=1}$ is multiplicative, $2^{w(n)}\delta_{(n,q)=1}$ is also multiplicative.

- Case 1: Let $p|q$. Then $(n,q)=1$ implies $p \nmid n$. Hence $(p^k, q) = 1$ only when $k=0$. Therefore, the local factor at a prime $p|q$ is $\sum_{k=0}^{\infty} \frac{2^{w\left(p^k\right)}\delta_{(p^k,q)=1}}{p^{ks}} = \frac{2^{w(1)}}{1^s} = 1$. Thus, no Euler factor appears for $p|q$.

- Case 2: Let $p \nmid q$. Then $(p^k, q) = 1$ for all integers $k \geq 0$, and hence $\delta_{(p^k,q)=1} = 1, \quad \forall k$. The local factor at a prime $p \nmid q$ is therefore

$$\sum_{k=0}^{\infty} \frac{2^{w(p^k)}}{p^{ks}} = 1 + \sum_{k=1}^{\infty} \frac{2}{p^{ks}} = \frac{1+p^{-s}}{1-p^{-s}}.$$

Therefore we have

$$\sum_{\substack{n\geq 1,\\ (n,q)=1}} \frac{2^{w(n)}}{n^s} = \prod_p \sum_{k=0}^{\infty} \frac{2^{w\left(p^k\right)}\,\delta_{(p^k,q)=1}}{p^{ks}} = \prod_{p\nmid q} \frac{1+p^{-s}}{1-p^{-s}} = \prod_p \frac{1+p^{-s}}{1-p^{-s}} \prod_{p\mid q} \frac{1-p^{-s}}{1+p^{-s}}$$

$$= \frac{\prod_p \frac{1}{(1-p^{-s})^2}}{\prod_p \frac{1}{1-p^{-2s}}} \prod_{p\mid q} \frac{1-p^{-s}}{1+p^{-s}} = \frac{\zeta^2(s)}{\zeta(2s)} \prod_{p\mid q} \frac{1-p^{-s}}{1+p^{-s}}.$$

□

Lemma 3.14. For any $x \geq 2$,

$$\sum_{n>x} \frac{2^{w(n)}}{n^5} \ll \frac{\log x}{x^4}.$$

Proof. Consider a dyadic interval $(2^k x, 2^{k+1}x]$, $k \geq 0$. Now

$$\begin{aligned}
\sum_{n>x} \frac{2^{w(n)}}{n^5} = \sum_{k\geq 0}^{d} \sum_{2^k x<n\leq 2^{k+1}x} \frac{2^{w(n)}}{n^5} &\leq \sum_{k\geq 0}^{d} \frac{1}{(2^k x)^5} \sum_{n\leq 2^{k+1}x} 2^{w(n)} \\
&\ll \frac{1}{x^5} \sum_{k\geq 0}^{d} \frac{1}{2^{5k}} \sum_{d\leq 2^{k+1}x} \frac{2^{k+1}x}{d} \\
&\ll \frac{1}{x^4} \sum_{k\geq 0}^{d} \frac{1}{2^{4k-1}} \sum_{d\leq 2^{k+1}x} \frac{1}{d} \\
&\ll \frac{1}{x^4} \sum_{k\geq 0}^{d} \frac{1}{2^{4k-1}} \log(2^{k+1}x) \\
&\ll \frac{1}{x^4} \sum_{k\geq 0}^{d} \frac{1}{2^{3k-2}} + \frac{\log x}{x^4} \sum_{k\geq 0}^{d} \frac{1}{2^{4k-1}} \\
&\ll \frac{\log x}{x^4}.
\end{aligned}$$

□

Chapter 4

Background Lemmas

This chapter presents the key lemmas required for analyzing the fourth moment of Dirichlet L-functions. These results supply the analytic tools needed to isolate the main and error terms in Soundararajan's proof.

4.1 Orthogonality of Primitive Characters

Lemma 4.1. If $(r,q)=1$, then

$$\sideset{}{^*}\sum_{\chi \pmod q} \chi(r) = \sum_{k|(q,r-1)} \mu\left(\frac{q}{k}\right)\varphi(k).$$

Proof. By orthogonality of Dirichlet characters,

$$\sum_{\chi \pmod q} \chi(r) = \begin{cases} \varphi(q) & \text{if } r \equiv 1 \pmod q, \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \varphi(q) & \text{if } q \mid (r-1), \\ 0 & \text{otherwise} \end{cases}.$$

Define, $h_r(k) = \sideset{}{^*}\sum_{\chi \pmod k} \chi(r)$. Let $(r,q)=1$ then

$$\sum_{k|q} h_r(k) = \sum_{k|q} \sideset{}{^*}\sum_{\chi \pmod k} \chi(r) = \sum_{\chi \pmod q} \chi(r) = \begin{cases} \varphi(q) & \text{if } q \mid (r-1), \\ 0 & \text{otherwise} \end{cases}.$$

Using Möbius inversion, we have

$$h_r(k) = \sum_{k|q} \mu\left(\frac{q}{k}\right) \begin{cases} \varphi(q) & \text{if } q \mid (r-1), \\ 0 & \text{otherwise} \end{cases} = \sum_{k|(q,r-1)} \mu\left(\frac{q}{k}\right)\varphi(k).$$

$\square$

4.2 Uniform Bounds for Off-Diagonal Solutions

Lemma 4.2. Let k be a positive integer, and let $Z_1,\ Z_2 \geq 2$ be real numbers. If $Z_1 Z_2 > k^{\frac{19}{10}}$, then

$$S = \sum_{\substack{a,b,c,d \geq 1, \\ Z_1 \leq ab \leq 2Z_1, \\ Z_2 \leq cd \leq 2Z_2, \\ (abcd,k)=1,\ ac \neq bd, \\ ac \equiv \pm bd \pmod{k}}} 1 \ll \frac{Z_1 Z_2}{k} \left(\log(Z_1 Z_2)\right)^3.$$

If $Z_1 Z_2 \leq k^{\frac{19}{10}}$, then for any $\epsilon > 0$, the above estimated quantity is $\ll \frac{(Z_1 Z_2)^{1+\varepsilon}}{k}$.

Proof. Without loss of generality, assume that $ac > bd$. The condition $ac \equiv \pm bd \pmod{k}$ implies that $kl \leq ac + bd$ for some $l \in \mathbb{Z}$. Consequently, we have

$$kl \leq ac + bd \leq 2ac \leq 2abcd \leq 8Z_1 Z_2. \tag{4.1}$$

Therefore,

$$1 \leq l \leq \frac{8Z_1 Z_2}{k}, \quad \text{and} \quad bd = \frac{8Z_1 Z_2}{2ac} \leq \frac{8Z_1 Z_2}{kl}.$$

Let $n = bd$, and $m = ac = kl \pm n$. The original sum S counts each quadruple (a, b, c, d) exactly once. Hence, the number of quadruples is equal to (number of ways to express $n = bd$) $\times$ (number of ways to express $m = ac$) $= d(n)d(m)$. Thus,

$$S \ll \sum_{1 \leq l \leq \frac{8Z_1 Z_2}{k}} \sum_{\substack{n \leq \frac{8Z_1 Z_2}{kl}, \\ n<m,\ (n,k)=1}} d(n)d(m) \ll \sum_{1 \leq l \leq \frac{8Z_1 Z_2}{k}} \sum_{n \leq \frac{8Z_1 Z_2}{kl}} d(n)d(m).$$

Since $(abcd, k) = 1$ and $ac \equiv \pm bd \pmod{k}$, we have $(n, m) = 1$. Therefore using Lemma 3.9, we have

$$d(n)d(m) = d(mn) = d(abcd) \ll d(Z_1 Z_2) \ll (Z_1 Z_2)^{\epsilon}, \quad \forall \epsilon > 0.$$

If $Z_1Z_2 \le k^{\frac{19}{10}}$, then

$$\log\left(\frac{Z_1Z_2}{k}\right) \ll \log k, \quad \text{and} \quad \frac{k}{Z_1Z_2} \le k.$$

Thus,

$$\begin{aligned}
S &\ll \sum_{1\le l\le \frac{8Z_1Z_2}{k}} \sum_{n\le \frac{8Z_1Z_2}{kl}} d(n)d(m) \ll (Z_1Z_2)^{\epsilon} \sum_{1\le l\le \frac{8Z_1Z_2}{k}} \frac{8Z_1Z_2}{kl} \\
&\ll \frac{(Z_1Z_2)^{\epsilon+1}}{k}\left(\log\left(\frac{Z_1Z_2}{k}\right) + O\left(\frac{k}{Z_1Z_2}\right) - O(1)\right) \\
&\ll \frac{(Z_1Z_2)^{\epsilon+1}}{k}.
\end{aligned}$$

Hence the second part of the lemma is done.

If $Z_1Z_2 > k^{\frac{19}{10}}$, then we break the analysis into two cases: $kl \le (Z_1Z_2)^{\frac{11}{20}}$, and $kl > (Z_1Z_2)^{\frac{11}{20}}$.

Case 1:

$$kl \le (Z_1Z_2)^{\frac{11}{20}} \le (Z_1Z_2)^{\frac{5}{9}} \Rightarrow kl \le \left(\frac{8Z_1Z_2}{kl}\right)^{\frac{4}{5}}.$$

Thus we can use Lemma 3.11. Let $L = \frac{(Z_1Z_2)^{\frac{11}{20}}}{k}$. Applying the Lemma 3.11, we have

$$\begin{aligned}
S &\ll \sum_{l\le L}\left(\frac{Z_1Z_2}{kl}\right)\left(\log\left(\frac{Z_1Z_2}{kl}\right)\right)^2 \sum_{d|l}\frac{1}{d} \ll \sum_{l\le L}\left(\frac{Z_1Z_2}{kl}\right)(\log Z_1Z_2)^2 \sum_{d|l}\frac{1}{d} \\
&= \left(\frac{Z_1Z_2}{k}\right)(\log Z_1Z_2)^2 \sum_{l\le L}\frac{1}{l}\sum_{d|l}\frac{1}{d} \\
&= \left(\frac{Z_1Z_2}{k}\right)(\log Z_1Z_2)^2 \sum_{d\le L}\frac{1}{d}\sum_{de\le L}\frac{1}{de} \\
&= \left(\frac{Z_1Z_2}{k}\right)(\log Z_1Z_2)^2 \sum_{d\le L}\frac{1}{d^2}\left(\log\left(\frac{L}{d}\right) + O(1)\right) \\
&\ll \left(\frac{Z_1Z_2}{k}\right)(\log Z_1Z_2)^2 \log L \sum_{d\le L}\frac{1}{d^2} \ll \left(\frac{Z_1Z_2}{k}\right)(\log Z_1Z_2)^3.
\end{aligned}$$

Case 2: Let $kl > (Z_1Z_2)^{\frac{11}{20}}$. Therefore for some $\delta > 0$, by equation (4.1), we have

$$\frac{(Z_1Z_2)^{\frac{11}{20}}}{2} \le m \le \frac{4Z_1Z_2}{n} \quad \text{and} \quad n \le \frac{8Z_1Z_2}{kl} \Rightarrow \frac{4Z_1Z_2}{n} \ge \frac{kl}{2} \ge k^{1+\delta}.$$

Then using Lemma 3.10 we have

$$S \ll \sum_{1\le l\le \frac{8Z_1Z_2}{k}} \sum_{\substack{n\le \frac{8Z_1Z_2}{kl},\\ n<kl\pm n,\ (n,k)=1}} d(n)d(kl\pm n) \ll \sum_{n\le \frac{8Z_1Z_2}{kl}} d(n) \sum_{\substack{\frac{(Z_1Z_2)^{\frac{11}{20}}}{2}\le m\le \frac{4Z_1Z_2}{n},\\ m\equiv \pm n \pmod k,\\ (m,n)=1}} d(m)$$

$$\ll \sum_{n\le \frac{8Z_1Z_2}{kl}} d(n)\frac{\varphi(k)}{k^2}\left(\frac{4Z_1Z_2}{n}\right)\log\left(\frac{4Z_1Z_2}{n}\right)$$

$$\ll \left(\frac{Z_1Z_2}{k}\right)\log(Z_1Z_2)\sum_{n\le \frac{8Z_1Z_2}{kl}}\frac{d(n)}{n}.$$

Now using Lemma 3.9,

$$S \ll \left(\frac{Z_1Z_2}{k}\right)\log(Z_1Z_2)\left(\log\left(\frac{8Z_1Z_2}{kl}\right)\right)^2$$

$$\ll \frac{Z_1Z_2}{k}\left(\log(Z_1Z_2)\right)^3.$$

□

4.3 Asymptotics of Coprime Harmonic Sums

Lemma 4.3. Let q be a positive integer and $x \ge 2$ be a real number. Then

$$\sum_{\substack{n\le x,\\ (n,q)=1}} \frac{1}{n} = \frac{\varphi(q)}{q}\left(\log x + \gamma + \sum_{p|q}\frac{\log p}{p-1}\right) + O\left(\frac{2^{w(q)}\log x}{x}\right).$$

In addition, we have

$$\sum_{p|q}\frac{\log p}{p-1} \ll 1+\log w(q).$$

Proof. Using a Möbius function identity, we have

$$\sum_{\substack{n\le x,\\ (n,q)=1}}\frac{1}{n} = \sum_{n\le x}\frac{1}{n}\,\delta_{(n,q)=1} = \sum_{n\le x}\frac{1}{n}\sum_{d|(n,q)}\mu(d) = \sum_{n\le x}\frac{1}{n}\sum_{d|n,\, d|q}\mu(d) = \sum_{d|q}\mu(d)\sum_{n\le x,\, d|n}\frac{1}{n}.$$

Since $d|n$, $\exists\, e \in \mathbb{Z}$ such that $n = de$. Using the property

$$\frac{\varphi(q)}{q} = \prod_{p|q}\left(1 - \frac{1}{p}\right) = \sum_{d|q}\frac{\mu(d)}{d}, \quad \text{and the Lemma 3.12, we have,}$$

$$\begin{aligned}
\sum_{\substack{n \le x,\\ (n,q)=1}} \frac{1}{n} &= \sum_{d|q}\frac{\mu(d)}{d}\sum_{e \le \frac{x}{d}}\frac{1}{e} = \sum_{d|q}\frac{\mu(d)}{d}\left(\log\left(\frac{x}{d}\right) + \gamma + O\left(\frac{d}{x}\right)\right) \\
&= (\log x + \gamma)\sum_{d|q}\frac{\mu(d)}{d} - \sum_{d|q}\frac{\mu(d)}{d}\log d + \sum_{d|q}\frac{\mu(d)}{d}O\left(\frac{d}{x}\right) \\
&= \frac{\varphi(q)}{q}\left(\log x + \gamma + \sum_{p|q}\frac{\log p}{p-1}\right) + \sum_{d|q}\frac{\mu(d)}{d}O\left(\frac{d}{x}\right).
\end{aligned}$$

Since $\exists\, 2^{w(q)}$ square-free divisors of q, we have

$$\sum_{d|q}\frac{\mu(d)}{d}O\left(\frac{d}{x}\right) = O\left(\sum_{d|q}|\mu(d)|\frac{1}{x}\right) = O\left(\frac{2^{w(q)}}{x}\right) = O\left(\frac{2^{w(q)}\log x}{x}\right).$$

Hence, the first part is proved. Let $q = \prod_{i=1}^{w(q)} p_i^{\alpha_i}$. Therefore

$$\sum_{p|q}\frac{\log p}{p-1} = \sum_{i=1}^{w(q)}\frac{\log p_i}{p_i - 1} = \sum_{i=1}^{w(q)}\frac{\log p_i}{p_i} + \sum_{i=1}^{w(q)}\frac{\log p_i}{p_i(p_i-1)} = I_1 + I_2.$$

Now I_2 is a convergent sum, and by Merten's theorem

$$I_1 = \log p_{w(q)} + O(1).$$

By prime number theorem, we have

$$p_{w(q)} \sim w(q)\log w(q).$$

Hence

$$\sum_{p|q}\frac{\log p}{p-1} \ll \log w(q) + \log\log w(q) + 1 \ll 1 + \log w(q).$$

□

4.4 Divisor-weighted Sum Asymptotics under Coprimality Constraints

Lemma 4.4. Let q, n be positive integers such that $q \geq 2$, and n is square-free. Then

$$\sum_{\substack{n \leq q, \\ (n,q)=1}} \frac{2^{w(n)}}{n} \ll \left(\frac{\varphi(q)}{q}\right)^2 (\log q)^2.$$

Furthermore, for $x \geq \sqrt{q}$ we have

$$\sum_{\substack{n \leq x, \\ (n,q)=1}} \frac{2^{w(n)}}{n} \left(\log \frac{x}{n}\right)^2 = \frac{(\log x)^4}{12\zeta(2)} \prod_{p|q} \left(\frac{1-p^{-1}}{1+p^{-1}}\right) \left(1 + O\left(\frac{1+\log w(q)}{\log q}\right)\right).$$

Proof. By Lemma 3.13, we have

$$G(s) := \sum_{\substack{n \geq 1, \\ (n,q)=1}} \frac{2^{w(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)} \prod_{p|q} \frac{1-p^{-s}}{1+p^{-s}}.$$

If $n \leq q$, then $\frac{1}{n} \leq \frac{e}{n^{1+\frac{1}{\log q}}}$. Let $\epsilon = \frac{1}{\log q}$, therefore

$$\sum_{\substack{n \leq q, \\ (n,q)=1}} \frac{2^{w(n)}}{n} \leq e\, G\left(1+\epsilon\right) = \frac{\zeta^2(1+\epsilon)}{\zeta(2+2\epsilon)} \prod_{p|q} \frac{1-p^{-(1+\epsilon)}}{1+p^{-(1+\epsilon)}}.$$

Now $\prod_{p|q} \frac{1-p^{-(1+\epsilon)}}{1+p^{-(1+\epsilon)}}$ becomes

$$\begin{aligned}
\prod_{p|q} \frac{1-\frac{e^{-\epsilon \log p}}{p}}{1+\frac{e^{-\epsilon \log p}}{p}} &= \prod_{p|q} \frac{1-\frac{1}{p}\left(1-\epsilon \log p + O(\epsilon^2 (\log p)^2)\right)}{1+\frac{1}{p}\left(1-\epsilon \log p + O(\epsilon^2 (\log p)^2)\right)} \\
&= \prod_{p|q} \frac{1-\frac{1}{p}+\frac{1}{p}+O\left(\frac{1}{p}\right)}{1+\frac{1}{p}-\frac{1}{p}+O\left(\frac{1}{p}\right)} = \prod_{p|q} \frac{\left(1-\frac{1}{p}\right)\left(1+\frac{1}{p-1}+O\left(\frac{1}{p-1}\right)\right)}{\left(1+\frac{1}{p}\right)\left(1-\frac{1}{p+1}+O\left(\frac{1}{p+1}\right)\right)} \\
&= \prod_{p|q} \frac{\left(1-p^{-1}\right)^2}{\left(1-p^{-2}\right)} \left(1+\frac{1}{p-1}+O\left(\frac{1}{p-1}\right)\right)\left(1+\frac{1}{p+1}+O\left(\frac{1}{p+1}\right)\right) \\
&\ll \left(\frac{\varphi(q)}{q}\right)^2 \zeta(2).
\end{aligned}$$

And $\frac{\zeta^2(1+\epsilon)}{\zeta(2+2\epsilon)}$ becomes

$$\frac{\frac{1}{\epsilon^2}+\frac{2\gamma}{\epsilon}+O(1)}{\zeta(2)+O(\epsilon)} = \left(\frac{1}{\epsilon^2}+\frac{2\gamma}{\epsilon}+\gamma^2+O(1)\right)\frac{1}{\zeta(2)}\left(1+O(\epsilon)\right)$$
$$\ll \frac{1}{\epsilon^2\zeta(2)} = \frac{(\log q)^2}{\zeta(2)}.$$

Hence, the first part is done.

For $\sigma = \Re(s) > 0$ and $c > 0$, $y \geq 1$, we know the Mellin identity,

$$(\log y)^2 = \frac{2}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{y^s}{s^3}ds.$$

Therefore

$$\sum_{\substack{n\leq x,\\(n,q)=1}}\frac{2^{w(n)}}{n}\left(\log\frac{x}{n}\right)^2 = \frac{2}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s^3}\sum_{\substack{n\leq x,\\(n,q)=1}}\frac{2^{w(n)}}{n^{1+s}}\,ds = \frac{2}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(1+s)\frac{x^s}{s^3}ds.$$

Now the factor $\zeta^2(1+s)$ in $F(1+s)$ has a pole of order 2 at $s=0$, while $\frac{1}{s^3}$ has a pole of order 3 at $s=0$. Hence, the integrand has a pole of order 5 at $s=0$. Since the non-trivial zeros of $\zeta(s)$ lie on the strip $0\leq\sigma\leq 1$, the non-trivial zeros of $\zeta(2+2s)$ lie on the strip $-1\leq\sigma\leq-\frac{1}{2}$. These zeros therefore give rise to poles of the integrand. To avoid these poles, we shift the contour to $\sigma=-\frac{1}{2}+\epsilon$, for some $\epsilon>0$. Thus

$$\sum_{\substack{n\leq x,\\(n,q)=1}}\frac{2^{w(n)}}{n}\left(\log\frac{x}{n}\right)^2 = \frac{2}{2\pi i}\int_{-\frac{1}{2}+\epsilon-i\infty}^{-\frac{1}{2}+\epsilon+i\infty}F(1+s)\frac{x^s}{s^3}ds \;+\; 2\,\mathbf{Res}_{s=0}\left(F(1+s)\frac{x^s}{s^3}\right) = I_1+I_2.$$

Now the part $\prod_{p|q}\frac{1-p^{-(1+s)}}{1+p^{-(1+s)}}$ of $F(1+s)$ becomes

$$\prod_{p|q}\left(\frac{1-p^{-1}}{1+p^{-1}}\right)\left(1+O\left(\frac{s\log p}{p}\right)\right) = \prod_{p|q}\left(\frac{1-p^{-1}}{1+p^{-1}}\right)\prod_{p|q}\left(1+O\left(\frac{s\log p}{p}\right)\right)$$
$$= \prod_{p|q}\left(\frac{1-p^{-1}}{1+p^{-1}}\right)\left(1+O\left(s\sum_{p|q}\frac{\log p}{p}\right)\right)$$
$$= \prod_{p|q}\left(\frac{1-p^{-1}}{1+p^{-1}}\right)\left(1+O\left(s(1+\log w(q))\right)\right).$$

Therefore $F(1+s)\,\frac{x^s}{s^3}$ becomes

$$\prod_{p|q}\left(\frac{1-p^{-1}}{1+p^{-1}}\right)\left(1+O\left(s(1+\log w(q))\right)\right)\left(\frac{1}{\epsilon^2}+\frac{2\gamma}{\epsilon}+\gamma^2+O(1)\right)\frac{1}{\zeta(2)}\left(1+O(\epsilon)\right)$$
$$\times\left(\frac{1}{s^3}+\frac{\log x}{s^2}+\frac{(\log x)^2}{s\,2!}+\frac{(\log x)^3}{3!}+\frac{s(\log x)^4}{4!}+O(s^2)\right).$$

Now the residue at $s=0$ is the coefficient of the factor $\frac{1}{s}$. Therefore the required residue is

$$\frac{1}{\zeta(2)}\prod_{p|q}\left(\frac{1-p^{-1}}{1+p^{-1}}\right)\left[\frac{(\log x)^4}{4!}+\frac{\gamma\,(\log x)^3}{3}+O((1+\log w(q))(\log x)^2)\right]$$
$$=\frac{1}{\zeta(2)}\prod_{p|q}\left(\frac{1-p^{-1}}{1+p^{-1}}\right)\frac{(\log x)^4}{4!}\left[1+\frac{4\gamma}{\log x}+O\left(\frac{1+\log w(q)}{(\log x)^2}\right)\right]$$
$$=\frac{1}{\zeta(2)}\prod_{p|q}\left(\frac{1-p^{-1}}{1+p^{-1}}\right)\frac{(\log x)^4}{4!}\left[1+O\left(\frac{1+\log w(q)}{\log x}\right)\right].$$

Since $x\geq\sqrt{q}$, we have

$$I_2=2\,\mathbf{Res}_{s=0}\left(F(1+s)\frac{x^s}{s^3}\right)=\frac{(\log x)^4}{12\zeta(2)}\prod_{p|q}\left(\frac{1-p^{-1}}{1+p^{-1}}\right)\left(1+O\left(\frac{1+\log w(q)}{\log q}\right)\right).$$

Let $s=\sigma+it$. The classical convexity bound for $\zeta(s)$ in the critical strip $0\leq\sigma\leq 1$ is

$$\zeta(\sigma+it)\ll|\,t\,|^{\frac{1-\sigma}{2}+\epsilon},\quad\forall\epsilon>0.$$

Thus we obtain

$$\zeta^2(1+s)\ll_\epsilon|t|^{-\sigma+2\epsilon}\quad\text{and}\quad\zeta(2+2s)=\zeta(2+2\sigma+i2t)\ll_\epsilon|2t|^{-\frac{1}{2}-\sigma+\epsilon}\ll_\epsilon|t|^{-\frac{1}{2}-\sigma+\epsilon}.$$

Since for $\Re(s)\geq 0$,

$$\prod_{p|q}\left|\frac{1-p^{-(1+s)}}{1+p^{-(1+s)}}\right|\leq\prod_{p|q}1=1,$$

we have

$$\left|F(1+s)\frac{x^s}{s^3}\right|=\left|\frac{\zeta^2(1+s)}{\zeta(2+2s)}\right|\prod_{p|q}\left|\frac{1-p^{-(1+s)}}{1+p^{-(1+s)}}\right|\frac{|x^s|}{|s^3|}\ll_\epsilon|t|^{-\frac{5}{2}-\epsilon}x^{-\frac{1}{2}+\epsilon}\ll_\epsilon x^{-\frac{1}{2}+\epsilon}.$$

Thus $I_1 = O\left(x^{-\frac{1}{2}+\epsilon}\right)$. Thus I_1 is negligible, since for small $\epsilon < \frac{1}{2}$, as $x \to \infty$, $x^{-\frac{1}{2}+\epsilon} \to 0$. Hence we have

$$\sum_{\substack{n \le x, \\ (n,q)=1}} \frac{2^{w(n)}}{n} \left(\log \frac{x}{n}\right)^2 = \frac{(\log x)^4}{12\zeta(2)} \prod_{p|q} \left(\frac{1-p^{-1}}{1+p^{-1}}\right) \left(1 + O\left(\frac{1+\log w(q)}{\log q}\right)\right).$$

$\square$

Chapter 5

Proposition 1

Claim: Let $\chi \pmod q$ be a primitive Dirichlet character and $Z = \frac{q}{2^{w(q)}}$, and $B(\chi) = \sum_{\substack{a,b\geq 1,\\ ab\leq Z}} \frac{\chi(a)\overline{\chi}(b)}{\sqrt{ab}} W_\alpha\left(\frac{\pi ab}{q}\right)$. Then

$$\sum_{\chi \pmod q}^{*} |B(\chi)|^2 = \frac{\varphi^*(q)}{8\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})} (\log q)^4 \left(1 + O\left(\frac{w(q)}{\log q}\right)\right).$$

5.1 Decomposition into Main and Error Terms

Since $|B(\chi)|^2 = B(\chi)\overline{B(\chi)}$, we have

$$\sum_{\chi \pmod q}^{*} |B(\chi)|^2 = \sum_{\substack{a,b,c,d\geq 1,\\ ab,cd\leq Z,\\ (abcd,q)=1}} \frac{W_\alpha\left(\frac{\pi ab}{q}\right) W_\alpha\left(\frac{\pi cd}{q}\right)}{\sqrt{abcd}} \sum_{\chi \pmod q}^{*} \chi(ac)\overline{\chi}(bd).$$

Now $\chi(-1) = 1, \text{ or } -1,$ Since $1 = \chi(1) = \chi(-1)\chi(-1)$. Thus $\chi(-1) = (-1)^\alpha$ such that $\alpha = 0, 1$ for odd and even characters respectively. Define the indicator function,

$$\begin{aligned} &\delta_{\chi(-1)=(-1)^\alpha} \\ &= \begin{cases} 1 & \text{if } \chi(-1) = (-1)^\alpha, \\ 0 & \chi(-1) \neq (-1)^\alpha \quad \text{i.e.} \quad \chi(-1) = -(-1)^\alpha \end{cases} \\ &= \frac{1 + (-1)^\alpha \chi(-1)}{2}. \end{aligned}$$

Then we have,

$$\sum_{\chi \pmod q}^{*} \chi(ac)\overline{\chi}(bd) = \sum_{\substack{\chi \pmod q, \\ \chi(-1)=(-1)^\alpha}}^{*} \chi(ac)\overline{\chi}(bd) = \sum_{\chi \pmod q}^{*} \chi(ac)\overline{\chi}(bd)\, \delta_{\chi(-1)=(-1)^\alpha}$$
$$= \frac{1}{2} \sum_{\chi \pmod q}^{*} \chi(ac)\overline{\chi}(bd) + \frac{(-1)^\alpha}{2} \sum_{\chi \pmod q}^{*} \chi(ac)\overline{\chi}(bd)\chi(-1).$$

Applying Lemma 4.1,

$$\sum_{\chi \pmod q}^{*} \chi(ac)\overline{\chi}(bd)$$
$$= \frac{1}{2} \sum_{k|(q,|ac-bd|)} \varphi(k)\,\mu\left(\frac{q}{k}\right) + \frac{(-1)^\alpha}{2} \sum_{k|(q,ac+bd)} \varphi(k)\,\mu\left(\frac{q}{k}\right)$$
$$= \frac{\varphi^*(q)}{2} + \frac{1}{2} \sum_{\substack{k|q, \\ k|(ac-bd), \\ ac\neq bd}} \varphi(k)\,\mu\left(\frac{q}{k}\right) + \frac{(-1)^\alpha}{2} \sum_{\substack{k|q, \\ k|(ac+bd), \\ ac\neq bd}} \varphi(k)\,\mu\left(\frac{q}{k}\right) + \frac{(-1)^\alpha}{2} \sum_{k|(q,2bd)} \varphi(k)\,\mu\left(\frac{q}{k}\right).$$

Since α can take value $0, 1$ for even and odd characters respectively, we have

$$\sum_{\chi \pmod q}^{*} |B(\chi)|^2 = \frac{\varphi^*(q)}{2} \sum_{\substack{a,b,c,d\geq 1, \\ ab,cd\leq Z, \\ ac=bd, \\ (abcd,q)=1}} \frac{W_0\left(\frac{\pi ab}{q}\right) W_0\left(\frac{\pi cd}{q}\right) + W_1\left(\frac{\pi ab}{q}\right) W_1\left(\frac{\pi cd}{q}\right)}{\sqrt{abcd}} + E = M + E,$$

$$\text{where,} \quad E \leq \sum_{k|q} \varphi(k)\mu^2\left(\frac{q}{k}\right) E(k), \text{ and where} \quad E(k) \ll \sum_{\substack{a,b,c,d\geq 1, \\ ab,cd\leq Z, \\ ac\neq bd,\, k|(ac\pm bd),\ (abcd,q)=1}} \frac{1}{\sqrt{abcd}}.$$

5.2 Estimation of the Error Term

In the beginning, we divide the range $1 \leq ab, cd \leq Z$ into dyadic blocks. Consider an arbitrary dyadic block as $Z_1 \leq ab \leq 2Z_1$, and $Z_2 \leq cd \leq 2Z_2$. Since $\sqrt{Z_1 Z_2} \leq Z < q$, for $Z_1 Z_2 > k^{\frac{19}{10}}$, the contribution of this dyadic block to $E(k)$ is, by Lemma 4.2, given as

$$\sum_{\substack{a,b,c,d\geq 1, \\ Z_1\leq ab\leq 2Z_1,\ Z_2\leq cd\leq 2Z_2, \\ ac\neq bd,\ k|(ac\pm bd),\ (abcd,q)=1}} \frac{1}{\sqrt{abcd}} \ll \frac{1}{\sqrt{Z_1 Z_2}} \frac{Z_1 Z_2}{k} (\log Z_1 Z_2)^3 \ll \frac{\sqrt{Z_1 Z_2}}{k} (\log q)^3.$$

Therefore, the total contribution from all dyadic blocks to $E(k)$ is bounded by,

$$\sum_{Z_1<Z}^{d}\sum_{Z_2<Z}^{d}\frac{\sqrt{Z_1Z_2}}{k}(\log q)^3=\frac{(\log q)^3}{k}\left(\sum_{Z_1<Z}^{d}\sqrt{Z_1}\right)\left(\sum_{Z_2<Z}^{d}\sqrt{Z_2}\right)\ll\frac{(\log q)^3}{k}\sqrt{Z}\sqrt{Z}=\frac{Z}{k}(\log q)^3.$$

Since there are $2^{w(q)}$ square-free divisors of q, we have

$$\sum_{\substack{k|q,\\ Z_1Z_2>k^{\frac{19}{10}}}}\varphi(k)\mu^2\left(\frac{q}{k}\right)E(k)\ll Z(\log q)^3\sum_{k|q}1=q\,(\log q)^3.$$

If $Z_1Z_2\leq k^{\frac{19}{10}}$ then by Lemma 4.2 the contribution of one dyadic block to $E(k)$ is

$$\ll\frac{(Z_1Z_2)^{\frac{1}{2}+\epsilon}}{k}\leq k^{\frac{19}{20}+\frac{19\epsilon}{10}-1}=k^{-\frac{1}{20}+\frac{19\epsilon}{10}}.$$

Therefore summing over all dyadic blocks when $Z_1Z_2\leq k^{\frac{19}{10}}$, the contribution is bounded by

$$\begin{aligned}\sum_{Z_1\leq k^{\frac{19}{10}}}^{d}\sum_{Z_2\leq\frac{k^{\frac{19}{10}}}{Z_1}}^{d}k^{-\frac{1}{20}+\frac{19\epsilon}{10}}=k^{-\frac{1}{20}+\frac{19\epsilon}{10}}\sum_{Z_1\leq k^{\frac{19}{10}}}^{d}\frac{k^{\frac{19}{10}}}{Z_1}=k^{-\frac{1}{20}+\frac{19\epsilon}{10}+\frac{19}{10}}\sum_{Z_1\leq k^{\frac{19}{10}}}^{d}\frac{1}{Z_1}\\ \ll k^{-\frac{1}{20}+\frac{19\epsilon}{10}+\frac{19}{10}}\log k\ll_{\delta}k^{-\frac{1}{20}+\frac{19\epsilon}{10}+\frac{19}{10}+\delta}\ll k^{-\frac{1}{20}+\epsilon_1}.\end{aligned}$$

Thus

$$\sum_{\substack{k|q,\\ Z_1Z_2\leq k^{\frac{19}{10}}}}\varphi(k)\mu^2\left(\frac{q}{k}\right)E(k)\leq k^{-\frac{1}{20}+\epsilon_1}\sum_{k|q}k\leq k^{\frac{19}{20}+\epsilon_1}d(q)\ll_{\beta}q^{\frac{19}{20}+\epsilon_1+\beta}=\frac{q}{q^{\frac{1}{20}-\epsilon_1-\beta}}.$$

One can always find some ϵ,δ,β such that $\frac{1}{20}-\epsilon_1-\beta>0$. Then $\frac{\frac{q}{q^{\frac{1}{20}-\epsilon_1-\beta}}}{q(\log q)^3}\longrightarrow 0$ as $q\longrightarrow\infty$. Therefore we have

$$E\ll q\,(\log q)^3.$$

5.3 Evaluation of the Main Term

Let $ac = bd$. Let $g = (a, b)$, $h = (c, d)$ such that $a = gr$, $b = gs$, $c = hs$, and $d = hr$ for some integers r, s with $(r, s) = 1$. So now, $ac = bd \Rightarrow g^2rs = h^2rs$ and $\sqrt{abcd} = ghrs$. Thus

$$
\begin{aligned}
M &= \frac{\varphi^*(q)}{2} \sum_{\substack{a,b,c,d\geq 1,\\ ab,cd\leq Z,\\ ac=bd,\\ (abcd,q)=1}} \frac{W_0\left(\frac{\pi ab}{q}\right) W_0\left(\frac{\pi cd}{q}\right) + W_1\left(\frac{\pi ab}{q}\right) W_1\left(\frac{\pi cd}{q}\right)}{\sqrt{abcd}} \\
&= \frac{\varphi^*(q)}{2} \sum_{\alpha=0,1} \sum_{\substack{g,h,r,s\geq 1,\\ g^2rs,h^2rs\leq Z,\\ (ghrs,q)=1,\\ (r,s)=1}} \frac{1}{ghrs} W_\alpha\left(\frac{\pi g^2rs}{q}\right) W_\alpha\left(\frac{\pi h^2rs}{q}\right).
\end{aligned}
$$

Since the number of ways of writing the square-free number $n = rs$ with $(r, s) = 1$ is $2^{w(n)}$, and $W(x) = 1 + O\left(x^{\frac{1}{2}-\epsilon}\right)$ we have

$$
\begin{aligned}
M &= \frac{\varphi^*(q)}{2} \sum_{\alpha=0,1} \sum_{\substack{(n,q)=1,\\ 1\leq n\leq Z}} \frac{2^{w(n)}}{n} \left(\sum_{\substack{1\leq g\leq\sqrt{\frac{Z}{n}},\\ (g,q)=1}} \frac{1}{g} W_\alpha\left(\frac{\pi g^2 n}{q}\right)\right) \left(\sum_{\substack{1\leq h\leq\sqrt{\frac{Z}{n}},\\ (h,q)=1}} \frac{1}{h} W_\alpha\left(\frac{\pi h^2 n}{q}\right)\right) \\
&= \frac{\varphi^*(q)}{2} \sum_{\alpha=0,1} \sum_{\substack{(n,q)=1,\\ 1\leq n\leq Z}} \frac{2^{w(n)}}{n} \left(\sum_{\substack{1\leq g\leq\sqrt{\frac{Z}{n}},\\ (g,q)=1}} \frac{1}{g} W_\alpha\left(\frac{\pi g^2 n}{q}\right)\right)^2 \\
&= \frac{\varphi^*(q)}{2} \sum_{\alpha=0,1} \sum_{\substack{(n,q)=1,\\ 1\leq n\leq Z}} \frac{2^{w(n)}}{n} \left(\sum_{\substack{1\leq g\leq\sqrt{\frac{Z}{n}},\\ (g,q)=1}} \frac{1}{g} + \sum_{\substack{1\leq g\leq\sqrt{\frac{Z}{n}},\\ (g,q)=1}} \frac{1}{g} O\left(\frac{g^{1-2\epsilon} n^{\frac{1}{2}-\epsilon}}{q^{\frac{1}{2}-\epsilon}}\right)\right)^2.
\end{aligned}
$$

Now since

$$
\sum_{\substack{1\leq g\leq\sqrt{\frac{Z}{n}},\\ (g,q)=1}} g^{-2\epsilon} \leq \int_1^{\sqrt{\frac{Z}{n}}} g^{-2\epsilon}\,dg \ll \left(\frac{Z}{n}\right)^{\frac{1}{2}-\epsilon},
$$

we have

$$\sum_{\substack{1\le g\le\sqrt{\frac{Z}{n}},\\(g,q)=1}} \frac{1}{g}\, O\left(\frac{g^{1-2\epsilon} n^{\frac{1}{2}-\epsilon}}{q^{\frac{1}{2}-\epsilon}}\right) = O\left(\left(\frac{Z}{q}\right)^{\frac{1}{2}-\epsilon}\right) = O(1).$$

Let us divide the region $n \le Z$ into two cases.

- Case 1: Let $n \le Z_0 = \frac{Z}{9^{w(q)}} = \frac{q}{18^{w(q)}}$. Now

$$n \le Z_0 \Longrightarrow 3^{w(q)} = \sqrt{\frac{Z}{Z_0}} \le \sqrt{\frac{Z}{n}} \Longrightarrow \frac{1}{\sqrt{\frac{Z}{n}}} \le \frac{1}{3^{w(q)}}.$$

Using Lemma 4.3, we have

$$\begin{aligned}
\sum_{\substack{1\le g\le\sqrt{\frac{Z}{n}},\\(g,q)=1}} \frac{1}{g} &= \frac{\varphi(q)}{q}\left(\log\sqrt{\frac{Z}{n}} + O(1+\log w(q))\right) + O\left(\frac{2^{w(q)}\,\log\sqrt{\frac{Z}{n}}}{\sqrt{\frac{Z}{n}}}\right)\\
&= \frac{\varphi(q)}{q}\left(\log\sqrt{\frac{Z}{n}} + O(1+\log w(q))\right) + O\left(\frac{2^{w(q)}\,\log\sqrt{\frac{Z}{n}}}{3^{w(q)}}\right)\\
&= \frac{\varphi(q)}{2q}\log\frac{Z}{n} + O(1+\log w(q)).
\end{aligned}$$

So the contribution of this case to M is

$$\begin{aligned}
S_1 &= \varphi^*(q) \sum_{\substack{n\le Z_0,\\(n,q)=1}} \frac{2^{w(n)}}{n}\left(\frac{\varphi(q)}{2q}\log\frac{Z}{n} + O(1+\log w(q))\right)^2\\
&= \varphi^*(q) \sum_{\substack{n\le Z_0,\\(n,q)=1}} \frac{2^{w(n)}}{n}\left(\frac{\varphi(q)}{2q}\left(\log\frac{Z_0}{n} + O(w(q))\right) + O(1+\log w(q))\right)^2\\
&= \varphi^*(q) \sum_{\substack{n\le Z_0,\\(n,q)=1}} \frac{2^{w(n)}}{n}\left(\frac{\varphi(q)}{2q}\log\frac{Z_0}{n} + O(w(q)) + O(1+\log w(q))\right)^2.
\end{aligned}$$

Now $\frac{Z_0}{n} \leq Z_0 < q$. So we have,

$$\begin{aligned}
\left(\frac{\varphi(q)}{2q}\log\frac{Z_0}{n} + O(w(q))\right)^2 &= \left(\frac{\varphi(q)}{2q}\log\frac{Z_0}{n}\right)^2 + O\left(\frac{\varphi(q)}{q}\log\frac{Z_0}{n}w(q)\right) + O(w(q)^2) \\
&= \left(\frac{\varphi(q)}{2q}\log\frac{Z_0}{n}\right)^2 + O(w(q)\log q).
\end{aligned}$$

In addition,

$$\begin{aligned}
&2\left(\frac{\varphi(q)}{2q}\log\frac{Z_0}{n} + O(w(q))\right) O(1+\log w(q)) \\
&= O(\log q \,+\, \log q\,\log w(q)) \,+\, O(w(q) + w(q)\log q) \\
&= O(w(q)\log q).
\end{aligned}$$

Therefore,

$$\begin{aligned}
S_1 &= \varphi^*(q)\left(\frac{\varphi(q)}{2q}\right)^2 \sum_{\substack{n\leq Z_0,\\ (n,q)=1}} \frac{2^{w(n)}}{n}\left(\left(\log\frac{Z_0}{n}\right)^2 + O(w(q)\log q)\right) \\
&= \varphi^*(q)\left(\frac{\varphi(q)}{2q}\right)^2 \sum_{\substack{n\leq Z_0,\\ (n,q)=1}} \frac{2^{w(n)}}{n}\left(\log\frac{Z_0}{n}\right)^2\left(1+O\left(\frac{w(q)}{\log q}\right)\right).
\end{aligned}$$

By Lemma 4.4, we have

$$\begin{aligned}
S_1 = \varphi^*(q)\frac{1}{4}\left(\prod_{p|q}(1-p^{-1})\right)^2 &\frac{(\log q)^4\left(1+O\left(\frac{w(q)}{\log q}\right)\right)}{12\frac{\pi^6}{6}}\prod_{p|q}\left(\frac{1-p^{-1}}{1+p^{-1}}\right) \\
&\left(1+O\left(\frac{w(q)}{\log q}\right)+O\left(\frac{1+\log w(q)}{\log q}\right)+O\left(\frac{w(q)+w(q)\log q}{(\log q)^2}\right)\right).
\end{aligned}$$

Since

$$w(q) \ll \frac{\log q}{\log\log q} \quad \text{and} \quad \frac{w(q)}{\log q} \sim \frac{w(q)+w(q)\log q}{(\log q)^2},$$

we have

$$S_1 = \frac{\varphi^*(q)}{8\pi^2}\prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})}(\log q)^4\left(1+O\left(\frac{w(q)}{\log q}\right)\right).$$

- Case 2: $Z_0 < n \leq Z$.

Applying Lemma 4.3, we have

$$\sum_{\substack{g\le\sqrt{\frac{Z}{n}},\\ (g,q)=1}} \frac{1}{g} \le \sum_{\substack{g\le 3^{w(q)},\\ (g,q)=1}} \frac{1}{g} = \frac{\varphi(q)}{q}\left(\log 3^{w(q)} + \gamma + O(1+\log w(q))\right) + O\left(\frac{2^{w(q)}\log 3^{w(q)}}{3^{w(q)}}\right) \ll \frac{\varphi(q)w(q)}{q}.$$

Thus the contribution of this case to M, using Lemma 4.3 is,

$$\begin{aligned}
S_2 &= \varphi^*(q) \sum_{Z_0<n\le Z} \frac{2^{w(n)}}{n}\left(\frac{\varphi(q)w(q)}{q}\right)^2 \\
&\ll \varphi^*(q)\left(\frac{\varphi(q)w(q)}{q}\right)^2\left(\left(\frac{\varphi(Z)\log Z}{Z}\right)^2 - \left(\frac{\varphi(Z_0)\log Z_0}{Z_0}\right)^2\right) \\
&= \varphi^*(q)\left(\frac{\varphi(q)w(q)}{q}\right)^2\left(\frac{\varphi(Z)\log Z}{Z}\right)^2\left(1-\left(\frac{\varphi(Z_0)\log Z_0}{\varphi(Z)\log Z}\right)^2\right) \\
&\ll \varphi^*(q)(w(q))^2(\log q)^2.
\end{aligned}$$

Therefore,

$$M = \frac{\varphi^*(q)}{8\pi^2}\prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})}(\log q)^4\left(1+O\left(\frac{w(q)}{\log q}\right)\right) + O\left(\varphi^*(q)(w(q))^2(\log q)^2\right).$$

5.4 Completion of the Proof

Hence,

$$\begin{aligned}
\sum_{\chi \pmod q}^{*} |B(\chi)|^2 = M+E =& \frac{\varphi^*(q)}{8\pi^2}\prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})}(\log q)^4\left(1+O\left(\frac{w(q)}{\log q}\right)\right) \\
&+ O\left(\varphi^*(q)(w(q))^2(\log q)^2\right) + O(q\log q)^3 \\
=& \frac{\varphi^*(q)}{8\pi^2}\prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})}(\log q)^4\left(1+O\left(\frac{w(q)}{\log q}\right)+O\left(\left(\frac{w(q)}{\log q}\right)^2\right)\right. \\
&\left. + O\left(\frac{q}{\varphi^*(q)\log q}\right)\right) \\
=& \frac{\varphi^*(q)}{8\pi^2}\prod_{p|q}\frac{(1-p^{-1})^3}{(1+p^{-1})}(\log q)^4\left(1+O\left(\frac{w(q)}{\log q}\right)\right).
\end{aligned}$$

Chapter 6

Proposition 2

Claim: Let $\chi \pmod q$ be a primitive Dirichlet character and $Z = \frac{q}{2^{w(q)}}$, and $C(\chi) = \sum_{\substack{a,b\geq 1,\\ ab>Z}} \frac{\chi(a)\bar{\chi}(b)}{\sqrt{ab}} W_\alpha\left(\frac{\pi ab}{q}\right)$. Then

$$\sum_{\chi \pmod q}^{*} |C(\chi)|^2 \ll \sum_{\chi \pmod q} |C(\chi)|^2 \ll q\left(\frac{\varphi(q)}{q}\right)^5 (w(q)\log q)^2 + q(\log q)^3.$$

6.1 Initial Decomposition

For $0 < x \leq 1$, Lemma 3.5 gives $W_\alpha(x) = 1 + O\left(x^{\frac{1}{2}-\epsilon}\right)$, and hence

$$|W_\alpha(x)| \ll 1.$$

For $x > 1$, Lemma 3.5 implies

$$|W_\alpha(x)| \ll x^{-c} \ll (1+x)^{-c}.$$

Letting $c = 2$, we have $|W_\alpha(x)| \ll (1+x)^{-2}$.

Therefore,

$$\sum_{\chi \pmod q} |C(\chi)|^2 \ll \varphi(q) \sum_{\substack{(abcd,q)=1,\\ ac\equiv \pm bd \pmod q,\\ ab,cd>Z}} \frac{1}{\sqrt{abcd}} \sum_{\alpha=0,1} \left| W_\alpha\left(\frac{\pi ab}{q}\right) W_\alpha\left(\frac{\pi cd}{q}\right) \right|$$

$$\ll \varphi(q) \sum_{\substack{(abcd,q)=1,\\ ac\equiv \pm bd \pmod q,\\ ab,cd>Z}} \frac{1}{\sqrt{abcd}} \left(1+\frac{\pi ab}{q}\right)^{-2} \left(1+\frac{\pi cd}{q}\right)^{-2}$$

$$\ll \varphi(q) \sum_{\substack{(abcd,q)=1,\\ ac\equiv \pm bd \pmod q,\\ ab,cd>Z}} \frac{1}{\sqrt{abcd}} \left(1+\frac{ab}{q}\right)^{-2} \left(1+\frac{cd}{q}\right)^{-2}.$$

Separating the diagonal and off-diagonal terms, we have

$$\sum_{\chi \pmod q} |C(\chi)|^2 \ll \varphi(q) \sum_{\substack{(abcd,q)=1,\\ ac=bd,\\ ab,cd>Z}} \frac{1}{\sqrt{abcd}} \left(1+\frac{ab}{q}\right)^{-2} \left(1+\frac{cd}{q}\right)^{-2} + E \;=\; D+E,$$

$$\text{where, } E \le \varphi(q) \sum_{\substack{(abcd,q)=1,\ ac\neq bd,\\ ac\equiv \pm bd \pmod q,\\ ab,cd>Z}} \frac{1}{\sqrt{abcd}} \left(1+\frac{ab}{q}\right)^{-2} \left(1+\frac{cd}{q}\right)^{-2}.$$

6.2 Evaluation of the Diagonal part

Let $ac = bd$. As in 5.3 of proposition 1, let $g = (a, b)$ and $h = (c, d)$, so that $a = gr$, $b = gs$, $c = hs$, and $d = hr$ for some integers r, s with $(r, s) = 1$. Thus

$$D = \varphi(q) \sum_{\substack{(ghrs,q)=1,\\ g^2rs,h^2rs>Z,\\ (r,s)=1}} \frac{1}{ghrs} \left(1+\frac{g^2rs}{q}\right)^{-2} \left(1+\frac{h^2rs}{q}\right)^{-2}$$

$$= \varphi(q) \sum_{\substack{(n,q)=1,\\ rs=n \text{ is square-free},}} \frac{2^{w(n)}}{n} \left(\sum_{\substack{g>\sqrt{\frac{Z}{n}},\\ (g,q)=1}} \frac{1}{g}\left(1+\frac{g^2n}{q}\right)^{-2} \right)^2.$$

To compute further, let us consider a few cases

Case 1: If $n > q$, then

$$\sum_{\substack{g>\sqrt{\frac{Z}{n}},\\ (g,q)=1}} \frac{1}{g}\left(1+\frac{g^2n}{q}\right)^{-2} \ll \sum_{\substack{g>\sqrt{\frac{Z}{n}},\\ (g,q)=1}} \frac{1}{g}\left(\frac{g^2n}{q}\right)^{-2} \ll \sum_{\substack{g>\sqrt{\frac{Z}{n}},\\ (g,q)=1}} \frac{q^2}{g^5n^2} \ll \zeta(5)\frac{q^2}{n^2} \ll \frac{q^2}{n^2}.$$

Now, by Lemma 3.14, the contribution of this case to D is

$$D_1 \ll \varphi(q) \sum_{n>q,\ (n,q)=1} \frac{2^{w(n)}}{n}\frac{q^4}{n^4} \ll \varphi(q)\log q.$$

Case 2: If $n \leq q$, then

$$\begin{aligned}\sum_{\substack{g>\sqrt{\frac{Z}{n}},\\ (g,q)=1}} \frac{1}{g}\left(1+\frac{g^2n}{q}\right)^{-2} &= \sum_{\substack{\sqrt{\frac{Z}{n}}<g\leq\sqrt{\frac{q}{n}},\\ (g,q)=1}} \frac{1}{g}\left(1+\frac{g^2n}{q}\right)^{-2} + \sum_{\substack{g>\sqrt{\frac{q}{n}},\\ (g,q)=1}} \frac{1}{g}\left(1+\frac{g^2n}{q}\right)^{-2}\\ &\ll \sum_{\substack{\sqrt{\frac{Z}{n}}<g\leq\sqrt{\frac{q}{n}},\\ (g,q)=1}} \frac{1}{g} + \sum_{\substack{g>\sqrt{\frac{q}{n}},\\ (g,q)=1}} \frac{1}{g}\left(\frac{g^2n}{q}\right)^{-2}.\end{aligned}$$

Now

$$\sum_{\substack{g>\sqrt{\frac{q}{n}},\\ (g,q)=1}} \frac{1}{g}\left(\frac{g^2n}{q}\right)^{-2} \ll \frac{q^2}{n^2}\int_{\sqrt{\frac{q}{n}}}^{\infty} \frac{1}{g^5}dg \ll 1.$$

Using Lemma 4.3, we obtain

$$\begin{aligned}\sum_{\substack{\sqrt{\frac{Z}{n}}<g\leq\sqrt{\frac{q}{n}},\\ (g,q)=1}} \frac{1}{g} &= \frac{\varphi(q)}{q}\left(\log\sqrt{\frac{q}{n}} - \log\sqrt{\frac{Z}{n}}\right) + O\left(\frac{2^{w(q)}\log\sqrt{\frac{q}{n}}}{\sqrt{\frac{q}{n}}} - \frac{2^{w(q)}\log\sqrt{\frac{Z}{n}}}{\sqrt{\frac{Z}{n}}}\right)\\ &\ll \frac{\varphi(q)}{2q}\log\frac{q}{Z} = \frac{\varphi(q)}{2q}\log 2^{w(q)}\\ &\ll \frac{\varphi(q)}{q}w(q).\end{aligned}$$

Now by Lemma 4.4, the contribution of this case to D is

$$D_2 \ll \varphi(q)\left(\frac{\varphi(q)}{q}w(q)\right)^2 \sum_{\substack{n\le q,\\(n,q)=1}} \frac{2^{w(n)}}{n} \ll \varphi(q)\left(\frac{\varphi(q)}{q}w(q)\right)^2\left(\frac{\varphi(q)}{q}\log q\right)^2$$

$$= \varphi(q)\left(\frac{\varphi(q)}{q}\right)^4 (w(q)\log q)^2$$

$$= q\left(\frac{\varphi(q)}{q}\right)^5 (w(q)\log q)^2.$$

Thus the total contribution of both cases to D is

$$q\left(\frac{\varphi(q)}{q}\right)^5 (w(q)\log q)^2 \;+\; \varphi(q)\log q \ll q\left(\frac{\varphi(q)}{q}\right)^5 (w(q)\log q)^2.$$

6.3 Estimation of the Remaining Part

We break the range $ab, cd > Z$ into dyadic blocks. Consider a typical dyadic block as $Z_1 \le ab < 2Z_1$, and $Z_2 \le cd < 2Z_2$ such that $Z_1, Z_2 > Z$. If $Z_1Z_2 \le q^{\frac{19}{10}}$ then by Lemma 4.2, the contribution of one dyadic block to E is

$$\ll \frac{(Z_1Z_2)^{\frac{1}{2}+\epsilon}}{q} \ll q^{-\frac{1}{20}+\frac{19\epsilon}{10}}.$$

As in the error estimation 5.2 of proposition 1, the total contribution of all dyadic blocks to E, when $Z_1Z_2 \le q^{\frac{19}{10}}$, is

$$\ll \sum_{Z_1\le q^{\frac{19}{10}}}^{d} \sum_{Z_2\le \frac{q^{\frac{19}{10}}}{Z_1}}^{d} q^{-\frac{1}{20}+\frac{19\epsilon}{10}} \ll q^{-\frac{1}{20}+\epsilon_1}.$$

If $Z_1Z_2 \ge Z^2 > q^{\frac{19}{10}}$ the by Lemma 4.2, the contribution of one dyadic block to E is

$$\ll \frac{\varphi(q)}{\sqrt{Z_1Z_2}}\left(1+\frac{Z_1}{q}\right)^{-2}\left(1+\frac{Z_2}{q}\right)^{-2}\frac{Z_1Z_2}{q}(\log Z_1Z_2)^3$$

$$\ll \frac{\varphi(q)}{q}\left(1+\frac{Z_1}{q}\right)^{-2}\left(1+\frac{Z_2}{q}\right)^{-2}\sqrt{Z_1Z_2}\,(\log Z_1Z_2)^3.$$

To sum of over all dyadic blocks, we need to consider a few cases. Without loss of generality assume that $Z_1 \leq Z_2$.

- Case 1: If $Z_1, Z_2 \leq q$, then

$$\left(1+\frac{Z_1}{q}\right)^{-2} = O(1), \quad \text{and} \quad \left(1+\frac{Z_2}{q}\right)^{-2} = O(1).$$

  Therefore the contribution is

$$R_1 \ll \sum_{Z_1,Z_2 \leq q}^{d} \frac{\varphi(q)}{q} \sqrt{Z_1 Z_2}\, (\log Z_1 Z_2)^3 \ll q\,(\log q)^3.$$

- Case 2: If $Z_1, Z_2 > q$, then

$$\left(1+\frac{Z_1}{q}\right)^{-2} = O\left(\left(\frac{Z_1}{q}\right)^{-2}\right), \quad \text{and} \quad \left(1+\frac{Z_2}{q}\right)^{-2} = O\left(\left(\frac{Z_2}{q}\right)^{-2}\right).$$

  The contribution of this case to the total dyadic sum is

$$\begin{aligned} R_2 \ll \sum_{Z_1,Z_2 > q}^{d} \frac{\varphi(q)}{q} \sqrt{Z_1 Z_2}\, (\log Z_1 Z_2)^3 \left(\frac{Z_1}{q}\right)^{-2} \left(\frac{Z_2}{q}\right)^{-2} &\ll \sum_{Z_1,Z_2>q}^{d} Z_2\,(\log Z_2)^3\, \frac{q^4}{Z_2^2}\frac{1}{Z_1^2} \\ &\ll q^3 \sum_{Z_2 > \max\{q,Z_1\}}^{d} \frac{(\log Z_2)^3}{Z_2^2}. \end{aligned}$$

Let $Z_2 = 2^s q$ for some positive integer s. Then we have

$$R_2 \ll q^3 \sum_{s > \frac{\log\left(\frac{\max\{q,Z_1\}}{q}\right)}{\log 2}} \frac{(\log q \,+\, s\log 2)^3}{2^{2s} q^2} \ll q\,(\log q)^3 \sum_{s > \frac{\log\left(\frac{\max\{q,Z_1\}}{q}\right)}{\log 2}} \frac{1}{2^{2s}} \ll q\,(\log q)^3.$$

Case 3: If $Z_1 \leq q < Z_2$, then

$$\left(1+\frac{Z_1}{q}\right)^{-2} = O\left(1\right), \quad \text{and} \quad \left(1+\frac{Z_2}{q}\right)^{-2} = O\left(\left(\frac{Z_2}{q}\right)^{-2}\right).$$

The contribution of this case to the total dyadic sum is

$$R_3 \ll \sum_{Z_1 \leq q < Z_2}^{d} \frac{\varphi(q)}{q} \sqrt{Z_1 Z_2}\, (\log Z_1 Z_2)^3 \left(\frac{Z_2}{q}\right)^{-2} \ll q^2 \sum_{Z_1 \leq q}^{d} \sqrt{Z_1} \sum_{Z_2 > q}^{d} \frac{(\log Z_1 Z_2)^3}{Z_2^{\frac{3}{2}}}.$$

Let $Z_2 = 2^t q$ for some positive integer t. Then we have

$$R_3 \ll q^2 \sqrt{q}\, \frac{1}{q^{\frac{3}{2}}} \sum_{t=0}^{\infty} \frac{(\log(Z_1 q)\ +\ t \log 2)^3}{2^{\frac{3t}{2}}} \ll q\,(\log q)^3.$$

Thus, the total contributions of all dyadic blocks from these three cases to E is $\ll q\,(\log q)^3$.

6.4 Completion of the Proof

Combining the outcomes for D and E, we have

$$\sideset{}{^*}\sum_{\chi \pmod q} |C(\chi)|^2 \ll \sum_{\chi \pmod q} |C(\chi)|^2 \ll q\left(\frac{\varphi(q)}{q}\right)^5 (w(q)\log q)^2 + q(\log q)^3.$$

Chapter 7

Discussion and Further Directions

7.1 Fourth Moment: From Prime to General Moduli

The study of moments of Dirichlet L-functions, particularly the fourth moment, has played a central role in analytic number theory due to its connections with the distribution of L-values, non-vanishing results, and conjectures such as Lindelöf and the generalized Riemann hypothesis. The work of Soundararajan[11] provides a significant breakthrough by establishing an asymptotic formula for the fourth moment of Dirichlet L-functions valid uniformly for all moduli, including large highly composite ones. As in (1.1), Heath-Brown[4] refined the asymptotic result of Ingham's[3] fourth moment for $\zeta(s)$ with an error term $O\left(T^{\frac{7}{8}+\epsilon}\right)$. Soundararajan[11] improved the error term, so that an asymptotic formula indeed holds for all muduli q. Then it was an open problem to obtain an asymptotic formula for $\sum_{\chi \pmod q}^{*} \left|L\left(\frac{1}{2},\chi\right)\right|^4$ with a error term $O(q^{1-\delta})$, for some $\delta > 0$. In 2005, J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein and N. C. Snaith[14], conjectured that

$$\sum_{\chi \pmod q}^{*} \left|L\left(\frac{1}{2},\chi\right)\right|^{2k} = \varphi^*(q)P_{k^2}(\log q) + O\left(q^{\frac{1}{2}+\epsilon}\right),$$

where $P_{k^2} = \sum_{i=0}^{k^2} a_i x^i$, with coefficients in terms of Euler product

$$a_i = C_i \prod_{p|q}(1 + f_j(p)) \ll (\log\log q)^{O(1)}$$

with explicit constants C_i and arithmetic functions $f_i(p) \ll p^{-1}$. In 2011, M. P. Young[15] obtained a breakthrough result. For prime $q \neq 2$, with some computable absolute constants c_i, he obtained

$$\sideset{}{^*}\sum_{\chi \pmod q} |L(\tfrac{1}{2}, \chi)|^4 = \varphi^*(q) \sum_{i=0}^{4} c_i (\log q)^i + O\left(q^{1-\frac{5}{512}+\epsilon}\right),$$

and using Ramanujan Conjecture, the error term can be of order $O\left(q^{1-\frac{1}{80}+\epsilon}\right)$. In 2017, V. Blomer, É. Fouvry, E. Kowalski, P. Michel, and D. Milićević[16] refined Young's work. They mentioned for any prime q and $\epsilon > 0$, with some computable absolute constants

$$\sideset{}{^*}\sum_{\chi \pmod q} |L(\tfrac{1}{2}, \chi)|^4 = \varphi^*(q) \sum_{i=0}^{4} b_i (\log q)^i + O\left(q^{1-\frac{1}{32}+\epsilon}\right).$$

They subsequently improved the error term to $O\left(q^{1-\frac{1}{20}+\epsilon}\right)$ which is the best known[17] result for prime moduli, till date. They also emphasis that if the Ramanujan–Petersson conjecture holds for Fourier coefficients of level-1 Hecke–Maass forms, then the exponent $\frac{1}{20}$ can be replaced by $\frac{1}{16}$. In 2023, for general moduli q, X. Wu[18] obtained an asymptotic formula with a power saving error term $O\left(q^{1-\frac{11}{448}+\epsilon}\right)$.

7.2 Higher Moments

In 1970, Huxley[19] provided upper bounds for fourth, sixth, and eight moment as below

$$\sum_{q \leq Q} \sideset{}{^*}\sum_{\chi \pmod q} \left| L\left(\frac{1}{2} + it, \chi\right)\right|^4 \ll (Q^2 + Q(|t|+2)) \log^4(Q(|t|+2)),$$

$$\sum_{q \leq Q} \sideset{}{^*}\sum_{\chi \pmod q} \left| L\left(\frac{1}{2} + it, \chi\right)\right|^6 \ll (Q^2 + (Q(|t|+2))^{\frac{3}{2}}) \log^9(Q(|t|+2)), \quad \text{and}$$

$$\sum_{q \leq Q} \sideset{}{^*}\sum_{\chi \pmod q} \left| L\left(\frac{1}{2} + it, \chi\right)\right|^8 \ll Q^2(|t|+2)^2 \log^{16}(Q(|t|+2)).$$

In Theorem 3 of his paper, he mentioned an upper bound for general even moments for the family of Riemann zeta function. For the family of Dirichlet L-functions with primitive characters, the upper bound for second and fourth moment can easily be turned into asymptotic, as discussed earlier, in fact these cases do not require additional averaging over modulus q. In 2006 Rudnick and Soundararajan[20] provided a lower bound for this family. They mentioned that if p is a large prime then for any positive rational k,

$$\sideset{}{^*}\sum_{\chi \pmod p} \left| L\left(\frac{1}{2}, \chi\right) \right|^{2k} \gg_k p(\log p)^{k^2}.$$

7.2.1 Sixth Moment

In 2012, J.B. Conrey, H. Iwaniec and K. Soundararajan[21] conjectured the following

Let $\quad a_3 = \prod_{p:\, primes} \left(1 - \frac{1}{p}\right)^4 \left(1 + \frac{4}{p} + \frac{1}{p^2}\right)$. Then, as $\quad q \to \infty \quad$ with $\quad q \not\equiv 2 \pmod 4$,

$$\frac{1}{\varphi^\flat(q)} \sideset{}{^\flat}\sum_{\chi \pmod q} \left| L\left(\frac{1}{2}, \chi\right) \right|^6 \sim 42 a_3 \prod_{p|q} \frac{\left(1 - \frac{1}{p}\right)^5}{\left(1 + \frac{4}{p} + \frac{1}{p^2}\right)} \frac{(\log q)^9}{9!},$$

where $\sideset{}{^\flat}\sum_{\chi \pmod q}$ is a sum over primitive, even Dirichlet characters, and $\varphi^\flat(q)$ is the number of such characters. Towards this conjecture, they obtained an asymptotic, provided that an additional short averaging in the t-aspect is included, that for large Q,

$$\sum_{q \le Q} \sideset{}{^\flat}\sum_{\chi \pmod q} \int_{-\infty}^{\infty} |\Lambda(\tfrac{1}{2} + it, \chi)|^6 \, dt$$
$$\sim 42 a_3 \sum_{q \le Q} \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{(1 + \frac{4}{p} + \frac{1}{p^2})} \phi^\flat(q) \frac{(\log q)^9}{9!} \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{\frac{1}{2} + it}{2}\right) \right|^6 dt.$$

Also their t-averaging eliminates the contribution of the so called unbalanced sums, that is sums of $d_3(n) d_3(m) \chi(m) \overline{\chi}(n)$. In 2024, V. Chandee and X. Li, and K. Matomäki, and M. Radziwiłł[22] managed to deal with the contribution of such sums using methods such as spectral theory of automorphic forms and bounds of Deligne for hyper-Kloosterman sums. As a result, they obtained an asymptotic formula for the sixth moment without any t-

averaging. They obtained, as $Q \to \infty$,

$$\sum_{q\leq Q} \ \sideset{}{^\flat}\sum_{\chi \pmod q} \left| L\left(\frac{1}{2}, \chi\right)\right|^6 \sim 42 a_3 \sum_{q\leq Q} \prod_{p|q} \frac{\left(1-\frac{1}{p}\right)^5}{\left(1+\frac{4}{p}+\frac{1}{p^2}\right)} \varphi^\flat(q) \frac{(\log q)^9}{9!},$$

where $a_3 = \prod_p \left(1-\frac{1}{p^4}\right)\left(1+\frac{4}{p}+\frac{1}{p^2}\right)$. There has been no further motivating work so far on sixth moment.

7.2.2 Eight Moment

From the work of J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein and N. C. Snaith[14], one may derive the conjecture for eight moment that as $q \to \infty$ with $q \neq 2 \pmod 4$,

$$\frac{1}{\varphi^\flat(q)} \ \sideset{}{^\flat}\sum_{\chi \pmod q} \left| L\left(\frac{1}{2}, \chi\right)\right|^8 \sim 24024 \ a_4 \prod_{p|q} \frac{\left(1-\frac{1}{p}\right)^7}{\left(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3}\right)} \frac{(\log q)^{16}}{16!},$$

where

$$a_4 := \prod_p \left(1-\frac{1}{p}\right)^9 \left(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3}\right).$$

In 2014, V. Chandee, and X. Li[23] obtained an asymptotic for the eight moment of Dirichlet L-functions averaged over primitive, even Dirichlet characters χ modulo q, over all moduli $q \leq Q$ and with a short average on the critical line, conditionally on generalized Riemann hypothesis. They proved the following,

$$\begin{aligned}\sum_{q\leq Q} \ \sideset{}{^\flat}\sum_{\chi \pmod q} \int_{-\infty}^{\infty} |\Lambda(\tfrac{1}{2}+it, \chi)|^8 \, dt \\ \sim 24024 \ a_4 \sum_{q\leq Q} \prod_{p|q} \frac{(1-\frac{1}{p})^7}{\left(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3}\right)} \varphi^\flat(q) \frac{(\log q)^{16}}{16!} \int_{-\infty}^{\infty} \left|\Gamma\left(\frac{\frac{1}{2}+it}{2}\right)\right|^8 dt.\end{aligned}$$

Recently in 2024, V. Chandee and X. Li, and K. Matomäki, and M. Radziwiłł[24] proved the same result but removing the GRH condition assumed in their earlier work.

7.2.3 Twelfth Moment

The study of this specific moment gained prominence with the work of Heath-Brown[6] in 1978. For many decades this stood as a standard result for the t-aspect growth of the Riemann zeta function. In 2021, D. Milícevíc and D. White[25] modified Heath-Brown's t-aspect result[6] to q-aspect for odd prime power moduli $q = p^n$, for the Dirichlet L-functions. For some $A > 0$, and odd prime p, they established the q-aspect twelfth moment as

$$\sum_{\chi \pmod q} \left| L\left(\frac{1}{2}, \chi\right) \right|^{12} \ll_\epsilon p^A q^{2+\epsilon}.$$

Similar to the Heath-Brown's result[6], here the Weyl-strength sun-convexity bound for Dirichlet L-functions in q-aspect is

$$L\left(\frac{1}{2}, \chi\right) \ll q^{\frac{1}{6}+\epsilon}.$$

7.3 Further Directions

- To obtain an unconditional asymptotic formula for the twelfth moment of Dirichlet L-functions for general composite moduli q, extending known results beyond prime power moduli.
- To obtain an asymptotic formula for the eight moment of Dirichlet L-functions without any t-averaging.
- To develop asymptotic formulas for higher moments.

Bibliography

[1] G. H. Hardy and J. E. Littlewood. Contributions to the theory of the riemann zeta-function and the theory of the distribution of primes. *Acta Mathematica*, 41:119–196, 1916.

[2] R. Backlund. Über die beziehung zwischen anwachsen und nullstellen der zeta-funktion. *Öfversigt af Finska Vetenskaps-Societetens Förhandlingar*, 61(9), 1918–1919.

[3] A. E. Ingham. Mean-value theorems in the theory of the riemann zeta-function. *Proceedings of the London Mathematical Society*, 27:273–300, 1926.

[4] D. R. Heath-Brown. The fourth power moment of the riemann zeta function. *Proceedings of the London Mathematical Society*, 38:385–422, 1979.

[5] H. Iwaniec. Fourier coefficients of cusp forms and the riemann zeta-function. In *Séminaire de Théorie des Nombres, 1979–1980*. Université de Bordeaux I, Talence, 1980. Exp. No. 18, 36 pp.

[6] D. R. Heath-Brown. The twelfth power moment of the riemann zeta-function. *Quarterly Journal of Mathematics*, 29(4):443–462, 1978. doi: 10.1093/qmath/29.4.443.

[7] Yoichi Motohashi. An explicit formula for the fourth power mean of the riemann zeta-function. *Acta Mathematica*, 170(2):181–220, 1993.

[8] Yoichi Motohashi. *Spectral Theory of the Riemann Zeta-Function*, volume 127 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1997. ISBN 9780521585328.

[9] K. Ramachandra. Some remarks on a theorem of montgomery and vaughan. *Journal of Number Theory*, 11:465–471, 1979. S. Chowla Anniversary Issue.

[10] D. R. Heath-Brown. The fourth power mean of dirichlet's L-functions. *Analysis*, 1: 25–32, 1981.

[11] K. Soundararajan. The fourth moment of dirichlet L-functions. In *Clay Mathematics Proceedings*, volume 7, pages 239–246. American Mathematical Society, 2007.

[12] A. I. Vinogradov and Yu. V. Linnik. Estimate of the sum of the number of divisors in a short segment of an arithmetic progression. *Uspekhi Mat. Nauk*, 12:277–280, 1957. Available at http://mi.mathnet.ru/eng/rm7688.

[13] P. Shiu. A brun–titchmarsh theorem for multiplicative functions. *Journal für die Reine und Angewandte Mathematik*, 313:161–170, 1980.

[14] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith. Integral moments of L-functions. *Proceedings of the London Mathematical Society*, 91:33–104, 2005.

[15] M. P. Young. The fourth moment of dirichlet L-functions. *Annals of Mathematics*, 173: 1–50, 2011.

[16] V. Blomer, É. Fouvry, E. Kowalski, P. Michel, and D. Miličević. On moments of twisted L-functions. *American Journal of Mathematics*, 139:707–768, 2017.

[17] V. Blomer, É. Fouvry, E. Kowalski, P. Michel, and D. Milićević. Some applications of smooth bilinear forms with Kloosterman sums. *Proc. Steklov Inst. Math.*, 296:18–29, 2017. English translation of Tr. Mat. Inst. Steklova 296 (2017), 24–35.

[18] X. Wu. The fourth moment of dirichlet L-functions at the central value. *Mathematische Annalen*, 387:1199–1248, 2023.

[19] M. N. Huxley. The large sieve inequality for algebraic number fields. II: Means and moments of hecke zeta-functions. *Proceedings of the London Mathematical Society*, 21: 108–128, 1970.

[20] Z. Rudnick and K. Soundararajan. Lower bounds for moments of L-functions. *Proceedings of the National Academy of Sciences of the USA*, 102:6837–6838, 2005.

[21] J. B. Conrey, H. Iwaniec, and K. Soundararajan. The sixth power moment of Dirichlet L-functions. *Geometric and Functional Analysis*, 22(5):1257–1288, 2012.

[22] V. Chandee, X. Li, K. Matomäki, and M. Radziwiłł. The sixth moment of Dirichlet L-functions at the central point. 2024. URL https://arxiv.org/abs/2409.01457. arXiv preprint.

[23] V. Chandee and X. Li. The eighth moment of Dirichlet L-functions. *Advances in Mathematics*, 259:339–375, 2014.

[24] V. Chandee, X. Li, K. Matomäki, and M. Radziwiłł. The eighth moment of Dirichlet L-functions II. *Duke Mathematical Journal*, 173(18), 2024.

[25] D. Milićević and D. White. The twelfth moment of Dirichlet L-functions to prime power moduli. *International Mathematics Research Notices*, 2021.