

Analytical and computational foundations of Peridynamics: Regularity  
theory and the resolution of discontinuities

by

Vu Quang Thinh Dang

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AN ABSTRACT OF A DISSERTATION

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requirements for the degree

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Department of Mathematics  
College of Arts and Sciences

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Manhattan, Kansas

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# Abstract

This dissertation develops analytical and computational foundations for the study of peridynamic and nonlocal models, with a particular focus on solution regularity, nonlocal-to-local limits, and the analysis and numerical treatment of discontinuities.

We first investigate the regularity of solutions to the nonlocal wave equation in the framework of periodic distributions. The spatial operator is given by a nonlocal Laplace operator with a compactly supported kernel, encompassing both integrable and singular interactions. Using a unified Fourier multiplier approach, we characterize the spatial and temporal regularity of solutions in terms of the initial data and forcing terms. This framework extends naturally to nonlocal pseudo-differential operators and applies in arbitrary spatial dimensions. Furthermore, we establish convergence of solutions to those of the classical wave equation under two limiting regimes: vanishing nonlocality and increasing kernel singularity toward a critical, dimension-dependent threshold.

Building on this scalar framework, we then study the peridynamic equilibrium and evolution equations for linear state-based models in multi-dimensional settings. Working in the space of periodic vector-distributions, we analyze both integrable and singular kernels through the asymptotic behavior of the Fourier multipliers associated with the peridynamic operator. This analysis yields explicit regularity results for solutions in terms of the data, kernel singularity, and spatial dimension. In addition, we establish nonlocal-to-local convergence results for both the operator and the corresponding solutions, again under regimes of vanishing horizon and increasing kernel singularity, thereby connecting peridynamic models to classical linear elasticity.

Finally, we investigate the formation and propagation of discontinuities in the one-dimensional nonlocal Poisson equation with discontinuous source terms. We show that the regularity of the solution is governed by the interplay between the smoothness of the kernel

and that of the source term. In particular, singularities of the kernel at the origin induce blow-up in derivatives of the solution, while discontinuities of the kernel or its derivatives at the horizon generate cascades of derivative discontinuities at shifted locations. Motivated by this analysis, we develop a semi-analytic spectral method for accurately resolving discontinuities. The method employs explicit smoothing transformations to remove the discontinuities, enabling high-order spectral approximation and reconstruction of the original solution. Numerical experiments demonstrate that this approach effectively mitigates Gibbs oscillations and significantly improves both accuracy and convergence.

Together, these results provide a unified analytical and computational framework for understanding regularity, singular behavior, and efficient numerical approximation in periodic and nonlocal models.

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# Chapter 1

## Introduction

### 1.1 Overview on peridynamics

In classical solid mechanics, many problems, such as fracture and crack propagation, involve the emergence of discontinuities that were not initially present. The standard continuum mechanics framework relies on partial derivatives to describe the forces and displacements between particles; however, these derivatives become undefined at discontinuities. Although various techniques have been developed to bypass this limitation, they typically require special treatment in the neighborhood of the crack. Consequently, the location of these discontinuities often must be predetermined, a requirement that is frequently unrealistic in complex modeling scenarios.

The peridynamics theory, as proposed by Silling<sup>2</sup>, reformulates continuum mechanics by replacing the differential operators of classical theory with integral operators. This shift ensures that the governing equations remain valid even where displacement derivatives are undefined, allowing a single mathematical framework to be applied across the entire domain. As a nonlocal theory, peridynamics allows particles within a finite distance, defined as the *horizon*, to interact without physical contact. This inherent consistency eliminates the need for a priori knowledge of crack locations or specialized treatments at singular points. As a result, complex phenomena such as crack initiation and propagation emerge naturally from

the solution without the need for tracking algorithms or predetermined failure zones.

Peridynamics has emerged as a versatile tool for engineering analysis, largely due to its inherent compatibility with discontinuities. It has been used in various fields, including the study of material deformation<sup>3–6</sup>, crack propagation<sup>7–10</sup>, dynamic brittle fracture<sup>11–14</sup>, impact fragmentation<sup>15–18</sup>. In addition, the nonlocal nature of the theory facilitates the modeling of corrosion damage<sup>19–22</sup>, composite failure<sup>23–26</sup>, and transport phenomena such as diffusion<sup>27–29</sup> and porous flow<sup>30–33</sup>. A thorough overview of these multifaceted applications can be found in several recent summaries<sup>34–36</sup>.

For a basic formulation of peridynamics, suppose that in the region of interest  $\Omega \subset \mathbb{R}^n$ , each pair of particles interact through a vector-valued function  $\mathbf{f}$ . Assume that at any point  $x \in \Omega$  and at any time  $t \geq 0$ , the force density  $\mathbf{L}$  is a function of the displacement vector field  $\mathbf{u}$ ,

$$\mathbf{L}\mathbf{u}(x, t) = \int_{\mathbb{R}^n} \mathbf{f}(\mathbf{u}(x', t) - \mathbf{u}(x, t), x' - x) dV_{x'}.$$

Then the peridynamic equation of motion is given by

$$\rho(x) \frac{\partial^2}{\partial t^2} \mathbf{u}(x, t) = \mathbf{L}\mathbf{u}(x, t) + \mathbf{b}(x, t), \quad x \in \Omega, \quad t \geq 0$$

and the associated peridynamic equilibrium is

$$\mathbf{L}\mathbf{u}(x) + \mathbf{b}(x) = 0, \quad x \in \Omega.$$

Here  $\rho$  is the mass density,  $\mathbf{f}$  is called the pairwise force function, while  $\mathbf{b}$  is the prescribed force loading density. Denoting by  $\delta$  the horizon (nonlocality) of the domain, we have the following peridynamic assumption

$$\|x' - x\| \geq \delta \implies \mathbf{f}(\mathbf{u}(x', t) - \mathbf{u}(x, t), x' - x) = 0, \quad \forall \mathbf{u}.$$

As a result, any point  $x$  in the material body interacts only with points  $x'$  in its neighborhood

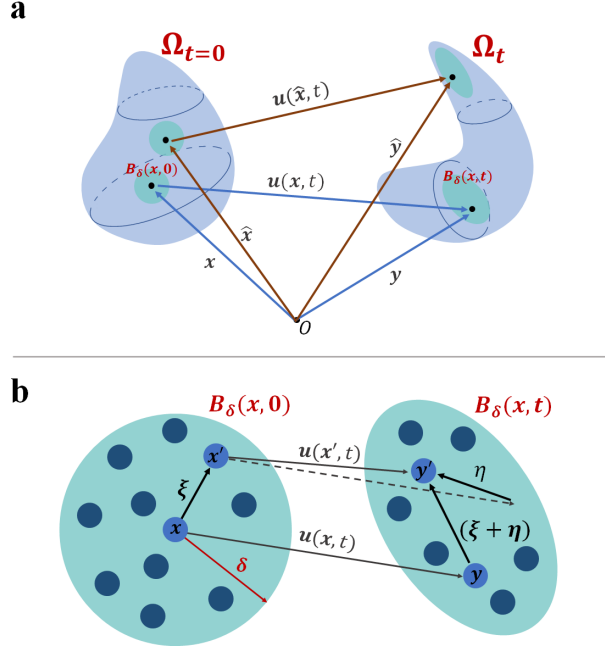


Figure 1.1: Interactions in peridynamics<sup>1</sup>

$B_\delta(x)$  (see Figure 1.1).

## 1.2 Overview on the peridynamic-type scalar nonlocal models

To establish a nonlocal analog for second-order scalar elliptic boundary-value problems, a nonlocal vector calculus was developed in<sup>37;38</sup>. As a rigorous foundation for nonlocal problems, this framework is designed to replicate the structure of classical vector calculus, where the nonlocal Laplacian operator is defined as

$$Lu(x) = \int_{\mathbb{R}^n} (u(y) - u(x))K(x, y)dy, \quad x \in \Omega \subset \mathbb{R}^n.$$

Here,  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  and  $K(x, y)$  represents a scalar kernel that dictates the interaction between points.

In the classical calculus, interactions are strictly local, allowing external influences on a

domain  $\Omega$  to be fully captured as a flux across the boundary  $\partial\Omega$ . In the nonlocal calculus, however, long-range interactions bypass the boundary, making traditional boundary integrals insufficient. Consequently, one must define a nonlocal interaction domain  $\Omega_I$ , representing the region outside  $\Omega$  that remains coupled to the interior as

$$\Omega_I := \{y \in \mathbb{R}^n \setminus \Omega \text{ such that } K(x, y) \neq 0 \text{ for some } x \in \Omega\}.$$

Because of this shift in the nature of particle coupling, standard boundary conditions are insufficient to ensure the well-posedness of the system; instead, a volumetric constraint must be prescribed over  $\Omega_I$ <sup>39</sup>.

Drawing inspiration from peridynamics, a physically significant special case occurs when  $K$  possesses compact support. Specifically, the interactions are restricted to a neighborhood of radius  $\delta$  (the horizon), such that

$$K(x, y) = 0 \quad \text{if} \quad |x - y| \geq \delta. \tag{1.1}$$

Given the compact support of the kernel  $K$ , the interaction domain  $\Omega_I$  is defined as

$$\Omega_I = \{y \in \mathbb{R}^n \setminus \Omega \mid \text{dist}(y, \Omega) < \delta\}.$$

This region characterizes a “collar” of thickness  $\delta$  surrounding the boundary  $\partial\Omega$  (see Figure 1.2).

Nonlocal models featuring this operator framework have been applied to diffusion<sup>39–41</sup>, mechanics<sup>42;43</sup>, and image processing<sup>44;45</sup>. Its mathematical flexibility also extends to multiscale modeling<sup>46;47</sup> and the analysis of stochastic processes<sup>48;49</sup>, demonstrating its efficacy in capturing long-range interactions across diverse physical and computational settings.

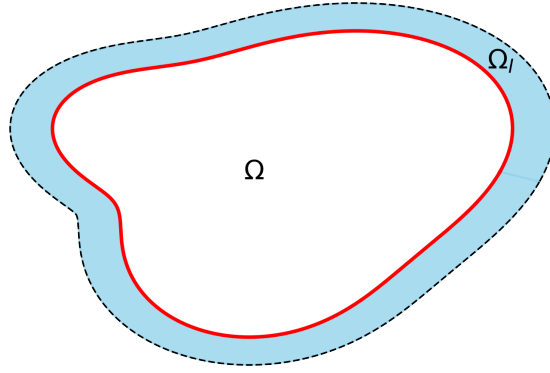


Figure 1.2: The domain  $\Omega$  and its corresponding nonlocal interaction domain  $\Omega_I$

### 1.3 Mathematical and computational challenges in non-local models

Despite the growing success of peridynamic and nonlocal models, several fundamental theoretical and computational challenges remain. While progress has been made in understanding solution regularity and establishing connections to classical local models, a comprehensive analytical theory is still under development. In particular, key issues such as the proper formulation of nonlocal boundary conditions and the mechanisms governing the formation and propagation of discontinuities are not yet fully understood.

From a computational perspective, designing numerical methods that accurately capture nonlocal interactions, especially in the presence of discontinuities, while maintaining computational efficiency remains a significant challenge. Existing approaches often struggle to balance accuracy, stability, and cost, particularly in high-dimensional or highly non-smooth settings.

### 1.4 Contributions of this dissertation

This dissertation addresses several of these challenges by developing both analytical and computational tools for peridynamic and nonlocal models. The main contributions are summarized as follows:

- In Chapter 2, we study the regularity of solutions to the nonlocal wave equation within the framework of periodic distributions. The nonlocal Laplacian is first analyzed as an integral operator with integrable and singular kernels and is then extended to a broader class of pseudo-differential operators. Using asymptotic analysis of Fourier multipliers, we characterize the spatial and temporal regularity of solutions and examine convergence to the classical wave equation under different limiting regimes.
- In Chapter 3, we extend this analysis to peridynamic models, establishing regularity results for both evolution and equilibrium equations. By formulating a state-based peridynamic operator with a kernel structure analogous to the scalar case, we derive rigorous regularity bounds and analyze convergence to classical linear elasticity.
- In Chapter 4, we investigate the formation and structure of discontinuities in the one-dimensional nonlocal Poisson equation with piecewise smooth source term. We show that the regularity of solutions is governed by the interplay between the smoothness of the kernel and that of the source term. To address the associated numerical challenges, we develop a semi-analytic spectral method that combines smoothing techniques with spectral approximations to achieve high-order accuracy in the presence of discontinuities.

# Chapter 2

## Regularity of solutions for the nonlocal wave equation on periodic distributions

This chapter is focused on the regularity of solutions to the nonlocal wave equation

$$\begin{cases} u_{tt}(x, t) = L_\gamma u(x, t) + b(x), & x \in \mathbb{T}^n, t > 0 \\ u(x, 0) = f(x), & x \in \mathbb{T}^n \\ u_t(x, 0) = g(x), & x \in \mathbb{T}^n \end{cases},$$

where  $\mathbb{T}^n$  is the periodic torus in  $\mathbb{R}^n$ . Here  $L_\gamma$  is a nonlocal Laplace operator of the form

$$L_\gamma u(x) = \int_{\mathbb{R}^n} (u(y) - u(x)) \gamma(y, x) dy, \quad (2.1)$$

with  $\gamma$  being a symmetric kernel with compact support. This type of operator, in the case when  $u$  is a vector field and  $\gamma$  being a second order tensor field, originated in peridynamics<sup>2;50;51</sup>. For the case of scalar fields  $u$ , this operator has been introduced in Nonlocal Vector Calculus<sup>38</sup>. Regularity of solutions for nonlocal equations associated with operators

of form (2.1) has been studied under different settings. The work in<sup>52</sup> studies a Dirichlet-type (volume-constraint) problem for scalar nonlocal equations and the work in<sup>53</sup> studies the periodic nonlocal Poisson equation. The regularity of the nonlocal diffusion equation over periodic distributions is studied in<sup>54</sup>. Peridynamics Dirichlet-type constraint problems have been studied in the works<sup>55–57</sup>. The work in<sup>58</sup> addresses nonlocal wave equations on bounded domains and the work in<sup>59</sup> discusses the solvability of peridynamics equations in  $\mathbb{R}^n$ . Regularity of solutions for peridynamics equations, associated with non-compactly supported integral kernels, is addressed in<sup>60</sup>. Dispersive effects in peridynamics and nonlocal wave equations have been investigated in<sup>61;62</sup>. The well-posedness of nonlinear peridynamic models has been studied in<sup>63–66</sup>. There are several numerical and computational studies of nonlocal wave equations and peridynamics equations including the works in<sup>67–73</sup>.

In this work, we focus on integral kernels of the form

$$\gamma(x, y) = \frac{c^{\delta, \beta}}{\|y - x\|^\beta} \chi_{B_\delta(x)}(y)$$

where  $\chi_{B_\delta(x)}$  denotes the indicator function of the ball centered at  $x$  with radius  $\delta > 0$ ,  $c^{\delta, \beta}$  a scaling constant, and the kernel exponent  $\beta < n + 2$ .

For  $\delta > 0$  and  $\beta < n + 2$ , the nonlocal Laplace operator  $L^{\delta, \beta}$  is defined by

$$L^{\delta, \beta} u(x) = c^{\delta, \beta} \int_{B_\delta(x)} \frac{u(y) - u(x)}{\|y - x\|^\beta} dy = c^{\delta, \beta} \int_{B_\delta(0)} \frac{u(x + z) - u(x)}{\|z\|^\beta} dz, \quad (2.2)$$

where the scaling constant  $c^{\delta, \beta}$  is given by

$$c^{\delta, \beta} = \frac{2(n + 2 - \beta)\Gamma\left(\frac{n}{2} + 1\right)}{\pi^{n/2}\delta^{n+2-\beta}}.$$

In<sup>53</sup>, the Fourier multiplier of this operator is shown to have a hypergeometric representation, which then is used to extend the multiplier, and consequently the operator, to the case in which  $\beta \geq n + 2$ , with  $\beta \neq n + 4, n + 6, \dots$ . We exploit the Fourier multipliers results that were developed in<sup>53</sup> to study the regularity of the nonlocal wave equation over the space

of periodic distributions  $H^q(\mathbb{T}^n)$ , with  $q \in \mathbb{R}$ .

## 2.1 Overview and some results on the nonlocal Laplacian

In this section, we present some previous as well as new results on the nonlocal Laplace operator (2.2). In addition, we provide some results on the space of periodic distributions that will be used in subsequent sections.

### 2.1.1 The nonlocal Laplace operator and its multiplier

The Fourier multiplier  $m^{\delta,\beta}$  of the operator  $L^{\delta,\beta}$ , in (2.2), are defined through the Fourier transform by

$$\widehat{L^{\delta,\beta}u} = m^{\delta,\beta} \hat{u},$$

where  $m^{\delta,\beta}$  is of the form

$$m^{\delta,\beta}(\nu) = c^{\delta,\beta} \int_{B_\delta(0)} \frac{\cos(\nu \cdot z) - 1}{\|z\|^\beta} dz. \quad (2.3)$$

In subsequent sections, we will make use of the following results, which are proved in<sup>53</sup>.

**Theorem 1.** *Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta < n + 2$ . The Fourier multipliers  $m^{\delta,\beta}$  have the following representation*

$$m^{\delta,\beta}(\nu) = -\|\nu\|^2 {}_2F_3 \left( 1, \frac{n+2-\beta}{2}; 2, \frac{n+2}{2}, \frac{n+4-\beta}{2}; -\frac{1}{4}\|\nu\|^2\delta^2 \right). \quad (2.4)$$

The hypergeometric function  ${}_2F_3$  on the right hand side is well-defined for any  $\beta \neq n+4, n+6, \dots$ , since for these values the parameter  $\frac{n+4-\beta}{2}$  becomes a non-positive integer, which leads to singularities in the definition of the generalized hypergeometric function due to poles in the Gamma functions appearing in its coefficients. Hence, using (2.4), the definition

of the multipliers is extended to the case when  $\beta \geq n + 2$  with  $\beta \neq n + 4, n + 6, \dots$ . Consequently, the operator  $L^{\delta, \beta}$  is extended to these larger values of  $\beta$  using the Fourier transform. In particular, for the case when  $\beta = n + 2$ , and since  $m^{\delta, n+2}(\nu)$  is equal to  $-\|\nu\|^2$ , the extended operator  $L^{\delta, \beta}$  coincides with the classical Laplace operator  $\Delta$ .

For clarity, we reiterate the fact that the operator  $L^{\delta, \beta}$  is an integral operator for  $\beta < n + 2$ , having an integral kernel when  $\beta < n$  and a singular kernel when  $n \leq \beta < n + 2$ . When  $\beta > n + 2$ , with  $\beta \neq n + 4, n + 6, \dots$ , the operator  $L^{\delta, \beta}$  can be recognized as an integral-differential operator, as we show in Section 2.1.2. In particular, when  $n + 2 < \beta < n + 4$ , the operator  $L^{\delta, \beta}$  can be interpreted as a super-diffusion operator in the setting of heat diffusion or conduction, as discussed in<sup>72</sup>.

**Corollary 1.** *Let  $n \geq 1$  and  $\nu \in \mathbb{R}^n$ . Then the Fourier multipliers  $m^{\delta, \beta}$  converge to the Fourier multipliers of the Laplacian as follows*

$$\lim_{\delta \rightarrow 0^+} m^{\delta, \beta}(\nu) = -\|\nu\|^2, \quad \text{for } \beta \in \mathbb{R} \setminus \{n + 4, n + 6, n + 8, \dots\}, \quad (2.5)$$

and

$$\lim_{\beta \rightarrow n+2} m^{\delta, \beta}(\nu) = -\|\nu\|^2, \quad \text{for } \delta > 0. \quad (2.6)$$

The asymptotic behavior of the multipliers for large  $\nu$  is described through the following result.

**Theorem 2.** *Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta \in \mathbb{R} \setminus \{n + 2, n + 4, n + 6, \dots\}$ . Then, as  $\|\nu\| \rightarrow \infty$ ,*

$$m^{\delta, \beta}(\nu) \sim \begin{cases} -\frac{2n(n+2-\beta)}{\delta^2(n-\beta)} + 2\left(\frac{2}{\delta}\right)^{n+2-\beta} \frac{\Gamma\left(\frac{n+4-\beta}{2}\right)\Gamma\left(\frac{n+2}{2}\right)}{(n-\beta)\Gamma\left(\frac{\beta}{2}\right)} \|\nu\|^{\beta-n} & \text{if } \beta \neq n, \\ -\frac{2n}{\delta^2} \left(2 \log \|\nu\| + \log\left(\frac{\delta^2}{4}\right) + \gamma - \psi\left(\frac{n}{2}\right)\right) & \text{if } \beta = n, \end{cases}$$

where  $\gamma$  is Euler's constant and  $\psi$  is the digamma function.

### 2.1.2 Explicit formula for the extended operator for $\beta < n + 4$

The extended operator  $L^{\delta,\beta}$  as described in Section 2.1.1 is only defined indirectly through the Fourier transform as a pseudo-operator for the case  $\beta > n + 2$ . In this section, we provide an explicit differential-integral formula for  $L^{\delta,\beta}$  when  $\beta < n + 4$ . We emphasize that such representation can be generalized in a similar way to  $\beta > n + 4$  with  $\beta \neq n + 6, n + 8, \dots$

**Theorem 3.** *Let  $\beta < n + 4$ . Then the nonlocal Laplace operator defined as the pseudo-differential operator through its multipliers given in (2.4) can be represented explicitly in the form*

$$L^{\delta,\beta}u(x) = \Delta u(x) + c^{\delta,\beta} \int_{B_\delta(0)} \frac{u(x+z) - u(x) - \nabla \nabla u(x) : \frac{z \otimes z}{2}}{\|z\|^\beta} dz, \quad (2.7)$$

for sufficiently differentiable fields  $u$ . Moreover, its multipliers can be expressed as

$$m^{\delta,\beta}(\nu) = -\|\nu\|^2 + c^{\delta,\beta} \int_{B_\delta(0)} \frac{\cos(\nu \cdot z) - 1 + \frac{(\nu \cdot z)^2}{2}}{\|z\|^\beta} dz. \quad (2.8)$$

*Proof.* We begin with representing  $u(x)$  through its Fourier transform as

$$u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}(\nu) e^{i\nu \cdot x} d\nu.$$

Applying the operator  $L^{\delta,\beta}$  gives

$$\begin{aligned}
L^{\delta,\beta}u(x) &= \Delta u(x) + c^{\delta,\beta} \int_{B_\delta(0)} \frac{u(x+z) - u(x) - \nabla \nabla u(x) : \frac{z \otimes z}{2}}{\|z\|^\beta} dz \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}(x) \Delta e^{i\nu \cdot x} d\nu \\
&\quad + \frac{c^{\delta,\beta}}{(2\pi)^n} \int_{B_\delta(0)} \|z\|^{-\beta} \left( \int_{\mathbb{R}^n} \hat{u}(\nu) e^{i\nu \cdot (x+z)} d\nu - \int_{\mathbb{R}^n} \hat{u}(\nu) e^{i\nu \cdot x} d\nu \right) dz \\
&\quad - \frac{c^{\delta,\beta}}{(2\pi)^n} \int_{B_\delta(0)} \|z\|^{-\beta} \left( \int_{\mathbb{R}^n} \hat{u}(\nu) \nabla \nabla e^{i\nu \cdot x} d\nu \right) : \frac{z \otimes z}{2} dz \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}(x) (-\|\nu\|^2) e^{i\nu \cdot x} d\nu \\
&\quad + \frac{c^{\delta,\beta}}{(2\pi)^n} \int_{B_\delta(0)} \|z\|^{-\beta} \int_{\mathbb{R}^n} \hat{u}(\nu) e^{i\nu \cdot x} (e^{i\nu \cdot z} - 1) d\nu dz \\
&\quad - \frac{c^{\delta,\beta}}{(2\pi)^n} \int_{B_\delta(0)} \|z\|^{-\beta} \left( \int_{\mathbb{R}^n} \hat{u}(\nu) (-\nu \otimes \nu) e^{i\nu \cdot x} d\nu \right) : \frac{z \otimes z}{2} dz \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left[ -\|\nu\|^2 + c^{\delta,\beta} \int_{B_\delta(0)} \frac{\cos(\nu \cdot z) - 1 + \frac{(\nu \cdot z)^2}{2}}{\|z\|^\beta} dz \right] \hat{u}(\nu) e^{i\nu \cdot x} d\nu \\
&:= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} m^{\delta,\beta}(\nu) \hat{u}(\nu) e^{i\nu \cdot x} d\nu.
\end{aligned}$$

Moreover, using the spherical coordinates  $\nu \cdot z = \|\nu\| r \cos \phi_1$ , we obtain

$$\begin{aligned}
m^{\delta,\beta}(\nu) &= -\|\nu\|^2 + c^{\delta,\beta} \int_{B_\delta(0)} \frac{\cos(\nu \cdot z) - 1 + \frac{(\nu \cdot z)^2}{2}}{\|z\|^\beta} dz \\
&= -\|\nu\|^2 + c^{\delta,\beta} \int_0^\delta r^{n-1-\beta} \int_{S^{n-1}} [\cos(\|\nu\| r \cos \phi_1) - 1 \\
&\quad + \|\nu\|^2 r^2 \cos^2 \phi_1 / 2] d_{S^{n-1}} V dr \\
&= -\|\nu\|^2 + c^{\delta,\beta} \int_0^\delta r^{n-1-\beta} \int_{S^{n-1}} [\cos(\|\nu\| r \cos \phi_1) - 1] d_{S^{n-1}} V dr \\
&\quad + \frac{c^{\delta,\beta} \|\nu\|^2}{2} \int_0^\delta r^{n+1+\beta} \int_0^\pi \cos^2 \phi_1 \sin^{n-2} \phi_1 d\phi_1 S_{n-2} dr.
\end{aligned}$$

Using<sup>53</sup> equation (15) we have

$$\begin{aligned}
m^{\delta,\beta}(\nu) &= -\|\nu\|^2 + c^{\delta,\beta} \int_0^\delta r^{n-1-\beta} 2\pi^{n/2} \sum_{k=1}^\infty \frac{\left(-\frac{1}{4}\|\nu\|^2 r^2\right)^k}{k! \Gamma\left(\frac{n}{2} + k\right)} dr \\
&\quad + \frac{c^{\delta,\beta} \|\nu\|^2}{2} \int_0^\delta r^{n+1-\beta} \sqrt{\pi} \left[ \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} - \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \right] \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} dr \\
&= -\|\nu\|^2 + 2\pi^{n/2} c^{\delta,\beta} \sum_{k=1}^\infty \frac{\left(-\frac{1}{4}\|\nu\|^2\right)^k}{k! \Gamma\left(\frac{n}{2} + k\right)} \frac{\delta^{n-\beta+2k}}{n-\beta+2k} + c^{\delta,\beta} \|\nu\|^2 \frac{\delta^{n+2-\beta}}{n+2-\beta} \frac{(n-1)\pi^{n/2}}{\Gamma\left(\frac{n+2}{2}\right)} \\
&= 2\pi^{n/2} c^{\delta,\beta} \sum_{k=1}^\infty \frac{\left(-\frac{1}{4}\|\nu\|^2\right)^k}{k! \Gamma\left(\frac{n}{2} + k\right)} \frac{\delta^{n-\beta+2k}}{n-\beta+2k}.
\end{aligned}$$

The series in the last equality was shown in<sup>53</sup> to coincide with the hypergeometric function given in (2.4), thus showing the result. □

**Proposition 1.** *Let  $\beta < n + 2$ . Then, the definition of  $L^{\delta,\beta}u$  given in (2.7) coincides with that in (2.2). Moreover, the multipliers given in (2.8) coincide with that in (2.3).*

*Proof.* This is a direct application of the following observation. For  $\beta < n + 2$  and sufficiently differentiable  $u$ , we have

$$\Delta u(x) = \frac{c^{\delta,\beta}}{2} \int_{B_\delta(0)} \frac{z \otimes z}{\|z\|^\beta} dz : \nabla \nabla u(x).$$

On the other hand, for  $\nu \in \mathbb{R}^n$

$$\|\nu\|^2 = I : (\nu \otimes \nu) = \left[ \frac{c^{\delta,\beta}}{2} \int_{B_\delta(0)} \frac{z \otimes z}{\|z\|^\beta} dz \right] : (\nu \otimes \nu) = \frac{c^{\delta,\beta}}{2} \int_{B_\delta(0)} \frac{(\nu \cdot z)^2}{\|z\|^\beta} dz.$$

□

In<sup>53</sup>, the convergence of the original operator  $L^{\delta,\beta}$  for  $\beta < n + 2$  in (2.4) to the Laplacian operator  $\Delta$  under two limits as  $\delta \rightarrow 0^+$  or as  $\beta \rightarrow (n + 2)^-$  has been established. We now generalize this result to the extended operator  $L^{\delta,\beta}$  in (2.7) under the assumption that  $\beta < n + 3$ . A similar result can be obtained when  $n + 3 \leq \beta < n + 4$  in requirement of more

regularity imposed on  $u(x)$ .

**Theorem 4.** *For  $u \in C^3(\mathbb{R}^n)$ , we have*

$$\lim_{\delta \rightarrow 0^+} L^{\delta, \beta} u(x) = \Delta u(x) \quad \text{for } \beta < n + 3,$$

and

$$\lim_{\beta \rightarrow n+2} L^{\delta, \beta} u(x) = \Delta u(x) \quad \text{for } \delta > 0.$$

*Proof.* Verifying that (2.7) is well-defined for  $\beta < n + 3$  and  $u \in C^3(\mathbb{R}^n)$  is straightforward.

Thanks to the Taylor's theorem, we get the following expansion of  $u$  around  $x$ :

$$u(x + z) = u(x) + \nabla u(x) : z + \frac{1}{2} \nabla \nabla u(x) : z \otimes z + R(u; x, z),$$

where the remainder  $R(u; x, z)$  is of the form

$$R(u; x, z) = \frac{1}{6} \nabla \nabla \nabla u(x + sz) : z \otimes z \otimes z,$$

for some scalar  $s \in [0, 1]$ . Observe that (2.7) now becomes

$$L^{\delta, \beta} u(x) - \Delta u(x) = c^{\delta, \beta} \nabla u(x) : \int_{B_\delta(0)} \frac{z}{\|z\|^\beta} dz + c^{\delta, \beta} \int_{B_\delta(0)} \frac{R(u; x, z)}{\|z\|^\beta} dz.$$

The first expression on the right hand side is identically zero due to symmetry. Hence, this shows that

$$|L^{\delta, \beta} u(x) - \Delta u(x)| = \left| c^{\delta, \beta} \int_{B_\delta(0)} \frac{R(u; x, z)}{\|z\|^\beta} dz \right| \leq H_x |c^{\delta, \beta}| \int_{B_\delta(0)} \frac{\|z\|^3}{\|z\|^\beta} dz.$$

Here  $H_x$  denotes

$$H_x := \frac{1}{6} \max_{\substack{s \in [0,1] \\ z \in \overline{B_\delta(0)}}} |\nabla \nabla \nabla u(x + sz)|.$$

A simple calculation yields that

$$\int_{B_\delta(0)} \frac{\|z\|^3}{\|z\|^\beta} dz = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \frac{\delta^{3+n-\beta}}{3+n-\beta}.$$

This observation indicates that

$$|L^{\delta,\beta}u(x) - \Delta u(x)| \leq H_x \frac{2n|2+n-\beta|}{3+n-\beta} \delta,$$

from which the convergence follows. □

### 2.1.3 Monotonicity of the multipliers

In this section, we include a result on the monotonicity of the multipliers as a function of  $\beta$ . This lemma will be essential to prove convergence of solutions of the nonlocal wave equation to the solution of the classical wave equation in the limit as  $\beta \rightarrow n+2$ , as given in Theorem 9.

**Lemma 1.** *Fixing  $\delta > 0$  and  $\nu \neq 0$ , the function  $\beta \mapsto m^{\delta,\beta}(\nu)$  is monotonically decreasing for  $\beta \in (\frac{n+4}{2}, n+4)$ .*

*Proof.* We express the multipliers as

$$m^{\delta,\beta}(\nu) = -\|\nu\|^2 {}_2F_3 \left( 1, a; 2, a+1, b; -\frac{1}{4}\|\nu\|^2 \delta^2 \right),$$

where

$$a = \frac{n+2-\beta}{2} \quad \text{and} \quad b = \frac{n+2}{2}.$$

For  $z < 0$  define the function

$$\phi(a) := {}_2F_3(1, a; 2, a+1, b; z).$$

Hence, if we let  $z = -\frac{1}{4}\delta^2\|\nu\|^2$ , then

$$m^{\delta, \beta}(\nu) = -\|\nu\|^2 \phi(a).$$

Note that if  $\phi(a)$  is strictly monotonically decreasing in  $a$  and  $\nu \neq 0$ , then  $m^{\delta, \beta}(\nu)$  is strictly monotonically decreasing in  $\beta$ .

We begin with an application of<sup>74</sup> Eq. (16.5.2), which provides the representation

$${}_2F_3(1, a; 2, a+1, b; z) = \int_0^1 {}_1F_2(a; a+1, b; zt) dt.$$

Thus, a sufficient condition for the monotonicity of  $\phi$  is that the function

$$\psi(a) := {}_1F_2(a; a+1, b; z)$$

is strictly monotonically decreasing for any  $z < 0$ . In general, differentiating hypergeometric functions with respect to their parameters is messy, but the special structure of the arguments in this case makes it relatively straightforward. Written as a series,

$$\psi(a) = \sum_{k=0}^{\infty} \frac{(a)_k}{(a+1)_k (b)_k} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{a}{a+k} \frac{1}{(b)_k} \frac{z^k}{k!}.$$

According to the Weierstrass M-test, the series converges uniformly. Thus,

$$\begin{aligned}
\psi'(a) &= \sum_{k=1}^{\infty} \frac{k}{(a+k)^2} \frac{1}{(b)_k} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{(a+1+k)^2} \frac{1}{(b)_{k+1}} \frac{z^{k+1}}{k!} \\
&= \frac{z}{b(a+1)^2} \sum_{k=0}^{\infty} \frac{(a+1)^2}{(a+1+k)^2} \frac{1}{(b+1)_k} \frac{z^k}{k!} \\
&= \frac{z}{b(a+1)^2} \sum_{k=0}^{\infty} \frac{(a+1)_k (a+1)_k}{(a+2)_k (a+2)_k (b+1)_k} \frac{z^k}{k!} \\
&= \frac{z}{b(a+1)^2} {}_2F_3(a+1, a+1; a+2, a+2, b+1; z).
\end{aligned}$$

Since  $z < 0$ , if we show that  ${}_2F_3(a+1, a+1; a+2, a+2, b+1; z) > 0$ , it follows that  $\psi'(a) < 0$ . Now consider<sup>75</sup>. As described in the introduction of that paper, we may start from the identity that

$$\begin{aligned}
{}_2F_3\left(a+1, a+1; a+1, a+\frac{3}{2}, 2(a+1); -x^2\right) &= {}_1F_2\left(a+1; a+\frac{3}{2}, 2(a+1); -x^2\right) \\
&= \frac{\Gamma\left(a+\frac{3}{2}\right)^2}{(x/2)^{2a+1}} J_{a+\frac{1}{2}}(x)^2 \geq 0.
\end{aligned}$$

Following that, we may use the transference principle (see<sup>75</sup> Prop. 2.1) to assert that, since  $a+2 > a+3/2 > a+1$  and  $b+1 > 2(a+1)$  (as long as  $\beta > \frac{n+4}{2}$ ), it follows that  $\psi'(a) < 0$  as desired.  $\square$

*Remark 1.* We note that a version of Lemma 1 has appeared in<sup>53</sup>, however the proof contained an error. This is fixed in Lemma 1 above, whose proof is based on the hypergeometric representation (2.4). In addition, a simpler proof based on the representation given by (2.8), but for a more restrictive condition on  $\beta$ , is included below for clarity of the presentation.

**Lemma 2.** *Let  $n+2 \leq \beta < n+4$  and assume  $\beta' < \beta$ . Then, for all  $\nu \neq 0$ ,*

$$m^{\delta, \beta}(\nu) < m^{\delta, \beta'}(\nu) < 0.$$

*Proof.* Obviously, when  $\beta < n+2$ ,  $c^{\delta, \beta} > 0$ . From (2.3) we have  $m^{\delta, \beta}(\nu) < 0$ . Otherwise,

when  $n + 2 \leq \beta < n + 4$ ,  $c^{\delta, \beta} \leq 0$ . From (2.8), we also get  $m^{\delta, \beta}(\nu) < 0$ .

- Case  $\beta' < n + 2 \leq \beta$ : We have  $c^{\delta, \beta'} > 0 \geq c^{\delta, \beta}$ . This leads to

$$m^{\delta, \beta}(\nu) \leq -\|\nu\|^2 < m^{\delta, \beta'}(\nu).$$

- Case  $n + 2 \leq \beta' < \beta < n + 4$ : By changing the variable  $z = \delta w$

$$\begin{aligned} m^{\delta, \beta}(\nu) &= -\|\nu\|^2 + c^{\delta, \beta} \delta^{n-\beta} \int_{B_1(0)} \frac{\cos(\nu \cdot \delta w) - 1 + \frac{(\nu \cdot \delta w)^2}{2}}{\|w\|^\beta} dw \\ &= -\|\nu\|^2 - \frac{2(\beta - (n + 2))\Gamma\left(\frac{n}{2} + 1\right)}{\pi^{n/2}\delta^2} \int_{B_1(0)} \frac{\cos(\nu \cdot \delta w) - 1 + \frac{(\nu \cdot \delta w)^2}{2}}{\|w\|^\beta} dw. \end{aligned}$$

Notice that when  $n + 2 \leq \beta' < \beta < n + 4$ , we have  $\beta - (n + 2) > \beta' - (n + 2) \geq 0$ . In addition,  $\|w\|^\beta < \|w\|^{\beta'}$  and  $\cos(\nu \cdot \delta w) - 1 + \frac{(\nu \cdot \delta w)^2}{2} \geq 0$  for  $w \in B_1(0)$ . Combining these facts yield

$$m^{\delta, \beta}(\nu) < m^{\delta, \beta'}(\nu).$$

□

#### 2.1.4 The periodic torus and the eigenvalues of $L^{\delta, \beta}$

Define the periodic torus as

$$\mathbb{T}^n = \prod_{j=1}^n [0, 2\pi], \quad \text{with } j = 1, 2, \dots, n.$$

The functions  $\{e^{ik \cdot x}\}_{k \in \mathbb{Z}^n}$  form a complete set in  $L^2(\mathbb{T}^n)$ . Moreover, when  $\beta < n + 2$ ,

$$L^{\delta, \beta} e^{ik \cdot x} = \left( c^{\delta, \beta} \int_{B_\delta(0)} \frac{e^{ik \cdot z} - 1}{\|z\|^\beta} dz \right) e^{ik \cdot x} = m^{\delta, \beta}(k) e^{ik \cdot x},$$

which implies that  $e^{ik \cdot x}$  is an eigenfunction of  $L^{\delta, \beta}$  with eigenvalue  $m^{\delta, \beta}(k)$ .

In fact, the above observation on the eigenvalues of  $L^{\delta, \beta}$  for  $\beta < n + 2$ , carries over to the extended operator with any  $\beta \geq n + 2$  with  $\beta \neq n + 4, n + 6, \dots$ . To see this, we give a proof for the case when  $\beta < n + 4$ .

**Proposition 2.** *Let  $\beta < n + 4$ . Then, for  $k \in \mathbb{Z}^n$ ,  $(m^{\delta, \beta}(k), e^{ik \cdot x})$  is an eigenpair for the operator defined in (2.7).*

*Proof.* We observe that

$$\begin{aligned} L^{\delta, \beta} e^{ik \cdot x} &= \Delta e^{ik \cdot x} + c^{\delta, \beta} \int_{B_\delta(0)} \frac{e^{ik \cdot (x+z)} - e^{ik \cdot x} - \nabla \nabla e^{ik \cdot x} : \frac{z \otimes z}{2}}{\|z\|^\beta} dz \\ &= -\|k\|^2 e^{ik \cdot x} + c^{\delta, \beta} \int_{B_\delta(0)} \frac{(e^{ik \cdot z} - 1)e^{ik \cdot x} + k \otimes k e^{ik \cdot x} : \frac{z \otimes z}{2}}{\|z\|^\beta} dz \\ &= \left( -\|k\|^2 + c^{\delta, \beta} \int_{B_\delta(0)} \frac{\cos(k \cdot z) - 1 + \frac{(k \cdot z)^2}{2}}{\|z\|^\beta} dz \right) e^{ik \cdot x} = m^{\delta, \beta}(k) e^{ik \cdot x}. \end{aligned}$$

□

Throughout the remaining part of this chapter, and to simplify notation, we use  $m_k$  to denote  $m^{\delta, \beta}(k)$  for any  $k \in \mathbb{Z}^n$ . We observe that for the case  $\beta = n + 2$ , the eigenvalue  $m_k$  is equal to  $-\|k\|^2$ .

### 2.1.5 Sobolev spaces on the torus

For  $q \in \mathbb{R}$ , let  $H^q(\mathbb{T}^n)$  denote the space of all periodic distributions  $g$  on  $\mathbb{T}^n$  such that

$$\sum_{k \in \mathbb{Z}^n} (1 + \|k\|^2)^q |\hat{g}_k|^2 < \infty.$$

Define the norm on  $H^q(\mathbb{T}^n)$  by

$$\|g\|_{H^q(\mathbb{T}^n)} = \left( \sum_{k \in \mathbb{Z}^n} (1 + \|k\|^2)^q |\hat{g}_k|^2 \right)^{1/2} < \infty,$$

for any  $g \in H^q(\mathbb{T}^n)$ .

**Lemma 3.** *Let  $f \in H^q(\mathbb{T}^n)$ . For any nonzero  $k \in \mathbb{Z}^n$ , there exists  $C > 0$  such that*

$$|\hat{f}_k| \leq \frac{C}{\|k\|^q}. \quad (2.9)$$

*Proof.* For any  $k \neq 0$  in  $\mathbb{Z}^n$ , there is a constant  $D > 0$  satisfying

$$(1 + \|k\|^2)^q \geq D\|k\|^{2q}.$$

Thus, using the fact that  $f$  is in  $H^q(\mathbb{T}^n)$ , we obtain

$$\infty > \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^q |\hat{f}_k|^2 \geq D \sum_{0 \neq k \in \mathbb{Z}^n} \|k\|^{2q} |\hat{f}_k|^2.$$

This implies that  $\|k\|^{2q} |\hat{f}_k|^2 \rightarrow 0$  as  $\|k\| \rightarrow \infty$ , from which the result follows.  $\square$

In subsequent sections, we consider maps from the real line to periodic distributions and use the following notion of derivative for these maps.

**Definition 1.** *Suppose  $T : [0, \infty) \rightarrow H^q(\mathbb{T}^n)$  maps  $t \geq 0$  to a periodic distribution  $T(t)$  in  $H^q(\mathbb{T}^n)$ . Define the derivative of  $T$  at  $t$  in the sense of Gâteaux derivative*

$$T'(t) = \frac{d}{dt}T(t) = \lim_{h \rightarrow 0} \frac{T(t+h) - T(t)}{h},$$

*where the limit is with respect to the norm in  $H^q(\mathbb{T}^n)$ .*

Let  $p \in \mathbb{N}$  and  $q \in \mathbb{R}$ , and denote by  $C^p([0, \infty), H^q(\mathbb{T}^n))$  the space of all maps from  $[0, \infty)$  to  $H^q(\mathbb{T}^n)$  that is  $p$  times differentiable.

**Lemma 4.** *Given a map  $T : [0, \infty) \rightarrow H^q(\mathbb{T}^n)$  and suppose that the derivative  $T'(t)$  exists. Then for any  $k \in \mathbb{Z}^n$ , the Fourier coefficient  $\hat{T}'_k(t)$  of  $T'(t)$  is the derivative, in the classical sense, of the Fourier coefficient  $\hat{T}_k(t)$ .*

*Proof.* Represent  $T(t)$  and  $T'(t)$  as

$$T(t) = \sum_{k \in \mathbb{Z}^n} \hat{T}_k(t) e^{ik \cdot x},$$

and

$$T'(t) = \sum_{k \in \mathbb{Z}^n} \hat{T}'_k(t) e^{ik \cdot x}.$$

From the derivative definition, we have

$$\begin{aligned} 0 &= \lim_{h \rightarrow 0} \left\| \frac{T(t+h) - T(t)}{h} - T'(t) \right\|_{H^q(\mathbb{T}^n)} \\ &= \lim_{h \rightarrow 0} \sum_{k \in \mathbb{Z}^n} (1 + \|k\|^2)^q \left| \frac{\hat{T}_k(t+h) - \hat{T}_k(t)}{h} - \hat{T}'_k(t) \right|^2. \end{aligned}$$

This proves that

$$\lim_{h \rightarrow 0} \frac{\hat{T}_k(t+h) - \hat{T}_k(t)}{h} = \hat{T}'_k(t),$$

for any  $k \in \mathbb{Z}^n$ , from which the result follows.  $\square$

**Definition 2.** Suppose  $T : [0, \infty) \rightarrow H^q(\mathbb{T}^n)$  is such that for any  $t \geq 0$ ,  $T(t)$  has the form

$$T(t) = \sum_{k \in \mathbb{Z}^n} \hat{T}_k(t) e^{ik \cdot x}.$$

Then, define the map  $L^{\delta, \beta} T : [0, \infty) \rightarrow H^q(\mathbb{T}^n)$  through the representation

$$L^{\delta, \beta} T(t) = \sum_{k \in \mathbb{Z}^n} m_k \hat{T}_k(t) e^{ik \cdot x},$$

for any  $t \geq 0$ .

In particular, when  $q < 0$ , the above notion extends the definition of the nonlocal Lapla-

cian  $L^{\delta,\beta}$  to periodic distributions. We emphasize that the discussion and results here apply for the extended nonlocal Laplace operator.

## 2.2 The nonlocal wave equation

In this section, we focus on the following nonlocal wave equation

$$\begin{cases} u_{tt}^{\delta,\beta}(x, t) &= L^{\delta,\beta} u^{\delta,\beta}(x, t) + b(x), & x \in \mathbb{T}^n, \ t > 0, \\ u^{\delta,\beta}(x, 0) &= f(x), \\ u_t^{\delta,\beta}(x, 0) &= g(x). \end{cases} \quad (2.10)$$

We emphasize that the results of this section hold for any nonlocal operator  $L^{\delta,\beta}$  with  $\beta < n+4$ . We recall from Sections 2.1.1 and 2.1.2 that  $\beta < n$  corresponds to an integrable kernel,  $n \leq \beta < n+2$  corresponds to a singular kernel,  $\beta = n+2$  corresponds to  $L^{\delta,\beta} = \Delta$  (the Laplacian), and  $n+2 < \beta < n+4$  corresponds to an integro-differential operator as given in (2.7).

In order to study the existence, uniqueness, and regularity of solutions to (2.10) over the space of periodic distributions, we consider the identification  $U(t) = u(\cdot, t)$ , with  $U : [0, \infty) \rightarrow H^q(\mathbb{T}^n)$ , for some  $q \in \mathbb{R}$ . In the following sections, spatial regularity refers to  $U(t)$  being in  $H^q(\mathbb{T}^n)$ , for some  $q \in \mathbb{R}$ , for a fixed  $t \geq 0$ . Temporal regularity refers to the differentiability of the map  $U$  as a function of time  $t$ .

### 2.2.1 Spatial and temporal regularity over periodic distributions

Fix  $b \in H^{s_0}(\mathbb{T}^n)$ ,  $f \in H^{s_1}(\mathbb{T}^n)$  and  $g \in H^{s_2}(\mathbb{T}^n)$ , with  $s_0, s_1$  and  $s_2$  in  $\mathbb{R}$ . For any  $t \geq 0$ , define

$$\begin{aligned} U^{\delta, \beta}(t) &:= \sum_{k \in \mathbb{Z}^n} \hat{U}_k(t) e^{ik \cdot x} \\ &= (\hat{f}_0 + \hat{g}_0 t + \hat{b}_0 \frac{t^2}{2}) \\ &\quad + \sum_{0 \neq k \in \mathbb{Z}^n} \left[ \hat{f}_k \cos(\sqrt{-m_k} t) + \frac{\hat{g}_k}{\sqrt{-m_k}} \sin(\sqrt{-m_k} t) + \frac{\hat{b}_k}{m_k} [\cos(\sqrt{-m_k} t) - 1] \right] e^{ik \cdot x}. \end{aligned} \quad (2.11)$$

We will show in Theorem 7 that  $U^{\delta, \beta}$  is the unique solution to the nonlocal wave equation.

We begin by studying the spatial and temporal differentiability of  $U^{\delta, \beta}$ .

**Theorem 5.** *Let  $n \geq 1$ ,  $\delta > 0$ , and  $\beta < n + 4$ . For any  $t \geq 0$ ,  $U^{\delta, \beta}(t) \in H^s(\mathbb{T}^n)$ , where  $U^{\delta, \beta}$  is given by (2.11),  $s = \min\{s_1, s_2 + \theta, s_0 + 2\theta\}$  with  $\theta = \max\{0, \frac{\beta - n}{2}\}$ .*

*Proof.* Observe that

$$\begin{aligned} \sum_{k \in \mathbb{Z}^n} (1 + \|k\|^2)^s |\hat{U}_k^{\delta, \beta}(t)|^2 &\leq \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^s |\hat{f}_k|^2 + \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^s \frac{|\hat{g}_k|^2}{|m_k|} \\ &\quad + 4 \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^s \frac{|\hat{b}_k|^2}{m_k^2} \\ &\quad + |\hat{f}_0|^2 + |\hat{g}_0|^2 t^2 + |\hat{b}_0|^2 \frac{t^4}{4}. \end{aligned}$$

Since  $f \in H^{s_1}$  and  $s \leq s_1$ , we get

$$\sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^s |\hat{f}_k|^2 \leq \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^{s_1} |\hat{f}_k|^2 < \infty.$$

Additionally, notice that

$$\begin{aligned}\sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^s \frac{|\hat{g}_k|^2}{|m_k|} &= \sum_{0 \neq k \in \mathbb{Z}^n} \frac{(1 + \|k\|^2)^{s-s_2}}{|m_k|} (1 + \|k\|^2)^{s_2} |\hat{g}_k|^2, \\ \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^s \frac{|\hat{b}_k|^2}{m_k^2} &= \sum_{0 \neq k \in \mathbb{Z}^n} \frac{(1 + \|k\|^2)^{s-s_0}}{m_k^2} (1 + \|k\|^2)^{s_0} |\hat{b}_k|^2.\end{aligned}$$

Hence, the proof will be complete provided that

$$\frac{(1 + \|k\|^2)^{s-s_2}}{|m_k|} \quad \text{and} \quad \frac{(1 + \|k\|^2)^{s-s_0}}{m_k^2}$$

are bounded for large values of  $\|k\|$ . To show this, we consider two cases. First, when  $\beta \leq n$ , then  $s \leq s_2$ ,  $s \leq s_0$  and by Theorem 2 there exists  $r_1 > 0$  and  $C_1 > 0$  such that  $|m_k| \geq C_1$  for  $\|k\| \geq r_1$ . This leads to

$$\frac{(1 + \|k\|^2)^{s-s_2}}{|m_k|} \leq \frac{1}{C_1} \quad \text{and} \quad \frac{(1 + \|k\|^2)^{s-s_0}}{m_k^2} \leq \frac{1}{C_1^2}$$

Similarly, for  $\beta > n$ , then  $s \leq s_2 + \frac{\beta-n}{2}$ ,  $s \leq s_0 + \beta - n$  and by Theorem 2 there exists  $r_2 > 0$  and  $C_2 > 0$  such that  $|m_k| \geq C_2 \|k\|^{\beta-n}$ , for  $\|k\| \geq r_2$ . This leads to

$$\frac{(1 + \|k\|^2)^{s-s_2}}{|m_k|} \leq \frac{1}{C_2} \left( \frac{1 + \|k\|^2}{\|k\|^2} \right)^{\frac{\beta-n}{2}} \quad \text{and} \quad \frac{(1 + \|k\|^2)^{s-s_0}}{m_k^2} \leq \frac{1}{C_2^2} \left( \frac{1 + \|k\|^2}{\|k\|^2} \right)^{\beta-n}$$

which are bounded. □

**Theorem 6.** *Let  $n \geq 1$ ,  $\delta > 0$ , and  $\beta < n + 4$ . Let  $U^{\delta, \beta}$  be the map given by (2.11) with  $s_0, s_1, s_2 \in \mathbb{R}$ , and  $s = \min\{s_1, s_2 + \theta, s_0 + 2\theta\}$  with  $\theta = \max\{0, \frac{\beta-n}{2}\}$ . Thus,*

1. *if  $\beta < n$ , then  $U^{\delta, \beta} \in C^\infty([0, \infty), H^q(\mathbb{T}^n))$ , for any  $q \leq s$ ,*
2. *if  $\beta = n$ , then  $U^{\delta, \beta} \in C^\infty([0, \infty), H^q(\mathbb{T}^n))$ , for any  $q < s$ , and*
3. *if  $\beta > n$ , then  $U^{\delta, \beta} \in C^{p+1}([0, \infty), H^q(\mathbb{T}^n))$ , for any  $q \in \mathbb{R}$  and any positive integer  $p$*

satisfying

$$q - s_1 + (p + 1)\frac{\beta - n}{2} \leq 0, \quad q - s_2 + p\frac{\beta - n}{2} \leq 0 \quad \text{and} \quad q - s_0 + (p - 1)\frac{\beta - n}{2} \leq 0.$$

*Proof.* We will show this result using induction. Suppose  $U$  is already differentiable up to  $p$  times, then by Lemma 4, we have

$$\frac{d^p}{dt^p} U^{\delta, \beta}(t) = \sum_{k \in \mathbb{Z}^n} \frac{d^p}{dt^p} \hat{U}_k^{\delta, \beta}(t) e^{ik \cdot x}.$$

where  $\frac{d^p}{dt^p} \hat{U}_k^{\delta, \beta}(t)$  is given by

$$\begin{aligned} \frac{d^p}{dt^p} \hat{U}_k^{\delta, \beta}(t) = & \hat{f}_k(\sqrt{-m_k})^p \cos\left(\sqrt{-m_k}t + \frac{p\pi}{2}\right) + \hat{g}_k(\sqrt{-m_k})^{p-1} \sin\left(\sqrt{-m_k}t + \frac{p\pi}{2}\right) \\ & - \hat{b}_k \sqrt{-m_k}^{p-2} \cos\left(\sqrt{-m_k}t + \frac{p\pi}{2}\right), \end{aligned}$$

for  $k \neq 0$  and  $\frac{d}{dt} \hat{U}_0^{\delta, \beta}(t) = \hat{g}_0 + \hat{b}_0 t$ ,  $\frac{d^2}{dt^2} \hat{U}_0^{\delta, \beta}(t) = \hat{b}_0$  and  $\frac{d^p}{dt^p} \hat{U}_0^{\delta, \beta}(t) = 0$  for all  $p \geq 3$ . We will show that

$$\frac{d^{p+1}}{dt^{p+1}} U^{\delta, \beta}(t) = \sum_{k \in \mathbb{Z}^n} \frac{d^{p+1}}{dt^{p+1}} \hat{U}_k^{\delta, \beta}(t) e^{ik \cdot x}.$$

Observe that using the mean value theorem,

$$\begin{aligned}
T_h &:= \left\| \frac{\frac{d^p}{dt^p} U^{\delta,\beta}(t+h) - \frac{d^p}{dt^p} U^{\delta,\beta}(t)}{h} - \sum_{k \in \mathbb{Z}^n} \frac{d^{p+1}}{dt^{p+1}} \hat{U}_k^{\delta,\beta}(t) e^{ik \cdot x} \right\|_{H^q(\mathbb{T}^n)}^2 \\
&= \sum_{k \in \mathbb{Z}^n} (1 + \|k\|^2)^q \left| \frac{\frac{d^p}{dt^p} \hat{U}_k^{\delta,\beta}(t+h) - \frac{d^p}{dt^p} \hat{U}_k^{\delta,\beta}(t)}{h} - \frac{d^{p+1}}{dt^{p+1}} \hat{U}_k^{\delta,\beta}(t) \right|^2 \\
&\leq \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^q |\hat{f}_k|^2 (-m_k)^{p+1} \left| \cos \left( \sqrt{-m_k}(t + \xi_{p,k}) + \frac{(p+1)\pi}{2} \right) \right. \\
&\quad \left. - \cos \left( \sqrt{-m_k}t + \frac{(p+1)\pi}{2} \right) \right|^2 \\
&\quad + \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^q |\hat{g}_k|^2 (-m_k)^p \left| \sin \left( \sqrt{-m_k}(t + \eta_{p,k}) + \frac{(p+1)\pi}{2} \right) \right. \\
&\quad \left. - \sin \left( \sqrt{-m_k}t + \frac{(p+1)\pi}{2} \right) \right|^2 \\
&\quad + \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^q |\hat{b}_k|^2 (-m_k)^{p-1} \left| \cos \left( \sqrt{-m_k}(t + \zeta_{p,k}) + \frac{(p+1)\pi}{2} \right) \right. \\
&\quad \left. - \cos \left( \sqrt{-m_k}t + \frac{(p+1)\pi}{2} \right) \right|^2,
\end{aligned}$$

for some  $\xi_{p,k}, \eta_{p,k}, \zeta_{p,k} \in (0, h)$ . This means that as  $h \rightarrow 0$ , each term of the summation will approach 0. The proof will be complete if we are able to pass the limit inside the summation.

This is possible provided that the following terms are uniformly bounded

$$\begin{aligned}
a_k &:= (1 + \|k\|^2)^{q-s_1} (-m_k)^{p+1}, \quad b_k := (1 + \|k\|^2)^{q-s_2} (-m_k)^p \\
&\text{and } c_k := (1 + \|k\|^2)^{q-s_0} (-m_k)^{p-1}.
\end{aligned}$$

To establish the bound for these terms, we will consider three cases when  $\beta < n$ ,  $\beta = n$  and  $\beta > n$ .

When  $\beta < n$ , from Theorem 2,  $-m_k$  is bounded, and this provides the uniform boundedness of  $a_k$ ,  $b_k$  and  $c_k$  provided that  $q \leq \min\{s_1, s_2, s_0\}$ .

When  $\beta = n$ , fix an  $\epsilon > 0$ . From Theorem 2, there exists  $C, r > 0$  such that  $-m_k \leq$

$C \log \|k\| \leq C(1 + \|k\|^2)^{\epsilon/2}$  whenever  $\|k\| \geq r$ . Then for large  $\|k\|$ ,

$$a_k \leq C(1 + \|k\|^2)^{q-s_1+(p+1)\epsilon/2}, \quad b_k \leq C(1 + \|k\|^2)^{q-s_2+p\epsilon/2}$$

$$\text{and} \quad c_k \leq C(1 + \|k\|^2)^{q-s_0+(p-1)\epsilon/2}.$$

This yields the uniform boundedness of  $a_k$ ,  $b_k$  and  $c_k$  as long as  $q - s_1 + (p + 1)\epsilon/2 \leq 0$ ,  $q - s_2 + p\epsilon/2 \leq 0$  and  $q - s_0 + (p - 1)\epsilon/2 \leq 0$ . So given  $q < \min\{s_1, s_2, s_0\}$ , we can always pick a sufficiently small  $\epsilon$  that satisfies these inequalities.

When  $\beta > n$ , from Theorem 2, there is  $C, r > 0$  such that  $-m_k \leq C\|k\|^{\beta-n} \leq C(1 + \|k\|^2)^{(\beta-n)/2}$  for  $\|k\| \geq r$ . Then for large  $\|k\|$ ,

$$a_k \leq C(1 + \|k\|^2)^{q-s_1+(p+1)\frac{\beta-n}{2}}, \quad b_k \leq C(1 + \|k\|^2)^{q-s_2+p\frac{\beta-n}{2}}$$

$$\text{and} \quad c_k \leq C(1 + \|k\|^2)^{q-s_0+(p-1)\frac{\beta-n}{2}}.$$

This shows that the uniform boundedness of  $a_k, b_k, c_k$  are established given the conditions

$$q - s_1 + (p + 1)\frac{\beta - n}{2} \leq 0, \quad q - s_2 + p\frac{\beta - n}{2} \leq 0$$

$$\text{and} \quad q - s_0 + (p - 1)\frac{\beta - n}{2} \leq 0.$$

□

The following theorem is a direct application of Theorem 5 and Theorem 6. The assumptions are to guarantee that the second derivative exists.

**Theorem 7.** *Let  $n \geq 1$ ,  $\delta > 0$ , and  $\beta < n + 4$ . Let  $b \in H^{s_0}(\mathbb{T}^n)$ ,  $f \in H^{s_1}(\mathbb{T}^n)$  and  $g \in H^{s_2}(\mathbb{T}^n)$ , with  $s_0, s_1, s_2 \in \mathbb{R}$ . Let  $q \in \mathbb{R}$  be such that*

1.  $q \leq \min\{s_1, s_2, s_0\}$ , in the case when  $\beta < n$ ,
2.  $q < \min\{s_1, s_2, s_0\}$ , in the case when  $\beta = n$ , and
3.  $q - s_1 + (\beta - n) \leq 0$ ,  $q - s_2 + (\beta - n)/2 \leq 0$  and  $q \leq s_0$ , in the case when  $\beta > n$ .

Then,  $U^{\delta,\beta}(t)$  defined by (2.11) is in  $H^q(\mathbb{T}^n)$ , for  $t \geq 0$ . Moreover,  $U^{\delta,\beta}$  is the unique solution to the nonlocal wave equation

$$\begin{cases} \frac{d^2}{dt^2} U^{\delta,\beta}(t) &= L^{\delta,\beta} U^{\delta,\beta}(t) + b, \quad t > 0 \\ U^{\delta,\beta}(0) &= f, \\ \frac{d}{dt} U^{\delta,\beta}(0) &= g. \end{cases} \quad (2.12)$$

Furthermore, when  $\beta \leq n$ ,  $U^{\delta,\beta} \in C^\infty([0, \infty); H^s(\mathbb{T}^n))$  where  $s = \min\{s_1, s_2, s_0\}$ , and when  $\beta > n$ ,  $U^{\delta,\beta} \in C^2([0, \infty); H^s(\mathbb{T}^n))$  where  $s = \min\{s_1, s_2 + \frac{\beta-n}{2}, s_0 + \beta - n\}$ .

### 2.2.2 Convergence to the classical wave equation solution

In this section, we will provide some results on the convergence of the solutions of the nonlocal wave equation (2.12) to the solution of the corresponding local wave equation for two types of limits: as  $\delta \rightarrow 0^+$ , with  $\beta < n + 4$  is being fixed, or as  $\beta \rightarrow n + 2$ , with  $\delta > 0$  is being fixed. The local wave equation is defined by

$$\begin{cases} \frac{d^2}{dt^2} U^\Delta(t) &= \Delta U^\Delta(t) + b, \quad t > 0 \\ U^\Delta(0) &= f, \\ \frac{d}{dt} U^\Delta(0) &= g, \end{cases} \quad (2.13)$$

and its solution is given by

$$\begin{aligned} U^\Delta(t) &:= (\hat{f}_0 + \hat{g}_0 t + \hat{b}_0 \frac{t^2}{2}) \\ &+ \sum_{0 \neq k \in \mathbb{Z}^n} \left[ \hat{f}_k \cos(\|k\|t) + \frac{\hat{g}_k}{\|k\|} \sin(\|k\|t) + \frac{\hat{b}_k}{-\|k\|^2} [\cos(\|k\|t) - 1] \right] e^{ik \cdot x}. \end{aligned} \quad (2.14)$$

**Theorem 8.** Let  $n \geq 1$ ,  $\delta > 0$ ,  $t \geq 0$ , and  $\beta < n + 4$ . Suppose  $b \in H^{s_0}(\mathbb{T}^n)$ ,  $f \in H^{s_1}(\mathbb{T}^n)$  and  $g \in H^{s_2}(\mathbb{T}^n)$  for some  $s_0, s_1, s_2 \in \mathbb{R}$ . Let  $U^\Delta(t) \in H^{s_\Delta}(\mathbb{T}^n)$  and  $U^{\delta,\beta}(t) \in H^s(\mathbb{T}^n)$  be the solutions to the local and nonlocal wave equations (2.13) and (2.12), respectively, where

$s_\Delta = \min\{s_1, s_2 + 1, s_0 + 2\}$  and  $s = \min\{s_1, s_2 + \theta, s_0 + 2\theta\}$  with  $\theta = \max\{0, \frac{\beta-n}{2}\}$ . Then, when  $\beta \leq n + 2$ ,

$$\lim_{\delta \rightarrow 0^+} U^{\delta, \beta}(t) = U^\Delta(t) \quad \text{in} \quad H^s(\mathbb{T}^n),$$

and when  $n + 2 < \beta < n + 4$

$$\lim_{\delta \rightarrow 0^+} U^{\delta, \beta}(t) = U^\Delta(t) \quad \text{in} \quad H^{s_\Delta}(\mathbb{T}^n).$$

*Proof.* It is easy to check that  $H^{s_\Delta}(\mathbb{T}^n) \subset H^s(\mathbb{T}^n)$  when  $\beta \leq n + 2$  and  $H^s(\mathbb{T}^n) \subset H^{s_\Delta}(\mathbb{T}^n)$  when  $n + 2 < \beta < n + 4$ . Hence all the limits make sense. From (2.11) and (2.14), the Fourier coefficients  $\hat{U}_k^{\delta, \beta}(t)$  in the nonlocal case and  $\hat{U}_k^\Delta(t)$  in the classical case ( $\beta = n + 2$ ) is determined by

$$\begin{aligned} \hat{U}_k^{\delta, \beta}(t) &= \hat{f}_k \cos(\sqrt{-m_k}t) + \frac{\hat{g}_k}{\sqrt{-m_k}} \sin(\sqrt{-m_k}t) + \frac{\hat{b}_k}{m_k} [\cos(\sqrt{-m_k}t) - 1], \\ \hat{U}_k^\Delta(t) &= \hat{f}_k \cos(\|k\|t) + \frac{\hat{g}_k}{\|k\|} \sin(\|k\|t) + \frac{\hat{b}_k}{-\|k\|^2} [\cos(\|k\|t) - 1], \end{aligned}$$

for nonzero  $k$  and  $\hat{U}_0^{\delta, \beta}(t) = \hat{U}_0^\Delta(t) = \hat{f}_0 + \hat{g}_0 t + \hat{g}_0 \frac{t^2}{2}$ . From these observation, for  $\tilde{s} = s$  when  $\beta \leq n + 2$  and  $\tilde{s} = s_\Delta$  when  $n + 2 < \beta < n + 4$ , one can get

$$\begin{aligned} \|U^{\delta, \beta}(t) - U^\Delta(t)\|_{H^{\tilde{s}}(\mathbb{T}^n)}^2 &= \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^{\tilde{s}} \|\hat{U}_k^{\delta, \beta}(t) - \hat{U}_k^\Delta(t)\|^2 \\ &\leq \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^{\tilde{s}} |\hat{f}_k|^2 |\cos(\sqrt{-m_k}t) - \cos(\|k\|t)|^2 \\ &\quad + \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^{\tilde{s}} |\hat{g}_k|^2 \left| \frac{\sin(\sqrt{-m_k}t)}{\sqrt{-m_k}} - \frac{\sin(\|k\|t)}{\|k\|} \right|^2 \\ &\quad + \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^{\tilde{s}} |\hat{b}_k|^2 \left| \frac{\cos(\sqrt{-m_k}t) - 1}{m_k} - \frac{\cos(\sqrt{\|k\|}t) - 1}{-\|k\|^2} \right|^2. \end{aligned}$$

Since  $b \in H^{s_0}(\mathbb{T}^n)$ ,  $f \in H^{s_1}(\mathbb{T}^n)$  and  $g \in H^{s_2}(\mathbb{T}^n)$ , one can pass the limit as  $\delta \rightarrow 0^+$  term by

term inside the summation provided the uniform boundedness of

$$(1 + \|k\|^2)^{\tilde{s}-s_1} |\cos(\sqrt{-m_k}t) - \cos(\|k\|t)|^2, \quad (1 + \|k\|^2)^{\tilde{s}-s_2} \left| \frac{\sin(\sqrt{-m_k}t)}{\sqrt{-m_k}} - \frac{\sin(\|k\|t)}{\|k\|} \right|^2$$

and

$$(1 + \|k\|^2)^{\tilde{s}-s_0} \left| \frac{\cos(\sqrt{-m_k}t) - 1}{m_k} - \frac{\cos(\sqrt{\|k\|}t) - 1}{-\|k\|^2} \right|^2,$$

for large  $\|k\|$ . It is obvious that given  $\tilde{s} \leq s_1$ ,

$$(1 + \|k\|^2)^{\tilde{s}-s_1} |\cos(\sqrt{-m_k}t) - \cos(\|k\|t)|^2 \leq 4.$$

For the second term, one gets

$$(1 + \|k\|^2)^{\tilde{s}-s_2} \left| \frac{\sin(\sqrt{-m_k}t)}{\sqrt{-m_k}} - \frac{\sin(\|k\|t)}{\|k\|} \right|^2 \leq (1 + \|k\|^2)^{\tilde{s}-s_2} \left( \frac{1}{\sqrt{-m_k}} + \frac{1}{\|k\|} \right)^2,$$

$$(1 + \|k\|^2)^{\tilde{s}-s_0} \left| \frac{\cos(\sqrt{-m_k}t) - 1}{m_k} - \frac{\cos(\sqrt{\|k\|}t) - 1}{-\|k\|^2} \right|^2 \leq 4(1 + \|k\|^2)^{\tilde{s}-s_0} \left( \frac{1}{-m_k} + \frac{1}{\|k\|^2} \right)^2.$$

We consider two cases. When  $\beta \leq n$ , then  $\tilde{s} - s_2 \leq 0$  and  $\tilde{s} - s_0 \leq 0$ . From Theorem 2, there exists  $r_1 > 0$  and  $C_1 > 0$  such that  $-m_k \geq C_1$  for  $\|k\| \geq r_1$ . When  $\beta > n$ , then  $\tilde{s} - s_2 \leq (\beta - n)/2$  and  $\tilde{s} - s_0 \leq \beta - n$ . From Theorem 2, there exists  $r_2 > 0$  and  $C_2 > 0$  such that  $-m_k \geq C_2\|k\|^{\beta-n}$  for  $\|k\| \geq r_2$ . These results lead to the uniform boundedness of the right hand sides. Now, taking the limit as  $\delta \rightarrow 0^+$  inside the summation term-wise and applying the pointwise convergence of (2.5) completes the proof.  $\square$

**Theorem 9.** *Let  $n \geq 1$ ,  $\delta > 0$ , and  $\beta < n + 4$ . Suppose  $b \in H^{s_0}(\mathbb{T}^n)$ ,  $f \in H^{s_1}(\mathbb{T}^n)$  and  $g \in H^{s_2}(\mathbb{T}^n)$  for some  $s_0, s_1, s_2 \in \mathbb{R}$ . Let  $U^\Delta(t) \in H^{s_\Delta}(\mathbb{T}^n)$  and  $U^{\delta,\beta}(t) \in H^s(\mathbb{T}^n)$  be the solutions to the local and nonlocal wave equations (2.13) and (2.12), respectively, where  $s_\Delta = \min\{s_1, s_2 + 1, s_0 + 2\}$  and  $s = \min\{s_1, s_2 + \theta, s_0 + 2\theta\}$  with  $\theta = \max\{0, \frac{\beta-n}{2}\}$ . Then, for any  $t \geq 0$  and for any  $\epsilon \in (0, 2)$  and  $\epsilon < n/2$ ,*

$$\lim_{\beta \rightarrow n+2} U^{\delta,\beta}(t) = U^\Delta(t) \quad \text{in} \quad H^{s_\epsilon}(\mathbb{T}^n),$$

where  $s_\epsilon = \min \left\{ s_1, s_2 + \frac{2-\epsilon}{2}, s_0 + 2 - \epsilon \right\}$ .

*Proof.* Fix an  $\epsilon \in (0, 2)$ ,  $\epsilon < n/2$  and define  $\beta' = n + 2 - \epsilon > \frac{n+4}{2}$ . For simplicity, we would still denote

$$m_k := m^{\delta, \beta}(k) \quad \text{and} \quad m'_k := m^{\delta, \beta'}(k).$$

Then for any  $\beta$  such that  $|\beta - (n + 2)| < \epsilon$  or  $\frac{n+4}{2} < \beta' < \beta < n + 2 + \epsilon < n + 4$ , we have  $\frac{\beta-n}{2} > \frac{2-\epsilon}{2}$ . This shows that  $s, s_\Delta > s_\epsilon$ , so the limit makes sense.

$$\begin{aligned} \|U^{\delta, \beta}(t) - U^\Delta(t)\|_{H^{s_\epsilon}(\mathbb{T}^n)}^2 &\leq \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^{s_\epsilon} |\hat{f}_k|^2 |\cos(\sqrt{-m_k}t) - \cos(\|k\|t)|^2 \\ &\quad + \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^{s_\epsilon} |\hat{g}_k|^2 \left| \frac{\sin(\sqrt{-m_k}t)}{\sqrt{-m_k}} - \frac{\sin(\|k\|t)}{\|k\|} \right|^2 \\ &\quad + \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^{s_\epsilon} |\hat{b}_k|^2 \left| \frac{\cos(\sqrt{-m_k}t) - 1}{m_k} - \frac{\cos(\sqrt{\|k\|}t) - 1}{-\|k\|^2} \right|^2. \end{aligned}$$

Similar to the previous theorem, we need to establish the uniform boundedness of

$$(1 + \|k\|^2)^{s_\epsilon - s_1} |\cos(\sqrt{-m_k}t) - \cos(\|k\|t)|^2, \quad (1 + \|k\|^2)^{s_\epsilon - s_2} \left| \frac{\sin(\sqrt{-m_k}t)}{\sqrt{-m_k}} - \frac{\sin(\|k\|t)}{\|k\|} \right|^2,$$

and

$$(1 + \|k\|^2)^{s_\epsilon - s_0} \left| \frac{\cos(\sqrt{-m_k}t) - 1}{m_k} - \frac{\cos(\sqrt{\|k\|}t) - 1}{-\|k\|^2} \right|^2,$$

for large  $\|k\|$ . Again, the first term is always bounded by 4 given  $s_\epsilon \leq s_1$ . For the second and third term, from Theorem 2 there exists  $r > 0$  and  $C, D > 0$  such that  $\sqrt{-m'_k} \geq$

$C(1 + \|k\|^2)^{\frac{2-\epsilon}{4}}$  and  $\|k\| \geq D(1 + \|k\|^2)^{1/2}$  for  $\|k\| \geq r$ . This implies that

$$\begin{aligned}
& (1 + \|k\|^2)^{s_\epsilon - s_2} \left| \frac{\sin(\sqrt{-m_k}t)}{\sqrt{-m_k}} - \frac{\sin(\|k\|t)}{\|k\|} \right|^2 \\
& \leq (1 + \|k\|^2)^{\frac{2-\epsilon}{2}} \left( \frac{1}{\sqrt{-m'_k}} + \frac{1}{\|k\|} \right)^2 \\
& \leq (1 + \|k\|^2)^{\frac{2-\epsilon}{2}} \left( \frac{1}{C(1 + \|k\|^2)^{\frac{2-\epsilon}{4}}} + \frac{1}{D(1 + \|k\|^2)^{1/2}} \right)^2 \\
& = \frac{1}{C^2} + \frac{2}{CD(1 + \|k\|^2)^{\epsilon/4}} + \frac{1}{D^2(1 + \|k\|^2)^{\epsilon/2}},
\end{aligned}$$

and

$$\begin{aligned}
& (1 + \|k\|^2)^{s_\epsilon - s_0} \left| \frac{\cos(\sqrt{-m_k}t) - 1}{m_k} - \frac{\cos(\sqrt{\|k\|}t) - 1}{-\|k\|^2} \right|^2 \\
& \leq 4(1 + \|k\|^2)^{2-\epsilon} \left( \frac{1}{-m'_k} + \frac{1}{\|k\|^2} \right)^2 \\
& \leq (1 + \|k\|^2)^{2-\epsilon} \left( \frac{1}{C(1 + \|k\|^2)^{\frac{2-\epsilon}{2}}} + \frac{1}{D(1 + \|k\|^2)} \right)^2 \\
& = \frac{1}{C^2} + \frac{2}{CD(1 + \|k\|^2)^{\epsilon/2}} + \frac{1}{D^2(1 + \|k\|^2)^\epsilon}.
\end{aligned}$$

This shows the uniform boundedness required to pass the limit inside the summation term-wise. The proof is then complete by using (2.6).  $\square$

### 2.2.3 Spatial and temporal regularity over $L^2(\mathbb{T}^n)$

The solution  $U^{\delta,\beta}(t) \in H^s(\mathbb{T}^n)$  given in (2.11) is a distribution when  $s < 0$ , and when  $s \geq 0$  defines a regular function

$$\begin{aligned} u^{\delta,\beta}(x, t) := U^{\delta,\beta}(t)(x) &= (\hat{f}_0 + \hat{g}_0 t + \hat{b}_0 \frac{t^2}{2}) \\ &+ \sum_{0 \neq k \in \mathbb{Z}^n} \left[ \hat{f}_k \cos(\sqrt{-m_k} t) + \frac{\hat{g}_k}{\sqrt{-m_k}} \sin(\sqrt{-m_k} t) + \frac{\hat{b}_k}{m_k} [\cos(\sqrt{-m_k} t) - 1] \right] e^{ik \cdot x}. \end{aligned} \quad (2.15)$$

Let the data be regular functions;  $b \in H^{s_0}(\mathbb{T}^n)$ ,  $f \in H^{s_1}(\mathbb{T}^n)$  and  $g \in H^{s_2}(\mathbb{T}^n)$  with  $s_0 \geq 0$ ,  $s_1 \geq 0$  and  $s_2 \geq 0$ . In this section, we focus on conditions that guarantee that  $u^{\delta,\beta}(x, t)$ , the solution of (2.10), is a regular function  $u^{\delta,\beta}(\cdot, t) \in L^2(\mathbb{T}^n)$ .

#### Functions with absolutely summable Fourier coefficients

In this section, we state some well-know results on the differentiability and summability of series in multidimensions. These results will be used in subsequent sections and are included here for completeness of the presentation.

The following theorem is a special case of Theorem 3.2.16 in <sup>76</sup>.

**Theorem 10.** *Suppose that  $f$  is a function defined on the torus  $\mathbb{T}^n$  satisfying the  $\zeta$ -Hölder continuity condition for some  $\zeta > n/2$ . Then, the Fourier coefficients of  $f$  are absolutely summable.*

The following fact on differentiating a series term by term will be used in subsequent sections.

**Lemma 5.** *Let  $T$  be an open subset of  $\mathbb{R}^+$ ,  $K$  is a measurable space equipped with the counting measure and  $f_k : T \rightarrow \mathbb{R}$ . Suppose that*

1.  $f_k(t)$  is summable over  $k$  for each  $t \in T$ .
2. For almost all  $k \in K$ , the derivative  $\frac{d}{dt} f_k(t)$  exists and is continuous for all  $t \in T$ .

3. There is a summable sequence  $\theta_k$  such that  $|\frac{d}{dt}f_k(t)| \leq \theta_k$  for all  $t \in T$  and almost all  $k \in K$ .

Then, for all  $t \in T$ ,

$$\frac{d}{dt} \sum_k f_k(t) = \sum_k \frac{d}{dt} f_k(t).$$

In addition, we include the following useful fact about the summability of certain series in multi-dimensions.

**Lemma 6.** In  $\mathbb{R}^n$ , the series

$$\sum_{0 \neq k \in \mathbb{Z}^n} \frac{1}{\|k\|^r} \tag{2.16}$$

converges if and only if  $r > n$ .

### Temporal regularity with respect to the Gâteaux derivative

The following Theorem 11 is a special case of Theorem 5 when  $s_0 \geq 0$ ,  $s_1 \geq 0$  and  $s_2 \geq 0$ . It is included here for the completeness of the section and for comparison with result in Theorem 12.

**Theorem 11.** Let  $n \geq 1$ ,  $\delta > 0$ , and  $\beta < n + 4$ . Let  $b \in H^{s_0}(\mathbb{T}^n)$ ,  $f \in H^{s_1}(\mathbb{T}^n)$  and  $g \in H^{s_2}(\mathbb{T}^n)$ , with  $s_0 \geq 0$ ,  $s_1 \geq 0$  and  $s_2 \geq 0$ . Suppose that

1.  $s_0, s_1, s_2 \geq 0$ , in the case when  $\beta < n$ ,
2.  $s_0, s_1, s_2 > 0$ , in the case when  $\beta = n$ , and
3.  $s_0 \geq 0$ ,  $s_1 \geq \beta - n$  and  $s_2 \geq (\beta - n)/2$ , in the case when  $\beta > n$ .

Then, the function  $u^{\delta, \beta}$ , where  $u^{\delta, \beta}(x, t) = U^{\delta, \beta}(t)(x)$ , with  $U^{\delta, \beta}$  defined in (2.11), is the unique solution to the nonlocal wave equation (2.10) in  $L^2(\mathbb{T}^n)$ . Moreover, when  $\beta \leq n$ , then  $u^{\delta, \beta}(x, t) \in C^\infty([0, \infty); H^s(\mathbb{T}^n))$ , where  $s = \min\{s_0, s_1, s_2\}$ , and when  $\beta > n$ , then  $u^{\delta, \beta}(x, t) \in C^2([0, \infty); H^s(\mathbb{T}^n))$ , where  $s = \min\{s_1, s_2 + \frac{\beta - n}{2}, s_0 + \beta - n\}$ .

## Temporal regularity with respect to the classical derivative

**Theorem 12.** *Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta < n$ . Let  $b \in H^{s_0}(\mathbb{T}^n)$ ,  $f \in H^{s_1}(\mathbb{T}^n)$  and  $g \in H^{s_2}(\mathbb{T}^n)$ , with  $s_0 \geq 0$ ,  $s_1 \geq 0$  and  $s_2 \geq 0$ . Suppose that  $b$ ,  $f$  and  $g$  satisfy the  $\alpha$ -Hölder continuity condition with  $\alpha > n/2$ . Then for  $u^{\delta,\beta}$  given by (2.15),  $u^{\delta,\beta}(x, \cdot) \in C^\infty[0, \infty)$ , for any  $x \in \mathbb{T}^n$ .*

*Proof.* Fix a value of  $p$  in  $\mathbb{Z}_{\geq 0}$ . Consider the following series

$$\begin{aligned} \sum_{0 \neq k \in \mathbb{Z}^n} \hat{f}_k \cos(\sqrt{-m_k}t) e^{ik \cdot x}, \quad \sum_{0 \neq k \in \mathbb{Z}^n} \frac{\hat{g}_k}{\sqrt{-m_k}} \sin(\sqrt{-m_k}t) e^{ik \cdot x} \\ \text{and} \quad \sum_{0 \neq k \in \mathbb{Z}^n} \frac{\hat{b}_k}{m_k} [\cos(\sqrt{-m_k}t) - 1] e^{ik \cdot x}. \end{aligned}$$

From Theorem 2,  $\sqrt{-m_k}$  is bounded for large value of  $\|k\|$ . Hence,

$$\begin{aligned} \left| \hat{f}_k \cos(\sqrt{-m_k}t) e^{ik \cdot x} \right| &\leq |\hat{f}_k|, \\ \left| \frac{d^p}{dt^p} (\hat{f}_k \cos(\sqrt{-m_k}t) e^{ik \cdot x}) \right| &\leq (\sqrt{-m_k})^p |\hat{f}_k| \leq C |\hat{f}_k|, \\ \left| \frac{\hat{g}_k}{\sqrt{-m_k}} \sin(\sqrt{-m_k}t) e^{ik \cdot x} \right| &\leq \frac{|\hat{g}_k|}{\sqrt{-m_k}} \leq C |\hat{g}_k|, \\ \left| \frac{d^p}{dt^p} \left( \frac{\hat{g}_k}{\sqrt{-m_k}} \sin(\sqrt{-m_k}t) e^{ik \cdot x} \right) \right| &\leq (\sqrt{-m_k})^{p-1} |\hat{g}_k| \leq C |\hat{g}_k|, \\ \left| \frac{\hat{b}_k}{m_k} [\cos(\sqrt{-m_k}t) - 1] e^{ik \cdot x} \right| &\leq \frac{|\hat{b}_k|}{|m_k|} \leq C |\hat{b}_k|, \\ \left| \frac{d^p}{dt^p} \frac{\hat{b}_k}{m_k} [\cos(\sqrt{-m_k}t) - 1] e^{ik \cdot x} \right| &\leq (\sqrt{-m_k})^{p-2} |\hat{b}_k| \leq C |\hat{b}_k|, \end{aligned}$$

as  $\|k\| \rightarrow \infty$ . From Theorem 10, the Fourier coefficients  $\hat{b}_k$ ,  $\hat{f}_k$  and  $\hat{g}_k$  are absolutely summable. Now applying Lemma 5 completes the proof.  $\square$

**Theorem 13.** *Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta = n$ . Let  $b \in H^{s_0}(\mathbb{T}^n)$ ,  $f \in H^{s_1}(\mathbb{T}^n)$  and  $g \in H^{s_2}(\mathbb{T}^n)$ , with  $s_0 \geq 0$ ,  $s_1 \geq 0$  and  $s_2 \geq 0$ . Suppose  $s_0 > n$ ,  $s_1 > n$  and  $s_2 > n$ . Then, for  $u^{\delta,\beta}$  given by (2.15),  $u^{\delta,\beta}(x, \cdot) \in C^\infty[0, \infty)$  for any  $x \in \mathbb{T}^n$ .*

*Proof.* Fix a value of  $p$  in  $\mathbb{Z}_{\geq 0}$  and choose  $\epsilon > 0$  such that  $s_1 - p\epsilon/2 > n$  and  $s_2 - (p-1)\epsilon/2 > n$

and  $s_0 - (p-2)\epsilon/2 > n$ . From Theorem 2 we have that for large value of  $\|k\|$ , there exists constant  $D, E > 0$  such that

$$-m_k \leq D \log \|k\| \leq E \|k\|^\epsilon.$$

Consider the following series

$$\begin{aligned} \sum_{0 \neq k \in \mathbb{Z}^n} \hat{f}_k \cos(\sqrt{-m_k}t) e^{ik \cdot x}, \quad \sum_{0 \neq k \in \mathbb{Z}^n} \frac{\hat{g}_k}{\sqrt{-m_k}} \sin(\sqrt{-m_k}t) e^{ik \cdot x} \\ \text{and} \quad \sum_{0 \neq k \in \mathbb{Z}^n} \frac{\hat{b}_k}{m_k} [\cos(\sqrt{-m_k}t) - 1] e^{ik \cdot x}. \end{aligned}$$

Note that from Theorem 2,  $-m_k$  is bounded from below by some constant for large  $\|k\|$ . Then use Lemma 2.9, we have

$$\begin{aligned} \left| \hat{f}_k \cos(\sqrt{-m_k}t) e^{ik \cdot x} \right| &\leq |\hat{f}_k| \leq \frac{C}{\|k\|^{s_1}}, \\ \left| \frac{d^p}{dt^p} (\hat{f}_k \cos(\sqrt{-m_k}t) e^{ik \cdot x}) \right| &\leq (\sqrt{-m_k})^p |\hat{f}_k| \leq \frac{C}{\|k\|^{s_1 - p\epsilon/2}}, \\ \left| \frac{\hat{g}_k}{\sqrt{-m_k}} \sin(\sqrt{-m_k}t) e^{ik \cdot x} \right| &\leq \frac{|\hat{g}_k|}{\sqrt{-m_k}} \leq \frac{C}{\|k\|^{s_2} \sqrt{-m_k}}, \\ \left| \frac{d^p}{dt^p} \left( \frac{\hat{g}_k}{\sqrt{-m_k}} \sin(\sqrt{-m_k}t) e^{ik \cdot x} \right) \right| &\leq (\sqrt{-m_k})^{p-1} |\hat{g}_k| \leq \frac{C}{\|k\|^{s_2 - (p-1)\epsilon/2}}, \\ \left| \frac{\hat{b}_k}{m_k} [\cos(\sqrt{-m_k}t) - 1] e^{ik \cdot x} \right| &\leq \frac{|\hat{b}_k|}{|m_k|} \leq C |\hat{b}_k|, \\ \left| \frac{d^p}{dt^p} \frac{\hat{b}_k}{m_k} [\cos(\sqrt{-m_k}t) - 1] e^{ik \cdot x} \right| &\leq (\sqrt{-m_k})^{p-2} |\hat{b}_k| \leq \frac{C}{\|k\|^{s_0 - \epsilon(p-2)/2}} \end{aligned}$$

as  $\|k\| \rightarrow \infty$ . Applying Lemma 6 and Lemma 5 completes the proof.  $\square$

**Theorem 14.** *Let  $n \geq 1$ ,  $\delta > 0$  and  $n < \beta < n + 4$ . Let  $b \in H^{s_0}(\mathbb{T}^n)$ ,  $f \in H^{s_1}(\mathbb{T}^n)$  and  $g \in H^{s_2}(\mathbb{T}^n)$ , with  $s_0 \geq 0$ ,  $s_1 \geq 0$  and  $s_2 \geq 0$ . Suppose that there exists  $p \in \mathbb{Z}_{\geq 0}$  such that  $s_1 > n + p \frac{\beta-n}{2}$  and  $s_2 > n + (p-1) \frac{\beta-n}{2}$  and  $s_0 > n + (p-2) \frac{\beta-n}{2}$ . Then, for  $u^{\delta, \beta}$  given by (2.15),  $u^{\delta, \beta}(x, \cdot) \in C^p[0, \infty)$ , for any  $x \in \mathbb{T}^n$ .*

*Proof.* From Theorem 2, for large  $\|k\|$ , there exists  $D_1, D_2 > 0$  such that  $D_1\|k\|^{\beta-n} \leq -m_k \leq D_2\|k\|^{\beta-n}$ . Consider the following series

$$\begin{aligned} \sum_{0 \neq k \in \mathbb{Z}^n} \hat{f}_k \cos(\sqrt{-m_k}t) e^{ik \cdot x}, \quad \sum_{0 \neq k \in \mathbb{Z}^n} \frac{\hat{g}_k}{\sqrt{-m_k}} \sin(\sqrt{-m_k}t) e^{ik \cdot x} \\ \text{and} \quad \sum_{0 \neq k \in \mathbb{Z}^n} \frac{\hat{b}_k}{m_k} [\cos(\sqrt{-m_k}t) - 1] e^{ik \cdot x}. \end{aligned}$$

Using Lemma 2.9, we get

$$\begin{aligned} \left| \hat{f}_k \cos(\sqrt{-m_k}t) e^{ik \cdot x} \right| &\leq |\hat{f}_k| \leq \frac{C}{\|k\|^{s_1}}, \\ \left| \frac{d^p}{dt^p} (\hat{f}_k \cos(\sqrt{-m_k}t) e^{ik \cdot x}) \right| &\leq (\sqrt{-m_k})^p |\hat{f}_k| \leq \frac{C}{\|k\|^{s_1 - p(\beta-n)/2}}, \\ \left| \frac{\hat{g}_k}{\sqrt{-m_k}} \sin(\sqrt{-m_k}t) e^{ik \cdot x} \right| &\leq \frac{|\hat{g}_k|}{\sqrt{-m_k}} \leq \frac{C}{\|k\|^{s_2 + (\beta-n)/2}}, \\ \left| \frac{d^p}{dt^p} \left( \frac{\hat{g}_k}{\sqrt{-m_k}} \sin(\sqrt{-m_k}t) e^{ik \cdot x} \right) \right| &\leq (\sqrt{-m_k})^{p-1} |\hat{g}_k| \leq \frac{C}{\|k\|^{s_2 - (p-1)(\beta-n)/2}}, \\ \left| \frac{\hat{b}_k}{m_k} [\cos(\sqrt{-m_k}t) - 1] e^{ik \cdot x} \right| &\leq \frac{|\hat{b}_k|}{|m_k|} \leq \frac{C}{\|k\|^{s_0 + (\beta-n)}}, \\ \left| \frac{d^p}{dt^p} \frac{\hat{b}_k}{m_k} [\cos(\sqrt{-m_k}t) - 1] e^{ik \cdot x} \right| &\leq (\sqrt{-m_k})^{p-2} |\hat{b}_k| \leq \frac{C}{\|k\|^{s_0 - (p-2)(\beta-n)/2}}, \end{aligned}$$

as  $\|k\| \rightarrow \infty$ . The summability of these series follows from Lemma 6. Applying Lemma 5 completes the proof.  $\square$

# Chapter 3

## Regularity of solutions for the peridynamic equilibrium and evolution equations on periodic distributions

In this chapter, we study the regularity of solutions to the peridynamic evolution equation

$$\begin{cases} \mathbf{u}_{tt}(x, t) = \mathcal{L}\mathbf{u}(x, t) + \mathbf{b}(x), & x \in \mathbb{T}^n, t > 0 \\ \mathbf{u}(x, 0) = \mathbf{f}(x), & x \in \mathbb{T}^n \\ \mathbf{u}_t(x, 0) = \mathbf{g}(x), & x \in \mathbb{T}^n, \end{cases} \quad (3.1)$$

and the peridynamic equilibrium equation

$$\mathcal{L}\mathbf{u}(x) = \mathbf{b}(x), \quad x \in \mathbb{T}^n. \quad (3.2)$$

Here  $\mathbb{T}^n$  is the periodic torus in  $\mathbb{R}^n$ , with  $n \geq 1$ ,  $\mathbf{u}$  is the displacement field,  $\mathbf{f} : \mathbb{T}^n \rightarrow \mathbb{R}^n$  and  $\mathbf{g} : \mathbb{T}^n \rightarrow \mathbb{R}^n$  are the initial displacement and the initial velocity, respectively, and

$\mathbf{b} : \mathbb{T}^n \rightarrow \mathbb{R}^n$  is the external force field. The operator  $\mathcal{L}$  is the state-based linear peridynamic operator defined by (3.4), see Section 3.1 for more details about this operator.

Several studies have addressed the well-posedness and regularity of solutions for peridynamics and nonlocal integral equations of the peridynamic type. The existence and uniqueness of solutions for the linear peridynamic equilibrium equation on bounded domains were established in<sup>77</sup>. Well-posedness for the peridynamic evolution equation with a bounded operator was demonstrated in<sup>78</sup> and<sup>46</sup>. For a nonlinear bond-based peridynamic evolution equation on  $\mathbb{R}^n$ , existence, uniqueness, and stability were proven in<sup>79</sup>.

Regularity of solutions for the nonlocal Poisson equation on bounded domains with integrable kernels was investigated in<sup>52;80</sup>. Additionally,<sup>81</sup> analyzed a linear nonlocal vector wave equation of peridynamic type defined on  $\mathbb{R}^n$  with a singular integral kernel. This work establishes existence, uniqueness, and regularity results under a “regularity gap” condition, where the initial displacement  $u_0 \in H^s(\mathbb{R}^n)$  and initial velocity  $v_0 \in H^{s-\alpha}(\mathbb{R}^n)$ , with  $s \in \mathbb{R}$  and  $n + 2\alpha$  representing the order of singularity for  $0 < \alpha < 1$ . Furthermore, it shows that, under this regularity gap, the distributional solutions of the nonlocal problem converge to the distributional solution of the classical vector wave equation.

The work in<sup>82</sup> provides regularity results for bond-based linear peridynamics equations on bounded domains in one-dimension. The works in<sup>60;83–85</sup> provide regularity results for the bond-based peridynamics evolution equation for different cases related to the type of the integral kernel and the spatial dimension.

In this work, we improve on previous results and provide a comprehensive description of the regularity of solutions for peridynamic equations over the space of periodic distributions in any spatial dimension. The results are presented for linear state-based peridynamics equilibrium and evolution equations. The analysis results apply to bond-based peridynamics as a special case of the more general state-based peridynamics.

The approach followed in this work builds on previous works on scalar peridynamic or nonlocal equations given in<sup>53;54;86</sup>. In<sup>53</sup>, the Fourier multipliers of the nonlocal Laplacian were derived and represented in an integral form as well as in terms of hypergeometric functions, which was key to characterize the multipliers asymptotic behavior and then to

develop regularity of solutions analysis for the periodic nonlocal Poisson equation.

This work builds also on the work in<sup>87</sup>, which studied the tensor multipliers of the state-based peridynamic operator. It was shown that the multipliers and the eigenvalues of the peridynamic operator have integral representations as well as hypergeometric representations. These representations are central to the current work and allow for studying the asymptotic behavior of the tensor multipliers and the eigenvalues, and their nonlocal-to-local limiting behavior.

The current work generalizes the approach in<sup>53</sup>, building on the explicit representations of the multipliers in<sup>87</sup>, to characterize the asymptotic behavior of the state-based peridynamic operator. We then utilize the asymptotic analysis to establish regularity of solutions for the equilibrium peridynamic equation as well as spatial and temporal regularity of solutions for the peridynamic evolution equation. Furthermore, we study the convergence of the solutions of peridynamic equations to their local counterparts governed by the Navier operator. The nonlocal-to-local convergence of solutions analysis is shown for two types of limiting behavior as the peridynamic horizon (nonlocality) vanishes or as the integral kernel becomes hyper-singular.

### 3.1 Multipliers and eigenvalues of the linear peridynamic operators

For a homogeneous isotropic solid, the linear peridynamic operator<sup>88;89</sup> has the form

$$\begin{aligned}\mathcal{L}\mathbf{u}(x) = & \rho \int_{\Omega} \frac{\gamma(\|y-x\|)}{\|y-x\|^2} (y-x) \otimes (y-x) (\mathbf{u}(y) - \mathbf{u}(x)) dy \\ & + \rho' \int_{\Omega} \int_{\Omega} \gamma(\|y-x\|) \gamma(\|z-x\|) (y-x) \otimes (z-x) (\mathbf{u}(z) - \mathbf{u}(x)) dz dy \\ & + \rho' \int_{\Omega} \int_{\Omega} \gamma(\|y-x\|) \gamma(\|z-y\|) (y-x) \otimes (z-y) (\mathbf{u}(z) - \mathbf{u}(y)) dz dy,\end{aligned}$$

where  $\gamma$  is a scalar field, and  $\rho, \rho'$  are scaling constants including the material properties. In the case  $\Omega = \mathbb{R}^n$ , due to the symmetry, the formula reduces to

$$\begin{aligned}\mathcal{L}\mathbf{u}(x) &= \rho \int_{\mathbb{R}^n} \frac{\gamma(\|y-x\|)}{\|y-x\|^2} (y-x) \otimes (y-x) (\mathbf{u}(y) - \mathbf{u}(x)) dy \\ &\quad + \rho' \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \gamma(\|y-x\|) \gamma(\|z-y\|) (y-x) \otimes (z-y) \mathbf{u}(z) dz dy.\end{aligned}$$

In this work, we will focus on radially symmetric kernels with compact support of the form

$$\gamma(\|y-x\|) = c^{\delta,\beta} \frac{1}{\|y-x\|^\beta} \chi_{B_\delta(x)}(y), \quad (3.3)$$

where  $c^{\delta,\beta}$  is given by

$$c^{\delta,\beta} := \frac{2(n+2-\beta)\Gamma\left(\frac{n}{2}+1\right)}{\pi^{n/2}\delta^{n+2-\beta}}.$$

Here,  $\chi_{B_\delta(x)}$  denotes the indicator function of the ball centered at  $x$  with radius  $\delta > 0$ , and the kernel exponent  $\beta$  satisfies  $\beta < n+2$ . The linear peridynamic operator, parameterized by the nonlocality parameter  $\delta$  and the integral kernel exponent  $\beta$  can then be represented as

$$\begin{aligned}\mathcal{L}^{\delta,\beta}\mathbf{u}(x) &= (n+2)\mu c^{\delta,\beta} \int_{B_\delta(x)} \frac{(y-x) \otimes (y-x)}{\|y-x\|^{\beta+2}} (\mathbf{u}(y) - \mathbf{u}(x)) dy \\ &\quad + (\lambda^* - \mu) \frac{(c^{\delta,\beta})^2}{4} \int_{B_\delta(x)} \int_{B_\delta(y)} \frac{y-x}{\|y-x\|^\beta} \otimes \frac{z-y}{\|z-y\|^\beta} \mathbf{u}(z) dz dy,\end{aligned} \quad (3.4)$$

where  $\mu$  and  $\lambda^*$  are Lamé parameters. (Here we use  $\lambda^*$  to denote the second Lamé parameter and save  $\lambda$  to denote the eigenvalues later).

Integral kernels of the form (3.3) appear in the original works on peridynamics, see for example<sup>50;90</sup>. In addition, this kernel's explicit form provides explicit results, in terms of the kernel's singularity and the spatial dimension, for the characterization of the peridynamic operators' Fourier multipliers<sup>87</sup> and, consequently, for the regularity of solutions for peridynamic equations.

For convenience, we will decompose  $\mathcal{L}^{\delta,\beta}$  into the sum of two operators  $\mathcal{L}_b$  and  $\mathcal{L}_s$ , where these operators refer to the first and second term in the summation respectively. After changing variables, we can rewrite those operators as following

$$\mathcal{L}_b^{\delta,\beta} \mathbf{u}(x) = (n+2)\mu c^{\delta,\beta} \int_{B_\delta(0)} \frac{w \otimes w}{\|w\|^{\beta+2}} (\mathbf{u}(x+w) - \mathbf{u}(x)) dw, \quad (3.5)$$

and

$$\mathcal{L}_s^{\delta,\beta} \mathbf{u}(x) = (\lambda^* - \mu) \frac{(c^{\delta,\beta})^2}{4} \int_{B_\delta(0)} \int_{B_\delta(0)} \frac{w}{\|w\|^\beta} \otimes \frac{q}{\|q\|^\beta} \mathbf{u}(x+w+q) dq dw. \quad (3.6)$$

We will be using the following results , Theorem 15 - Theorem 19, on the multipliers of  $\mathcal{L}_b, \mathcal{L}_s$ , and  $\mathcal{L}^{\delta,\beta}$ , which were presented in<sup>87</sup>.

**Theorem 15.** *Expressing  $\mathbf{u}$  through its Fourier transform as*

$$\mathbf{u}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{\mathbf{u}}(\nu) e^{i\nu \cdot x} d\nu,$$

*we have the representations*

$$\begin{aligned} \mathcal{L}_b^{\delta,\beta} \mathbf{u}(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} M_b^{\delta,\beta}(\nu) \widehat{\mathbf{u}}(\nu) e^{i\nu \cdot x} d\nu, \\ \mathcal{L}_s^{\delta,\beta} \mathbf{u}(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} M_s^{\delta,\beta}(\nu) \widehat{\mathbf{u}}(\nu) e^{i\nu \cdot x} d\nu, \end{aligned}$$

*where*

$$M_b^{\delta,\beta}(\nu) = (n+2)\mu c^{\delta,\beta} \int_{B_\delta(0)} \frac{w \otimes w}{\|w\|^{\beta+2}} (\cos(\nu \cdot w) - 1) dw,$$

*and*

$$M_s^{\delta,\beta}(\nu) = -(\lambda^* - \mu) \frac{(c^{\delta,\beta})^2}{4} \left( \int_{B_\delta(0)} \frac{w}{\|w\|^\beta} \sin(\nu \cdot w) dw \right) \otimes \left( \int_{B_\delta(0)} \frac{w}{\|w\|^\beta} \sin(\nu \cdot w) dw \right).$$

In addition, the multipliers  $M^{\delta,\beta}$  of  $\mathcal{L}^{\delta,\beta}$  is given by  $M^{\delta,\beta} = M_b^{\delta,\beta} + M_s^{\delta,\beta}$ , which satisfies  $\widehat{\mathcal{L}^{\delta,\beta} \mathbf{u}} = M^{\delta,\beta} \widehat{\mathbf{u}}$ .

**Theorem 16.** *The hypergeometric representation of  $M_b(\nu)$  is given by*

$$M_b^{\delta,\beta}(\nu) = \alpha_{b1}^{\delta,\beta}(\nu)I + \alpha_{b2}^{\delta,\beta}(\nu)\nu \otimes \nu,$$

where  $I$  denotes the identity matrix and

$$\begin{aligned}\alpha_{b1}^{\delta,\beta}(\nu) &= -\mu \|\nu\|^2 {}_2F_3 \left( 1, \frac{n+2-\beta}{2}; 2, \frac{n+4}{2}, \frac{n+4-\beta}{2}; -\frac{1}{4} \|\nu\|^2 \delta^2 \right), \\ \alpha_{b2}^{\delta,\beta}(\nu) &= -2\mu {}_1F_2 \left( \frac{n+2-\beta}{2}; \frac{n+4}{2}, \frac{n+4-\beta}{2}; -\frac{1}{4} \|\nu\|^2 \delta^2 \right).\end{aligned}$$

**Theorem 17.** *The hypergeometric representation of  $M_s(\nu)$  is given by*

$$M_s^{\delta,\beta}(\nu) = \alpha_s^{\delta,\beta}(\nu)\nu \otimes \nu,$$

where

$$\alpha_s^{\delta,\beta}(\nu) = -(\lambda^* - \mu) {}_1F_2 \left( \frac{n+2-\beta}{2}; \frac{n+2}{2}, \frac{n+4-\beta}{2}; -\frac{1}{4} \|\nu\|^2 \delta^2 \right)^2.$$

**Theorem 18.** *The hypergeometric representation of the multiplier  $M^{\delta,\beta}(\nu)$  is given by*

$$M^{\delta,\beta}(\nu) = \alpha_{b1}^{\delta,\beta}(\nu)I + (\alpha_{b2}^{\delta,\beta}(\nu) + \alpha_s^{\delta,\beta}(\nu))\nu \otimes \nu.$$

This shows that  $M^{\delta,\beta}(\nu)$  is a real symmetric matrix. Furthermore,  $\nu$  is an eigenvector of  $M^{\delta,\beta}(\nu)$ , that is

$$M^{\delta,\beta}(\nu)\nu = \lambda_1^{\delta,\beta}(\nu)\nu,$$

where

$$\begin{aligned}\lambda_1^{\delta,\beta}(\nu) &= \alpha_{b1}^{\delta,\beta}(\nu) + (\alpha_{b2}^{\delta,\beta}(\nu) + \alpha_s^{\delta,\beta}(\nu))\|\nu\|^2 \\ &= -\|\nu\|^2 \left( 3\mu {}_3F_4 \left( 1, \frac{5}{2}, \frac{n+2-\beta}{2}; 2, \frac{3}{2}, \frac{n+4}{2}, \frac{n+4-\beta}{2}; -\frac{1}{4}\|\nu\|^2\delta^2 \right) \right. \\ &\quad \left. + (\lambda^* - \mu) {}_1F_2 \left( \frac{n+2-\beta}{2}; \frac{n+2}{2}, \frac{n+4-\beta}{2}; -\frac{1}{4}\|\nu\|^2\delta^2 \right)^2 \right).\end{aligned}$$

The other  $n-1$  eigenvectors are orthogonal to  $\nu$ . Denote any vector in  $\mathbb{R}^n$  orthogonal to  $\nu$  as  $\nu^\perp$ , then

$$M^{\delta,\beta}(\nu)\nu^\perp = \lambda_2^{\delta,\beta}(\nu)\nu^\perp,$$

where

$$\lambda_2^{\delta,\beta}(\nu) = \alpha_{b1}^{\delta,\beta}(\nu) = -\mu\|\nu\|^2 {}_2F_3 \left( 1, \frac{n+2-\beta}{2}; 2, \frac{n+4}{2}, \frac{n+4-\beta}{2}; -\frac{1}{4}\|\nu\|^2\delta^2 \right).$$

**Theorem 19.** For any  $\nu$  in  $\mathbb{R}^n$ , the eigenvalues  $\lambda_1^{\delta,\beta}(\nu)$  and  $\lambda_2^{\delta,\beta}(\nu)$  of  $M^{\delta,\beta}(\nu)$ , corresponding to eigenvectors that are parallel to  $\nu$  and orthogonal to  $\nu$ , respectively, have the following integral representations

$$\begin{aligned}\lambda_1^{\delta,\beta}(\nu) &= (n+2)\mu c^{\delta,\beta} \int_{B_\delta(0)} \frac{(\nu \cdot w)^2}{\|\nu\|^2 \|w\|^{\beta+2}} (\cos(\nu \cdot w) - 1) dw \\ &\quad - (\lambda^* - \mu) \left( \frac{c^{\delta,\beta}}{2} \int_{B_\delta(0)} \frac{\nu \cdot w}{\|\nu\| \|w\|^\beta} \sin(\nu \cdot w) dw \right)^2,\end{aligned}\tag{3.7}$$

and

$$\lambda_2^{\delta,\beta}(\nu) = (n+2)\mu c^{\delta,\beta} \int_{B_\delta(0)} \frac{\nu \cdot w}{\|\nu\|^2 \|w\|^{\beta+2}} (\sin(\nu \cdot w) - \nu \cdot w) dw.\tag{3.8}$$

From (3.8), one can claim that  $\lambda_2^{\delta,\beta}(\nu) < 0$  for any nonzero  $\nu$ , while from (3.7), the sign of  $\lambda_1^{\delta,\beta}(\nu)$  depends on the values of parameters  $\lambda^*$  and  $\mu$ . Throughout the remainder of this

chapter, we will assume that some additional conditions are imposed on  $\lambda^*$  and  $\mu$  such that  $\lambda_1^{\delta,\beta}(\nu)$  is also negative for any non-zero  $\nu$ . With this assumption, we can easily show that the multiplier matrices  $M^{\delta,\beta}(\nu)$  are invertible for any nonzero  $\nu$ . This is obvious due to the fact that

$$\det M^{\delta,\beta}(\nu) = \lambda_1^{\delta,\beta}(\nu)(\lambda_2^{\delta,\beta}(\nu))^{n-1}.$$

### 3.1.1 Eigenvalues' asymptotic analysis

The asymptotic analysis for the eigenvalues of the peridynamic operator  $\mathcal{L}^{\delta,\beta}$ , provided in this section, is key to studying the regularity of solutions of peridynamic equations in subsequent sections.

Following a similar approach to that used in<sup>53</sup>, we analyze the asymptotic behavior of the eigenvalues  $\lambda_1^{\delta,\beta}(\nu)$  and  $\lambda_2^{\delta,\beta}(\nu)$  as  $\|\nu\| \rightarrow \infty$ . The asymptotic behavior of  $\lambda_2^{\delta,\beta}(\nu)$  is summarized in Theorem 20 below.

**Theorem 20.** *Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta < n + 2$ . Then, as  $\|\nu\| \rightarrow \infty$ ,*

$$\lambda_2^{\delta,\beta}(\nu) \sim \begin{cases} -\mu \frac{2(n+2-\beta)(n+2)}{\delta^2(n-\beta)} - \mu \frac{\Gamma(\frac{n+4}{2})\Gamma(\frac{n+4-\beta}{2})}{\frac{\beta-n}{2}\Gamma(\frac{\beta+2}{2})} \left(\frac{2}{\delta}\right)^{n+2-\beta} \|\nu\|^{\beta-n} & \text{if } \beta \neq n, \\ -\frac{2\mu}{\delta^2}(n+2) \left(2 \log \|\nu\| + \log\left(\frac{\delta^2}{4}\right) + \gamma - \psi\left(\frac{n+2}{2}\right)\right) & \text{if } \beta = n, \end{cases}$$

where  $\gamma$  is Euler's constant and  $\psi$  is the digamma function.

*Proof.* Let  $a = \frac{n+2-\beta}{2}$ ,  $b = \frac{n+4}{2}$  and  $z = \frac{\delta\|\nu\|}{2}$  then

$$\lambda_2^{\delta,\beta}(\nu) = -\mu\|\nu\|^2 {}_2F_3(1, a; 2, b, a+1; -z^2). \quad (3.9)$$

For large  $|z|$ ,<sup>74</sup> Eq. (16.11.8) states that

$${}_2F_3(1, a; 2, b, a+1; -z^2) \sim a\Gamma(b) \left( H_{2,3}(z^2) + E_{2,3}(z^2 e^{-i\pi}) + E_{2,3}(z^2 e^{i\pi}) \right),$$

where  $E_{2,3}$  and  $H_{2,3}$  are formal series defined in<sup>74</sup> Eq. (16.11.1) and<sup>74</sup> Eq. (16.11.2) respectively. From those definitions, it yields that

$$E_{2,3}(z^2 e^{\pm i\pi}) = (2\pi)^{-1/2} 2^{b+1} e^{\pm 2iz} \sum_{k=0}^{\infty} c_k (\pm 2iz)^{-\frac{2b+3}{2}-k}.$$

Since  $2b = n + 4$  and  $c_0 = 1$ , the  $E_{2,3}$  terms decay asymptotically like  $|z|^{-\frac{n+7}{2}}$ , and do not contribute the the asymptotic behavior described in the theorem. For the remaining term that relates to the  $H_{2,3}$  function, from the remark below<sup>74</sup> Eq. (16.11.5),  $H_{2,3}$  can be recognized as the sum of the residues of certain poles of the integrand in<sup>74</sup> Eq. (16.5.1). The integrand for this setting is as following:

$$f(s) := \frac{\Gamma(1+s)\Gamma(a+s)}{\Gamma(2+s)\Gamma(b+s)\Gamma(a+1+s)} \Gamma(-s)(z^2)^s = \frac{\Gamma(-s)}{(s+1)(s+a)\Gamma(b+s)} z^{2s}.$$

The two poles of interest are at  $s = -1$  and  $s = -a$ . The restriction that  $\beta \notin \{n+2, n+4, n+6, \dots\}$  ensures that  $-a$  is not a nonnegative integer and, therefore, that  $s = -a$  is not a pole of  $\Gamma(-s)$ . Thus, there are only two cases to consider, either  $\beta \neq n$ , which implies that  $s = -1$  and  $s = -a$  are distinct simple poles of  $f$ , or  $\beta = n$ , yielding a double pole at  $s = -1$ .

In the case where  $\beta \neq n$ , we have

$$H_{2,3}(z^2) = \text{Res}(f, -1) + \text{Res}(f, -a) = \frac{1}{(a-1)\Gamma(b-1)} z^{-2} + \frac{\Gamma(a)}{(1-a)\Gamma(b-a)} z^{-2a}.$$

When  $\beta = n$ , the double pole makes the residue more complicated (and gives rise to the logarithmic term):

$$H_{2,3}(z^2) = \text{Res}(f, -1) = \frac{d}{ds} \left( \frac{\Gamma(-s)}{\Gamma(b+s)} z^{2s} \right) \Big|_{s=-1} = \frac{2 \log z - \psi(b-1) - \psi(1)}{\Gamma(b-1)} z^{-2}.$$

Finally, using (3.9) and substituting completes the theorem. □

For the asymptotic behavior of  $\lambda_1(\nu)$ , we first decompose  $\lambda_1(\nu)$  into

$$\lambda_1^{\delta,\beta}(\nu) = \lambda_{1,1}^{\delta,\beta}(\nu) + \lambda_{1,2}^{\delta,\beta}(\nu),$$

where

$$\begin{aligned}\lambda_{1,1}^{\delta,\beta}(\nu) &= -3\mu\|\nu\|^2 {}_3F_4\left(1, \frac{5}{2}, \frac{n+2-\beta}{2}; 2, \frac{3}{2}, \frac{n+4}{2}, \frac{n+4-\beta}{2}; -\frac{1}{4}\|\nu\|^2\delta^2\right), \\ \lambda_{1,2}^{\delta,\beta}(\nu) &= -\|\nu\|^2(\lambda^* - \mu) {}_1F_2\left(\frac{n+2-\beta}{2}; \frac{n+2}{2}, \frac{n+4-\beta}{2}; -\frac{1}{4}\|\nu\|^2\delta^2\right)^2,\end{aligned}$$

and then obtain results for  $\lambda_{1,1}(\nu)$  in Theorem 21 and for  $\lambda_{1,2}(\nu)$  in Theorem 22.

**Theorem 21.** *Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta < n + 2$ . Then, as  $\|\nu\| \rightarrow \infty$ ,*

$$\lambda_{1,1}^{\delta,\beta}(\nu) \sim \begin{cases} -\mu \frac{2(n+2-\beta)(n+2)}{\delta^2(n-\beta)} - 2\mu \frac{n-\beta-1}{n-\beta} \frac{\Gamma(\frac{n+4}{2})\Gamma(\frac{n+4-\beta}{2})}{\Gamma(\frac{\beta+2}{2})} \left(\frac{2}{\delta}\right)^{n+2-\beta} \|\nu\|^{\beta-n} & \text{if } \beta \neq n, \\ -\frac{2\mu}{\delta^2}(n+2) \left(2\log\|\nu\| + \log\left(\frac{\delta^2}{4}\right) + \gamma + 2 - \psi\left(\frac{n+2}{2}\right)\right) & \text{if } \beta = n, \end{cases}$$

where  $\gamma$  is Euler's constant and  $\psi$  is the digamma function.

*Proof.* Let  $a = \frac{n+2-\beta}{2}$ ,  $b = \frac{n+4}{2}$  and  $z = \frac{\delta\|\nu\|}{2}$  then

$$\lambda_{1,1}^{\delta,\beta}(\nu) = -3\mu\|\nu\|^2 {}_3F_4\left(1, \frac{5}{2}, a; 2, \frac{3}{2}, b, a+1; -z^2\right). \quad (3.10)$$

From the setting,<sup>74</sup> Eq. (16.11.8) states that for large  $|z|$

$${}_3F_4\left(1, \frac{5}{2}, a; 2, \frac{3}{2}, b, a+1; -z^2\right) \sim \frac{2}{3}a\Gamma(b) \left(H_{3,4}(z^2) + E_{3,4}(z^2e^{-i\pi}) + E_{3,4}(z^2e^{i\pi})\right),$$

where  $E_{3,4}$  and  $H_{3,4}$  are formal series defined in<sup>74</sup> Eq. (16.11.1) and<sup>74</sup> Eq. (16.11.2) respec-

tively. From those definitions, it yields that

$$E_{3,4}(z^2 e^{\pm i\pi}) = (2\pi)^{-1/2} 2^b e^{\pm 2iz} \sum_{k=0}^{\infty} c_k(\pm 2iz)^{-\frac{2b+1}{2}-k}.$$

Since  $2b = n + 4$  and  $c_0 = 1$ , the  $E_{3,4}$  terms decay asymptotically like  $|z|^{-\frac{n+5}{2}}$ , and do not contribute to the asymptotic behavior described in the theorem. For the remaining term that relates to the  $H_{3,4}$  function, from the remark below<sup>74</sup> Eq. (16.11.5),  $H_{3,4}$  can be recognized as the sum of the residues of certain poles of the integrand in<sup>74</sup> Eq. (16.5.1). The integrand for this setting is as following:

$$f(s) := \frac{\Gamma(1+s)\Gamma(\frac{5}{2}+s)\Gamma(a+s)}{\Gamma(2+s)\Gamma(\frac{3}{2}+s)\Gamma(b+s)\Gamma(a+1+s)}\Gamma(-s)(z^2)^s = \frac{(\frac{3}{2}+s)\Gamma(-s)}{(s+1)(s+a)\Gamma(b+s)}z^{2s}.$$

The two poles of interest are at  $s = -1$  and  $s = -a$ . The restriction that  $\beta \notin \{n+2, n+4, n+6, \dots\}$  ensures that  $-a$  is not a nonnegative integer and, therefore, that  $s = -a$  is not a pole of  $\Gamma(-s)$ . Thus, there are only two cases to consider, either  $\beta \neq n$ , which implies that  $s = -1$  and  $s = -a$  are distinct simple poles of  $f$ , or  $\beta = n$ , yielding a double pole at  $s = -1$ .

In the case where  $\beta \neq n$ , we have

$$H_{3,4}(z^2) = \text{Res}(f, -1) + \text{Res}(f, -a) = \frac{\frac{1}{2}}{(a-1)\Gamma(b-1)}z^{-2} + \frac{(\frac{3}{2}-a)\Gamma(a)}{(1-a)\Gamma(b-a)}z^{-2a}.$$

When  $\beta = n$ , the double pole makes the residue more complicated (and gives rise to the logarithmic term):

$$\begin{aligned} H_{3,4}(z^2) = \text{Res}(f, -1) &= \frac{d}{ds} \left( \frac{(\frac{3}{2}+s)\Gamma(-s)}{\Gamma(b+s)}z^{2s} \right) \Big|_{s=-1} \\ &= \left[ \frac{\Gamma(-s)}{\Gamma(b+s)}z^{2s} + \left( \frac{3}{2}+s \right) \frac{d}{ds} \left( \frac{\Gamma(-s)}{\Gamma(b+s)}z^{2s} \right) \right] \Big|_{s=-1} \\ &= \frac{2 \log z - \psi(b-1) - \psi(1) + 2}{2\Gamma(b-1)}z^{-2}. \end{aligned}$$

Finally, using (3.10) and substituting completes the theorem.  $\square$

**Theorem 22.** *Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta < n + 2$ . Then, as  $\|\nu\| \rightarrow \infty$ ,*

$$\lambda_{1,2}^{\delta,\beta}(\nu) \sim -(\lambda^* - \mu) \left[ \frac{\Gamma(\frac{n+2}{2})\Gamma(\frac{n+4-\beta}{2})}{\Gamma(\frac{\beta}{2})} \right]^2 \left( \frac{2}{\delta} \right)^{2(n+2-\beta)} \|\nu\|^{2(\beta-(n+1))}.$$

*Proof.* Let  $a = \frac{n+2-\beta}{2}, b = \frac{n+2}{2}$  and  $z = \frac{\delta\|\nu\|}{2}$  then

$$\lambda_{1,2}^{\delta,\beta}(\nu) = -\|\nu\|^2(\lambda^* - \mu) {}_1F_2(a; b, a + 1; -z^2)^2. \quad (3.11)$$

For large  $|z|$ ,<sup>74</sup> Eq. (16.11.8) states that

$${}_1F_2(a; b, a + 1; -z^2) \sim a\Gamma(b) (H_{1,2}(z^2) + E_{1,2}(z^2 e^{-i\pi}) + E_{1,2}(z^2 e^{i\pi})),$$

where  $E_{1,2}$  and  $H_{1,2}$  are formal series defined in<sup>74</sup> Eq. (16.11.1) and<sup>74</sup> Eq. (16.11.2) respectively. From those definitions, it yields that

$$E_{1,2}(z^2 e^{\pm i\pi}) = (2\pi)^{-1/2} 2^{b+1} e^{\pm 2iz} \sum_{k=0}^{\infty} c_k (\pm 2iz)^{-\frac{2b+1}{2}-k}.$$

Since  $2b = n + 2$  and  $c_0 = 1$ , the  $E_{1,2}$  terms decay asymptotically like  $|z|^{-\frac{n+3}{2}}$ , and do not contribute to the asymptotic behavior described in the theorem. For the remaining term that relates to the  $H_{1,2}$  function, from the remark below<sup>74</sup> Eq. (16.11.5),  $H_{1,2}$  can be recognized as the sum of the residues of certain poles of the integrand in<sup>74</sup> Eq. (16.5.1). The integrand for this setting is as following:

$$f(s) := \frac{\Gamma(a+s)}{\Gamma(b+s)\Gamma(a+1+s)} \Gamma(-s)(z^2)^s = \frac{\Gamma(-s)}{(s+a)\Gamma(b+s)} z^{2s}.$$

The only pole of interest is at  $s = -a$ . The restriction that  $\beta \notin \{n+2, n+4, n+6, \dots\}$  ensures that  $-a$  is not a nonnegative integer and, therefore, that  $s = -a$  is not a pole of

$\Gamma(-s)$ . Thus

$$H_{1,2}(z^2) = \text{Res}(f, -a) = \frac{\Gamma(a)}{\Gamma(b-a)} z^{-2a}.$$

Finally, using (3.11) and substituting completes the theorem.  $\square$

The asymptotic behavior of  $\lambda_1^{\delta,\beta}(\nu)$  is summarized in Theorem 23 below.

**Theorem 23.** *Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta < n + 2$ . Then, as  $\|\nu\| \rightarrow \infty$ ,*

$$\lambda_1^{\delta,\beta}(\nu) \sim -(\lambda^* - \mu) \left[ \frac{\Gamma(\frac{n+2}{2})\Gamma(\frac{n+4-\beta}{2})}{\Gamma(\frac{\beta}{2})} \right]^2 \left( \frac{2}{\delta} \right)^{2(n+2-\beta)} \|\nu\|^{2(\beta-(n+1))}$$

$$+ \begin{cases} -\mu \frac{(n+2-\beta)(n+2)}{\delta^2(n-\beta)} - 2\mu \frac{n-\beta-1}{n-\beta} \frac{\Gamma(\frac{n+4}{2})\Gamma(\frac{n+4-\beta}{2})}{\Gamma(\frac{\beta+2}{2})} \left( \frac{2}{\delta} \right)^{n+2-\beta} \|\nu\|^{\beta-n} & \text{if } \beta \neq n, \\ -\frac{2\mu}{\delta^2}(n+2) \left( 2 \log \|\nu\| + \log \left( \frac{\delta^2}{4} \right) + \gamma + 2 - \psi\left(\frac{n+2}{2}\right) \right) & \text{if } \beta = n, \end{cases}$$

where  $\gamma$  is Euler's constant and  $\psi$  is the digamma function.

### 3.1.2 Local limits of the multipliers and the peridynamic operator

In this section, we compute the limits of the Fourier multipliers and the peridynamic operator  $\mathcal{L}^{\delta,\beta}$  in (3.4). Two types of limits are considered as  $\delta \rightarrow 0$  with fixed  $\beta < n + 2$  and as  $\beta \rightarrow n + 2^-$  with fixed  $\delta > 0$ . The results of this section will be used to study the limiting behavior of the solutions of peridynamic equations in subsequent sections.

Denoting by  $\mathcal{N}$  the Navier operator of linear elasticity. For any homogeneous medium, it has the form

$$\mathcal{N}\mathbf{u} = (\lambda^* + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu\Delta\mathbf{u}. \quad (3.12)$$

Similar to the peridynamic operator, its multiplier is defined through Fourier transform as

$\widehat{\mathcal{N}\mathbf{u}} = M^{\mathcal{N}}\widehat{\mathbf{u}}$  and can be explicitly expressed as

$$M^{\mathcal{N}}(\nu) = -(\lambda^* + \mu)\nu \otimes \nu - \mu\|\nu\|^2 I.$$

The multiplier of the peridynamic operator  $M^{\delta,\beta}$  converges to the multiplier of the Navier operator  $M^{\mathcal{N}}$  in the following sense (see<sup>87</sup>):

**Proposition 3.** *Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta < n + 2$ . Then for fixed  $\beta$ ,*

$$\lim_{\delta \rightarrow 0^+} M^{\delta,\beta}(\nu) = M^{\mathcal{N}}(\nu).$$

Moreover, for fixed  $\delta$ ,

$$\lim_{\beta \rightarrow n+2^-} M^{\delta,\beta}(\nu) = M^{\mathcal{N}}(\nu).$$

One corollary of this result is the convergence of the eigenvalues  $\lambda_1^{\delta,\beta}(\nu)$  and  $\lambda_2^{\delta,\beta}(\nu)$  of  $M^{\delta,\beta}$  to the eigenvalues  $\lambda_1^{\mathcal{N}}(\nu)$  and  $\lambda_2^{\mathcal{N}}(\nu)$  of  $M^{\mathcal{N}}$ , respectively.

**Corollary 2.** *Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta < n + 2$ . Then for fixed  $\beta$ ,*

$$\lim_{\delta \rightarrow 0^+} \lambda_1^{\delta,\beta}(\nu) = -(\lambda^* + \mu)\|\nu\|^2 = \lambda_1^{\mathcal{N}}(\nu),$$

$$\lim_{\delta \rightarrow 0^+} \lambda_2^{\delta,\beta}(\nu) = -\mu\|\nu\|^2 = \lambda_2^{\mathcal{N}}(\nu),$$

and for fixed  $\delta$ ,

$$\lim_{\beta \rightarrow n+2^-} \lambda_1^{\delta,\beta}(\nu) = -(\lambda^* + \mu)\|\nu\|^2 = \lambda_1^{\mathcal{N}}(\nu),$$

$$\lim_{\beta \rightarrow n+2^-} \lambda_2^{\delta,\beta}(\nu) = -\mu\|\nu\|^2 = \lambda_2^{\mathcal{N}}(\nu).$$

The corresponding convergence of the peridynamic operator  $\mathcal{L}$  to the Navier operator  $\mathcal{N}$  as  $\delta \rightarrow 0$  or as  $\beta \rightarrow n + 2^-$  is given in the following results.

**Theorem 24.** Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta < n + 2$ . Assume that  $\mathbf{u}(x) \in (C^3(\Omega))^n$ , then for fixed  $\beta$ ,

$$\lim_{\delta \rightarrow 0^+} \mathcal{L}_b^{\delta, \beta} \mathbf{u}(x) = \mu \Delta \mathbf{u}(x) + 2\mu \nabla(\nabla \cdot \mathbf{u})(x),$$

and for fixed  $\delta$ ,

$$\lim_{\beta \rightarrow n+2^-} \mathcal{L}_b^{\delta, \beta} \mathbf{u}(x) = \mu \Delta \mathbf{u}(x) + 2\mu \nabla(\nabla \cdot \mathbf{u})(x).$$

*Proof.* Using Taylor's theorem,  $\mathbf{u}$  is expanded about  $x$  as

$$\mathbf{u}(x + w) = \mathbf{u}(x) + \nabla \mathbf{u}(x) : w + \frac{1}{2} \nabla \nabla \mathbf{u}(x) : w \otimes w + R_\sigma(\mathbf{u}; x, w),$$

where the remainder  $R_\sigma(\mathbf{u}; x, w)$  is of the form

$$R_\sigma(\mathbf{u}; x, w) = \frac{1}{6} \nabla \nabla \nabla \mathbf{u}(x + \sigma w) : w \otimes w \otimes w,$$

for some scalar  $\sigma \in [0, 1]$ . By substituting, (3.5) becomes

$$\begin{aligned} \mathcal{L}_b^{\delta, \beta} \mathbf{u}(x) &= (n+2) \mu c^{\delta, \beta} \int_{B_\delta(0)} \frac{w \otimes w \otimes w}{\|w\|^{\beta+2}} dw : \nabla \mathbf{u}(x) \\ &\quad + (n+2) \mu c^{\delta, \beta} \int_{B_\delta(0)} \frac{w \otimes w \otimes w \otimes w}{\|w\|^{\beta+2}} dw : \frac{1}{2} \nabla \nabla \mathbf{u}(x) \\ &\quad + (n+2) \mu c^{\delta, \beta} \int_{B_\delta(0)} \frac{w \otimes w}{\|w\|^{\beta+2}} R_\sigma(\mathbf{u}; x, z) dw. \end{aligned}$$

Due to symmetry, the integral in the first term is identically zero, which makes the whole term vanishes. For the second term, using spherical coordinates, it is straightforward to

show that

$$\begin{aligned}
& (n+2)c^{\delta,\beta} \int_{B_\delta(0)} \frac{w_i w_j w_k w_l}{\|w\|^{\beta+2}} dw \\
&= \begin{cases} 6, & \text{for } i = j = k = l, \\ 2, & \text{for } i = j \neq k = l, \text{ or } i = k \neq j = l, \text{ or } i = l \neq j = k, \\ 0, & \text{otherwise.} \end{cases} \tag{3.13}
\end{aligned}$$

Utilizing (3.13), observe that for any component  $i$ , we have

$$\begin{aligned}
(n+2)c^{\delta,\beta} \int_{B_\delta(0)} \frac{w_i w_j w_k w_l}{\|w\|^{\beta+2}} dw \frac{1}{2} \frac{\partial^2 \mathbf{u}_j}{\partial x_i \partial x_k} &= \frac{1}{2} \left[ 6 \frac{\partial^2 \mathbf{u}_i}{\partial x_i^2} + 2 \sum_{k \neq i} \frac{\partial^2 \mathbf{u}_i}{\partial x_k^2} + 4 \sum_{j \neq i} \frac{\partial^2 \mathbf{u}_j}{\partial x_i \partial x_j} \right] \\
&= \sum_k \frac{\partial^2 \mathbf{u}_i}{\partial x_k^2} + 2 \sum_j \frac{\partial^2 \mathbf{u}_j}{\partial x_i \partial x_j} \\
&= (\Delta \mathbf{u})_i(x) + 2 \nabla(\nabla \cdot \mathbf{u})_i(x).
\end{aligned}$$

This shows that the second term is  $\mu \Delta \mathbf{u}(x) + 2\mu \nabla(\nabla \cdot \mathbf{u})(x)$ . Finally, for the third term, we have the following bound

$$\left| (n+2)\mu c^{\delta,\beta} \int_{B_\delta(0)} \frac{w \otimes w}{\|w\|^{\beta+2}} R_\sigma(\mathbf{u}; x, z) dw \right| \leq M_x \delta(n+2-\beta),$$

where  $M_x$  is a constant dependent on  $x$  only. All these arguments in combination complete the proof.  $\square$

**Theorem 25.** *Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta < n+2$ . Assume that  $\mathbf{u} \in (C^3(\Omega))^n$ , then for fixed  $\beta$ ,*

$$\lim_{\delta \rightarrow 0^+} \mathcal{L}_s^{\delta,\beta} \mathbf{u}(x) = (\lambda^* - \mu) \nabla(\nabla \cdot \mathbf{u})(x) \quad \text{for } \beta < n+2,$$

and for fixed  $\delta$ ,

$$\lim_{\beta \rightarrow n+2^-} \mathcal{L}_s^{\delta, \beta} \mathbf{u}(x) = (\lambda^* - \mu) \nabla(\nabla \cdot \mathbf{u})(x) \quad \text{for } \delta > 0,$$

for all  $x \in \mathbb{R}^n$ .

*Proof.* Firstly, rewrite (3.6) as

$$\mathcal{L}_s^{\delta, \beta} \mathbf{u}(x) = (\lambda^* - \mu) \frac{(c^{\delta, \beta})^2}{4} \int_{B_\delta(0)} \frac{w}{\|w\|^\beta} \left[ \int_{B_\delta(0)} \frac{q}{\|q\|^\beta} \cdot \mathbf{u}(x + w + q) dq \right] dw. \quad (3.14)$$

Then  $\mathbf{u}$  is expanded about  $x + w$  using Taylor's expansion as

$$\mathbf{u}(x + w + q) = \mathbf{u}(x + w) + \nabla \mathbf{u}(x + w) : q + \frac{1}{2} \nabla \nabla \mathbf{u}(x + w) : q \otimes q + R_{s_1}(\mathbf{u}; x + w, q),$$

where the remainder  $R_{s_1}(\mathbf{u}; x + w, q)$  is of the form

$$R_{s_1}(\mathbf{u}; x + w, q) = \frac{1}{6} \nabla \nabla \nabla \mathbf{u}(x + s_1 w) : q \otimes q \otimes q,$$

for some scalar  $s_1 \in [0, 1]$ . The inner integral in (3.14) now becomes

$$\begin{aligned} \int_{B_\delta(0)} \frac{q}{\|q\|^\beta} \cdot \mathbf{u}(x + w + q) dq &= \int_{B_\delta(0)} \frac{q}{\|q\|^\beta} dq \cdot \mathbf{u}(x + w) + \int_{B_\delta(0)} \frac{q \otimes q}{\|q\|^\beta} dq : \nabla \mathbf{u}(x + w) \\ &+ \int_{B_\delta(0)} \frac{q \otimes q \otimes q}{\|q\|^\beta} dq : \frac{1}{2} \nabla \nabla \mathbf{u}(x + w) + \int_{B_\delta(0)} \frac{q}{\|q\|^\beta} \cdot R_{s_1}(\mathbf{u}; x + w, q) dq \end{aligned}$$

Note that due to symmetry, the integral in the first and third term in the expression above vanish. In addition, since

$$\int_{B_\delta(0)} \frac{q \otimes q}{\|q\|^\beta} dq = \frac{2}{c^{\delta, \beta}} I,$$

then

$$\mathcal{L}_s^{\delta,\beta} \mathbf{u}(x) = (\lambda^* - \mu) \frac{c^{\delta,\beta}}{2} \int_{B_\delta(0)} \frac{w}{\|w\|^\beta} \nabla \cdot \mathbf{u}(x+w) dw \quad (3.15)$$

$$+ (\lambda^* - \mu) \frac{(c^{\delta,\beta})^2}{4} \int_{B_\delta(0)} \frac{w}{\|w\|^\beta} \left[ \int_{B_\delta(0)} \frac{q}{\|q\|^\beta} \cdot R_{s_1}(\mathbf{u}; x+w, q) dq \right] dw \quad (3.16)$$

Using Taylor's theorem, we have the following expansion

$$\nabla \cdot \mathbf{u}(x+w) = \nabla \cdot \mathbf{u}(x) + \nabla(\nabla \cdot \mathbf{u})(x) : w + R_{s_2}(\mathbf{u}; x, w),$$

where the remainder  $R_{s_2}(\mathbf{u}; x, w)$  is of the form

$$R_{s_2}(\mathbf{u}; x, w) = \frac{1}{2} \nabla \nabla(\nabla \cdot \mathbf{u})(x + s_2 w) : w \otimes w,$$

for some scalar  $s_2 \in [0, 1]$ . The integral in (3.15) becomes

$$\int_{B_\delta(0)} \frac{w}{\|w\|^\beta} \nabla \cdot \mathbf{u}(x+w) dw = \int_{B_\delta(0)} \frac{w}{\|w\|^\beta} dw \nabla \cdot \mathbf{u}(x) + \int_{B_\delta(0)} \frac{w \otimes w}{\|w\|^\beta} dw : \nabla(\nabla \cdot \mathbf{u})(x) \quad (3.17)$$

$$+ \int_{B_\delta(0)} \frac{w \otimes w}{\|w\|^\beta} R_{s_2}(\mathbf{u}; x, w) dw \quad (3.18)$$

In the above expression, the first term in (3.17) is identically 0 due to symmetry. About the second term, we have

$$\int_{B_\delta(0)} \frac{w \otimes w}{\|w\|^\beta} dw : \nabla(\nabla \cdot \mathbf{u})(x) = \frac{2}{c^{\delta,\beta}} \nabla(\nabla \cdot \mathbf{u})(x).$$

We then get from (3.15)

$$\frac{c^{\delta,\beta}}{2} \int_{B_\delta(0)} \nabla \cdot \mathbf{u}(x+w) \frac{w}{\|w\|^\beta} dw = \nabla(\nabla \cdot \mathbf{u})(x) + \frac{c^{\delta,\beta}}{2} \int_{B_\delta(0)} \frac{w \otimes w}{\|w\|^\beta} R_{s_2}(\mathbf{u}; x, w). \quad (3.19)$$

The second expression in (3.19) vanishes under the limit of  $\delta \rightarrow 0^+$  or  $\beta \rightarrow n+2^-$ , where

we have used the bounds

$$\left| \frac{c^{\delta,\beta}}{2} \int_{B_\delta(0)} \frac{w \otimes w}{\|w\|^\beta} R_{s_2}(\mathbf{u}; x, w) dw \right| \leq M_1 \delta (2 + n - \beta)$$

A similar result applies to (3.16) as well, from which the statement of the theorem follows.  $\square$

## 3.2 The Sobolev space of periodic distributions

Consider  $\mathcal{L}^{\delta,\beta}$  as the periodic operator on the torus

$$\mathbb{T}^n = \prod_{i=1}^n [0, 2\pi], \quad \text{with } i = 1, 2, \dots, n.$$

For simplicity, from now on, for any  $k \in \mathbb{Z}^n$ , we will write  $M_k^{\delta,\beta}$  and  $M_k^{\mathcal{N}}$  in replacement for  $M^{\delta,\beta}(k)$  and  $M^{\mathcal{N}}(k)$ , respectively. In addition, since the matrix space is finite dimensional, all matrix norms are equivalent. Throughout this chapter, we will use the matrix operator norm induced by the vector 2-norm.

The eigenvalues and eigenvectors of the operator  $\mathcal{L}^{\delta,\beta}$  are determined through the multipliers developed in the previous section. The following result, Theorem 26, has been proved in<sup>87</sup>.

**Theorem 26.** *Let  $\gamma$  be a fixed vector in  $\mathbb{R}^n$ . For any  $k \in \mathbb{Z}^n$ , define*

$$\psi_k(x) = e^{ik \cdot x} \gamma.$$

*Then,*

$$\mathcal{L}^{\delta,\beta} \psi_k(x) = M_k^{\delta,\beta} \psi_k(x) = M_k^{\delta,\beta} \psi_k(x).$$

The regularity of solutions for periodic peridynamic equations considered in this work is studied over the Sobolev space of periodic vector distributions, which is defined below.

**Definition 3.** For  $q \in \mathbb{R}$ , define by  $\mathcal{H}^q(\mathbb{T}^n)$  the space of periodic vector distributions  $\mathbf{g} : \mathbb{T}^n \rightarrow \mathbb{R}^n$  such that

$$\|\mathbf{g}\|_{\mathcal{H}^q(\mathbb{T}^n)}^2 := \sum_{k \in \mathbb{Z}^n} (1 + \|k\|^2)^q \|\widehat{\mathbf{g}}_k\|^2 < \infty.$$

From this definition, we see that if  $\mathbf{g} \in \mathcal{H}^q(\mathbb{T}^n)$ , then for any nonzero  $k \in \mathbb{Z}^n$

$$\|\widehat{\mathbf{g}}_k\| \leq \frac{\|\mathbf{g}\|_{\mathcal{H}^q(\mathbb{T}^n)}}{\|k\|^q}. \quad (3.20)$$

In subsequent sections, we consider maps from the real line to periodic distributions and use the following notion of derivative for these maps.

**Definition 4.** Suppose  $\mathbf{U} : [0, \infty) \rightarrow \mathcal{H}^q(\mathbb{T}^n)$  maps  $t \geq 0$  to a periodic vector distribution  $\mathbf{U}(t)$  in  $\mathcal{H}^q(\mathbb{T}^n)$ . The derivative of  $\mathbf{U}$  at  $t$  is defined in the sense of Gâteaux derivative

$$\mathbf{U}'(t) = \frac{d}{dt} \mathbf{U}(t) = \lim_{h \rightarrow 0} \frac{\mathbf{U}(t+h) - \mathbf{U}(t)}{h},$$

where the limit is with respect to the norm of  $\mathcal{H}^q(\mathbb{T}^n)$  space.

We will denote by  $C^p([0, \infty), \mathcal{H}^q(\mathbb{T}^n))$  the space of all maps from  $[0, \infty)$  to  $\mathcal{H}^q(\mathbb{T}^n)$  that is  $p$  times differentiable.

**Lemma 7.** Given a map  $\mathbf{U} : [0, \infty) \rightarrow \mathcal{H}^q(\mathbb{T}^n)$  and suppose that the derivative  $\mathbf{U}'(t)$  exists. Then for any  $k \in \mathbb{Z}^n$ , the Fourier coefficient  $\widehat{\mathbf{U}}'_k(t)$  of  $\mathbf{U}'(t)$  is the derivative in the classical sense of the Fourier coefficient  $\widehat{\mathbf{U}}_k(t)$  of  $\mathbf{U}(t)$ .

*Proof.* Represent  $\mathbf{U}(t)$  and  $\mathbf{U}'(t)$  through their Fourier series

$$\mathbf{U}(t) = \sum_{k \in \mathbb{Z}^n} \widehat{\mathbf{U}}_k(t) e^{ik \cdot x},$$

and

$$\mathbf{U}'(t) = \sum_{k \in \mathbb{Z}^n} \widehat{\mathbf{U}}'_k(t) e^{ik \cdot x}.$$

From the derivative definition, observe that

$$\begin{aligned} 0 &= \lim_{h \rightarrow 0} \left\| \frac{\mathbf{U}(t+h) - \mathbf{U}(t)}{h} - \mathbf{U}'(t) \right\|_{\mathcal{H}^q(\mathbb{T}^n)}^2 \\ &= \lim_{h \rightarrow 0} \sum_{k \in \mathbb{Z}^n} (1 + \|k\|^2)^q \left| \frac{\widehat{\mathbf{U}}_k(t+h) - \widehat{\mathbf{U}}_k(t)}{h} - \widehat{\mathbf{U}}'_k(t) \right|^2. \end{aligned}$$

The statement implies that

$$\lim_{h \rightarrow 0} \frac{\widehat{\mathbf{U}}_k(t+h) - \widehat{\mathbf{U}}_k(t)}{h} = \widehat{\mathbf{U}}'_k(t).$$

for any  $k \in \mathbb{Z}^n$ . This is the definition for derivative in the classical sense.  $\square$

**Definition 5.** Suppose  $\mathbf{U} : [0, \infty) \rightarrow \mathcal{H}^q(\mathbb{T}^n)$  such that for any  $t \geq 0$ ,  $\mathbf{U}(t)$  has the form

$$\mathbf{U}(t) = \sum_{k \in \mathbb{Z}^n} \widehat{\mathbf{U}}_k(t) e^{ik \cdot x}.$$

Let  $\mathcal{L}^{\delta, \beta}$  and  $\mathcal{N}$  be the peridynamic operator and the Navier operator respectively, then define the maps  $\mathcal{L}^{\delta, \beta} \mathbf{U}$ ,  $\mathcal{N} \mathbf{U} : [0, \infty) \rightarrow \mathcal{H}^q(\mathbb{T}^n)$  through the representation

$$\mathcal{L}^{\delta, \beta} \mathbf{U}(t) = \sum_{k \in \mathbb{Z}^n} M_k^{\delta, \beta} \widehat{\mathbf{U}}_k(t) e^{ik \cdot x}, \quad \text{and} \quad \mathcal{N} \mathbf{U}(t) = \sum_{k \in \mathbb{Z}^n} M_k^{\mathcal{N}} \widehat{\mathbf{U}}_k(t) e^{ik \cdot x},$$

for any  $t \geq 0$ .

In particular, when  $q < 0$ , the above notion extends the definition of the peridynamic operator  $\mathcal{L}^{\delta, \beta}$  to periodic distributions.

### 3.3 The peridynamic equilibrium equation

In this section, we present regularity results for the solutions of the peridynamic equilibrium equation (3.2) and study their nonlocal-to-local convergence. The equilibrium problem (3.2) defined over the space of periodic distributions is given by

$$\mathcal{L}^{\delta,\beta} \mathbf{u}^{\delta,\beta} = \mathbf{b}. \quad (3.21)$$

**Theorem 27.** *Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta < n + 2$ . If the vector field  $\mathbf{b} \in \mathcal{H}^{s_0}(\mathbb{T}^n)$  with  $s_0 \in \mathbb{R}$  is such that  $\widehat{\mathbf{b}}_0 = \mathbf{0}$ , then there is a unique vector field  $\mathbf{u}^{\delta,\beta} \in \mathcal{H}^s(\mathbb{T}^n)$  satisfying  $\mathcal{L}^{\delta,\beta} \mathbf{u}^{\delta,\beta} = \mathbf{b}$  and  $\widehat{\mathbf{u}}_0^{\delta,\beta} = \mathbf{0}$ , where  $s = s_0 + \max\{0, \beta - n\}$ .*

*Proof.* Represent  $\mathbf{b}$  and  $\mathbf{u}^{\delta,\beta}$  through their Fourier series and then substitute into (3.21) yields

$$\sum_{k \in \mathbb{Z}^n} \widehat{\mathbf{b}}_k e^{ik \cdot x} = \mathcal{L}^{\delta,\beta} \left( \sum_{k \in \mathbb{Z}^n} \widehat{\mathbf{u}}_k^{\delta,\beta} e^{ik \cdot x} \right) = \sum_{k \in \mathbb{Z}^n} M_k^{\delta,\beta} \widehat{\mathbf{u}}_k^{\delta,\beta} e^{ik \cdot x}.$$

This shows that for any nonzero  $k$ , the coefficient vector  $\widehat{\mathbf{u}}_k$  is uniquely determined through the formula

$$\widehat{\mathbf{u}}_k^{\delta,\beta} = (M_k^{\delta,\beta})^{-1} \widehat{\mathbf{b}}_k. \quad (3.22)$$

It remains to show that  $\mathbf{u}$  is in  $\mathcal{H}^s(\mathbb{T}^n)$  where  $s = s_0 + \max\{0, \beta - n\}$ . Observe that

$$\sum_{k \in \mathbb{Z}^n} (1 + \|k\|^2)^s \|\widehat{\mathbf{u}}_k^{\delta,\beta}\|^2 \leq \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^{s_0} \|\widehat{\mathbf{b}}_k\|^2 (1 + \|k\|^2)^{s-s_0} \|(M_k^{\delta,\beta})^{-1}\|^2.$$

Since  $\mathbf{b} \in \mathcal{H}^{s_0}(\mathbb{T}^n)$ , it suffices to prove the uniform boundedness of

$$(1 + \|k\|^2)^{s-s_0} \|(M_k^{\delta,\beta})^{-1}\|^2,$$

for large value of  $\|k\|$ . Since  $\lambda_1^{\delta,\beta}(k)$  and  $\lambda_2^{\delta,\beta}(k)$  are eigenvalues of  $M_k^{\delta,\beta}$ , the eigenvalues of

$(M_k^{\delta,\beta})^{-1}$  are  $1/\lambda_1^{\delta,\beta}(k)$ ,  $1/\lambda_2^{\delta,\beta}(k)$ , for  $k \neq 0$ . This implies that

$$\|(M_k^{\delta,\beta})^{-1}\| = \max \left\{ \frac{1}{|\lambda_1^{\delta,\beta}(k)|}, \frac{1}{|\lambda_2^{\delta,\beta}(k)|} \right\} = \frac{1}{\min\{|\lambda_1^{\delta,\beta}(k)|, |\lambda_2^{\delta,\beta}(k)|\}}.$$

We then consider two cases. First, when  $\beta \leq n$ , then  $s = s_0$ . Using Theorem 20 and Theorem 23, there exists  $r_1 > 0$  and  $C_1 > 0$  such that  $\|(M_k^{\delta,\beta})^{-1}\| \leq C_1$  whenever  $\|k\| \geq r_1$ . This shows that

$$(1 + \|k\|^2)^{s-s_0} \|(M_k^{\delta,\beta})^{-1}\|^2 \leq C_1^2.$$

For the case when  $\beta > n$ , we have  $s = s_0 + \beta - n$ . Again from Theorem 20 and Theorem 23, there exists  $r_2 > 0$  and  $C_2 > 0$  such that  $\|(M_k^{\delta,\beta})^{-1}\| \leq \frac{C_2}{\|k\|^{\beta-n}}$  whenever  $\|k\| \geq r_2$ . This shows that

$$(1 + \|k\|^2)^{s-s_0} \|(M_k^{\delta,\beta})^{-1}\|^2 \leq C_2^2 \left( \frac{1 + \|k\|^2}{\|k\|^2} \right)^{\beta-n},$$

which is uniformly bounded, completing the proof.  $\square$

It is worth observing that this regularity theorem implies that the solution operator is non-smoothing when the kernel is integrable ( $\beta \leq n$ ).

For the next two theorems, we provide some results on the convergence of the equilibrium peridynamic equation to the corresponding equilibrium equation of linear elasticity under two types of limits as  $\delta \rightarrow 0$  with fixed  $\beta < n + 2$  or as  $\beta \rightarrow (n + 2)^-$  with fixed  $\delta > 0$ .

**Theorem 28.** *Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta < n + 2$ . Suppose  $\mathbf{b} \in \mathcal{H}^{s_0}(\mathbb{T}^n)$  with  $s_0 \in \mathbb{R}$  satisfying  $\widehat{\mathbf{b}}_0 = \mathbf{0}$ . Let  $\mathbf{u}^\mathcal{N} \in \mathcal{H}^{s_0+2}(\mathbb{T}^n)$  be the solution of the Navier equation  $\mathcal{N}\mathbf{u}^\mathcal{N} = \mathbf{b}$  with  $\widehat{\mathbf{u}}_0^\mathcal{N} = \mathbf{0}$ , where  $\mathcal{N}$  is given by (3.12). Let  $\mathbf{u}^{\delta,\beta} \in \mathcal{H}^s(\mathbb{T}^n)$  be the solution of  $\mathcal{L}^{\delta,\beta}\mathbf{u}^{\delta,\beta} = \mathbf{b}$  with  $\widehat{\mathbf{u}}_0^{\delta,\beta} = \mathbf{0}$  where  $s = s_0 + \max\{0, \beta - n\}$ . Then, for fixed  $\beta < n + 2$ ,*

$$\lim_{\delta \rightarrow 0^+} \mathbf{u}^{\delta,\beta} = \mathbf{u}^\mathcal{N}, \quad \text{in } \mathcal{H}^s(\mathbb{T}^n).$$

*Proof.* From 3.22, we see that

$$\begin{aligned}\|\mathbf{u}^{\delta,\beta} - \mathbf{u}^{\mathcal{N}}\|_{\mathcal{H}^s(\mathbb{T}^n)}^2 &= \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^s \|\widehat{\mathbf{u}}_k^{\delta,\beta} - \widehat{\mathbf{u}}_k^{\mathcal{N}}\|^2 \\ &\leq \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^s \|(M_k^{\delta,\beta})^{-1} - (M_k^{\mathcal{N}})^{-1}\|^2 \|\widehat{\mathbf{b}}_k\|^2.\end{aligned}$$

Each term in the summation converges to 0 under the limit using Proposition 3. The proof is then complete if we can pass the limit inside the summation. Since  $\mathbf{b} \in \mathcal{H}^{s_0}(\mathbb{T}^n)$ , it remains to show that  $(1 + \|k\|^2)^{s-s_0} \|(M_k^{\delta,\beta})^{-1} - (M_k^{\mathcal{N}})^{-1}\|^2$  or equivalently,  $\|k\|^{s-s_0} \|(M_k^{\delta,\beta})^{-1} - (M_k^{\mathcal{N}})^{-1}\|$  is uniformly bounded for large value of  $\|k\|$ . Note that

$$\|(M_k^{\delta,\beta})^{-1} - (M_k^{\mathcal{N}})^{-1}\| = \frac{1}{\min\{|\lambda_1(k) - \lambda_1^{\mathcal{N}}(k)|, |\lambda_2(k) - \lambda_2^{\mathcal{N}}(k)|\}}.$$

Using Theorem 20 and Theorem 23, one can show that there exists  $r, C_1, C_2, C_3 > 0$  such that whenever  $\|k\| \geq r$

$$|\lambda_1^{\delta,\beta}(k)|, |\lambda_2^{\delta,\beta}(k)| \leq \begin{cases} C_1 & \text{if } \beta < n \\ C_2 \log \|k\| & \text{if } \beta = n \\ C_3 \|k\|^{\beta-n} & \text{if } n < \beta < n+2 \end{cases}.$$

Since  $|\lambda_1^{\mathcal{N}}(k)| = |\lambda^* + \mu| \|k\|^2$  and  $|\lambda_2^{\mathcal{N}}(k)| = \mu \|k\|^2$ , respectively, there is  $C > 0$  such that for large  $\|k\|$ ,

$$\|(M_k^{\delta,\beta})^{-1} - (M_k^{\mathcal{N}})^{-1}\| \leq \frac{C}{\|k\|^2}. \quad (3.23)$$

This combined with the fact that

$$\|k\|^{s-s_0} = \|k\|^{\max\{0, \beta-n\}} \leq \|k\|^2,$$

yields the desired uniform bound. □

**Theorem 29.** Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta < n + 2$ . Suppose  $\mathbf{b} \in \mathcal{H}^{s_0}(\mathbb{T}^n)$  with  $s_0 \in \mathbb{R}$  satisfying  $\widehat{\mathbf{b}}_0 = \mathbf{0}$ . Let  $\mathbf{u}^{\mathcal{N}} \in \mathcal{H}^{s_0+2}(\mathbb{T}^n)$  be the solution the the Navier equation  $\mathcal{N}\mathbf{u}^{\mathcal{N}} = \mathbf{b}$  with  $\widehat{\mathbf{u}}_0^{\mathcal{N}} = \mathbf{0}$ , where  $\mathcal{N}$  is given by (3.12). Let  $\mathbf{u}^{\delta,\beta} \in H^s(\mathbb{T}^n)$  be the solution of the peridynamic equation  $\mathcal{L}^{\delta,\beta}\mathbf{u}^{\delta,\beta} = \mathbf{b}$  with  $\widehat{\mathbf{u}}_0^{\delta,\beta} = \mathbf{0}$ , where  $s = s_0 + \max\{0, \beta - n\}$ . Then, for  $\delta > 0$  and  $0 < \epsilon < 2$ ,

$$\lim_{\beta \rightarrow n+2^-} \mathbf{u}^{\delta,\beta} = \mathbf{u}^{\mathcal{N}}, \quad \text{in } \mathcal{H}^{s_\epsilon}(\mathbb{T}^n),$$

where  $s_\epsilon = s_0 + 2 - \epsilon$ .

*Proof.* Fix an  $\epsilon \in (0, 2)$ . Then for any  $n + 2 - \epsilon < \beta < n + 2$ , both  $s_0 + 2$  and  $s = s_0 + \beta - n$  are larger than  $s_\epsilon$ , hence the limit makes sense. Similar to Theorem 28, we have that

$$\begin{aligned} \|\mathbf{u}^{\delta,\beta} - \mathbf{u}^{\mathcal{N}}\|_{\mathcal{H}^{s_\epsilon}(\mathbb{T}^n)}^2 &= \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^{s_\epsilon} \|\widehat{\mathbf{u}}_k^{\delta,\beta} - \widehat{\mathbf{u}}_k^{\mathcal{N}}\|^2 \\ &\leq \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^{s_\epsilon} \|(M_k^{\delta,\beta})^{-1} - (M_k^{\mathcal{N}})^{-1}\|^2 \|\widehat{\mathbf{b}}_k\|^2. \end{aligned}$$

The proof is completed by showing the uniform boundedness of

$$\|k\|^{2-\epsilon} \|(M_k^{\delta,\beta})^{-1} - (M_k^{\mathcal{N}})^{-1}\|.$$

Using (3.23), we obtain

$$\|k\|^{2-\epsilon} \|(M_k^{\delta,\beta})^{-1} - (M_k^{\mathcal{N}})^{-1}\| \leq \frac{C}{\|k\|^\epsilon},$$

which is uniformly bounded. □

### 3.4 The peridynamic evolution equation

In this section, we study regularity results for the solution of the peridynamic evolution equation (3.1) and their nonlocal-to-local convergence. To extend the problem to the space of periodic distributions, we consider the identification  $\mathbf{U}(t) = \mathbf{u}(\cdot, t)$ , with  $\mathbf{U} : [0, \infty) \rightarrow$

$\mathcal{H}^q(\mathbb{T}^n)$ , for some  $q \in \mathbb{R}$ . Following this identification, equation (3.1) becomes

$$\begin{cases} \frac{d^2}{dt^2} \mathbf{U}^{\delta, \beta}(t) &= \mathcal{L}^{\delta, \beta} \mathbf{U}^{\delta, \beta}(t) + \mathbf{b}, \quad t > 0 \\ \mathbf{U}^{\delta, \beta}(0) &= \mathbf{f}, \\ \frac{d}{dt} \mathbf{U}^{\delta, \beta}(0) &= \mathbf{g}. \end{cases} \quad (3.24)$$

In the following sections, spatial regularity refers to  $\mathbf{U}^{\delta, \beta}(t)$  being in  $\mathcal{H}^q(\mathbb{T}^n)$ , for some  $q \in \mathbb{R}$ , for a fixed  $t \geq 0$ . Temporal regularity refers to the differentiability of  $\mathbf{U}^{\delta, \beta}$  as a function of  $t$ . For the matrix notation used in this section, such as  $\cos(\sqrt{-M_k^{\delta, \beta}} t)$ , see Section 3.4.1.

### 3.4.1 Tensor Fourier multipliers

From Theorem 19 and the assumption that  $\lambda_1^{\delta, \beta}(k) < 0$  and  $\lambda_2^{\delta, \beta}(k) < 0$  for any nonzero  $k$ , we have that  $M_k^{\delta, \beta}$  is a real symmetric invertible matrix. Hence,  $M_k^{\delta, \beta}$  can be orthogonally diagonalized as

$$M_k^{\delta, \beta} = P_k \begin{bmatrix} \lambda_1^{\delta, \beta}(k) & 0 & \cdots & 0 \\ 0 & \lambda_2^{\delta, \beta}(k) & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \lambda_2^{\delta, \beta}(k) \end{bmatrix} P_k^T,$$

where  $P_k$  is the matrix whose columns are the orthogonal eigenvectors of  $M_k^{\delta, \beta}$ . Then we can define the following matrices  $\sqrt{-M_k^{\delta, \beta}}$ ,  $\cos(\sqrt{-M_k^{\delta, \beta}} t)$  and  $\sin(\sqrt{-M_k^{\delta, \beta}} t)$  as

$$\sqrt{-M_k^{\delta, \beta}} := P_k \begin{bmatrix} \sqrt{-\lambda_1^{\delta, \beta}(k)} & 0 & \cdots & 0 \\ 0 & \sqrt{-\lambda_2^{\delta, \beta}(k)} & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \sqrt{-\lambda_2^{\delta, \beta}(k)} \end{bmatrix} P_k^T := P_k \Lambda_k^{\delta, \beta} P_k^T,$$

and

$$\cos(\sqrt{-M_k^{\delta,\beta}}t) := P_k \begin{bmatrix} \cos(\sqrt{-\lambda_1^{\delta,\beta}(k)t}) & 0 & \cdots & 0 \\ 0 & \cos(\sqrt{-\lambda_2^{\delta,\beta}(k)t}) & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \cos(\sqrt{-\lambda_2^{\delta,\beta}(k)t}) \end{bmatrix} P_k^T,$$

and

$$\sin(\sqrt{-M_k^{\delta,\beta}}t) := P_k \begin{bmatrix} \sin(\sqrt{-\lambda_1^{\delta,\beta}(k)t}) & 0 & \cdots & 0 \\ 0 & \sin(\sqrt{-\lambda_2^{\delta,\beta}(k)t}) & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \sin(\sqrt{-\lambda_2^{\delta,\beta}(k)t}) \end{bmatrix} P_k^T.$$

From these definitions, one can show that for any  $k \in \mathbb{Z} \setminus \{0\}$  and  $t \geq 0$ ,

$$\|(\sqrt{-M_k^{\delta,\beta}})^{-1}\| = \max \left\{ \frac{1}{\sqrt{-\lambda_1^{\delta,\beta}(k)}}, \frac{1}{\sqrt{-\lambda_2^{\delta,\beta}(k)}} \right\} = \frac{1}{\sqrt{\min\{|\lambda_1^{\delta,\beta}(k)|, |\lambda_2^{\delta,\beta}(k)|\}}},$$

and

$$\begin{aligned} \|\cos(\sqrt{-M_k^{\delta,\beta}}t)\| &= \max\{|\cos(\sqrt{-\lambda_1^{\delta,\beta}(k)t})|, |\cos(\sqrt{-\lambda_2^{\delta,\beta}(k)t})|\}, \\ \|\sin(\sqrt{-M_k^{\delta,\beta}}t)\| &= \max\{|\sin(\sqrt{-\lambda_1^{\delta,\beta}(k)t})|, |\sin(\sqrt{-\lambda_2^{\delta,\beta}(k)t})|\}. \end{aligned}$$

These observations imply that  $\|\cos(\sqrt{-M_k^{\delta,\beta}}t)\| \leq 1$  and  $\|\sin(\sqrt{-M_k^{\delta,\beta}}t)\| \leq 1$  for any  $k \in \mathbb{Z} \setminus \{0\}$  and  $t \geq 0$ . Taking the classical derivative with respect to  $t$ , we have that

$$\begin{aligned} & \frac{d^p}{dt^p} \begin{bmatrix} \cos(\sqrt{-\lambda_1^{\delta,\beta}(k)t}) & 0 & \cdots & 0 \\ 0 & \cos(\sqrt{-\lambda_2^{\delta,\beta}(k)t}) & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \cos(\sqrt{-\lambda_2^{\delta,\beta}(k)t}) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\sqrt{-\lambda_1^{\delta,\beta}(k)t + \frac{p\pi}{2}}) & 0 & \cdots & 0 \\ 0 & \cos(\sqrt{-\lambda_2^{\delta,\beta}(k)t + \frac{p\pi}{2}}) & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \cos(\sqrt{-\lambda_2^{\delta,\beta}(k)t + \frac{p\pi}{2}}) \end{bmatrix} (\Lambda_k^{\delta,\beta})^p. \end{aligned}$$

This implies that

$$\frac{d^p}{dt^p} \cos(\sqrt{-M_k^{\delta,\beta}}t) = \cos(\sqrt{-M_k^{\delta,\beta}}t + p\pi/2)(\sqrt{-M_k^{\delta,\beta}})^p.$$

Similarly, we also have

$$\frac{d^p}{dt^p} \sin(\sqrt{-M_k^{\delta,\beta}}t) := \sin(\sqrt{-M_k^{\delta,\beta}}t + p\pi/2)(\sqrt{-M_k^{\delta,\beta}})^p.$$

### 3.4.2 Spatial and temporal regularity over periodic distributions

Consider the vector fields  $\mathbf{b} \in \mathcal{H}^{s_0}(\mathbb{T}^n)$ ,  $\mathbf{f} \in \mathcal{H}^{s_1}(\mathbb{T}^n)$ , and  $\mathbf{g} \in \mathcal{H}^{s_2}(\mathbb{T}^n)$  with  $s_0, s_1, s_2 \in \mathbb{R}$ . Then, for any  $t \geq 0$ , define

$$\mathbf{U}^{\delta,\beta}(t) := \widehat{\mathbf{f}}_0 + \widehat{\mathbf{g}}_0 t + \widehat{\mathbf{b}}_0 \frac{t^2}{2} + \sum_{0 \neq k \in \mathbb{Z}^n} \widehat{\mathbf{U}}_k^{\delta,\beta}(t) e^{ik \cdot x}, \quad (3.25)$$

with

$$\begin{aligned}\widehat{\mathbf{U}}_k^{\delta,\beta}(t) &= \cos(\sqrt{-M_k^{\delta,\beta}}t)\widehat{\mathbf{f}}_k + \sin(\sqrt{-M_k^{\delta,\beta}}t)(\sqrt{-M_k^{\delta,\beta}})^{-1}\widehat{\mathbf{g}}_k \\ &\quad + \left[ \cos\left(\sqrt{-M_k^{\delta,\beta}}t\right) - I \right] (M_k^{\delta,\beta})^{-1}\widehat{\mathbf{b}}_k.\end{aligned}\tag{3.26}$$

We will show in Theorem 32 that  $\mathbf{U}^{\delta,\beta}$  in (3.25)-(3.26) is the unique solution to the peridynamic equation (3.24). We proceed by studying the spatial and temporal regularity of  $\mathbf{U}^{\delta,\beta}$ .

We recall that we are using the matrix operator norm throughout the chapter.

**Theorem 30** (Spatial regularity). *Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta < n + 2$ . Assume that  $\mathbf{b} \in \mathcal{H}^{s_0}(\mathbb{T}^n)$ ,  $\mathbf{f} \in \mathcal{H}^{s_1}(\mathbb{T}^n)$ , and  $\mathbf{g} \in \mathcal{H}^{s_2}(\mathbb{T}^n)$  with  $s_0, s_1, s_2 \in \mathbb{R}$ . Then, for any  $t \geq 0$ ,  $\mathbf{U}^{\delta,\beta}(t) \in \mathcal{H}^s(\mathbb{T}^n)$ , where  $\mathbf{U}^{\delta,\beta}$  is given by (3.25)-(3.26),  $s = \min\{s_1, s_2 + \theta, s_0 + 2\theta\}$ , and  $\theta = \max\{0, \frac{\beta-n}{2}\}$ .*

*Proof.* It is straightforward to see that (check Section 3.4.1)

$$\left\| \cos\left(\sqrt{-M_k^{\delta,\beta}}t\right) - I \right\| = \max\{|\cos(\sqrt{-\lambda_1(k)}t) - 1|, |\cos(\sqrt{-\lambda_2(k)}t) - 1|\} \leq 2.$$

Using (3.26), it follows that

$$\begin{aligned}\sum_{k \in \mathbb{Z}^n} (1 + \|k\|^2)^s \|\widehat{\mathbf{U}}_k^{\delta,\beta}(t)\|^2 &\leq \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^s \|\widehat{\mathbf{f}}_k\|^2 \\ &\quad + \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^s \left\| (\sqrt{-M_k^{\delta,\beta}})^{-1} \right\|^2 \|\widehat{\mathbf{g}}_k\|^2 \\ &\quad + \sum_{0 \neq k \in \mathbb{Z}^n} 4(1 + \|k\|^2)^s \|(M_k^{\delta,\beta})^{-1}\|^2 \|\widehat{\mathbf{b}}_k\|^2 \\ &\quad + \|\widehat{\mathbf{f}}_0\|^2 + \|\widehat{\mathbf{g}}_0\|^2 t^2 + \|\widehat{\mathbf{b}}_0\|^2 \frac{t^4}{4}.\end{aligned}$$

Since  $\mathbf{f} \in \mathcal{H}^{s_1}$  and  $s \leq s_1$ , then

$$\sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^s \|\widehat{\mathbf{f}}_k\|^2 \leq \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^{s_1} \|\widehat{\mathbf{f}}_k\|^2 < \infty.$$

Similarly, observe that

$$\begin{aligned}
\sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^s \left\| (\sqrt{-M_k^{\delta, \beta}})^{-1} \right\|^2 \|\widehat{\mathbf{g}}_k\|^2 \\
= \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^{s-s_2} \left\| (\sqrt{-M_k^{\delta, \beta}})^{-1} \right\|^2 (1 + \|k\|^2)^{s_2} \|\widehat{\mathbf{g}}_k\|^2, \\
\sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^s \|(M_k^{\delta, \beta})^{-1}\|^2 \|\widehat{\mathbf{b}}_k\|^2 = \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^{s-s_0} \|(M_k^{\delta, \beta})^{-1}\|^2 (1 + \|k\|^2)^{s_0} \|\widehat{\mathbf{b}}_k\|^2.
\end{aligned}$$

Hence,  $\mathbf{g} \in \mathcal{H}^{s_1}$  and  $\mathbf{b} \in \mathcal{H}^{s_0}$ , the proof will be complete provided that

$$(1 + \|k\|^2)^{s-s_2} \left\| (\sqrt{-M_k^{\delta, \beta}})^{-1} \right\|^2 \quad \text{and} \quad (1 + \|k\|^2)^{s-s_0} \|(M_k^{\delta, \beta})^{-1}\|^2,$$

are bounded for large values of  $\|k\|$ .

To show this, we will consider two cases. First, when  $\beta \leq n$ , then  $s \leq s_2$  and  $s \leq s_0$ . By Theorem 20 and Theorem 23, there exists  $r_1 > 0$  and  $C_1 > 0$  such that  $\left\| (\sqrt{-M_k^{\delta, \beta}})^{-1} \right\|^2 \leq C_1$  and  $\|(M_k^{\delta, \beta})^{-1}\|^2 \leq C_1$  for  $\|k\| \geq r_1$ . This leads to

$$(1 + \|k\|^2)^{s-s_2} \left\| (\sqrt{-M_k^{\delta, \beta}})^{-1} \right\|^2 \leq C_1 \quad \text{and} \quad (1 + \|k\|^2)^{s-s_0} \|(M_k^{\delta, \beta})^{-1}\|^2 \leq C_1.$$

Similarly, for  $\beta > n$ , then  $s \leq s_2 + \frac{\beta-n}{2}$  and  $s \leq s_0 + \beta - n$ . By Theorem 20 and Theorem 23, there exists  $r_2 > 0$  and  $C_2 > 0$  such that  $\left\| (\sqrt{-M_k^{\delta, \beta}})^{-1} \right\|^2 \leq C_2/(\|k\|^{\beta-n})$  and  $\|(M_k^{\delta, \beta})^{-1}\|^2 \leq C_2/\|k\|^{2(\beta-n)}$  for  $\|k\| \geq r_2$ . This leads to

$$\begin{aligned}
(1 + \|k\|^2)^{s-s_2} \left\| (\sqrt{-M_k^{\delta, \beta}})^{-1} \right\|^2 &\leq C_2 \left( \frac{1 + \|k\|^2}{\|k\|^2} \right)^{\frac{\beta-n}{2}}, \text{ and} \\
(1 + \|k\|^2)^{s-s_0} \|(M_k^{\delta, \beta})^{-1}\|^2 &\leq C_2 \left( \frac{1 + \|k\|^2}{\|k\|^2} \right)^{\beta-n},
\end{aligned}$$

which is bounded.

□

**Theorem 31.** *Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta < n + 2$ . Assume that  $\mathbf{b} \in \mathcal{H}^{s_0}(\mathbb{T}^n)$ ,  $\mathbf{f} \in \mathcal{H}^{s_1}(\mathbb{T}^n)$  and*

$\mathbf{g} \in \mathcal{H}^{s_2}(\mathbb{T}^n)$  with  $s_0, s_1, s_2 \in \mathbb{R}$ . Let  $\mathbf{U}^{\delta, \beta}$  be the map given by (3.25)-(3.26). Then,

1. if  $\beta < n$ , then  $\mathbf{U}^{\delta, \beta} \in C^\infty([0, \infty), \mathcal{H}^q(\mathbb{T}^n))$ , for any  $q \leq \min\{s_1, s_2, s_0\}$ ,
2. if  $\beta = n$ , then  $\mathbf{U}^{\delta, \beta} \in C^\infty([0, \infty), \mathcal{H}^q(\mathbb{T}^n))$ , for any  $q < \min\{s_1, s_2, s_0\}$ , and
3. if  $\beta > n$ , then  $\mathbf{U}^{\delta, \beta} \in C^{p+1}([0, \infty), \mathcal{H}^q(\mathbb{T}^n))$ , for any  $q \in \mathbb{R}$  and any positive integer  $p$  satisfying

$$q - s_1 + (p+1)\frac{\beta - n}{2} \leq 0, \quad q - s_2 + p\frac{\beta - n}{2} \leq 0, \quad \text{and} \quad q - s_0 + (p-1)\frac{\beta - n}{2} \leq 0.$$

*Proof.* We will prove this result by induction. Suppose  $\mathbf{U}^{\delta, \beta}$  is already differentiable up to  $p$  times, then by Lemma 7, we have

$$\frac{d^p}{dt^p} \mathbf{U}^{\delta, \beta}(t) = \sum_{k \in \mathbb{Z}^n} \frac{d^p}{dt^p} \widehat{\mathbf{U}}_k^{\delta, \beta}(t) e^{ik \cdot x},$$

where  $\frac{d^p}{dt^p} \widehat{\mathbf{U}}_k(t)$  is given by

$$\begin{aligned} \frac{d^p}{dt^p} \widehat{\mathbf{U}}_k(t) = & \cos\left(\sqrt{-M_k^{\delta, \beta}}t + \frac{p\pi}{2}\right) (\sqrt{-M_k^{\delta, \beta}})^p \widehat{\mathbf{f}}_k + \sin\left(\sqrt{-M_k^{\delta, \beta}}t + \frac{p\pi}{2}\right) (\sqrt{-M_k^{\delta, \beta}})^{p-1} \widehat{\mathbf{g}}_k \\ & + \cos\left(\sqrt{-M_k^{\delta, \beta}}t + \frac{p\pi}{2}\right) (\sqrt{-M_k^{\delta, \beta}})^{p-2} \widehat{\mathbf{b}}_k, \end{aligned}$$

for  $k \neq 0$ , and for  $k = 0$ ,

$$\frac{d^p}{dt^p} \widehat{\mathbf{U}}_0(t) = \begin{cases} \widehat{\mathbf{g}}_0 + \widehat{\mathbf{b}}_0 t, & p = 1, \\ \widehat{\mathbf{b}}_0, & p = 2, \\ \mathbf{0}, & p \geq 3. \end{cases}$$

We will show that

$$\frac{d^{p+1}}{dt^{p+1}} \mathbf{U}(t) = \sum_{k \in \mathbb{Z}^n} \frac{d^{p+1}}{dt^{p+1}} \widehat{\mathbf{U}}_k(t) e^{ik \cdot x}.$$

Using the mean value theorem, we have that

$$\begin{aligned}
T_h &:= \left\| \frac{\frac{d^p}{dt^p} \mathbf{U}(t+h) - \frac{d^p}{dt^p} \mathbf{U}(t)}{h} - \sum_{k \in \mathbb{Z}^n} \frac{d^{p+1}}{dt^{p+1}} \widehat{\mathbf{U}}_k(t) e^{ik \cdot x} \right\|_{\mathcal{H}^q(\mathbb{T}^n)}^2 \\
&= \sum_{k \in \mathbb{Z}^n} (1 + \|k\|^2)^q \left| \frac{\frac{d^p}{dt^p} \widehat{\mathbf{U}}_k(t+h) - \frac{d^p}{dt^p} \widehat{\mathbf{U}}_k(t)}{h} - \frac{d^{p+1}}{dt^{p+1}} \widehat{\mathbf{U}}_k(t) \right|^2 \\
&\leq \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^q \left\| \cos \left( \sqrt{-M_k^{\delta, \beta}}(t + \xi_{p,k}) + \frac{(p+1)\pi}{2} \right) \right. \\
&\quad \left. - \cos \left( \sqrt{-M_k^{\delta, \beta}}t + \frac{(p+1)\pi}{2} \right) \right\|^2 \left\| \sqrt{-M_k^{\delta, \beta}} \right\|^{2(p+1)} \|\widehat{\mathbf{f}}_k\|^2 \\
&\quad + \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^q \left\| \sin \left( \sqrt{-M_k^{\delta, \beta}}(t + \eta_{p,k}) + \frac{(p+1)\pi}{2} \right) \right. \\
&\quad \left. - \sin \left( \sqrt{-M_k^{\delta, \beta}}t + \frac{(p+1)\pi}{2} \right) \right\|^2 \left\| \sqrt{-M_k^{\delta, \beta}} \right\|^{2p} \|\widehat{\mathbf{g}}_k\|^2 \\
&\quad + \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^q \left\| \cos \left( \sqrt{-M_k^{\delta, \beta}}(t + \zeta_{p,k}) + \frac{(p+1)\pi}{2} \right) \right. \\
&\quad \left. - \cos \left( \sqrt{-M_k^{\delta, \beta}}t + \frac{(p+1)\pi}{2} \right) \right\|^2 \left\| \sqrt{-M_k^{\delta, \beta}} \right\|^{2(p-1)} \|\widehat{\mathbf{b}}_k\|^2,
\end{aligned}$$

for some  $\xi_{p,k}, \eta_{p,k}, \zeta_{p,k} \in (0, h)$ . As  $h \rightarrow 0$ , each term of the summation will approach 0, the proof will be complete by showing that we can pass the limit inside the summation. To show this, it suffices to prove the uniform boundedness of each the following three expressions

$$\begin{aligned}
a_k &:= (1 + \|k\|^2)^{q-s_1} \left\| \sqrt{-M_k^{\delta, \beta}} \right\|^{2(p+1)}, \\
b_k &:= (1 + \|k\|^2)^{q-s_2} \left\| \sqrt{-M_k^{\delta, \beta}} \right\|^{2p}, \\
c_k &:= (1 + \|k\|^2)^{q-s_0} \left\| \sqrt{-M_k^{\delta, \beta}} \right\|^{2(p-1)}.
\end{aligned}$$

To uniformly bound each of these expressions, we consider three cases  $\beta < n$ ,  $\beta = n$ , and  $\beta > n$ .

For  $\beta < n$  and from Theorem 20 and Theorem 23,  $\left\| \sqrt{-M_k^{\delta, \beta}} \right\|$  is bounded, and this provides the uniform boundedness of  $a_k$ ,  $b_k$  and  $c_k$  provided that  $q \leq \min\{s_0, s_1, s_2\}$ .

For  $\beta = n$ , let us fix an  $\epsilon > 0$ . Theorem 20 and Theorem 23 imply that there exists constants  $C, r > 0$  such that  $\left\| \sqrt{-M_k^{\delta, \beta}} \right\|^2 \leq C \log \|k\| \leq C(1 + \|k\|^2)^{\epsilon/2}$  whenever  $\|k\| \geq r$ . Then for large  $\|k\|$ ,

$$\begin{aligned} a_k &\leq C(1 + \|k\|^2)^{q-s_1+(p+1)\epsilon/2}, \\ b_k &\leq C(1 + \|k\|^2)^{q-s_2+p\epsilon/2}, \text{ and} \\ c_k &\leq C(1 + \|k\|^2)^{q-s_0+(p-1)\epsilon/2}. \end{aligned}$$

This provides the uniform boundedness of  $a_k$ , provided that  $q - s_1 + (p+1)\epsilon/2 \leq 0$ , and the uniform boundedness of  $b_k$ , provided that  $q - s_2 + p\epsilon/2 \leq 0$ , and the uniform boundedness of  $c_k$ , provided that  $q - s_0 + (p-1)\epsilon/2 \leq 0$ . Thus, given  $q < \min\{s_1, s_2, s_0\}$ , we can pick  $\epsilon$  satisfying these conditions.

For  $\beta > n$ , from Theorem 20 and Theorem 23, there is  $C, r > 0$  such that  $\left\| \sqrt{-M_k^{\delta, \beta}} \right\|^2 \leq C\|k\|^{\beta-n} \leq C(1 + \|k\|^2)^{(\beta-n)/2}$  for  $\|k\| \geq r$ . Then, for large  $\|k\|$

$$\begin{aligned} a_k &\leq C(1 + \|k\|^2)^{q-s_1+(p+1)\frac{\beta-n}{2}}, \\ b_k &\leq C(1 + \|k\|^2)^{q-s_2+p\frac{\beta-n}{2}}, \text{ and} \\ c_k &\leq C(1 + \|k\|^2)^{q-s_0+(p-1)\frac{\beta-n}{2}}. \end{aligned}$$

This shows that the uniform boundedness of  $a_k, b_k, c_k$  is established provided that the following conditions are satisfied

$$q - s_1 + (p+1)\frac{\beta-n}{2} \leq 0, \quad q - s_2 + p\frac{\beta-n}{2} \leq 0, \quad \text{and} \quad q - s_0 + (p-1)\frac{\beta-n}{2} \leq 0.$$

□

The next theorem is a direct application of Theorem 30 and Theorem 31. The assumptions are to guarantee that the second derivative exists.

**Theorem 32.** *Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta < n + 2$ . Assume  $\mathbf{f} \in \mathcal{H}^{s_1}(\mathbb{T}^n)$ ,  $\mathbf{g} \in \mathcal{H}^{s_2}(\mathbb{T}^n)$ , and*

$\mathbf{b} \in \mathcal{H}^{s_0}(\mathbb{T}^n)$  with  $s_1, s_2, s_0 \in \mathbb{R}$ . Suppose further that

1.  $q \leq \min\{s_1, s_2, s_0\}$  in the case  $\beta < n$ ,
2.  $q < \min\{s_1, s_2, s_0\}$  in the case  $\beta = n$ ,
3.  $q - s_1 + (\beta - n) \leq 0$  and  $q - s_2 + (\beta - n)/2 \leq 0$  and  $q \leq s_0$  in the case  $\beta > n$ .

Then,  $\mathbf{U}^{\delta, \beta}$  in (3.25)-(3.26) is the unique solution to (3.24). Moreover, when  $\beta \leq n$ , then  $\mathbf{U}^{\delta, \beta} \in C^\infty([0, \infty); \mathcal{H}^s(\mathbb{T}^n))$ , where  $s = \min\{s_1, s_2, s_0\}$ . Furthermore, when  $\beta > n$ , then  $\mathbf{U}^{\delta, \beta} \in C^2([0, \infty); \mathcal{H}^s(\mathbb{T}^n))$ , where  $s = \min\{s_1, s_2 + \frac{\beta - n}{2}, s_0 + \beta - n\}$ .

### 3.4.3 Convergence to the transient Navier-Cauchy equation solution

In this section, we will provide some results on the convergence of the solutions of the peridynamic equation (3.24) to the solution of the corresponding Navier-Cauchy equation for two types of limits as  $\delta \rightarrow 0^+$ , with  $\beta < n + 2$  being fixed, or as  $\beta \rightarrow (n + 2)^-$ , with  $\delta > 0$  being fixed. The transient Navier-Cauchy equation is given by

$$\begin{cases} \frac{d^2}{dt^2} \mathbf{U}^{\mathcal{N}}(t) &= \mathcal{N} \mathbf{U}^{\mathcal{N}}(t) + \mathbf{b}, \quad t > 0 \\ \mathbf{U}^{\mathcal{N}}(0) &= \mathbf{f}, \\ \frac{d}{dt} \mathbf{U}^{\mathcal{N}}(0) &= \mathbf{g}, \end{cases} \quad (3.27)$$

and its solution is given by

$$\begin{aligned} \mathbf{U}^{\mathcal{N}}(t) := & \widehat{\mathbf{f}}_0 + \widehat{\mathbf{g}}_0 t + \widehat{\mathbf{b}}_0 \frac{t^2}{2} + \sum_{0 \neq k \in \mathbb{Z}^n} \left[ \cos \left( \sqrt{-M_k^{\mathcal{N}}} t \right) \widehat{\mathbf{f}}_k \right. \\ & \left. + \sin \left( \sqrt{-M_k^{\mathcal{N}}} t \right) \left( \sqrt{-M_k^{\mathcal{N}}} \right)^{-1} \widehat{\mathbf{g}}_k + \left( \cos \left( \sqrt{-M_k^{\mathcal{N}}} t \right) - I \right) (M_k^{\mathcal{N}})^{-1} \widehat{\mathbf{b}}_k \right] e^{ik \cdot x}. \end{aligned} \quad (3.28)$$

**Theorem 33.** Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta < n + 2$ . Suppose  $\mathbf{f} \in \mathcal{H}^{s_1}(\mathbb{T}^n)$ ,  $\mathbf{g} \in \mathcal{H}^{s_2}(\mathbb{T}^n)$ , and  $\mathbf{b} \in \mathcal{H}^{s_0}(\mathbb{T}^n)$ , for  $s_1, s_2, s_0 \in \mathbb{R}$ . Let  $t \geq 0$ ,  $\mathbf{U}^{\delta, \beta}(t) \in \mathcal{H}^s(\mathbb{T}^n)$  and  $\mathbf{U}^{\mathcal{N}}(t) \in \mathcal{H}^{s_{\mathcal{N}}}(\mathbb{T}^n)$  be the solutions to the peridynamic equation (3.24) and Navier-Cauchy equation (3.27), respectively, with  $s = \min\{s_1, s_2 + \theta, s_0 + 2\theta\}$ ,  $\theta = \max\{0, \frac{\beta-n}{2}\}$ , and  $s_{\mathcal{N}} = \min\{s_1, s_2 + 1, s_0 + 2\}$ . Then, for  $t \geq 0$  and fixed  $\beta < n + 2$ , we have

$$\lim_{\delta \rightarrow 0^+} \mathbf{U}^{\delta, \beta}(t) = \mathbf{U}^{\mathcal{N}}(t), \quad \text{in } \mathcal{H}^s(\mathbb{T}^n).$$

*Proof.* From (3.25), (3.26) and (3.28), one can show that

$$\begin{aligned} \|\mathbf{U}^{\delta, \beta}(t) - \mathbf{U}^{\mathcal{N}}(t)\|_{\mathcal{H}^s(\mathbb{T}^n)}^2 &\leq \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^s \left\| \cos(\sqrt{-M_k^{\delta, \beta}} t) - \cos(\sqrt{-M_k^{\mathcal{N}}} t) \right\|^2 \|\widehat{\mathbf{f}}_k\|^2 \\ &+ \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^s \left\| \sin(\sqrt{-M_k^{\delta, \beta}} t)(\sqrt{-M_k^{\delta, \beta}})^{-1} - \sin(\sqrt{-M_k^{\mathcal{N}}} t)(\sqrt{-M_k^{\mathcal{N}}})^{-1} \right\|^2 \|\widehat{\mathbf{g}}_k\|^2 \\ &+ \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^s \left\| [\cos(\sqrt{-M_k^{\delta, \beta}} t) - I](M_k^{\delta, \beta})^{-1} - [\cos(\sqrt{-M_k^{\mathcal{N}}} t) - I](M_k^{\mathcal{N}})^{-1} \right\|^2 \|\widehat{\mathbf{b}}_k\|^2 \end{aligned}$$

Using Proposition 3, each term tends to 0 under the limits. Hence, the proof is complete if we can pass the limit inside the summation. This can be accomplished using the facts that  $\mathbf{f} \in \mathcal{H}^{s_1}(\mathbb{T}^n)$ ,  $\mathbf{g} \in \mathcal{H}^{s_2}(\mathbb{T}^n)$ , and  $\mathbf{b} \in \mathcal{H}^{s_0}(\mathbb{T}^n)$ , and by establishing the uniform boundedness of the following expressions

$$\|k\|^{s-s_1} \left\| \cos(\sqrt{-M_k^{\delta, \beta}} t) - \cos(\sqrt{-M_k^{\mathcal{N}}} t) \right\|,$$

$$\|k\|^{s-s_2} \left\| \sin(\sqrt{-M_k^{\delta, \beta}} t)(\sqrt{-M_k^{\delta, \beta}})^{-1} - \sin(\sqrt{-M_k^{\mathcal{N}}} t)(\sqrt{-M_k^{\mathcal{N}}})^{-1} \right\|,$$

and

$$\|k\|^{s-s_0} \left\| [\cos(\sqrt{-M_k^{\delta, \beta}} t) - I](M_k^{\delta, \beta})^{-1} - [\cos(\sqrt{-M_k^{\mathcal{N}}} t) - I](M_k^{\mathcal{N}})^{-1} \right\|.$$

as  $\|k\| \rightarrow \infty$ .

It is straightforward to see that

$$\left\| \cos(\sqrt{-M_k^{\delta,\beta}}t) - \cos(\sqrt{-M_k^{\mathcal{N}}}t) \right\| = \max_{i=1,2} \left| \cos(\sqrt{-\lambda_i(k)}t) - \cos(\sqrt{-\lambda_i^{\mathcal{N}}(k)}t) \right| \leq 2.$$

This provides the uniform bound for the first term since  $s \leq s_1$ . From Theorem 20 and Theorem 23, there exists  $r, C_1, C_2 > 0$  such that

$$|\lambda_1^{\delta,\beta}(k)|, |\lambda_2^{\delta,\beta}(k)| \geq \begin{cases} C_1 & \text{if } \beta \leq n \\ C_2 \|k\|^{\beta-n} & \text{if } \beta > n \end{cases},$$

for any  $\|k\| \geq r$ . Note that

$$\begin{aligned} & \left\| \sin(\sqrt{-M_k^{\delta,\beta}}t)(\sqrt{-M_k^{\delta,\beta}})^{-1} - \sin(\sqrt{-M_k^{\mathcal{N}}}t)(\sqrt{-M_k^{\mathcal{N}}})^{-1} \right\| \\ &= \max_{i=1,2} \left| \frac{\sin(\sqrt{-\lambda_i^{\delta,\beta}(k)}t)}{\sqrt{-\lambda_i^{\delta,\beta}(k)}} - \frac{\sin(\sqrt{-\lambda_i^{\mathcal{N}}(k)}t)}{\sqrt{-\lambda_i^{\mathcal{N}}(k)}} \right| \\ &\leq \max_{i=1,2} \left( \frac{1}{\sqrt{-\lambda_i^{\delta,\beta}(k)}} + \frac{1}{\sqrt{-\lambda_i^{\mathcal{N}}(k)}} \right) \\ &\leq \begin{cases} \frac{D}{\|k\|} + C & \text{if } \beta \leq n \\ \frac{D}{\|k\|} + \frac{C}{\|k\|^{\frac{\beta-n}{2}}} & \text{if } \beta > n \end{cases}. \end{aligned} \tag{3.29}$$

This result combined with

$$\|k\|^{s-s_2} \leq \begin{cases} 1 & \text{if } \beta \leq n \\ \|k\|^{\frac{\beta-n}{2}} & \text{if } \beta > n \end{cases},$$

provide a uniform bound for the second term. Observe that

$$\begin{aligned}
& \left\| [\cos(\sqrt{-M_k^{\delta,\beta}}t) - I](M_k^{\delta,\beta})^{-1} - [\cos(\sqrt{-M_k^{\mathcal{N}}}t) - I](M_k^{\mathcal{N}})^{-1} \right\| \\
&= \max_{i=1,2} \left| \frac{\cos(\sqrt{\lambda_i^{\delta,\beta}(k)}) - 1}{-\lambda_i^{\delta,\beta}(k)} - \frac{\cos(\sqrt{\lambda_i^{\mathcal{N}}(k)}) - 1}{-\lambda_i^{\mathcal{N}}(k)} \right| \\
&\leq \max_{i=1,2} \left( \frac{2}{|\lambda_i^{\delta,\beta}(k)|} + \frac{2}{|\lambda_i^{\mathcal{N}}(k)|} \right) \\
&\leq \begin{cases} \frac{D}{\|k\|^2} + C & \text{if } \beta \leq n \\ \frac{D}{\|k\|^2} + \frac{C}{\|k\|^{\beta-n}} & \text{if } \beta > n \end{cases}. \tag{3.30}
\end{aligned}$$

Moreover, since

$$\|k\|^{s-s_0} \leq \begin{cases} 1 & \text{if } \beta \leq n \\ \|k\|^{\beta-n} & \text{if } \beta > n \end{cases},$$

then these facts imply the uniform boundedness for the last term. □

**Theorem 34.** *Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta < n + 2$ . Suppose  $\mathbf{f} \in \mathcal{H}^{s_1}(\mathbb{T}^n)$  and  $\mathbf{g} \in \mathcal{H}^{s_2}(\mathbb{T}^n)$  and  $\mathbf{b} \in \mathcal{H}^{s_0}(\mathbb{T}^n)$  for  $s_1, s_2, s_0 \in \mathbb{R}$ . Let  $\mathbf{U}^{\delta,\beta}(t) \in \mathcal{H}^s(\mathbb{T}^n)$  and  $\mathbf{U}^{\mathcal{N}}(t) \in \mathcal{H}^{s_{\mathcal{N}}}(\mathbb{T}^n)$  be the solutions to the peridynamic equation (3.24) and Navier-Cauchy equation (3.27), respectively, with  $s = \min\{s_1, s_2 + \theta, s_0 + 2\theta\}$  with  $\theta = \max\{0, \frac{\beta-n}{2}\}$  and  $s_{\mathcal{N}} = \min\{s_1, s_2 + 1, s_0 + 2\}$ . Let  $t \geq 0$ , then for  $0 < \epsilon < 2$ ,*

$$\lim_{\beta \rightarrow n+2^-} \mathbf{U}^{\delta,\beta}(t) = \mathbf{U}^{\mathcal{N}}(t), \quad \text{in } \mathcal{H}^{s_{\epsilon}}(\mathbb{T}^n),$$

where  $s_{\epsilon} = \min\{s_1, s_2 + \frac{2-\epsilon}{2}, s_0 + 2 - \epsilon\}$ .

*Proof.* Fix  $\epsilon$ , with  $0 < \epsilon < 2$ , and for any  $\beta < n + 2$  such that  $n + 2 - \beta < \epsilon$ , both  $\mathbf{U}^{\delta,\beta}(t)$  and  $\mathbf{U}^{\mathcal{N}}(t)$  belong to the space  $\mathcal{H}^{s_{\epsilon}}(\mathbb{T}^n)$ . With the same approach as in the proof of Theorem

33, we can show that

$$\begin{aligned}
\|\mathbf{U}^{\delta,\beta}(t) - \mathbf{U}^{\mathcal{N}}(t)\|_{\mathcal{H}^{s_\epsilon}(\mathbb{T}^n)}^2 &\leq \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^{s_\epsilon} \left\| \cos(\sqrt{-M_k^{\delta,\beta}}t) - \cos(\sqrt{-M_k^{\mathcal{N}}}t) \right\|^2 \|\widehat{\mathbf{f}}_k\|^2 \\
&+ \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^{s_\epsilon} \left\| \sin(\sqrt{-M_k^{\delta,\beta}}t)(\sqrt{-M_k^{\delta,\beta}})^{-1} - \sin(\sqrt{-M_k^{\mathcal{N}}}t)(\sqrt{-M_k^{\mathcal{N}}})^{-1} \right\|^2 \|\widehat{\mathbf{g}}_k\|^2 \\
&+ \sum_{0 \neq k \in \mathbb{Z}^n} (1 + \|k\|^2)^{s_\epsilon} \left\| [\cos(\sqrt{-M_k^{\delta,\beta}}t) - I](M_k^{\delta,\beta})^{-1} - [\cos(\sqrt{-M_k^{\mathcal{N}}}t) - I](M_k^{\mathcal{N}})^{-1} \right\|^2 \|\widehat{\mathbf{b}}_k\|^2.
\end{aligned}$$

Next we establish uniform bounds for each of the following expressions

$$\|k\|^{s_\epsilon - s_1} \left\| \cos(\sqrt{-M_k^{\delta,\beta}}t) - \cos(\sqrt{-M_k^{\mathcal{N}}}t) \right\|, \quad (3.31)$$

$$\|k\|^{s_\epsilon - s_2} \left\| \sin(\sqrt{-M_k^{\delta,\beta}}t)(\sqrt{-M_k^{\delta,\beta}})^{-1} - \sin(\sqrt{-M_k^{\mathcal{N}}}t)(\sqrt{-M_k^{\mathcal{N}}})^{-1} \right\|, \quad (3.32)$$

and

$$\|k\|^{s_\epsilon - s_0} \left\| [\cos(\sqrt{-M_k^{\delta,\beta}}t) - I](M_k^{\delta,\beta})^{-1} - [\cos(\sqrt{-M_k^{\mathcal{N}}}t) - I](M_k^{\mathcal{N}})^{-1} \right\|. \quad (3.33)$$

The bound for (3.31) is the same as in Theorem 33. For (3.32) and (3.33), and since  $\beta > n$ , then from (3.29) and (3.30), we have

$$\begin{aligned}
\left\| \sin(\sqrt{-M_k^{\delta,\beta}}t)(\sqrt{-M_k^{\delta,\beta}})^{-1} - \sin(\sqrt{-M_k^{\mathcal{N}}}t)(\sqrt{-M_k^{\mathcal{N}}})^{-1} \right\| &\leq \frac{D}{\|k\|} + \frac{C}{\|k\|^{\frac{\beta-n}{2}}}, \\
\left\| [\cos(\sqrt{-M_k^{\delta,\beta}}t) - I](M_k^{\delta,\beta})^{-1} - [\cos(\sqrt{-M_k^{\mathcal{N}}}t) - I](M_k^{\mathcal{N}})^{-1} \right\| &\leq \frac{D}{\|k\|^2} + \frac{C}{\|k\|^{\beta-n}}.
\end{aligned}$$

Since  $\|k\|^{s_\epsilon - s_2} \leq \|k\|^{\frac{2-\epsilon}{2}}$ , this implies that

$$\|k\|^{s_\epsilon - s_2} \left\| \sin(\sqrt{-M_k^{\delta,\beta}}t)(\sqrt{-M_k^{\delta,\beta}})^{-1} - \sin(\sqrt{-M_k^{\mathcal{N}}}t)(\sqrt{-M_k^{\mathcal{N}}})^{-1} \right\| \leq \frac{D}{\|k\|^{\frac{\epsilon}{2}}} + \frac{C}{\|k\|^{\frac{\epsilon+\beta-(n+2)}{2}}},$$

which is bounded, establishing a uniform bound for (3.32). We also have  $\|k\|^{s_\epsilon - s_0} \leq \|k\|^{2-\epsilon}$ , hence

$$\begin{aligned} \|k\|^{s_\epsilon - s_0} & \left\| [\cos(\sqrt{-M_k^{\delta,\beta}} t) - I](M_k^{\delta,\beta})^{-1} - [\cos(\sqrt{-M_k^N} t) - I](M_k^N)^{-1} \right\| \\ & \leq \|k\|^{2-\epsilon} \left( \frac{D}{\|k\|^2} + \frac{C}{\|k\|^{\beta-n}} \right) \\ & = \frac{D}{\|k\|^\epsilon} + \frac{C}{\|k\|^{\epsilon+\beta-(n+2)}}, \end{aligned}$$

which is bounded, establishing a uniform bound for (3.33).  $\square$

### 3.4.4 Spatial and temporal regularity over $(L^2(\mathbb{T}^n))^n$

Let  $\mathbf{b} \in \mathcal{H}^{s_0}(\mathbb{T}^n)$ ,  $\mathbf{f} \in \mathcal{H}^{s_1}(\mathbb{T}^n)$  and  $\mathbf{g} \in \mathcal{H}^{s_2}(\mathbb{T}^n)$  with  $s_0, s_1, s_2 \in \mathbb{R}$ . Let  $s = \min\{s_1, s_2 + \theta, s_0 + 2\theta\}$  and  $\theta = \max\{0, \frac{\beta-n}{2}\}$ , the solution  $\mathbf{U}^{\delta,\beta}(t) \in \mathcal{H}^s(\mathbb{T}^n)$  given in (3.25)-(3.26) is a distribution when  $s < 0$ , and when  $s \geq 0$  defines a regular vector field

$$\begin{aligned} \mathbf{u}^{\delta,\beta}(x, t) &= \mathbf{U}^{\delta,\beta}(t)(x) \\ &= (\widehat{\mathbf{f}}_0 + \widehat{\mathbf{g}}_0 t) + \sum_{0 \neq k \in \mathbb{Z}^n} \left( \cos(\sqrt{-M_k^{\delta,\beta}} t) \widehat{\mathbf{f}}_k + \sin(\sqrt{-M_k^{\delta,\beta}} t) (\sqrt{-M_k^{\delta,\beta}})^{-1} \widehat{\mathbf{g}}_k \right) e^{ik \cdot x}. \end{aligned} \tag{3.34}$$

Let the initial data and the forcing term be regular vector fields, that is,  $s_0 \geq 0$ ,  $s_1 \geq 0$  and  $s_2 \geq 0$ . In this section, we focus on conditions that guarantee that  $\mathbf{u}^{\delta,\beta}(x, t)$ , the solution of (3.24), is a regular vector field  $\mathbf{u}^{\delta,\beta}(\cdot, t) \in (L^2(\mathbb{T}^n))^n$ .

### Functions with absolutely summable Fourier coefficients

We state some well-known results on the differentiability and summability of series in multi-dimensions. These results are used in proving the regularity results in Section 3.4.4 and are included here for completeness of the presentation.

We recall that a vector field  $\mathbf{f}$  on  $\mathbb{R}^n$  is called  $\alpha$ -Hölder continuous, where  $\alpha \in [0, 1]$ , if

there exists a constant  $C$  such that

$$\|\mathbf{f}(x) - \mathbf{f}(y)\| \leq C\|x - y\|^\alpha,$$

for any  $x, y$  in the domain of  $\mathbf{f}$ .

**Lemma 8.** *Suppose that  $\mathbf{f}$  is a vector field defined on the torus  $\mathbb{T}^n$  satisfying the  $\alpha$ -Hölder continuity condition for some  $\alpha > n/2$ . Then*

$$\sum_{k \in \mathbb{Z}^n} \|\widehat{\mathbf{f}}_k\| < \infty.$$

A version of the Lemma 10 for scalar functions can be found in<sup>76</sup>.

**Lemma 9.** *Let  $T$  be an open subset of  $\mathbb{R}^+$ ,  $K$  is a measurable space equipped with the counting measure and  $\mathbf{f}_k : T \rightarrow \mathbb{C}^n$ . Suppose that*

1.  $\sum_{k \in K} \|\mathbf{f}_k(t)\| < \infty$  for each  $t \in T$ .
2. For almost all  $k \in K$ , the derivative  $\frac{d}{dt}\mathbf{f}_k(t)$  exists and is continuous for all  $t \in T$ .
3. There is a summable sequence  $\theta_k$  such that  $\|\frac{d}{dt}\mathbf{f}_k(t)\| \leq \theta_k$  for all  $t \in T$  and almost all  $k \in K$ .

Then, for all  $t \in T$ ,

$$\frac{d}{dt} \sum_k \mathbf{f}_k(t) = \sum_k \frac{d}{dt} \mathbf{f}_k(t).$$

A version of Lemma 9 for scalar functions can be found in<sup>91</sup>.

### Temporal regularity with respect to the Gâteaux derivative

**Theorem 35.** *Let  $n \geq 1$ ,  $\delta > 0$ , and  $\beta < n + 2$ . Let  $\mathbf{f} \in \mathcal{H}^{s_1}(\mathbb{T}^n)$ ,  $\mathbf{g} \in \mathcal{H}^{s_2}(\mathbb{T}^n)$  and  $\mathbf{b} \in \mathcal{H}^{s_0}(\mathbb{T}^n)$ , with  $s_0 \geq 0$ ,  $s_1 \geq 0$  and  $s_2 \geq 0$ . Suppose that*

1.  $s_0, s_1, s_2 \geq 0$ , in the case when  $\beta < n$ ,
2.  $s_0, s_1, s_2 > 0$ , in the case when  $\beta = n$ , and
3.  $s_1 \geq \beta - n$  and  $s_2 \geq (\beta - n)/2$  and  $s_0 \geq 0$ , in the case when  $\beta > n$ .

Then, the vector field  $\mathbf{u}^{\delta, \beta}$ , where  $\mathbf{u}^{\delta, \beta}(x, t) = \mathbf{U}^{\delta, \beta}(t)(x)$ , with  $\mathbf{U}^{\delta, \beta}$  defined in (3.25)-(3.26), is the unique solution to the peridynamic equation (3.24) in  $(L^2(\mathbb{T}^n))^n$ . Moreover, when  $\beta \leq n$ , then  $\mathbf{u}^{\delta, \beta} \in C^\infty([0, \infty); \mathcal{H}^s(\mathbb{T}^n))$ , where  $s = \min\{s_1, s_2, s_0\}$ . Furthermore, when  $\beta > n$ , then  $\mathbf{u}^{\delta, \beta} \in C^2([0, \infty); \mathcal{H}^s(\mathbb{T}^n))$ , where  $s = \min\{s_1, s_2 + \frac{\beta - n}{2}, s_0 + \beta - n\}$ .

Theorem 35 is a special case of Theorem 32 with  $s_0 \geq 0, s_1 \geq 0$ , and  $s_2 \geq 0$ . We have included it in this section for completeness and for comparison with the next result Theorem 36.

### Temporal regularity with respect to the classical derivative

**Theorem 36.** Let  $n \geq 1, \delta > 0$  and  $\beta < n$ . Let  $\mathbf{f} \in \mathcal{H}^{s_1}(\mathbb{T}^n)$ ,  $\mathbf{g} \in \mathcal{H}^{s_2}(\mathbb{T}^n)$  and  $\mathbf{b} \in \mathcal{H}^{s_0}(\mathbb{T}^n)$ , with  $s_0 \geq 0, s_1 \geq 0$  and  $s_2 \geq 0$ . Suppose that each of  $\mathbf{f}, \mathbf{g}$  and  $\mathbf{b}$  satisfies the  $\alpha$ -Hölder continuity condition with  $\alpha > n/2$ . Then, for  $\mathbf{u}^{\delta, \beta}$  given in (3.34),  $\mathbf{u}^{\delta, \beta}(x, \cdot) \in C^\infty[0, \infty)$ , for all  $x \in \mathbb{T}^n$ .

*Proof.* Fix a value of  $p$  in  $\mathbb{Z}_{\geq 0}$ . Consider the following three series

$$\begin{aligned} & \sum_{0 \neq k \in \mathbb{Z}^n} \cos(\sqrt{-M_k^{\delta, \beta}} t) \widehat{\mathbf{f}}_k e^{ik \cdot x}, \quad \sum_{0 \neq k \in \mathbb{Z}^n} \sin(\sqrt{-M_k^{\delta, \beta}} t) (\sqrt{-M_k^{\delta, \beta}})^{-1} \widehat{\mathbf{g}}_k e^{ik \cdot x}, \\ \text{and} \quad & \sum_{0 \neq k \in \mathbb{Z}^n} \left[ \cos\left(\sqrt{-M_k^{\delta, \beta}} t\right) - I \right] (M_k^{\delta, \beta})^{-1} \widehat{\mathbf{b}}_k e^{ik \cdot x}. \end{aligned}$$

From Theorem 20 and Theorem 23, there exists  $C > 0$  such that  $\left\| \sqrt{-M_k^{\delta, \beta}} \right\| \leq C$  and

$\left\| \left( \sqrt{-M_k^{\delta,\beta}} \right)^{-1} \right\| \leq C$  as  $\|k\| \rightarrow \infty$ . It follows that

$$\begin{aligned}
& \left\| \cos(\sqrt{-M_k^{\delta,\beta}} t) \widehat{\mathbf{f}}_k e^{ik \cdot x} \right\| \leq \|\widehat{\mathbf{f}}_k\|, \\
& \left\| \frac{d^p}{dt^p} \cos(\sqrt{-M_k^{\delta,\beta}} t) \widehat{\mathbf{f}}_k e^{ik \cdot x} \right\| \leq \left\| \sqrt{-M_k^{\delta,\beta}} \right\|^p \|\widehat{\mathbf{f}}_k\| \leq C^p \|\widehat{\mathbf{f}}_k\|, \\
& \left\| \sin(\sqrt{-M_k^{\delta,\beta}} t) (\sqrt{-M_k^{\delta,\beta}})^{-1} \widehat{\mathbf{g}}_k e^{ik \cdot x} \right\| \leq \left\| (\sqrt{-M_k^{\delta,\beta}})^{-1} \right\| \|\widehat{\mathbf{g}}_k\| \leq C \|\widehat{\mathbf{g}}_k\|, \\
& \left\| \frac{d^p}{dt^p} \sin(\sqrt{-M_k^{\delta,\beta}} t) (\sqrt{-M_k^{\delta,\beta}})^{-1} \widehat{\mathbf{g}}_k e^{ik \cdot x} \right\| \leq \left\| (\sqrt{-M_k^{\delta,\beta}}) \right\|^{p-1} \|\widehat{\mathbf{g}}_k\| \leq C^{p-1} \|\widehat{\mathbf{g}}_k\|, \\
& \left\| \left[ \cos \left( \sqrt{-M_k^{\delta,\beta}} t \right) - I \right] (M_k^{\delta,\beta})^{-1} \widehat{\mathbf{b}}_k e^{ik \cdot x} \right\| \leq 2 \|(M_k^{\delta,\beta})^{-1}\| \|\widehat{\mathbf{b}}_k\| \leq 2C^2 \|\widehat{\mathbf{b}}_k\|, \\
& \left\| \frac{d^p}{dt^p} \left[ \cos \left( \sqrt{-M_k^{\delta,\beta}} t \right) - I \right] (M_k^{\delta,\beta})^{-1} \widehat{\mathbf{b}}_k e^{ik \cdot x} \right\| \leq \left\| \sqrt{-M_k^{\delta,\beta}} \right\|^{p-2} \|\widehat{\mathbf{b}}_k\| \leq C^{p-2} \|\widehat{\mathbf{b}}_k\|,
\end{aligned}$$

as  $\|k\| \rightarrow \infty$ . From Lemma 8 in Section 3.4.4, the Fourier coefficients of  $\widehat{\mathbf{f}}_k$ ,  $\widehat{\mathbf{g}}_k$  and  $\widehat{\mathbf{b}}_k$  are absolutely summable. The proof is now complete using Lemma 9 from Section 3.4.4.  $\square$

**Theorem 37.** *Let  $n \geq 1$ ,  $\delta > 0$  and  $\beta = n$ . Let  $\mathbf{f} \in \mathcal{H}^{s_1}(\mathbb{T}^n)$ ,  $\mathbf{g} \in \mathcal{H}^{s_2}(\mathbb{T}^n)$  and  $\mathbf{b} \in \mathcal{H}^{s_0}(\mathbb{T}^n)$ , with  $s_0 \geq 0$ ,  $s_1 \geq 0$  and  $s_2 \geq 0$ . Suppose  $s_1 > n$ ,  $s_2 > n$  and  $s_0 > n$ . Then for  $\mathbf{u}^{\delta,\beta}$  as in (3.34),  $\mathbf{u}^{\delta,\beta}(x, \cdot) \in C^\infty[0, \infty)$ , for any  $x \in \mathbb{T}^n$ .*

*Proof.* Fix a value of  $p$  in  $\mathbb{Z}_{\geq 0}$  and choose  $\epsilon > 0$  such that  $s_1 - p\epsilon > n$  and  $s_2 - (p-1)\epsilon > n$  and  $s_0 - (p-2)\epsilon > n$ . Consider the following three series

$$\begin{aligned}
& \sum_{0 \neq k \in \mathbb{Z}^n} \cos(\sqrt{-M_k^{\delta,\beta}} t) \widehat{\mathbf{f}}_k e^{ik \cdot x}, \quad \sum_{0 \neq k \in \mathbb{Z}^n} \sin(\sqrt{-M_k^{\delta,\beta}} t) (\sqrt{-M_k^{\delta,\beta}})^{-1} \widehat{\mathbf{g}}_k e^{ik \cdot x}, \\
& \text{and} \quad \sum_{0 \neq k \in \mathbb{Z}^n} \left[ \cos \left( \sqrt{-M_k^{\delta,\beta}} t \right) - I \right] (M_k^{\delta,\beta})^{-1} \widehat{\mathbf{b}}_k e^{ik \cdot x}.
\end{aligned}$$

From Theorem 20 and Theorem 23, there exists  $C > 0$  such that

$$\left\| \sqrt{-M_k^{\delta,\beta}} \right\| \leq C(\log \|k\|)^{1/2} \leq C\|k\|^\epsilon, \quad \text{and} \quad \left\| (\sqrt{-M_k^{\delta,\beta}})^{-1} \right\| \leq C,$$

as  $\|k\| \rightarrow \infty$ . As  $\mathbf{f} \in \mathcal{H}^{s_1}(\mathbb{T}^n)$ ,  $\mathbf{g} \in \mathcal{H}^{s_2}(\mathbb{T}^n)$  and  $\mathbf{b} \in \mathcal{H}^{s_0}(\mathbb{T}^n)$ , using (3.20), it follows that

$$\begin{aligned}
& \left\| \cos(\sqrt{-M_k^{\delta,\beta}t}) \widehat{\mathbf{f}}_k e^{ik \cdot x} \right\| \leq \|\widehat{\mathbf{f}}_k\|, \\
& \left\| \frac{d^p}{dt^p} \cos(\sqrt{-M_k^{\delta,\beta}t}) \widehat{\mathbf{f}}_k e^{ik \cdot x} \right\| \leq \left\| \sqrt{-M_k^{\delta,\beta}} \right\|^p \|\widehat{\mathbf{f}}_k\| \leq \frac{C^p}{\|k\|^{s_1-p\epsilon}}, \\
& \left\| \sin(\sqrt{-M_k^{\delta,\beta}t}) (\sqrt{-M_k^{\delta,\beta}})^{-1} \widehat{\mathbf{g}}_k e^{ik \cdot x} \right\| \leq \left\| (\sqrt{-M_k^{\delta,\beta}})^{-1} \right\| \|\widehat{\mathbf{g}}_k\| \leq \frac{C}{\|k\|^{s_2}}, \\
& \left\| \frac{d^p}{dt^p} \sin(\sqrt{-M_k^{\delta,\beta}t}) (\sqrt{-M_k^{\delta,\beta}})^{-1} \widehat{\mathbf{g}}_k e^{ik \cdot x} \right\| \leq \left\| (\sqrt{-M_k^{\delta,\beta}}) \right\|^{p-1} \|\widehat{\mathbf{g}}_k\| \leq \frac{C^{p-1}}{\|k\|^{s_2-(p-1)\epsilon}}, \\
& \left\| \left[ \cos\left(\sqrt{-M_k^{\delta,\beta}t}\right) - I \right] (M_k^{\delta,\beta})^{-1} \widehat{\mathbf{b}}_k e^{ik \cdot x} \right\| \leq 2 \left\| (M_k^{\delta,\beta})^{-1} \right\| \|\widehat{\mathbf{b}}_k\| \leq \frac{2C^2}{\|k\|^{s_0}}, \\
& \left\| \frac{d^p}{dt^p} \left[ \cos\left(\sqrt{-M_k^{\delta,\beta}t}\right) - I \right] (M_k^{\delta,\beta})^{-1} \widehat{\mathbf{b}}_k e^{ik \cdot x} \right\| \leq \left\| \sqrt{-M_k^{\delta,\beta}} \right\|^{p-2} \|\widehat{\mathbf{b}}_k\| \leq \frac{C^{p-2}}{\|k\|^{s_0-(p-2)\epsilon}},
\end{aligned}$$

as  $\|k\| \rightarrow \infty$ . The proof is complete by using Lemma 9 from Section 3.4.4.  $\square$

**Theorem 38.** *Let  $n \geq 1$ ,  $\delta > 0$  and  $n < \beta < n + 2$ . Let  $\mathbf{f} \in \mathcal{H}^{s_1}(\mathbb{T}^n)$ ,  $\mathbf{g} \in \mathcal{H}^{s_2}(\mathbb{T}^n)$  and  $\mathbf{b} \in \mathcal{H}^{s_0}(\mathbb{T}^n)$ , with  $s_0 \geq 0$ ,  $s_1 \geq 0$  and  $s_2 \geq 0$ . Suppose that there exist a nonnegative integer  $p$  such that*

$$s_1 > n + p \frac{\beta - n}{2} \quad \text{and} \quad s_2 > n + (p-1) \frac{\beta - n}{2} \quad \text{and} \quad s_0 > n + (p-2) \frac{\beta - n}{2}.$$

Then, for  $\mathbf{u}^{\delta,\beta}$  given in (3.34),  $\mathbf{u}^{\delta,\beta}(x, \cdot) \in C^p[0, \infty)$ , for any  $x \in \mathbb{T}^n$ .

*Proof.* Consider the following three series

$$\begin{aligned}
& \sum_{0 \neq k \in \mathbb{Z}^n} \cos(\sqrt{-M_k^{\delta,\beta}t}) \widehat{\mathbf{f}}_k e^{ik \cdot x} \quad \sum_{0 \neq k \in \mathbb{Z}^n} \sin(\sqrt{-M_k^{\delta,\beta}t}) (\sqrt{-M_k^{\delta,\beta}})^{-1} \widehat{\mathbf{g}}_k e^{ik \cdot x}, \\
& \text{and} \quad \sum_{0 \neq k \in \mathbb{Z}^n} \left[ \cos\left(\sqrt{-M_k^{\delta,\beta}t}\right) - I \right] (M_k^{\delta,\beta})^{-1} \widehat{\mathbf{b}}_k e^{ik \cdot x}.
\end{aligned}$$

From Theorem 20 and Theorem 23, there exists  $C > 0$  such that

$$\left\| \sqrt{-M_k^{\delta,\beta}} \right\| \leq C \|k\|^{(\beta-n)/2}, \quad \text{and} \quad \left\| (\sqrt{-M_k^{\delta,\beta}})^{-1} \right\| \leq C \|k\|^{-(\beta-n)/2},$$

as  $\|k\| \rightarrow \infty$ . As  $\mathbf{f} \in \mathcal{H}^{s_1}(\mathbb{T}^n)$ ,  $\mathbf{g} \in \mathcal{H}^{s_2}(\mathbb{T}^n)$  and  $\mathbf{b} \in \mathcal{H}^{s_0}(\mathbb{T}^n)$ , using (3.20), it follows that

$$\begin{aligned}
& \left\| \cos(\sqrt{-M_k^{\delta,\beta}} t) \widehat{\mathbf{f}}_k e^{ik \cdot x} \right\| \leq \|\widehat{\mathbf{f}}_k\|, \\
& \left\| \frac{d^p}{dt^p} \cos(\sqrt{-M_k^{\delta,\beta}} t) \widehat{\mathbf{f}}_k e^{ik \cdot x} \right\| \leq \left\| \sqrt{-M_k^{\delta,\beta}} \right\|^p \|\widehat{\mathbf{f}}_k\| \leq \frac{C^p}{\|k\|^{s_1-p(\beta-n)/2}}, \\
& \left\| \sin(\sqrt{-M_k^{\delta,\beta}} t) (\sqrt{-M_k^{\delta,\beta}})^{-1} \widehat{\mathbf{g}}_k e^{ik \cdot x} \right\| \leq \left\| (\sqrt{-M_k^{\delta,\beta}})^{-1} \right\| \|\widehat{\mathbf{g}}_k\| \leq \frac{C}{\|k\|^{s_2+(\beta-n)/2}}, \\
& \left\| \frac{d^p}{dt^p} \sin(\sqrt{-M_k^{\delta,\beta}} t) (\sqrt{-M_k^{\delta,\beta}})^{-1} \widehat{\mathbf{g}}_k e^{ik \cdot x} \right\| \leq \left\| (\sqrt{-M_k^{\delta,\beta}})^{-1} \right\|^{p-1} \|\widehat{\mathbf{g}}_k\| \leq \frac{C^{p-1}}{\|k\|^{s_2-(p-1)(\beta-n)/2}}, \\
& \left\| \left[ \cos(\sqrt{-M_k^{\delta,\beta}} t) - I \right] (M_k^{\delta,\beta})^{-1} \widehat{\mathbf{b}}_k e^{ik \cdot x} \right\| \leq 2 \left\| (M_k^{\delta,\beta})^{-1} \right\| \|\widehat{\mathbf{b}}_k\| \leq \frac{2C^2}{\|k\|^{S+\beta-n}}, \\
& \left\| \frac{d^p}{dt^p} \left[ \cos(\sqrt{-M_k^{\delta,\beta}} t) - I \right] (M_k^{\delta,\beta})^{-1} \widehat{\mathbf{b}}_k e^{ik \cdot x} \right\| \leq \left\| \sqrt{-M_k^{\delta,\beta}} \right\|^{p-2} \|\widehat{\mathbf{b}}_k\| \leq \frac{C^{p-2}}{\|k\|^{S-(p-2)(\beta-n)/2}},
\end{aligned}$$

as  $\|k\| \rightarrow \infty$ . The proof is complete by using Lemma 9 from Section 3.4.4.  $\square$

# Chapter 4

## Discontinuity analysis and semi-analytic spectral approximation for the nonlocal Poisson equation

We study a one-dimensional nonlocal Poisson problem with discontinuous source and investigate both the analytical structure of its solutions and the design of accurate numerical methods for resolving the resulting discontinuities. Specifically, we consider

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = b & \text{in } \Omega_I, \end{cases} \quad (4.1)$$

where  $b$  is a prescribed constraint on the interaction region, and the source term  $f$  has one or more jump discontinuities in  $\Omega$  while being otherwise differentiable. Here  $\Omega = (a_1, a_2)$  is a bounded open interval

$$\Omega_I = (a_1 - \delta, a_1] \cup [a_2, a_2 + \delta)$$

denotes the associated  $\delta$ -collar region, where  $\delta > 0$  is the horizon. The nonlocal Laplace operator is defined by

$$Lu(x) = \int_{\Omega \cup \Omega_I} (u(y) - u(x))K(x, y) dy, \quad (4.2)$$

where  $K$  is the interaction kernel.

Nonlocal diffusion and peridynamic models have attracted considerable attention because of their ability to represent long-range interactions and to naturally accommodate defects, interfaces, and fracture. In contrast to classical local models, nonlocal formulations incorporate interactions over a finite distance, which makes them especially suitable for describing processes in which discontinuities play an essential role. For the nonlocal Poisson problem (4.1), well-posedness under standard assumptions on  $K$  is classical. For example, if  $f \in L^2(\Omega)$  and  $b \in L^2(\Omega_I)$ , then there exists a unique solution  $u \in L^2(\Omega \cup \Omega_I)$ ; see, for instance,<sup>39;92</sup>. Regularity, however, depends sensitively on both the kernel and the data. For integrable kernels,<sup>52</sup> proved that if  $f \in C^1(\overline{\Omega})$ , then the corresponding solution belongs to  $W^{1,2}(\Omega)$ .

The present work is motivated by the observation that discontinuous data arise naturally in nonlocal and peridynamic models, particularly in the study of fracture, interfaces, and material heterogeneities. In such situations, it is not enough to know that a solution exists: one must also understand how discontinuities in the source term influence the continuity, differentiability, and singular structure of the solution. This leads to the first objective of the chapter, which is analytical in nature. Assuming that  $f$  is  $C^N$  away from its jump set, we determine how discontinuities propagate through the nonlocal operator and how the behavior of the kernel near the origin and at the horizon endpoints  $\pm\delta$  controls the regularity of the solution. Our analysis identifies three principal mechanisms:

- *Discontinuity inheritance:* under mild conditions ensuring that  $u * K$  is continuous,  $f$  is continuous at  $x$  if and only if  $u$  is continuous at  $x$  (Lemma 10).
- *Singular behavior at the origin:* if  $K$ , or one of its derivatives  $K^{(j)}$ , is unbounded at the

origin, then the corresponding derivative  $u^{(j+1)}$  blows up at the jump location (Theorems 40–42).

- *Horizon effects:* jump discontinuities of  $K$ , or of one of its derivatives, at  $\pm\delta$  generate a cascade of derivative jumps in the solution at translated locations (Theorems 43–44).

The second objective of the chapter is computational. From the numerical point of view, a variety of methods have been developed for nonlocal problems of peridynamic type. The work in<sup>71</sup> provides an extensive discussion of numerical methods, including finite element, finite difference, and spectral methods, for approximating solutions of nonlocal models. Other notable contributions to the computational treatment of peridynamics include the peridynamic differential operator method<sup>36;93</sup>, the boundary element method for peridynamics<sup>94</sup>, meshfree methods<sup>95–97</sup>, and spectral methods<sup>72;98–104</sup>.

Modeling and resolving fracture and discontinuities is one of the key strengths of peridynamic models. At the same time, the presence of discontinuities requires special care in order to obtain accurate numerical approximations. It is well known that, in the presence of jump discontinuities, standard high-order methods lose their nominal convergence rates unless the discontinuity is explicitly incorporated into the discretization. In particular, Fourier and spectral approximations suffer from the Gibbs phenomenon, while standard finite difference and finite element discretizations for interface and jump problems typically reduce to first-order behavior near the discontinuity unless special interface-aware corrections are used<sup>105–107</sup>. Several studies have therefore focused on finite element approximations for peridynamic models with jump discontinuities; see, for example,<sup>108–111</sup>. In particular, the work in<sup>108</sup> demonstrated that a discontinuous Galerkin method can achieve  $\mathcal{O}((\Delta x)^2)$  accuracy both when the location of the jump is known and, through iterative refinement, when it is unknown. However, finite element methods become inefficient when the grid size is much smaller than the horizon  $\delta$ .

The main difficulty in applying spectral methods to fracture and discontinuity problems in peridynamics arises from two distinct sources. On the one hand, the Gibbs phenomenon degrades accuracy whenever Fourier-based methods are applied directly to discontinuous

data. On the other hand, standard damage models do not always possess a convolutional structure and therefore do not reduce to simple pointwise multiplication in the frequency domain. The latter issue was addressed in<sup>101</sup>, where damage models based on pairwise bond failure were replaced by a local *material integrity* indicator. It was shown in<sup>101</sup> that this formulation enables spectral solvers to reproduce benchmark fracture results accurately, thereby providing an elegant framework for dynamic fracture computations. Nevertheless, because the method still requires the application of the FFT to discontinuous data, its convergence rate remains limited by Gibbs oscillations.

To overcome this limitation, we introduce a framework that combines the regularity and discontinuity analysis developed in this work, as described in Sections 4.1 and 4.2, with an accurate and efficient semi-analytic spectral approach. The guiding idea is that the analytical description of the discontinuity structure should not only explain the behavior of solutions, but should also be incorporated directly into the numerical method.

Motivated by the regularity theory developed here, the computational part of the chapter introduces a semi-analytic spectral framework for discontinuous nonlocal problems. The central idea is to transform the original problem into an auxiliary *smoothed problem* by successively subtracting explicitly constructed correction functions that capture the discontinuity structure of the solution. The resulting smoothed problem has improved regularity and can therefore be solved much more accurately by a spectral method. The approximation to the original discontinuous solution is then reconstructed by combining the numerical solution of the smoothed problem with the analytic correction terms. In this way, the regularity analysis is used not only to explain the behavior of discontinuous solutions, but also to design an efficient computational method that restores high-order accuracy while retaining the efficiency of spectral solvers.

The remainder of the chapter is organized as follows. In Section 4.1, we state the standing assumptions and derive a useful convolution representation of the operator. Section 4.2 contains the main regularity and discontinuity results, together with examples illustrating the sharpness of the theory for several commonly used kernels. In Section 4.3, we introduce the successive smoothing procedure and the resulting semi-analytic spectral method. Section 4.4

presents numerical experiments demonstrating the accuracy and convergence of the proposed approach.

## 4.1 Preliminaries

### 4.1.1 Model assumptions

In this section, we state the standing assumptions on the kernel  $K$  and the source term  $f$ . These assumptions on  $K$  are inspired by peridynamics and its applications to material fracture and discontinuities.

**Assumption (A1).** *The kernel  $K$  is a positive, radial and integrable kernel that is compactly supported. Specifically,*

$$\begin{cases} K(x, y) = K(|x - y|) \geq 0, & \text{for all } x, y \in \Omega \cup \Omega_I, \\ K(|x - y|) = 0, & \text{whenever } |x - y| \geq \delta, \\ K(\cdot) \in L^1(\mathbb{R}). \end{cases} \quad (4.3)$$

In addition, we assume that  $K$  is smooth away from the origin and the horizon endpoints  $\pm\delta$ .

**Assumption (A2).** *The kernel  $K$  is continuous on  $[-\delta, 0) \cup (0, \delta]$  and  $C^\infty$  on the intervals  $(-\delta, 0)$  and  $(0, \delta)$ .*

The behavior of  $K$  at 0 and  $\pm\delta$  will be discussed in Sections 4.1.2 and 4.2 where we will consider several cases of whether  $K$  is bounded/unbounded at 0 and whether  $K$  is continuous or has jumps at  $\pm\delta$ .

Without loss of generality, we assume that the source term  $f$  has a jump discontinuity at 0 and is otherwise differentiable.

**Assumption (B).**  *$f$  has a jump discontinuity at 0.  $f$  is  $C^N$ , in the intervals  $(a_1, 0)$  and  $(0, a_2)$ , for some  $N \in \mathbb{N}$ . Moreover,  $f$  and all of its derivatives are essentially bounded on*

$(a_1, a_2)$ .

The case of multiple jump discontinuities can be treated similarly by applying the same analysis locally near each jump point. In particular, Assumption (B) implies that  $f \in L^2(\Omega)$ .

### 4.1.2 Convolution structure of the operator $L$

Before analyzing discontinuities in the solution, it is useful to first examine the structure of the nonlocal operator when applied to functions with jump discontinuities. In particular, the operator admits a convolution-type representation that isolates the contribution of the kernel near singular points. This representation separates the effect of discontinuities in the source term from the regularity properties of the kernel itself, and it will play a central role in the analysis below.

This section develops the mentioned representation and establishes several auxiliary results that will be used throughout the remainder of the chapter.

Since  $K \in L^1(\mathbb{R})$ , we use the decomposition

$$Lu = u * K - \alpha u, \tag{4.4}$$

where  $\alpha := \|K\|_{L^1(\mathbb{R})}$ , and  $u$  is extended by zero outside  $\Omega \cup \Omega_I$ .

**Lemma 10.** *Let  $f \in L^2(\Omega)$  and  $u \in L^2(\Omega \cup \Omega_I)$  be such that  $Lu = f$  in  $\Omega$ . Assume that  $u * K \in C(\Omega)$ . Then,  $f$  is continuous at  $x \in \Omega$  if and only if  $u$  is continuous at  $x$ .*

*Proof.* This is a direct application of the decomposition in (4.4). □

Thus, whenever  $u * K \in C(\Omega)$ , the solution  $u$  and the source  $f$  have exactly the same points of continuity and discontinuity. The following are examples in which the hypothesis of the lemma is satisfied.

- If  $K \in L^2(\mathbb{R})$  and  $u \in L^2(\Omega \cup \Omega_I)$ , then  $u * K \in C(\Omega)$ .
- If  $K \in L^1(\mathbb{R})$  and  $u \in L^\infty(\mathbb{R})$ , then  $u * K \in C(\Omega)$ .

The following is a result from<sup>52</sup>.

**Theorem 39.** *Suppose that  $K$  satisfies Assumption (A1). Let  $u \in L^2(\Omega \cup \Omega_I)$  and  $f \in L^2(\Omega)$  such that  $Lu = f$  in  $\Omega$ .*

1. *If  $f \in L^\infty(\Omega)$ , then  $u \in L^\infty(\Omega)$ .*
2. *If  $f \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ , then  $u \in W^{1,2}(\Omega)$ .*

Part (1) is Theorem 3.1 of<sup>52</sup>, and Part (2) is a mild relaxation of Theorem 4.3 therein. In Part (2), the original assumption is  $f \in C^1(\overline{\Omega})$ ; however, the same proof remains valid under the weaker hypothesis  $f \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ .

For  $k \in \mathbb{N}$ , define

$$\varphi_k(x) = \frac{x^k}{k!} \chi_{[0,\infty)}(x), \quad x \in \mathbb{R}.$$

These functions serve as elementary building blocks for isolating jump contributions. Observe that

$$\varphi'_{k+1}(x) = \varphi_k(x), \quad x \neq 0, \tag{4.5}$$

and therefore, by (4.4),

$$(L\varphi_{k+1})'(x) = L\varphi_k(x), \quad x \neq 0. \tag{4.6}$$

For a given kernel  $K$ , the functions  $L\varphi_k$  can often be computed explicitly, and they will be used repeatedly in the sequel.

**Lemma 11.** *Let  $K$  satisfy Assumption (A1). Then,*

1.  *$L\varphi_k \in L^\infty_{loc}(\mathbb{R})$  for all  $k$ .*
2.  *$(L\varphi_0)' = K$  almost everywhere.*

*Proof.* 1. Observe that

$$\begin{aligned}\varphi_k * K(x) &= \int_{-\infty}^{\infty} \varphi_k(y) K(x-y) dy = \int_0^{\infty} \frac{y^k}{k!} K(y-x) dy = \int_{-x}^{\infty} \frac{(z+x)^k}{k!} K(z) dz \\ &= \int_{-x}^{\delta} \frac{(z+x)^k}{k!} K(z) dz.\end{aligned}\tag{4.7}$$

Since  $K \in L^1(\mathbb{R})$ , it follows from (4.7) that  $\varphi_k * K \in L^\infty(\mathbb{R})$ . Note that  $\varphi_k \in L^\infty_{\text{loc}}(\mathbb{R})$  by definition. From the decomposition,  $L\varphi_k = \varphi_k * K - \alpha\varphi_k$ , we obtain that  $L\varphi_k \in L^\infty_{\text{loc}}(\mathbb{R})$ .

2. In the case that  $k = 0$ ,

$$\varphi_0 * K(x) = \int_{-x}^{\delta} K(z) dz.\tag{4.8}$$

Since  $K \in L^1(\mathbb{R})$ , from the Fundamental theorem of Calculus,  $(\varphi_0 * K)' = K$  almost everywhere. Note that  $\varphi_0'(x) = 0$  for all  $x \neq 0$ . Differentiating the identity  $L\varphi_0 = \varphi_0 * K - \alpha\varphi_0$  for  $x \neq 0$  completes the proof. □

We denote the jump of a function  $\Phi$  at  $p$  by  $[\Phi]_p$ , specifically,

$$[\Phi]_p = \lim_{x \rightarrow p^+} \Phi(x) - \lim_{x \rightarrow p^-} \Phi(x).$$

### 4.1.3 Differentiation under the integral

The next two lemmas provide the differentiation tools needed in the proofs below. Lemma 12 is the classical Leibniz rule on  $\mathbb{R}$ , while Lemma 13 gives a measure-theoretic version suitable for differentiating convolution terms under minimal regularity assumptions.

**Lemma 12** (Leibniz's theorem on  $\mathbb{R}$ ). *Let  $g(x, y)$  be a function such that both  $g$  and  $\frac{\partial g}{\partial x}$  are continuous in  $x$  and  $y$  in some  $xy$ -domain containing  $\xi(x) \leq y \leq \zeta(x)$ ,  $c \leq x \leq d$ . Suppose that  $\xi(x)$  and  $\zeta(x)$  are continuous and have continuous derivatives for  $c \leq x \leq d$ . Then for*

$$c \leq x \leq d,$$

$$\frac{d}{dx} \left( \int_{\xi(x)}^{\zeta(x)} g(x, y) dy \right) = g(x, \zeta(x)) - g(x, \xi(x)) + \int_{\xi(x)}^{\zeta(x)} \frac{\partial g}{\partial x}(x, y) dy.$$

**Lemma 13** (Leibniz's theorem: Measure theory statement). *Let  $X$  be an open set of  $\mathbb{R}$  and  $D$  be a measure space. Suppose that a function  $g : X \times D \rightarrow \mathbb{R}$  satisfies the following conditions:*

1.  *$g(x, \omega)$  is jointly measurable in  $(x, \omega)$ , and is integrable over  $\omega$ , for almost  $x \in X$  held fixed.*
2. *For almost  $\omega \in D$ ,  $g(x, \omega)$  is an absolutely continuous function of  $x$ .*
3.  *$\partial g / \partial x$  is locally integrable, that is for all compact intervals  $[a, b]$  contained in  $X$*

$$\int_a^b \int_D \left| \frac{\partial g}{\partial x}(x, \omega) \right| d\omega dx < \infty.$$

*Then  $\int_D g(x, \omega) d\omega$  is an absolutely continuous function of  $x$ , and for almost  $x \in X$ , its derivative exists and is given by*

$$\frac{d}{dx} \int_D g(x, \omega) d\omega = \int_D \frac{dg}{dx}(x, \omega) d\omega.$$

## 4.2 Effects of the regularity of the kernel on the solution

In this section, we investigate how the smoothness and singularity of the kernel influence the behavior of the solution near discontinuity points of the source term.

### 4.2.1 Unboundedness of the kernel and its higher derivatives

We first consider the case in which the kernel itself is unbounded at the origin. In this setting, the singular behavior of  $K$  is transferred directly to the first derivative of the solution at the jump point.

**Theorem 40.** *Assume that the kernel  $K$  satisfies Assumptions (A1)–(A2), that  $K$  is unbounded at 0, and that  $K \in L^2(\mathbb{R})$ .*

1. *Suppose that  $f$  satisfies Assumption (B) and  $b \in W^{1,2}(\Omega_I)$ . Then, for the solution  $u \in L^2(\Omega \cup \Omega_I)$  of (4.1), one has*

$$|u'(0^\pm)| = \infty.$$

2. *Suppose that  $u$  satisfies Assumption (B). Then, for  $f := Lu$ , one has*

$$|f'(0^\pm)| = \infty.$$

*Proof.* 1. Since  $u \in L^2(\mathbb{R})$  by zero extension and  $K \in L^2(\mathbb{R})$ , the convolution  $u * K$  is continuous on  $\mathbb{R}$ . From the decomposition

$$u * K - \alpha u = f \quad \text{in } \Omega,$$

it follows that  $u$  has a jump discontinuity at 0 of magnitude

$$c_0 := [u]_0 = -\frac{1}{\alpha}[f]_0,$$

where  $0 < |c_0| < \infty$ . Moreover,  $u$  is continuous on  $\Omega \setminus \{0\}$ . Define

$$u_0 := u - c_0 \varphi_0. \tag{4.9}$$

Then  $u_0 \in C^0(\Omega)$  and  $u_0$  solves

$$\begin{cases} Lu_0 = f_0 := f - c_0 L\varphi_0 & \text{in } \Omega, \\ u_0 = b - c_0 \varphi_0 & \text{in } \Omega_I. \end{cases} \quad (4.10)$$

We now show that  $u_0$  has enough regularity to differentiate the convolution term. Since  $f_0 = f - c_0 L\varphi_0$ ,  $f \in L^\infty(\Omega)$  by Assumption (B) and  $L\varphi_0 \in L^\infty(\Omega)$  by Lemma 11, it follows that  $f_0 \in L^\infty(\Omega)$ . Moreover,

$$f'_0 = f' - c_0 K$$

almost everywhere in  $\Omega$ . Since  $f'$  is essentially bounded on each side of the origin and  $K \in L^2(\mathbb{R})$ , it follows that  $f'_0 \in L^2(\Omega)$ . Hence  $f_0 \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ . By Theorem 39, we obtain  $u_0 \in W^{1,2}(\Omega)$ . Since  $b \in W^{1,2}(\Omega_I)$ , it follows that

$$u_0 \in W^{1,2}(\Omega \cup \Omega_I).$$

In particular,  $u_0$  and its weak derivative  $u'_0$  both belong to  $L^2(\Omega \cup \Omega_I)$ .

Consider

$$u_0 * K(x) = \int_{-\delta}^{\delta} u_0(x-y)K(y)dy := \int_{-\delta}^{\delta} g(x,y)dy,$$

where  $g(x,y) := u_0(x-y)K(y)$ . For each fixed  $x \in \Omega$ , Holder's inequality gives

$$\|g(x, \cdot)\|_{L^1(-\delta,\delta)} \leq \|u_0(x - \cdot)\|_{L^2(-\delta,\delta)} \|K\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\Omega \cup \Omega_I)} \|K\|_{L^2(\mathbb{R})} < \infty.$$

For fixed  $y \in (-\delta, \delta)$ , the pointwise derivative

$$\frac{\partial g}{\partial x} = u'_0(x-y)K(y)$$

exists for almost every  $x$ , and is integrable in  $x$  because  $u'_0 \in L^2(\Omega \cup \Omega_I)$ . Thus  $g$  is absolutely continuous in  $x$ . Furthermore, for every compact interval  $[a, b] \subset \Omega$ ,

$$\int_a^b \int_{-\delta}^{\delta} \left| \frac{\partial g}{\partial x}(x, y) \right| dy dx = \int_a^b (|u'_0| * K)(x) dx \leq (b - a) \|u'_0\|_{L^2(\Omega \cup \Omega_I)} \|K\|_{L^2(\mathbb{R})} < \infty.$$

Therefore, by Lemma 13,

$$\frac{d}{dx}(u_0 * K)(x) = \int_{-\delta}^{\delta} \frac{\partial g}{\partial x}(x, y) dy = u'_0 * K(x)$$

for almost every  $x \in \Omega$ . Since  $u'_0 \in L^2(\mathbb{R})$  and  $K \in L^2(\mathbb{R})$ , the convolution  $u'_0 * K$  is continuous on  $\mathbb{R}$ . On the other hand, from (4.10) and Lemma 11,

$$f'_0 = f' - c_0 K$$

almost everywhere. Because  $K$  is unbounded at 0 and  $0 < |c_0| < \infty$ , we obtain

$$|f'_0(0^\pm)| = \infty.$$

Using the decomposition

$$u_0 * K - \alpha u_0 = f_0,$$

and the continuity of  $(u_0 * K)'$ , it follows that

$$|u'_0(0^\pm)| = \infty.$$

Finally, since  $u = u_0 + c_0 \varphi_0$  by (4.9), we conclude that

$$|u'(0^\pm)| = \infty.$$

2. As in Part (1), since  $u \in L^2(\mathbb{R})$  by zero extension and  $K \in L^2(\mathbb{R})$ , the convolution

$u * K$  is continuous on  $\mathbb{R}$ . From,

$$u * K - \alpha u = f \quad \text{in } \Omega,$$

it follows that  $f$  has a jump discontinuity at 0 of magnitude

$$[f]_0 = -\alpha[u]_0,$$

and  $f$  is continuous on  $\Omega \setminus \{0\}$ .

Let

$$c_0 := [u]_0 = -\frac{1}{\alpha}[f]_0,$$

and define  $u_0 := u - c_0\varphi_0$ . Since  $u$  satisfies Assumption (B), it follows that  $u_0$  is absolutely continuous and hence belongs to  $W^{1,2}(\Omega \cup \Omega_I)$ . Therefore, by Lemma 13,

$$(u_0 * K)' = u_0' * K$$

almost everywhere. Since  $u_0' \in L^2(\mathbb{R})$  and  $K \in L^2(\mathbb{R})$ , the function  $(u_0 * K)'$  is continuous on  $\mathbb{R}$ .

Let  $f_0 := f - c_0L\varphi_0$ . From the decomposition

$$u_0 * K - \alpha u_0 = f_0,$$

it follows that

$$|f_0'(0^\pm)| < \infty.$$

On the other hand, Lemma 11 gives

$$f_0' = f' - c_0K$$

almost everywhere. Since  $K$  is unbounded at 0 and  $0 < |c_0| < \infty$ , we conclude that

$$|f'(0^\pm)| = \infty.$$

□

Theorem 40 is illustrated by the following singular kernel:

$$S(x, y) = S(|x - y|) = \frac{c_S}{|x - y|^\beta} \chi_{(-\delta, \delta)}(x - y), \quad c_S = \frac{3 - \beta}{\delta^{3-\beta}}, \quad 0 < \beta < 1.$$

A direct computation gives

$$L_S \varphi_0(x) = \begin{cases} \frac{c_S}{1-\beta} (x^{1-\beta} - \delta^{1-\beta}), & 0 \leq x < \delta, \\ \frac{c_S}{1-\beta} (\delta^{1-\beta} - (-x)^{1-\beta}), & -\delta < x < 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is immediate that

$$|(L_S \varphi_0)'(0^\pm)| = \infty.$$

We next consider kernels that remain bounded at the origin, but whose first derivative is unbounded there. In this case, the singularity is transferred to the second derivative of the solution.

**Theorem 41.** *Assume that the kernel  $K$  satisfies Assumptions (A1)–(A2), that  $K \in L^\infty(\mathbb{R})$ , that  $K'$  is unbounded at 0, and that  $K' \in L^2(\mathbb{R})$ .*

1. *Suppose that  $f$  satisfies Assumption (B) and  $b \in W^{1,2}(\Omega_I)$ . Then, for the solution  $u \in L^2(\Omega \cup \Omega_I)$  of (4.1), one has*

$$|u''(0^\pm)| = \infty.$$

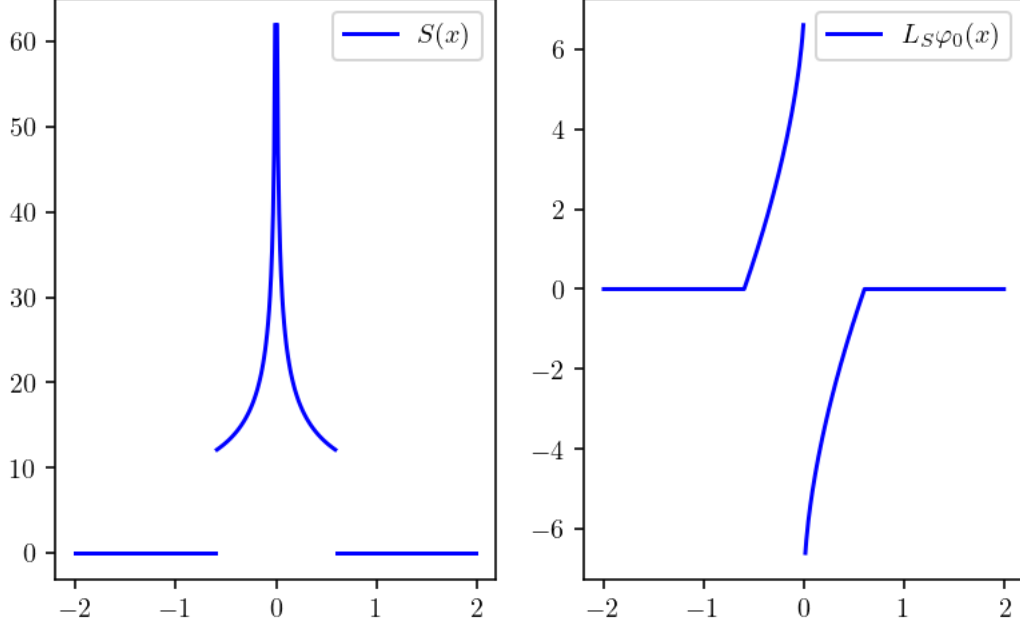


Figure 4.1: Graphs of the kernel  $S(x)$  (left) and  $L_S\varphi_0(x)$  (right) for  $\delta = 0.6$  and  $\beta = 0.4$

2. Suppose that  $u$  satisfies Assumption (B). Then, for  $f := Lu$ , one has

$$|f''(0^\pm)| = \infty.$$

*Proof.* 1. Since  $u \in L^2(\mathbb{R})$  by the zero extension and  $K \in L^2(\mathbb{R})$  because  $K$  being bounded and compactly supported, the convolution  $u * K$  is continuous on  $\mathbb{R}$ . From the decomposition

$$u * K - \alpha u = f \quad \text{in } \Omega,$$

it follows that  $u$  is continuous on  $\Omega \setminus \{0\}$ , and that  $u$  has a jump discontinuity at 0 of magnitude

$$c_0 := [u]_0 = -\frac{1}{\alpha}[f]_0,$$

where  $0 < |c_0| < \infty$ .

Define

$$u_0 := u - c_0 \varphi_0. \quad (4.11)$$

Then  $u_0 \in C^0(\Omega)$  and solves

$$\begin{cases} Lu_0 = f_0 := f - c_0 L\varphi_0 & \text{in } \Omega, \\ u_0 = b - c_0 \varphi_0 & \text{in } \Omega_I. \end{cases} \quad (4.12)$$

Since  $K$  is continuous on  $[-\delta, 0) \cup (0, \delta]$ , it extends continuously to  $[-\delta, 0]$  and  $[0, \delta]$ .

Writing

$$u_0 * K(x) = \int_{x-\delta}^{x+\delta} u_0(y) K(x-y) dy = \int_{x-\delta}^{x+\delta} g(x, y) dy,$$

and applying Lemma 12 to each subintervals, we obtain

$$\begin{aligned} \frac{d}{dx} \int_{x-\delta}^x g(x, y) dy &= g(x, x) - g(x, x-\delta) + \int_{x-\delta}^x \frac{\partial g}{\partial x}(x, y) dy \\ &= u_0(x) K(0^+) - u_0(x-\delta) K(\delta^-) + \int_{x-\delta}^x u_0(y) K'(x-y) dy. \end{aligned} \quad (4.13)$$

and similarly,

$$\begin{aligned} \frac{d}{dx} \int_x^{x+\delta} g(x, y) dy &= g(x, x+\delta) - g(x, x) + \int_x^{x+\delta} \frac{\partial g}{\partial x}(x, y) dy \\ &= u_0(x+\delta) K(-\delta^+) - u_0(x) K(0^-) + \int_x^{x+\delta} u_0(y) K'(x-y) dy. \end{aligned} \quad (4.14)$$

Adding (4.13) and (4.14) together, we find

$$\begin{aligned} \frac{d}{dx} (u_0 * K)(x) &= K(-\delta^+) u_0(x+\delta) - K(\delta^-) u_0(x-\delta) + (K(0^+) - K(0^-)) u_0(x) \\ &\quad + (u_0 * K)'(x) \end{aligned} \quad (4.15)$$

Since  $u_0 \in L^2(\mathbb{R})$  and  $K' \in L^2(\mathbb{R})$ , the convolution  $u_0 * K'$  is continuous on  $\mathbb{R}$ . Therefore,  $(u_0 * K)'$  is continuous on  $\Omega$ .

From (4.12) and by Lemma 11, we have

$$f'_0 = f' - c_0 K$$

almost everywhere. Since  $f'$  is bounded on each side of the origin and  $K \in L^\infty(\mathbb{R})$ , it follows that  $0 \leq |[f'_0]_0| < \infty$ . Let

$$c_1 := [u'_0]_0 = -\frac{1}{\alpha} [f'_0]_0,$$

so that  $0 \leq |c_1| < \infty$ , and define

$$u_1 := u_0 - c_1 \varphi_1. \tag{4.16}$$

Then  $u_1 \in C^0(\Omega)$ ,  $u_1$  is continuously differentiable on  $\Omega \setminus \{\pm\delta\}$ , and  $u_1$  solves

$$\begin{cases} Lu_1 = f_1 := f_0 - c_1 L\varphi_1 & \text{in } \Omega, \\ u_1 = b - c_0 \varphi_0 - c_1 \varphi_1 & \text{in } \Omega_I. \end{cases} \tag{4.17}$$

Applying Lemma 12 and proceeding as above, we obtain

$$\begin{aligned} \frac{d^2}{dx^2}(u_1 * K)(x) &= K(-\delta^+)u'_1(x + \delta) - K(\delta^-)u'_1(x - \delta) + (K(0^+) - K(0^-))u'_1(x) \\ &\quad + \frac{d}{dx}(u_1 * K')(x) \end{aligned} \tag{4.18}$$

We now show that  $u_1 \in W^{1,2}(\Omega \cup \Omega_I)$ . Since

$$f_1 = f - c_0 L\varphi_0 - c_1 L\varphi_1,$$

and  $f \in L^\infty(\Omega)$  by Assumption (B), while  $L\varphi_0, L\varphi_1 \in L^\infty(\Omega)$  by Lemma 11, it follows that  $f_1 \in L^\infty(\Omega)$ . Also,

$$f'_1 = f' - c_0 K - c_1 L\varphi_0$$

almost everywhere, so  $f'_1 \in L^2(\Omega)$ . Hence

$$f_1 \in W^{1,2}(\Omega) \cap L^\infty(\Omega).$$

By Theorem 39, this implies  $u_1 \in W^{1,2}(\Omega)$ . Since  $b \in W^{1,2}(\Omega_I)$ , it follows that

$$u_1 \in W^{1,2}(\Omega \cup \Omega_I).$$

Therefore  $u'_1 \in L^2(\mathbb{R})$ .

By Lemma 13,

$$\frac{d}{dx}(u_1 * K')(x) = u'_1 * K'(x)$$

almost everywhere. Since  $u'_1 \in L^2(\mathbb{R})$  and  $K' \in L^2(\mathbb{R})$ , the convolution  $u'_1 * K'$  is continuous on  $\mathbb{R}$ . Hence (4.18) implies

$$|(u_1 * K)''(0^\pm)| < \infty.$$

On the other hand, by Lemma 11,

$$f''_1 = f'' - c_0 K' - c_1 K$$

almost everywhere. Since  $K'$  is unbounded at 0 and  $0 < |c_0| < \infty$ , it follows that

$$|f''_1(0^\pm)| = \infty.$$

Using the decomposition

$$u_1 * K - \alpha u_1 = f_1,$$

we conclude that

$$|u_1''(0^\pm)| = \infty.$$

Finally, by (4.11) and (4.16),

$$|u''(0^\pm)| = \infty.$$

2. The argument is analogous to that of Part (1). Since  $u * K$  is continuous on  $\mathbb{R}$ , the decomposition

$$u * K - \alpha u = f \quad \text{in } \Omega$$

shows that  $f$  has a jump discontinuity at 0 of magnitude

$$[f]_0 = -\alpha[u]_0,$$

and is continuous on  $\Omega \setminus \{0\}$ .

Let

$$c_0 := [u]_0 = -\frac{1}{\alpha}[f]_0,$$

and define  $u_0 := u - c_0\varphi_0$ . Since  $u$  satisfies Assumption (B), it is  $C^N$  on  $(a_1, 0)$  and on  $(0, a_2)$ , with a single jump discontinuity at 0. By construction the jump at 0 is removed. Hence  $u_0$  is continuous on  $\Omega$  and remains  $C^N$  on each of the intervals  $(a_1, 0)$  and  $(0, a_2)$ . In particular,  $u_0$  is absolutely continuous on every compact subinterval of  $\Omega$ , and its weak derivative agrees almost everywhere with the classical derivative on  $(a_1, 0) \cup (0, a_2)$ . Since  $u_0$  and  $u_0'$  are both square-integrable on  $\Omega \cup \Omega_I$ , it follows that

$$u_0 \in W^{1,2}(\Omega \cup \Omega_I).$$

Lemma 13 then implies  $(u_0 * K)' = u'_0 * K$  almost everywhere. As in (4.15), we have

$$\begin{aligned} \frac{d}{dx}(u_0 * K)(x) &= K(-\delta^+)u_0(x + \delta) - K(\delta^-)u_0(x - \delta) + (K(0^+) - K(0^-))u_0(x) \\ &\quad + (u_0 * K')(x). \end{aligned} \tag{4.19}$$

Since  $u_0 \in L^2(\mathbb{R})$  and  $K' \in \mathbb{R}$ , the convolution  $u_0 * K'$  is continuous on  $\mathbb{R}$ . Hence  $(u_0 * K)'$  is continuous on  $\Omega$ .

Define  $f_0 := f - c_0 L\varphi_0$ . From

$$u_0 * K - \alpha u_0 = f_0,$$

it follows that  $f'_0$  has a jump discontinuity at 0 of magnitude

$$[f'_0]_0 = -\alpha[u'_0]_0.$$

Let

$$c_1 := [u'_0]_0 = -\frac{1}{\alpha}[f'_0]_0,$$

and define

$$u_1 := u_0 - c_1 \varphi_1.$$

Since  $0 < |c_0| < \infty$ , we may differentiate once more, as in (4.18),

$$\begin{aligned} \frac{d^2}{dx^2}(u_1 * K)(x) &= K(-\delta^+)u'_1(x + \delta) - K(\delta^-)u'_1(x - \delta) + (K(0^+) - K(0^-))u'_1(x) \\ &\quad + \frac{d}{dx}(u_1 * K')(x). \end{aligned} \tag{4.20}$$

By Lemma 13,

$$(u_1 * K')' = u_1' * K',$$

almost everywhere. Since  $u_1' \in L^2(\mathbb{R})$  and  $K' \in L^2(\mathbb{R})$ , the convolution  $u_1' * K'$  is continuous on  $\mathbb{R}$ . Therefore, from (4.20),

$$|(u_1 * K)''(0^\pm)| < \infty.$$

Defining  $f_1 := f_0 - c_1 L\varphi_1$ , the decomposition

$$u_1 * K - \alpha u_1 = f_1$$

implies that

$$|f_1''(0^\pm)| < \infty.$$

By Lemma 11,

$$f_1'' = f'' - c_0 K' - c_1 K$$

almost everywhere. Since  $K'$  is unbounded at 0 and  $0 < |c_0| < \infty$ , it follows that

$$|f''(0^\pm)| = \infty.$$

□

To illustrate Theorem 41, consider the kernel

$$P(x, y) = P(|x - y|) = c_P \left(1 - \frac{|x - y|^\beta}{\delta^\beta}\right) \chi_{(-\delta, \delta)}(x - y), \quad c_P = \frac{3(3 + \beta)}{\beta \delta^3}, \quad 0 < \beta < 1,$$

which satisfies

$$|P'(0^\pm)| = \infty.$$

A direct computation gives

$$L_P\varphi_0(x) = \begin{cases} c_P(x - \delta) - \frac{3(3+\beta)}{\beta(1+\beta)\delta^{3+\beta}}(x^{1+\beta} - \delta^{1+\beta}), & 0 \leq x < \delta, \\ c_P(x + \delta) - \frac{3(3+\beta)}{\beta(1+\beta)\delta^{3+\beta}}(\delta^{1+\beta} - (-x)^{1+\beta}) & -\delta < x < 0, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that  $|(L_P\varphi_0)''(0^\pm)| = \infty$ .

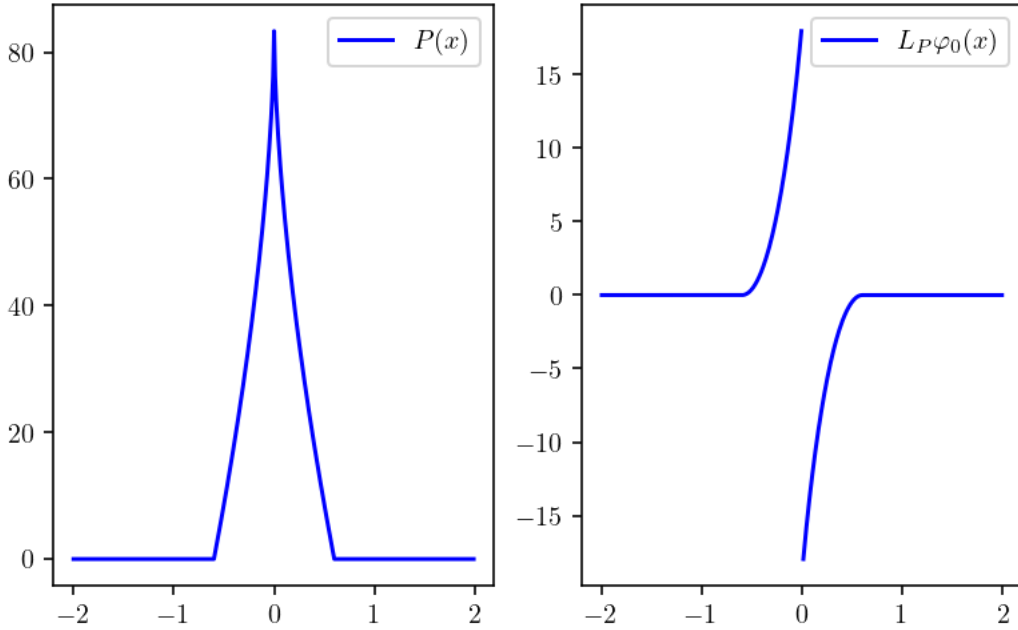


Figure 4.2: Graphs of the kernel  $P(x)$  (left) and  $L_P\varphi_0(x)$  (right) for  $\delta = 0.6$  and  $\beta = 0.6$

Theorems 40 and 41 admit the following generalization.

**Theorem 42.** *Assume that the kernel  $K$  satisfies Assumptions (A1)–(A2), that*

$$K, K', \dots, K^{(j-1)} \in L^\infty(\mathbb{R}),$$

*that  $K^{(j)}$  is unbounded at 0, and that  $K^{(j)} \in L^2(\mathbb{R})$ .*

1. *Suppose that  $f$  satisfies Assumption (B) and  $b \in W^{1,2}(\Omega_I)$ . Then, for the solution*

$u \in L^2(\Omega \cup \Omega_I)$  of (4.1), one has

$$|u^{(j+1)}(0^\pm)| = \infty.$$

2. Suppose that  $u$  satisfies Assumption (B). Then, for  $f := Lu$ , one has

$$|f^{(j+1)}(0^\pm)| = \infty.$$

The proof of Theorem 42 follows by iterating the argument used in Theorem 41. At each step, one subtracts the appropriate jump component  $c_m \varphi_m$ , differentiates the convolution representation, and uses the singularity of  $K^{(j)}$  at the origin to conclude blow-up of the next derivative.

These results show that the regularity of the kernel at the origin determines the order of derivative blow-up in the solution. In particular, the stronger the singularity of the kernel or its derivatives at 0, the higher the order at which singular behavior appears in the solution.

### 4.2.2 Jump discontinuities at $\pm\delta$

We next consider the effect of jump discontinuities of the kernel at the horizon endpoints  $\pm\delta$ . In contrast to the singular behavior at the origin studied in the previous subsection, these jumps generate a cascade of discontinuities in higher derivatives of the solution at translated locations.

**Theorem 43.** *Assume that the kernel  $K$  satisfies Assumptions (A1)–(A2), that*

$$K, K', \dots, K^{(j)}, \dots \in L^\infty(\mathbb{R})$$

*for all  $j$ , and that  $K$  has jump discontinuities at  $\pm\delta$ .*

1. Suppose that  $f$  satisfies Assumption (B) and  $b \in W^{1,2}(\Omega_I)$ . Then, for the solution  $u \in L^2(\Omega \cup \Omega_I)$  of (4.1), and for every integer  $k \geq 1$  such that the points  $\pm k\delta$  belong

to  $\Omega$ , one has

$$0 < |[u^{(k)}]_{\pm k\delta}| < \infty.$$

2. Suppose that  $u$  satisfies Assumption (B). Then, for  $f := Lu$ , one has

$$0 < |[f']_{\pm\delta}| < \infty,$$

and

$$f \in C^\infty(\Omega \setminus \{0, \pm\delta\}).$$

The jump propagation described below is understood only at translated points that remain inside the spatial domain  $\Omega$ . Thus, whenever we refer to jumps at  $\pm k\delta$ , we implicitly assume that these points belong to  $\Omega$ .

*Proof.* 1. Since  $u \in L^2(\mathbb{R})$  by the zero extension and  $K \in L^2(\mathbb{R})$  because  $K$  being bounded and compactly supported, the convolution  $u * K$  is continuous on  $\mathbb{R}$ . From

$$u * K - \alpha u = f \quad \text{in } \Omega,$$

it shows that  $u$  has a jump discontinuity at 0 of magnitude

$$c_0 := [u]_0 = -\frac{1}{\alpha}[f]_0,$$

where  $0 < |c_0| < \infty$ . Moreover,  $u$  is continuous on  $\Omega \setminus \{0\}$ .

Define

$$u_0 := u - c_0 \varphi_0. \tag{4.21}$$

Then  $u_0 \in C^0(\Omega)$  and  $u_0$  solves

$$\begin{cases} Lu_0 = f_0 := f - c_0 L\varphi_0 & \text{in } \Omega, \\ u_0 = b - c_0 \varphi_0 & \text{in } \Omega_I. \end{cases}$$

As in (4.15), applying Lemma 12 yields

$$\begin{aligned} \frac{d}{dx}(u_0 * K)(x) &= K(-\delta^+)u_0(x + \delta) - K(\delta^-)u_0(x - \delta) + (K(0^+) - K(0^-))u_0(x) \\ &\quad + (u_0 * K')(x). \end{aligned} \tag{4.22}$$

Since  $u_0 \in L^2(\mathbb{R})$  and  $K' \in L^2(\mathbb{R})$ , the convolution  $u_0 * K'$  is continuous on  $\mathbb{R}$ . Hence  $(u_0 * K)'$  is continuous on  $\Omega$ .

By Lemma 11,

$$f'_0 = f' - c_0 K,$$

almost everywhere. Because  $K$  has jump discontinuities at  $\pm\delta$ , it follows that  $f'_0$  has jump discontinuities at 0 and  $\pm\delta$ . Since  $(u_0 * K)'$  is continuous, the decomposition

$$u_0 * K - \alpha u_0 = f_0$$

implies that  $u'_0$  has jump discontinuities at the same points. In particular, using (4.21), we obtain

$$0 < |[u']_{\pm\delta}| < \infty.$$

Next, let

$$c_1 := [u'_0]_0 = -\frac{1}{\alpha}[f'_0]_0,$$

where  $0 \leq |c_1| < \infty$ , and define

$$u_1 := u_0 - c_1 \varphi_1. \quad (4.23)$$

Then  $u_1 \in C^0(\Omega)$ ,  $u_1$  is continuously differentiable on  $\Omega \setminus \{\pm\delta\}$ , and  $u_1$  solves

$$\begin{cases} Lu_1 = f_1 := f_0 - c_1 L\varphi_1 & \text{in } \Omega, \\ u_1 = b - c_0 \varphi_0 - c_1 \varphi_1 & \text{in } \Omega_I. \end{cases}$$

Applying Lemma 12 once more, as in (4.18), we obtain

$$\begin{aligned} \frac{d^2}{dx^2}(u_1 * K)(x) &= K(-\delta^+)u_1'(x + \delta) - K(\delta^-)u_1'(x - \delta) + (K(0^+) - K(0^-))u_1'(x) \\ &\quad + K'(-\delta^+)u_1(x + \delta) - K'(\delta^-)u_1(x - \delta) + (K'(0^+) - K'(0^-))u_1(x) \\ &\quad + (u_1 * K'')(x) \end{aligned} \quad (4.24)$$

Since  $u_1 \in L^2(\mathbb{R})$  and  $K'' \in L^2(\mathbb{R})$ , the convolution  $u_1 * K''$  is continuous on  $\mathbb{R}$ . Moreover, by assumption,  $K(0^+) = K(0^-)$ , while  $K(-\delta^+)$  and  $K(\delta^-)$  are nonzero. Since  $u_1 \in C^0(\Omega)$ ,  $u_1' \in C^0(\Omega \setminus \{\pm\delta\})$ , and  $u_1'$  has jump discontinuities at  $\pm\delta$ , it follows from (4.24) that  $(u_1 * K)''$  has jump discontinuities at 0 and  $\pm 2\delta$ .

On the other hand, Lemma 11 gives

$$f_1'' = f'' - c_0 K' - c_1 K$$

almost everywhere, so  $f_1''$  has jump discontinuities at 0 and  $\pm\delta$ . Therefore, from the decomposition

$$u_1 * K - \alpha u_1 = f_1,$$

we conclude that  $u_1''$  has jump discontinuities at 0,  $\pm\delta$ , and  $\pm 2\delta$ . Using (4.21) and

(4.23), we obtain

$$0 < |[u'']_{\pm 2\delta}| < \infty.$$

Repeating the same argument inductively shows that each further differentiation produces jump discontinuities at the next translated locations  $\pm k\delta$ , which proves the claim.

2. The argument begins as in Part (1). Since  $u * K$  is continuous on  $\mathbb{R}$ , the decomposition

$$u * K - \alpha u = f \quad \text{in } \Omega,$$

shows that  $f$  has a jump discontinuity at 0 of magnitude

$$[f]_0 = -\alpha[u]_0,$$

and that  $f$  is continuous on  $\Omega \setminus \{0\}$ .

Let

$$c_0 := [u]_0 = -\frac{1}{\alpha}[f]_0,$$

where  $0 < |c_0| < \infty$ , and define

$$u_0 := u - c_0 \varphi_0.$$

Then  $u_0 \in C^0(\Omega)$ . As in (4.22), we have

$$\begin{aligned} \frac{d}{dx}(u_0 * K)(x) &= K(-\delta^+)u_0(x + \delta) - K(\delta^-)u_0(x - \delta) + (K(0^+) - K(0^-))u_0(x) \\ &\quad + (u_0 * K')(x). \end{aligned} \tag{4.25}$$

Since  $u_0 \in L^2(\mathbb{R})$  and  $K' \in L^2(\mathbb{R})$ , the convolution  $u_0 * K'$  is continuous on  $\mathbb{R}$ . Hence

$(u_0 * K)'$  is continuous on  $\Omega$ .

Let  $f_0 := f - c_0 L\varphi_0$ . From the decomposition

$$u_0 * K - \alpha u_0 = f_0,$$

it follows that  $f'_0$  has a jump discontinuity at 0 of magnitude

$$[f'_0]_0 = -\alpha[u'_0]_0,$$

and  $f'_0 \in C^0(\Omega \setminus \{0\})$ . By Lemma 11,

$$f'_0 = f' - c_0 K$$

almost everywhere. Since  $K$  has jumps only at  $\pm\delta$ , we conclude that

$$f' \in C^0(\Omega \setminus \{0, \pm\delta\}) \quad \text{and} \quad 0 < |[f']_{\pm\delta}| < \infty.$$

Next, let

$$c_1 := [u'_0]_0 = -\frac{1}{\alpha}[f'_0]_0,$$

where  $0 \leq |c_1| < \infty$ , and define  $u_1 := u_0 - c_1 \varphi_1$ . Then  $u_1 \in C^1(\Omega)$ . Let  $f_1 := f_0 - c_1 L\varphi_1$ .

As in (4.24), we have

$$\begin{aligned} \frac{d^2}{dx^2}(u_1 * K)(x) &= K(-\delta^+)u'_1(x + \delta) - K(\delta^-)u'_1(x - \delta) + (K(0^+) - K(0^-))u'_1(x) \\ &\quad + K'(-\delta^+)u_1(x + \delta) - K'(\delta^-)u_1(x - \delta) + (K'(0^+) - K'(0^-))u_1(x) \\ &\quad + (u_1 * K'')(x). \end{aligned} \tag{4.26}$$

Since  $u_1 \in L^2(\mathbb{R})$  and  $K'' \in L^2(\mathbb{R})$ , the convolution  $u_1 * K''$  is continuous on  $\mathbb{R}$ . Because  $u_1 \in C^1(\Omega)$ , it follows from (4.26) that  $(u_1 * K)'' \in C^0(\Omega)$ .

From the decomposition

$$u_1 * K - \alpha u_1 = f_1,$$

we obtain  $f_1'' \in C^0(\Omega \setminus \{0\})$ . Using Lemma 11,

$$f_1'' = f'' - c_0 K' - c_1 K$$

almost everywhere, which implies

$$f'' \in C^0(\Omega \setminus \{0, \pm\delta\}).$$

Repeating the argument inductively yields

$$f \in C^\infty(\Omega \setminus \{0, \pm\delta\}).$$

□

Theorem 43 is illustrated by the truncated Gaussian kernel

$$G(x, y) = G(|x - y|) = c_G \exp\left(-\frac{|x - y|^2}{\delta^2}\right) \chi_{(-\delta, \delta)}(x - y), \quad c_G = \frac{4e}{\delta^3(e\sqrt{\pi} \operatorname{erf}(1) - 2)}.$$

A direct computation gives

$$L_G \varphi_0(x) = \begin{cases} \frac{-\delta\sqrt{\pi}c_G}{2} \left( \operatorname{erf}\left(\frac{-x}{\delta}\right) - \operatorname{erf}(1) \right), & -\delta < x < 0, \\ \frac{\delta\sqrt{\pi}c_G}{2} \left( \operatorname{erf}\left(\frac{x}{\delta}\right) - \operatorname{erf}(1) \right), & 0 \leq x < \delta, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$(L_G\varphi_0)'(x) = \begin{cases} c_G \exp(\frac{-x^2}{\delta^2}), & -\delta < x < \delta, x \neq 0, \\ 0, & x < -\delta \quad \text{or} \quad x > \delta. \end{cases}$$

This kernel satisfies the hypotheses of Theorem 43. Therefore,

$$L_G\varphi_0 \in C^\infty(\mathbb{R} \setminus \{0, \pm\delta\}) \quad \text{and} \quad 0 < |[(L_G\varphi_0)']_{\pm\delta}| < \infty.$$

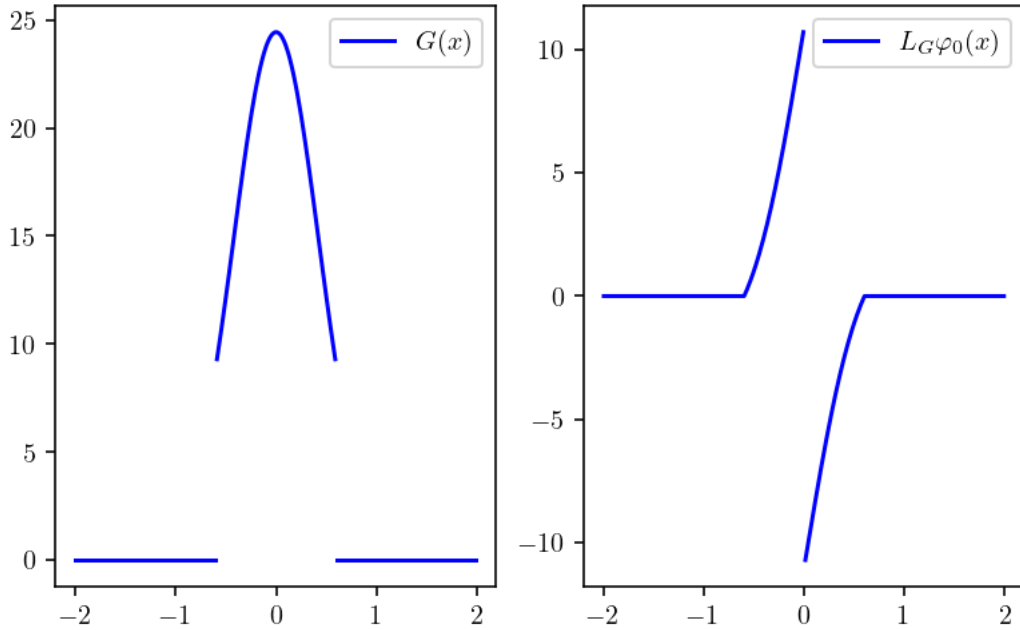


Figure 4.3: Graphs of the kernel  $G(x)$  (left) and  $L_G\varphi_0(x)$  (right) for  $\delta = 0.6$

A closely related variant arises when  $K$  is continuous at  $\pm\delta$ , but  $K'$  has jump discontinuities there. In that case, the first nontrivial jump appears one derivative later.

**Theorem 44.** *Assume that the kernel  $K$  satisfies Assumptions (A1)–(A2), that*

$$K, K', \dots, K^{(j)}, \dots \in L^\infty(\mathbb{R})$$

*for all  $j$ , that  $K$  is continuous at  $\pm\delta$ , and that  $K'$  has jump discontinuities at  $\pm\delta$ .*

1. Suppose that  $f$  satisfies Assumption (B) and  $b \in W^{1,2}(\Omega_I)$ . Then, for the solution  $u \in L^2(\Omega \cup \Omega_I)$  of (4.1), and for every integer  $k \geq 1$  such that the points  $\pm k\delta$  belong to  $\Omega$ , one has

$$0 < |[u^{(k+1)}]_{\pm k\delta}| < \infty.$$

2. Suppose that  $u$  satisfies Assumption (B). Then, for  $f := Lu$ ,

$$0 < |[f'']_{\pm\delta}| < \infty,$$

and

$$f \in C^\infty(\Omega \setminus \{0, \pm\delta\}).$$

### 4.2.3 Compactly supported smooth kernel

We finally consider the case of a compactly supported smooth kernel. In this setting, no new singular behavior is created by the operator, and the source term and the solution have equivalent piecewise smooth regularity.

**Theorem 45.** Assume that the kernel  $K$  satisfies Assumptions (A1)–(A2) and, in addition, that  $K \in C_c^\infty(\mathbb{R})$ . Let  $f \in L^2(\Omega)$  and  $u \in L^2(\Omega \cup \Omega_I)$  satisfy  $Lu = f$  in  $\Omega$ . Then  $f$  satisfies Assumption (B) if and only if  $u$  satisfies Assumption (B). Furthermore,

$$[u^{(k)}]_0 = -\frac{1}{\alpha}[f^{(k)}]_0, \quad k = 0, 1, 2, \dots \quad (4.27)$$

*Proof.* Since  $K \in C_c^\infty(\mathbb{R})$ , the convolution  $u * K$  belongs to  $C^\infty(\mathbb{R})$ . The decomposition

$$u * K - \alpha u = f$$

therefore implies that  $u$  and  $f$  have the same piecewise smooth regularity. The jump relation (4.27) follows immediately by comparing the left and right limits of the derivatives at 0.  $\square$

An example illustrating Theorem 45 is the kernel

$$H(x, y) = H(|x - y|) = c_H \exp \left( -\frac{1}{1 - |x - y|^2/\delta^2} \right) \chi_{(-\delta, \delta)}(x - y). \quad (4.28)$$

### 4.3 Semi-analytic spectral method

In this section, we introduce a semi-analytic spectral method for the accurate resolution of jump discontinuities in the solution of (4.1). The main idea is to transform the original discontinuous problem into a sequence of smoother problems that are better suited for spectral approximation.

We begin with a kernel  $K \in C_c^\infty(\mathbb{R})$  satisfying Assumptions (A1) and (A2) and assume that the source term  $f$  satisfies Assumption (B). Then,  $u$  has a jump at  $x = 0$  by Theorem 45. To simplify notation, in this section, we denote a jump of a function  $g$  at 0 by  $[g] = [g]_0 = g(0^+) - g(0^-)$ .

By Theorem 45, the solution  $u$  also satisfies Assumption (B), and

$$[u^{(k)}] = -\frac{1}{\alpha} [f^{(k)}],$$

for  $k = 0, 1, 2, \dots$ . Let

$$c_0 := [u] = -\frac{1}{\alpha} [f],$$

and define

$$u_0 := u - c_0 \varphi_0$$

on  $\mathbb{R}$ . Since both  $u$  and  $\varphi_0$  belong to  $C^\infty(\Omega \setminus \{0\})$ , it follows that  $u_0 \in C^\infty(\Omega \setminus \{0\})$ .

Moreover, the jump at 0 is removed because

$$[u_0] = [u] - c_0 [\varphi_0] = [u] - c_0 = 0.$$

Hence  $u_0 \in C(\Omega)$ , and therefor  $Lu_0 := f_0 \in C(\Omega)$  by Lemma 10. Since

$$f_0 = f - c_0 L\varphi_0,$$

Theorem 45 and Lemma 11 imply that

$$c_1 := [u'] = [u'_0] = -\frac{1}{\alpha}[f'_0] = -\frac{1}{\alpha}[f'].$$

We then define

$$u_1 := u_0 - c_1 \varphi_1$$

on  $\mathbb{R}$ . Since  $u'_1 = u' - c_1 \varphi_0$  for all  $x \neq 0$ , we have  $u_1 \in C(\Omega)$  and  $u'_1 \in C(\Omega \setminus \{0\})$ . The jump in the derivative is removed because

$$[u'_1] = [u'] - c_1[\varphi_0] = [u'] - c_1 = 0.$$

Thus  $u'_1 \in C(\Omega)$ , which implies that both  $u_1$  and  $f_1 := Lu_1$  belong to  $C^1(\Omega)$ .

Proceeding inductively, we define the sequences  $\{c_k\}$ ,  $\{u_k\}$ , and  $\{f_k\}$  by

$$\begin{cases} c_k := -\frac{1}{\alpha} \left[ f_{k-1}^{(k)} \right], \\ u_k := u_{k-1} - c_k \varphi_k, \\ f_k := f_{k-1} - c_k L\varphi_k, \end{cases} \quad (4.29)$$

for  $k \geq 0$ , with  $u_{-1} \equiv u$  and  $f_{-1} \equiv f$ . By Theorem 45, and by repeating the above argument inductively, it follows that

$$u_k, f_k \in C^k(\Omega), \quad k \geq 0.$$

This motivates the two successive algorithms, Algorithms 1 and 2, for resolving the jump discontinuities in  $u$ .

---

**Algorithm 1** Successive smoothing procedure for constructing the function  $f_M$  and the coefficients  $c_0, \dots, c_M$  in (4.29).

---

**function** SMOOTH( $f, M$ )

$f_{-1} \leftarrow f$

**for**  $k=0,1,2,\dots,M$  **do**

    Compute  $c_k$  in (4.29) symbolically

$\triangleright$  Magnitude of  $[u^{(k)}]$

    Compute  $L\varphi_k$  symbolically

$f_k \leftarrow f_{k-1} - c_k L\varphi_k$

$\triangleright f_k \in C^k(\Omega)$

**end for**

**return**  $f_M, c_0, \dots, c_M$

**end function**

---



---

**Algorithm 2** Construction of the semi-analytic approximation to the solution of (4.1).

---

**function** RESOLVE( $f, b, M$ )

$f_M, c_0, c_1, \dots, c_M \leftarrow \text{SMOOTH}(f, M)$

$b_M \leftarrow b - c_0\varphi_0 - c_1\varphi_1 - \dots - c_M\varphi_M$

Solve  $Lu_M = f_M$  in  $\Omega$ ,  $u_M = b_M$  in  $\Omega_I$

$\triangleright$  Using a spectral solver

$u \leftarrow u_M + \sum_{k=0}^M c_k \varphi_k$

$\triangleright$  Semi-analytic reconstruction

**return**  $u$

**end function**

---

One limitation of using kernels in  $C_c^\infty(\mathbb{R})$  is that their Fourier multipliers  $m$ , defined through

$$\widehat{Lu} = m\widehat{u},$$

are often difficult to compute explicitly. As noted in <sup>53</sup>, this multiplier is a central component of the spectral method. As a practical trade-off, we therefore reduce the regularity of  $K$  while retaining enough smoothness for Algorithms 1 and 2 to remain valid up to a finite level  $M$ .

For example, consider the kernel

$$K(x, y) = K(|x - y|) = \frac{105}{\delta^3} \left(1 - \left|\frac{x - y}{\delta}\right|\right)^4 \chi_{(-\delta, \delta)}(x - y). \quad (4.30)$$

Then,  $K \in C^3(\mathbb{R} \setminus \{0\})$ , and one can verify that Algorithms 1 and 2 apply up to level  $M = 4$ . Higher levels of smoothing can be achieved by choosing a smoother kernel  $K$  by a slight modification to (4.30).

## 4.4 Numerical results

In this section, we present numerical experiments illustrating the performance of the proposed semi-analytic spectral method. We consider both manufactured solutions, for which the exact solution is known, and a problem with a prescribed discontinuous source term, for which the solution is computed numerically. In all cases, the results demonstrate that successive smoothing significantly improves both accuracy and convergence.

### 4.4.1 Manufactured solutions

We begin with manufactured solutions  $u$  that are continuous across the interface between  $\Omega$  and  $\Omega_I$ . For each such choice of  $u$ , the corresponding source term  $f$  and constraint  $b$  in (4.1) are generated from the exact solution. More precisely,  $f$  is computed symbolically from the definition of the operator  $L$  in (4.2), while  $b$  is obtained by restricting  $u$  to  $\Omega_I$ .

In Examples 1 and 2, we take  $\Omega = (-8, 8)$ ,  $\delta = 1.6$ , and use the kernel  $K$  given by (4.30). We denote by  $N$  the number of grid points in  $\Omega$ , by  $M$  the smoothing level used in Algorithms 1 and 2, and by  $u_N^M$  the corresponding numerical approximation.

*Example 1.* Consider the manufactured solution

$$u(x) = \begin{cases} e^x, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

This function satisfies Assumption (B). The corresponding source term  $f = Lu$  is computed symbolically using MATHEMATICA; its explicitly given by

$$f(x) = \begin{cases} 0, & x \leq -\delta, \\ -\frac{105}{\delta^7}(24 - 24e^{x+\delta} + 24x + 12x^2 + 4x^3 + x^4 + 24\delta + 24x\delta + 12x^2\delta + 4x^3\delta \\ \quad + 12\delta^2 + 12x\delta^2 + 6x^2\delta^2 + 4\delta^3 + 4x\delta^3 + \delta^4), & -\delta < x < 0, \\ -\frac{21}{\delta^7}(120 - 120e^{x+\delta} + 120x + 60x^2 + 20x^3 + 5x^4 - 120\delta + 240\delta e^x - 120\delta x \\ \quad - 60x^2\delta - 20x^3\delta + 60\delta^2 + 60x\delta^2 + 30x^2\delta^2 - 20\delta^3 + 40\delta^3 e^x - 20x\delta^3 \\ \quad + 5\delta^4 + 2\delta^5 e^x), & 0 \leq x < \delta, \\ \frac{42}{\delta^7}e^{x-\delta}(-60 + 60e^{2\delta} - 120e^\delta - 20\delta^3 e^\delta - \delta^5 e^\delta), & x \geq \delta. \end{cases}$$

We then solve (4.1) using the semi-analytic spectral method described in Algorithms 1 and 2.

For each pair  $(N, M)$ , we define the error by

$$E_N^M = \|u_{\text{exact}} - u_N^M\|_\infty.$$

The observed order of convergence is computed from the errors corresponding to consecutive values of  $N$ .

Table 4.1 and Figure 4.4 show the effect of smoothing on the numerical approximation. Without smoothing, the method exhibits essentially first-order convergence, reflecting the influence of the jump discontinuity. A substantial improvement is already visible at smoothing level  $M = 0$ , where the method becomes approximately second order. For  $M = 1$  and  $M = 2$ , the observed convergence rate is close to fourth order, while for  $M = 3$  and  $M = 4$  the method exhibits very rapid convergence, with errors reaching near machine precision on

relatively modest grids. These results clearly illustrate the effectiveness of the smoothing procedure in restoring high-order accuracy.

| $N$ | No smoothing     |       | Smoothing level $M = 0$ |       |
|-----|------------------|-------|-------------------------|-------|
|     | $L^\infty$ error | Order | $L^\infty$ error        | Order |
| 41  | 12.09            | -     | 0.84                    | -     |
| 61  | 8.31             | 0.93  | 0.38                    | 1.95  |
| 81  | 6.33             | 0.95  | 0.22                    | 1.97  |
| 101 | 5.11             | 0.96  | 0.14                    | 1.98  |
| 121 | 4.28             | 0.97  | 0.10                    | 1.98  |

| $N$ | Smoothing level $M = 1$ |       | Smoothing level $M = 2$ |       |
|-----|-------------------------|-------|-------------------------|-------|
|     | $L^\infty$ error        | Order | $L^\infty$ error        | Order |
| 41  | $3.26 \cdot 10^{-3}$    | -     | $2.52 \cdot 10^{-3}$    | -     |
| 61  | $5.94 \cdot 10^{-4}$    | 4.20  | $4.58 \cdot 10^{-4}$    | 4.20  |
| 81  | $1.86 \cdot 10^{-4}$    | 4.03  | $1.45 \cdot 10^{-4}$    | 3.99  |
| 101 | $7.6 \cdot 10^{-5}$     | 3.99  | $6.0 \cdot 10^{-5}$     | 3.99  |
| 121 | $3.7 \cdot 10^{-5}$     | 4.00  | $2.9 \cdot 10^{-5}$     | 3.99  |

| $N$ | Smoothing level $M = 3$ |       | Smoothing level $M = 4$ |       |
|-----|-------------------------|-------|-------------------------|-------|
|     | $L^\infty$ error        | Order | $L^\infty$ error        | Order |
| 41  | $9.59 \cdot 10^{-4}$    | -     | $9.60 \cdot 10^{-4}$    | -     |
| 61  | $1.62 \cdot 10^{-5}$    | 10.07 | $1.62 \cdot 10^{-5}$    | 10.07 |
| 81  | $8.77 \cdot 10^{-7}$    | 10.13 | $8.81 \cdot 10^{-7}$    | 10.12 |
| 101 | $9.05 \cdot 10^{-8}$    | 10.18 | $9.14 \cdot 10^{-8}$    | 10.15 |
| 121 | $1.64 \cdot 10^{-8}$    | 9.36  | $1.43 \cdot 10^{-8}$    | 10.16 |

Table 4.1: Errors and observed convergence rates for Example 1 at different smoothing levels.

*Example 2.* Consider the piecewise smooth manufactured solution

$$u(x) = \begin{cases} \sin(x+2), & x \leq -2 \\ 0, & -2 < x < 4 \\ e^{x-4}, & x \geq 4 \end{cases}.$$

This function has jump discontinuities at  $x = -2$  and  $x = 4$ . As in Example 1, the source term  $f = Lu$  is computed symbolically using MATHEMATICA, and its explicit expression,

assuming that  $\delta < 3$ , is given by

$$f(x) = \begin{cases} -\frac{42}{\delta^7}(-60 \cos(2+x-\delta) + 60 \cos(2+x+\delta) + 120\delta \sin(2+x) \\ \quad -20\delta^3 \sin(2+x) + \delta^5 \sin(2+x)), & x \leq -2-\delta, \\ -\frac{21}{\delta^7}(-40 + 80x + 60x^2 + 40x^3 + 5x^4 - 80\delta + 120\delta x + 120\delta x^2 + 20\delta x^3 \\ \quad + 60\delta^2 + 120\delta^2 x + 30\delta^2 x^2 + 40\delta^3 + 20\delta^3 x + 5\delta^4 - 120 \cos(2+x-\delta) \\ \quad + 240\delta \sin(2+x) - 40\delta^3 \sin(2+x) + 2\delta^5 \sin(2+x)), & -2-\delta < x \leq -2, \\ -\frac{105}{\delta^7}(-8 - 16x + 12x^2 + 8x^3 + x^4 + 16\delta - 24x\delta - 24x^2\delta - 4x^3\delta \\ \quad + 12\delta^2 + 24x\delta^2 + 6x^2\delta^2 - 8\delta^3 - 4x\delta^3 + \delta^4 - 24 \cos(2+x-\delta)), & -2 < x < -2+\delta, \\ 0, & -2+\delta \leq x \leq 4-\delta, \\ -\frac{105}{e^4\delta^7}(120e^4 - 24e^{x+\delta} - 136e^4x + 60e^4x^2 - 12e^4x^3 + e^4x^4 - 136e^4\delta \\ \quad + 120e^4x\delta - 36e^4x^2\delta + 4e^4x^3\delta + 60e^4\delta^2 - 36e^4x\delta^2 + 6e^4x^2\delta^2 \\ \quad - 12e^4\delta^3 + 4e^4x\delta^3 + e^4\delta^4), & 4-\delta < x < 4, \\ -\frac{21}{e^4\delta^7}(600e^4 - 120e^{x+\delta} - 680e^4x + 300e^4x^2 - 60e^4x^3 + 5e^4x^4 \\ \quad + 680e^4\delta + 240e^x\delta - 600e^4x\delta + 180e^4x^2\delta - 20e^4x^3\delta + 300e^4\delta^2 \\ \quad - 180e^4x\delta^2 + 30e^4x^2\delta^2 + 60e^4\delta^3 + 40e^x\delta^3 - 20e^4x\delta^3 + 5e^4\delta^4 + 2e^x\delta^5), & 4 \leq x < 4+\delta, \\ \frac{42}{\delta^7}e^{x-\delta-4}(-60 + 60e^{2\delta} - 120\delta e^\delta - 20\delta^3 e^\delta - \delta^5 e^\delta), & x \geq 4+\delta. \end{cases}$$

The semi-analytic spectral method is then applied, with the smoothing procedure carried

out separately at each jump.

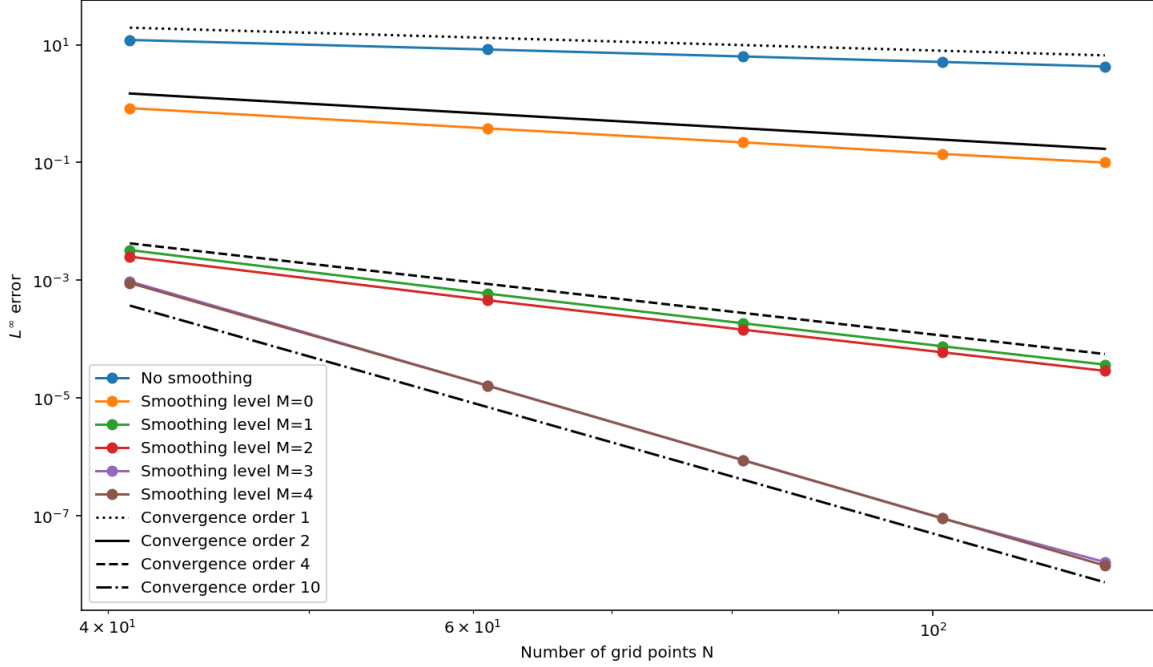


Figure 4.4: Observed convergence rates for Example 1 at different smoothing levels.

The corresponding results are presented in Table 4.2 and Figure 4.5. The same general pattern observed in Example 1 is present here as well. Without smoothing, the method again behaves essentially as a first-order scheme. Once smoothing is introduced, the accuracy improves markedly. At levels  $M = 1$  and  $M = 2$ , the method achieves approximately fourth-order convergence, while at levels  $M = 3$  and  $M = 4$  the method again displays much faster convergence. These experiments confirm that the proposed approach remains highly effective even in the presence of multiple jump discontinuities.

#### 4.4.2 Arbitrary source term

We next consider the case in which the source term  $f$  is prescribed directly and satisfies Assumption (B), rather than being generated from a known exact solution. In this setting, the choice of the constraint  $b$  on  $\Omega_I$  becomes important. If  $b$  is imposed arbitrarily, for example by setting  $b \equiv 0$ , then the corresponding solution generally develops jump discontinuities at the transition between  $\Omega$  and  $\Omega_I$ ; see<sup>104;112</sup>. Since Algorithms 1 and 2 are designed to

| $N$ | No smoothing     |       | Smoothing level $M = 0$ |       |
|-----|------------------|-------|-------------------------|-------|
|     | $L^\infty$ error | Order | $L^\infty$ error        | Order |
| 21  | 16.15            | -     | 2.92                    | -     |
| 41  | 9.48             | 0.77  | 0.47                    | 2.64  |
| 81  | 4.88             | 0.96  | 0.12                    | 1.97  |
| 121 | 3.29             | 0.98  | $5.32 \cdot 10^{-2}$    | 1.99  |
| 161 | 2.48             | 0.98  | $3.0 \cdot 10^{-2}$     | 1.99  |

| $N$ | Smoothing level $M = 1$ |       | Smoothing level $M = 2$ |       |
|-----|-------------------------|-------|-------------------------|-------|
|     | $L^\infty$ error        | Order | $L^\infty$ error        | Order |
| 21  | $2.60 \cdot 10^{-2}$    | -     | $2.32 \cdot 10^{-2}$    | -     |
| 41  | $2.72 \cdot 10^{-3}$    | 3.26  | $3.0 \cdot 10^{-3}$     | 2.95  |
| 81  | $1.76 \cdot 10^{-4}$    | 3.95  | $1.93 \cdot 10^{-4}$    | 3.96  |
| 121 | $3.5 \cdot 10^{-5}$     | 3.99  | $3.8 \cdot 10^{-5}$     | 3.98  |
| 161 | $1.1 \cdot 10^{-5}$     | 4.00  | $1.2 \cdot 10^{-5}$     | 3.99  |

| $N$ | Smoothing level $M = 3$ |       | Smoothing level $M = 4$ |       |
|-----|-------------------------|-------|-------------------------|-------|
|     | $L^\infty$ error        | Order | $L^\infty$ error        | Order |
| 21  | $1.53 \cdot 10^{-2}$    | -     | $1.53 \cdot 10^{-2}$    | -     |
| 41  | $1.69 \cdot 10^{-5}$    | 9.83  | $1.72 \cdot 10^{-5}$    | 9.80  |
| 81  | $1.44 \cdot 10^{-7}$    | 6.87  | $7.98 \cdot 10^{-8}$    | 7.75  |
| 121 | $1.25 \cdot 10^{-8}$    | 6.03  | $6.87 \cdot 10^{-9}$    | 6.05  |
| 161 | $2.21 \cdot 10^{-9}$    | 6.04  | $1.27 \cdot 10^{-9}$    | 5.88  |

Table 4.2: Errors and observed convergence rates for Example 2 at different smoothing levels.

address discontinuities generated by the interior source term, but not those induced at the transition layer, such incompatibilities lead to a noticeable loss of accuracy.

To avoid this issue, we construct a compatible constraint  $b$  numerically as follows. Let  $f_4$  denote the level-4 smoothing of  $f$  produced by Algorithm 1, with corresponding coefficients  $c_0, c_1, c_2, c_3, c_4$ . We enlarge the original domain in (4.1) to

$$\Omega' = (-2 - \delta, 2 + \delta),$$

with associated collar

$$\Omega'_I = (-2 - 2\delta, -2 - \delta] \cup [2 + \delta, 2 + 2\delta).$$

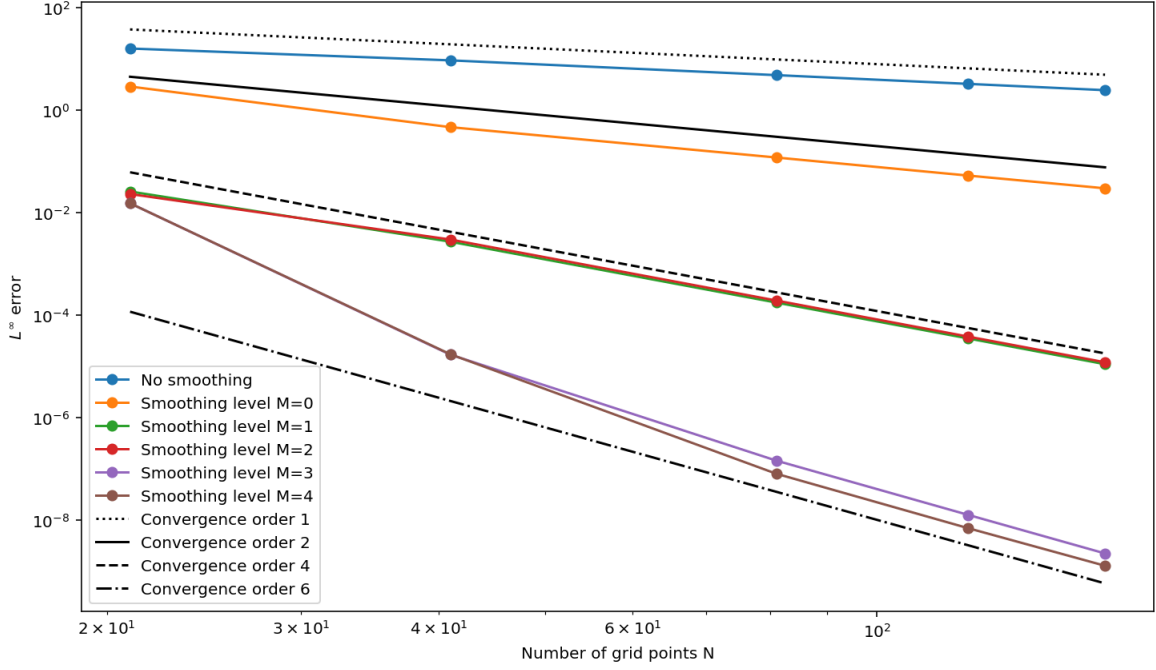


Figure 4.5: Observed convergence rates for Example 2 at different smoothing levels.

The set  $\Omega' \cup \Omega'_I$  is then extended periodically, and the problem

$$Lw_4 = f_4$$

is solved on the torus  $\mathbb{T}$ . By<sup>53</sup>, since  $f_4 \in C^4(\mathbb{T})$ , the corresponding solution satisfies  $w_4 \in C(\mathbb{T})$ . We then define

$$u_{\text{approx}} = w_4 + \sum_{k=0}^4 c_k \varphi_k,$$

and set

$$b = u_{\text{approx}}|_{\Omega_I}.$$

This enables the smoothing procedure to focus on the interior jumps of the solution and allows the spectral approximation to recover high-order behavior.

*Example 3.* We consider the source term

$$f(x) = \begin{cases} \frac{10}{6x^2 + 0.8}, & x \geq 0 \\ \tanh(3x) + 1, & x < 0 \end{cases}.$$

This function satisfies Assumption (B). We take  $\Omega = (-2, 2)$ ,  $\delta = 0.4$ , and again use the kernel  $K$  from (4.30). We denote by  $N$  the number of grid points in  $\Omega$ , and by  $M$  the smoothing level used in Algorithms 1 and 2. The corresponding numerical approximation is denoted by  $u_N^M$ .

Since no exact solution is available in this case, we use the following error measurement strategy. In Table 4.3, we compute a reference solution  $u_{\text{approx}}$  on a fine grid with  $N = 401$  points in  $\Omega$ , and define

$$E_N^M = \|u_N^M - u_{\text{approx}}\|_{\infty}.$$

The associated convergence rate is estimated by

$$\frac{\log(E_{N_1}^M/E_{N_2}^M)}{\log(N_2/N_1)},$$

where  $N_1$  and  $N_2$  are two consecutive values of  $N$  listed in the table.

We observe the following trend: higher smoothing levels lead to smaller errors and significantly improved convergence. In particular, the transition from no smoothing to even a modest amount of smoothing produces a substantial gain, while higher smoothing levels restore the high-order accuracy expected from the spectral solver once the dominant discontinuity effects have been removed.

| $N$ | No smoothing            |       | Smoothing level $M = 0$ |       |
|-----|-------------------------|-------|-------------------------|-------|
|     | $L^\infty$ error        | Order | $L^\infty$ error        | Order |
| 21  | 1.124                   | -     | $1.48 \cdot 10^{-2}$    | -     |
| 41  | 0.569                   | 1.02  | $2.72 \cdot 10^{-3}$    | 2.53  |
| 51  | 0.457                   | 1.01  | $1.68 \cdot 10^{-3}$    | 2.22  |
| 81  | 0.287                   | 1.00  | $6.33 \cdot 10^{-4}$    | 2.11  |
| 101 | 0.230                   | 1.00  | $4.03 \cdot 10^{-4}$    | 2.05  |
| $N$ | Smoothing level $M = 1$ |       | Smoothing level $M = 2$ |       |
|     | $L^\infty$ error        | Order | $L^\infty$ error        | Order |
| 21  | $6.57 \cdot 10^{-3}$    | -     | $2.05 \cdot 10^{-3}$    | -     |
| 41  | $3.81 \cdot 10^{-4}$    | 4.26  | $7.21 \cdot 10^{-5}$    | 5.00  |
| 51  | $1.49 \cdot 10^{-4}$    | 4.29  | $2.74 \cdot 10^{-5}$    | 4.44  |
| 81  | $2.27 \cdot 10^{-5}$    | 4.07  | $3.87 \cdot 10^{-6}$    | 4.23  |
| 101 | $9.23 \cdot 10^{-6}$    | 4.08  | $1.56 \cdot 10^{-6}$    | 4.11  |
| $N$ | Smoothing level $M = 3$ |       | Smoothing level $M = 4$ |       |
|     | $L^\infty$ error        | Order | $L^\infty$ error        | Order |
| 21  | $1.28 \cdot 10^{-3}$    | -     | $3.89 \cdot 10^{-4}$    | -     |
| 41  | $1.66 \cdot 10^{-5}$    | 6.50  | $1.90 \cdot 10^{-6}$    | 7.95  |
| 51  | $4.12 \cdot 10^{-6}$    | 6.37  | $4.37 \cdot 10^{-7}$    | 6.73  |
| 81  | $2.21 \cdot 10^{-7}$    | 6.32  | $2.79 \cdot 10^{-8}$    | 5.95  |
| 101 | $5.46 \cdot 10^{-8}$    | 6.34  | $8.02 \cdot 10^{-9}$    | 5.65  |

Table 4.3: Errors and observed convergence rates for Example 3, using a reference solution computed on a fine grid.

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