Orientable genera of knots in 4-manifolds

by

Jacob Pichelmeyer

B.S., University of Wisconsin-Oshkosh, 2010

M.S., University of Wisconsin-Milwaukee, 2012

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2021

Abstract

We study a branch of knot invariants that assign an integer to a knot based on the properties of associated orientable surfaces. Let K be a knot and M a smooth closed 4-manifold. The M-genus of K, denoted $g_M(K)$, is the least genus among all orientable surfaces S embedded smoothly and properly in $M \setminus B^4$ with boundary $\partial S = K$. We study the M-genus of knots in the cases where M is $S^2 \times S^2$, S^4, and $\mathbb{C}P^2$.

In Chapter 1, we provide historical background and an overview of our work. In Chapter 2, we introduce the myriad of concepts that will appear in later chapters, including mathematical surgery and cobordisms. In Chapter 3, we study the $S^2 \times S^2$-genera of knots. There we detail the argument showing that for every knot K, the $S^2 \times S^2$-genus is 0. In Chapter 4, we study the S^4-genera of knots. We examine the Milnor and Fox-Milnor Conjectures before laying out Piccirillo's argument that the Conway knot is not slice. In Chapter 5, we provide the results of our attempt to systematically compute the $\mathbb{C}P^2$-genus of all prime knots up to 8-crossings. We show the upper bounds on the $\mathbb{C}P^2$-genus that we obtained using coherent band surgery; followed by the lower bounds we obtained through the obstruction of homological degrees of potential slice discs. The obstructions are pulled from a variety of sources in low-dimensional topology and adapted to $\mathbb{C}P^2$. There are 27 prime knots and distinct mirrors up to 7-crossings. We now know the $\mathbb{C}P^2$-genus of all these knots. There are 64 prime knots and distinct mirrors up to 8-crossings. We now know the $\mathbb{C}P^2$-genus of all but 6 of these knots. Where the $\mathbb{C}P^2$-genus was not determined explicitly, it has been narrowed down to 2 possibilities. As a further consequence of this work, we show an infinite family of knots such that the $\mathbb{C}P^2$-genus of each knot differs from that of it's mirror.

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Dedication

To all my family and friends. The appreciation I have for your love and support is beyond words. Without you I would be nothing.

Chapter 1

Introduction

Knot theory is a branch of mathematics that has a lot going for it. At the very least, it is beautiful, accessible, deep, and applicable. Examine an ancient Celtic drawing and see the inherent beauty knots exude. Show a knot diagram and the Reidemeister moves to someone with no mathematical knowledge and watch them delve into manipulating that diagram with excitement and curiosity. Ask a research-level mathematician working in any area related to geometric topology if knot theory is just for amateurs and they will scoff. Outside of pure mathematics, one will find knot theory being used in a wide array of applications such as protein knotting and generating new cryptological ciphers.

Mathematics is seen by many people to be somewhat impenetrable in general. Go deeper to the level at which a specialist operates and it is downright mystifying. It is often the case that high level research mathematicians only have a dozen or so people in world who can really speak with them fluently about their work. Ask them to explain their work to a non-mathematician and they will ask "how many months do we have?"

One of the most attractive aspects of knot theory in the author's opinion is that it retains the mystery and complexity of research-level mathematics while also being immediately accessible. There are interesting open questions about knots that motivated high school students and undergraduates can tackle after learning a few introductory concepts. At the same time, there are other questions about knots that have long eluded even elite-level research mathematicians. And above all, the questions themselves are generally easy to understand.

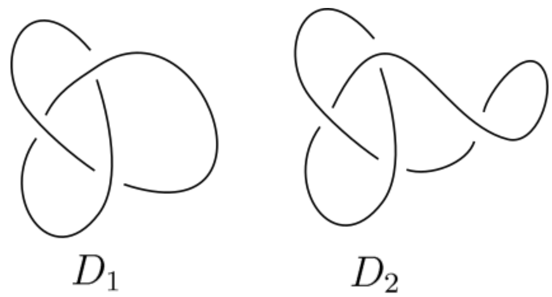


Figure 1.1: Two diagrams representing the same mathematical knot.

The mathematical field of knot theory was originally concerned with classifying knots. The knot diagrams in Figure 1.1 represent the same knot because a half twist on one portion of the left diagram will generate the diagram on the right. The diagrams in Figure 1.2 do not represent the same knot because no amount of intuitive manipulations to the diagram on the left will yield the diagram on the right.

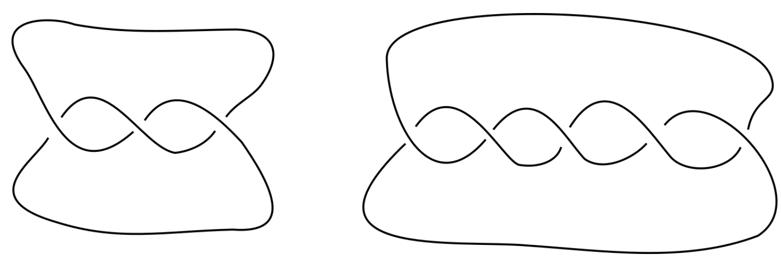


Figure 1.2: Two knot diagrams for two distinct knots.

The problem of classification is to take any two arbitrary knot diagrams and determine if they represent the same knot. If we consider the diagrams in Figure 1.4, we see two different knot diagrams that the reader may be surprised to find represent the same knot. In general, even the determination of whether an arbitrary knot is the unknot is a highly nontrivial problem.

Though classification is still an important part of the field, a new focus has arguably taken center stage: knot invariants. Though knot invariants were originally used as a tool to tackle the classification problem, they have since taken on a life of their own. A knot

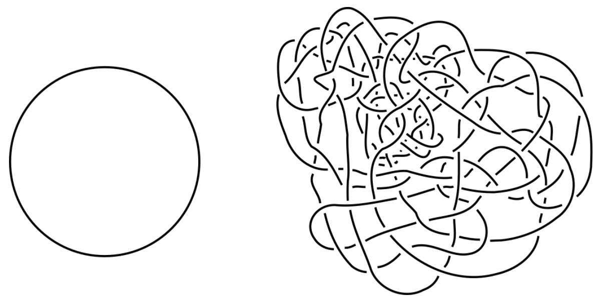


Figure 1.3: Do these diagrams represent the same knot?

invariant is a means of associating to each knot an algebraic object. One can think of each knot invariant as a function whose input is a knot and output is an algebraic object such as an integer or polynomial. Note that we are saying a knot invariant takes *knots* as inputs and not *knot diagrams*. This is an important distinction. It's true that many knot invariants are computed *using* a knot diagram, but the algebraic object assigned to the diagram should be the same no matter what knot diagram is used to represent that knot. This is why they are called *invariants* and it is exact their invariance that makes them so useful.

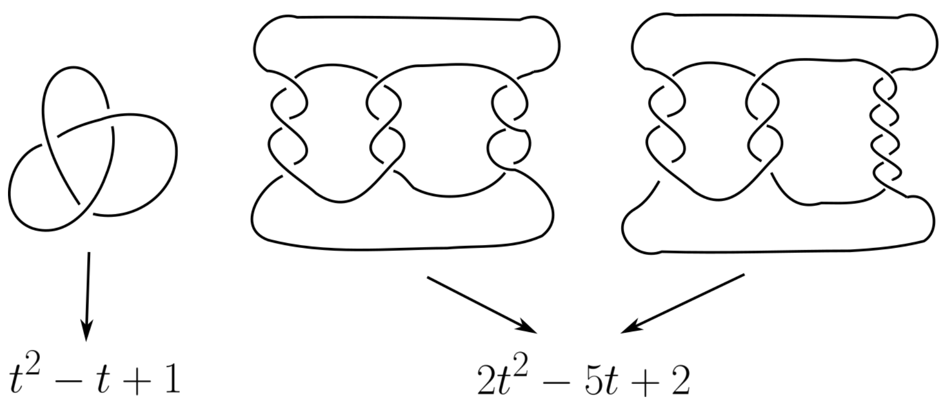


Figure 1.4: The Alexander polynomial is a function that maps knot diagrams to polynomials with integer coefficients.

At the time of this writing, there exist a multitude of knot invariants. In the future, it is likely that many more will be discovered. A look at Knotinfo (cite) - an online database maintained by well-known knot theorists[1] - shows knot invariants divided into broad classes such as 3D numeric, 4D numeric, polynomial, Heegard Floer, Khovanov, and hypberbolic invariants. Each of these classes contains between seven and thirteen sub-invariants. Yet this is far from complete.

There are even still many more invariants to be be found in the literature that are variations of those listed. One such variation will be the focus of this thesis: orientable knot genera.

There is a broad family of numeric knot invariants which each assign a number to a knot based on the properties of surfaces bounded by the knot. In particular, they are each a map $\{\text{set of knots}\} \to \mathbb{Z}_{\geq 0}$. There are invariants of many flavors in this family, each characterized according restrictions placed on the types of surfaces considered. In this thesis, we restrict ourselves to surfaces which are *orientable* and *properly embedded* in punctured smooth 4-dimensional manifolds. For each smooth closed 4-dimensional manifold there is a separate such invariant. For instance, the 4-sphere S^4 is a smooth closed 4-manifold. Thus, there is a S^4 invariant in this family. Likewise, $S^2 \times S^2$ is another smooth closed 4-manifold, so there is a $S^2 \times S^2$ invariant in this family as well. For the reader unfamiliar with the spaces S^4 and $S^2 \times S^2$, be content for now that there are well-known and relatively simple 4-dimensional spaces.

There are infinitely many smooth closed 4-manifolds but we will focus on three in particular. The manifolds S^4 and $S^2 \times S^2$ are in some sense bounding manifolds in this family of invariants. It happens to be that the $S^2 \times S^2$-invariant is the "smallest" function in this family in that it maps all knots to 0. On the other hand, the S^4-invariant is the "largest" function in that every other invariant in this family maps a knot to either the same integer as the S^4-invariant or to a lesser one. In general, every knot invariant in this family assigns a number to a knot that is inclusively between the $S^2 \times S^2$-invariant value (which is always 0) and the S^4-invariant value (some non-negative integer).

The $S^2 \times S^2$-invariant is completely understood. It assigns every knot to 0. Note that this is not an arbitrary assignment. The $S^2 \times S^2$-invariant, like all knot invariants, is based on the geometry of the knot and the ambient space in which the knot or associated object lives. It just so happens that based on the geometric definition that this number is always 0. In Chapter 3, we present a geometric argument showing why invariant always assigns the lowest possible value.

The S^4-invariant, while not fully understood, is certainly pretty well understood. There are 2,978 distinct prime knots up to 12 crossings. As of this writing, the S^4 invariant is known for at least all such knots - and for several infinite families beyond this. In general, there has been much work done on this invariant. In Chapter 4, we present several major results about this invariant, intended to give the reader some of the highlights and to provide foundation for future work that may branch off of these results into work on the final invariant of interest.

Arguably, the simplest closed smooth 4-manifold after S^4 and $S^2 \times S^2$ is the complex projective plane $\mathbb{C}P^2$. Though a lot was known about the S^4 and $S^2 \times S^2$ invariants at the time the author started working on this problem, very little was known in the case of $\mathbb{C}P^2$. The lone results stem from work by Yasuhara[2] and Ait Nouh[3] and is limited to $T(2, 2m+1)$ torus knots where $m \geq 0$. Interestingly, Yasuhara found that a knot K and it's mirror imagine mK are not necessarily assigned the same value by the $\mathbb{C}P^2$-invariant. This stands in stark contrast to the S^4-invariant. The dearth of results about the $\mathbb{C}P^2$-invariant was one thing that attracted the author to working in this area. As the reader will see in Chapter 5, considering both a knot K and it's mirror image mK, there are 64 distinct prime knots up to 8-crossings. It can been seen relatively that many of these knots have $\mathbb{C}P^2$-invariant 0 by considering other known invariants such as the unknotting number and the S^4-invariant. After such an examination, 25 knots are left with unknown $\mathbb{C}P^2$-invariant. As a result of the author's work, this list of unknowns has been reduced to just 6.

We end this introduction with a word about who this document is written for. Dissertations are notorious in academia for going largely unread and unused. Many doctoral candidates pour their heart into the document only to realize that no one is really going to do anything with it. Others, hearing stories like this, put minimal effort into and perhaps rightly invest more of their energy into future projects that will help them get a job or secure tenure. The author has chosen to write this work with three goals in mind.

The first concerns mentorship. The author thoroughly enjoys mentorship. So much so, that they founded an undergraduate mentorship program at their doctoral institution. This program routinely paired graduate students with undergraduates with the goal of working through a paper or text in order for the undergraduate to gain exposure to mathematics that they would not have normally had the opportunity to see. This has inspired more than one student to become a math major and has been a fulfilling program for all involved. The author has also spent nine summers instructing courses for an academic program where gifted middle and high school students can take rigorous enrichment classes. The author has designed a course in topology from scratch, taught it, and was asked by one of the students to be their mentor for a year-long self-study program the following year. The author enjoys teaching thoroughly and would love for mentorship to be a strong focus in their career. As stated earlier, knot theory is a highly versatile topic in terms of accessability. There will be certain parts of this thesis that are accessible only to a specialist in the field. Other parts will be accessible to a motivated high school or undergraduate. So much so that they may attempt to push some of the graphic results even further. Indeed, the author hopes to work with at least one undergraduate in the future on doing just this. From this point of view, it would be nice to hand a copy of this thesis to said student as primary document they can use to get up to speed on the area. Accordingly, much effort is devoted to the preliminaries in Chapter 2 for the sake of such potential future undergraduate or undergraduates. A specialist who happens to read this can skip most if not all of Chapter 1.

The second goal concerns review. The author expects that various projects will take their attention away from doing research directly on this topic, whether it be due to the draw of collaborating on another problem, the time investment of a teaching-focused job, or whatever else. Nonetheless, the author hopes to continue working on this problem as sees several avenues to development. The document is written thus to act as a reference to help the author get back up to speed after some time away.

The third goal concerns extension. As mentioned above, the author sees more than one way in which these results could be extended. In particular, the author believes that it will be fruitful to adapat tools used in the case of the S^4-invariant. By explaining those tools thoroughly, it gives the author better proficiency and hope of extending the work presented.

Chapter 2

Preliminaries

This chapter is a survey of the geometric and topological concepts that are used in the subsequent chapters. There is no expectation of it being complete or self-contained. References are provided where the author expects the reader may want further explanation. This chapter is particularly aimed at undergraduates and junior mathematicians who do not have a long history with geometric topology and so could benefit from a softer introduction to these concepts.

2.1 Manifolds & Handlebodies

The defining property of a manifold is that it is locally Euclidean. When you zoom in close to any spot on a m-manifold, it is indistinguishable from $\mathbb{R}^m$, much as the surface of the earth seems flat to us despite the earth being a relatively large sphere. The language of handlebodies gives us a way of decomposing manifolds into sensible and manageable bits.

2.1.1 Manifolds

The invariants discussed in this thesis depends largely on the ambient space into which surfaces are embedded. All the ambient spaces will be manifolds. Even a knot is a manifold. Thus, there is probably no better place to begin than by stating what a manifold is. Let's start with the intuitive definition before proceeding to the formal one.

In the modern era, we take for granted that the surface of the earth is a 2-dimensional sphere (the surface of a ball, if you will). However, this fact caused historical consternation. And in some sense it's not hard to see why. Zoomed in really close to the surface of the Earth and discounting local geography, the surface of the earth appears to be flat. Only after moving far away from the surface of the Earth does it's curvature become apparent. The fact that the earth appears flat when viewed up close but is not flat in general is exactly what determines a manifold.

Intuitively, an n-dimensional manifold is a space that everywhere locally looks like $\mathbb{R}^n$. In Figure 2.1 we see examples of $\mathbb{R}^1$ (the real line), $\mathbb{R}^2$ (the real plane), and $\mathbb{R}^3$ (commonly called '3-space' in multivariable calculus classes).

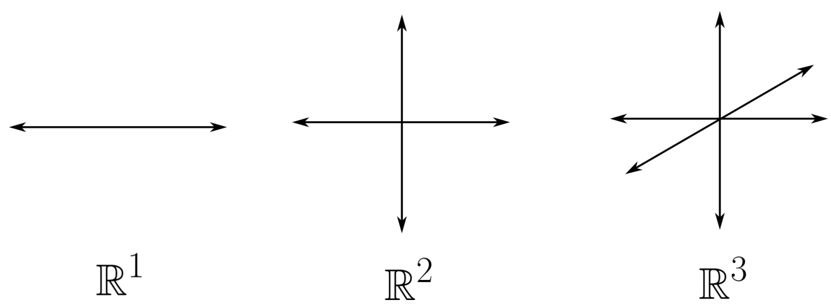


Figure 2.1: Euclidean spaces $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

What we mean by saying that a n-dimensional manifolds looks "locally" like $\mathbb{R}^n$ is that if you pick any point on the manifold and "zoom in" far enough toward that point, the space will essentially look like $\mathbb{R}^n$. This is also called being 'locally Euclidean.' In Figure 2.2 we see an 2-dimensional plane on the left and the 2-dimensional torus on the right. These are both examples of 2-dimensional manifolds.

Without a picture of Earth from space or some clever use of astronomy and mathematics, there is nothing to immediately suggest that the surface of the Earth couldn't be a shape besides a sphere - say a 2-dimensional torus. That statement is of course laughable since most of us have seen beautiful pictures of a spherical earth (and other celestial bodies) from space. But without such gifts how would we convince ourselves that the right 2-dimensional

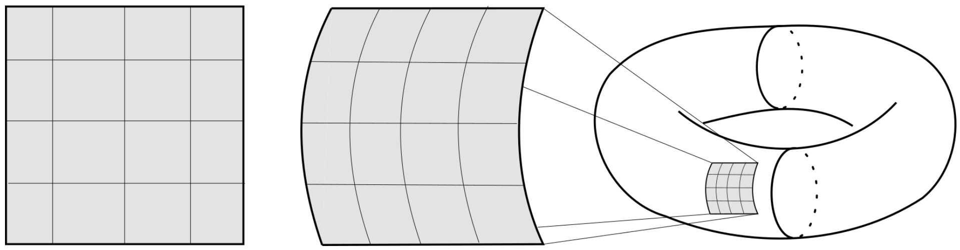


Figure 2.2: Up close, the 2-dimensional torus looks just like $\mathbb{R}^2$.

manifold is the 2-dimensional sphere and not the 2-dimensional torus?

There is a higher-dimensional analog of this question that is being tackled by the scientific community right now. It is believed that the universe itself is a 3-dimensional manifold[4]. Now the question is: Which one? Is it the 3-dimensional sphere? The 3-dimensioal torus? Some other 3-dimensional manifold?

It is worth pointing out that there is a difference between a manifold existing abstractly without reference to any ambient space and an embedding of the same manifold into a Euclidean space. In Figure 2.2 above, we see an embedding of the 2-dimensional plane $\mathbb{R}^2$ and 2-dimensional torus $S^1 \times S^2$ into 3-dimensional Euclidean space $\mathbb{R}^3$. However, there are often many ways to embed one manifold into another, such as the two ways in which the plane $\mathbb{R}^2$ is embedded into $\mathbb{R}^3$ as shown in Figure 2.3. They could potentially be embedded into other 3-dimensional manifolds. An interesting question for the reader to consider is under what conditions can an M-manifold M can be embedded into an n-manifold N?

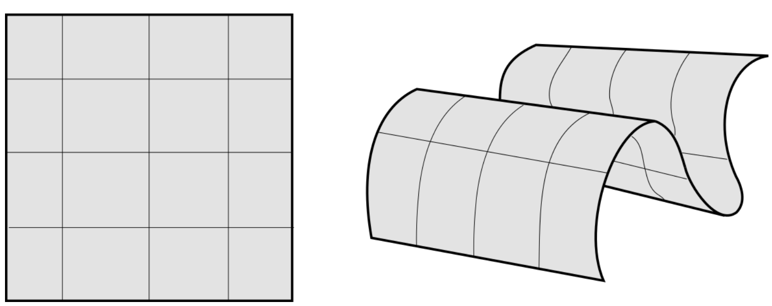


Figure 2.3: Embedding a 2-dimensional sheet into $\mathbb{R}^3$.

Having provided some intuition, we now present the formal definition of a manifold.

Definition 2.1.1. A *topological m-dimensional manifold M* is a second countable Hausdorff space that is locally homemorphic to $\mathbb{R}^m$. That is, for every point $x \in M$, there exists an open neighborhood $U \ni x$ and homeomorphism $\phi : U \to \mathbb{R}^m$. The map ϕ is called a *chart*. A collection of $\{(U_\alpha, \phi_\alpha : \alpha \in A\}$ such that every element $x \in M$ is contained in some U_α is called an *atlas* of M.

The terms *second countable*, *Hausdorff*, *open*, and *homeomorphism / homeomorphic* can all be found in any text on elementary point-set topology such as Topology Without Tears by Morris[5] or Topology by Munkres[6]. The uninformed reader who wishes to continue unabated may safely forgo looking up the terms second countable and Hausdorff. Knowing them won't provide any illumination on the rest of this work. A homeomorphism f is a bijective function between topological spaces such that both f and it's inverse f^{-1} are continuous. Homeomorphisms preserve the topology between spaces. When there exists a homeomorphism between two topological spaces such as M and $\mathbb{R}^m$ in the definition above, it means that the spaces have the same topological structure. And yet, despite all this formalism, the working definition that even advanced researchers use is that manifolds are locally euclidean. That is, in small patches they look just like euclidean space. When you "zoom out," or consider the space "globally" they can look much different than Euclidean space (like the 2-torus). However, up close they are indistinguishable.

We will concern ourselves with manifolds that have additional structure beyond that presented in the definition above. The additional structure is called a *smooth* structure and the types of manifolds are called smooth manifolds. The description that follows requires knowledge of multivariable calculus. In particular, partial derivatives. For the reader unfamiliar with multivariable calculus, you can skip this explanation and file *smoothness* away in their mind as an additional requirement involving derivatives. For those who are familiar with multivariable calculus, we will look at a *smooth* manifold as a special limiting case of *differentiable* manifolds.

Definition 2.1.2. A map $f : \mathbb{R}^m \to \mathbb{R}^n$ is defined as a vector valued function

$$[x_1, ..., x_m] \mapsto [f_1(x_1, ..., x_m), ..., f_n(x_1, ..., x_m)].$$

The reader who has taken an elementary calculus (single variable) class or studied any analysis/calculus on their own has likely seen the limit definition of the derivative of a single-variable function $g : \mathbb{R} \to \mathbb{R}$. Intuitively it is the slope of the tangent line to the graph of the curve of that function. It is not too difficult to see how the intuitive definition and the formal limit definition are saying the same thing. On the other hand, for the derivative of a multivariable function, both the limit definition of the derivative and attempts at understanding the multivariable derivative geometrically are quite a bit more mystifying. The *components* of the multivariable derivative, partial derivaties, are more intuitive in special cases. But when all the component partial derivaties are assembled into the full derivative, the reader is wished good luck in trying to picture what that actually looks like. We forgo stating formal definitions for the multivariable derivative or even partial derivatives. However, we do make use of partial derivatives in the following definition:

Definition 2.1.3. A map $f : \mathbb{R}^m \to \mathbb{R}^n$ is *smooth* if the partial derivatives of all orders exist and are continuous.

Definition 2.1.4. Let M be a m-manifold and let $(U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta)$ be charts. A function $f : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ such that $f = \phi_\beta|_{U_\alpha \cap U_\beta} \circ \phi_\alpha^{-1}|_{\phi_\alpha(U_\alpha \cap U_\beta)}$ is called a *transition map.*

What looks like a mess of notation is much easier to sort out by considering the diagram in Figure 2.4.

Definition 2.1.5. A *smooth m-manifold M* is a topological m-manifold such that for every transition map f is a smooth bijection with smooth inverse f^{-1}.

If this is all a bit much for the reader, suffice it to say that working with smooth manifolds simply involves partial derivatives. More is required to be a smooth manifold than

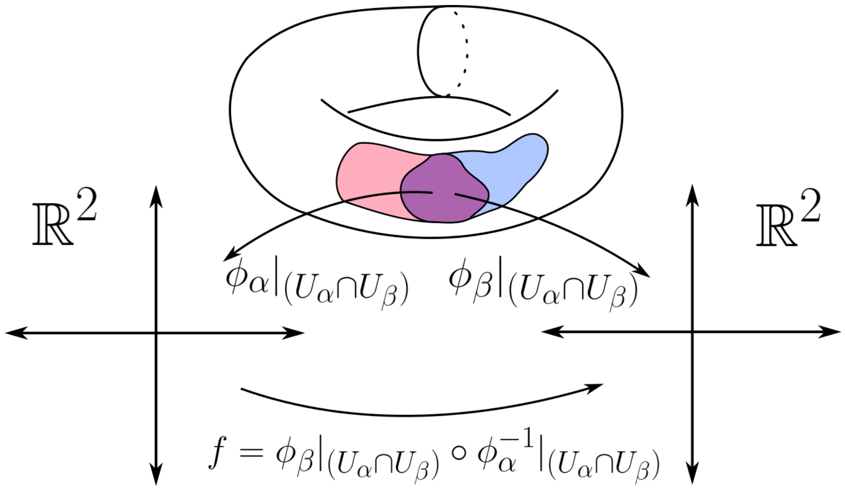


Figure 2.4: In a smooth manifold, all such maps f are smooth.

a topological manifold. Sometimes working with smooth manifolds is more difficult, but oftentimes the greater specificity of smooth manifolds grants one powerful tools.

Going back to Definition 2.1.1, there are some natural questions worth asking. First, what is an *embedding*? Second, why should we care about embeddings and specifically why should we care about embeddings of manifolds? An embedding is a one-to-one (or *injective*) function mapping one topological space into another such that the topological structure of the embedded space is preserved. More intuitively, an embedding maintains the essential properties of a space as it sits inside another space without allowing the space to "collapse." Formally,

Definition 2.1.6. Let M^m, N^n be m-dimensional and n-dimensional topological (or smooth) manifolds. A topological (or smooth) embedding of M into N is a map $\phi : M \to N$ that is injective, continuous (or smooth), and with continuous (or smooth) map $\phi' : \phi(M) \to M$ such that $\phi' \circ \phi = \mathrm{id}|_M$.

There are some manifolds which are utterly ubiquitous in geometric topology (and many other areas of mathematics) such as *spheres*, *discs*, *tori*, and *projective spaces*. Each of these manifolds has a standard smooth structure.

In Figure 2.5, we see an example of a 1-dimensional sphere S^1 embedded into $\mathbb{R}^2$ in two different ways. The leftmost diagram is the "standard" embedding that we all think of when we hear or read the word "circle." The middle diagram is an embedding of S^1 into $\mathbb{R}^2$ such that the topological structure is preserved, but not the smooth structure (which is more complicated than the topological structure). The right diagram shows S^1 *mapped* but *not* embedded into $\mathbb{R}^1$. It fails to be an embedded because there is at least one pair of points (highlighted in blue and yellow) in S^1 that are mapped to the same point $\mathbb{R}^1$ (shown in green). This is what is meant by the phrase "collapsing."

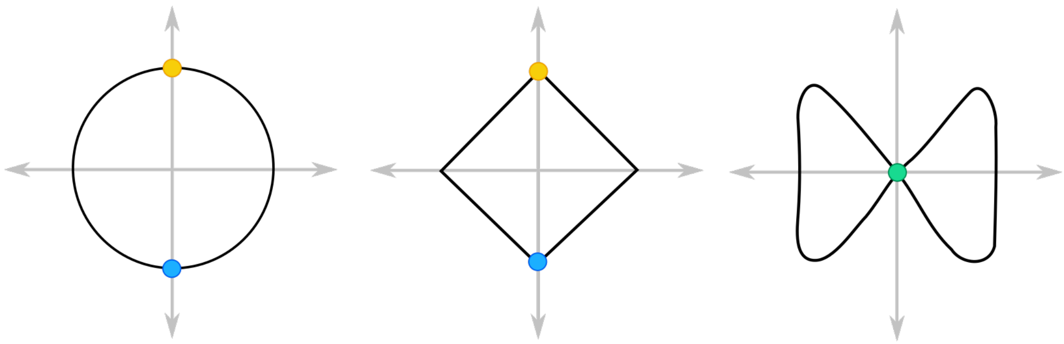


Figure 2.5: Maps of S^1 into $\mathbb{R}^2$. The left-most is a smooth embedding, the middle a topological embedding, and the rightmost fails to be an embedding (it is instead called an *immersion*.

One reason that embeddings are useful is that they sometimes give us a way to visualize manifolds. For instance, describing even a 1-sphere (a circle) without being able to draw it might be pretty challenging. However, by embedding S^1 into the real plane $\mathbb{R}^2$, the global structure of S^1 becomes immediately clear. Note that even the weaker cousin of embeddings - immersions - can still be useful for this. When most people see the Klein bottle (either a diagram or even an actual physical model), they are not seeing the Klein bottle exactly, but rather an immersion of it into $\mathbb{R}^3$. We aren't seeing the Klein bottle's true shape. Rather we are seeing a crushed version. Another reason we care about embeddings of manifolds is that the degree to which an embedding is or is not possible tells us about not only the manifold we are trying to embed but also about the manifold *into* which we are trying to embed. Thanks to the WHitney Embedding Theorem, we have some information about the minimality of dimension needed to embed a manifold into Euclidean space.

Theorem 2.1.7 (Whitney Embedding Theorem)**.** *If M^m is a smooth, real n-dimensional manifold, then there is an embedding of M^m into $\mathbb{R}^{2m}$.*

Note that this theorem guarantees one doesn't need to go any higher than dimension $2m$ in order to embed a m-manifold. There is nothing saying that an embedding can exist for some $p < 2m$. Spheres are characterized solely by their dimension k where k is a nonnegative integer. We call S^k the *k-sphere* and it has a standard embedding into $\mathbb{R}^{k+1}$. Examples of this standard embedding are shown in Figure 2.6.

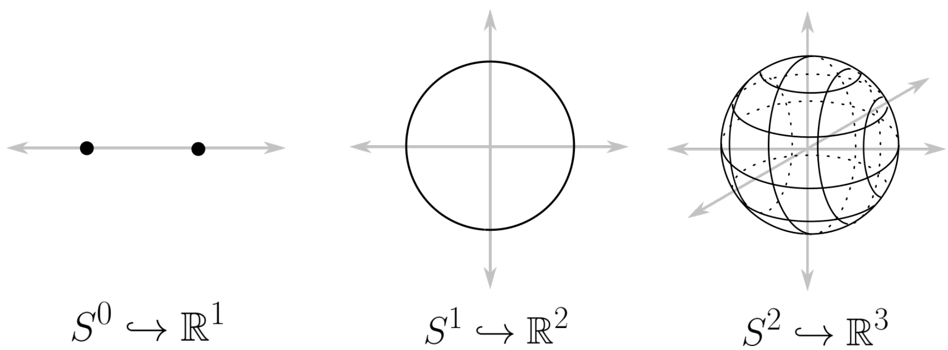


Figure 2.6: Examples of S^n embedding into $\mathbb{R}^{n+1}$ in a standard way.

Discs and spheres are intimately related. A S^k-sphere can be obtained by gluing two discs D^k together along their boundary. At the same time, the boundary of a disc D^k is itself a sphere S^{k-1}.

The big takeaways from this subsection are:

- m-manifolds look locally like $\mathbb{R}^m$,
- smooth manifolds are a special type of manifold that require the existence of all orders of continuous partial derivatives,
- good examples of manifolds to keep in mind are the spheres S^0, S^1, S^2, discs D^0, D^1, D^2, D^3, and the 2-torus $S^1 \times S^1$.

The definitions of *topological manifold* and *embedding* can be found in Munkres' Topology[6], which is a very common undergrad / early grad text on topology. For more in depth treatments of manifolds at the same level, see Lee's Introduction to Topological Manifolds[7] or his Introduction to Smooth Manifolds[8]. If the reader would like something more advanced, including an initial treatment of manifolds in the abstract, see Kosinski's Differential Manifolds[9] or Milnor's Topology from a Differentiable Viewpoint[10]. These texts are aimed at the advanced undergraduate and graduate levels.

2.1.2 Handlebodies

Handlebody decompositions are a way of describing compact manifolds by decomposing them into sensible bits and noting the way these bits fit together.

The bits themselves are as simple as one could hope for. For a n-dimensional manifold, the bits are simply n-dimensional discs. In the case of a 1-dimensional manifold, the bits are wavy line segments. In dimension 2, they are 2-dimensional discs like flattened pancakes. In dimensional 3, they are akin to solid balls of clay. However, though all the bits are the same topologically, we do distinguish between them using an index. This index molds the bit so that it has a particular type of boundary. The aforementioned bits are what we call *handles*.

Definition 2.1.8. Let n be a nonzero integer and k also a non-zero integer such that $k \leq n$. A *n-dimensional k-handle H^k* is a copy of $D^k \times D^{n-k}$.

No matter what k is, the dimension of $D^k \times D^{n-k}$ is n. This follows from the fact that the dimension of the cartesian product of two manifolds is the product of the dimension of it's factors[11]. Seeing this, it might not be hard to believe that there is an obvious diffeomorphism $(D^k \times D^{n-k}) \to D^n$. This is what was meant by saying all the "bits" - i.e. "handles" - are just n-dimensional discs.

If all the handles of various index are topologically the same, then why bother to distinguish them at all? That's because we want to take a collection of handles and fit them together to create manifolds. Or alternatively, look at how manifolds could be decomposed into a collection of handles. The way in which these handles are fit together completely determine the manifold. Thus, one can take two handles H^k and H^l and obtain two different manifolds simply by attaching the two handles together in two different ways. Handles are attaching along their boundaries and if $k \neq l$, then $\partial H^k \neq \partial H^l$.

The boundary of a cartesian product $M \times N$ of manifolds M and N follows a Leibniz-type rule. That is

$$\partial(M \times N) = (\partial M \times N) \cup_{\partial M \times \partial N} (M \times \partial N).$$

Applied to a n-dimensional k-handle H^k, we have that

$$\begin{aligned}
\partial H^k &= \partial(D^k \times D^{n-k}) \\
&= (\partial D^k \times D^{n-k}) \cup_{\partial D^k \times \partial D^{n-k}} (D^k \times \partial D^{n-k}) \\
&= (S^{k-1} \times D^{n-k}) \cup_{S^{k-1} \times S^{n-k-1}} (D^k \times S^{n-k-1}).
\end{aligned}$$

Observe that the boundary ∂H^k has two components: $S^{k-1} \times D^{n-k}$ and $D^k \times S^{n-k-1}$. We build up handlebodies by successively attaching handles along their $S^{k-1} \times D^{n-k}$ component.

Definition 2.1.9. Let M be a compact n-dimensional manifold. A *handlebody decomposition of* M is a sequence $M_0 \subset M_1 \subset \cdots \subset M_p$ such that each for $i = 0, 1, ..., p-1$, M_{i+1} is obtained from M_i by attaching a k_{i+1}-handle to M_i via an attaching map $\phi_{i+1} : (S^{k_{i+1}-1} \times D^{n-k_{i+1}}) \to \partial M_i$. That is

$$M_{i+1} = M_i \cup_\phi H^{k_{i+1}}.$$

The maps ϕ_{i+1} are called *attaching maps*. The sequence $M_0 \subset \cdots \subset M_p$ is called a *handlebody*.

Thinking of manifolds in terms of their handlebody decompositions makes it much easier to get a handle on them as their process now becomes one of

1. Knowing how many handles there are in the handlebody decomposition and

2. Understanding the attaching maps.

One particularly nice feature of handlebodies is handle rearrangement. It so happens that given any handlebody decomposition of a manifold, one can rearrange the handles so that they proceed in terms of increasing index. That is, all the 0-handles come first in the sequence, followed by all the 1-handles, followed by all the 2-handles, and so on. Thus, one can think of a handlebody decomposition as an ascending tower of handles, as shown in Figure 2.7.

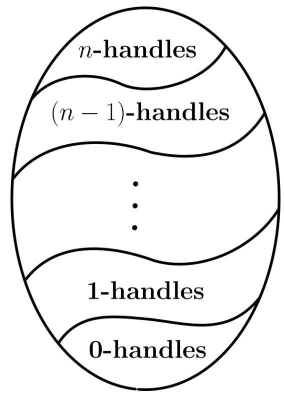


Figure 2.7: A handlebody decomposition of a manifold.

The biggest takeaway for us regarding handlebodies is how to decompose our primary objects of study: $S^4, S^2 \times S^2$, and $\mathbb{C}P^2$. Each of these has a very simple handlebody decomposition. S^4 is made up of a 0-handle and a 4-handle. $S^2 \times S^2$ is made up of a 0-handle, two 2-handles, and a 4-handle. $\mathbb{C}P^2$ is made up of a 0-handle, a 2-handle, and a 4-handle. We can see that each of these 4-manifolds has a single 0-handle and a single 4-handle. The handle-decomposition of every connected manifold has just a single 0-handle.

Later when we "puncture" these 4-manifolds to define the M-genus, this will amount to removing their single 4-handle. In some sense, this means that all of the relevant information about each manifold in it's number of 2-handles and how those 2-handles are attached to the single base 0-handle.

Handlebody decompositions are a way of describing manifolds. Yet many students won't encounter manifolds in an explicit sense until at least their first class in topology. Since topology courses usually take a back-seat to algebra and analysis courses, it's very likely that many students wouldn't encounter handlebody decompositions until graduate school. As such, well-known texts that talk about handlebodies tend to be graduate-level texts. Nevertheless, the first chapter of Kirby's Topology of 4-Manifolds[12] is a relatively accessible introduction to handlebodies for undergraduates. As is Akbalut's 4-Manifolds[13]. Note that both of these texts have an obvious focus on 4-dimensional manifolds. Don't be fooled. Handlebody decompositions are used extensively in other dimensions as well.

2.2 Knots & Knot invariants

The central object of study in this work is essence a function: one that takes a tame knot input and assigns a non-negative integer output. When we talk about knot invariants, we are talking about functions. In this section, we provide the fundamentals of both knots and knot invariants.

2.2.1 Knots

In a colloquial sense, a mathematical knot is a "circle" sitting inside 3-dimensional space in a potentially interesting way. It does not self intersect but it may loop around itself. Importantly, there are no free "ends." Such an object can be physically modeled in our 3-dimensional world using material such as malleable wire or string so long as one ties the "ends" together.

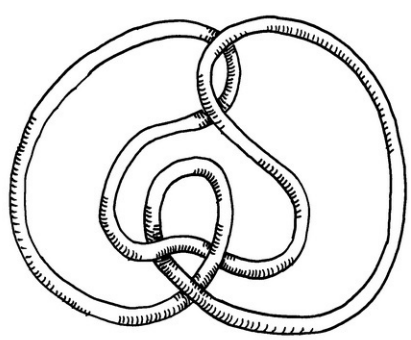


Figure 2.8: A physical model of a mathematical knot. Credit for picture to Juris Steprans, York University.

While it can be fun to construct such physical objects and play with them, there is a way to encode all a knot's information into a package that is both convenient to store and easy to manipulate.

Thinking of a knot as a 1-dimensional thing existing in 3-dimensional space, we can imagine projecting the knot into the 2-dimensional plane as if we had cast a light on the knot and looked at it's shadow. So long as we do not happen to project in a very unlucky way (such as where 3 strands cross at the same point), the projection will contain everything we would want to know about the knot with the exception of what happens at the crossings. If necessary, an unlucky projection can be easily avoided by giving the knot a slight perturbation before we project. In the projection, the crossings will appear as self-intersections.

In the projection, at each crossing we lose a sense of which strand is "above" the other. This is rectified by creating "breaks" in the projection. Once all the self-intersections have been resolved with a diagrammatic break that recaptures the over- and under-crossings of the original model, we have a *knot diagram*. This diagram contains all the information that the original physical model did but can now be stored in the plane (that is, on some flat medium like paper).

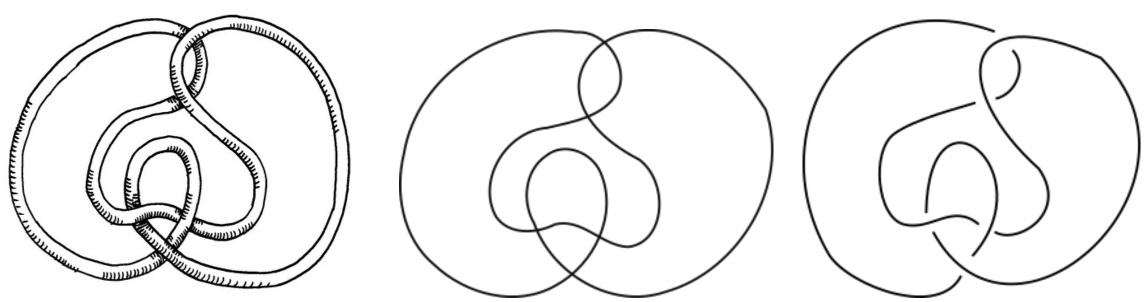


Figure 2.9: From left to right: the physical model, the projection, and the knot diagram. Physical model diagram credit to Juris Steprans, York University.

With a physical model of a knot in hand (such as a shoe lace with ends tied together), one would be hard pressed to resist tinkering with it. Pull this strand over that or try to untwist that portion over there. Human curiosity would lead to the discovery that some physical models are really the same as others. That is, by moving strands around in 3-dimensional space, some physical models can be transformed into others.

What's nice about physical models is that they also model the exact type of transformations we want to allow mathematically. In an intuitive sense, an *isotopy* of a mathematical knot consists exactly of those movements we find to be possible in the physical realm, along with the addition of stretching and compressing (but not breaking or gluing).

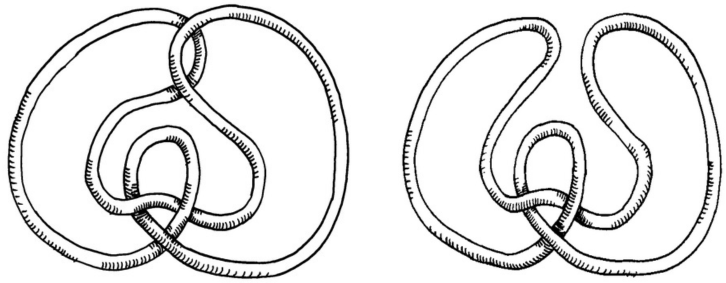


Figure 2.10: Transforming a physical model simply by moving one strand a little past another. Credit for pictures to Juris Steprans, York University.

There is a one-to-one correspondence between transformations of physical models and transformations of knot diagrams. This is formalized in the so-called Reidemeister moves RI, RII, and RIII, shown below. It is a fact that each isotopy or transformation taking one knot diagram to another is obtained by some sequence of Reidemeister moves.

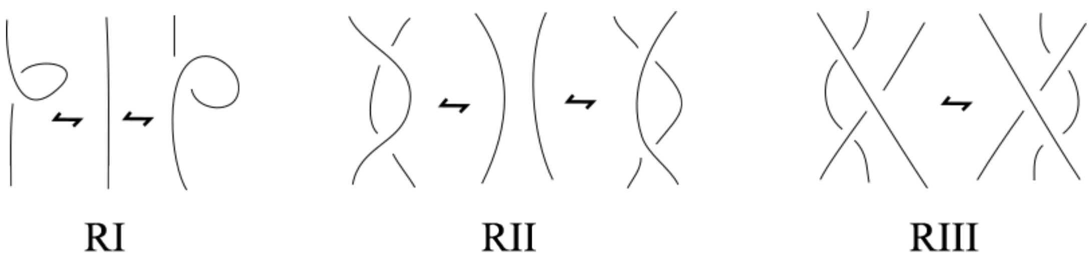
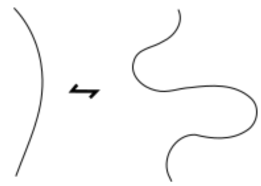


Figure 2.11: The three Reidemeister moves.

Observe that the first Reidemeister move involves one crossing, the second involves two crossings, and the third involves three crossings. Also note that the third Reidemeister move is not presented in it's entirety. There are analogous diagrams where the strand that "moves" sits above the other two and where the crossing in the middle is changed. However, the diagrams are analogous to the one presented.

The three standard Reidemeister moves don't capture the bending and stretching that is allowed. This allowance is sometimes represented by a "0th" Reidemeister move.

As a simple example, we see in Figure 2.13 that the knot diagram of the original physical model we considered can be changed by a Reidemeister II followed by a series of Reidemeister



R0

Figure 2.12: An informal "0th" Reidemeister move.

0 moves.

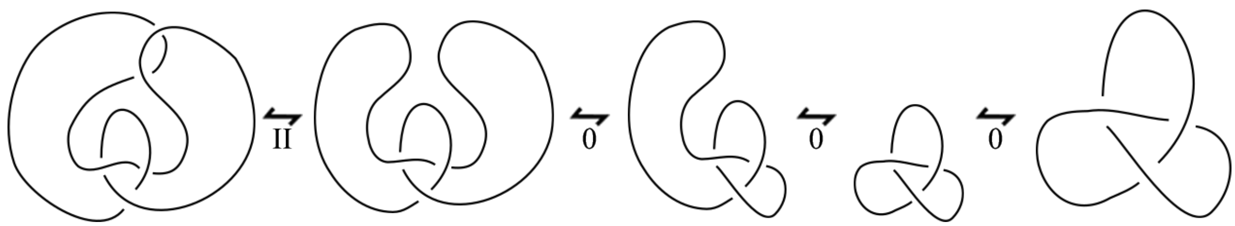


Figure 2.13: A sequence of Reidemesiter moves from the knot diagram on far left to the diagram on far right.

We now have both a way to conveniently store information about knots (via knot diagrams) and a way to transform those diagrams (Reidemeister moves). This forms the most essential set of ideas in knot theory. We present these now with formalism.

Definition 2.2.1. A *knot* is a smooth embedding $K : S^1 \to S^3$.

Definition 2.2.2. Let X_0, X_1 be topological spaces embedded in an ambient space Y. An *ambient isotopy of Y taking X_0 to X_1* is an embedding $F : Y \times I \to Y$ such that $F(X_0, 1) = X_1$.

When we speak of a knot in a mathematical sense, we don't actually mean some specific embedding $K : S^1 \to S^3$. Rather, we mean the family of *all* embeddings $J : S^1 \to S^3$ such that if $J_1 : S^1 \to S^3$ and $J_2 : S^1 \to S^3$ are such embeddings, then there is an ambient isotopy of S^3 taking $J_1(S^1)$ to $J_2(S^1)$. From now on, when we talk about a *knot*, what we truly mean is

Definition 2.2.3. A *knot* K is a set of topological subspaces $X \subset \mathbb{R}^3 \subset S^3$ such that for each $X \in K$, there exists an embedding $K' : S^1 \to S^3$ with $K'(S^1) = X$ and for every $X_1, X_2 \in K$, there exists an ambient isotopy $F : S^3 \times I \to S^3$ taking X_1 to X_2.

When we draw a knot diagram D, it is a representation of a single knot K (which is a set of spaces each thought to be sitting inside S^3). A knot diagram D' that is related by a series of Reidemeister moves to D also represents the knot K.

A generally excellent introduction to knot theory is The Knot Book by Colin Adams[14]. This is easily accessible to high school students but with enough substance to be engaging for undergraduates (and even graduates!). Perhaps the most classic advanced text on knot theory is Knots and Links by Dale Rolfsen[15]. This is aimed at students with more advanced mathematical knowledge such as that usually encountered at the advanced undergraduate or graduate level.

2.2.2 Knot Invariants

One of the most fundamental problems in knot theory is distinguishing knots. For example, consider the two knot diagrams D_1, D_2 in Figure 2.14. By performing a Reidemeister I move on D_1, diagram D_2 is obtained. Thus, D_1 and D_2 represent the same knot - the left-handed trefoil. On the other hand, all attempts to perform a series of Reidemeister moves to turn D_3 into D_4 as in Figure 2.15 will fail.

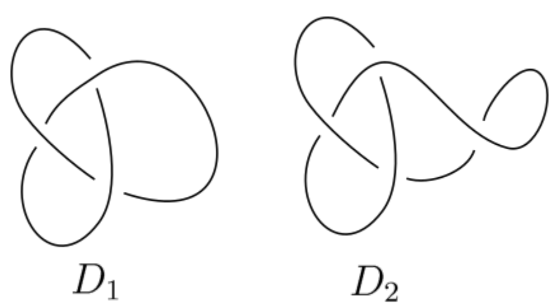


Figure 2.14: D_1 and D_2 are related by a single Reidemeister I move.

We see that diagrammatics is some sense not sufficient to fully distinguish knots. This stems from the fact that the space of possible sequences of Reidemeister moves is infinite.

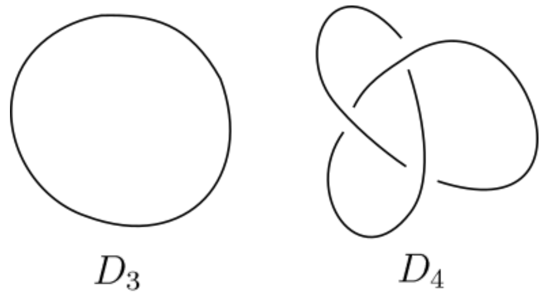


Figure 2.15: No series of Reidemeister moves will change D_3 into D_4 (or vice versa).

Even after performing one hundred Reidemeister moves on one knot diagram to try to get it to looks like another, it's not as if we can just give up and say "well, we tried 100 moves and couldn't get these diagrams to look the same, it must be that they don't represent the same knot." No, in fact, it could be that with just two, or three, or one thousand more moves, the diagrams *then* would look the same. Or it could be that they will never look the same. The point is that the space of possible sequences of Reidemeister moves cannot be exhausted. This shows us that we want something that is either finite or exhaustable in nature.

One such method is to construct so-called knot invariants. A knot invariant is a method of assigning algebraic objects to knot diagrams such that the method is well-defined under Reidemeister moves or even independent of them altogether. The former works in the following way: if a knot diagram D is assigned the algebraic object A and D' is related to D by some series of Reidemeister moves, then D' should also be assigned the algebraic object A. This type of variant is very useful in helping to distinguish knots. Perhaps the most classical example of this type of knot invariant is the Alexander Polynomial. For a given knot K, this is usually denoted by $\Delta(K)$.

The Alexander Polynomial invariant is a method of assigning polynomials with integer coefficients to knot diagrams. If D is a knot diagram, then by following a particular algorithm, a polynomial $\Delta \in \mathbb{Z}[t, t^{-1}]$ is assigned to D. As discussed, it is necessary that if D' is a knot diagram representing the same knot K as D, then both D and D' should be assigned the same polynomial (in the case of the Alexander polynomial, up to multiplication by a unit). Another way of saying this is that the algebraic object should be the same no matter which diagrammatic representation is used to compute it.

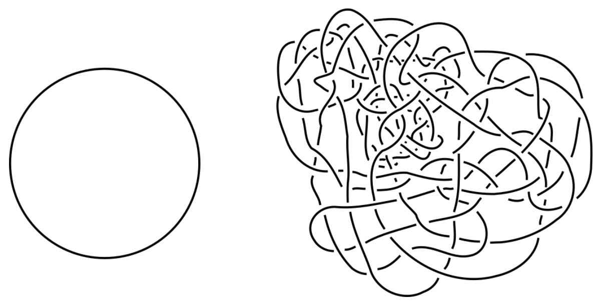


Figure 2.16: Surprisingly, the two knot diagrams above represent the same knot.

Several knots are shown in Figure 2.17 along with their corresponding Alexander polynomials.

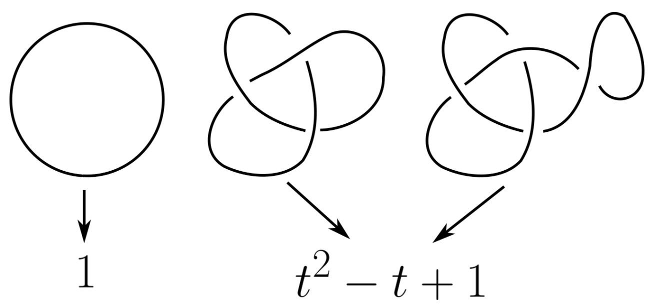
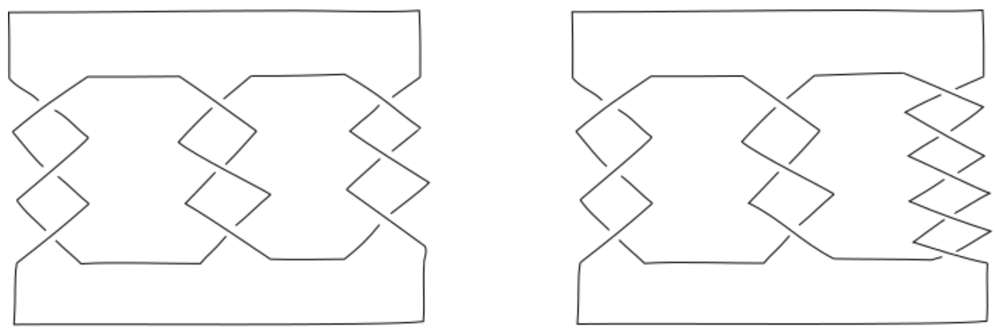


Figure 2.17: The unknot and two diagrams for the left-handed trefoil with corresponding Alexander polynomials.

Though we don't prove it here, the Alexander polynomial is invariant with respect to Reidemeister moves. That is, once the Alexander polynomial is computed for a diagram D, then every other diagram D' related to D by a series of Reidemeister moves would be assigned the same polynomial.

Thus, it should be no surprise that the two diagrams on the right in Figure 2.17 have the same Alexander polynomial since they differ only by a single Reidemeister I move. At the same time, since the Alexander polynomial doesn't change between Reidemeister moves, it follows that the left and middle diagrams in Figure 2.17 cannot represent the knot. If they did represent the same knot, then there would be some series of Reidemeister moves taking one to the other and hence they would have the same Alexander polynomial. And though the Alexander polynomial is a wonderful tool (as are the many other knot polynomial invariants), it is not perfect. Consider the knot diagrams in Figure ??. Through inspection, the reader may be able to convince themselves that the diagrams represent different knots. It is critical to note that two different knots can have the same Alexander polynomial, but a single knot cannot be assigned two different Alexander polynomials.



$2 - 5t + 2t^2$ $\quad$ $2 - 5t + 2t^2$

The Alexander polynomial is not enough to detect that the two knots above are distinct. One needs to compute the Jones polynomial to see this[16].

Invariants tend to walk a line between computability and distinguishment. One knot invariant may be easy to compute, but unable to distinguish knot diagrams well. That is, if almost knot diagrams - even those representing different knots - are assigned the same algebraic object, then the invariant may not do much good in distinguishing the knots. Alternatively, another knot invariant may easily distinguish between arbitrary knot diagrams, but have the barrier of being exceptionally difficult to compute. Mathematicians have responded to this reality by developing a wide array of invariants, forming a veritable tool box. Like a tool box, the type of invariant a mathematician reaches for depends on the situation.

The wide range of knot invariants can be seen by visiting KNOTINFOCITE. We present an organized list of some of the invariants here to give a reader a sense of their numeracy in a digestible format. Note that this listing is far from complete.

- **Polynomial invariants.** Alexander Polynomial (described earlier in this section), Conway Polynomial, Jones Polynomial, Kaufmann Polynomial, HOMFLY-PT Polynomial

- **3D Numeric invariants.** Bridge index, Crossing number, Determinant, Nonorientable 3-genus, Seifert genus (or 3-genus), Stick number, Unknotting number
- **4D Numeric invariants.** Arf invariant, Concordance order, Signature, Slice genus (or 4-genus), Smooth Non-orientable 4-genus, Topological 4-genus
- **Homological invariants.** Hom's ϵ-invariant (Heegard-Floer), Ozvath-Szabo ν-invariant (Heegard-Floer), Ozvath-Szabo τ-invariant (Heegard-Floer), Rasmussen's s-invariant (Khovanov)

The focus of this dissertation will be squarely in the realm of 4D numeric invariants. In particular, we focus our gaze on 4D *genus* invariants. This family is indexed by

- All 4-dimensional closed manifolds,
- Orientability (orientable or not)
- Smoothness (smooth or not)

Knot invariants are not just a tool for sorting knots. They are also useful for gaining a deeper understanding of the knots themselves. That knowledge can in turn be used to aid investigations which do not even involve knots. For example, by looking at a knot K embedded in a higher dimensional manifold M and looking at the surfaces that the knot bounds within M, we may be able to gain information about M.

For each 4-dimensional closed manifold M, there are at least four genus knot invariants, namely *orientable smooth*, *orientable topological*, *nonorientable smooth*, and *nonorientable topological*. As there are an infinite number of 4-dimensional closed manifolds, it follows that there are an infinite number of 4D genus invariants. However large this range is in theory, what has been studied in practice is an entirely different matter. Before the work of this dissertation, the range of manifolds M which had been studied regarding genus knot invariants was limited almost exclusively to S^4 and $S^2 \times S^2$, with some preliminary notions about $\mathbb{C}P^2$. This work greatly expands what we know about the case of $M = \mathbb{C}P^2$.

For the sake of keeping the scope reasonable, we will restrict ourselves to the 4D genus knot invariant involving smooth and orientable surfaces. In order to understand this type of invariant, it will help us to follow the historical from the 3D world.

2.3 Knot genera

Knot genera are a class of knot invariants that assign integers to knots based on the surfaces those knot bounds. The original such invariant was the Seifert genus. This has spawned a number of offshoots, but we will focus specifically on M-genera, which involve smooth and orientable surfaces.

2.3.1 Seifert Genus

In 1935, Seifert showed that given any knot K with knot diagram D of K, one can construct an orientable surface embedded in S^3 bounded by K.

Definition 2.3.1. Let K be a knot. A *Seifert surface for K* is a compact, connected, oriented 2-dimensional manifold S embedded in S^3 with $\partial S = K$.

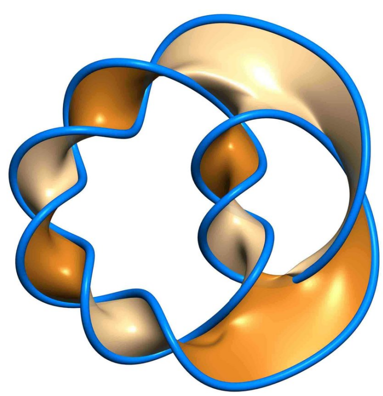


Figure 2.18: A 3d graphical model of a seifert surface bound by a blue knot. Model made using Jack van Wijik's Seifert View Software.

As long as one has a diagram D of a knot K, Seifert's algorithm allows one to find at least one such surface. First, let K be a particular knot with an arbitrary diagram D.

1. Resolve each crossing as explained below these steps.

2. Color the resulting bounded regions red if they have ”clockwise” boundary and blue if they have ”counter clockwise” boundary.

3. Add back in the twisted bands that were ”removed” as the result of resolving the crossings.

Crossings are resolved in the following way: In a knot diagram, every crossing is either ’positive’ or ’negative,’ as shown in Figure 2.19. However, no matter the sign of the crossing, a small cut is introduced, called a *resolution* of the crossing, is introduced. The resolution is also shown in Figure 2.19. An example of this entire process is shown in Figure 2.20.

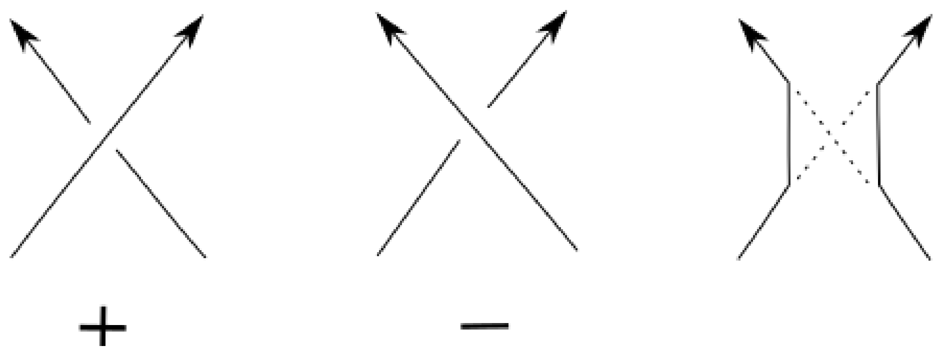


Figure 2.19: Positive, negative, and resolved crossing.

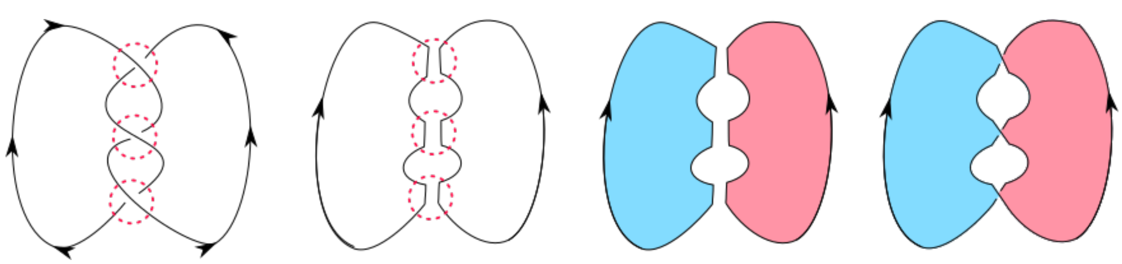


Figure 2.20: The Seifert algorithm applied to a certain diagram of the trefoil.

Armed with the Seifert algorithm, we may now construct at least one Seifert surface S_K for each tame knot.

In particular, the algorithm guarantees that the surface is orientable. To see this is the case, we start by considering all the ways in which it could fail to be orientable. In the context of the Seifert algorithm, there are only two such ways. Either (a) there are two adjacent circles (non-concentric) that would be connected by a band but have the same coloring or (b) there are two concentric circles connected by a band that would have different colorings. If a band is removed between two resulting circles and they end up being non-concentric, then the two circles will have opposite orientations. Hence they will have opposite colorings. Alternatively, if a band is removed and it leaves the two circles concentric, then their orientations will be the same. Hence their colorings will be the same. With no way for the surface to be non-orientable, the only possibility is that it is orientable.

Since the surface is embedded in $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$, we can compute it's Euler charateristic $\chi(S_k)$, which will be odd (since it has one boundary component). By the *classification theorem for compact surfaces* it follows that S_K is either homeomorphic to a punctured connect sum $\big(\#_g(S^1 \times S^1)\big) \setminus B^2$ of some positive integer g number of tori or homeomorphic to a once-punctured sphere $S^2 \setminus B^2$. We say that the *genus* of the surface S_K is g in the former case and 0 in the later case. In the later case, the knot K would bound a surface homeomorphic to a disc D^2.

Once we have a Seifert surface for a knot K, we can easily generate more surfaces bounded by the same knot that are of ever-higher genus. We do this by cutting out a small 2-disc D from S_K and gluing in a punctured torus - increasing the genus by 1 each time we do this. This is shown in Figure 2.21.

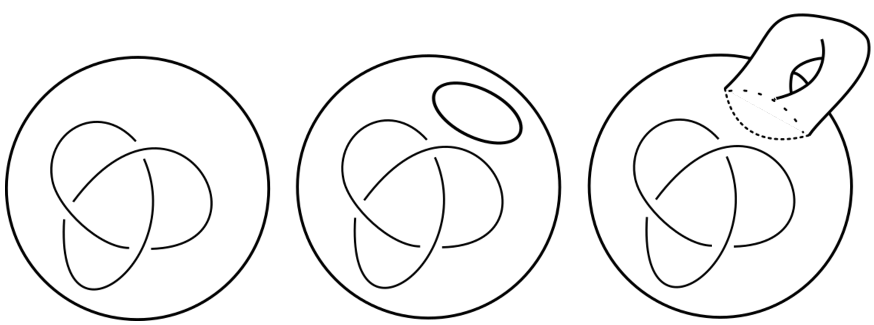


Figure 2.21: The trefoil bounds a surface S on the left of genus g. By construction, it also bounds a genus $g+1$ surface S as shown on the right.

It follows that for any knot K, we can find a surface S_K bounded by K that is of arbitrarily large genus. Perhaps it is clear that there is no upper bound on the genus in this regard. Thus, for a given K, it does not make sense to ask the following question

What is the *largest possible genus that a surface bounded by K could have?*

The answer to that would be "there is none." On the other hand, there is a clear lower bound on the genus of such surfaces. Namely, they cannot have genus less than 0. Thus, it *does* make sense to ask the question

What is the *smallest* possible genus that a surface bounded by K could have?

Consider the Seifert surface constructed using the Seifert algorithm for the left-handed trefoil as shown in Figure 2.20. This surface has genus 1, which can be shown by finding a homotopy equivalence between the surface and a cell complex made up of 2 points and 3 edges. Such a cell complex has Euler characteristic -1 (using the classic formula with 2 vertices, 3 edges, 0 faces gives $2 - 3 + 0 = -1$). Since Euler characteristic is invariant under homotopy (Theorem 2.44 of[17]), it follows that the Euler characteristic of S_K is also -1. We also know that S_K is an orientable surface embedded in S^3. By the classification of compact surfaces, any orientable surface embedded in S^3 is characterized solely by it's Euler characteristic and number of boundary components. That is, any two such surfaces with same Euler characteristic and same number of boundary components are the same surfaces topologically. By inspection, one can easily check that the once-punctured torus $(S^1 \times S^1) \setminus B^2$ is also an orientable surface embedded in S^3 with Euler characteristic -1. Thus S_K is homeomorphic to $(S^1 \times S^1) \setminus B^4$ and thus has genus 1.

Per the argument related to Figure 2.21, since we have a surface of genus 1 by the left-handed trefoil, we can easily construct further surfaces of genus 2, 3, 4, and so on that are also bounded by the left-handed trefoil. But what about genus 0? Is it possible for the left-handed trefoil to bound a 2-dimensional disc that is embedded in S^3? It turns out that

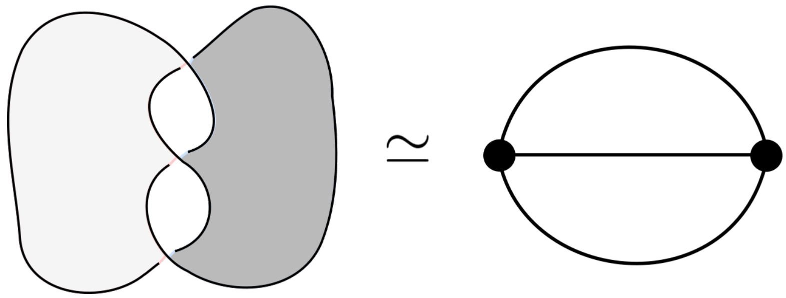


Figure 2.22: A Seifert surface and a graph that the surface is homotopy equivalent to.

the answer is no and this leads us naturally to what will be a foundational definition in this thesis.

Definition 2.3.2. Let K be a knot in S^3. The *Seifert genus of* K, denoted $g_3(K)$, is the least genus among all orientable surfaces S embedded in S^3 with boundary K.

Seifert gave an algorithm to generate a Seifert surface for a knot K from any given knot diagram D_K of K. However, there is no promise that this surface is the "best" in the sense of having the smallest genus. Of course, the absolute "best" surface we could ever hope for a knot to bound would be a 2-dimensional disc - that is, a surface of genus 0. It turns out that the only knot which bounds an embedded disc in S^3 is the unknot. The argument for this being the case is simple and elegant enough that we present it here shortly. But first, let us provide a formal definition for what we mean when we use the term *unknot.*

Definition 2.3.3. The *standard unknot* is the map $U : S^1 \to \mathbb{R}^3 \subset S^3$ defined by $\theta \mapsto (\cos\theta, \sin\theta, 0)$. We call a knot $K : S^1 \to S^3$ *a unknot* or often *the unknot* when K is ambient isotopic to U.

Proposition 2.3.4. The Seifert genus of a knot K is 0 if and only if K is the unknot.

Proof. ($\Rightarrow$) Suppose that K bounds a 2-dimensional disc D in S^3. There is a tiny, flat disc D' around the center point of D. Removing the interior of D' leaves an annulus $A = D \setminus \mathrm{int}D'$. There is a clear isotopy of A taking the inner boundary of A to the outer boundary of A. This shows that the inner and outer boundary are isotopic and hence are the same knot.

The inner boundary, which is the boundary of D', is clearly ambient isotopic to U in the definition above. Thus, so too is K. It follows that K is the unknot.

($\Leftarrow$) Let S^1 be the equatorial sphere of S^2 and S^2 be the equatorial sphere of S^3. Suppose K is the unknot. Then there is an ambient isotopy of K to S^1 (which is the image of U as listed in the definition above). As such, there is a diffeomorphism *phi* taking (K, S^3) to (S^1, S^3). Since S^1 is an equatorial sphere of S^2, it bounds a hemisphere D of S^2, which is a 2-dimensional disc. The image $\phi(D)$ is also disc bounded by K. This shows that K has genus 0. $\square$

We've remarked that a knot invariant is a means of assigning an algebraic object to a knot. In many cases, computing such an invariant involves taking a diagram representing the knot and following a given algorithm. For example, when computing the Alexander polynomial of the left-handed trefoil, the associated algorithm will yield the polynomial $t^2 - t + 1$ no matter what representative knot diagram is used during the algorithm. Does this apply to the minimal genus? Consider knot diagrams for the unknot as shown in Figure 2.23

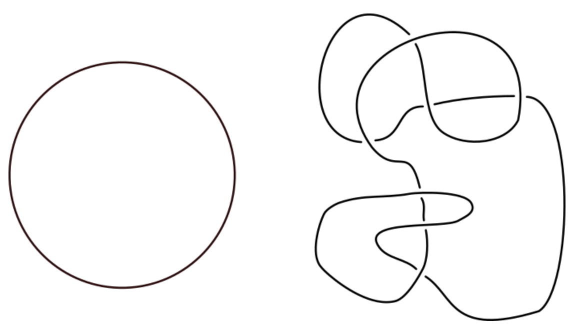


Figure 2.23: Two knot diagrams for the unknot.

Applying the Seifert algorithm to the left diagram of the unknot in Figure 2.23 yields a disc. On the other hand, applying the Seifert algorithm on the right-most diagram yields a surface of genus 2. One can check that the diagrams in Figure 2.23 do indeed both represent the unknot as they are related by a series of Reidemeister moves. It is true that the Seifert surface obtained by the Seifert algorithm is not invariant under Reidemeister moves. However, the minimal genus is not computed from a single knot diagram. It is computed rather from the entire (infinite) set of surfaces embedded in S^3 that are bounded by the knot. Thus, the invariant is in a general sense is completely independent of a knot's diagrams. In a practical sense though, one can actually use Seifert's algorithm to obtain a quick upper bound for the minimal genus. Once an orientable surface S_K embedded in S^3 is constructed for a knot K and the genus g of S_K determined, we have an immediate upper bound on $g_3(K)$. Namely, $g_3(K) \leq g_3(S_K)$.

The Seifert genus has been well studied. In partciular, it is known for at least all 2,978 prime knots up to 12 crossings[1]. Anyway, the Seifert genus is not the invariant we are particularly interested in. Rather, we want to focus on a higher-dimensional analog about two steps beyond it, the $\mathbb{C}P^2$-genus. In order to get there, we will consider an infinite family of 4-dimensional genus knot invariants and then $\mathbb{C}P^2$ as a particular, under-studied member of that family. And so now we move to the 4-dimensional world.

2.3.2 M-genus

The topological, piecewise-linear, and smooth categories are all equivalent in three dimensions. Unfortunately (or fortunately), this is not the case in the 4-dimensional world. For the rest of this thesis, we will be working in the smooth category. All our manifolds (including knots) and all maps between them will be smooth.

The reader may be unfamiliar with proper maps so we provide a definition here.

Definition 2.3.5. Let M_1 and M_2 be manifolds of dimension $\dim(M_1) \leq \dim(M_2)$. A map $f : M_1 \to M_2$ is *proper* if $f(\partial M_1) \subseteq f(\partial M_2)$ and $f(\text{int}M_1) \subseteq f(\text{int}M_2)$.

In 1966, Fox and Milnor extended the Seifert genus in a very natural way.

Definition 2.3.6. Let K be a knot. The *4-genus of K*, denoted $g_4(K)$, is the least genus among all orientable surfaces S smoothly and properly embedded in D^4 with $\partial S = K$.

Since it can be difficult (but not impossible) to imagine objects in 4-dimensional space, we will use cartoons to visually illustrate this idea. In particular, we will draw objects in reduced dimension. In Figure 2.24 below, we draw D^4 as a 2-dimensional disc with boundary $\partial D^4 = S^3$ represented as a 1-dimensional sphere. We represent K as a 0-dimensional sphere in the S^3 that D^4 bounds. We represent a surface S_K bounded by K and embedded properly in D^4 as a curve with end points K.

We call the diagrams in Figure 2.24 cartoons because they are not drawn according to any formal rigorous system. In particular, they do not accurately reflect the dimension of the objects discussed. However, they do perfectly encapsulate the relationships between objects in terms of boundary, inclusion, and embedding. This is often all we need. As such, we will alternate between these reduced-dimensional cartoons and full-dimensional diagrams.

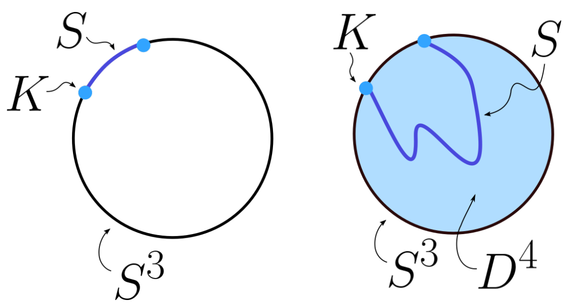


Figure 2.24: A surface S in S^3 bounded by a knot K may be "pushed into" the interior of D^4.

Because of reduced-dimensionality, S_K appears in Figure 2.24 as a line segment. Nevertheless, the reader should keep in mind that what appears as a line segment is actually a surface. In the right cartoon of Figure 2.24, we again have the knot K sitting in S^3, but now the interior of the surface S_K has "drifted" or "floated" into the interior of D^4. The fact that any surface S in S^3 with boundary K can be pushed into the interior of D^4 follows from the fact that for every compact, closed manifold S, there is a Morse function $f : S \to [0, 1]$.

Since there is a diffeomorphism between S_K and S'_K (the one "pushed into" the D^4), the two surfaces have the same genus. If S_K is a surface of minimal genus $g_3(K)$ for K in S^3, then S'_K also has genus $g_3(K)$. However, it is still possible that there is some other surface that embeds properly into D^4 of even less genus than $g_3(K)$. That is, for every tame knot K, $g_4(K) \leq g_3(K)$.

Earlier we mentioned that the only knot with Seifert genus 0 is the unknot. That distinction no longer holds when we move to the 4D world.

Definition 2.3.7. A knot K is *slice* if and only if $g_4(K) = 0$.

There are many slice knots. The smallest-crossing slice knot in the Rolfsen knot table notation is 6_1. We can see that moving to the 4-dimensional world is immediately interesting. As such, it is not surprising that much work has been done in studying the 4-genus. So much so that the 4-genus is currently known for all 2,978 prime knots up to 12 crossings (and many more beyond that). While the 4-genus has not been computed for every single knot (a potentially impossible task), it clearly is not wanting for attention. It is partially for this reason that the author choose to look beyond the 4-genus.

Let M be a smooth, closed 4-manifold. Closed means that it is compact and has no boundary. Being a manifold, we can choose a general point $x \in M$ and find a small open ball B^4 around x. By removing the ball B^4 from M, we now have a 4-manfiold $M \setminus B^4$ with boundary $\partial(M \setminus B^4) \cong S^3$. We present a cartoon example of this in Figure 3.7. A closed 4-manifold M with open 4-ball removed is often called a *punctured* manifold.

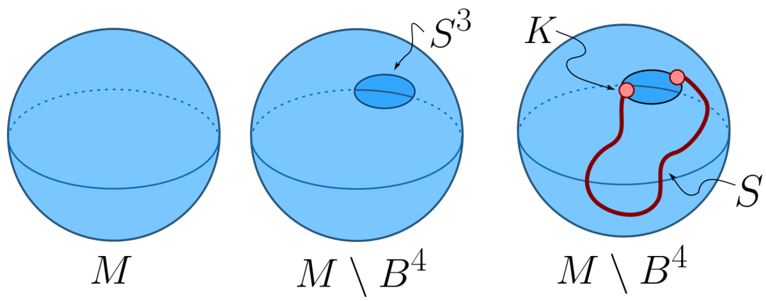


Figure 2.25: Cartoon of M, punctured M, and a properly embedded surface in puctured M.

Removing the open 4-ball leaves punctured M with a boundary diffeomorphic to S^3. If K is a knot, we think of K as sitting inside this S^3 boundary in the same way as with the 4-disc D^4. We now define a generalized version of the 4-genus.

Definition 2.3.8. Let K be a knot and M a smooth, closed 4-manifold. The *M-genus of* K, denoted $g_M(K)$, is the least genus among all surfaces S smoothly and properly embedded in $M \setminus B^4$ with $\partial S = K$.

If $g_M(K) = 0$, we say that K is *slice in M*.

In Figure 3.7 we show a cartoon of a knot K bounding a surface S in punctured M. Remember that the diagram is indeed a cartoon, where relationships are accurately represented but dimensions are not.

The manifold $M \setminus B^4$ may be very complicated and interesting. We've drawn it as a figurative S^2 only for simplicity. The reader should keep in mind that there may be very intricate and involved structure in what currently looks very plain. Likewise, the surface S may figuratively wind around the interior of $M \setminus B^4$ in a complicated and interesting way. It is drawn only as a wobbly line segment again for simplicity.

At this point, we are considering a pretty wide class of knot invariants. Though we've restricted ourselves to smooth and orientable surfaces (as opposed to topological and/or nonorientable), the M parameter is still wide open. The set of all smooth closed 4-manifolds is infinite. The family $\#_k(S^2 \times S^2)$ with $k = 1, 2, 3, ...$ is just one infinite subset of it. It would be helpful to organize these manifolds in such a way as to get a better grapple on this infinitude. One way to do this is to classify manifolds M according to how often knots are slice within them.

Every smooth, compact, connected 4-manifold is obtained by attaching one or more 4-dimensional handles of varying index to a base 4-disc $h^0 \cong D^4$[13]. Let's think about just h^0 for a moment. Since h^0 is a 4-disc D^4, it has a boundary $\partial h^0 \cong S^3$. In constructing a connected 4-dimensional handlebody M, all the subsequent handles are attached to this ∂h^0. After all the constituent handles are added to form M, there is no remaining boundary (since M is closed). When we puncture M, we introduce a boundary component to M. Let's consider that puncture to occur in the h^0 portion of M, away from ∂h^0. Restricting our attention to just h^0 for a moment, we see that before the handle attachments are made, a punctured $h^0 \setminus B^4$ would have two boundary components. One component the original boundary ∂h^0 and the second component stemming from the puncture. A cartoon diagram of this is presented in Figure 2.26.

We take our knots K to be contained in the punctured boundary, as shown in Figure XX below. Consider $h^0 \setminus B^4$ as a trivial S^3 bundle over the interval $[0, 2]$ with S^3_0 being the S^3 resulting from the puncture and S^3_2 being the original S^3 boundary of h^0. If we take $\{S^3_t :$

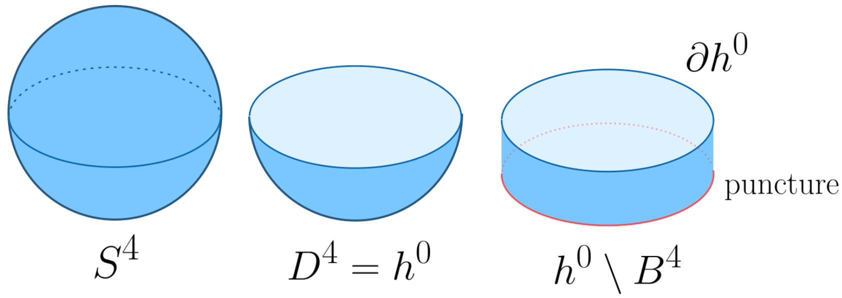


Figure 2.26: From left to right, S^4 with two successive punctures.

$t \in [0,1]\}$ together with the 4-disc D_1^4 bounded by S_1^3, we have a 4-disc $D = D_1^4 \cup \bigcup_{t\in[0,1]} S_t^3$ contained in $M \setminus B^4$ that contains K. Thus,any surface S that can be smoothly and properly embedded in D^4 with boundary K can also be smoothly and properly embedded in $M \setminus B^4$ with boundary K. Keeping in mind that $g_{S^4}(K) = g_4(K)$, we have now shown that for every smooth, connected, closed 4-manifold M,

$$g_M(K) \leq g_{S^4}(K).$$

In terms of the M-genus, S^4 is a bounding manifold. For any knot K, the highest M-genus it could have is in S^4. The eager reader might hope for a bound in the other direction and indeed, one such bound exists. It happens to be that every knot is slice in $S^2 \times S^2$. The lowest possible genus a knot could have in a given manifold M is 0. Thus

$$g_{S^2\times S^2}(K) \leq g_M(K).$$

Combining the two inequalities, we have

$$g_{S^2\times S^2}(K) \leq g_M(K) \leq g_{S^4}(K)$$

.

The g_{S^4}-genus is exactly the 4-genus g_4. This has been thoroughly studied. On the other hand, in Chapter 3 we will completely determine the $(S^2 \times S^2)$-genus for all tame knots. This leaves one to wonder about all the other smooth closed 4-manifolds. In general, the M-genus of this set remains almost entirely untouched. The only 4 manifold for which something appreciable was known about this type of knot genus invariant was $\mathbb{C}P^2$. The new work presented in this thesis expands on what little was known.

For a further introduction to Seifert surfaces and general knot genus, see section 4.3 of Colin Adams' The Knot Book[14], titled *Genus and Seifert Surfaces.*

2.4 Cobordisms & Surgery

A Cobordism is a manifold between manifolds. Surgery is a method of taking one manifold and modifying it to construct a new manifold. This process naturally generates a cobordism. This relationship between corbordism and surgery will be essential to our main work.

2.4.1 Cobordisms

In simplest terms, a cobordism is a manifold between manifolds. In an informal sense, think of a cobordism as a "tube" with one manifold on each of it's two "ends."

Definition 2.4.1. Let M and N be n-dimensional manifolds. A *cobordism W between M and N* is a $(n+1)$-dimensional manifold W such that $\partial W = M \sqcup N$.

That the boundary of W is the disjoint union of two manifolds is what we mean informally by saying that there is a manifold on each of W's two "ends." A cobordism is perhaps better shown than explained, so we point to Figure 2.27 as an example of a cobordism between the manifolds S^1 and $S^1 \sqcup S^1$ - the infamous "pair of pants".

2.4.2 Surgery

In a moment we will see the definition of surgery. Before looking at it, note that is is something of a notational mess which largely obscures a relatively simple idea. Put in simplest terms, mathematical surgery is nothing more than the process of removing one piece of a manifold and then gluing back in another piece.

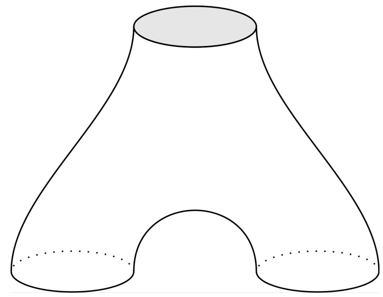


Figure 2.27: The "pair of pants" surface is a cobordism between the manifolds S^1 (at top) and $S^1 \sqcup S^1$ at bottom.

Definition 2.4.2. Let M, M' be manifolds of dimension $n = p + q$, let ϕ be an embedding $S^p \times D^q \to M$. We say that M' *is produced via surgery from* M if and only if

$$M' = (M \setminus \mathrm{int}(\mathrm{im}\phi)) \cup_{\phi|_{S^p \times S^{q-1}}} (D^{p+1} \times S^{q-1}).$$

Let's consider a simple example. Let $M = S^1$, with $p = 0$ and $q = 1$. Let $\phi : S^0 \times D^1 \to S^1$ be the embedding shown in FIGURE 2.28. Removing the interior of image of ϕ leaves a space of two-components. Having removed a copy of $S^0 \times D^1$ from S^1, we now glue in a complementary $D^1\ timesS^0$. The result of this gluing is $S^1 \sqcup S^1$, a disjoint union of two S^1s.

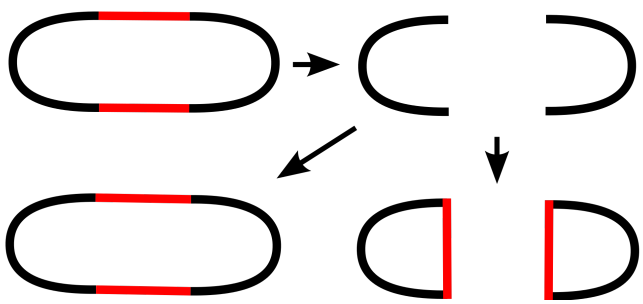


Figure 2.28: Performing surgery on S^1 to produce $S^1 \sqcup S^1$.

Performing surgery on M to produce M' has a pleasant side effect of generating a cobordism W between M and M'. In particular,

$$W := (M \times I) \cup_{S^p \times S^q \times 1} D^{p+1} \times D^q.$$

One can think of W as being primarily the $M \times I$ in the following sense. The $M \times \{0\}$ slice of $M \times I$ is clearly a copy of M. The $M \times \{1\}$ end is modified by attaching the $(n+1) = (p+q+1)$-dimensional piece $D^{p+1} \times D^q$. Observe that $D^{p+1} \times D^q$ has boundary $(S^p \times D^q) \cup (D^{p+1} \times S^{q-1})$. The $S^p \times D^q$ portion of the boundary is absorbed into the $M \times \{1\}$, leaving the $D^{p+1} \times S^{q-1}$ portion free to form M' along with the remaining portion of $M \times \{1\}$.

The simple example we used to show how surgery works is actually highly relevant to the work of this document. The technique we use to construct specific surfaces bounded by knots involves surgery on knots and links. As we see, surgery on 1-dimensional knots and links will produce a 2-dimensional cobordism. The 2-dimensional cobordisms of subsequent surgeries are glued together and capped off to form surfaces bound by knots. These surfaces are used to provide upper bounds for the $\mathbb{C}P^2$-genus.

It should be noted that Surgery Theory (with capital letters) is a significant field within geometric topology that was developed to study higher-dimensional manifolds. Going down that path would lead to quite a time investment. Readers who simply want to know a little more about surgery would better serve themselves by exploring some of the ideas that led to the development of the field, rather than jumping into the formal field itself. To this end, the author recommends A procedure for killing homotopy groups of differentiable manifolds by John Milnor[18].

2.5 Mutation

Mutation appears in our Chapter 4, where we recount Piccirillo's excellent proof showing that the Conway knot is not slice. Since the basic concept is relatively simple and the non-specialist reader is probably unfamiliar with it, we introduce the idea here.

Mutation is somewhat similar to surgery in the sense that it is a sequence of removal and replacement. It is also similar in the sense that the process sometimes produces something new and sometimes does not. Whereas surgery can be (and is) performed on manifolds of various dimensions, mutation is used most often in dimension 1 with classical knots. However,

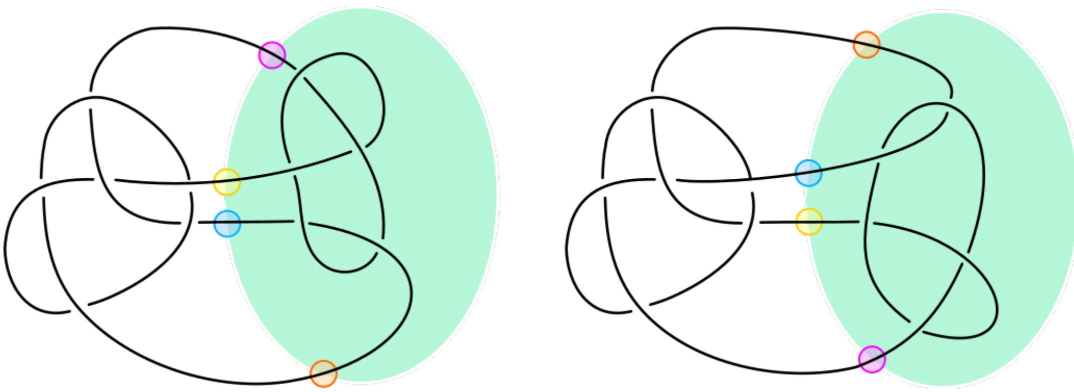


Figure 2.29: (Left) The Conway knot C ($11n_{34}$ in KnotInfo (cite)) (Right) The Kinoshita-Teresaka knot $11n_{42}$.

it should be noted that interesting work is being done with mutation on 3-dimensional manifolds.

Mutation uses the notion of a Conway sphere to dictate what is removed. A Conway sphere S is a copy of S^2 embedded in $\mathbb{R}^3$ such that S intersects a knot K in exactly 4 distinct spots. That is, $|S \cap K| = 4$. An informal argument should convince the reader that each Conway sphere then contains exactly two disjoint strands. Now, instead of removing these strands entirely from the knot and gluing in something new (as was the case with surgery), one can think of mutation as rigidly "rotating" the Conway sphere so that the strands align back up with the knot and another knot is obtained. This last point is easily missed when first encountering mutation (at least, the author missed it initially). The only rotations allowed are those that preserve the number of components. So if there is a knot to begin with (1 component), there should be a knot at the end.

The most famous pair of mutant knots are the Conway and Kinoshita-Teresaka knots, as shown in Figure 2.29. Mutant knots are pretty interesting in general, but this pair will be of particular interest to us in Chapter 4.

Like many foundational notions in knot theory, Colin Adams' The Knot Book[14] provides an excellent introduction to mutation.

Chapter 3

Genera of knots in $S^2 \times S^2$

When considering the M-genus over all smooth closed 4-manifolds, $S^2 \times S^2$ and S^4 act as bounds of sorts. If M is an arbitrary smooth closed 4-manifold and K is an arbitrary knot, then

$$g_{S^2\times S^2}(K) \leq g_M(K) \leq g_{S^4}(K).$$

While this would not be notable if it was true for only a single knot, in fact it holds for *all* tame knots. By definition, the lowest value $g_M(K)$ could ever be is zero since the minimal possible genus of an orientable surface with boundary is zero. This of course doesn't inherently guarantee that there even is a manifold M such that $g_M(K) = 0$. Or even more - that there is some M such that $g_M(K) = 0$ for *all* knots. And yet $S^2 \times S^2$ is just such a manifold. This is why we call $g_{S^2\times S^2}$ a lower bound.

Here is a rough outline for our argument that every tame knot is slice in $S^2 \times S^2$. Every tame knot can be turned into the unknot via a finite number of crossing changes. The two S^2 factors ($S^2 \times \{x\}$ and $\{x\} \times S^2$) in $S^2 \times S^2$ allows one to make these changes. After moving crossings over the factors as required by the unknotting sequence, the result is an unknot and the unknot naturally bounds a disc. This is not fully rigorous of course, but it is simple, easy to remember, and captures the essence of the argument enough to hopefully make the upcoming rigor easier to digest.

The constructions in both this chapter and the penultimate Chapter 5 will involve co-

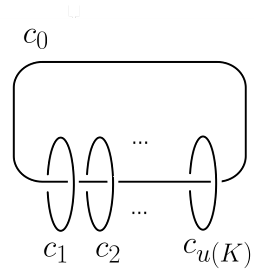


Figure 3.1: The link $D = \cup_{i=0}^{u(K)} c_i$.

herent band surgery, so we define it here.

Definition 3.0.1. Let L be a link in S^3. A *band surgery* on L is an embedding $b : I \times I \to S^3$ such that $L \cap b(I \times I) = b(I \times \partial I)$. The surgery is *coherent* if the link

$$L' = \big[L \setminus \mathrm{int}[b(I \times \partial I)]\big] \cup_{\partial I \times \partial I} b(\partial I \times I)$$

is oriented. Otherwise, the surgery is *non-coherent*. We say that the link L' is obtained from L via the band surgery b.

Remark 3.0.2. A band surgery may also be referred to as a *band move* in the literature.

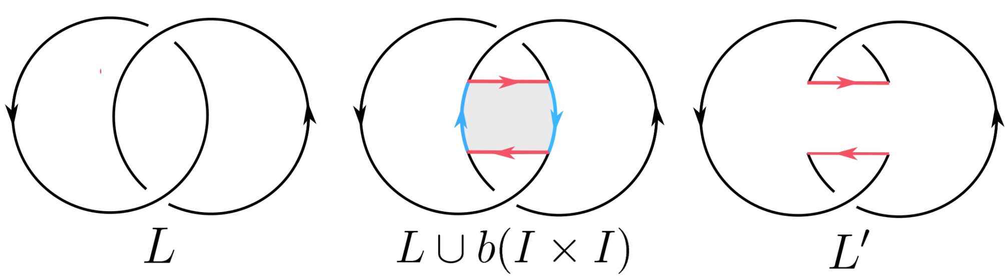


Figure 3.2: A coherent band surgery from link L to link L'.

Let L' be a link obtained from L via band surgery b. If $b(I \times I)$ intersects only a single component of L, then L' will have exactly one more component than L. If $b(I \times I)$ intersects two components of L, then L' will have exactly one less component than L. A band surgery either joins two components of L into one or splits one component of L into two. All components of L that do not intersect $b(I \times I)$ remain unchanged. It follows from elementary surgery theory that there is a genus 0 cobordism $C \subset S^3 \times I$ between L and L' where C is a "pair of pants" between the component(s) that are changed and a series of disjoint cylinders elsewise.

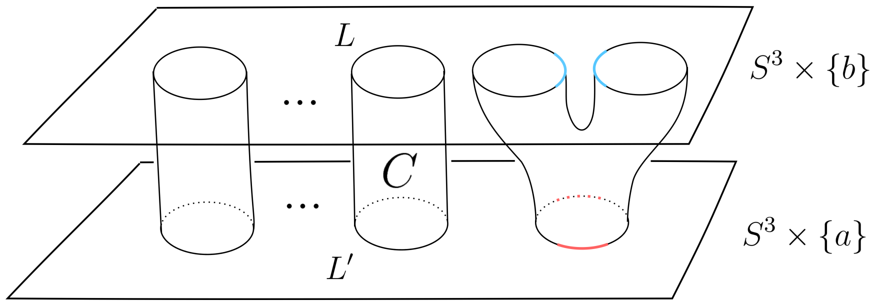


Figure 3.3: A cartoon of a genus 0 cobordism C in $S^3 \times I$ with boundary components $L \subset S^3 \times \{b\}$ and $L' \subset S^3 \times \{a\}$, where $0 \leq a < b \leq 1$. Such a corbordism would be the result of performing a coherent band surgery to obtain L' from L.

Lemma 3.0.3. *Let K be a nontrivial smooth (and hence tame) knot and let $u(K)$ be the unknotting number of K. Let $D = \bigcup_{i=0}^{u(K)} c_i$ be the union of disjoint unknotted circles $c_0, c_1, ...c_{u(K)}$ embedded in S^3 as shown in Figure 3.1. Then K is obtainable from D via a series of $u(K)$ coherent band surgeries.*

We will call D together with the required surgery bands $b_1, ...b_{u(K)}$ a *pre surgery presentation* for a knot. This is the author's labeling for further ease of reference during this chapter. As an example of Lemma 3.0.3 in action, we present in Figure 3.4 a pre surgery presentation for the knot listed as 6_3 in the Rolfsen knot table[15]. We then show the result post-surgery. Using Reidemeister moves, this knot can be reduced to 6-crossings. Checking the Alexander polynomial against the prime knots of 6-crossings and below show it to be the knot 6_3.

Lemma 3.0.3 will be essential for proving the following central theorem of this chapter.

Theorem 3.0.4. *Let K be a smooth (and hence tame) knot. Then $g_{S^2\times S^2}(K) = 0$.*

Theorem 3.0.4 follows naturally from Lemma 3.0.3 in the following way.

Each of the components of D in Lemma 3.0.3 bound an obvious 2-dimensional disc that is embedded in S^3. Let $D_0, D_1, ...D_{u(k)}$ be the discs bounded by $c_0, c_1, ..., c_{u(k)}$ respectively,

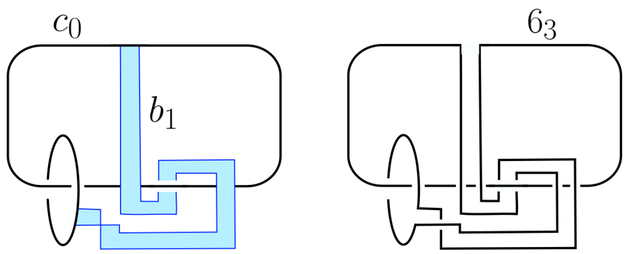


Figure 3.4: The knot 6_3 before and after surgery.

as shown in Figure 3.5. Taking the boundary connect sum between $D_0, D_1, ..., D_{u(K)}$ and the bands $b_1, ..., b_{u(K)}$ one obtains a single *immersed* disc in S^3 with boundary K. Locally, this disc is embedded in $\mathbb{R}^3$. However, due to intersection between the discs $D_1, ..., D_{u(K)}$ and D_0, it is not *globally* embedded. Computing the $S^2 \times S^2$-genus requires embedded surfaces. In some sense, the M-genus is a calculation of how easily the aforementioned self-intersections can be resolved. In this way, there is something special about the geometry of $S^2 \times S^2$ that allows for the self-intersections to be completely avoided in the case of every tame knot.

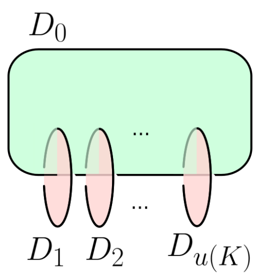


Figure 3.5: Discs $D_0, D_1, ..., D_{u(K)}$ embedded in S^3 and bounded by components $c_0, c_1, ..., c_{u(K)}$.

Proof of Theorem 3.0.4. Let K be a smooth (and hence tame) knot with unknotting number $u(K)$. By Lemma 3.0.3, there is a pre surgery presentation for K, as shown in Figure 3.1. Let $S_\alpha := S^2 \times \{pt\}$ and $S_\beta = \{pt\} \times S^2$. We know that $S^2 \cong D^2 \cup_{S^1} D^2$. Let D_α be one disc in this decomposition of S_α and D_β likewise be one disc in the decomposition of S_β. The product $D_\alpha \times D_\beta$ is diffeomorphic to a 4-dimensional disc. Removing the interior of $D_\alpha \times D_\beta$ is diffeomorphically equivalent to removing a 4-ball from $S^2 \times S^2$. Hence,

$S^2 \times S^2 \setminus \mathrm{int}(D_\alpha \times D_\beta) \cong S^2 \times S^2 \setminus B^4$. It follows that $S^2 \times S^2 \setminus \mathrm{int}(D_\alpha \times D_\beta)$ has one S^3 boundary component. Let K be contained in this boundary, as shown in Figure 3.6. Let D'_α be the disc in S_α such that $S_\alpha = D_\alpha \cup_{S^1} D'_\alpha$. Since $S^2 \times S^2$ is a trivial S^2-bundle over S^2, we can take a parallel copy D_0 of D'_α such that c_0 bounds D_0. In the same way, if $S_\beta = D_\beta \cup_{S^1} D'_\beta$, then we can take $u(K)$ parallel copies $D_1, ..., D_{u(K)}$ of D'_β such that c_i bounds D_i for each $i \in \{1, ..., u(K)\}$. Due to the triviality of the bundle, these paralell copies can be taken to be non-intersecting (and non-linking). Taking the boundary connect sum of $D_0, D_1, ..., D_{u(K)}$ with the respective bands $b_1, ..., b_{u(K)}$ provides a properly embedded disc in $S^2 \times S^2 \setminus \mathrm{int}(D_\alpha \times D_\beta) \cong S^2 \times S^2 \setminus B_4$ with boundary the knot K. This shows that $g_{S^2 \times S^2}(K) = 0$. $\square$

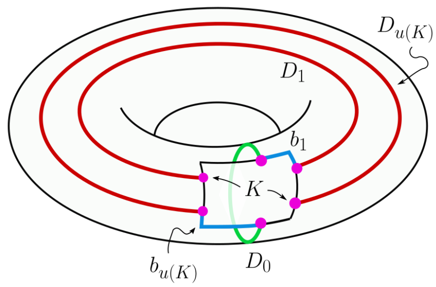


Figure 3.6: Reduced-dimensional picture of punctured $S^2 \times S^2$ with corresponding discs and bands.

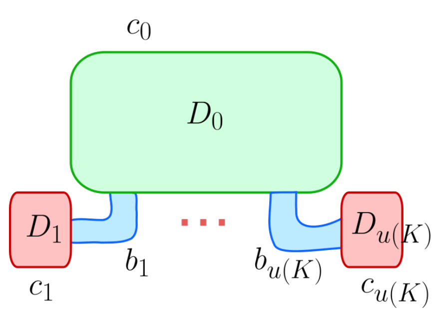


Figure 3.7: Full dimensional picture of the discs $D_0, D_1, ..., D_{u(K)}$ together with surgery bands $b_1, ..., b_{u(K)}$.

Chapter 4

Genera of knots in S^4

When Fox and Milnor introduced the S^4-genus in 1966, the focus was not on the extended family to which we now consider it a part of, but rather solely on the invariant itself. Even when Suzuki showed in 1969 that the concept of the S^4-genus could be extended to include other smooth 4-manifolds, there was no great rush by mathematicians to explore orientable genera across the spectrum of 4-manifolds. Indeed, almost all efforts were concentrated in a quest to completely understand the S^4-genus. How much of this was intentional, the author cannot say. Either way, after Suzuki's 1969 paper, it would take until 1992 for any significant work to appear that involved orientable genera in smooth closed 4-manifolds besides S^4.

Much work has been done in investigating the S^4-genus. As of the time of this writing, the S^4-genus is known for all prime knots through 12-crossings (though it is notable that there was a singular holdout for many years). That is 2,978 knots for which we now know the S^4-genus. We also know the S^4-genus of knots beyond this, such as for infinite families of knots like family of general torus knots.

In this chapter, we explore some highlights of the work done in trying to determine the S^4-genus of low crossing knots.

4.1 The S^4-genus is an upper bound on the M-genus

When the M-genus is considered across all smooth closed 4-manifolds M, the manifolds $S^2 \times S^2$ and S^4 have special distinction. For every tame knot K and smooth 4-manifold M,

$$g_{S^2 \times S^2}(K) \leq g_M(K) \leq g_{S^4}(K).$$

In Chapter 3 we showed the left part of the inequality. Now we show the right part.

Lemma 4.1.1. *Let K be a knot with S^4-genus $g_{S^4}(K)$. Then*

$$g_M(K) \leq g_{S^4}(K)$$

Proof. Let K be a knot in $\partial(M \setminus B^4) \cong S^3$. Since M is a connected manifold, it's handlebody decomposition has exactly one 0-handle h^0. By ambient isotopy H of $\partial(M \setminus B^4)$, we can shrink K to be as small as we want and move it through $\partial(M \setminus B^4)$ until it resides entirely in the boundary S^3 of h^0. By the definition of smooth 4-genus, there is a surface S_K of genus $g_4(K)$ smoothly and properly embedded in h^0 such that $(S_K, \partial S_K = K) \subset (h^0, \partial h^0 \cong S^3)$. At this point, K is in the boundary of h^0 but not necessarily in the boundary of $M \setminus B^4$. There is a cobordism C between the shrunken K in the boundary of h^0 (the image $H(K, 1)$) and K in the boundary of $M \setminus B^4$. Gluing C to S_K gives a surface S of genus $g_{S^4}(K)$ that is smoothly and properly embedded in $M \setminus B^4$ with boundary K.

□

4.2 The unknotting number is an upper bound on the S^4-genus

In order to determine the S^4-genus of a knot, one needs an upper bound, a lower bound, and for those bounds to agree. The unknotting number invariant gives us an easy upper bound. The unknotting number of a knot is the minimal number of crossing changes required to convert the knot into the unknot. For example, the left-handed trefoil has unknotting number 1. Changing zero crossings will not turn it into the unknot. Yet by changing any crossing in a minimal-crossing diagram, the unknot is obtained. Computation of the unknotting number is still an open problem for prime knots up to 12-crossings in general. The lowest crossing knot for which the unknotting number is not definitively known is 10_{11} (in the notation of KnotInfo[1]). Nonetheless, the unknotting number is known for the vast majority of prime knots up to 12-crossings and it provides an upper bound on the S^4-genus that is devilishly easy to compute.

Theorem 4.2.1. *Let K be a knot with unknotting number $u(K)$. Then*

$$g_{S^4}(K) \leq u(K).$$

Proof. If K is the unknot, then $u(K) = 0$ and clearly $g_{S^4}(K) = 0$. Thus, the inequality holds. Now suppose that K is a nontrivial knot and hence $u \in \mathbb{Z}^+$. Fix a sequence of u crossing changes $c_1, ..., c_u$ for K that turns K into the unknot. Such a sequence exists by the definition of unknotting number.

Let $K_0 = K$ and for each $i = 1, ..., u$, let K_i be the knot obtained from K_{i-1} by performing the crossing change at c_i. For each remaining c_i, we perform two coherent band surgeries as shown in Figure 4.1. The first of the surgeries will generate a cobordism $C_{i,a}$ between $K_{i-1} \subset S^3 \times \{(u-i+1)/u\}$ and $K_i \# L2a1$ where $L2a1$ is the link $L2a1\{0\}$ or $L2a1\{1\}$ depending on whether c_i is a positive or negative crossing. The second of the surgeries will create a cobordism $C_{i,b}$ between $K_i \# L2a1$ and $K_i \subset S^3 \times \{(u-1)/u\}$. The stacking of $C_{i,a}$ and $C_{i,b}$ gives a genus 1 cobordism C_i between $K_{i-1} \subset S^3 \times \{(u-i+1)/u\}$ and $K_i \subset S^3 \times \{(u-i)/u\}$. The cobordism C_i and the necessary coherent band surgeries needed to create it are shown in Figure 4.1.

Stacking the cobordisms $C_1, C_2, ...C_u$, we obtain a genus u cobordism between $K = K_0$ and K_u. Since K has unknotting number u and we chose a sequence of crossing changes $c_1, ..., c_u$ specifically to unknot K, it follows that K_u is the unknot. The unknot is easily seen to be capped off with an embedded 2-disc, thus generating a genus u properly embedded surface $\bigcup_{i=1}^{u} C_i$ in $\mathbb{C}P^2 \setminus B^4$ with boundary K. This shows that $g_{S^4}(K) \leq u(K)$, as desired.

$\square$

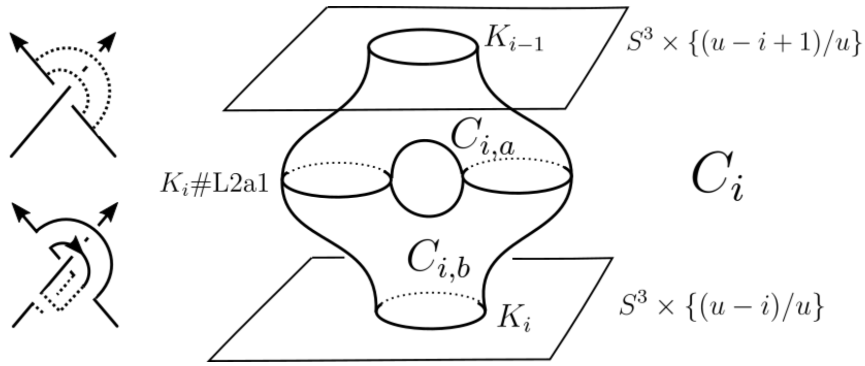


Figure 4.1: Two coherent band surgeries are performed in succession to generate a genus 1 cobordism C_i between $K_{i-1} \subset S^3 \times \{(u-i+1)/u\}$ and $K_1 \subset S^3 \times \{(u-i)/u\}$.

4.3 Fox-Milnor Theorem

One of the most satisfying results obtained so far regarding the S^4-genus is the Fox-Milnor Theorem. It is a sliceness characterization that involves one of the most fundamental knot invariants - the Alexander polynomial. We will use the notation $\Delta(K)$ to stand for the Alexander polynomial of a knot K.

Theorem 4.3.1 (Fox-Milnor Theorem). *Let K be a tame knot. If $g_{S^4}(K) = 0$ then there exists some integral polynomial $f(t)$ such that $\Delta(K) \equiv f(t)f(t^{-1})$.*

The equivalence $\equiv$ represents equality up to multiplication by $\pm t^n$ for any $n \in \mathbb{Z}$.

Example 4.3.2. Consider the knot 6_1 (as notated in the Rolfsen knot table). This knot is slice in S^4. Thus, by the Fox-Milnor Theorem, there should be some polynomial $f(t)$ in t with integral coefficients such that $\Delta(6_1) = f(t)f(t^{-1})$. The Alexander polynomial of 6_1, up to multiplication by $\pm t^n$, is $\Delta(6_1) = -2t + 5 - 2t^{-1}$. Simple algebraic computation shows that this polynomial factors as $(-t+2)(-t^{-1}+2)$ so we see that $f(t) = t - 2$.

The lowest-crossing knot after the unknot is the trefoil. Even though 6_1 is slice, the trefoil - perhaps surprisingly - is not.

Proposition 4.3.3. The S^4-genus of the trefoil is 1.

Proof. It's Alexander polynomial is $\Delta(3_1) = -t + 1 - t^{-1}$. Suppose for contradiction that there exists some $f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$ such that $\Delta(3_1) = f(t)f(t^{-1})$. Then

$$\begin{aligned} 1 + (-1)t + (-1)t^{-1} &= (a_0 + a_1 t + \cdots + a_n t^n) \cdot (a_0 + a_1 t^{-1} + \cdots + a_n t^{\pm n}) \\ &= (a_0^2 + a_1^2 + \cdots + a_n^2) + (a_0 a_1 + a_2 a_1 + a_4 a_3 + \cdots)t + (a_0 a_1 + a_3 a_2 + a_5 a_4 + \cdots)t^{-1} \end{aligned}$$

This shows that $a_0^2 + a_1^2 + \cdots + a_n^2 = 1$. Since $a_0, a_1, ..., a_n$ are all integers, then all but one of $a_0, a_1, ..., a_n$ are equal to zero. If that is the case, then the coefficient $a_0 a_1 + a_2 a_1 + a_4 a_3 + \cdots$ is equal to 0, a contradiction (since it should be -1). By the Fox-Milnor Theorem, 3_1 is not slice in S^4. This shows that $g_{S^4}(3_1) \geq 1$. The surface shown in Figure 2.13 is bounded by the trefoil and embedded in S^3 with genus 1. It is easy enough to "push" that surface into the interior of D^4 so that 3_1 bounds the same surface now properly embedded in D^4. Thus, $g_{S^4}(3_1) \leq 1$. Combining these two inequalities gives $g_{S^4}(3_1) = 1$. $\square$

For the mathematician who wants to compute the S^4-genus of a knot, the Fox-Milnor theorem offers an obstruction to sliceness. This can be pretty helpful in of itself. Upper bounds on the genus of a knot are relatively easy to obtain simply by constructing a surface. Obtaining lower bounds is often much more difficult and usually relies heavily on algebraic topology. The Fox-Milnor theorem only obstructs a knot from having S^4-genus 0, but given that the S^4-genus is generally pretty small, this is still quite helpful. Of the 2,978 prime knots up to 12-crossings, 1,866 of them have S^4-genus equal to 1. For those knots, obstructing the S^4-genus from being 0 is a big step as it only leaves one to construct a genus 1 surface bounded by the knot.

A key ingredient of the proof of the Fox-Milnor Theorem is a "torsion invariant" as defined by Milnor[19]. We give a characterization of the invariant.

Proposition 4.3.4 (Milnor 1962)**.** Let M be an oriented m-manifold and let $\widetilde{M}$ be the infinite cyclic cover of M. There is a rational function $\Delta(\widetilde{M}) = a(t)/b(t)$, consisting of non-zero integral polynomials $a(t), b(t)$ and defined up to multiplication by $\pm t^n, n\mathbb{Z}$ such that the following hold:

(a) $\Delta(\widetilde{\partial M}) = \Delta(\widetilde{M}) \cdot \overline{\Delta}(\widetilde{M})$ where $\overline{\Delta}(\widetilde{M})$ is $\Delta(\widetilde{M})$ under the operation $t \mapsto t^{-1}$,

(b) If $H_n(M;\mathbb{Z}) = H_n(S^1;\mathbb{Z})$, then $\Delta(\widetilde{M}) = \Delta(\pi_1(M))/(t-1)$, where $\Delta(\pi_1(M))$ is the Alexander polynomial of the fundamental group of M,

(c) If N is a submanifold of M, then there is a relative torsion invariant $\Delta(\widetilde{M}, \widetilde{N})$ with the property that

$$\Delta(\widetilde{M}) = \Delta(\widetilde{N})\Delta(\widetilde{M}, \widetilde{N}).$$

(d) If D is a 2-disc, then $\Delta(D \times \widetilde{\partial D^2}, K \times \widetilde{\partial D^2}) = 1/(t-1)$.

Note that the proposition above is a compilation of several results from Milnor's 1962 paper, assembled to make it easier to prove the Fox-Milnor Theorem here. So as to avoid straying too far afield, we forgo giving a proof of Proposition 4.3.4. The interested reader is encouraged to read Milnor's 1962 paper for more details. Relying on Proposition 4.3.4 we now prove the Fox-Milnor Theorem.

Proof of Fox-Milnor Theorem. Let K be a tame knot that is slice in S^4. By definition, there is a 2-disc D_1^2, properly embedded in a 4-disc D_1^4 such that $\partial D_1^2 = K$ and $K \subset \partial D_1^4$. Likewise, the mirror mK of K bounds a 2-disc D_2^2, properly embedded in a 4-disc D_2^4. Gluing the 4-discs D_1^4 and D_2^4 along their boundary S^3 with an orientation reversing diffeomorphism yields $D_1^4 \cup_{S^3} D_2^4$ diffeomorphic to S^4. Contained in this S^4 is a 2-sphere $D_1^2 \cup_K D_2^2$.

Let V be a tubular neighborhood of D_1. Since D_1^2 is contractible, V is a trivial D^2 bundle over D_1^2. That is, $V = D_1^2 \times D^2$. Let $Q := \mathrm{cl}(D_1^4 \setminus V)$ and $W := \mathrm{cl}[S^3 \setminus (K \times D^2)]$, where the S^3 here is the gluing area used to construct $D_1^4 \cup_{S^3} D_2^4$. Observe that $\partial Q = W \cup_{K \times \partial D^2} (D_1^2 \times \partial D^2)$.

By Proposition 4.3.4(a), we have that

$$\Delta(\widetilde{\partial Q}) = \Delta(\widetilde{Q}) \cdot \overline{\Delta}(\widetilde{Q}). \tag{4.1}$$

We defined W to be $\mathrm{cl}[S^3 \setminus (K \times D^2)]$. This is homotopy equivalent to $S^3 \setminus K$ via a deformation retract. The fundamental group $\pi_1(S^3 \setminus K)$ is known as the knot group of

K and is given by the Wirtinger presentation[15]. The Wirtinger presentation for a tame knot is of the form $\langle x_1, ..., x_n; r_1, ..., r_n \rangle$ where $x_1, ..., x_n$ are classes of generating loops in $S^3 \setminus K$ and $r_1, ..., r_n$ are relations. Each relation r_i is either of the form $x_k x_i = x_{i+1} x_k$ or $x_i x_k = x_k x_{i+1}$ with subscripts modulo n. Abelianizing these relations gives a series of equalities showing that all the generators are the same in homology. Further argument shows that $H_\star(W; \mathbb{Z}) = H_\star(S^1; \mathbb{Z})$. By Proposition 4.3.4(b), we have that

$$\Delta(\widetilde{W}) = \Delta(K)/(t-1) \tag{4.2}$$

where $\Delta(K)$ is the Alexander polynomial of the knot K.

Since W is a submanifold of ∂Q, we can take the relative paring $(\partial Q, W)$. By removing the interior of W from both ∂Q and W, we are left with the relative pair $(D_1 \times \partial D^2, K \times \partial D^2)$. By excision and Proposition 4.3.4(d), we have that

$$\begin{aligned} \Delta(\widetilde{\partial Q}, \widetilde{W}) &= \Delta(D_1 \times \widetilde{\partial D^2}, K \times \widetilde{\partial D^2}) \\ &= 1/(t-1). \end{aligned} \tag{4.3}$$

Substituting Equations 4.2, 4.3, and 4.1 into the equation $\Delta(\widetilde{\partial Q}) = \Delta(\widetilde{W})\Delta(\widetilde{\partial Q}, \widetilde{W})$ (Proposition 4.3.4(c)) , we have

$$\begin{aligned} \Delta(\widetilde{Q}) \cdot \overline{\Delta}(\widetilde{Q}) &= \Delta(K)/(t-1)^2 \\ \Delta(K) &= \left[\Delta(\widetilde{Q}) \cdot (t-1)\right] \cdot \left[\overline{\Delta}(\widetilde{Q}) \cdot (t-1)\right] \\ &\equiv \left[\Delta(\widetilde{Q}) \cdot (t-1)\right] \cdot \left[\overline{\Delta}(\widetilde{Q}) \cdot (t^{-1}-1)\right] \\ &= \left[\frac{a(t)}{b(t)} \cdot (t-1)\right] \cdot \left[\frac{a(t^{-1})}{b(t^{-1})} \cdot (t^{-1}-1)\right] \\ \Delta(K) &= c(t) \cdot c(t^{-1}) \end{aligned} \tag{4.4}$$

where $c(t)$ is a rational expression made up of polynomials with integer coefficients. It is worth noting that $\Delta(K)$ is nearly in the form we want. It is a product, but the factors are not quite of the form we would like. The remaining argument rewriting the product appropriately.

Since $\mathbb{Z}[t]$ is a unique factorization domain and $c(t)$ is a ratio of elements of this domain, we may write $c(t)$ in "lowest terms." That is, we may write $c(t) = p(t)/q(t)$ where $p(t), q(t)$ are relatively prime elements of $\mathbb{Z}[t]$. Now we have that $p(t), q(t)$ are in lowest terms, but not that $p(t^{-1}), q(t)$ necessary are. Let $d(t) \in \mathbb{Z}[t]$ be the greatest common divisor of both $p(t^{-1})$ and $q(t)$. Then $p(t^{-1}) = f(t^{-1})d(t)$ and $q(t) = g(t)d(t)$ for some $f(t), g(t) \in \mathbb{Z}[t]$, where $f(t^{-1})$ and $g(t)$ are relatively prime. Substituting appropriately into equation 4.4, we have

$$\begin{aligned}
\Delta(K) &= c(t) \cdot c(t^{-1}) \\
&= \frac{p(t) \cdot p(t^{-1})}{q(t) \cdot q(t^{-1})} \\
&= \frac{f(t) \cdot d(t^{-1}) \cdot f(t^{-1}) \cdot d(t)}{g(t) \cdot d(t) \cdot g(t^{-1}) \cdot d(t^{-1})} \\
&= \frac{f(t) \cdot f(t^{-1})}{g(t) \cdot g(t^{-1})}
\end{aligned}$$

Since $p(t)$ and $g(t)$ were relatively prime, then so are $f(t)$ and $g(t)$. By definition $f(t^{-1})$ and $g(t)$ are relatively prime. Thus $g(t)$ is relatively prime to both $f(t)$ and $f(t^{-1})$. Thus, the ratio $f(t) \cdot f(t^{-1})/g(t) \cdot g(t^{-1})$ is in lowest terms (up to multiplication by $\pm t^n, n \in \mathbb{Z}$). Since $\Delta(K)$ is an integral polynomial, it follows that $g(t) \equiv 1$ and thus

$$\Delta(K) \equiv f(t) \cdot f(t^{-1})$$

as desired. $\square$

4.4 Milnor conjecture

Torus knots are an important and oft-studied family of tame knots. Luckily, mathematicians have determined exactly what the S^4 genus is for any such knot.

Theorem 4.4.1 (Milnor conjecture)**.** *For every pair* $(p, q) \in \mathbb{Z}^+ \times \mathbb{Z}^+$*,*

$$g_{S^4}(T(p,q)) = \frac{(p-1)(q-1)}{2}$$

As usual with knot genera, this can be proved by showing the inequalities $g_{S^4}(T(p,q)) \leq \frac{(p-1)(q-1)}{2}$ and $g_{S^4}(T(p,q)) \geq \frac{(p-1)(q-1)}{2}$ are both true. The later inequality was shown by Williams[20], Murasugi, and Pryztycki[21]. The former inequality was proved for the first time in 1993 by Kronheimer and Mrowka using Gauge Theoretic techniques[22].

Gauge Theory is quite difficult to both learn and use. Proving the Milnor conjecture using these techniques is well beyond the scope of this work. However, we are fortunate in the sense that there is a more readily accessible method of proving the Milnor conjecture. In 2004, Rasmussen used Lee homology - an alternative version of Khovanov homology - to re-prove the conjecture. We provide exposition for this proof, beginning with an explanation of Lee homology.

Lee Homology

Let L be a link and D a minimal crossing diagram of L with initial orientation θ_0. In a full computation, an s_i is computed for each possible orientation θ_i, where $i = 0, ..., 2^c - 1$ and c is the number of components of L. We will explain how to compute s_0 for θ_0. The other computations would be analogous. In the diagram D with orientation θ_0, resolve each crossing according to the rule shown in Figure 4.2.

To resolve a crossing diagrammatically is to:

1. Remove a small neighborhood around the crossing the intersects the link L in exactly 4 points.

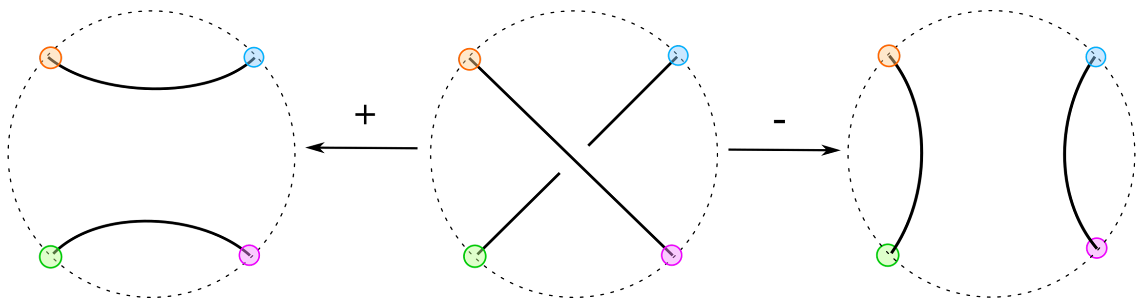


Figure 4.2: Each crossing in the diagram D is resolved according to it's sign.

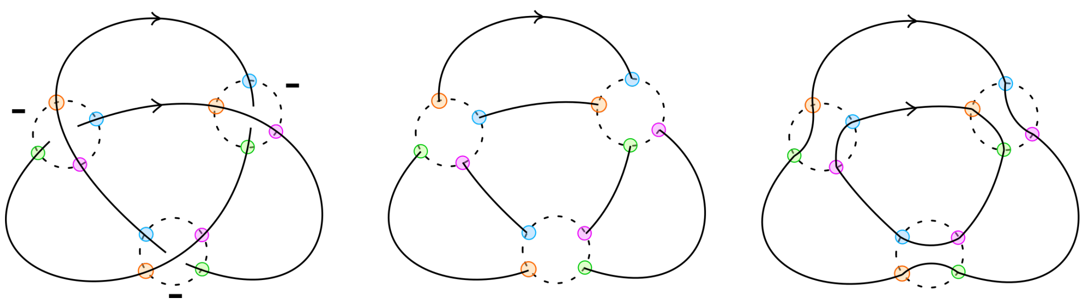


Figure 4.3: A canoncial resolution in Lee homology of the left-handed trefoil with given orientation.

2. Replace the neighborhood with a one containing 2 components and no crossing.

An example of this can be seen in Figure 4.3, where the left-handed trefoil is resolved according to the rule given in Figure 4.2.

Once D with given orientation θ_0 is resolved, each component C_i in the resolution is assigned a monomial $x + 1$ or $x - 1$ according to the table below. When we write even, we mean that the component is separated from ∞ by an even number of components (here, think of ∞ as the edges of this paper). For example, the center component in Figure 4.3 is separated from ∞ by the singular component which surrounds it and no other. Thus, the inner component is separated from infinity by an odd number of components (namely 1). Since the inner component also has a clockwise orientation, it is assigned $x + 1$. Let $v_1, ..., v_k$ be the polynomials assigned to the k components of the diagram resolution under orientation θ_0. We define s_0 to be the tensor product of $v_1, ..., v_k$. That is

$$s_0 = v_1 \otimes v_2 \otimes \cdots \otimes v_k.$$

	Even	Odd
Clockwise	$x-1$	$x+1$
Anti-clockwise	$x+1$	$x-1$

Table 4.1: Polynomial assignment rule

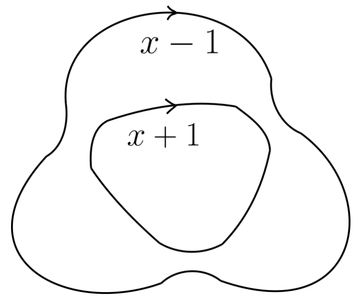


Figure 4.4: Example of assignment

The elements x and 1 are generators of a Frobenius algebra $A = \mathbb{Q}\{x, 1\}$. Due to the simplicity of Lee homology, we need not concern ourselves with the details of A. For our purposes, we need only keep in mind the following facts:

- The degree of x, denoted $\deg(x)$, is -1,
- The degree of 1, denoted $\deg(1)$, is 1, and
- $\deg(a \otimes b) = \deg(a) + \deg(b)$ when $a, b \in \{x, 1\}$.

The reader may notice that the degree of the polynomials $x+1$ and $x-1$ are not defined by the rules above. Indeed, this reveals an issue that arises in Lee homology but is not found in Khovanov homology, namely that elements lack a homogeneous degree. This will be addressed shortly. For now, let us take stock of what it is that we have done so far. We started with a link L and a diagram D of L with an orientation θ_0. We resolved D in a canonical way and labeled it's components with monomials of the form $x+1$ and $x-1$. Then we defined an element $s_0 = v_1 \otimes v_2 \otimes \cdots \otimes v_k$. This element s_0 is a homology class that acts as a generator for the Lee cohomology $Lee^*(L)$. The diagram D of L has 2^c distinct orientations $\theta_0, ..., \theta_{2^c-1}$, where c is the number of components of L. For each orientation θ_i, we can define an element s_i in the same way as we did for s_0. The set of s_i fully describes the Lee cohomology of L. In particular

$$Lee^*(L) = \mathbb{Q}\{[s_i] : i = 0, ..., 2^c - 1\}.$$

Note that each s_i is a homology class with $[s_i]$ being a *cohomology* class in $Lee^*(L)$ represented by s_i. It is helpful to keep this in mind as we define the following integers.

Since $s_0 = v_1 \otimes \cdots \otimes v_k$ and each v_i is either $x+1$ or $x-1$, s_0 can be alternatively written as

$$s_0 = \sum_{k=0}^{c} \delta_k \binom{c}{k} \bigotimes^{c-k} x \bigotimes^{k} 1$$

where c is the number of components in the canonical resolution of D with orientation θ_0 and $\delta_k = \pm 1$ for each $k = 0, 1, ..., c$. Rasmussen[23] defines the *quantum degree* of s_0 to be

$$q(s_0) = -c + w(D)$$

where $w(D)$ is the writhe of D (the number of positive crossings minus the number of negative crossings).

In the case where L is a knot K and $\alpha \in Lee^*(K)$, Rasmussen[23] defines the following values.

$$s(\alpha) = \max\{q(v) : [v] = \alpha\}$$
$$s_{\min}(K) = \min\{s(\alpha) : \alpha \in Lee^0(K), \alpha \neq 0\}$$
$$s(K) = s_{\min}(K) + 1$$

Let $\alpha \in Lee^*(K)$ be arbitrary and nonzero. Since a knot consists of only a single component, there are only two orientations to consider, say θ_0 and θ_1, with respective homology classes s_0 and s_1. We know from Lee's paper that $[s_0], [s_1]$ generate $Lee^*(K)$ as $\mathbb{Q}$-linear combinations. Thus, $\alpha = a[s_0] + b[s_1]$ for $a, b \in \mathbb{Q}$. In order to find $s_{\min}(K)$, we want to choose α in such a way that it minimizes $s(\alpha)$. A minimizing choice is $\alpha = s_0$. This gives that $s_{\min}(K) = -c + w(D)$ and ultimately that $s(K) = -c + w(D) + 1$.

We now want to show that for an general torus knot $T(p, q)$, we have $-c + w(D) + 1 = (p-1)(q-1)$. We start by letting $T(p, q)$ be a torus knot with $p, q \in \mathbb{Z}^+$. This can be modeled as in Figure 4.5.

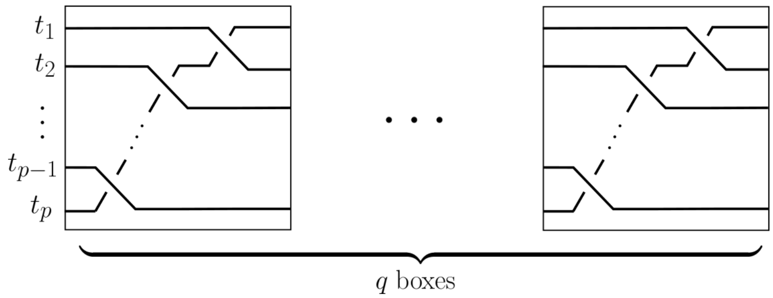


Figure 4.5: A braid diagram for $T(p, q)$.

Observe that in each of the q boxes of the braid diagram, there are $p-1$ crossings as one of the p strands passes underneath the $p-1$ other strands. Since $p, q \in \mathbb{Z}^+$, all the crossings in the braid diagram are positive. Thus

$$w(D) = (p-1)q.$$

Furthermore, since each crossing is positive,, it is resolved canonically as shown in Figure 4.6. This leaves the identity braid, which is a set of p unlinked components. Thus, $c = p$. We now have that

$$\begin{aligned} s(T(p,q)) &= -c + w(D) + 1 \\ &= -p + (p-1)q + 1 \\ &= pq - p - q + 1 \\ &= (p-1)(q-1) \end{aligned}$$

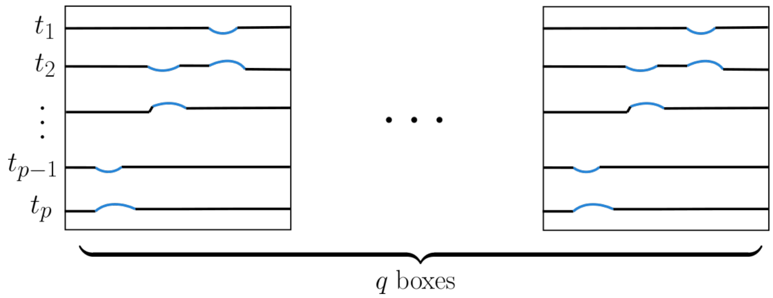


Figure 4.6: A braid diagram for $T(p, q)$ after the canonical resolutions of Lee homology.

Proof of Milnor Conjecture

Proof of Theorem 4.4.1. We have shown that $s(T(p, q)) = (p - 1)(q - 1)$. By Rasmussen[23],

$$\begin{aligned} g_{S^4}(T(p, q)) &\geq \frac{|s(T(p, q))|}{2} \\ &= \frac{(p - 1)(q - 1)}{2} \end{aligned}$$

Combining this inequality with the inequality from Section 4.2 gives the desired equality. $\square$

4.5 The Conway knot is not slice

With a single short paper, Suzuki completely determined the $S^2 \times S^2$-genus for all tame knots. That was in 1969. Though the S^4-genus was introduced in 1966, as of 2020, it is still not completely understood. Indeed, until recently it was not even understood for all prime knots through 12-crossings. As of 2018, the S^4-genus was known for all 2,977 prime knots up to 12-crossings with one exception - the Conway knot. However, that single exception was not to last. In the same year, Lisa Piccirillo showed that the Conway Knot has S^4-genus equal to 1. What follows is exposition of Piccirillo's work.

In section 4.2, we showed that the unknotting number of a knot is an upper bound on it's S^4-genus. By inspection, we find that the unknotting number of the Conway knot C is 1. Thus $g_{S^4}(C) \leq 1$. It is left to show that C is not slice (hence $g_{S^4}(C) \geq 1$).

Mutation is a means of constructing new knots from old ones. In this sense, the Conway

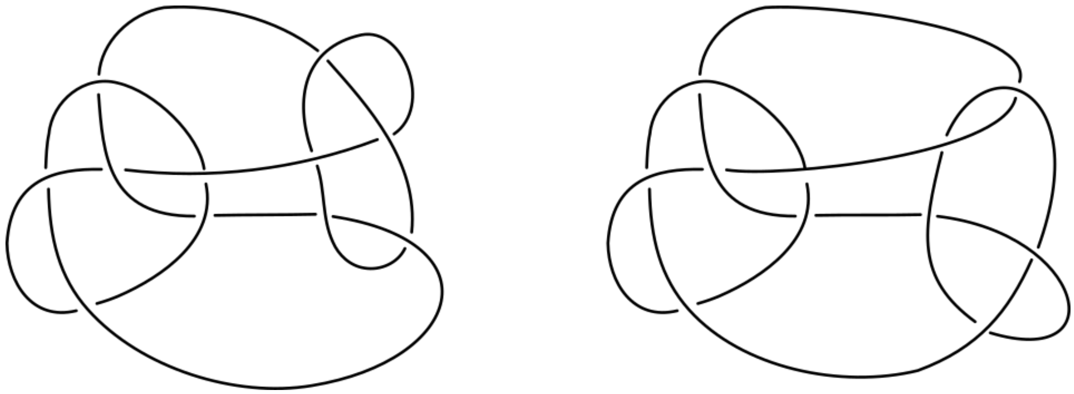


Figure 4.7: (Left) The Conway knot C ($11n_{34}$ in KnotInfo (cite)) (Right) The Kinoshita-Teresaka knot $11n_{42}$.

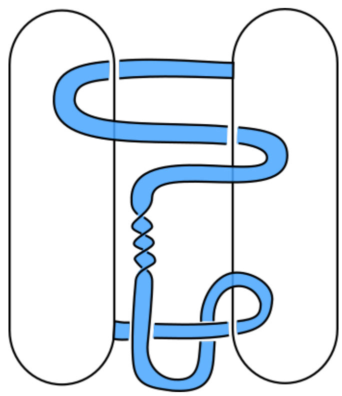


Figure 4.8: A band surgery between two unknots, resulting in the the Kinoshita-Teresaka knot.

knot can be thought of as being constructed from the Kinoshita-Teresaka knot via mutation, as shown in Figure 4.7. The Kinoshita-Teresaka knot can be seen to be slice in S^4 by consulting Figure 4.8. Since there is a coherent band surgery on the unlink that results in the Kinoshita-Teresaka knot, there is an orientable cobordism in D^4 between the unlink in the interior of D^4 and the Kinoshita-Teresaka knot KT in the boundary S^3 of D^4. Since the unlink can be capped off in the interior of D^4 with two discs, it follows that the Kinoshita-Teresaka knot bounds a properly embedded slice disc in D^4.

Mathematicians often use knot invariants to obtain lower bounds on know genera. However, mutation preserves many knot invariants. That is, for many knot invariants, the value assigned to the Conway knot will be the same as the one assigned to the Kinoshita-Teresaka

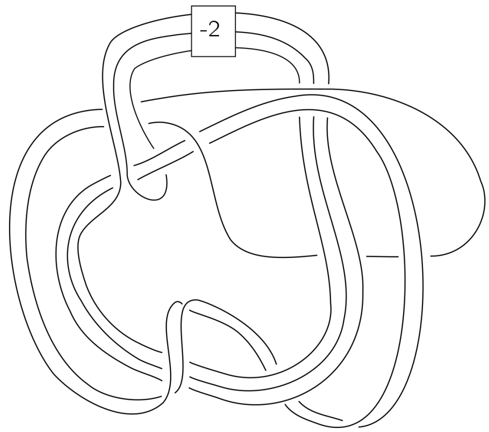


Figure 4.9: Knot K'.

knot. Since the Kinoshita-Teresaka knot is slice, none of it's invariants will satisfy a sliceness obstruction. Hence, the invariants shared between the Conway knot and the Kinoshita-Teresaka knot will also fail to satisfy any sliceness obstruction. Despite the evasiveness of the Conway knot in avoiding sliceness obstruction, there was some early hint that the Conway knot was not slice. In 1986, Gabai[24] showed that the 3-genus of the Conway knot and the Kinoshita-Teresaka knot were distinct (the former being 3 and the later being 2). If the Conway knot had 3-genus 1 greater than the Kinoshita-Teresaka knot, it was reasonable to guess that it's 4-genus was also 1 greater. The critical insight of Piccirillo was to forgo looking at the Conway knot itself (or the Kinoshita-Teresaka knot) and instead construct another knot that failed to be slice yet shared enough information with the Conway knot that the failure could be passed to the Conway knot. That knot we will denote K' and is shown in Figure 4.9.

There are two reasons that we are about K'.

1. It is related to the Conway knot in a critical way, and

2. it is not slice.

First let us show that it is not slice. To do so, we will use Lee homology.

Theorem 4.5.1 ([23]). *For any knot K, the generators of Lee homology are located in gradings $(i, j) = (0, s(K) \pm 1)$. If K is slice, then $s(K) = 0$.*

By computing the Khovnaov homology table of K' and examining the spectral sequence of the associated Lee homology, the Rasmussen s-invariant of K' is determined to be 2. Thus, since $s(K') \neq 0$ it follows by Rasmussen's theorem that K' is not slice. Hence the following lemma:

Lemma 4.5.2 ([25]). *K' is not slice.*

We now see that K' is a knot which is not slice and we have thus far trusted that there is a relationship between K' and the Conway knot C. To see that relationship, let us consider the following definition.

Definition 4.5.3. The *trace* of a knot K, denoted $X(K)$, is the smooth 4 manifold resulting from attaching a 0-framed, 4-dimensional 2-handle to a 4-dimensional 0-handle along the knot K.

The next lemma allows us to turn a question of sliceness into one of embedding.

Lemma 4.5.4 ([25]). *K is slice if and only if $X(K)$ smoothly embeds into S^4.*

Proof. ($\Rightarrow$) Choose a S^3 inside of S^4 that decomposes S^4 into 4-discs D_1 and D_2. That is, $S^4 = D_1 \cup_{S^3} D_2$. Let K rest in S^3. Since K is slice it bounds a slice disc D_K properly embedded in D^1. Since D_K is homologous to a point, it has trivial disc bundle $\nu(D_K)$. The handlebody $D^2 \cup \nu(D_K)$ is diffeomorphic to $X(K)$ since it is by definition a 0-framed 2-handle attached to a 0-handle along K. We observe that since $D^2 \cup \nu(D_K)$ embeds into S^4, then so does $X(K)$.

($\Leftarrow$) Let D be the core disc of the 2-handle h^2 of $X(K)$. Due to the way that $X(K)$ is constructed, we observe that D is bounded by K. Now let $C(K)$ be the cone of K, which we can take to sit in the interior of the 0-handle $h^0 \cong D^4$ of $X(K)$. We observe that $S = D \cup_K C(K)$ is the image of a piece-wise linear embedding $F : S^2 \hookrightarrow X(K)$. By hypothesis, there exists a smooth embedding $i : X(K) \hookrightarrow S^4$. Let p be the cone point of $C(K)$ in $X(K)$ and let p_S be the point in S^2 such that $F(p_S) = p$. Meanwhile, let K_S be the S^1 embedded in S^2 such that $F(K_S) = K \in X(K)$ and let D_S be the 2-disc of S^2 with boundary K_S and not containing p_S. We observe that $i \circ F$ is not a smooth embedding at p_S. Let N be an arbitrarily small tubular neighborhood of p_S in S^2 that does not intersect K_S and hence does not intersect D_S. We observe that $i \circ F|_{S^2 \;\; N}$ is a smooth embedding into S^4 that embeds both K_S into S^4 as K and D_S into S^4 as a disc bounded by K. Decomposing S^4 appropriately shows that K bounds a properly embedded disc in D^4 and hence is slice. $\square$

We now have all the tools needed to prove the following:

Theorem 4.5.5 (Piccirillo CITE)**.** *The Conway knot C is not slice.*

Proof. Suppose for contradiction that C is slice. By Theorem 4.5.4, $X(C)$ embeds into S^4. Using Kirby calculus, Piccirillo showed diagrammatically that $X(K') \cong X(C)$. Since $X(C)$ embeds into S^4 and $X(K')$ is diffeomorphic to $X(C)$, then $X(K')$ embeds into S^4. By a further application of 4.5.4, we have that K' is slice, contradicting Lemma 4.5.2. $\square$

Since the Conway knot is not slice in S^4, then $g_{S^4}(C) \geq 1$. By the Section 4.2, since the unknotting number of C is 1, we have that $g_{S^4}(C) \leq 1$. The compound inequality gives us that $g_{S^4}(C) = 1$.

The Conway knot was the last prime knot of 12-crossings or less to reveal its S^4-genus. The revelation greatly surprised the mathematical community and neatly ties up that chapter in mathematical history.

We now know the the S^4-genus of all 2,978 prime knots through 12-crossings. Yet, there are still many knots whose S^4 genus remains unknown; not least of which is the set of prime knots of 13-crossings, of which there are nearly 10,000. Not to mention prime knots of 16-crossings, of which there are over 1 million. Given that it is has taken over 50 years and a wide ray of techniques to compute less than 3,000 knots, the task of continuing to compute the S^4-genus seems ever more daunting.

Chapter 5

Genera of knots in $\mathbb{C}P^2$

Throughout this chapter, we work in the smooth category. All manifolds are considered to be connected, orientable, oriented, and compact unless otherwise stated. D^n stands for a n-disc with boundary S^{n-1}, while B^n stands for an open n-ball with no boundary. We use general notation and orientation conventions that are consistent with Livingston and Naik's excellent text on knot concordance[26]. In particular, if K is a knot in S^3, then mK stands for it's mirror, which is the same knot in S^3 but with all positive crossings changed to negative crossings and vice versa. The knot which is the same as K but with reverse string orientation is called the *reverse* and is denoted rK. The knot which has the reverse string orientation of mK is denoted by $rmK = mrK = -K$ and is called the *inverse*. The same holds for links. We use notation for specific knots that aligns with Knotinfo[1]. In particular, when one clicks on a knot K at the Knotinfo site, we take the diagram shown on the left to be the knot K and the diagram shown on the right to be the knot mK. For links we use the notation of Linkinfo[27]. The top left corner diagram is L, while the bottom right diagram is mL. We call a knot K *chiral* if K fails to be isotopic to either mK or $m(rK) = -K$.

In $S^2 \times S^2$, the genus of all knots is known (as they are all slice). In S^4, the genus is now known for all prime knots up through 12-crossings. Before the original work of this thesis, much less was known in the case of $\mathbb{C}P^2$. What was known is limited to the following:

- All knots that are slice in S^4 are slice in $\mathbb{C}P^2$ or have unknotting number 1 are slice

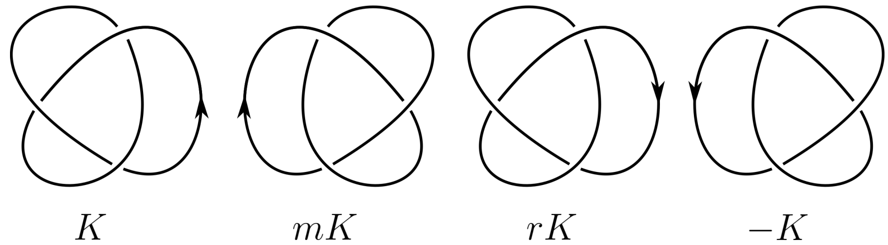


Figure 5.1: From left to right: a knot K, it's mirror mK, it's reverse rK, and it's inverse $r(mK) = m(rK) = -K$.

in $\mathbb{C}P^2$,

- A knot K may have a different $\mathbb{C}P^2$-genus than it's mirror mK,
- $g_{\mathbb{C}P^2}(T(2, 2x+1)) = x - 1$ when $1 \leq x \leq 8$,
- $g_{\mathbb{C}P^2}(T(-2, 2x+1)) = 0$ when $1 \leq x \leq 4$,
- $g_{\mathbb{C}P^2}(T(-2, 2x+1)) = 1$ when $x = 5$.

The primary goal of the author's dissertation work has been to methodically compute the $\mathbb{C}P^2$-genus of low-crossing knots. Let's see how the known information above advanced - or failed to advance - that goal.

There are 2,978 knots up to 12-crossings (not counting mirrors distinctly). The first bullet point covers 505 of these knots. So at first blush, this seems to have made our job much easier. This feeling vanishes however when we consider the second bullet point. Unlike in S^4, in $\mathbb{C}P^2$ knots may have different genera than their mirrors. The lowest crossing example of this is 5_1. The 5_1 knot (cinquefoil with all positive crossings) has $\mathbb{C}P^2$-genus 1, while $m5_1$ (the cinquefoil with all negative crossings) has $\mathbb{C}P^2$-genus 0. Now some knots do not have distinct mirrors - the figure-eight knot 4_1 is one such example - but the vast majority do. So while we were fortunate to be able to cross off all the 1-unknotting number knots from our list, that fortune is completely negated by this additional complication. The later three bullet points concern a finite number of torus knots. Very few prime knots up to 12-crossings are torus knots.

Since computing the $\mathbb{C}P^2$-genus for all prime knots up to 12-crossings seems wildly unattainable for a single author's dissertation work, let's restrict our attention to prime knots up to 8-crossings. Counting distinct mirrors, there are 64 such knots. Below is a table showing the state of affairs at the start of the author's work.

Crossings	Total distinct	Previously unknown $\mathbb{C}P^2$-genus
Up to 6	13	0
7	14	6
8	37	19

To whittle down the list of the list of knots for which the genus was still unknown, the author established the following results.

Theorem 5.0.1. . *Let K be a knot such that mK is obtained from one of the links below via coherent band surgery. Then K bounds a properly embedded disc D_K in $\mathbb{C}P^2 \setminus B^4$ with $[D_K] = d\gamma$. In particular, K is slice in $\mathbb{C}P^2$.*

$\lvert d \rvert$			Links				
0		mL2a1{1}		L5a1{0}		L7n2{0}{1}	
1	Unlink		L4a1{0}		mL7a4{0}{1}	mL7n1{1}	$3_1\#m$L4a1{1}
2		L2a1{1}		mL5a1{0}		mL7n2{0}{1}	
3			L4a1{1}		L7a3{0}{1}	mL7n1{0}	$3_1\#m$L4a1{0}

The notation for the links above is taken from Link Info[27]. For the reader without easy access to a computer or internet, diagrams for the links are provided in Appendix A.

Theorem 5.0.2. *Let K be an alternating knot with $|\sigma(K)| = 4$. Then either K or mK fails to be slice in $\mathbb{C}P^2$.*

Theorem 5.0.3. *Let K be an alternating knot with $\sigma(K) = 4$, $g_4(K) \leq 2$, and Arf$(K) = 0$. Then K is not slice in $\mathbb{C}P^2$.*

With these three results, the author was able to determine the $\mathbb{C}P^2$-genus of 5 of the 6 unknown 7-crossing knots and 13 of the 19 unknown 8-crossing knots. Work done by Daemi, Scaduto[28] in the midst of writing this thesis found the remaining unknown 7-crossing knot genus. The results as they stand at the writing of this thesis follow.

Crossings	Total distinct	Currently unknown $\mathbb{C}P^2$-genus
Up to 6	13	0
7	14	0
8	37	6

5.1 Upper bounds

There are two preliminaries which the reader will find helpful in order to understand the sliceness conditions in in Subsection 5.1.2 and the proof of Theorem 5.0.1 in Subsection 5.1.3. The first - coherent band surgery - was covered in Chapter 3. The second - the handlebody decomposition of $\mathbb{C}P^2$ - was mentioned in Chapter 2, but we reemphasize it again here.

5.1.1 Handlebody decomposition of $\mathbb{C}P^2 \setminus B^4$

Basic decomposition

The handlebody decomposition of $\mathbb{C}P^2 \setminus B^4$ is a single 4-dimensional 0-handle $h^0 \cong D^4$ and a single 4-dimensional 2-handle $h^2 \cong D^2 \times D^2$ together with a gluing map $\phi : \partial D^2 \times D^2 \to \partial h^0$. Thinking of $S^1 \times D^2$ as a trivial D^2-fiber bundle over S^1, ϕ maps the 0-section of $S^1 \times D^2$ to an unknot in ∂h^0, while mapping the D^2 fibers of $S^1 \times D^2$ into ∂h^0 so that the fibers have exactly one full positive twist. Details of this construction can be found in pages 119-124 of Gompf and Stipsicz's excellent text on 4-manifolds[29].

Labeling of slices in h^0

Any disc D^n is diffeomorphic to the cone $C(S^{n-1})$ smoothed out over the singular point. Using this fact, it is not hard to see that the cylinder $S^3 \times [0, 1]$ embeds smoothly into h^0 via a map ψ so that $S^3 \times \{1\}$ maps to ∂h^0 and $S^3 \times \{0\}$ maps into the interior of h^0. We fix such an embedding ψ once and for all. We label the boundary ∂h^0 as $S^3 \times \{1\}$ and the image $\psi(S^3 \times \{x\})$ in the interior of h^0 as $S^3 \times \{x\}$ for each $x \in [0, 1)$. We recognize that

$h^0 \setminus \psi(S^3 \times (0,1])$ is diffeomorphic to D^4. A diagram of this labeling is provided in Figure 5.2.

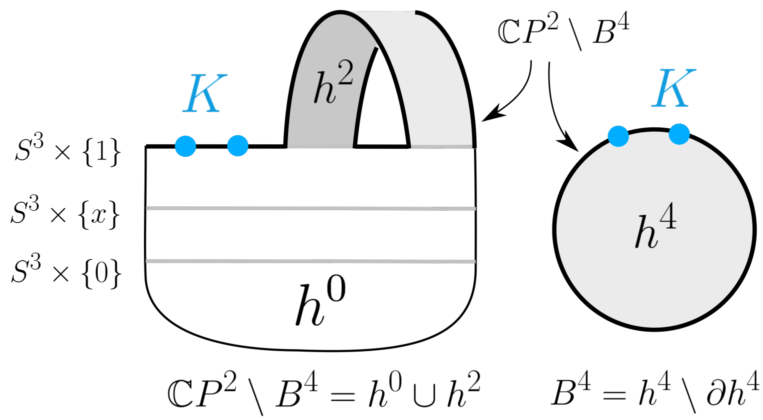


Figure 5.2: A flattened (lower-dimensional) handlebody diagram of $\mathbb{C}P^2 \setminus B^4$ with labels given by ψ.

Torus link $T(n,n)$

The core disc of the 2-handle h^2 is $D := D^2 \times 0$. We may push any number of parallel copies $D_1, ..., D_n$ off of D and they will be embedded in h^2 without pair-wise or self-intersections. Since $D_1, ..., D_n$ are 2-dimensional discs, their boundaries $\partial D_1, ..., \partial D_n$ are 1-dimensional spheres. Due to the $+1$-twist of the 2-handle attachment, the boundaries $\partial D_1, ..., \partial D_n$ form a $T(n,n)$ torus link first in the attaching region $S^1 \times D^2$ and then in turn in the boundary $S^3 \times \{1\}$ to which $S^1 \times D^2$ is attached via ϕ. A diagram of this is shown in Figure 5.3.

Homological degree

When pushing off a parallel copy D_i of the core disc D, the orientation may be chosen to be either compatible or incompatible with the orientation of D. The boundary component ∂D_i of $T(n,n)$ inherits an orientation from D_i. Thus, the link $T(n,n)$ has oriented components $\partial D_1, ..., \partial D_n$. All the constructions we perform will involve creating a cobordism C between a knot K and a link $T(n,n)$. Since the link $T(n,n)$ may be capped off with n discs $D_1, ..., D_n$, it follows that $S_K = C \cup_{T(n,n)} \bigcup_{i=1}^n D_i$ is a surface in $\mathbb{C}P^2 \setminus B^4$ with boundary $K \subset \partial(\mathbb{C}P^2 \setminus B^4)$.

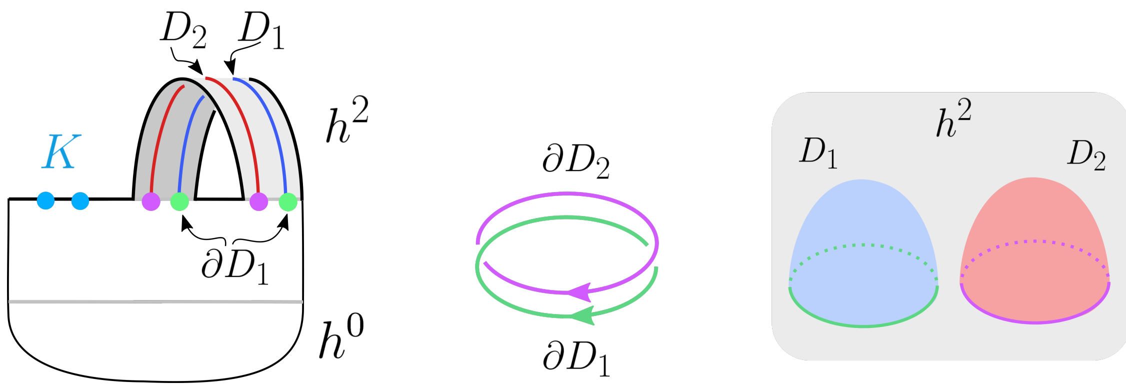
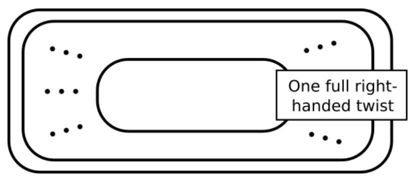


Figure 5.3: (Left) A flattened (lower-dimensional) handlebody diagram of $\mathbb{C}P^2 \setminus B^4$. (Center) The link $T(2,2) = \partial D_1 \cup \partial D_2$ sitting in the boundary $S^3 \times \{1\}$ of h^0 with particular orientation. (Right) Copies D_1, D_2 of the core disc $D^2 \times 0$ of h^2 sitting inside h^2, pictured in full dimension with unlinked boundaries for the sake of visualization.



$$T(n,n) = \bigcup_{i=1} \partial D_i$$

Figure 5.4: The link $T(n,n)$ is unoriented in the diagram. Choice of orientation will determine which coherent band surgeries are possible and the homological degree of a surface with components $D_1, ..., D_n$.

In particular, S_K represents the class $d\gamma \in H_2(\mathbb{C}P^2 \setminus B^4, \partial; \mathbb{Z})$, where γ is the generator of $H_2(\mathbb{C}P^2 \setminus B^4, \partial; \mathbb{Z}) \cong \mathbb{Z}$ represented by $\mathbb{C}P^1$ in $\mathbb{C}P^2 \setminus B^4$.

The degree $|[S_K]| = d$ is determined solely by the orientations of the discs $D_1, ..., D_n$ and hence by the orientations of the components $\partial D_1, ..., \partial D_n$ of $T(n,n)$. The integer d is exactly $d_+ - d_-$, where d_+ is the number of components of $T(n,n)$ with orientation matching the orientation of ∂D and d_- is the number of components of $T(n,n)$ with orientation opposed to the orientation of ∂D.

5.1.2 Knots quickly seen to be slice in $\mathbb{C}P^2$

If a knot K satisfies any of the following conditions, then it is slice in $\mathbb{C}P^2$.

1. $g_{S^4}(K) = 0$.

2. $u(K) = 1$, where $u(K)$ is the unknotting number of K.

The author makes no claim of being the first to develop any of these arguments. Exposition is provided only for the sake of readability and reference.

The first condition follows from the fact (proved in Chapter 4) that $g_M(K) \leq g_{S^4}(K)$ for all tame knots K and all smooth closed 4-manifolds M. We justify the second condition now.

Unknotting number as an upper bound

Lemma 5.1.1. *Let K be a knot with unknotting number $u \in \mathbb{Z}^+$ (therefore, K is not the unknot). Then*

$$g_{\mathbb{C}P^2}(K) \leq u - 1$$

Proof. Let K be a knot with unknotting number $u \in \mathbb{Z}^+$. Fix a sequence of u crossing changes $c_1, ..., c_u$ for K that turns K into the unknot. Such a sequence exists by the definition of unknotting number. For c_1, we use two parallel copies D_1, D_2 of the core disc of the 2-handle in $\mathbb{C}P^2$ to realize the crossing change. Specifically, we perform coherent band surgeries between $\partial D_1, \partial D_2$ and K to generate a genus 0 cobordism C_1 between $K \subset S^3 \times \{1\}$ and $K_1 \subset S^3 \times \{(u-1)/u\}$, where K_1 differs from K only by the crossing change c_1. The link with components $\partial D_1, \partial D_2$ is a $T(2,2)$ link equivalent to either L2a1$\{0\}$ or L2a1$\{1\}$ depending on whether c_1 is a change from positive crossing to negative (L2a1$\{0\}$) or negative to positive (L2a1$\{1\}$). The necessary coherent band surgeries are shown in Figure 5.5 and the resulting genus 0 cobordism C_1 is shown in Figure 5.6.

There remain $u - 1$ necessary crossing changes $c_2, c_3, ..., c_u$ to turn K_1 into the unknot. Let K_m be the knot obtained from K_{m-1} by performing the crossing change c_m. For each remaining c_i, where $2 \leq i \leq n$, we perform two coherent band surgeries as shown in Figure 5.7. The first of the surgeries will generate a cobordism $C_{i,a}$ between $K_{i-1} \subset S^3 \times \{(u - i + 1)/u\}$ and $K_i \# L2a1$ where $L2a1$ is the link $L2a1\{0\}$ or $L2a1\{1\}$ depending on whether c_i is a positive or negative crossing. The second of the surgeries will create a cobordism $C_{i,b}$ between $K_i \# L2a1$ and $K_i \subset S^3 \times \{(u - 1)/u\}$. The stacking of $C_{i,a}$ and $C_{i,b}$ gives a genus

1 cobordism C_i between $K_{i-1} \subset S^3 \times \{(u-i+1)/u\}$ and $K_i \subset S^3 \times \{(u-i)/u\}$. The cobordism C_i and the necessary coherent band surgeries needed to create it are shown in Figure 5.7.

Stacking the cobordisms $C_1, C_2, ...C_u$, we obtain a genus $u-1$ cobordism between K and K_u. Since K has unknotting number u and we chose a sequence of crossing changes $c_1, ..., c_u$ specifically to unknot K, it follows that K_u is the unknot. The unknot is easily seen to be capped off with an embedded 2-disc, thus generating a genus $u-1$ properly embedded surface $\bigcup_{i=1}^{u} C_i$ in $\mathbb{C}P^2 \setminus B^4$ with boundary K. This shows that $g_{\mathbb{C}P^2}(K) \leq u-1$, as desired. $\square$

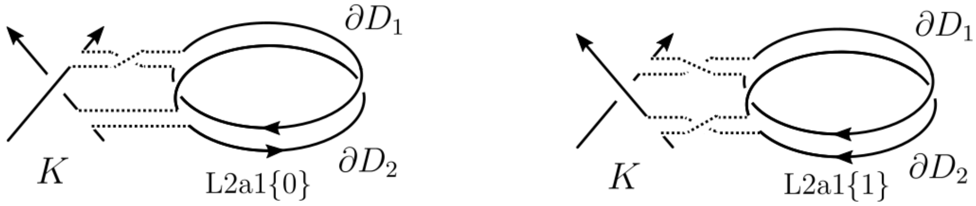


Figure 5.5: A crossing may be changed from positive to negative (left) or from negative to positive (right) using the boundaries ∂D_1 and ∂D_2.

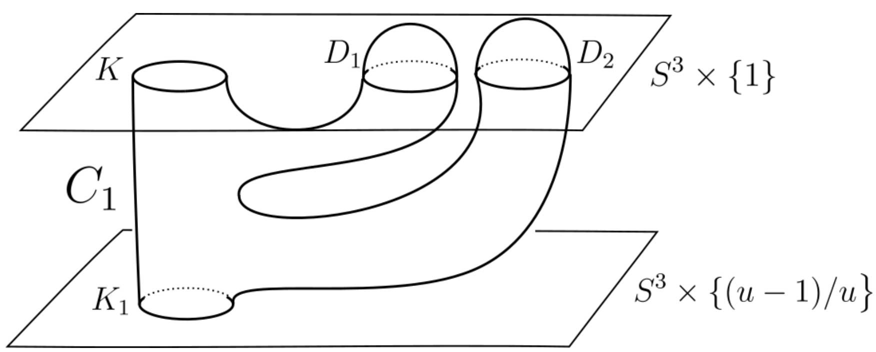


Figure 5.6: The genus 0 cobordism C_1 between K and K_1. The knot K_1 differs from K by a single crossing change.

Remark 5.1.2. Neither Lemma 4.1.1 nor Lemma 5.1.1 are strictly better than the other in terms of finding slice knots in $\mathbb{C}P^2$ quickly. Lemma 5.1.1 is able to detect that 3_1 is slice in $\mathbb{C}P^2$ but not 8_8, while Lemma 4.1.1 is able to detect that knot 8_8 is slice in $\mathbb{C}P^2$ but not 3_1.

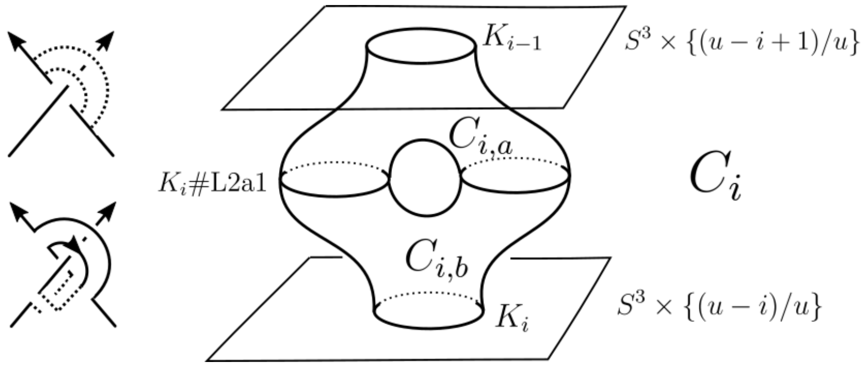


Figure 5.7: Two coherent band surgeries are performed in succession to generate a genus 1 cobordism C_i between $K_{i-1} \subset S^3 \times \{(u-i+1)/u\}$ and $K_1 \subset S^3 \times \{(u-i)/u\}$.

5.1.3 Proof of Theorem 5.0.1

Proof of Theorem 5.0.1. Let L be a link that is listed in the statement of the theorem. Let K be a knot such that mK, and hence rmK, is obtained from L via a coherent band surgery.

Every knot is concordant to itself. By Theorem 3.3.2[26], since K is concordant to itself, there is a genus 0 cobordism C_K between $K \subset S^3 \times \{1\}$ and $K \subset S^3 \times \{0\}$. The link $L \in S^3 \times \{1/2\}$ is obtained from some link $T(n,n) = \cup_{i=1}^{n}(\partial D_i) \subset S^3 \times \{1\}$ via a series of coherent band surgeries such that the resulting cobordism $C_{L,a}$ between $T(n,n)$ and L has only Morse critical points of the form $-x_1^2 + x_2^2$ and hence $C_{L,a}$ is a genus 0 cobordism. A proof of this for each specific link in the statement of the theorem is given diagrammatically in Appendix B. By hypothesis, there is a genus 0 cobordism $C_{L,b}$ between $L \subset S^3 \times \{1/2\}$ and $rmK \subset S^3 \times \{0\}$. Since the cobordism $C_{L,b}$ only has a single Morse critical point of the form $-x_1^2 + x_2^2$, stacking $C_{L,a}$ and $C_{L,b}$ to obtain $C_L = C_{L,a} \cup_L C_{L,b}$ will not introduce any genus. We cap off C_L with the core discs $D_1, ..., D_n$ in the 2-handle h^2 to obtain a disc C_{rmK} bounded by $rmK \subset S^3 \times \{0\}$. We take the boundary connect sum $C_K \natural C_{rmK}$ over the K and rmK boundary components in $S^3 \times \{0\}$. We now have a surface $C_K \natural C_{rmK}$ with boundary components $K \subset S^3 \times \{1\}$ and $K\#rmK \subset S^3 \times \{0\}$ as shown in Figure 5.8. By Theorem 3.1.1 of[26], $K\#rmK$ is a slice knot. Thus, we may cap off the $K\#rmK$ boundary component of $C_K \natural C_{rmK}$ with a disc D' to get $S = C_K \natural C_{rmK} \cup_{K\#rmK} D'$. This yields the existence of an orientable and properly embedded surface in $\mathbb{C}P^2 \setminus B^4$ with no genus and boundary K. Thus, K is slice in $\mathbb{C}P^2$. $\square$

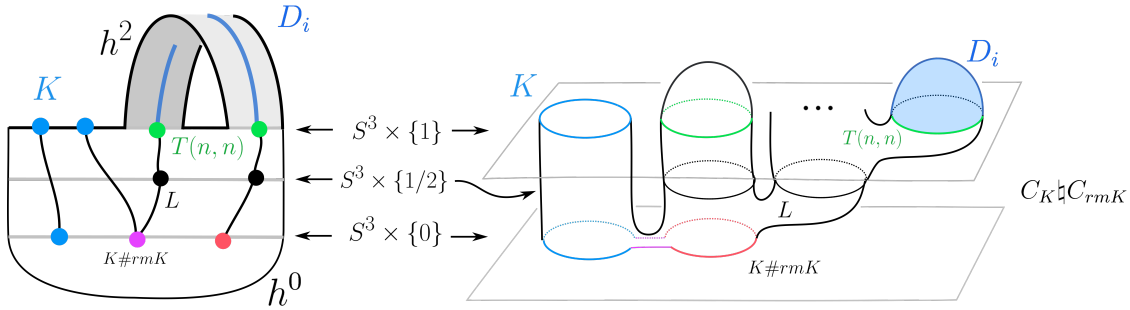


Figure 5.8: $C_K \natural C_{rmK}$ is the cobordism between $K \subset S^3 \times \{1\}$ and $K\#rmK \subset S^3 \times \{0\}$.

5.2 Lower bounds

5.2.1 Obstructing homological degrees

We adapt several results from low-dimensional topology to the world of knots and surfaces in $\mathbb{C}P^2 \setminus B^4$. In particular, we rely on Corollary 5.2.3, Corollary 5.2.6, Corollary 5.2.8, Corollary 5.2.12, and Corollary 5.2.15 to obstruct all possible slice degrees for knots with certain characteristics. We list the utility of each Corollary and remind the reader that we are considering slice degrees in absolute value. For example, if we say that degree 5 is obstructed, we really mean that both degrees 5 and -5 are obstructed.

- Corollary 5.2.3. Positive signatures of alternating knots obstruct small slice degrees.
- Corollary 5.2.6. The smooth 4-genus obstructs large slice degrees.
- Corollary 5.2.8. The signature obstructs almost all even slice degrees.
- Corollary 5.2.12. Knots with odd slice degree obstruct their mirrors from having odd slice degrees.
- Corollary 5.2.15. The Arf invariant obstructs half the remaining odd slice degrees.

Positive signatures of alternating knots obstruct small degrees

Theorem 5.2.1 makes use of Ozvath and Szabo's Tau invariant, which is derived from Knot Floer homology. For our calculations it is only important to note that when K is an alternating knot, the Tau invariant $\tau(K)$ can be expressed in terms of it's signature $\sigma(K)$. More on the Tau invariant can be found in Ozvath and Szabo's paper[30].

Theorem 5.2.1 (Ozvath, Szabo[30]). *Let X^4 be a smooth, oriented four-manifold with $b_2^+(X^4) = b_1(X^4) = 0$ and $\partial X^4 = S^3$. If S_K is any properly embedded surface in X^4 such that $\partial S_K = K$ for some knot K, then*

$$2\tau(K) + \big|[S_K]\big| + [S_K]\cdot[S_K] \leq 2g(S_k)$$

Lemma 5.2.2. *Let K be an alternating knot that bounds a properly embedded surface S_K in $\mathbb{C}P^2 \setminus B^4$ with $[S_K] = d\gamma \in H_2(\mathbb{C}P^2 \setminus B^4, \partial; \mathbb{Z})$. Then*

$$g(S_K) \geq \frac{\sigma(K)}{2} + \frac{|d|(1-|d|)}{2}$$

.

Proof. Let S_K be a properly embedded surface in $\mathbb{C}P^2 \setminus B^4$ bounded by K with $[S_K] = d\gamma \in H_2(\mathbb{C}P^2 \setminus B^4, \partial; \mathbb{Z})$. Then rmK bounds a surface $\overline{S_K}$ in $\overline{\mathbb{C}P^2} \setminus B^4$ with $[\overline{S_K}] = d\overline{\gamma} \in H_2(\overline{\mathbb{C}P^2} \setminus B^4, \partial; \mathbb{Z})$. By Theorem 5.2.1,

$$\begin{aligned} g(\overline{S_K}) &\geq \tau(rmK) + \frac{|[\overline{S_K}]| + [\overline{S_K}]\cdot[\overline{S_K}]}{2} \\ g(\overline{S_K}) &\geq \tau(rmK) + \frac{|d| - |d|^2}{2} \\ g(\overline{S_K}) &\geq \tau(rmK) + \frac{|d|(1-|d|)}{2} \end{aligned} \tag{5.1}$$

Ozvath, Szabo proved that $\tau(K) = \frac{-\sigma(K)}{2}$ for alternating knots[30]. Further, they showed that $-\tau(K) = \tau(rmK)$. Clearly, $g(S_K) = g(\overline{S_K})$. Making these substitutions into (5.1) gives the desired inequality.

$\square$

Corollary 5.2.3. Let K be an alternating knot with signature $\sigma(K)$ that bounds a properly embedded 2-disc D_K in $\mathbb{C}P^2 \setminus B^4$ with $[D_K] = d\gamma \in H_2(\mathbb{C}P^2 \setminus B^4, \partial; \mathbb{Z})$.

1. If $\sigma(K) \geq 2$, then $|d| \notin \{0, 1\}$,
2. If $\sigma(K) \geq 4$, then $|d| \notin \{0, 1, 2\}$

Proof. We prove (2). Part (1) is analogous. Suppose K is an alternating knot with $\sigma(K) \geq 4$ that bounds a properly embedded 2-disc D_K in $\mathbb{C}P^2 \backslash B^4$ such that $[D_K] = d\gamma \in H_2(\mathbb{C}P^2 \backslash B^4)$. By Lemma 5.2.2,

$$\begin{aligned} 0 &\geq \sigma(K) + |d|(1 - |d|) \\ |d|(|d| - 1) &\geq \sigma(K) \\ |d|(|d| - 1) &\geq 4 \\ |d|^2 - |d| - 4 &\geq 0. \end{aligned}$$

By simple algebra, $|d| \geq \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4 \cdot 4} > \frac{1}{2} + 2 = \frac{5}{2}$. $\square$

The smooth 4-genus obstructs large degrees.

Theorem 5.2.4 (Kronheimer, Mrowka[31])**.** *Let S be an oriented 2-manifold smoothly embedded in $\mathbb{C}P^2$ such that $[S] = d\gamma \in H_2(\mathbb{C}P^2; \mathbb{Z})$ with $d \geq 0$ and g is the genus of S. Then $2g \geq (d-1)(d-2)$.*

The reason for the $d \geq 0$ condition is because the Thom Conjecture associates the embedded surface S with an algebraic curve. The degree of an algebraic curve may not be negative.

Lemma 5.2.5. *Let K be a knot that bounds a properly embedded 2-disc D_K in $\mathbb{C}P^2 \setminus B^4$ with $[D_K] = d\gamma \in H_2(\mathbb{C}P^2 \setminus B^4, \partial; \mathbb{Z})$ and $d \geq 0$. Then*

$$2g_4(K) \geq (d-1)(d-2)$$

where $g_4(K)$ is the slice genus of K.

Proof. Suppose that K bounds a properly embedded 2-disc D_K in $\mathbb{C}P^2 \setminus B^4$ with $[D_K] = d\gamma \in H_2(\mathbb{C}P^2 \setminus B^4, \partial; \mathbb{Z})$, where $d \geq 0$. By definition, mK bounds an orientable surface S_{mK} of genus $g_4(K)$ in D^4. We glue D^4 to $\mathbb{C}P^2 \setminus B^4$ via an orientation-reversing diffeomorphism $\phi : \partial D^4 \hookrightarrow \partial(\mathbb{C}P^2 \setminus B^4)$ that identifies K to the image $\phi(mK) = K$. In doing so, we obtain an embedded closed surface $S = D_k \cup_K S_{mK}$ in $\mathbb{C}P^2$ with $g(S) = g(D_K) + g(S_{mK}) = g(S_{mk}) = g_4(K)$. By Theorem 5.2.4, we have the desired inequality for $d \geq 0$. $\square$

If we are ever able to obstruct slice degree d for a knot, then we also have an obstruction to the slice degree $-d$. The argument for this follows:

Let K be a knot such that it cannot bound a properly embedded disc in $\mathbb{C}P^2 \setminus B^4$ with degree $d \in \mathbb{Z}$. Now suppose for contradiction that K bounds a properly embedded disc D_K in $\mathbb{C}P^2 \setminus B^4$ with $[D_K] = -d\gamma \in H_2(\mathbb{C}P^2 \setminus B^4, \partial; \mathbb{Z})$. By changing the string-orientation of K, we get rK. This changes the orientation of D_K and thereby the sign of $[D_K]$. We now have that rK bounds a properly embedded disc $\overline{D_K}$ with $[\overline{D_K}] = -[D_K] = -(-d) = d\gamma \in H_2(\mathbb{C}P^2 \setminus B^4, \partial; \mathbb{Z})$. Forgetting about the orientation of rK, we just have an unoriented K. Now we have that K bounds a properly embedded disc $\overline{D_K}$ with degree d, a contradiction.

Corollary 5.2.6. Let K be a knot with smooth four genus $g_4(K)$ that bounds a properly embedded disc D_K in $\mathbb{C}P^2 \setminus B^4$ with $[D_K] = d\gamma \in H_2(\mathbb{C}P^2 \setminus B^4, \partial; \mathbb{Z})$.

1. If $g_4(K) \leq 2$, then $d \in \{0, 1, 2, 3\}$,
2. If $g_4(K) \leq 5$, then $d \in \{0, 1, 2, 3, 4\}$.

The signature obstructs almost all even degrees.

Theorem 5.2.7 (Gilmer[32], Viro[33], Yasuhara[34])**.** *Let K be a knot in $\partial(\mathbb{C}P^2 \setminus B^4) \cong S^3$. Suppose that K bounds a properly embedded surface S_K in $\mathbb{C}P^2 \setminus B^4$ and $[S_K] = d\gamma \in$*

$H_2(\mathbb{C}P^2 \setminus B^4, \partial; \mathbb{Z})$ *is divisible by 2. Then*

$$4g(S_K) + 2 \geq \left| d^2 - 2 - 2\sigma(K) \right|$$

Corollary 5.2.8. Let K be a knot with signature $\sigma(K)$.

1. If $\sigma(K) \leq -4$ or $\sigma(K) = 4$ then K does not bound a properly embedded 2-disc in $\mathbb{C}P^2 \setminus B^4$ with even degree,
2. If $\sigma(K) = -2$ and K bounds a properly embedded 2-disc in $\mathbb{C}P^2 \setminus B^4$ of even degree d, then $d = 0$,
3. If $\sigma(K) = 0$ and K bounds a properly embedded 2-disc in $\mathbb{C}P^2 \setminus B^4$ of even degree d, then $|d| \in \{0, 1\}$,
4. If $\sigma(K) = 2$ and K bounds a properly embedded 2-disc in $\mathbb{C}P^2 \setminus B^4$ of even degree d, then $|d| = 2$,
5. If $\sigma(K) = 6$ or $\sigma(K) = 8$ and K bounds a properly embedded 2-disc in $\mathbb{C}P^2 \setminus B^4$ of even degree d, then $|d| = 4$

Proof. Substitute $g(S_K) = 0$ into Theorem 5.2.7 and consider the case for each signature. □

Knots with odd slice degree obstruct their mirrors from having odd slice degrees.

The next few results will focus on characteristic surfaces. In order to characterize characteristic surfaces, we use Wu's Formula, which can be found on page 168 of Scorpan's Wild World of 4-manifolds[35].

Theorem 5.2.9 (Wu's Formula)**.** *Let X^4 be a simply-connected 4-manifold. An oriented surface Σ is characteristic if and only if*

$$\Sigma \cdot S = S \cdot S \mod 2$$

for all oriented surfaces S in X^4.

In the next lemma, we consider the second integral homology group $H_2(\mathbb{C}P^2;\mathbb{Z})$ with canonical generator $[\mathbb{C}P^1]$. We say a surface Σ in $\mathbb{C}P^2$ has degree d if and only if $[\Sigma] = d[\mathbb{C}P^1]$.

Lemma 5.2.10. *A surface Σ in $\mathbb{C}P^2$ is characteristic if and only if the degree d of Σ is odd.*

Proof. ($\Rightarrow$) If Σ is characteristic, then by Wu's Formula, we have that, modulo 2

$$\Sigma \cdot \mathbb{C}P^1 = \mathbb{C}P^1 \cdot \mathbb{C}P^1$$

$$d(\mathbb{C}P^1 \cdot \mathbb{C}P^1) = 1$$

$$d = 1.$$

Since we're working mod 2, this shows that d must be odd. ($\Leftarrow$) Suppose that d is odd. If S has even degree, then $\Sigma \cdot S = 0 = S \cdot S \mod 2$. If S has odd degree, then $\Sigma \cdot S = 1 = S \cdot S \mod 2$. It follows by Wu's Formula that Σ is characteristic. $\square$

A nearly identical proof shows that the characteristic surfaces of $\overline{\mathbb{C}P^2}$ are also exactly those with odd degree. When taking connect sums of $\mathbb{C}P^2$ of $\overline{\mathbb{C}P^2}$, a very slight extension of this argument shows that characteristic surfaces of the connect sum are simply connect sums of characteristic surfaces from each of the summands. That is, a surface Σ is characteristic in $m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ if and only if $[\Sigma] = d_1[\mathbb{C}P^1_1] + \cdots + d_m[\mathbb{C}P^1_m] + d_{m+1}[\overline{\mathbb{C}P^1_1}] + \cdots + d_{m+n}[\overline{\mathbb{C}P^1_n}]$ and $d_1, ...d_m, d_{m+1}, ..., d_{m+n}$ are all odd.

Theorem 5.2.11 (Lawson[36])**.** *Let S be a characteristic embedded 2-sphere in $2\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ (respectively $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$). Then $[S]\cdot[S] = 1$ (respectively $[S]\cdot[S] = -1$).*

Corollary 5.2.12. Let K be a knot that bounds a properly embedded disc D_K in $\mathbb{C}P^2 \setminus B^4$ with $[D_K] = d\gamma \in H_2(\mathbb{C}P^2 \setminus B^4, \partial; \mathbb{Z})$.

1. Let d be of odd degree and $|d| \geq 3$. Then mK does not bound a properly embedded 2-disc in $\mathbb{C}P^2 \setminus B^4$ of odd degree.

2. Let $|d| = 1$. If mK bounds a properly embedded disc D'_K in $\mathbb{C}P^2 \setminus B^4$ with $[D'_K] = d'\gamma \in H_2(\mathbb{C}P^2 \setminus B^4, \partial; \mathbb{Z})$ of odd degree d', then $|d'| = 1$.

Proof. (1) Suppose for contradiction that mK bounds a properly embedded 2-disc D'_K in $\mathbb{C}P^2 \setminus B^4$ with $[D'_K] = d'\gamma_2 \in H_2(\mathbb{C}P^2 \setminus B^4, \partial; \mathbb{Z})$ and d' odd. By taking a single parallel copy $\widetilde{D_K}$ of the core disc in the (-1)-twisted 2-handle of $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \setminus B^4$ we may perform surgery to absorb $\widetilde{D_K}$ into D'_K. This gives us that mK bounds a properly embedded 2-disc D''_K in $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \setminus B^4$ with $[D''_K] = d'\gamma_2 + \overline{\gamma}_3 \in H_2(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \setminus B^4)$. Gluing $\mathbb{C}P^2 \setminus B^4$ to $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \setminus B^4$ via an orientation reversing diffeomorphism, we have an embedded characteristic 2- sphere $S = D_K \cup_K D''_K$ in $2\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ with $[S] = d\gamma_1 + d'\gamma_2 + \overline{\gamma}_3 \in H_2(2\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}; \mathbb{Z})$. By Theorem 5.2.11,

$$\begin{aligned} 1 &= [D_K] \cdot [D_K] \\ &= d^2 + (d')^2 - 1 \\ &\geq 9 + (d')^2 - 1 \\ -7 &\geq (d')^2 \end{aligned}$$

a contradiction.

(2) Following the proof of (1) above, we have

$$\begin{aligned} 1 &= [D_K] \cdot [D_K] \\ &= 1 + d^2 - 1 \\ 1 &= d^2 \end{aligned}$$

as desired. $\square$

The Arf invariant obstructs half the remaining odd slice degrees.

Definition 5.2.13 (Robertello[37]). Let $f : S^2 \to X^4$ be a combinatorial embedding of the 2-sphere S^2 into a closed, oriented, simply connected, differentiable 4-manifold X^4. Let

f be differentiable and regular except at one point $x_0 \in S^2$, and suppose there exists a differentiably embedded 4-disk D^4 such that $D^4 \subset X^4$, $f(x_0)$ (the singularity of f) is at the center of D^4, and $f(S^2) \cap D^4$ is a knot in $S^3 = \partial D^4$. Let $\xi = [f(S^2)] \in H_2(X^4, \mathbb{Z})$ be characteristic. Then

$$Arf(K) \equiv \frac{\xi \cdot \xi - \sigma(X^4)}{8} \mod 2.$$

Lemma 5.2.14. *Let K be a knot with Arf invariant Arf(K) and let D_K be a properly embedded disc in $\mathbb{C}P^2 \setminus B^4$ with $\partial D_K = K$ and odd homological degree d. Then*

$$\mathit{Arf}(K) \equiv \frac{d^2 - 1}{8} \mod 2.$$

Proof. Suppose that K is a knot in $\partial(\mathbb{C}P^2 \setminus B^4) \cong S^3$ and K bounds a properly embedded disc D_K with degree d. We may cap off $\mathbb{C}P^2 \setminus B^4$ with a D^4 and D_K with the cone $C(K)$ of K, as shown in Figure 5.9. This gives us an embedding of S^2 into $\mathbb{C}P^2$ with only one singularity point, the cone point of $C(K)$. Call the image of this embedding S, which is the union over K of D_K and $C(K)$. Since the degree d of $[S] \in H_2(\mathbb{C}P^2; \mathbb{Z})$ is determined completely by the number of times it traverses the 2-handle h^2 of $\mathbb{C}P^2$, the degree of $[S]$ is equal to the degree d of $[D_K] \in H_2(\mathbb{C}P^2 \setminus B^4)$. Since d is odd, it means that $[D_K]$ is characteristic. By substitution into definition 5.2.13, we have the desired result. $\square$

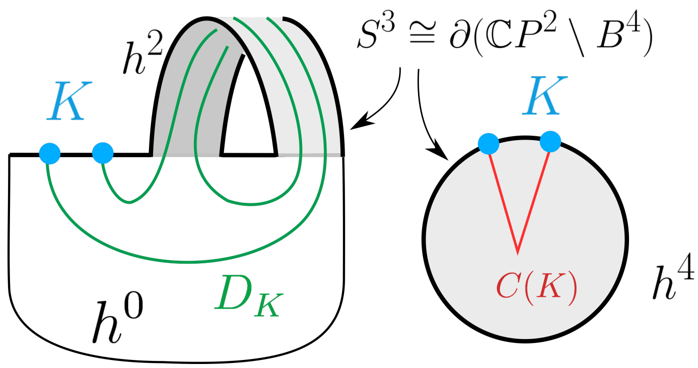


Figure 5.9: (Left) A knot K in $S^3 = \partial(\mathbb{C}P^2 \setminus B^4) = \partial(h^0 \cup h^2)$ bounding a properly embedded disc D_K. (Right) K in $S^3 = \partial h^4$ bounding $C(K)$.

Corollary 5.2.15. Let K be a knot and D_K a properly embedded disc in $\mathbb{C}P^2 \setminus B^4$ with characteristic (odd) degree d.

1. If $\mathrm{Arf}(K) = 0$, then $|d|$ is not 3.

2. If $\mathrm{Arf}(K) = 1$, then $|d|$ is not 1.

Maximal Thurston-Bennequin number

Lemma 5.2.16. *Let K be a knot in $\partial(\mathbb{C}P^2 \setminus B^4)$. If $tb(mK) > d^2$, then K does not bound a properly embedded disc of degree d in $\mathbb{C}P^2 \setminus B^4$.*

Proof. Let h^0 be a 4-dimensional 0-handle and let mK be in the boundary $S^3 = \partial h^0 \cong \partial D^4$. Attach a 4-dimensional 2-handle h to ∂h^0 along mK with framing $\mathrm{tb}(mK) - 1$, where $\mathrm{tb}(mK)$ is the maximal Thurston-Bennequin number of mK. By Theorem 11.2.8 of Gompf and Stipsicz[29], $W = h^0 \cup h$ is Stein. By Theorem 1.1 of Lisca and Matić[38], there exists a minimal complex surface X of general type with $b_2^+(X) > 1$ and a holomorphic embedding of W as a domain inside X. By Friedman[39], the Seiberg-Witten invariants of X are nontrivial. Let $\widetilde{X} = X \# \overline{\mathbb{C}P^2}$. We make several observations. The first is that by Corollary 4.6.8 of Nicolaescu[40], the Seiberg-Witten invariants of $\widetilde{X}$ are still nontrivial. The second is that $b_2^+(\widetilde{X}) > 1$. The third is that $\widetilde{X}$ contains a copy of $\overline{\mathbb{C}P^2} \setminus B^4$.

Suppose for contradiction that K bounds a properly embedded disc D_K of degree d in $\mathbb{C}P^2 \setminus B^4$, where $\mathrm{tb}(mK) > d^2$. Then mK bounds a properly embedded disc $\overline{D_K}$ in $\overline{\mathbb{C}P^2} \setminus B^4$ with degree d. We observe that mK also bounds the core disc D of the 2-handle h that was used to construct W. Taken together, we have a 2-sphere $\Sigma = D \cup_{mK} \overline{D_K}$ embedded in $\widetilde{X}$. It is clear that $[\Sigma] = H + dE$, where H is the generator of $H_2(W; \mathbb{Z})$ and E is the generator of $H_2(\overline{\mathbb{C}P^2} \setminus B^4; \mathbb{Z})$. We observe that

$$\begin{aligned}
\Sigma \cdot \Sigma &= (H \cdot H) + 2d(H \cdot E) + d^2(E \cdot E) \\
&= \mathrm{tb}(mK) - 1 + 2d(0) + d^2(-1) \\
&= \mathrm{tb}(mK) - 1 - d^2 \\
&\geq 0.
\end{aligned}$$

By Nicolescu Corollary 4.6.9[40], the Seiberg-Witten invariants of $\widetilde{X}$ are trivial, a contradiction.

□

5.2.2 Proofs of Theorems 5.0.2 and 5.0.3

With Corollaries 5.2.3, 5.2.6, 5.2.8, 5.2.12, and 5.2.15, we may now easily prove Theorems 5.0.2 and 5.0.3.

Proof of Theorem 5.0.2. Let K be an alternating knot with $\sigma(K) = 4$, $g_4(K) \leq 2$, and $\mathrm{Arf}(K) = 0$. Suppose for contradiction that K bounds a properly embedded disc D_K in $\mathbb{C}P^2 \setminus B^4$ with slice degree d. By Corollary 5.2.6, $|d| \in \{0, 1, 2, 3\}$. By Corollary 5.2.3, $|d|$ is not 0, 1, or 2. By Corollary 5.2.15, $|d|$ is not 3. Since each properly embedded surface in $\mathbb{C}P^2 \setminus B^4$ has some homological degree, we have reached a contradiction. It follows that such a disc D_K cannot exist. □

Proof of Theorem 5.0.3. Let K be an alternating knot with $|\sigma(K)| = 4$. Suppose for contradiction that both K and mK bound properly embedded discs D_K and D'_K in $\mathbb{C}P^2 \setminus B^4$ respectively with slice degrees d and d'. Without loss of generality, suppose that $\sigma(K) = 4$ and $\sigma(-K) = -4$. By Corollary 5.2.8 we know that d and d' are not even. Hence both d and d' are odd. By Corollary 5.2.3, we know that d cannot be 1. Thus, d is an odd number greater than or equal to 3. It follows by Corollary 5.2.12 that d' is not odd, a contradiction. □

5.3 Additional computations

In this section we provide computations of the $\mathbb{C}P^2$-genus for knots that were beyond the intended scope of this work. Namely, we compute the $\mathbb{C}P^2$-genus for a finite set of prime knots of 9- and 10-crossings and show an infinite family $\{K_n\}$ of knots such that K_n and mK_n have differing $\mathbb{C}P^2$-genus for each $n \in \mathbb{N}$.

5.3.1 Genera computed using Theorem 5.0.1

One can show that a knot K is slice in $\mathbb{C}P^2$ by showing a coherent band surgery taking mK to one of the links listed in Theorem 5.0.1 or vice versa (taking one of the links to mK). We have done this for the following prime knots of 9− and 10−crossings

$$m9_4,\ \ 9_5,\ \ m9_{13},\ \ m9_{15},\ \ 9_{29},\ \ m9_{35},\ \ m10_{11},\ \ m10_{12},\ \ 10_{37}.$$

The required coherent band surgeries are provided in Appendix C. In addition to the finite list of knots above, we have computed the $\mathbb{C}P^2$-genus of an infinite family $\{K_n\}$ of knots along with their mirrors. Consider the family $\{K_n\}_{n\geq 0}$ shown in Figure 5.10. In the same figure, we see the coherent band surgery required to take each K_n to the link L4a1$\{1\}$. Thus, by Theorem 5.0.1, mK_n is slice in $\mathbb{C}P^2$ for each $n \in \mathbb{N}$.

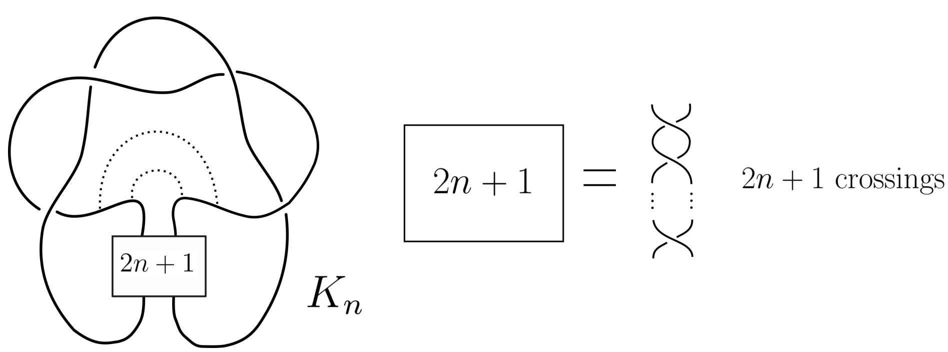


Figure 5.10: Coherent band surgery between K_n ($n \geq 0$) and L4a1$\{1\}$.

5.3.2 Genera computed using Theorem 5.0.2

From the preceeding subsection we found the knots $m9_4$ and $m9_{13}$ to be slice in $\mathbb{C}P^2$. It follows from Theorem 5.0.2 that their mirrors

$$9_4, \quad , 9_{13},$$

are not slice in $\mathbb{C}P^2$. The author has constructed an explicit genus 1 surface in $\mathbb{C}P^2 \setminus B^4$ bounding each knot. It follows that the $\mathbb{C}P^2$-genus of both knots is 1.

We may also apply Theorem 5.0.2 to the infinite family $\{K_n\}_{n\geq 0}$. In 2017, McCoy proved that if an alternating knot K has unknotting number 1, then it has an unknotting crossing in every alternating diagram[41]. We observe that the diagram of K_n shown in Figure 5.10 is alternating. Suppose for contradiction that $u(K_n) = 1$. Then there is some unknotting crossing in the shown diagram. Inspection shows this not to be the case. Thus $u(K_n) \neq 1$. The Alexander polynomial of K_n is nontrivial. Thus K_n is not the unknot. Since the only knot with unknotting number 0 is the unknot, it follows that $u(K_n) \neq 0$. This leaves $u(K_n) \geq 2$. By inspection, for each $n \geq 0$ one can always find 2 unknotting crossings. Thus $u(K_n) = 2$ for all $n \geq 0$ and by Lemma 5.1.1, we have that $g_{\mathbb{C}P^2}(K_n) \leq 1$. To determine the explicit $\mathbb{C}P^2$-genus of K_n it is left to show that $g_{\mathbb{C}P^2}(K_n) \geq 1$. To use Theorem 5.0.2, we need to show both that K_n is alternating and $|\sigma(K_n)| = 4$ for each $n \geq 0$.

That K_n is alternating may be confirmed by an informal combination of inspection and induction. Starting with $n = 0$, we see that K_0 (which is the knot 5_1) is alternating. Moving from K_n to K_{n+1} we see that two crossings are added in such a way that it maintains the alternating nature.

From the papers of Rasmussen[23] and Kawamura[42] we have:

Theorem 5.3.1 (Rasmussen[23])**.** *If K is an alternating knot, then $s(K) = \sigma(K)$, where $\sigma(K)$ is the classical signature of K.*

Theorem 5.3.2 (Kawamura[42])**.** *Let K be a knot in the three-sphere S^3. If K can be repre-*

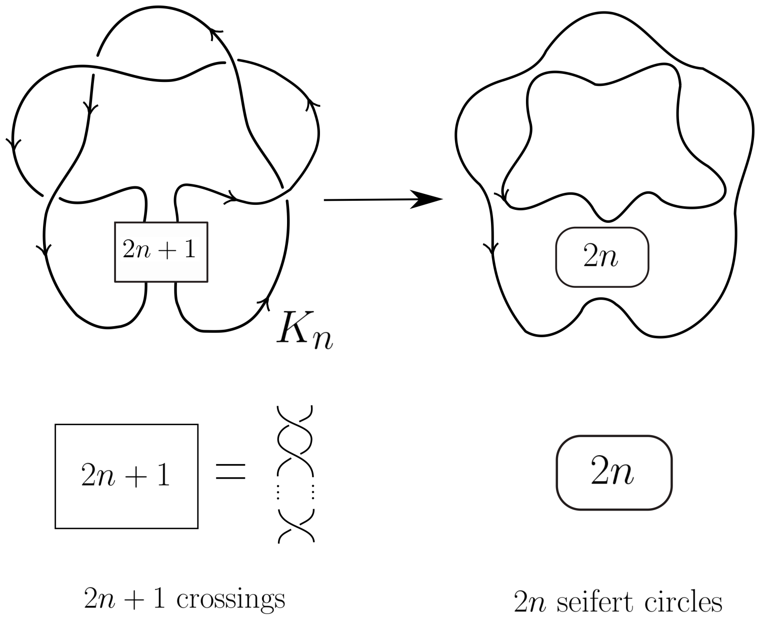


Figure 5.11

sented by a positive diagram D, then

$$-s(K) = x(D) - O(D) + 1$$

where $x(D)$ is the number of crossings in the diagram D and $O(D)$ is the number of Seifert circles in the Seifert resolution of D.

We observe that $x = 2n+1+4 = 2n+5$ and $O = 2n+2$. The diagram shown in Figure 5.11 is both alternating and positive. Putting this all together, we have:

$$\begin{aligned}
\sigma(K) &= s(K) \\
&= -x(D) + O(D) - 1 \\
&= -(2n+5) + (2n+2) - 1 \\
&= -4.
\end{aligned}$$

5.3.3 Genera computed using Theorem 5.0.3

A search of Knotinfo[1] shows that there are 126 alternating prime knots up to 12-crossings with signature 4, smooth four genus less than 3, Arf invariant 0, and unknotting number 2. By Theorem 5.0.3 and Lemma 5.1.1, these knots all have $\mathbb{C}P^2$-genus equal to 1. Beyond the prime knots up to 8-crossings, we have not listed these since they can be easily identified.

5.4 Explanation of appendices

5.4.1 Appendix A

A display of the different links listed in Theorem 5.0.1

5.4.2 Appendix B

Each figure shows a link and a coherent band surgery. The link is either (a) a $T(n, n)$ torus link or (b) the result of performing a coherent band surgery on two oppositely-oriented components of a $T(n, n)$ link to obtain an extra unlinked component. The band surgery shown in the diagram is that which will give the resulting 2-component link listed in the figure caption. These correspond to the cobordism $C_{L,a}$ described in the proof of Theorem 5.0.1.

5.4.3 Appendix C

These diagrams show the coherent band surgeries required for the knots listed in the introduction to satisfy Theorem 5.0.1.

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Appendix A

Links listed in Theorem 5.0.1

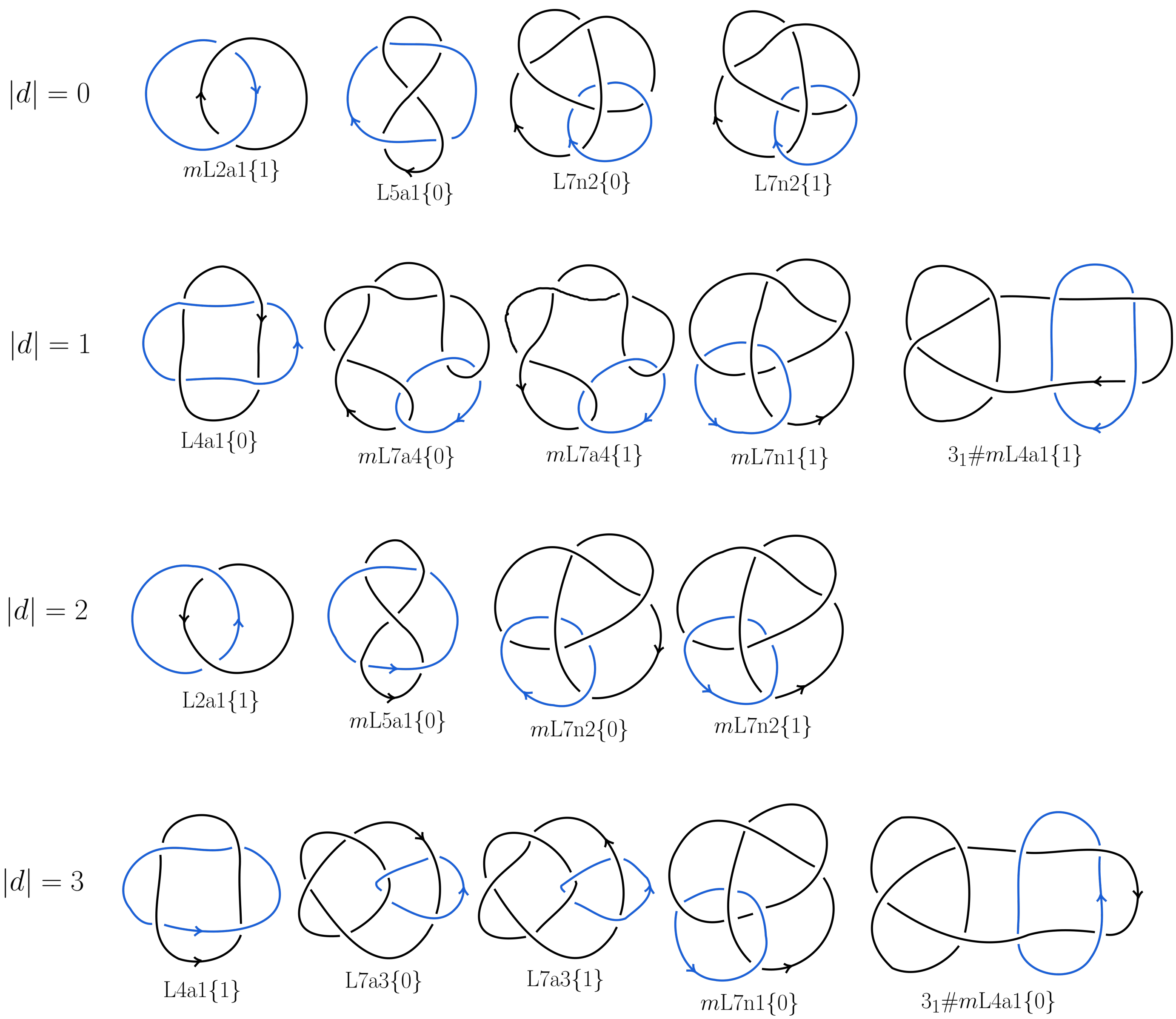

$|d| = 0$
mL2a1{1}
L5a1{0}
L7n2{0}
L7n2{1}
$|d| = 1$
L4a1{0}
mL7a4{0}
mL7a4{1}
mL7n1{1}
$3_1\#m$L4a1{1}
$|d| = 2$
L2a1{1}
mL5a1{0}
mL7n2{0}
mL7n2{1}
$|d| = 3$
L4a1{1}
L7a3{0}
L7a3{1}
mL7n1{0}
$3_1\#m$L4a1{0}

Figure A.1

Appendix B

Coherent band surgeries required for Theorem 5.0.1

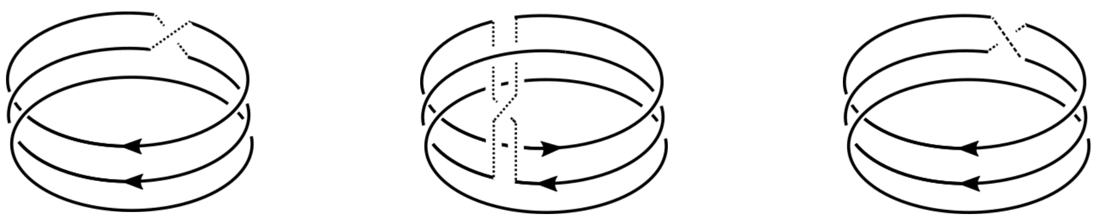


Figure B.1: (Left) L4a1{0}{1} degree $1, 3$, (Middle) mL7a4{0}{1} degree 1, (Right) mL7n1{0}{1} degree $1, 3$

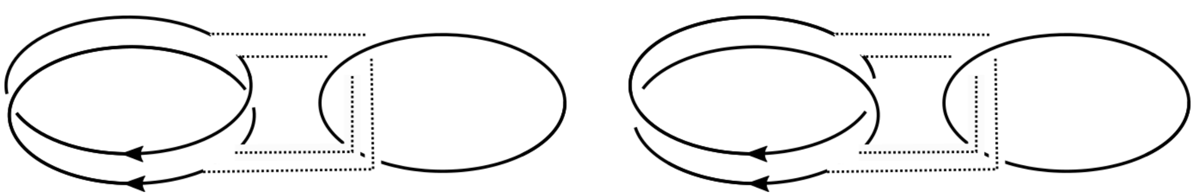
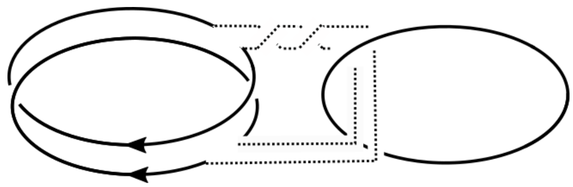


Figure B.2: (Left) mL5a1{0} degree 2, (Right) L5a1{0} degree 0



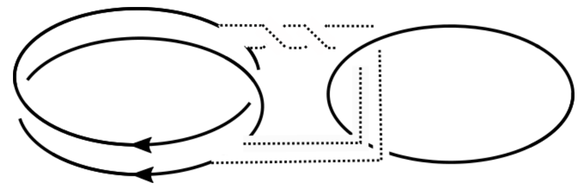
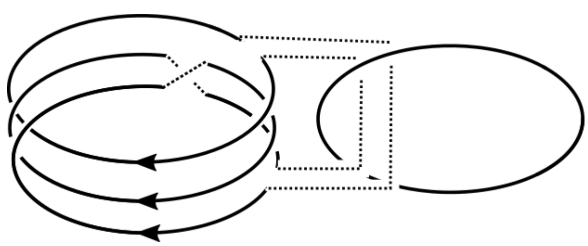


Figure B.3: (Left) mL7n2{0}{1} degree 2, (Right) L7n2{0}{1} degree 0



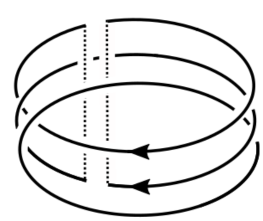


Figure B.4: (Left) L7a3{0}{1} degree 3, (Right) $3_1\#m$L4a1{0}{1} degree $1, 3$

Appendix C

Coherent band surgeries from specific knots to links in Theorem 5.0.1

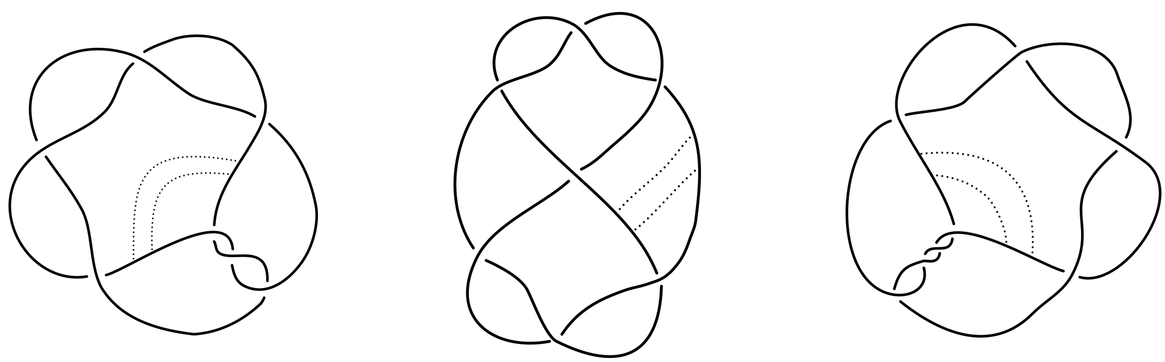


Figure C.1: (Left) $7_3 \longrightarrow$ L4a1{1}, (Middle) $m7_4 \longrightarrow$ L4a1{1}, (Right) $m8_3 \longrightarrow$ L4a1{0}

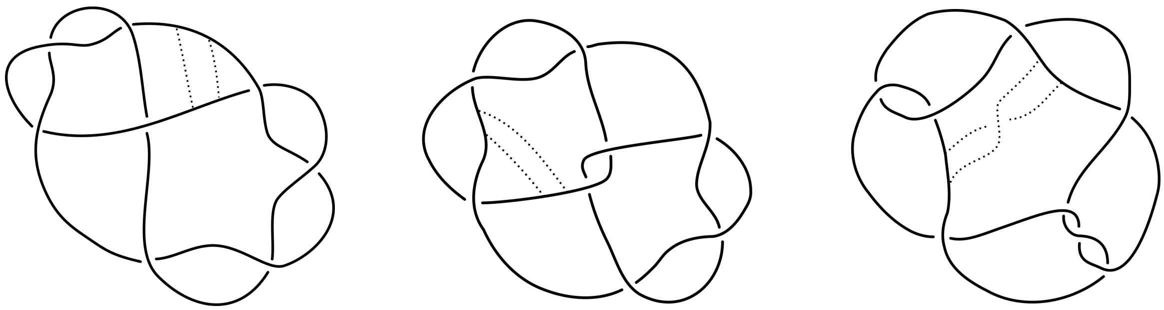


Figure C.2: (Left) $m8_4 \longrightarrow$ L4a1{1}, (Middle) $8_5 \longrightarrow$ L4a1{1}, (Right) $m8_6 \longrightarrow m$L2a1{1}

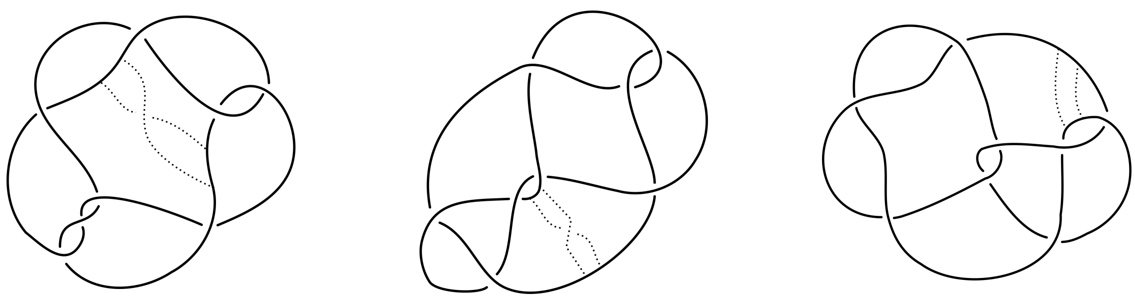


Figure C.3: (Left) $8_6 \longrightarrow$ L2a1{1}, (Middle) $m8_{12} \longrightarrow$ L5a1{0}, (Right) $8_{19} \longrightarrow m$L7n1{0}

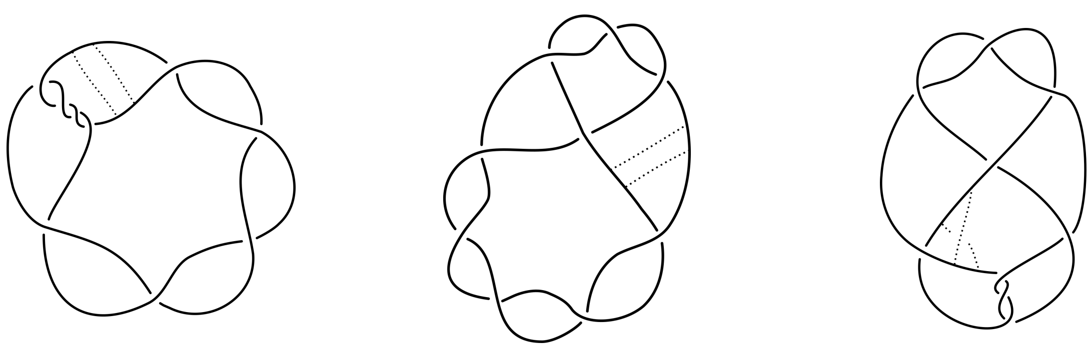


Figure C.4: (Left) $9_4 \longrightarrow$ L4a1{1}, 0.5cm (Middle) $m9_5 \longrightarrow$ L4a1{0}, 0.5cm (Right) $9_{13} \longrightarrow$ $3_1\#m$L4a1{0}

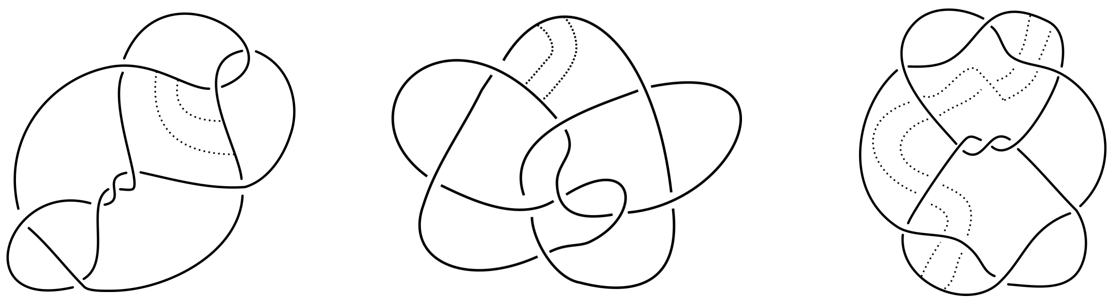


Figure C.5: (Left) $9_{15} \longrightarrow$ L7a4{0}, (Middle) $m9_{29} \longrightarrow$ L7a3{0}, 0.5cm (Right) $9_{35} \longrightarrow 3_1\#m$L4a1{0}