EVALUATION OF ONE CLASSICAL AND TWO BAYESIAN ESTIMATORS OF SYSTEM AVAILABILITY USING MULTIPLE ATTRIBUTE DECISION MAKING TECHNIQUES

by

CYNTHIA S. MCCAHON

B.S.I.M., Industrial Management, Purdue University, West Lafayette, Indiana
1978

A MASTER'S THESIS

submitted in partial fulfillment of the
requirements for the degree

MASTER OF SCIENCE

College of Engineering
Department of Industrial Engineering
Kansas State University
Manhattan, Kansas
1980

Approved by:

Co-Major Professor

Co-Major Professor

THIS BOOK CONTAINS NUMEROUS PAGES WITH THE ORIGINAL PRINTING BEING SKEWED DIFFERENTLY FROM THE TOP OF THE PAGE TO THE BOTTOM.

THIS IS AS RECEIVED FROM THE CUSTOMER.

ACKNOWLEDGEMENTS

At this time I wish to thank my two co-major professors, Dr. C. L. Hwang, and Dr. F. A. Tillman, for all of their invaluable assistance, encouragement, and guidance throughout this study. Also, a special thanks goes to Dr. D. L. Grosh for revealing the insights needed for reliability-availability study.

Most importantly, I am deeply grateful to my husband, Thomas, for providing constant encouragement and unwavering support during the course of this project.

This study was partially supported by the U.S. Office of Naval Research, Contract No. N00014-76-C-0842.

ACKNOWLEDGEMENTS

At this time I wish to thank my two co-major professors, Dr. C. L. Hwang, and Dr. F. A. Tillman, for all of their invaluable assistance, encouragement, and guidance throughout this study. Also, a special thanks goes to Dr. D. L. Grosh for revealing the insights needed for reliability-availability study.

Most importantly, I am deeply grateful to my husband, Thomas, for providing constant encouragement and unwavering support during the course of this project.

This study was partially supported by the U.S. Office of Naval Research, Contract No. N00014-76-C-0842.

TABLE OF CONTENTS

TABLE OF CONTENTS (cont).

Page

LIST OF FIGURES

LIST OF TABLES

LIST OF TABLES (cont.)

CHAPTER 1 - INTRODUCTION

System availability is defined as the probability a system is operating satisfactorily at any point in time when used under stated conditions, where the [total] time considered is operating time and active repair time [22]. Because of this useful combination of reliability and maintainability measures, availability is increasingly being used as a measure of system performance. But why, specifically, the increase? Presently, more emphasis is being placed on the facets of system maintenance along with the reliability, rather than solely on the reliability, which is due to an increased awareness of operational and maintenance constraints (of which costs and time are just a few). Availability measures, alone, take into account a growing desire to decrease system maintenance while increasing system reliability.

The estimation of system availability has been approached in many different manners. Most require the accumulation of data of the on and off times of the system with, usually, the more data accumulated, the better the estimate. But, can the system availability be accurately estimated if not much data are available? What if no data are available and an availability measure is still desired? (This is especially true in the case of nuclear power plants, when data of system failures are definitely <u>not</u> desired, even if available.) In these cases, Bayesian estimation approaches have proven most fruitful.

But Bayesian approaches have been slow in appearing, possibly due to the many criticisms of incorporating what is essentially subjective prior information about the parameters with the data information, if it is available. And even though the use of Bayesian theory is usually thought of as one technique in and of itself, numerous estimation methods have recently evolved that could all be considered Bayesian. Hence, the confusion and the controversy continues.

1.1 Reasons for Study

The underlying theme of this study is the comparison of three estimation methods of system availability, one classical and two Bayesian, to determine which is "best" in terms of closeness to steady-state availability, variability between samples, computer execution time, ease of programming and ease of understanding. Comparisons will be made between the classical estimator and the Bayesian estimators, but of major concern will be the comparisons among the Bayesian estimators themselves. Not much work has been done in terms of distinguishing better Bayesian methods from others. (Probably because of the controversy still existing as to whether Bayesian techniques should be utilized at all! [6] [4a] But this is not the purpose of the study.) Hopefully, this study will identify which estimators prove most helpful under certain sampling conditions.

1.2 Bayesian Treatment of System Availability

System availability from the non-Bayesian viewpoint has been widely studied. Many definitions of availability are available and many distributions are associated with the operative and repair intervals of the system. Numerous approaches have been derived and various system configurations have been explored. For a thorough literature survey of these and other topics of non-Bayesian system availability see [13].

The impetus for using Bayesian approaches stems from a desire to incorporate when available, prior information about the system and its parameters under study. Most commonly, engineers extremely familiar with a specific system feel this way, not wanting to waste <u>any</u> information, no matter how informal. And, in cases where this "informal" prior information can be expressed more formally in the form of specific probability distributions, Bayesian inference is most productive.

Brender [3] was the first to use Bayesian theory to predict and measure system availability. The model considered was a basic single system configuration involving an alternating sequence of independent and exponentially distributed operative and repair intervals. The intervals' respective rate parameters were described by gamma distributions. Brender showed the steady-state point availability had a Euler distribution from which he derived his availability estimate. The transient case (where availability is time-dependent), along with other broader availability cases, was then derived from the steady-state case. In his second paper [2], Brender removed the restrictions of exponential operative and repair intervals and gamma prior distributions. Applications were then made to cases involving:

(1) prior distributions composed of linear combinations of gamma distributions;

(2) gamma-distributed repair intervals with uncertain location and shape parameters;

(3) random demands within an initial interval, demands repeated at intervals, redundant configurations; and,

(4) measures of performance other than availability.

Gaver and Mazumdar [7], using the same model as Brender, derived Bayesian estimators of long run availability using two different sampling techniques:

(1) "snapshot" - observations made at points in time to determine merely if the system is up or down at that point in time; and,

(2) "patch" - a sequence of continuous observations recording the duration of up and down times of the system.

They also explored cases with different loss functions.

Thompson and Springer [21] calculated a Bayesian prediction interval for an N-series system. Snapshot data was accumulated for each of the N components of the series and a posterior density function and availability estimate were determined for each component. The system availability, essentially a product of the component availabilities was then calculated through the use of the Mellin integral transform. The components, however, did not have the two-state configuration as previously described.

Later, Thompson and Palico [20] incorporated the two-state configuration with exponential on and off times and gamma prior distributions into each of the components and used Brender's Euler distribution to express each of the components' availability. The system availabilities for N-series and N-parallel systems were then calculated by using a method of successive approximatic of the cumulative distribution function given the sequence of integer moments, in lieu of the Mellin integral transform.

All of the above references, with the exception of Brender, dealt solely with the estimation of the steady-state availability of a system. The models did not take into account any dependency on time. Kuo [12] filled this void by introducing Bayesian estimation for a time-dependent system availability model. The system was of single component configuration, and was represented as a two-state stochastic process,the two states being the on state and the off state. The operative (on) and repair (off) times were gamma distributed and the prior distributions of the non-fixed parameters were exponential. The time-dependent availability expression was derived via renewal theory.

Kuo calculated his Bayesian estimate by taking the expected value of the posterior distribution, because he assumed a squared error loss function. He also calculated a classical maximum likelihood estimate, and Brender's estimate for comparative purposes.

Kuo was the first to compare different estimation methods in terms of Bayesian versus classical and Bayesian versus Bayesian. His criteria, although not specifically stated, were closeness to steady state availability and variability between samples. Calculating system availability estimates for a data set with negative exponentially distributed on and off times using different samples, priors and time intervals, he reached the following conclusions:

(1) for small sample sizes, the maximum likelihood estimate was not useful due to wide variation between samples;

(2) the choice of priors did not have much affect on the Bayesian estimates;

(3) with smaller samples, the Bayesian approaches showed less variability;

(4) Bayesian approaches with good or bad priors gave better results than the maximum likelihood estimate for a biased sample; and

(5) when no data are available, only Bayesian approaches work.

1.3 Methodology of Study

This study consists of two main portions: first, the estimation of system availability and, second, the selection of the best estimate.

Chapter Two outlines how the system availability will be represented. The model used here is based on renewal theory, since the system is a two-state stochastic process with the two states being the on and the off state. Also, specific representations will be given for the cases when on and off times have a gamma distribution and when they have an exponential distribution.

Introduced in Chapter Three will be the three estimates: the classical maximum likelihood estimate, the traditional Bayesian estimate with squared error loss function and Brender's Bayesian estimate. All will be derived for a general time dependent system availability expression along with expressions for the two special cases mentioned above.

Actual calculations of these three estimates will be made for two separate data sets in Chapter Four. Sensitivity analyses will also be performed with different sizes and types of samples, different priors and different time horizons. Also, the data-no data cases will be explored.

The remainder of the study will be devoted to the selection of the best estimate. In Chapter Five, after a brief introduction to multiple-criteria and multiple-attribute decision making (MADM), the attributes for the best estimate will be given along with the five MADM methods used: dominance, simple additive weighting, linear assignment, ELECTRE and TOPSIS. Finally, the best estimation method for system availability will be chosen based on the results of the MADM analyses of the two example data sets.

CHAPTER 2 REPRESENTATION OF AVAILABILITY

Availability is generally known as the probability the system is operating satisfactorily at any point in time under stated conditions. But many categories and classifications are defined in the literature, with no uniformity of terms. Therefore, a short review of definitions and terms is presented along with the statement and derivation of the two-state stochastic system via renewal theory.

2.1 Definitions

The major reason availability is enjoying a wider useage as a measure of system performance is the fact that it combines the measures of reliability and maintainability. Reliability is the probability a system will perform satisfactorily for at least a given period of time ("up time") whereas maintainability is the probability a system is restored to an operable condition within a specified time ("down time"). It is this incorporation of maintainability that makes availability more attractive than reliability alone as a measure of system performance.

Depending on the time interval considered, availability is classified as either: (1) instantaneous availability, (2) average uptime availability, or (3) steady-state availability [17]. Instantaneous availability, $g(t)$, is defined as the probability the system is operational at any random time t, where $0 \leq t < \infty$. Average uptime availability, $\bar{g}(T)$, is the proportion of time in a specified time interval (0, T) the system is available for use. It is expressed as

$$\bar{g}(T) = \frac{1}{T} \int_0^T g(t)\, dt \tag{1}$$

Steady-state availability is the instantaneous availability at time $t = \infty$ and, therefore, the limiting case of instantaneous availability. It is easily estimated from sample data as the ratio of mean up time to mean total time:

$$\lim_{t\to\infty} g(t) = \frac{E[T_{on}]}{E[T]} \tag{2}$$

where: T_{on} is on time

T is total cycle time

The choice of availability class is dependent upon the system mission and its conditions of use. For systems which are required to perform a function at any random time, instantaneous availability would be the best measure. A good example would be a data-processing system used in air traffic control which is called upon to process flight paths and then remain idle for a length of time. The average uptime availability would be the most appropriate measure for systems whose usage is defined by a duty cycle, such as a tracking radar system which is called upon only after an object has been detected and is expected to track continuously for a given time period. Finally, the steady-state availability would be the most satisfactory measure for systems which are operating continuously, as a detection radar system.

Note that the average uptime and steady-state availabilities are special cases of the instantaneous availability. Therefore, considerable importance is attached to the development and understanding of instantaneous availability.

2.2 System Representation using Renewal Theory

Several approaches are available to derive and represent system availability. Here, however, renewal theory is chosen because of the two-state stochastic nature of the system.

Renewal theory, which has its origins in the study of self-renewing aggregates [1], was not applied directly to availability problems until 1962 when Parzen [14] derived the steady-state availability using renewal theory. He considered a simple two-state stochastic process with the two states being the on state and the off state. He presented the expected number of renewals at a random time t in a complex form assuming a gamma distributed inter-arrival time. Unfortunately, due to the complex form, it is not very practical to use, except in a few special cases. Therefore, Kuo [12], also using renewal theory, derived a much more useful analytical expression without the complex terms. Again, gamma distributed inter-arrival times are assumed, since analytical solutions for system availability are extremely difficult, if not impossible, to obtain when the underlying on times and cycle times are not gamma distributed. But Kuo also provided a computer simulation approach useful when the on and cycle times are other than gamma distributed, or when the empirical data are based on general renewal theory.

2.2.1 Statement of the System and its General Analytical Solution

The System

Consider a system which can be in one of two states, either on or off. In the on state the system is operating, while in the off state, the system is failed and under repair. Assume at $t=0$ the system is on, and is in service for a random time T_{on} until it fails. T_{on} has the probability density function $f_{on}(t)$ and the cumulative distribution function $F_{on}(t)$. When the system fails, it is then off and under repair for a random time T_{off} with probability density function $f_{off}(t)$ and cumulative distribution function $F_{off}(t)$. The system then repeats these on and off states of random duration. Successive times to breakdown and repair are assumed to be independent and the events of operative or inoperative are independent of time.

A complete cycle time, T, is also a random variable, composed of the addition of the random variables T_{on} and T_{off} (See Figure 2-1). Then T is a random variable of the time from just the beginning of an operative state through a breakdown and repair to the time the system is just restored to operative again. It has the probability density function f(t) and the cumulative distribution function F(t).

<u>A Renewal Equation</u>

The instantaneous availability, g(t), is defined as

$$g(t) \equiv \underline{\text{probability the system is operative at any random time } t}$$

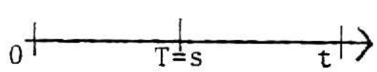
$$= \int_0^\infty \Pr[\text{system on at } t \mid T=s] f(s)ds$$

$$= \int_0^\infty \Pr[\text{system on at } t \mid T=s] dF(s) \qquad (3)$$

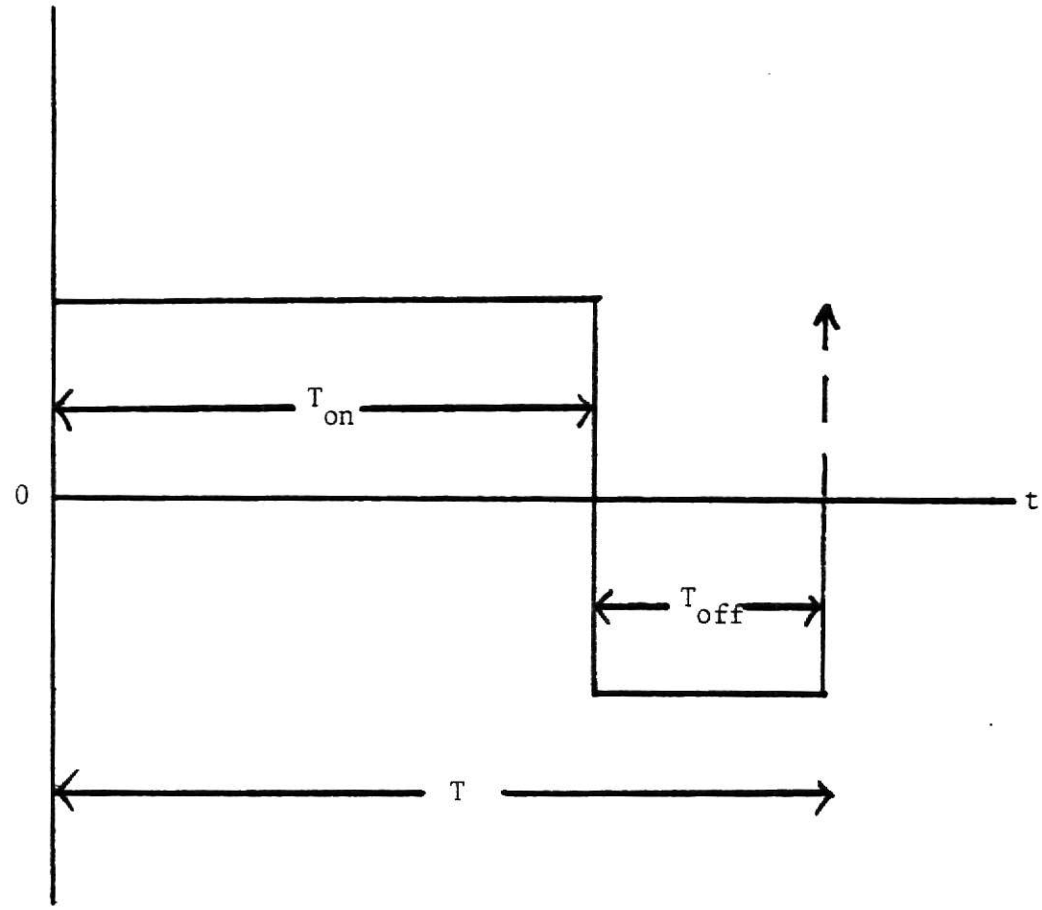
where s is a time index in the time interval $[0,\infty]$.

But the Pr[system on at t | T=s] gives different values depending upon whether $t < s$ or not.

Case 1: $T = s \leq t$



A complete cycle has terminated at $T \leq t$, so the conditional probability of the system being on at t given T=s is exactly the <u>un</u>conditional probability of the system being on when starting at a point which excludes the completed cycle, i.e., the availability at (t-s).



T_{on} : on time

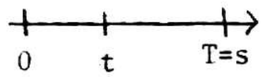
T_{off}: off time

T : total cycle time

Note that $T_{on} + T_{off} = T$

Figure 2-1: A pictorial representation of one cycle of the two state system. T_{on}, T_{off}, and T are all random variables.

Case 2: $T=s > t$



A complete cycle has terminated at $T > t$, so the conditional probability of the system being on at t given $T=s$ is equivalent to the conditional probability that $t < T_{on}$ given $T=s$.

Summarizing, $Pr[\text{system on at } t \mid T=s]$

$$= \begin{cases} g(t-s), & \text{when } s < t \\ Pr[t < T_{on} \mid T=s], & \text{otherwise} \end{cases} \tag{4}$$

The renewal equation for instantaneous availability can now be expressed by substituting eq. (4) into eq. (3):

$$g(t) = \int_0^\infty Pr[\text{system on at } t \mid T=s]dF(s)$$

$$= \int_0^t Pr[\text{system on at } t \mid T=s]dF(s)$$

$$+ \int_t^\infty Pr[\text{system on at } t \mid T=s]dF(s)$$

$$= \int_0^t g(t-s)dF(s) + \int_t^\infty Pr[t < T_{on} \mid T=s]dF(s)$$

$$= \int_0^t g(t-s)dF(s) + Pr[t < T_{on} \text{ and } t < T]$$

$$= \int_0^t g(t-s)dF(s) + Pr[t < T_{on}]$$

$$= \int_0^t g(t-s)dF(s) + Pr[T_{on} > t]$$

$$g(t) = \int_0^t g(t-s)dF(s) + [1-F_{on}(t)] \tag{5}$$

$\equiv$ the probability the system is operative a time t

Recall, $F_{on}(t)$ is the cumulative distribution function of T_{on}.

Eq. (5) is a renewal equation of availability at any random time t to which a general solution can be obtained. Note that eq. (5) is derived without any assumptions on the cycle, on or off time distributions. To reach a general solution of eq. (5), the total number of renewals at time t, N(t), and the counting process $[N(t), t \geq 0]$ and its distribution must be derived.

<u>The Counting Process and its Distribution</u>

Let the cycling events in the interval $[0,\infty]$ be denoted by the successive inter-arrival times T_1, T_2, ... defined as:

T_1 = the time from 0 to the first cycle

T_i = the time from the $(i-1)^{st}$ cycle to the i^{th} cycle, i = 2,3,...

All T_i, i=1,2,... have the same distribution as T.

Also consider the waiting time to the i^{th} cycle, W_i, defined as the time it takes to observe the i^{th} cycle finished in a series of cycles occurring in a time span (See Figure 2-2).

From Figure 2-2, note that the inter-arrival times, T_i, can be conveniently defined in terms of the waiting times, W_i:

$$T_1 = W_1$$

$$T_2 = W_2-W_1$$

$$T_3 = W_3-W_2$$

$$\vdots$$

$$T_n = W_n-W_{n-1} \tag{6}$$

$$\vdots$$

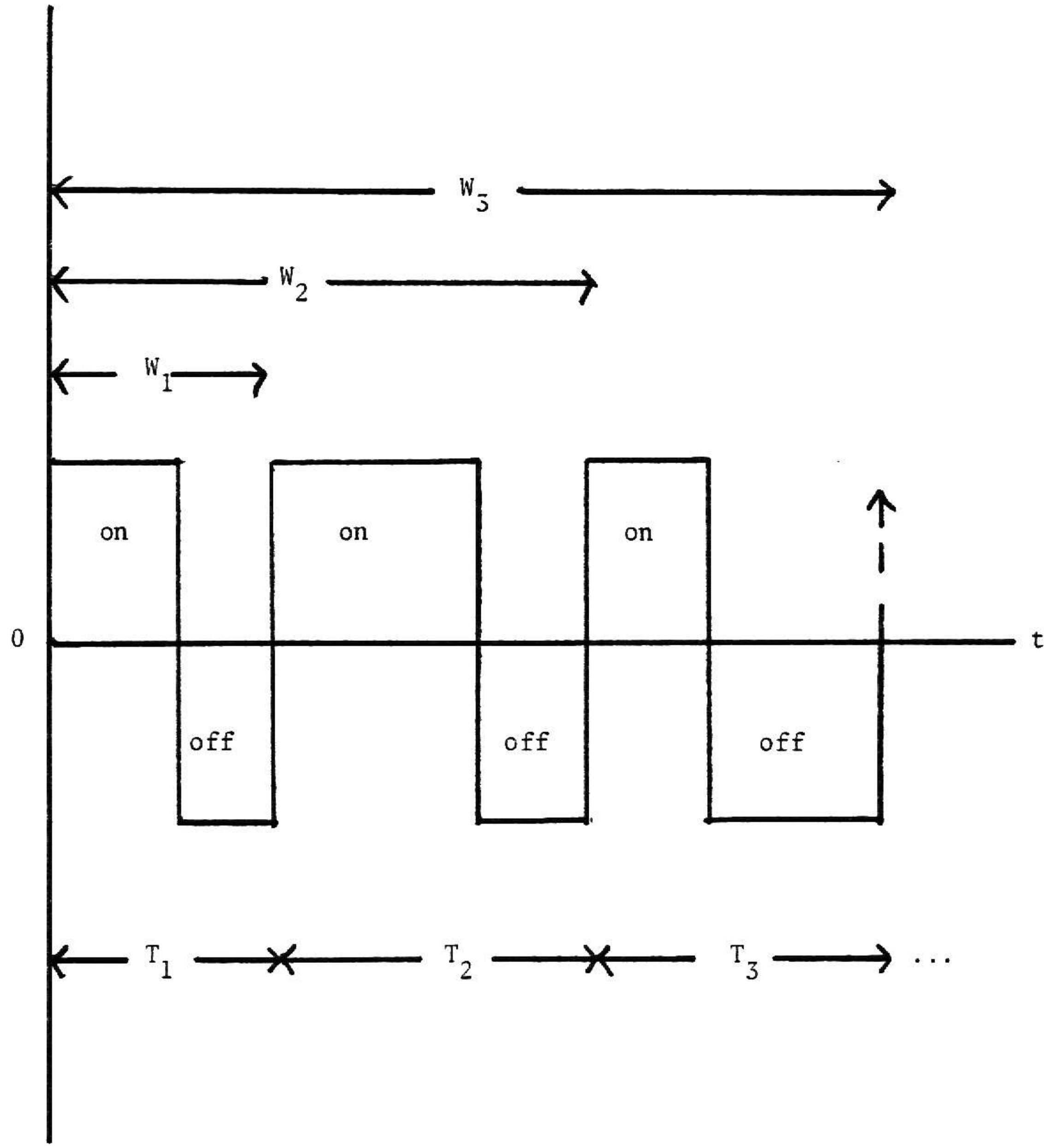


Figure 2-2: A pictorial representation of inter-arrival times, T_i, and waiting times, W_i.

Similarly, the waiting times, W_i, can be expressed in terms of the inter-arrival times, T_i:

$$W_1 = T_1$$

$$W_2 = T_1 + T_2$$

$$W_3 = T_1 + T_2 + T_3$$

$$\vdots$$

$$W_n = \sum_{i=1}^{n} T_i \qquad \text{for } n > 1 \tag{7}$$

$$\vdots$$

For $t \geq 0$, let $N(t)$ represent the number of cycles' end points lying in the interval $[0,t]$. The counting process, defined as $\{N(t), t \geq 0\}$, can then be related to a corresponding sequence of waiting times, W_n. Note that $N(t)$ is a discrete random variable while W_n is a continuous random variable. For any $t > 0$ and $n=1,2,\ldots$, the number of cycles occurring in the interval $[0,t]$ is less than or equal to n if and only if the waiting time to the $(n+1)^{st}$ event is greater than t, i.e.,

$$N(t) \leq n \quad \text{iff } W_{n+1} > t \tag{8}$$

From eq. (8) it directly follows that exactly n cycles occur if and only if the waiting time to the n^{th} event is less than or equal to t plus the waiting time to the $(n+1)^{st}$ event is greater than t, i.e.,

$$N(t) = n \text{ iff } W_n \leq t \text{ and } W_{n+1} > t \tag{9}$$

From eqs. (8) and (9), two probability relationships fall out,

A. $Pr[N(t) \leq n] = Pr[W_{n+1} > t]$

$$Pr[N(t) \leq n] = 1 - Pr[W_{n+1} \leq t],\ n = 0,1,2,... \quad (10)$$

B. Since $Pr[N(t) < n] = Pr[W_n > t] = 1 - Pr[W_n \leq t]$

and $Pr[N(t)=n] = Pr[N(t) \leq n] - Pr[N(t) < n]$

$$= \{1-Pr[W_{n+1} \leq t]\} - \{1-Pr[W_n \leq t]\}$$

then,

$$Pr[N(t)=n] = Pr[W_n \leq t] - Pr[W_{n+1} \leq t],\ n=1,2,... \quad (11)$$

As a special case, when n=0:

$$Pr[N(t)=0] = Pr[W_0 \leq t] - Pr[W_1 \leq t]$$

$$Pr[N(t)=0] = 1 - Pr[W_1 \leq t] \quad (12)$$

Eqs. (10), (11) and (12) can also be stated in terms of the cumulative distribution function of the waiting times:

$$F_{N(t)}(n) = 1-F_{W_{n+1}}(t), \qquad n = 0,1,... \quad (13)$$

$$P_{N(t)}(n) = F_{W_n}(t) - F_{W_{n+1}}(t), \qquad n = 1,2,... \quad (14)$$

$$P_{N(t)}(0) = 1-F_{W_1}(t) \quad (15)$$

Eqs. (13), (14) and (15) describe the distribution of the counting process.

A General Solution to the Renewal Equation

Let m(t) be the expected instantaneous renewal rate and M(t) be the mean value function of a renewal counting process corresponding to independently identical distribution times, T, with nonlattice distribution functions F(t) and finite mean μ. Since N(t) is the total number of renewals at time t, let

$$M(t) = E[N(t)] \tag{16}$$

$$= \sum_{n=0}^{\infty} nP_{N(t)}(n)$$

$$= \sum_{n=0}^{\infty} n[F_{W_n}(t) - F_{W_{n+1}}(t)] \tag{17}$$

$$M(t) = \sum_{n=1}^{\infty} F_{W_n}(t) \tag{18}$$

by expansion of eq. (17).

Recall the waiting time, W_n, can be expressed as a sum of the inter-arrival times, $\sum_{i=1}^{n} T_i$, each with probability density $f_{T_i}(t)$. Since

$$F_{W_n}(t) = \int_0^t f_{W_n}(x)dx$$

and

$$f_{W_n}(t) = [f_{T_i}(t)]^n$$

due to the additivity of independent random variables, then

$$F_{W_n}(t) = \int_0^t [f_{T_i}(x)]^n dx \tag{19}$$

Substituting eq. (19) into eq. (18) and taking the Laplace transform of both sides of eq. (18),

$$M^*(\theta) = \sum_{n=1}^{\infty} \frac{1}{\theta} [f^*_{T_i}(\theta)]^n \tag{20}$$

Note that the above sum is the sum of an infinite geometric series and can be further simplified† to

$$M^*(\theta) = \frac{1}{\theta} \frac{f^*_{T_i}(\theta)}{1-f^*_{T_i}(\theta)} \tag{21}$$

Whenever $\frac{dM(t)}{dt}$ exists, this derivative is denoted by $m(t)$ and it follows from eq. (21) that its Laplace transform (also see Rau [16a] for derivation) is

$$m^*(\Theta) = \frac{f^*_{T_i}(\Theta)}{1 - f^*_{T_i}(\Theta)} \tag{22}$$

The $m(t)$ is referred to as the expected instantaneous renewal rate since $m(t)dt$ denotes the probability of at least one renewal occurring in the interval $[t, t+dt]$. It is also sometimes called the renewal density, but this is misleading since $m(t)$ is not necessarily a probability density, i.e., $\int_0^\infty m(t)\, dt \neq 1$.

Now, to solve the renewal equation, eq. (5), take the Laplace transform of both sides, noting that the first term on the right side is a convolution $g(t)*f(t)$:

$$g^*(\Theta) = g^*(\Theta)f^*(\Theta) + [1 - F_{on}(\Theta)]^* \tag{23}$$

---

†Recall, the sum of a geometric series:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Solving for $g^*(\theta)$,

$$g^*(\Theta) = \frac{[1 - F_{on}(\Theta)]^*}{1 - f^*(\Theta)}$$

$$= [1 - F_{on}(\Theta)]^* + [1 - F_{on}(\Theta)]^* \left[\frac{f^*(\Theta)}{1 - f^*(\Theta)} \right]$$

$$g^*(\Theta) = [1 - F_{on}(\Theta)]^* + [1 - F_{on}(\Theta)]^* \, m^*(\Theta) \qquad (24)$$

Taking the inverse Laplace transform of eq. (24) gives the general solution to the renewal equation,

$$g(t) = [1 - F_{on}(t)] + \int_0^t [1 - F_{on}(t-s)] \, m(s)ds$$

$$g(t) = [1 - F_{on}(t)] + \int_0^t [1 - F_{on}(t-s)] \, dMs \qquad (25)$$

Note this solution does <u>not</u> take into account the distributions of T, T_{on} and T_{off}.

To show that eq. (25) is reasonable, take $\lim_{t\to\infty} g(t)$ and compare it to the expression for the steady-state availability

$$\frac{E[T_{on}]}{E[T]},$$

as stated in eq . (2)

Since

(i) $1 - F_{on}(t) \geq 0$ for all $t \geq 0$

(ii) $\int_0^\infty [1 - F_{on}(t)]dt = \mu < \infty$, and

(iii) $[1 - F_{on}(t)]$ is nonincreasing, then,

$$\lim_{t\to\infty} g(t) = 0 + \frac{1}{\mu}\int_0^\infty [1 - F_{on}(s)]ds$$

$$= \frac{1}{\mu}\int_0^\infty Pr[T_{on} > s]ds = \frac{1}{\mu}\int_0^\infty R_{on}\, ds$$

$$= \frac{E[T_{on}]}{E[T_{on} + T_{off}]}$$

$$\lim_{t\to\infty} g(t) = \frac{E[T_{on}]}{E[T]}$$

due to Theorem 2.9 in Barlow and Proschan [1].

2.2.2 An Analytical Solution Assuming Gamma Distributed T and T_{on}

Assume inter-arrival time, T, is *gamma distributed* with density function

$$f_T(t) = \begin{cases} \frac{\lambda}{(k-1)!}(\lambda t)^{k-1} e^{-\lambda t}, & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

and on time, T_{on}, is gamma distributed with density function

$$f_{T_{on}}(t) = \begin{cases} \frac{\beta}{(\alpha-1)!}(\beta t)^{\alpha-1} e^{-\beta t}, & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

where k, λ, α, $\beta > 0$.

To obtain the analytical solution of eq. (25) using the above densities, first the expression *for the mean* value function of renewals, M(t), is found.

Since the inter-arrival time is gamma distributed, the renewal counting process $\{n(t) \geq 0,\ t \geq 0\}$ is a Poisson process with intensity λ, where $n(t) = k \cdot N(t)$, for $t \geq 0$. For proof, see Appendix A. Again, n(t) is discrete.

Then,

$$1 - F_{W_n}(t) = Pr[W_n > t]$$

$$= Pr[N(t) < n]$$

$$= Pr[\frac{n(t)}{k} < n]$$

$$= Pr[n(t) < nk]$$

$$= \sum_{m=0}^{nk-1} e^{-\lambda t} \frac{(\lambda t)^m}{m!} \tag{28}$$

$$1 - F_{W_n}(t) = \int_t^{\infty} \frac{\lambda^{nk} x^{nk-1}}{\Gamma(nk)} e^{-\lambda x} dx \tag{29}$$

and $$F_{W_n}(t) = \int_0^t \frac{\lambda^{nk} \lambda^{nk-1}}{\Gamma(nk)} e^{-\lambda x} dx \tag{30}$$

The equivalence of the gamma function and the cumulative Poisson distribution is shown in Appendix B.

Therefore, N(t) has the probability mass function

$$P_{N(t)}(n) = F_{W_n}(t) - F_{W_{n+1}}(t)$$

$$= \sum_{m=nk}^{\infty} \frac{e^{-\lambda t} (\lambda t)^m}{m!} - \sum_{m=(n+1)k}^{\infty} \frac{e^{-\lambda t} (\lambda t)^m}{m!}$$

$$P_{N(t)}(n) = \sum_{m=nk}^{(n+1)k-1} \frac{e^{-\lambda t} (\lambda t)^m}{m!} \tag{31}$$

Computing the probability generating function of N(t) as

$$\psi(z,t) = \sum_{n=0}^{\infty} z^n \Pr[N(t) = n] \tag{32}$$

Then,

$$\psi(z,t) = \Pr[N(t) = 0] + \sum_{n=1}^{\infty} z^n \,[F_{W_n}(t) - F_{W_{n+1}}(t)]$$

$$= [1 - F_{W_1}(t)] + z[F_{W_1}(t) - F_{W_2}(t)]$$

$$+ z^2[F_{W_2}(t) - F_{W_3}(t)] + z^3[F_{W_3}(t) - F_{W_4}(t)] + \ldots$$

$$= 1 + (z-1)\, F_{W_1}(t) + z(z-1)\, F_{W_2}(t) + z^2(z-1)\, F_{W_3}(t) + \ldots$$

$$= 1 + (z-1) \sum_{n=1}^{\infty} z^{n-1} F_{W_n}(t)$$

$$\psi(z,t) = 1 + (z-1) \sum_{n=1}^{\infty} z^{n-1} \int_0^t \frac{\lambda^{nk} x^{nk-1}}{\Gamma(nk)} e^{-\lambda x}\, dx \tag{34}$$

and

$$\frac{\partial\, \psi(z,t)}{\partial z} = \sum_{n=1}^{\infty} z^{n-1} \int_0^t \frac{\lambda^{nk} x^{nk-1}}{\Gamma(nk)} e^{-\lambda x}\, dx$$

$$+ (z-1) \sum_{n=1}^{\infty} (n-1) z^{n-2} \int_0^t \frac{\lambda^{nk} x^{nk-1}}{\Gamma(nk)} e^{-\lambda x}\, dx \tag{35}$$

Evaluating eq. (35) at z=1 gives the expected number of renewals at time t, M(t):

$$M(t) = E[N(t)]$$

$$= \left.\frac{\partial\psi\ (z,t)}{\partial z}\right|_{z=1}$$

$$M(t) = \sum_{n=1}^{\infty} \int_0^t \frac{\lambda^{nk}\ x^{nk-1}}{\Gamma(nk)}\ e^{-\lambda x}\ dx \tag{36}$$

and

$$\frac{dM(t)}{dt} = \sum_{n=1}^{\infty} \frac{\lambda^{nk}\ t^{nk-1}}{\Gamma(nk)}\ e^{-\lambda t} \tag{37}$$

Hence,

$$dM(t) = [\sum_{n=1}^{\infty} \frac{(\lambda t)^{nk-1}}{(nk-1)!}\ \lambda e^{-\lambda t}]\ dt \tag{38}$$

Since,

$$F_{on}(t) = \int_0^t \frac{\beta}{(\alpha-1)!}(\beta x)^{\alpha-1}\ e^{-\beta x}\ dx$$

$$= \sum_{\ell=\alpha}^{\infty} P_0(\ell;\ \beta t) \tag{39}$$

where $P_0(\ell;\ \beta t)$ indicates the Poisson probability

$$\frac{e^{-\beta t}(\beta t)^{\ell}}{\ell!}$$

as indicated in Appendix B, then,

$$1 - F_{on}(t) = 1 - \sum_{\ell=\alpha}^{\infty} P_0(\ell;\ \beta t)$$

$$1 - F_{on}(t) = \sum_{\ell=0}^{\alpha-1} P_0(\ell;\ \beta t) \tag{40}$$

and

$$1 - F_{on}(t-s) = \sum_{\ell=0}^{\alpha-1} P_0[\ell;\ \beta(t-s)] \tag{41}$$

Substituting eqs. (38), (40), and (41) into eq. (25), the expression for the instantaneous availability when T and T_{on} are gamma distributed is

$$g(t) = \sum_{\ell=0}^{\alpha-1} P_0(\ell;\ \beta t) + \int_0^t \{ \sum_{\ell=0}^{\alpha-1} P_0[\ell;\ \beta(t-s)]\} \cdot \{ \sum_{n=1}^{\infty} \frac{(\lambda s)^{nk-1}}{(nk-1)!} \lambda e^{-\lambda s}\}\ ds \tag{42}$$

To simplify the second term,

$$\int_0^t [1 - F_{on}(t-s)]dM(s) = \int_0^t \lambda \{ \sum_{\ell=0}^{\alpha-1} \frac{e^{-\beta(t-s)}[\beta(t-s)]^{\ell}}{\ell!}\} \cdot \{ \sum_{n=1}^{\infty} \frac{(\lambda s)^{nk-1}}{(nk-1)!} e^{-\lambda s}\}\ ds \tag{43}$$

Let nk-1 = q, then

when n=1, q=k-1

n=2, q=2k-1

n=3, q=3k-1

.
.
.
etc.

and

$$\int_0^t [1 - F_{on}(t-s)]\ dM(s) = \int_0^t \lambda \{ \sum_{\ell=0}^{\alpha-1} \frac{e^{-\beta(t-s)}[\beta(t-s)]^{\ell}}{\ell!}\} \cdot \{ \sum_{\substack{q=k-1\\2k-1\\ \vdots}}^{\infty} \frac{(\lambda s)^q e^{-\lambda s}}{q!}\}\ ds$$

$$= \int_0^t \lambda \{ \sum_{\ell=0}^{\alpha-1} P_0[\ell;\ \beta(t-s)]\} \cdot \{ \sum_{\substack{q=k,\\2k,\\3k,}}^{\infty} P_0[(q-1);\lambda s] \tag{44}$$

Therefore, the availability function in a real form is

$$g(t) = \sum_{\ell=0}^{\alpha-1} P_0(\ell;\beta t) + \lambda \int_0^t \Big\{ \sum_{\ell=0}^{\alpha-1} P_0[\ell;\beta(t-s)]\Big\}\cdot\Big\{ \sum_{\substack{q=k\\2k\\3k\\ \vdots}}^{\infty} P_0[(q-1);\lambda s]\Big\} ds \tag{45}$$

An expression for the availability function involving complex numbers is:

$$g(t) = \sum_{\ell=0}^{\alpha-1} P_0(\ell;\beta t) + \sum_{\ell=0}^{\alpha-1} \frac{\lambda}{\beta k} \cdot \frac{1}{\ell!}\Gamma_{\beta t}(\ell+1) +$$

$$\frac{\lambda}{k} \sum_{\ell=0}^{\alpha-1} \sum_{r=1}^{k-1} \frac{\varepsilon^r \beta^\ell e^{-\lambda t(1-\varepsilon^r)}\,\Gamma(\ell+1)}{\ell!\,[\beta + \lambda(1-\varepsilon^r)]^{\ell+1}}$$

$$\cdot \sum_{n=\ell+1}^{\infty} P_0(n;\ t[\beta + \lambda(1-\varepsilon^r)]) \tag{46}$$

where

$$\Gamma_x(\alpha) = \int_0^x t^{\alpha-1} e^{-t}\, dt$$

$$\varepsilon = \exp(2\pi i/k)$$

For the derivation of eq.(46), see Kuo [12].

2.2.3 Analytical Solutions Assuming Exponentially Distributed T and T_{on} or T_{on} and T_{off}

Analytical solutions for two cases of exponentially distributed data combinations (T and T_{on} or T_{on} and T_{off}) are presented. The case using exponentially distributed T and T_{on} is presented to show continuity with the previously derived g(t) using gamma distributed T and T_{on} The case using exponentially distributed T_{on} and T_{off} is presented because this derivation is more commonly known in the literature, and will also prove of use later in the study.

Using T and T_{on}

Assume inter-arrival time, T, is exponentially distributed (eq. (26) with k=1) with probability density function

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0 \\ 0 \quad , & \text{otherwise} \end{cases} \tag{47}$$

and on time, T_{on} is exponentially distributed (eq. (27) with α=f) with probability function

$$f_{T_{on}}(t) = \begin{cases} \beta e^{-\beta t}, & t \geq 0 \\ 0 \quad , & \text{otherwise} \end{cases} \tag{48}$$

where $\lambda, \beta > 0$.

Assume the renewal counting process $\{N(t), t \geq 0\}$ corresponding to exponentially distributed cycle time, T, with mean $\frac{1}{\lambda}$, has mean value function $M(t) = E[N(t)] = \lambda t$. Note this is a linear function of t. Also note the mean of on time is $1/\beta$ (since it is exponentially distributed also), and

$$1/\lambda \geq 1/\beta.$$

Since M(t) is a linear function of t, eq. (25) can be simplified to

$$g(t) = [1 - F_{on}(t)] + \lambda \int_0^t [1 - F_{on}(t-y)]dy \tag{49}$$

Also,

$$F_{on}(t) = \int_0^t \beta e^{-\beta t} \, dt$$

$$= 1 - e^{-\beta t}$$

so $\quad [1 - F_{on}(t)] = e^{-\beta t}$ (50)

and $\quad [1 - F_{on}(t-y)] = e^{-\beta(t-y)}$ (51)

Therefore, the simplified expression for instantaneous availability at time t is:

$$g(t) = e^{-\beta t} + \lambda \int_0^t e^{-\beta(t-y)} dy$$

$$= e^{-\beta t} + \lambda e^{-\beta t} \int_0^t e^{\beta y} \, dy$$

$$= e^{-\beta t} + \frac{\lambda}{\beta} e^{-\beta t} \, [e^{\beta t} - 1]$$

$$= e^{-\beta t} + \frac{\lambda}{\beta} [1 - e^{-\beta t}]$$

$$g(t) = \frac{\lambda}{\beta} + (1 - \frac{\lambda}{\beta}) \, e^{-\beta t} \tag{52}$$

Again, as a check on the reasonability of the expression, take its limit and compare to the steady state expression:

$$\lim_{t\to\infty} g(t) = \frac{\lambda}{\beta}$$

which is equivalent to the steady-state expression

$$\frac{E[T_{on}]}{E[T]}$$

where $E[T_{on}]$ is $1/\beta$ and $E[T]$ is $1/\lambda$.

The instantaneous availability can also be derived by using eq. (45) with $k=\alpha=1$

$$g(t) = P_0(0;\ \beta t) + \lambda\int_0^t P_0[0;\ \beta(t-s)]\cdot \{\sum_{q=0}^{\infty} P_0(q;\ \lambda s)\}ds$$

$$= \frac{e^{-\beta t}(\beta t)^0}{0!} + \lambda\int_0^t \frac{e^{-\beta(t-s)}\ [\beta(t-s)]^0}{0!}\ :1\ ds$$

$$= e^{-\beta t} + \lambda \int_0^t e^{-\beta(t-s)}\ ds$$

$$= e^{-\beta t} + \lambda\ e^{-\beta t} \int_0^t e^{\beta s}\ ds$$

$$= e^{-\beta t} + \frac{\lambda}{\beta} e^{-\beta t}\ [e^{-\beta t} - 1]$$

$$g(t) = \frac{\lambda}{\beta} + (1 - \frac{\lambda}{\beta})\ e^{-\beta t} \qquad (53)$$

which is identical to eq. (52).

This special case shows that if the underlying distributions of T and T_{on} are exponential, which is not uncommon, the instantaneous availability can be found from eqs. (25) and (52). However, if T and T_{on} are gamma distributed with shape parameters not equal to 1, the instantaneous availability must be either simulated from eq. (25) or calculated from eq. (45). A computer routine for the simulation of eq. (25) can be found in Kuo [12]. This simulation is extremely useful especially when the underlying distributions are not gamma distributed.

Using T_{on} and T_{off} [17]

The probability density function of T_{on} has already been stated in eq. (48) and will be used again here,

$$f_{T_{on}}(t) = \begin{cases} \beta e^{-\beta t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (54)$$

Similarly, T_{off} has a probability density function

$$f_{T_{off}}(t) = \begin{cases} \eta e^{-\eta t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (55)$$

where $\eta > 0$.

Consequently, the cumulative distribution functions for T_{on} and T_{off} are

$$F_{T_{on}}(t) = 1 - e^{-\beta t} \quad (56)$$

$$F_{T_{off}}(t) = 1 - e^{-\eta t} \quad (57)$$

where η, β, $t > 0$.

To solve for the instantaneous availability, g(t), recall the system is represented as a two-state stochastic process, so use the theory of Markov processes. Let state 0 represent an operating system and state 1 represent a failed system under repair. Let β be the probability of a failure and η be the probability of a repair.

The conditional probability of a failure in the time interval [t, t + dt] is βdt and the conditional probability of a repair in the time interval [t, t+dt] is ηdt, where dt is a very small unit of time. Therefore, the transition matrix for the system is

$$P = \begin{array}{c} \text{State} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 1-\beta & \beta \\ \eta & 1-\eta \end{bmatrix} \end{array} \tag{58}$$

where $1 - \beta$ is the probability of no failures and $1 - \eta$ is the probability of no repairs.

The differential equations giving the probabilities of being in a certain state are

$$P_0(t + dt) = P_0(t)\ (1 - \beta dt) + P_1(t)\ \eta dt + 0dt \tag{59}$$

which is the probability of being at state 0 at the time (t + dt), and

$$P_1(t + dt) = P_0(t)\ \beta dt + P_1(t)\ (1 - \eta dt) + 0dt \tag{60}$$

which is the probability of being at state 1 at the time (t + dt). The 0dt represents the probability of two events occurring in dt.

The limit of the ratio is then defined for each equation

$$\frac{P_0(t + dt) - P_0(t)}{dt} = -P_0(t)\beta + P_1(t)\eta$$

or

$$\frac{dP_0(t)}{dt} = -\beta P_0(t) + \eta P_1(t) \tag{61}$$

and

$$\frac{P_1(t + dt) - P_1(t)}{dt} = P_0(t)\beta - P_1(t)\eta$$

or

$$\frac{dP_1(t)}{dt} = \beta P_0(t) - \eta P_1(t) \tag{62}$$

To solve the above differential equations with the initial conditions of $P_0(0) = 1$ and $P_1(0) = 0$, first take the Laplace transforms of both sides which yields, after simplification,

$$(\theta + \beta)\, P_0(\theta) - \eta P_1(\theta) = 1 \tag{63}$$

$$(\theta + \eta)\, P_1(\theta) - \beta P_0(\theta) = 0 \tag{64}$$

or

$$P_0(\theta) = \frac{\theta + \eta}{\theta(\theta + \beta + \eta)} \tag{65}$$

$$P_1(\theta) = \frac{\beta}{\theta(\theta + \beta + \eta)} \tag{66}$$

The availability, g(t), is simply the probability the system is operating, or the probability of being in state 0 $(P_0(t))$. Note that $P_0(t)$ is the inverse Laplace transform of $P_0(\theta)$. So to find the availability, simply take the inverse transform of eq. (65):

$$\mathcal{L}^{-1}\{P_0(\theta)\} = \mathcal{L}^{-1}\{\frac{\theta + \eta}{\theta(\theta + \beta + \eta)}\}$$

$$= \mathcal{L}^{-1}\{\frac{\eta/(\beta+\eta)}{\theta} + \frac{\beta/(\beta+\eta)}{\theta + \beta + \eta}\}$$

so

$$g(t) = P_0(t) = \frac{\eta}{\beta+\eta} + (\frac{\beta}{\beta+\eta})\, e^{-(\beta+\eta)t}$$

or

$$g(t) = \frac{\eta}{\beta+\eta} + (1 - \frac{\eta}{\beta+\eta})\, e^{-(\beta+\eta)t} \tag{67}$$

Chapter 3 - THE ESTIMATORS

The three estimation methods explored in this study are: the maximum likelihood estimate, the traditional Bayesian estimate with squared error loss function, and Brender's Bayesian estimate. For the latter two, estimates for the data and no data cases are both derived. Gamma distributed on, off and cycle times are assumed for all the derivations. Also, where applicable, simplified forms are derived for exponentially distributed on, off and cycle times.

Assumptions

Assume cycle time, T, on time, T_{on}, and off time, T_{off}, are gamma distributed with probability density functions:

$$X \sim G(k,\lambda): \quad f_T(x) = \frac{\lambda}{(k-1)!} (\lambda x)^{k-1} e^{-\lambda x} \tag{1}$$

$$\text{with } E(T) = k/\lambda \text{ , } VAR(T) = k/\lambda^2$$

$$Y \sim G(\alpha,\beta): \quad f_{T_{on}}(y) = \frac{\beta}{(\alpha-1)!} (\beta y)^{\alpha-1} e^{-\beta y} \tag{2}$$

$$\text{with } E(T_{on}) = \alpha/\beta, \ VAR(T_{on}) = \alpha/\beta^2$$

$$Z \sim G(m,\eta): \quad f_{T_{off}}(z) = \frac{\eta}{(m-1)!} (\eta z)^{m-1} e^{-\eta z} \tag{3}$$

$$\text{with } E(T_{off}) = m/\eta, \ VAR(T_{off}) = m/\eta^2$$

where λ, β, $\eta > 0$; k, α, m are positive integers; and x, y, z > 0, are the sample values. Note that these three distributions are never used all at once. They are only used in pairs of T and T_{on} or T_{on} and T_{off}. (i.e. no assumptions are made on the additivity of the time distributions.)

Also assume n independent samples of x and y (or x and z or y and z) are drawn from T and T_{on} (T and T_{off}, T_{on} and T_{off}) respectively. Let the observations be denoted by (x_i, y_i) $[(x_i, z_i), (y_i, z_i)]$, i = 1,2,...,n. The density functions of x_i, y_i, and z_i are eqs. (1), (2) and (3) respectively. The objective is to use the observed sample data to estimate the system availability. The time-dependent representation of system availability derived in Chapter 2 using independent T and T_{on} will be used:

$$G(t;\, k,\, \lambda\,,\, \alpha\,,\, \beta) = \sum_{\ell=0}^{\alpha-1} P_0(\ell;\, \beta t) + \lambda \int_0^t \{\sum_{\ell=0}^{\alpha-1} P_0[\ell;\, \beta(t-s)]\} \cdot \{\sum_{\substack{q=k\\2k\\3k\\ \vdots}}^{\infty} P_0[(q-1);\, \lambda s]\}ds \tag{4}$$

All three densities are given, even though only two are needed, because some estimates and their derivations require on and off times while others require on and cycle times. Substitutions for the parameters can be made, however, so the user can adapt the estimate to the data available. From Figure 1 in Chapter 2 note

$$T_{on} + T_{off} = T \tag{5}$$

so

$$y_i + z_i = x_i \tag{6}$$

for each cycle i

Because of eq. (5)

$$E(T_{on}) + E(T_{off}) + E(T) \tag{7}$$

or

$$\frac{\alpha}{\beta} + \frac{m}{\eta} = \frac{k}{\lambda} \tag{8}$$

Hence, when the two parameters of any two of the densities are known, the third set can be found if one of the parameter set is given.

3.1 Maximum Likelihood Estimate

This method of estimation provides the maximum likelihood estimate, $\hat{g}_{MLE}(t;\, k,\, \lambda,\, \alpha,\, \beta)$ simply by substituting the maximum likelihood estimates of each of the parameters, $\hat{k}_{MLE}$, $\hat{\lambda}_{MLE}$, $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$, into the expression for the availability, $g(t;\, k,\, \lambda,\, \alpha,\, \beta)$,

i.e.,

$$\hat{g}_{MLE}(t;\, \hat{k}_{MLE},\, \hat{\lambda}_{MLE},\, \hat{\alpha}_{MLE},\, \hat{\beta}_{MLE}) = \sum_{\ell=0}^{\hat{\alpha}_{MLE}-1} P_0(\ell;\, \hat{\beta}_{MLE}t)$$

$$+\, \hat{\lambda}_{MLE} \int_0^t \{ \sum_{\ell=0}^{\hat{\alpha}_{MLE}-1} P_0[\ell;\, \hat{\beta}_{MLE}(t-s)]\} \cdot \{ \sum_{q=\hat{k}_{MLE},\, 2\hat{k}_{MLE},\, \ldots}^{\infty} P_0[(q-1);\, \hat{\lambda}_{MLE}s]\} ds \tag{9}$$

So, basically, the problem becomes one of deriving the maximum likelihood estimates for each of the parameters.

3.1.1 Assuming Gamma Distributed T and T_{on}

First, the likelihood* functions of x and y are expressed, given that the $(x_i,\, y_i)$, $i = 1,2,\ldots,n$, have density functions stated in eqs. (1) and (2). They are simply the products of the probabilities of each x (or y) occurring, i.e.,

$$L(x_1,\, x_2,\, \ldots,\, x_n\; ;\, k,\lambda) = \prod_{i=1}^{n} f_T(x_i)$$

$$= [\frac{\lambda^k}{(k-1)!}]^n \left(\prod_{i=1}^{n} x_i\right)^{k-1} e^{-\lambda \sum_{i=1}^{n} x_i} \tag{10}$$

---

* "Likelihood" meaning the probability of each value (x_i or y_i) happening.

and

$$L(y_1, y_2, \ldots, y_n; \alpha, \beta) = \prod_{i=1}^{n} f_{T_{on}}(y_i) = \left[\frac{\beta^{\alpha}}{(\alpha-1)!}\right]^n \left(\prod_{i=1}^{n} y_i\right)^{\alpha-1} e^{-\beta \sum_{i=1}^{n} y_i} \tag{11}$$

Realize that some value exists for each of the parameters k and λ that maximizes the value of the likelihood function for the x_i stated in eq. (10). Hence, the name "maximum likelihood estimate" is used to describe each of these parameter values, and they are represented by $\hat{k}_{MLE}$ and $\hat{\lambda}_{MLE}$. Likewise, the values $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$ are the values of those parameters which maximize the value of eq. (11).

Maximization of eq. (10) is equivalent to the maximization of its logarithm, which is

$$\ell n\, L(x_1, x_2, \ldots, x_n; k, \lambda) = n\,[k\,\ell n\,\lambda - \ell n\,(k-1)!] + (k-1)\,\ell n\left(\prod_{i=1}^{n} x_i\right) - \lambda \sum_{i=1}^{n} x_i \tag{12}$$

Note eq. (12) is a function of two variables: the discrete variable k and the continuous variable λ. To obtain the maximum values of each, use the necessary conditions of calculus:

$$\frac{\partial\, \ell n\, L(x_1, x_2, \ldots, x_n; k, \lambda)}{\partial \lambda} = 0 \tag{13}$$

$$\ell n\, L(x_1, x_2, \ldots, x_n; (k-1), \lambda) \le \ell n\, L(x_1, x_2, \ldots, x_n; k, \lambda) \tag{14}$$

$$\ell n\, L(x_1, x_2, \ldots, x_n; (k+1), \lambda) \le \ell n\, L(x_1, x_2, \ldots, x_n; k, \lambda) \tag{15}$$

Eq. (13) is the necessary condition for the continuous variable λ and will hold true when λ is at its maximum. Eqs. (14) and (15) are the necessary conditions for the discrete variable k and will hold true when k is at its maximum.

To solve for $\hat{\lambda}_{MLE}$, substitute eq. (12) into eq. (13)

$$\frac{\partial \ \ell n \ L \ (x_1, x_2, \ldots, x_n; k, \lambda)}{\partial \lambda} = \frac{nk}{\lambda} - \sum_{i=1}^{n} x_i = 0 \tag{16}$$

The solution of λ in eq. (16) is the maximum likelihood estimates λ_{MLE} or

$$\hat{\lambda}_{MLE} = \frac{nk}{\sum_{i=1}^{n} x_i} = \frac{k}{\bar{x}} \tag{17}$$

where $\bar{x} = \sum_{i=1}^{n} x_i/n$

To solve for $\hat{k}_{MLE}$, substitute eq. (12) into eqs. (14) and (15). Solving eq. (14) first:

$$n[(k-1)\ell n \ \lambda - \ell n \ (k-2)!] + (k-2) \ \ell n \ (\prod_{i=1}^{n} x_i)$$

$$-\lambda \sum_{i=1}^{n} x_i < n[k \ \ell n \ \lambda - \ell n(k-1)!] + (k-1) \ \ell n (\prod_{i=1}^{n} x_i) - \lambda \sum_{i=1}^{n} x_i$$

or

$$n \ k \ \ell n \ \lambda - n \ \ell n \lambda - n \ \ell n(k-2)! + (k-2) \ \ell n \ (\prod_{i=1}^{n} x_i)$$

$$< n \ k \ \ell n \ \lambda - n \ \ell n \ (k-1) - n \ \ell n(k-2)! + \ell n(\prod_{i=1}^{n} x_i) + (k-2) \ \ell n(\prod_{i=1}^{n} x_i)$$

or

$$- n \ \ell n \ \lambda < - n \ \ell n(k-1) + \ell n(\prod_{i=1}^{n} x_i)$$

or

$$\ell n \ (k-i) < \ell n[\lambda \ (\prod_{i=1}^{n} x_i)^{\frac{1}{n}}]$$

Therefore,

$$k < 1 + \lambda(\prod_{i=1}^{n} x_i)^{\frac{1}{n}} \tag{18}$$

Solving eq. (15):

$$n[(k+1)\,\ln\lambda - \ln(k)!] + k\,\ln\Big(\prod_{i=1}^{n} x_i\Big) - \lambda\sum_{i=1}^{n} x_i$$

$$\leq n\,[k\,\ln\lambda - \ln(k-1)!] + (k-1)\,\ln\Big(\prod_{i=1}^{n} x_i\Big) - \lambda\sum_{i=1}^{n} x_i$$

or

$$n\,k\,\ln\lambda + n\,\ln\lambda - n\,\ln k - n\,\ln(k-1)! + \ln\prod_{i=1}^{n} x_i$$

$$+ (k-1)\,\ln\Big(\prod_{i=1}^{n} x_i\Big) \leq n\,k\,\ln\lambda - n\,\ln(k-1)! + (k-1)\,\ln\Big(\prod_{i=1}^{n} x_i\Big)$$

or $$n\,\ln\lambda - n\,\ln k + \ln\Big(\prod_{i=1}^{n} x_i\Big) \leq 0$$

or

$$\ln k \geq \ln\lambda + \frac{1}{n}\ln\Big(\prod_{i=1}^{n} x_i\Big)$$

or

$$\ln k \geq \ln\Big[\lambda\Big(\prod_{i=1}^{n} x_i\Big)^{\frac{1}{n}}\Big]$$

Therefore

$$k \geq \lambda\Big(\prod_{i=1}^{n} x_i\Big)^{\frac{1}{n}} \qquad (1$$

Combining eqs. (18) and (19), the maximum k, and therefore the $\hat{k}_{MLE}$ is

$$\lambda\Big(\prod_{i=1}^{n} x_i\Big)^{\frac{1}{n}} \leq \hat{k}_{MLE} < 1 + \lambda\Big(\prod_{i=1}^{n} x_i\Big)^{\frac{1}{n}} \qquad (2$$

where $\hat{k}_{MLE}$ is an integer.

Note that since λ appears in the solution of $\hat{k}_{MLE}$ and k appears in the solution of $\hat{\lambda}_{MLE}$, the maximum likelihood estimates are not obtained analytically. They can only be obtained numerically through the solution of simultaneous eqs. (17) and (20).

Similar procedures can be applied to eq. (11) to derive $\hat{\beta}_{MLE}$ and $\hat{\alpha}_{MLE}$:

$$\hat{\beta}_{MLE} = \frac{n\alpha}{\sum_{i=1}^{n} y_i} = \frac{\alpha}{\bar{y}} \qquad (2$$

where $\bar{y} = \sum_{i=1}^{n} y_i/n$

and

$$\beta\left(\prod_{i=1}^{n} y_i\right)^{\frac{1}{n}} \leq \hat{\alpha}_{MLE} \leq 1 + \beta\left(\prod_{i=1}^{n} y_i\right)^{\frac{1}{n}} \qquad (2$$

where $\hat{\alpha}_{MLE}$ is an integer

Simultaneous solution of eqs. (21) and (22) will give numerical estimates of $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$.

Therefore, the maximum likelihood estimates of system availability, $\hat{g}_{MLE}$, when cycle time and on time are gamma distributed is simply eqs. (17), (20), (21) and (22) substituted into eq. (9).

3.1.2 Assuming Exponentially Distributed T and T_{on} or T_{on} and T_{off}

When T and T_{on} are exponentially distributed, the density functions $f_T(x)$ and $f_{T_{on}}(y)$ are exactly eqs. (1) and (2) with $k=\alpha=1$, i.e.,

$$X \sim G(1, \lambda) : \quad f_T(x) = \lambda\, e^{-\lambda x} \qquad (2$$

with $E(T) = \frac{1}{\lambda}$, $VAR(T) = \frac{1}{\lambda^2}$

$$Y \sim G(1, \beta) : \quad f_{T_{on}}(y) = \beta\, e^{-\beta y} \qquad (2$$

with $E(T_{on}) = \frac{1}{\beta}$, $VAR(T_{on}) = \frac{1}{\beta^2}$

Since $k=\alpha=1$ is known, the maximum likelihood estimates are only needed for parameters λ and β. No simultaneous solutions are needed. Substituting $k=1$ into eq. (17) leaves

$$\hat{\lambda}_{MLE} = \frac{n}{\sum_{i=1}^{n} x_i} \quad \text{or} \quad \frac{1}{\bar{x}} \tag{25}$$

Substituting $\alpha=1$ into eq. (21) leaves

$$\hat{\beta}_{MLE} = \frac{n}{\sum_{i=1}^{n} y_i} \quad \text{or} \quad \frac{1}{\bar{y}} \tag{26}$$

Using the simplified expression for availability found in Chapter 2 [eq. (53)]

$$g(t;\ \lambda\ \ \beta) = \frac{\lambda}{\beta} + (1 - \frac{\lambda}{\beta})\ e^{-\beta t}, \tag{27}$$

and substituting in the MLE's for λ and β, the maximum likelihood estimate of system availability when cycle time and on time are exponentially distributed is:

$$\hat{g}_{MLE}(t;\ \hat{\lambda}_{MLE},\ \hat{\beta}_{MLE}) = \frac{\hat{\lambda}_{MLE}}{\hat{\beta}_{MLE}} + (1 - \frac{\hat{\lambda}_{MLE}}{\hat{\beta}_{MLE}})\ e^{-\hat{\beta}_{MLE} t}$$

$$= \frac{\bar{y}}{\bar{x}} + (1 - \frac{\bar{y}}{\bar{x}})\ e^{\frac{t}{\bar{y}}}$$

$$\hat{g}_{MLE}(t;\ \hat{\lambda}_{MLE},\ \hat{\beta}_{MLE}) = \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i} + (1 - \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i})e^{-\frac{nt}{\sum_{i=1}^{n} y_i}} \tag{28}$$

Note that as $n \to \infty$ or as $t \to \infty$, or both,

$$\hat{g}_{MLE} \to \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i}$$

When T_{on} and T_{off} are exponentially distributed, the density functions $f_{T_{on}}(y)$ and $f_{T_{off}}(z)$ are exactly eqs. (2) and (3) with α=m=1, i.e.,

$$Y \sim G(1, \beta): f_{T_{on}}(x) = \beta\ e^{-\beta y} \tag{29}$$

$$\text{with } E[T_{on}] = \frac{1}{\beta}\ ,\ VAR[T_{on}] = \frac{1}{\beta^2}$$

$$Z \sim G(1, \eta): f_{T_{off}}(z) = \eta\ e^{-\eta z} \tag{30}$$

$$\text{with } E[T_{off}] = \frac{1}{\eta},\ VAR[T_{off}] = \frac{1}{\eta^2}$$

Similar to the previous example, the MLE's for β and η are

$$\hat{\beta}_{MLE} = \frac{n}{\sum_{i=1}^{n} y_i} \quad \text{or} \quad \frac{1}{\bar{y}} \tag{31}$$

$$\hat{\eta}_{MLE} = \frac{n}{\sum_{i=1}^{n} z_i} \quad \text{or} \quad \frac{1}{\bar{z}} \tag{32}$$

therefore, the $\hat{g}_{MLE}$ when on and off times are used (using the availability expression in eq. (67) of Chapter 2) is,

$$\hat{g}_{MLE}(t;\ \hat{\beta}_{MLE},\ \hat{\eta}_{MLE}) = \frac{\hat{\eta}_{MLE}}{\hat{\beta}_{MLE} + \hat{\eta}_{MLE}} + \left(1 - \frac{\hat{\eta}_{MLE}}{\hat{\beta}_{MLE} + \hat{\eta}_{MLE}}\right) \cdot$$

$$e^{-(\hat{\beta}_{MLE} + \hat{\eta}_{MLE})t}$$

$$= \frac{\frac{1}{\bar{z}}}{\frac{1}{\bar{y}} + \frac{1}{\bar{z}}} + \left(1 - \frac{\frac{1}{\bar{z}}}{\frac{1}{\bar{y}} + \frac{1}{\bar{z}}}\right) e^{\left\{\left(\frac{1}{\bar{y}} + \frac{1}{\bar{z}}\right)\right\}t}$$

$$\hat{g}_{MLE}(t;\ \hat{\beta}_{MLE},\ \hat{\eta}_{MLE}) = \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} z_i} + \left(1 - \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} z_i}\right) \cdot$$

$$\exp\left\{-n\left[\frac{\left(\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} z_i\right)}{\sum_{i=1}^{n} y_i \sum_{i=1}^{n} z_i}\right] t\right\} \tag{33}$$

3.1.3 Classical Confidence Intervals

To obtain a classical confidence interval for $g(t;\ k,\ \lambda,\ \alpha,\ \beta)$ simply obtain confidence intervals for each of the parameters, k, λ, α, β, and substitute them into the availability expression, much like what was done for the $\hat{g}_{MLE}$. But, since confidence intervals, unlike point estimates, are extremely hard, if not impossible, to solve for simultaneously, assume two of the parameters are fixed. Then obtain the confidence intervals for the remaining two parameters and substitute these values to form the confidence interval for the instantaneous availability function.

Let k and α in eqs. (1) and (2) be constant positive integers. Since X is a gamma random variable with parameters k and λ, $2\lambda X$ is a chi-square variable with 2k degrees of freedom, i.e.,

$$2\lambda X \sim \chi^2(2k) \tag{34}$$

For proof, see Appendix C.

In general let $\chi_a^2(r)$ denote the value of a chi-square variable having r degrees of freedom such that

$$\Pr[\chi^2(r) \leq \chi_a^2(r)] = a \tag{35}$$

Therefore,

$$\Pr[\chi^2_{\frac{\gamma}{2}}(2k) \leq 2\lambda X \leq \chi^2_{1-\frac{\gamma}{2}}(2k)] = 1-\gamma$$

or

$$\Pr\left[\frac{\chi^2_{\frac{\gamma}{2}}(2k)}{2X} \leq \lambda \leq \frac{\chi^2_{1-\frac{\gamma}{2}}(2k)}{2X}\right] = 1-\gamma$$

The 100 $(1-\gamma)$% confidence interval for λ is then

$$\frac{\chi^2_{\frac{\gamma}{2}}(2k)}{2X} \leq \lambda \leq \frac{\chi^2_{1-\frac{\gamma}{2}}(2k)}{2X} \tag{36}$$

where $\chi^2_{\frac{\gamma}{2}}(2k)$ and $\chi^2_{1-\frac{\gamma}{2}}(2k)$ are obtained from a chi-square table.

For X use E[X].

Similarly, since Y is a gamma random variable with parameters α and β, and $2\beta Y \sim \chi^2(2\alpha)$, the $100(1-\gamma)\%$ confidence interval for β is

$$\frac{\chi^2_{\frac{\gamma}{2}}(2\alpha)}{2Y} \leq \beta \leq \frac{\chi^2_{1-\frac{\gamma}{2}}(2\alpha)}{2Y} \qquad (37)$$

Substituting eqs. (36) and (37) into eq. (4), and knowing k and α are constant positive integers, the $100(1-\gamma)\%$ confidence interval for $g(t;\, k, \lambda, \alpha, \beta)$ is

$$\sum_{\ell=0}^{\alpha-1} P_0\left(\ell;\ \frac{\chi^2_{1-\frac{\gamma}{2}}(2\alpha)}{2Y}\, t\right) + \frac{\chi^2_{\frac{\gamma}{2}}(2k)}{2X} \cdot \int_0^t \left\{\sum_{\ell=0}^{\alpha-1} P_0\left[\ell;\ \frac{\chi^2_{1-\frac{\gamma}{2}}(2\alpha)}{2Y}(t-s)\right]\right\} \cdot$$

$$\left\{\sum_{\substack{q=k\\2k\\ \vdots}}^{\infty} P_0\left[(q-1);\ \frac{\chi^2_{1-\frac{\gamma}{2}}(2k)}{2X}\, s\right]\right\} ds \leq g(t;\, k, \lambda, \alpha, \beta) \leq$$

$$\sum_{\ell=o}^{\alpha-1} P_0\left(\ell;\ \frac{\chi^2_{\frac{\gamma}{2}}(2\alpha)}{2Y}\, t\right. + \frac{\chi^2_{1-\frac{\gamma}{2}}(2k)}{2X} \cdot \int_0^t \left\{\sum_{\ell=0}^{\alpha-1} P_0\ \ell;\ \frac{\chi^2_{\frac{\gamma}{2}}(2\ )}{2Y}(t-s)\right\} \cdot$$

$$\left\{\sum_{\substack{q=k\\2k\\ \vdots}}^{\infty} P_0\left[(q-1);\ \frac{\chi^2_{\gamma/2}(2k)}{2X}\right]\right\} ds \qquad (38)$$

The $100(1-\gamma)\%$ confidence interval for system availability when T and T_{on} are exponentially distributed (i.e., $k=\alpha=1$) is:

$$\frac{Y \cdot \chi^2_{\frac{\gamma}{2}}(2)}{X \cdot \chi^2_{1-\frac{\gamma}{2}}(2)} + \left[1 - \frac{Y \cdot \chi^2_{\frac{\gamma}{2}}(2)}{X \cdot \chi^2_{1-\frac{\gamma}{2}}(2)}\right] e^{-\left(\frac{\chi^2_{\frac{\gamma}{2}}(2)}{2Y}\right)t} \leq g(t;\lambda,\beta)$$

$$\leq \frac{Y \cdot \chi^2_{1-\frac{\gamma}{2}}(2)}{X \cdot \chi^2_{\frac{\gamma}{2}}(2)} + \left[1 - \frac{Y \cdot \chi^2_{1-\frac{\gamma}{2}}(2)}{X \cdot \chi^2_{\frac{\gamma}{2}}(2)}\right] e^{-\left(\frac{\chi^2_{\frac{\gamma}{2}}(2)}{2Y}\right)t} \qquad (39$$

And when T_{on} and T_{off} exponentially distributed data are used, the $100\,(1-\gamma)\%$ confidence interval (using eq. (67) in Chapter 2 for g(t)) is,

$$\frac{Y \cdot \chi^2_{\frac{\gamma}{2}}(2)}{2(Y+Z)\,\chi^2_{1-\frac{\gamma}{2}}(2)} + \left(1 - \frac{Y \cdot \chi^2_{\frac{\gamma}{2}}(2)}{2(Y+Z)\cdot \chi^2_{1-\frac{\gamma}{2}}(2)}\right) e^{-\left[\chi^2_{\frac{\gamma}{2}}(2)\left(\frac{Y+Z}{2YZ}\right)\right]t}$$

$$\leq g(t;\beta,\eta) \leq \frac{Y \cdot \chi^2_{1-\frac{\gamma}{2}}(2)}{2(Y+Z)\chi^2_{\frac{\gamma}{2}}(2)} + \left(1 - \frac{Y \cdot \chi^2_{1-\frac{\gamma}{2}}}{2(Y+Z)\cdot\chi^2_{\frac{\gamma}{2}}(2)}\right) \cdot \qquad (40$$

$$e^{-\left[\chi^2_{\frac{\gamma}{2}}(2)\left(\frac{Y+Z}{2YZ}\right)\right]t}$$

3.2 Traditional Bayesian Estimate

The primary mathematical tool used in Bayesian analysis is Bayes' theorem, named after Thomas Bayes who studied this topic in the mid-18th century. Crellin [4] discusses the philosophy and mathematics of the theorem along with its reliability applications.

Basically, Bayes' theorem incorporates two sources of information about the parameters of a model. The first source, called a priori information, represents the totality of knowledge available about the parameters before any observation of data takes place. This information is mathematically summarized into a prior distribution or model. The second source is simply the observed data. Bayes' theorem combines the prior model with the observed sample data to form a posterior model, upon which various inferences are made about the parameter Note that this posterior model can, subsequently be used as the prior if, another data observation takes place to form another posterior model, and so on. Summarizing, if any decision or inferences are made when using the posterior model this implies both the prior model and sample data information influenced the decision.

Statement of Bayes' Theorem

Let $f(t_i \mid \theta)$ denote the data model for an observation t_i on a variable T given θ is the parameter used in describing T. θ is a random variable. If $p(\theta)$ is the prior model for the parameter vector θ, and if a sample $(t_1, t_2, \ldots, t_n)$ of n independent observations on T is observed, then, given the observations, the posterior model for θ using Bayes theorem is defined as:

$$h(\theta \mid t_1, t_2, \ldots, t_n) = \frac{p(\theta) \prod_{i=1}^{n} f(t_i \mid \theta)}{\int_{k \in \Omega} p(k) \sum_{i=1}^{n} f(t_i \mid k)\, dk} \tag{41}$$

where k is the integration variable representing the different values of θ and Ω is the parameter space of θ. The numerator, $p(\theta) \prod_{i=1}^{n} f(t_i \mid \theta)$ is also known as th joint distribution composed of the prior distribution and the "conditional" data distribution. The denominator, $\int_{k \epsilon \Omega} p(k)\, f(t_i \mid k) dk$, is also known as the marginal distribution composed of the joint distribution integrated over all possible values of k.

3.2.1 Assuming Gamma Distributed T and T_{on}

To implement Bayesian analysis in the estimation of the availability function g(t), the joint distributions of the parameters k, λ, α and β must be assigned. However, in practice, this assignment is too complicated to derive analytically. Therefore, it is usually possible to fix one of the two parameters in a gamma distribution while leaving the other floating with certain variation. So, to approach this problem, let k and α be fixed constant positive integers, with λ and β varying.

The λ and β will vary according to negative exponential distributions

$$f_\lambda(\lambda) = \mu e^{-\mu\lambda} \tag{4:}$$

$$f_\beta(\beta) = \nu e^{-\nu\beta} \tag{4.}$$

where μ and ν are undetermined positive constants and λ and β are positive and independent random numbers. The $f_\lambda(\lambda)$ and $f_\beta(\beta)$ will be known as the prior distributions of λ and β, respectively.

To find the posterior distributions of λ and β, use Bayes' theorem to combine eqs. (10) and (42) for λ and eqs. (11) and (43) for β. Letting $f_\lambda(\lambda;\, x_1, x_2, \ldots, x_n)$ represent the posterior distribution of λ given the pooled sample of $(x_1, x_2, \ldots x_n)$;

$$f_\lambda(\lambda;\, x_1, x_2, \ldots, x_n) = \frac{f_\lambda(\lambda) \cdot L(x_1, x_2, \ldots, x_n)}{\int_0^\infty f_\lambda(\lambda) \cdot L(x_1, x_2, \ldots, x_n)\, d\lambda}$$

$$= \frac{\mu e^{-\mu\lambda} \left(\frac{\lambda^k}{(k-1)!}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{k-1} e^{-\lambda \sum_{i=1}^{n} x_i}}{\int_0^\infty \mu e^{-\mu\lambda} \cdot \left[\frac{\lambda^k}{(k-1)!}\right]^n \left(\prod_{i=1}^{n} x_i\right)^{k-1} e^{-\lambda \sum_{i=1}^{n} x_i} d\lambda}$$

$$= \frac{\lambda^{kn} e^{-\lambda(\mu + \sum_{i=1}^{n} x_i)}}{\int_0^\infty \lambda^{kn} e^{-\lambda(\mu + \sum_{i=1}^{n} x_i)} d\lambda} \qquad (4$$

To simplify further, let

$$w = \lambda(\mu + \sum_{i=1}^{n} x_i)$$

then $\lambda = \dfrac{w}{\mu + \sum_{i=1}^{n} x_i}$, $\quad d\lambda = \dfrac{dw}{\mu + \sum_{i=1}^{n} x_i}$

and when $\lambda = 0 \rightarrow w = 0$

$\lambda = \infty \rightarrow w = \infty$

Eq. (44) then becomes

$$f_\lambda(\lambda; x_1, x_2, \ldots, x_n) = \frac{\lambda^{kn} e^{-\lambda(\mu + \sum_{i=1}^{n} x_i)}}{\int_0^\infty \left(\frac{w}{\mu + \sum_{i=1}^{n} x_i}\right)^{kn} e^{-w} \frac{dw}{\mu + \sum_{i=1}^{n} x_i}}$$

$$= \frac{(\mu + \sum_{i=1}^{n} x_i)^{kn+1} \lambda^{kn} e^{-\lambda(\mu + \sum_{i=1}^{n} x_i)}}{\int_0^\infty w^{kn} e^{-w} dw}$$

Note the denominator is the definition of a gamma function, so

$$f_\lambda(\lambda;\, x_1, x_2, \ldots, x_n) = \frac{(\mu + \sum_{i=1}^{n} x_i)}{\Gamma(kn+1)} \left[\lambda(\mu + \sum_{i=1}^{n} x_i)\right]^{kn} e^{-\lambda(\mu + \sum_{i=1}^{n} x_i)} \tag{45}$$

which is a gamma pdf with parameters

$$(kn+1,\ \mu + \sum_{i=1}^{n} x_i)$$

Similarly, the posterior distribution function of β given the pooled sample $(y_1, y_2, \ldots, y_n)$ is

$$f_\beta(\beta;\, y_1, y_2, \ldots, y_n) = \frac{(\nu + \sum_{i=1}^{n} y_i)}{\Gamma(\alpha n + 1)} \left[\beta(\nu + \sum_{i=1}^{n} y_i)\right]^{\alpha n} e^{-\beta(\nu + \sum_{i=1}^{n} y_i)} \tag{46}$$

which is a gamma pdf with parameters

$$(\alpha n+1,\ \nu + \sum_{i=1}^{n} y_i)$$

See Table 3-1 for a summary of all the distributions used.

Now that the posterior distributions are established, what is the Bayesian availability estimate? Here, a squared error loss function is assumed, so via Bayesian analysis [8] the Bayesian estimator, $\hat{g}_B(t;\, k, \lambda, \alpha, \beta)$, is the function which minimizes the expected value of the loss function with respect to the posterior distributions (or prior distributions when no data are available) of λ and β. In terms of the availability, the availability estimate $\hat{g}_B(t;\, k, \lambda, \alpha, \beta)$, is the mean or expected value of $g(t;\, k, \lambda\ \alpha\ \beta)$, i.e.,

TABLE 3-1: The Gamma Density Functions Used for the Traditional Bayesian Estimator

	Time	Distribution	Density Function
PRIORS	cycle	$\lambda \sim G(1,\frac{1}{\mu})$	$f_\lambda(\lambda) = \mu e^{-\mu\lambda}$
	on	$\beta \sim G(1,\frac{1}{\nu})$	$f_\beta(\beta) = \nu e^{-\nu\beta}$
	off	$\eta \sim G(1,\frac{1}{\gamma})$	$f_\eta(\eta) = \gamma e^{-\gamma\eta}$
CONDITIONALS	cycle	$X \sim G(k,\frac{1}{\lambda})$	$f_T(x) = \frac{\lambda}{(k-1)!} (\lambda x)^{k-1} e^{-\lambda x}$
	on	$Y \sim G(\alpha,\frac{1}{\beta})$	$f_{T_{on}}(y) = \frac{\beta}{(\alpha-1)!} (\beta y)^{\alpha-1} e^{-\beta y}$
	off	$Z \sim G(m,\frac{1}{\eta})$	$f_{T_{off}}(z) = \frac{\eta}{(m-1)!} (\eta z)^{m-1} e^{-\eta z}$
POSTERIORS	cycle	$\lambda \sim G(kn+1,\frac{1}{\mu+\sum_{i=1}^{n} x_i})$	$f_\lambda(\lambda; x_1,\ldots,x_n) = \frac{(\mu + \sum_{i=1}^{n} x_i)^{kn+1} \lambda^{kn} e^{-\lambda(\mu+\sum_{i=1}^{n} x_i)}}{\Gamma(kn+1)}$
	on	$\beta \sim G(\alpha n+1,\frac{1}{\nu+\sum_{i=1}^{n} y_i})$	$f_\beta(\beta; y_1,\ldots,y_n) = \frac{(\nu + \sum_{i=1}^{n} y)^{\alpha n+1} \beta^{\alpha n} e^{-\beta(\nu+\sum_{i=1}^{n} y_i)}}{\Gamma(\alpha n+1)}$

TABLE 3-1 (cont).

off	$\eta \sim G\left(mn+1, \dfrac{1}{\gamma + \sum_{i=1}^{n} z_i}\right)$	$f_\eta(\eta;\, z_1, \ldots, z_n) = \dfrac{\left(\gamma + \sum_{i=1}^{n} z_i\right)^{mn+1} \eta^{mn}\, e^{-\eta\left(\gamma + \sum_{i=1}^{n} z_i\right)}}{\Gamma(mn+1)}$

where $X \sim G(a,b)$ is defined as $\dfrac{b^a x^{a-1} e^{-bx}}{\Gamma(a)}$

where $\mu = a/b$ and $\sigma^2 = a/b^2$

Note: These three distributions are listed together for convenience. They are always used in pairs of T and T_{on} or T_{on} and T_{off} only. No assumptions are made on the additivity of these distributions.

$$\hat{g}_B(t;\ k,\alpha) = \int_0^\infty \int_0^\infty g(t;\ k,\ \lambda,\alpha,\beta)\ f_\lambda(\lambda)\ f_\beta(\beta)\ d\lambda d\beta \tag{47}$$

when no data are available and

$$\hat{g}_B(t;\ k,\ \lambda\ \alpha\ \beta) = \int_\theta^\infty \int_0^\infty g(t;\ k,\ \lambda,\alpha,\beta)\ f_\lambda(\lambda;\ x_1,\ x_2,\ \ldots,\ x_n)\ \cdot$$

$$f_\beta(\beta;\ y_1,\ y_2,\ \ldots,\ y_n)\ d\lambda d\beta \tag{48}$$

when data in the form of samples

$(x_i,\ y_i)$ i = 1, 2, ..., n are available.

Substituting in eqs. (4), (42) and (43), eq. (47) becomes

$$\hat{g}_B(t;\ k,\ \alpha) = \int_0^\infty \int_0^\infty \left[\sum_{\ell=0}^{\alpha-1} P_0(\ell;\beta t) + \lambda \int_0^t \{ \sum_{\ell=0}^{\alpha-1} P_0[\ell;\ \beta(t-s)]\} \cdot \right.$$

$$\left. \{ \sum_{q=k,2k,\ldots}^{\infty} P_0[(q-1);\ \lambda s]\}\ ds \right] \mu e^{-\mu\lambda}\ \nu e^{-\nu\beta}\ d\lambda d\beta$$

$$= \int_0^\infty \int_0^\infty \mu\nu e^{-\mu\lambda}\ e^{-\nu\beta} \sum_{\ell=0}^{\alpha-1} P_0(\ell;\ \beta t)\ d\lambda d\beta\ +$$

$$\int_0^\infty \int_0^\infty \mu\nu\lambda e^{-\mu\lambda} e^{-\nu\beta} \int_0^t \{ \sum_{\ell=0}^{\alpha-1} P_0[\ell;\ \beta(t-s)]\} \cdot$$

$$\{ \sum_{q=k,2k,\ldots}^{\infty} P_0\ [(q-1);\ \lambda s]\}\ ds\ d\lambda d\beta$$

$$= \int_0^\infty \sum_{\ell=0}^{\alpha-1} P_0(\ell;\ \beta t) \nu e^{-\nu\beta}\ d\beta \int_0^\infty \mu e^{-\mu\lambda} d\lambda$$

$$+ \mu\nu \int_0^\infty \int_0^\infty \int_0^t \lambda \cdot e^{-\mu\lambda} e^{-\nu\beta}\ \{ \sum_{\ell=0}^{\alpha-1} P_0\ [\ell;\ \beta(t-s)]\} \cdot$$

$$\{ \sum_{q=k,\ 2k,\ \ldots}^{\infty} P_0[(q-1);\ \lambda s]\}\ ds d\lambda d\beta$$

Therefore the traditional Bayesian estimate assuming squared error loss function with no data observed is

$$\hat{g}_B(t;\ k,\alpha) = \int_0^{\infty} \sum_{\ell=0}^{\alpha-1} P_0(\ell;\ \beta t)\ \ \nu e^{-\nu\beta} d\beta\ +$$

$$\mu\nu \int_0^t \left[\int_0^{\infty} \lambda e^{-\mu\lambda} \sum_{q=k,\ 2k\ \ldots}^{\infty} P_0[(q-1);\ \lambda s]\ d\lambda \right] \cdot$$

$$\int_0^{\infty} e^{-\nu\beta} \sum_{\ell=0}^{\alpha-1} P_0[\ell;\ \beta(t-s)]\ d\beta]\ ds \tag{49}$$

Now to find the traditional Bayesian estimate when observed data are available substitute eqs. (4), (45) and (46) into eq. (48)

$$\hat{g}_B(t;\ k,\ \lambda,\alpha,\beta) = \int_0^{\infty} \int_0^{\infty} \sum_{\ell=0}^{\alpha-1} P_0(\ell;\ \beta t) + \lambda \int_0^t \{ \sum_{\ell=0}^{\alpha-1} P_0[\ell;\ \beta(t-s)]\}$$

$$\cdot \{ \sum_{\substack{q=k, \\ 2k\ldots}}^{\infty} P[(q-1);\lambda s]\}\ ds] \cdot \frac{\mu + \sum_{i=1}^{n} x_i}{\Gamma(kn+1)} [\lambda(\mu + \sum_{i=1}^{n} x_i)]^{kn}$$

$$\cdot\ e^{-\lambda(\mu + \sum_{i=1}^{n} x_i)} \cdot \frac{\nu + \sum_{i=1}^{n} y_i}{\Gamma(\alpha n+1)} [\beta(\nu + \sum_{i=1}^{n} y_i)]^{-\alpha n}\ e^{-\beta(\nu + \sum_{i=1}^{n} y_i)}\ d\lambda d\beta$$

Letting $d = \mu + \sum_{i=1}^{n} x_i$ $\qquad f = \Gamma(kn+1)$

$h = \nu + \sum_{i=1}^{n} y_i$ $\qquad a = \Gamma(\alpha n+1);$

$$\hat{g}_B(t;\ k,\ \lambda,\alpha,\beta) = \int_0^\infty \int_0^\infty \frac{d}{f}\ (\lambda d)^{kn}\ e^{-\lambda d} \cdot \frac{h}{a}\ (\beta h)^{\alpha n}\ e^{-\beta h}\ \cdot$$

$$\sum_{\ell=0}^{\alpha-1} P\ (\lambda;\ \beta t)\ d\lambda d\beta\ +$$

$$\int_0^\infty \int_0^\infty \frac{d}{f}\ (\lambda d)^{kn}\ e^{-\lambda d}\ \frac{h}{a}\ (\beta h)^{\beta h} e \lambda \int_0^t \{\sum_{\ell=0}^{\alpha-1} P\ [\ell;\ \beta(t-s)]\} \cdot$$

$$\{\sum_{q=k,2k,\ldots}^{\infty} P_0\ [(q-1);\lambda s]\}\ ds\ d\lambda d\beta$$

But, since $\frac{1}{d} \int_0^\infty [\lambda d]^{kn}\ e^{-\lambda d}(d)d\lambda = \frac{\Gamma\ (kn+1)}{d} = \frac{f}{d}$

due to the definition of a gamma function, the traditional Bayesian estimate assuming squared error loss function when independent samples (x_i, y_i), $i = 1,2,\ldots,n$ are observed is

$$\hat{g}_B(t;\ k,\ \lambda,\alpha,\beta) = \frac{h}{a} \int_0^\infty (\beta h)^{\alpha n}\ e^{-\beta h} \sum_{\ell=0}^{\alpha-1} P_0(\ell;\ \beta t)\ d\beta\ +$$

$$\frac{dh}{fa} \int_0^t \{\int_0^\infty \lambda\ (\lambda d)^{kn}\ e^{-\lambda d} \sum_{q=k,2k,\ldots}^{\infty} P_0[(q-1);\lambda s]\ d\lambda\} \cdot$$

$$\{\int_0^\infty (\beta h)^{\alpha n}\ e^{-\beta h} \sum_{\ell=0}^{\alpha-1} P_0[\ell;\ \beta(t-s)]\ d\beta\}\ ds \tag{50}$$

3.2.2 Assuming Exponentially Distributed T_{on} and T_{off}

When the on and off times of a system are exponentially distributed, the density functions are eqs. (29) and (30), respectively. The prior distributions on β and η are

$$f_\beta(\beta) = \nu e^{-\nu\beta} \tag{51}$$

$$f_\eta(\eta) = \gamma e^{-\gamma\eta} \tag{52}$$

using the same Bayesian approach as before, keeping in mind $\alpha=m=1$, the posterior distributions of β and η are

$$f_\beta(\beta;\, y_1,\, \ldots,\, y_n) = \frac{\nu + \sum_{i=1}^{n} y_i}{\Gamma\,(n+1)} \left[\beta(\nu + \sum_{i=1}^{n} y_i) \right]^n e^{-\beta(\nu + \sum_{i=1}^{n} y_i)} \tag{53}$$

$$f_\eta(\eta;\, z_1,\, \ldots,\, z_n) = \frac{\gamma + \sum_{i=1}^{n} z_i}{\Gamma\,(n+1)} \left[\eta(\gamma + \sum_{i=1}^{n} z_i) \right]^n e^{-\eta(\gamma + \sum_{i=1}^{n} z_i)} \tag{54}$$

To derive the Bayesian estimates of the data and no-data cases, use the availability function of eq. (67) in Chapter 2, and find its expected value with respect to either the posterior or prior distributions.

Recall,

$$g(t;\beta,\eta\,) = \frac{\eta}{\eta+\beta} + \left(1 - \frac{\eta}{\eta+\beta} \right) e^{-(\eta+\beta)t} \tag{55}$$

so,

$$\hat{g}_B(t) = \int_0^\infty \int_0^\infty g(t;\beta,\eta) \;\; f_\beta(\beta) \;\; f_\eta(\eta) \;\; d\beta d\eta \tag{56}$$

for the no-data case, and

$$\hat{g}_B(t;\, \beta,\eta) = \int_0^\infty \int_0^\infty g(t;\, \beta,\eta)\; f_\beta(\beta;\, y_1,\, \ldots,\, y_n) \cdot$$

$$f_\eta(\eta;\, z_1,\, \ldots,\, z_n)\; d\beta d\eta \tag{57}$$

when data in the form of samples $(y_1,\, z_i)$, $i = 1,\, \ldots,\, n$ are available.

To find the analytical form of the no-data case of the Bayesian estimate, substitute in eqs. (51), (52) and (55) into eq. (56):

$$\hat{g}_B(t) = \int_0^\infty \int_0^\infty \left[\frac{\eta}{\eta + \beta} + (1 - \frac{\eta}{\eta + \beta}) \, e^{-(\eta+\beta)t} \right] \nu e^{-\nu\beta} \, \gamma e^{-\gamma\eta} \, d\eta d\beta \qquad (5$$

Let $\beta + \eta = \beta'$

which means $\beta = \beta' - \eta, \quad d\beta = d\beta'$

and when $\quad \beta = 0 \quad \rightarrow \quad \beta' = \eta$

$\qquad\qquad \beta = \infty \quad \rightarrow \quad \beta' = \infty$

After the above transformation, eq. (58) becomes

$$\hat{g}_B(t) = \int_0^\infty \left\{ \int_\eta^\infty \left[\frac{\eta}{\beta'} + (1 - \frac{\eta}{\beta'}) \, e^{-\beta' t} \right] \nu\gamma e^{-\nu(\beta' - \eta)} \, . \, e^{-\gamma\eta} \, d\beta' \right\} d\eta$$

$$= \nu\gamma \int_0^\infty \left\{ \int_\eta^\infty \left[e^{-\beta' t} + \frac{\eta}{\beta'} \, (1 - e^{-\beta' t}) \right] e^{-\nu(\beta' - \eta)} \, . \, e^{-\gamma\eta} \, d\beta' \right\} d\eta$$

$$= \nu\gamma \int_0^\infty \left\{ \int_\eta^\infty e^{-\beta' t} \, e^{-\nu(\beta' - \eta)} \, e^{-\gamma\eta} \, d\beta' \right\} d\eta \; +$$

$$\nu\gamma \int_0^\infty \left\{ \int_\eta^\infty \frac{\eta}{\beta'} \, (1 - e^{-\beta' t}) \, e^{-\,(\beta' - \eta)} \, e^{-\gamma\eta} \, d\beta' \right\} d\eta$$

$$\hat{g}_B(t) = A + B \qquad (59)$$

where

$$A = \nu\gamma \int_0^\infty \left\{ \int_\eta^\infty e^{-\beta' t} e^{-\nu(\beta' - \eta)} e^{-\gamma\eta} \, d\beta' \right\} d\eta$$

$$= \nu\gamma \int_0^\infty e^{-\eta(\gamma-\nu)} \left\{ \int_\eta^\infty e^{-\beta'(t+\nu)} \, d\beta' \right\} d\eta$$

$$= \nu\gamma \int_0^\infty e^{-\eta(\gamma-\nu)} \left\{ \frac{-1}{t+\nu} [e^{-\beta'(t+\nu)}]_\eta^\infty \right\} d\eta$$

$$= \frac{\nu\gamma}{t+\nu} \int_0^\infty e^{-\eta(\gamma-\nu)} e^{-\eta(t+\nu)} \, d\eta$$

$$= \frac{\nu\gamma}{t+\nu} \int_0^\infty e^{-\eta(\gamma+t)} \, d\eta$$

$$A = \frac{\nu\gamma}{(t+\nu)\ (\gamma+t)} \qquad (60)$$

and

$$B = \nu\gamma \int_0^\infty \left\{ \int_0^\infty \frac{\eta}{\beta'} (1 - e^{-\beta' t}) \, e^{-\nu(\beta' - \eta)} \, . \, e^{-\gamma\eta} \, d\beta' \right\} d\eta$$

$$= \nu\gamma \int_0^\infty \int_\eta^\infty \eta \, e^{-\eta(\gamma-\nu)} \, \frac{1}{\beta'} \, (1 - e^{-\beta' t}) \, e^{-\nu\beta'} \, . \, d\beta' d\eta \qquad (61)$$

which converges to a value. The proof of convergence is given in Appendix D.

So, substituting eqs. (60) and (61) into eq. (59), the traditional Bayesian estimate of availability when a system has exponentially distributed on and off times, given there is no data is

$$\hat{g}_B(t) = \frac{\nu\gamma}{(t+\nu)(t+\gamma)} + \nu\gamma \int_0^\infty \int_\eta^\infty \eta \, e^{-\eta(\gamma-\nu)} \, \frac{1}{\beta} \, (1 - e^{-\beta' t}) \, e^{-\nu\beta'} \, . \, d\beta' d\eta \qquad (62)$$

When sample data are available, substitute eqs. (53), (54) and (55) into eq. (57) to find the traditional Bayesian estimate:

$$\hat{g}_B(t;\ \beta,\eta) = \int_0^\infty \int_0^\infty \left[\left[\frac{\eta}{\eta + \beta}\right] + \left(1 - \frac{\eta}{\eta + \beta}\right) e^{-(\eta+\beta)t}\right.\ .$$

$$\left\{\frac{\nu + \sum_{i=1}^{n} y_i}{\Gamma(n+1)} \left[\beta(\nu + \sum_{i=1}^{n} y_i)\right]^n e^{-\beta(\nu + \sum_{i=1}^{n} y_i)}\right\}\ .$$

$$\left\{\frac{\gamma + \sum_{i=1}^{n} z_i}{\Gamma(n+1)} \left[\eta(\gamma + \sum_{i=1}^{n} z_i)\right]^n e^{-\eta(\gamma + \sum_{i=1}^{n} z_i)}\right\}\cdot d\beta d\eta \tag{63}$$

Let $h = \nu + \sum_{i=1}^{n} y_i$

$k = \gamma + \sum_{i=1}^{n} z_i$

and

$f = \Gamma(n+1)$

so

$$\hat{g}_B(t;\ \beta,\eta) = \frac{(hk)^{n+1}}{f^2} \int_0^\infty \int_0^\infty \left[\frac{\eta}{\eta + \beta} + (1 - \frac{\eta}{\eta + \beta})\ e^{-(\eta+\beta)t}\right]\cdot$$

$$\eta^n \beta^n\ e^{-h\beta}\ e^{-k\eta}\ d\beta d\eta \tag{64}$$

$$= \frac{(hk)^{n+1}}{f^2} \int_0^\infty \int_0^\infty \left[e^{-(\eta+\beta)t} + \frac{\eta}{\eta + \beta}\ (1-e^{-(\eta+\beta)t})\right]\cdot$$

$$\eta^n \beta^n\ e^{-h\beta}\ e^{-k\eta}\ d\beta d\eta$$

$$= \frac{(hk)^{n+1}}{f^2} \int_0^\infty \int_0^\infty e^{-(\eta+\beta)t} \eta^n \beta^n e^{-h\beta} e^{-k\eta} \, d\beta d\eta +$$

$$\frac{(hk)^{n+1}}{f^2} \int_0^\infty \int_0^\infty \frac{\eta}{\eta + \beta} (1 - e^{-(\eta+\beta)t}) \eta^n \beta^n e^{-h\beta} e^{-k\eta} \, d\beta d\eta$$

or,

$$\hat{g}_B(t;\ \beta,\eta) = C + D \tag{65}$$

where

$$C = \frac{(hk)^{n+1}}{f^2} \int_0^\infty \int_0^\infty e^{-(\eta+\beta)t} \eta^n \beta^n e^{-h\beta} e^{-k\eta} \, d\beta d\eta$$

$$= \frac{(hk)^{n+1}}{f^2} \int_0^\infty e^{-\beta(t+h)} \beta^n \, d\beta \int_0^\infty e^{-\eta(t+k)} \eta^h \, d\eta$$

$$= \frac{(hk)^{n+1}}{f^2} \cdot \frac{1}{(t+h)^{n+1}} \int_0^\infty [(t+h)\beta]^n e^{-\beta(t+h)} \, d[(t+h)\beta] \; .$$

$$\frac{1}{(t+k)^{n+1}} \int_0^\infty [(t+k)\eta]^n e^{-\eta(t+k)} \, d[(t+k)\eta]$$

$$= \frac{(hk)^{n+1}}{f^2} \cdot \frac{\Gamma(n+1)}{(t+h)^{n+1}} \cdot \frac{\Gamma\ (n+1)}{(t+k)^{n+1}}$$

$$C = \left[\frac{hk}{(t+h)(t+k)} \right]^{n+1} \tag{66}$$

and $D = \dfrac{(hk)^{n+1}}{f^2} \displaystyle\int_0^\infty\int_0^\infty \frac{\eta}{\eta + \beta} (1 - e^{-(\eta+\beta)t}) \eta^n \beta^n e^{-h\beta} \cdot e^{-k\eta} \, d\beta d\eta$

Again, let

$\beta + \eta = \beta'$

which means $\beta = \beta' - \eta \qquad d\beta = d\beta'$

and when $\beta = 0 \rightarrow \beta' = \eta$

$\beta = \infty \rightarrow \beta' = \infty$

So D becomes

$$D = \frac{(hk)^{n+1}}{f^2} \int_0^\infty \int_\eta^\infty \frac{\eta}{\beta'} (1-e^{-\beta' t}) \eta^n (\beta'-\eta)^n \cdot e^{-h(\beta'-\eta)} e^{-k\eta} \, d\beta' d\eta \tag{67}$$

which converges due to the convergence of eq. (61).

Therefore, the traditional Bayesian estimate when sample data are available is

$$\hat{g}_B(t;\beta,\eta) = \left[\frac{hk}{(t+h)(t+k)}\right]^{n+1} + \frac{(hk)^{n+1}}{f^2} \int_0^\infty \int_\eta^\infty \eta^{n+1} e^{-\eta(k-h)} \cdot \left[\frac{1}{\beta'} (1-e^{-\beta' t}) e^{-h\beta'}\right] (\beta'-\eta)^n \, d\beta' d\eta \tag{68}$$

3.2.3. Bayesian Probability Intervals

To develop a probability interval for system availability, first the probability intervals for the parameters must be developed. In general, for any gamma random variable θ with density function

$$f_\theta(\theta) = \frac{w}{\Gamma(u)} (w\theta)^{u-1} e^{-w\theta}, \qquad \theta > 0 \tag{69}$$

u is a positive integer $2w\theta$ is a chi-square variable with 2u degrees of freedom, as proven in Appendix C.

Specifically, when sample data are not available and the prior distributions of λ, β and η are given by eqs. (42), (43) and (52) respectively, it follows that

$$2\mu\lambda \sim \chi^2(2) \tag{70}$$

$$2\nu\beta \sim \chi^2(2) \tag{71}$$

$$2\gamma\eta \sim \chi^2(2) \tag{72}$$

Therefore the 100(1-γ)% Bayesian probability intervals* for λ, β, and η are

$$\lambda_{\frac{\gamma}{2}} \leq \lambda \leq \lambda_{1-\frac{\gamma}{2}} \tag{73}$$

$$\beta_{\frac{\gamma}{2}} \leq \beta \leq \beta_{1-\frac{\gamma}{2}} \tag{74}$$

$$\eta_{\frac{\gamma}{2}} \leq \eta \leq \eta_{1-\frac{\gamma}{2}} \tag{75}$$

where

$$\lambda_{\frac{\gamma}{2}} = \frac{\chi^2_{\frac{\gamma}{2}}(2)}{2\mu} \tag{76}$$

$$\lambda_{1-\frac{\gamma}{2}} = \frac{\chi^2_{1-\frac{\gamma}{2}}(2)}{2\mu} \tag{77}$$

$$\beta_{\frac{\gamma}{2}} = \frac{\chi^2_{\frac{\gamma}{2}}(2)}{2\nu} \tag{78}$$

$$\beta_{1-\frac{\gamma}{2}} = \frac{\chi^2_{1-\frac{\gamma}{2}}(2)}{2\nu} \tag{79}$$

$$\eta_{\frac{\gamma}{2}} = \frac{\chi^2_{\frac{\gamma}{2}}(2)}{2\gamma} \quad * \tag{80}$$

$$\eta_{1-\frac{\gamma}{2}} = \frac{\chi^2_{1-\frac{\gamma}{2}}(2)}{2\gamma} \quad * \tag{81}$$

---

*Note that the γ used in defining the significance level in the probability intervals are not the same γ used as the prior parameter for η.

When sample data are available, the posterior distributions as defined in eqs. (45), (46) and (54) for λ, β and η respectively are used. Again, it follows

$$2\lambda(\mu + \sum_{i=1}^{n} x_i) \sim \chi^2(2kn+2) \tag{82}$$

$$2\beta(\nu + \sum_{i=1}^{n} y_i) \sim \chi^2(2\alpha n+2) \tag{83}$$

$$2\eta(\gamma + \sum_{i=1}^{n} z_i) \sim \chi^2(2mn+2) \tag{84}$$

and the $100(1-\gamma)\%$ Bayesian probability intervals for λ, β and η are defined by eqs. (73), (74) and (75) with the following limit values

$$\lambda_{\frac{\gamma}{2}} = \frac{\chi_{\frac{\gamma}{2}}(2kn+2)}{2(\mu + \sum_{i=1}^{n} x_i)} \tag{85}$$

$$\lambda_{1-\frac{\gamma}{2}} = \frac{\chi^2_{1-\frac{\gamma}{2}}(2kn+2)}{2(\mu + \sum_{i=1}^{n} x_i)} \tag{86}$$

$$\beta_{\frac{\gamma}{2}} = \frac{\chi^2_{\frac{\gamma}{2}}(2\alpha n+2)}{2(\nu + \sum_{i=1}^{n} y_i)} \tag{87}$$

$$\beta_{1-\frac{\gamma}{2}} = \frac{\chi^2_{1-\frac{\gamma}{2}}(2\alpha n+2)}{2(\nu + \sum_{i=1}^{n} y_i)} \tag{88}$$

$$\eta_{\frac{\gamma}{2}} = \frac{\chi^2_{\frac{\gamma}{2}}(2mn+z)}{2(\gamma + \sum_{i=1}^{n} z_i)^{*}} \tag{89}$$

$$\eta_{1-\frac{\gamma}{2}} = \frac{\chi^2_{1-\frac{\gamma}{2}}(2mn+z)}{2(\gamma + \sum_{i=1}^{n} z_i)^{*}} \tag{90}$$

The Bayesian probability interval for g(t) when T and T_{on} are gamma distributed is simply eq. (4) with the appropriate values of eqs. (76)-(90) substituted in for the parameters λ and β. Eqs. (76)-(79) are used when no data are available and eqs. (85)-(88) are used when sample data is available. Therefore, the 100(1-γ)% Bayesian probability interval for system availability is

$$\sum_{\ell=0}^{\alpha-1} P_0(\ell;\, \beta_{1-\frac{\gamma}{2}} t) + \lambda_{\frac{\gamma}{2}} \int_0^t \left\{ \sum_{\ell=0}^{\alpha-1} P_0\left[\ell;\, \beta_{1-\frac{\gamma}{2}}(t-s)\right] \right\}$$

$$\cdot \left\{ \sum_{\substack{q=k,\\ 2k,\\ \vdots}}^{\infty} P_0[(q-1);\, \lambda_{1-\frac{\gamma}{2}} s] \right\} ds \le g(t;\, k,\lambda,\alpha,\beta) \le$$

$$\sum_{\ell=0}^{\alpha-1} P_0(\ell;\, \beta_{\frac{\gamma}{2}} t) + \lambda_{1-\frac{\gamma}{2}} \int_0^t \left\{ \sum_{\ell=0}^{\alpha-1} P_0[\ell;\, \beta_{\frac{\gamma}{2}}(t-s)] \right\} \cdot$$

$$\left\{ \sum_{\substack{q=k\\ 2k,\\ \vdots}}^{\infty} P_0[(q-1);\, \lambda_{\frac{\gamma}{2}} s] \right\} ds \tag{91}$$

* See note on p. 60.

The Bayesian probability interval for g(t) when T_{on} and T_{off} are exponentially distributed is eq. (55) with the appropriate values of eqs. (78)-(90), with α=m=1, substituted in for the parameters β and η. Eqs. (78)-(81) are used when no data are available and eqs. (87)-(90) are used when sample data are available. Thus, the 100(1-γ)% Bayesian probability interval for g(t) for the special exponentially distributed case is

$$\frac{\eta_{\frac{\gamma}{2}}}{\eta_{1-\frac{\gamma}{2}} + \beta_{1-\frac{\gamma}{2}}} + \left[1 - \frac{\eta_{\frac{\gamma}{2}}}{\eta_{1-\frac{\gamma}{2}} + \beta_{1-\frac{\gamma}{2}}}\right] e^{-(\eta_{1-\frac{\gamma}{2}} + \beta_{1-\frac{\gamma}{2}})t} \leq$$

$$g(t;\, \beta,\eta) \leq \frac{\eta_{1-\frac{\gamma}{2}}}{\eta_{\frac{\gamma}{2}} + \beta_{\frac{\gamma}{2}}} + \left[1 - \frac{\eta_{1-\frac{\gamma}{2}}}{\eta_{\frac{\gamma}{2}} + \beta_{\frac{\gamma}{2}}}\right] \cdot e^{-(\eta_{\frac{\gamma}{2}} + \beta_{\frac{\gamma}{2}})t} \qquad (92)$$

Also, note that the parameter limits can also be defined in the following manners.

For eqs. (76)-(81):

$$\int_0^{\lambda_{\frac{\gamma}{2}}} f_\lambda(\lambda)d\lambda = \frac{\gamma}{2} \qquad (93)$$

$$\int_0^{\lambda_{1-\frac{\gamma}{2}}} f_\lambda(\lambda)d\lambda = 1 - \frac{\gamma}{2} \qquad (94)$$

$$\int_0^{\beta_{\frac{\gamma}{2}}} f_\beta(\beta)\ d\beta = \frac{\gamma}{2} \tag{95}$$

$$\int_0^{\beta_{1-\frac{\gamma}{2}}} f_\beta(\beta) d\beta = 1 - \frac{\gamma}{2} \tag{96}$$

$$\int_0^{\eta_{\frac{\gamma}{2}}} f_\eta(\eta)\ d\eta = \frac{\gamma}{2} \tag{97}$$

$$\int_0^{\eta_{1-\frac{\gamma}{2}}} f_\eta(\eta)\ d\eta = 1 - \frac{\gamma}{2} \tag{98}$$

And for eqs. (85)-(90):

$$\int_0^{\lambda_{\frac{\gamma}{2}}} f_\lambda(\lambda;\ x_1,\ \dots\ x_n)\ d\lambda = \frac{\gamma}{2} \tag{99}$$

$$\int_0^{\lambda_{1-\frac{\gamma}{2}}} f_\lambda(\lambda;\ x_1,\ \dots\ x_n)\ d\lambda = 1 - \frac{\gamma}{2} \tag{100}$$

$$\int_0^{\beta_{\frac{\gamma}{2}}} f_\beta(\beta;\ y_1,\ \dots\ y_n)\ d\beta = \frac{\gamma}{2} \tag{101}$$

$$\int_0^{\beta_{1-\frac{\gamma}{2}}} f_\beta(\beta;\ y_1,\ \dots\ y_n)\ d\beta = 1 - \frac{\gamma}{2} \tag{102}$$

$$\int_0^{\eta_{\frac{\gamma}{2}}} f\ (\ ;\ z_1,\ \dots\ z_n)\ d\eta = \frac{\gamma}{2} \tag{103}$$

$$\int_{0}^{\eta_{1-\frac{\gamma}{2}}} f_{\eta}(\eta;\ z_1,\ \ldots\ z_n)\ d\eta = 1 - \frac{\gamma}{2} \tag{104}$$

3.3 Brender's Bayesian Estimate

Brender [2], [3] was the first to apply Bayesian analysis to the prediction and measurement of system availability. His approach, however, differs somewhat from the traditional Bayesian approach. The traditional approach with squared error loss function used the expected value of the availability with respect to the parameters' posterior (or prior) distributions calculated directly from time-dependent joint and marginal distributions. In his method, Brender first derives the steady-state expression for availability, calculates the estimate from the first moment (i.e.; expected value) and then transforms this result to a time-dependent case.

3.3.1 Statement of the System and Derivation of the Model

In this derivation, on and off times are used as opposed to cycle and on times. The system is basically the same as previously mentioned, one with an alternating sequence of independent operation and repair intervals. Here, the T_{on} and T_{off} are exponentially distributed. Later, in his second paper [3], Brender relaxes this assumption to include the more general gamma distributed T_{on} and T_{off}.

As stated before, the term for availability is first derived then the first moment of this expression is taken for the availability estimate. In general, the term for availability can be expressed as

$$A_1(t;\ \beta,\eta) = \Pr\{S(t)=1 | I(\beta,\eta)\} \qquad (105)$$

where β and η are the parameters of on and off time, respectively; $S(t)$ indicates the state of the system (0 representing off, 1 representing on); and $I(\beta,\eta)$ represents the totality of required information about the system.

$A_1(t;\ \beta\ \eta)$ is also known as "availability of the first kind," that is, no prior information is used yet.

The first moment (expected value) of this term, via the definition of expected value and the use of a priori information about the parameters, is

$$E\{A_1(t;\ \beta,\eta)\} = A_2(t;\ \nu,\gamma) =$$

$$\int_0^\infty \int_0^\infty A_1(t;\ \beta,\eta)\ g(\beta|\nu)h(\eta|\gamma)d\beta\ d\eta \tag{106}$$

where $g(\beta|\nu)$ and $h(\eta|\gamma)$ represent the prior information on the parameters β and η, respectively. $A_2(t;\ \nu,\gamma)$ is known as "availability of the second kind" and is the estimate of system availability.

Steady-State Point Availability (SSPA)

To derive the estimate for time-dependent availability, Brender first derives the steady state availability of the second kind and then transforms it to the time dependent case through an extension theorem.

The availability of the first kind for the steady-state case is well known:

$$A_1(\beta,\eta) = \frac{\eta}{\eta+\beta}, \qquad \beta,\eta > 0 \tag{107}$$

By using the fact that the steady state point availability has a Euler density function [2], the product moment of the SSPA is expressed as:

$$\mu_{ik}(r,s;w,u)=E\{A^i(\beta,\eta).\ B^k(\beta,\eta)\,|\,r,s;w,u\};\ i,k=0,1,2,\ldots$$

$$= \frac{(1-z)^w}{b(w,r)} \int_0^1 \frac{a^{w+i-1}(1-a)^{r+k-1}}{(1-az)^{r+w}} \, da, \quad 0 \le z \le 1;\ w,r, \ge 0 \tag{108}$$

where r,s and w,u are the gamma parameters* of β and η, respectively, $B(\beta,\eta)$ represents the unavailability of the system, b(w,r) is the beta function with parameters w and r, and $z=(1-\frac{u}{s})$.

A more computable form of eq. (108) is derived using Theorems 16 and 21 of Rainville [15],

$$\mu_{ik}(r,s;\ w,u) = (1-z)^x \ \frac{[\Gamma(w+i)/\Gamma(w)]\ [\Gamma(r+k)/\Gamma(r)]}{[\Gamma(w+r+i+k)/\Gamma(w+r)]} \ \cdot$$

$$\sum_{j=0}^{\infty} \left\{ \frac{[\Gamma(x+r+j\ /\Gamma(x+r)][\Gamma(x+i+j)/\Gamma(x+i)]}{[\Gamma(w+r+i+k+j)/\Gamma(w+r+i+k)]} \cdot \frac{z^j}{j!} \right\} \tag{109}$$

$r,w > 0,\ 0 \le z < 1$

where x=w or x=k depending on the relative magnitudes of w and k. If w is somewhat greater than k, use x=k and the series will converge more rapidly; otherwise, use x=w.

Therefore, the SSPA of the second kind is simply the product moment with i=1 and k=0 (conversely, the SSPUA would be the product moment with i=0 and k=1),

$$E\{A_1(\beta,\eta)\} = A_2(r,s;\ w,u) = \frac{w}{w+r} \sum_{j=0}^{\infty} \left\{ \frac{[\Gamma(r+j)/\Gamma(r)]}{[\Gamma(w+r+j+1)/\Gamma(w+r+1)]} \ \cdot \right.$$

$$\left. (1-\frac{u}{s})^j \right\} \tag{110}$$

*Brender defines the gamma function as $g(\beta\,|r,s) = s^r\beta^{r-1}e^{-\beta s}/\Gamma(r)$ with mean r/s and variance r/s^2.

The Extension Theorem

This extension theorem enables the expansion of the steady-state case to the time dependent case. It states, simply, that the expectation of the product of dependent variables can be expressed as a product of expectations. In univariate form, the extension theorem is

$$E\{\beta^a e^{-\beta t_a} \cdot \phi(\beta) | r,s\} = E\{\beta^a e^{-\beta t_a} | r,s\} \cdot E\{\phi(\beta) | r+a,\ s+t_a\} \tag{111}$$

In bivariate form,

$$E\{\beta^a e^{-\beta t_a} \cdot \eta^b e^{-\eta t_b} \cdot \phi(\eta,\beta) | r,s;\ w,u\} =$$

$$E\{\beta^a e^{-\beta t_a} | r,s\} \cdot E\{\eta^b e^{-\eta t_b} | w,u\} \cdot E\{\phi(\beta,\eta) | r+a,\ s+t_a;\ w+b,\ u+t_b\} \tag{112}$$

The proof follows from the repeated use of Bayes' theorem [2]. Note that this theorem is restricted to gamma distributions only.

3.3.2 The No-Data Case

Assuming exponentially distributed on and off times and at time 0 the system is on, the availability of the first kind is

$$A_1(t|\beta,\eta) = \Pr\{S(t) = 1 | S(0) = 1;\ I(\beta,\eta)\}$$

$$= \frac{\eta}{\eta+\beta} + \left(1 - \frac{\eta}{\eta+\beta}\right) e^{-(\eta+\beta)t} \tag{113}$$

as derived in Section 2.2.3.

The availability of the second kind, given gamma parameters of the system parameters, is

$$A_2(t|r,s;\ w,u) = E\{A_1(t|\beta,\eta)\}$$

$$= E\ \{\frac{\eta}{\eta+\beta} + (1 - \frac{\eta}{\eta+\beta})\ e^{-(\eta+\beta)t}\}$$

$$= E\ \{\frac{\eta}{\eta+\beta}\} + E\ \{(1- \frac{\eta}{\eta+\beta})\ e^{-\eta t}e^{-\beta t}\}$$

$$= A_2(r,s;\ w,u) + [1-A_2(r,s+t;\ w,u+t)]\cdot$$

$$(\frac{s}{s+t})^r\ (\frac{u}{u+t})^w \tag{114}$$

using the extension theorem on the second term and knowing that if

$$R(t|\lambda) = e^{-\lambda t}, \qquad t \geq 0 \tag{115}$$

then $R(t|r,s) = (\frac{s}{s+t})^r, \qquad r,s > 0$ (116)

according to Brender [2]. Recall that

$$A_2(r,s;\ w,u) = \frac{w}{w+r} \sum_{j=0}^{\infty} \left\{\frac{[\Gamma(r+j)/\Gamma(r)]}{[\Gamma(w+r+j+1)\quad (w+r+j)]} \cdot (1-\frac{u}{s})^j \right\} \tag{117}$$

so

$$A_2(r,s+t;\ w,u+t) = \frac{w}{w+r} \sum_{j=0}^{\infty} \left\{\frac{[\Gamma(r+j)/\Gamma(r)]}{[\Gamma(w+r+j+1)/\Gamma(w+r+j)]}\right.$$

$$\left. \cdot\ (1 - \frac{u+t}{s+t})^j \right\} \tag{118}$$

When the underlying prior distributions of the on and off parameters are exponential with parameters $(1, \frac{1}{\nu})$ and $(1, \frac{1}{\gamma})$ for β and η respectively, as they are for the model introduced in Sec. 3.2 (See Table 3-1 for priors), eq. (114) simplies to

$$\hat{g}_{BBND}(t) = A_2(t|1,\nu;\ 1,\gamma) = \sum_{j=0}^{\infty} \frac{\Gamma(1+j)}{\Gamma(3+j)}\ (1 - \frac{\gamma}{\nu})^j +$$

$$[1 - \sum_{j=0}^{\infty} \frac{\Gamma(1+j)}{\Gamma(3+j)}\ (1 - \frac{\gamma+t}{\nu+t})^j]\ (\frac{\nu}{\nu+t})\ (\frac{\gamma}{\gamma+t}) \tag{119}$$

3.3.3. The Data Case

It is well known that by using Bayes' theorem, a gamma prior distribution $g(\beta|r,s)$ is transformed, with the collection of some data, into a posterior distribution of the gamma form $g(\beta|r + N,\ s + \sum_{i=1}^{N} y_i)$ where N is the number of observations and $\sum_{i=1}^{N} y_i$ is the total observation time.

Therefore, the estimate for system availability when some data are available is simple eq. (114) with the following substitution of parameters:

$$
\begin{aligned}
r &\rightarrow r + N_1 \\
s &\rightarrow s + \sum_{i=1}^{N_1} y_i \\
w &\rightarrow w + N_2 \\
u &\rightarrow u + \sum_{i=1}^{N_2} z_i
\end{aligned}
\tag{120}
$$

where N_1 and N_2 represent the number of failures and repairs observed, respectively, and $\sum_{i=1}^{N_1} y_i$ and $\sum_{i=1}^{N_2} z_i$ represent the total operation and repair times, respectively.

The availability estimate using sample data, assuming gamma distributed parameters is

$$
\begin{aligned}
&A_2(t|r+N_1,\ s + \sum_{i=1}^{N_1} y_i;\ w + N_2,\ u + \sum_{i=1}^{N_2} z_i) = \\
&\quad A_2(r + N_1\ s + \sum_{i=1}^{N_1} y_i;\ w + N_2,\ u + \sum_{i=1}^{N_2} z_i) + \\
&\quad [1 - A_2(r + N_1,\ s + \sum_{i=1}^{N_1} y_i + t;\ w + N_2,\ u + \sum_{i=1}^{N_2} z_i + t)] \ . \\
&\quad \left(\frac{s + \sum_{i=1}^{N_1} y_i}{s + \sum_{i=1}^{N_1} y_i + t}\right)^{r+N_1} \cdot \left(\frac{u + \sum_{i=1}^{N_2} z_i}{u + \sum_{i=1}^{N_2} z_i + t}\right)^{w+N_2}
\end{aligned}
\tag{121}
$$

The availability estimate using the exponential posterior distributions of β and η, eqs. (53) and (54), respectively, assuming $N_1 = N_2 = N$, is

$$\hat{g}_{BBD}(t) = A_2(t|N+1, h; N+1, k) = A_2(N+1, h; N+1, k) +$$

$$[1 - A_2(N+1; h+t; N+1, k+t] \cdot \left(\frac{h}{h + t}\right)^{N+1} \left(\frac{k}{k + t}\right)^{N+1} \qquad (122)$$

where

$$h = \nu + \sum_{i=1}^{N} y_i$$

$$k = \gamma + \sum_{i=1}^{N} z_i$$

$$A_2(N+1, h; N+1, h) = \frac{\Gamma(2N+3)}{2\Gamma(N+1)} \sum_{j=0}^{\infty} \frac{\Gamma(N+1+j)}{\Gamma(2N+3+j)} (1 - \frac{k}{h})^j$$

$$A_2(N+1, h+t; N+1, k+t) = \frac{\Gamma(2N+3)}{2\Gamma(N+1)} \sum_{j=0}^{\infty} \frac{\Gamma(N+1+j)}{\Gamma(2N+3+j)} \left(1 - \frac{k + t}{h + t}\right)^j$$

All of the estimates for an exponentially distributed system in analytical form, are listed in Table 3-2 for quick reference.

TABLE 3-2: The Availability Estimates in Analytical Form for an Exponentially Distributed System

Estimate	No Data Case	Data Case
Maximum Likelihood		1) *Using on & off times:* $\hat{g}_{MLE}(t) = \frac{\Sigma y}{\Sigma y + \Sigma z} + (1 - \frac{\Sigma y}{\Sigma y + \Sigma z})\, e^{-[\frac{n(\Sigma y + \Sigma z)}{\Sigma y \Sigma z}]t}$ 2) Using cycle & on times: $\hat{g}_{MLE}(t) = \frac{\Sigma y}{\Sigma x} + (1 - \frac{\Sigma y}{\Sigma x})\, e^{-(\frac{n}{\Sigma y})t}$ where Σ denotes $\sum_{i=1}^{n}$
Traditional Bayesian	$\hat{g}_B(t) = \frac{\nu\gamma}{(t+\nu)(t+\gamma)} + \nu\gamma \int_0^\infty \int_\eta^\infty \eta e^{-\eta(\gamma+\nu)}$ $\cdot \left[\frac{1}{\beta'} + (1-e^{-\beta' t})\, e^{-\nu\beta'}\right] d\beta' d\eta$ where $\beta' = \eta+\beta$	$\hat{g}_B(t;\, \eta,\beta) = \left[\frac{hk}{(t+h)(t+k)}\right]^{n+1} + \frac{(hk)^{n+1}}{f^2} \int_0^\infty \int_\eta^\infty \eta^{n+1}\, e^{-\eta(k-h)}\, \cdot$ $\left[\frac{1}{\beta'}\, (1-e^{-\beta' t})\, e^{-h\beta'}\right] (\beta'-\eta)^n\, d\beta'\, d\eta$ where $\beta' = \eta+\beta$, $h=\nu+\Sigma y$, $k=\gamma+\Sigma z$, n=sample size, $f=\Gamma(n+1)$
Brender's Bayesian	$\hat{g}_{BB}(t;\, \nu,\gamma) = \sum_{j=0}^{\infty} \frac{\Gamma(1+j)}{\Gamma(3+j)} (1-\frac{\gamma}{\nu})^j +$ $\left[1 - \sum_{j=0}^{\infty} \frac{\Gamma(1+j)}{\Gamma(3+j)} (1 - \frac{\gamma+t}{\nu+t})^j\right] (\frac{\nu}{\nu+t}) (\frac{\gamma}{\gamma+t})$	$\hat{g}_{BB}(t;\, N,h,k) = \frac{\Gamma(2N+3)}{2\Gamma(N+1)} \sum_{j=0}^{\infty} \frac{\Gamma(N+1+j)}{\Gamma(2N+3+j)} (1-\frac{k}{h})^j +$ $\left[1 - \frac{\Gamma(2N+3)}{2\Gamma(N+1)} \sum_{j=0}^{\infty} \frac{\Gamma(N+1+j)}{\Gamma(2N+3+j)} (1 - \frac{k+t}{h+t})^j\right] (\frac{h}{h+t})^{N+1} (\frac{k}{k+t})^{N+1}$ where $h=\nu+\sum_{i=1}^{N} y_i$, $k=\gamma+\sum_{i=1}^{N} z_i$

CHAPTER 4 - TWO NUMERICAL EXAMPLES

Here, the three estimation methods previously described are used to estimate the system availability of two data sets which represent two distinct exponentially distributed systems. Approximately 220 estimates are calculated for each system using six different sample size and type combinations along with three different sets of prior parameters, where applicable, at four different time horizons. The results and a partial analysis of these findings are presented in this chapter, and a more thorough analysis, along with the final selection of the best method, will be presented in Chapter Five.

4.1 Background Information

4.1.1 Data Sets

Each data set is comprised of two negative exponentially distributed sets of values, designated as the on and off times of the system. These on and off times are assumed independent of one another. Data Set 1, listed in Table 4-A, has been proven to be negatively exponentially distributed by Epstein [5] and has 49 values each for the system on times and off times.

Data Set 2, listed in Table 4-B, was constructed from two independent sets of gamma random variables generated by a SAS (Statistical Analysis System) program. Appendix E outlines the computer routine used, along with the original sets of gamma random variables generated having parameters (1, 1/4) and (1, 1/2). Data Set 2 has 40 values each for the system on times and off times. These values were then scaled (here, multiplied by 100) to conform to the computer routine.

4.1.2 Types of Samples

Six samples from each data set were drawn and, from these samples, in conjunction with specified prior parameters and time horizons, the system

Table 4-A: Data Set 1 - Two Independent Sets of 49 Exponentially Distributed System On and Off Times [5]

On Time	Off Time	On Time	Off Time
12.0	1.2	951.0	95.1
22.0	2.2	979.0	97.9
49.0	4.9	996.0	99.6
50.0	5.0	1028.0	102.8
68.0	6.8	1055.0	108.5
70.0	7.0	1227.0	128.7
121.0	12.1	1256.0	133.6
137.0	13.7	1351.0	144.1
151.0	15.1	1426.0	147.6
152.0	15.2	1491.0	150.6
239.0	23.9	1516.0	151.6
243.0	24.3	1526.0	152.6
251.0	25.1	1592.0	164.2
358.0	35.8	1668.0	166.8
389.0	38.9	1746.0	178.6
479.0	47.9	1852.0	185.2
484.0	48.4	1871.0	187.1
493.0	49.3	2031.0	203.0
532.0	53.2	2043.0	204.3
556.0	55.6	2295.0	229.5
627.0	62.7	2591.0	253.1
734.0	72.4	3041.0	304.1
736.0	73.6	3427.0	341.7
768.0	76.8	3544.0	354.4
858.0	83.3		
	Means	1042.49	104.89

Steady State Availability: 0.9086

Table 4-B: Data Set 2 - Two Independent Sets of 40 Exponentially Distributed System On and Off Times

On Time	Off Time	On Time	Off Time
4.75	3.44	357.32	86.49
8.59	5.41	366.68	90.03
20.69	5.48	387.42	121.87
22.98	6.41	426.77	125.29
39.50	7.16	432.18	126.60
61.86	8.20	453.03	134.21
64.72	9.74	459.08	142.17
90.34	10.00	462.66	159.55
97.28	12.02	595.80	183.70
138.14	12.91	730.25	190.79
142.77	13.82	765.68	208.02
179.69	14.15	778.70	212.62
191.68	17.06	923.80	243.07
249.32	23.67	936.76	275.30
256.10	29.29	968.24	350.08
260.00	30.10	1041.90	351.56
275.96	34.58	1160.31	373.08
289.95	47.49	1180.43	409.50
292.94	58.62	1415.78	510.97
311.17	86.04	1544.31	816.34
	Means	459.64	138.67

Steady State Availability: 0.7682

availability was estimated using the three estimation methods previously discussed. Relatively small samples (of sizes three, five, and eight) were drawn, because it was assumed that the data from each of the systems was extremely expensive or impossible to obtain. Because of this assumed unavailability of large amounts of data, the use of Bayesian-type estimation methods seemed logical and warranted. Therefore, this examination of small sample sizes will help test the Bayesian methods' effectiveness versus the classical maximum likelihood estimate.

In addition to size, the biasedness of the sample contributes to the efficiency of any estimation method. To see how the biasedness of a sample affects an estimate, both random and biased samples were drawn. The random samples were determined with the use of a random number table [19]. The biased samples were formed with the authors' discretion.

To test each combination of biasedness and size, 3 x 2 or six samples were needed. The samples for each data set, along with their characteristics are illustrated in Tables 4-C(1)-(6) and 4-D(1)-(6).

4.1.3 Prior Information

For the Bayesian estimation methods, several priors were explored to determine their effects on the availability estimates. The prior parameter sets were denoted by the pair (V, U). These prior parameter sets represent the experimenter's expectations of the mean on and off times which are usually always subjective as they may be arrived at in any manner. V is the mean of the negative exponential on time distribution represented by eq. (51) in Chapter 3, while U is the mean of the negative exponential off time distribution represented by eq. (52) in Chapter 3.

The prior parameter sets used are listed in Table 4-E. Three prior parameter sets were used for each data set (population). One represented the population's mean on time and off time, while the other two either underestimated or overestimated both mean on time and mean off time.

Table 4-C(1): Random Sample of Size Three (Sample No. 1) From Data Set 1

Observation	On Time	Off Time
1	556.00	187.10
2	1668.00	204.30
3	68.00	23.90
Means	764.00	138.43

(Note: mean on time is somewhat low and mean off time is a bit high when compared to the population means, so any availability estimate is expected to be low when compared to the steady state availability)

Table 4-C(2): Biased Sample of Size Three (Sample No. 2) From Data Set 1

Observation	On Time	Off Time
1	1351.00	62.70
2	3427.00	72.40
3	3544.00	48.40
Means	2774.00	61.17

(Note: mean on time is too high, while mean off time is too low so any availability estimate is expected to be too high)

Table 4-C(3): Random Sample of Size Five (Sample No. 3) From Data Set 1

Observation	On Time	Off Time
1	2043.00	99.60
2	152.00	47.90
3	1516.00	38.90
4	1871.00	47.90
5	22.00	204.30
Means	1120.80	87.72

(Note: mean on time is about right and mean off time is a bit low, so any availability estimate is expected to be a bit high)

Table 4-C(4): Biased Sample of Size Five (Sample No. 4) From Data Set 1

Observation	On Time	Off Time
1	627.00	76.80
2	239.00	253.10
3	493.00	187.10
4	239.00	55.60
5	137.00	187.10
Means	347.00	151.94

(Note: mean on time is too low, while mean off time is too high, so any availability estimate is expected to be too low)

Table 4-C(5) Random Sample of Size Eight (Sample No. 5) From Data Set 1

Observation	On Time	Off Time
1	121.00	128.70
2	2043.00	38.90
3	1871.00	5.00
4	1668.00	62.70
5	70.00	24.30
6	768.00	187.10
7	1351.00	25.10
8	151.00	253.10
Means	1005.38	90.61

(Note: mean on time and mean off time are about right so any availability estimate should be close to the steady state availability-this is the most representative sample of the six)

Table 4-C(6) Biased Sample of Size Eight (Sample No. 6) From Data Set 1

Observation	On Time	Off Time
1	996.00	187.10
2	484.00	144.10
3	479.00	97.90
4	484.00	164.20
5	151.00	166.80
6	1055.00	55.60
7	556.00	102.30
8	137.00	354.40
Means	542.75	159.11

(Note: mean on time is too low while mean off time is too high, so any availability estimate is expected to be too low)

Table 4-D(1): Random Sample of Size Three (Sample No. 1) From Data Set 2

Observation	On Time	Off Time
1	39.50	29.29
2	432.18	6.41
3	968.24	159.55
Means	479.97	65.08

(Note: mean on time is just about right and mean off time is too low when compared to the population means, so any availability estimate is expected to be high when compared to the steady state availability)

Table 4-D(2): Biased Sample of Size Three (Sample No. 2) From Data Set 2

Observation	On Time	Off Time
1	61.86	190.79
2	256.10	208.02
3	453.03	275.30
Means	257.00	224.70

(Note: mean on time is too low while mean off time is too high, so any availability estimate is expected to be too low)

Table 4-D(3): Random Sample of Size Five (Sample No. 3) From Data Set 2

Observation	On Time	Off Time
1	97.28	8.20
2	366.68	183.70
3	142.77	351.56
4	387.42	212.62
5	778.70	14.15
Means	354.57	154.05

(Note: mean on time is somewhat low and mean off time is somewhat high, so any availability estimate is expected to be a bit low-however, this is the most representative sample of the six)

Table 4-D(4): Biased Sample of Size Five (Sample No. 4) From Data Set 2

Observation	On Time	Off Time
1	968.24	7.16
2	459.08	12.91
3	1041.90	5.41
4	292.94	134.21
5	1544.31	58.62
Means	861.29	43.66

(Note: mean on time is too high while mean off time is too low, so any availability estimate is expected to be too high)

Table 4-D(5): Random Sample of Size Eight (Sample No. 5) From Data Set 2

Observation	On Time	Off Time
1	426.97	23.67
2	90.34	7.16
3	765.68	12.91
4	923.80	275.30
5	311.17	3.44
6	730.25	350.08
7	249.32	47.49
8	459.08	5.48
Means	494.58	90.69

(Note: mean on time is somewhat high while mean off time is a bit how, so any availability estimate is expected to be a bit too high)

Table 4-D(6): Biased Sample of Size Eight (Sample No. 6) From Data Set 2

Observation	On Time	Off Time
1	453.03	86.49
2	595.80	30.10
3	387.42	373.08
4	936.76	29.29
5	1180.43	10.00
6	765.68	34.58
7	249.32	125.29
8	289.95	243.07
Means	607.30	116.49

(Note: mean on time is too high while mean off time is somewhat low, so any availability estimate is expected to be too high)

Table 4-E: Prior Parameter Sets Used in Estimating System Availability

	Data Set 1	Data Set 2
Prior 1:	V = 1042.00	V = 460.00
	U = 104.20	U = 139.00
Prior 2:	V = 500.00	V = 250.00
	U = 50.00	U = 50.00
Prior 3:	V = 1600.00	V = 600.00
	U = 160.00	U = 250.00

V = the mean of the negative exponential on time distribution

U = the mean of the negative exponential off time distribution

4.2 Computation Methods

The five system availability estimates, listed in Table 3-2, were calculated through the use of a Fortran computer routine. The flow diagrams and the routine itself are found in Appendix F.

The maximum likelihood and Brenders' Bayesian estimates are fairly straightforward, merely substituting desired parameters into the formulas. The traditional Bayesian estimate, however, is not so easily calculated. The indefinite integrals must be estimated, and care must be taken in the selection of the upper and lower limits. For these examples, a numerical approximation using Simpson's Rule is used. Note that more advanced numerical integration techniques may improve the results of the traditional Bayesian estimation technique.

4.3 General Results and Observations

The system availability estimates using the maximum likelihood, traditional Bayesian, and Brender's Bayesian techniques are listed in Tables 4-1 to 4-6. Tables 4-1, 4-2, and 4-3 refer to Data Set 1 found in Table 4-A while Tables 4-4, 4-5, and 4-6 refer to Data Set 2 found in Table 4-B. Note the variations in the availability estimates due to the different samples, prior parameter sets, time horizons, and methods. These differences will be explored and analyzed shortly.

Keep in mind that the results presented in Tables 4-1, to 4-6 will be analyzed, for now, with only the following two criteria in mind:

(1) Closeness to steady-state availability

(2) Variability between samples (where small variability is best, because it signifies a lesser dependence on the sample).

Later, when the final selection of the estimation method is being made, additional criteria will be used. These additional criteria are not discussed here, however, because they are not based upon the computational results.

TABLE 4-1: MAXIMUM LIKELIHOOD ESTIMATES OF SYSTEM AVAILABILITY FOR DATA SET 1

	TIMES T=100	T=200	T=300	T=400
MAXIMUM LIKELIHOOD ESTIMATE				
SAMPLE 1 (N= 3)	0.9120	0.8744	0.8585	0.8517
MAXIMUM LIKELIHOOD ESTIMATE OF BETA IS	0.001309			
MAXIMUM LIKELIHOOD ESTIMATE OF ETA IS	0.007224			
SAMPLE 2 (N= 3)	0.9825	0.9792	0.9786	0.9785
MAXIMUM LIKELIHOOD ESTIMATE OF BETA IS	0.000360			
MAXIMUM LIKELIHOOD ESTIMATE OF ETA IS	0.016349			
SAMPLE 3 (N= 5)	0.9486	0.9336	0.9292	0.9279
MAXIMUM LIKELIHOOD ESTIMATE OF BETA IS	0.000892			
MAXIMUM LIKELIHOOD ESTIMATE OF ETA IS	0.011400			
SAMPLE 4 (N= 5)	0.8137	0.7414	0.7133	0.7024
MAXIMUM LIKELIHOOD ESTIMATE OF BETA IS	0.002882			
MAXIMUM LIKELIHOOD ESTIMATE OF ETA IS	0.006582			
SAMPLE 5 (N= 8)	0.9421	0.9248	0.9196	0.9180
MAXIMUM LIKELIHOOD ESTIMATE OF BETA IS	0.000995			
MAXIMUM LIKELIHOOD ESTIMATE OF ETA IS	0.011036			
SAMPLE 6 (N= 8)	0.8739	0.8179	0.7931	0.7821

```
MAXIMUM LIKELIHOOD ESTIMATE OF BETA IS   0.001842
MAXIMUM LIKELIHOOD ESTIMATE OF  ETA IS   0.006285
```

TABLE 4-2: TRADITIONAL BAYESIAN ESTIMATES OF SYSTEM AVAILABILITY FOR DATA SET 1

	TIMES			
	T=100	T=200	T=300	T=400
TRADITIONAL BAYESIAN ESTIMATE				
SAMPLE 1 (N= 3)				
PARAMETER SET 1	0.9091	0.8713	0.8529	0.8431
PARAMETER SET 2	0.8860	0.8459	0.8280	0.8192
PARAMETER SET 3	0.9229	0.8868	0.8679	0.8574
SAMPLE 2 (N= 3)				
PARAMETER SET 1	0.8037	0.7817	0.7774	0.7760
PARAMETER SET 2	0.6703	0.6410	0.6367	0.6356
PARAMETER SET 3	0.8881	0.8720	0.8679	0.8663
SAMPLE 3 (N= 5)				
PARAMETER SET 1	0.9032	0.8801	0.8722	0.8688
PARAMETER SET 2	0.8653	0.8402	0.8330	0.8302
PARAMETER SET 3	0.9260	0.9039	0.8952	0.8911
SAMPLE 4 (N= 5)				
PARAMETER SET 1	0.8555	0.7950	0.7676	0.7544
PARAMETER SET 2	0.8262	0.7611	0.7341	0.7221
PARAMETER SET 3	0.8754	0.8186	0.7909	0.7767
SAMPLE 5 (N= 8)				
PARAMETER SET 1	0.9234	0.9019	0.8945	0.8916
PARAMETER SET 2	0.9076	0.8853	0.8784	0.8758
PARAMETER SET 3	0.9337	0.9126	0.9047	0.9015
SAMPLE 6 (N= 8)				
PARAMETER SET 1	0.8886	0.8377	0.8132	0.8007

PARAMETER SET 2	0.8785	0.8258	0.8015	0.7897
PARAMETER SET 3	0.8969	0.8474	0.8225	0.8093

NO SAMPLE DATA

PARAMETER SET 1	0.9399	0.9107	0.8929	0.8810
PARAMETER SET 2	0.8724	0.8416	0.8265	0.8177
PARAMETER SET 3	0.9658	0.9453	0.9314	0.9211

PARAMETER SET 1:	PARAMETER SET 2:	PARAMETER SET 3:
V=1042.00	V= 500.00	V=1600.00
U= 104.20	U= 50.00	U= 160.00

TABLE 4-3: BRENDERS BAYESIAN ESTIMATES OF SYSTEM AVAILABILITY FOR DATA SET 1

	TIMES			
	T=100	T=200	T=300	T=400
BRENDERS BAYESIAN ESTIMATE				
SAMPLE 1 (N= 3)				
PARAMETER SET 1	0.9201	0.8842	0.8665	0.8571
PARAMETER SET 2	0.9089	0.8716	0.8545	0.8459
PARAMETER SET 3	0.9289	0.8945	0.8764	0.8662
SAMPLE 2 (N= 3)				
PARAMETER SET 1	0.9762	0.9684	0.9653	0.9637
PARAMETER SET 2	0.9772	0.9711	0.9688	0.9678
PARAMETER SET 3	0.9757	0.9666	0.9625	0.9604
SAMPLE 3 (N= 5)				
PARAMETER SET 1	0.9464	0.9278	0.9203	0.9171
PARAMETER SET 2	0.9444	0.9269	0.9204	0.9177
PARAMETER SET 3	0.9485	0.9289	0.9206	0.9167
SAMPLE 4 (N= 5)				
PARAMETER SET 1	0.8588	0.7997	0.7733	0.7608
PARAMETER SET 2	0.8320	0.7681	0.7419	0.7303
PARAMETER SET 3	0.8783	0.8234	0.7972	0.7840
SAMPLE 5 (N= 8)				
PARAMETER SET 1	0.9413	0.9216	0.9142	0.9113
PARAMETER SET 2	0.9395	0.9205	0.9139	0.9113
PARAMETER SET 3	0.9431	0.9227	0.9147	0.9114
SAMPLE 6 (N= 8)				
PARAMETER SET 1	0.8861	0.8347	0.8105	0.7966

PARAMETER SET 2	0.8759	0.8222	0.7979	0.7865
PARAMETER SET 3	0.8949	0.8456	0.8215	0.8093

NO SAMPLE DATA

PARAMETER SET 1	0.9377	0.9071	0.8685	0.8760
PARAMETER SET 2	0.9051	0.8743	0.8592	0.8504
PARAMETER SET 3	0.9538	0.9265	0.9081	0.8949

PARAMETER SET 1:	PARAMETER SET 2:	PARAMETER SET 3:
V=1042.00	V= 500.00	V=1600.00
U= 104.20	U= 50.00	U= 160.00

CORE USAGE OBJECT CODE= 12056 BYTES,ARRAY AREA= 20856

DIAGNOSTICS NUMBER OF ERRORS= 0, NUMBER OF WARNINGS=

COMPILE TIME= 0.96 SEC,EXECUTION TIME= 211.46 SEC, 11 .

TABLE 4-4: MAXIMUM LIKELIHOOD ESTIMATES OF SYSTEM AVAILABILITY FOR DATA SET 2

	TIMES			
	T=100	T=200	T=300	T=400
MAXIMUM LIKELIHOOD ESTIMATE				
SAMPLE 1 (N= 3)	0.9015	0.8842	0.8812	0.8807
MAXIMUM LIKELIHOOD ESTIMATE OF BETA IS 0.002083				
MAXIMUM LIKELIHOOD ESTIMATE OF ETA IS 0.015365				
SAMPLE 2 (N= 3)	0.7361	0.6215	0.5717	0.5501
MAXIMUM LIKELIHOOD ESTIMATE OF BETA IS 0.003891				
MAXIMUM LIKELIHOOD ESTIMATE OF ETA IS 0.004450				
SAMPLE 3 (N= 5)	0.8165	0.7442	0.7157	0.7044
MAXIMUM LIKELIHOOD ESTIMATE OF BETA IS 0.002820				
MAXIMUM LIKELIHOOD ESTIMATE OF ETA IS 0.006492				
SAMPLE 4 (N= 5)	0.9561	0.9521	0.9518	0.9518
MAXIMUM LIKELIHOOD ESTIMATE OF BETA IS 0.001161				
MAXIMUM LIKELIHOOD ESTIMATE OF ETA IS 0.022903				
SAMPLE 5 (N= 8)	0.8871	0.8564	0.8481	0.8459
MAXIMUM LIKELIHOOD ESTIMATE OF BETA IS 0.002022				
MAXIMUM LIKELIHOOD ESTIMATE OF ETA IS 0.011026				
SAMPLE 6 (N= 8)	0.8969	0.8599	0.8465	0.8417

```
MAXIMUM LIKELIHOOD ESTIMATE OF BETA IS   0.001647
MAXIMUM LIKELIHOOD ESTIMATE OF  ETA IS   0.008585
```

TABLE 4-5: TRADITIONAL BAYESIAN ESTIMATES OF SYSTEM AVAILABILITY FOR DATA SET 2

	TIMES			
	T=100	T=200	T=300	T=400
TRADITIONAL BAYESIAN ESTIMATE				
SAMPLE 1 (N= 3)				
PARAMETER SET 1	0.7583	0.7122	0.6983	0.6927
PARAMETER SET 2	0.5773	0.5291	0.5194	0.5163
PARAMETER SET 3	0.8451	0.7963	0.7765	0.7674
SAMPLE 2 (N= 3)				
PARAMETER SET 1	0.7797	0.6833	0.6381	0.6158
PARAMETER SET 2	0.7469	0.6491	0.6075	0.5885
PARAMETER SET 3	0.7953	0.6981	0.6496	0.6243
SAMPLE 3 (N= 5)				
PARAMETER SET 1	0.8228	0.7506	0.7188	0.7040
PARAMETER SET 2	0.8101	0.7404	0.7122	0.6999
PARAMETER SET 3	0.8261	0.7489	0.7125	0.6945
SAMPLE 4 (N= 5)				
PARAMETER SET 1	0.6577	0.6215	0.6155	0.6140
PARAMETER SET 2	0.4159	0.3737	0.3690	0.3682
PARAMETER SET 3	0.8316	0.8023	0.7944	0.7916
SAMPLE 5 (N= 8)				
PARAMETER SET 1	0.8586	0.8195	0.8068	0.8022
PARAMETER SET 2	0.8261	0.7872	0.7763	0.7727
PARAMETER SET 3	0.8736	0.8319	0.8166	0.8105
SAMPLE 6 (N= 8)				
PARAMETER SET 1	0.8925	0.8522	0.8358	0.8298

PARAMETER SET 2	0.8870	0.8485	0.8340	0.8282
PARAMETER SET 3	0.8942	0.8510	0.8321	0.8234

NO SAMPLE DATA

PARAMETER SET 1	0.8630	0.8019	0.7685	0.7480
PARAMETER SET 2	0.7975	0.7560	0.7388	0.7299
PARAMETER SET 3	0.8786	0.8104	0.7677	0.7369

PARAMETER SET 1:	PARAMETER SET 2:	PARAMETER SET 3:
V= 460.00	V= 250.00	V= 600.00
U= 139.00	U= 50.00	U= 250.00

TABLE 4-6: BRENDERS BAYESIAN ESTIMATES OF SYSTEM AVAILABILITY FOR DATA SET 2

	TIMES			
	T=100	T=200	T=300	T=400
BRENDERS BAYESIAN ESTIMATE				
SAMPLE 1 (N= 3)				
PARAMETER SET 1	0.8651	0.8505	0.8379	0.8326
PARAMETER SET 2	0.8688	0.8639	0.8564	0.8536
PARAMETER SET 3	0.8803	0.8356	0.8165	0.8076
SAMPLE 2 (N= 3)				
PARAMETER SET 1	0.7830	0.6875	0.6423	0.6197
PARAMETER SET 2	0.7530	0.6566	0.6154	0.5964
PARAMETER SET 3	0.7970	0.6997	0.6503	0.6240
SAMPLE 3 (N= 5)				
PARAMETER SET 1	0.8264	0.7559	0.7252	0.7112
PARAMETER SET 2	0.8161	0.7478	0.7203	0.7085
PARAMETER SET 3	0.8306	0.7562	0.7217	0.7049
SAMPLE 4 (N= 5)				
PARAMETER SET 1	0.9396	0.9262	0.9223	0.9210
PARAMETER SET 2	0.9467	0.9385	0.9366	0.9360
PARAMETER SET 3	0.9329	0.9131	0.9061	0.9034
SAMPLE 5 (N= 8)				
PARAMETER SET 1	0.8828	0.8463	0.8338	0.8292
PARAMETER SET 2	0.8828	0.8501	0.8399	0.8363
PARAMETER SET 3	0.8809	0.8396	0.8241	0.8130
SAMPLE 6 (N= 8)				
PARAMETER SET 1	0.8932	0.8523	0.8356	0.8284

```
        PARAMETER SET 2   0.8925    0.8541    0.8394    0.8334
        PARAMETER SET 3   0.8924    0.8481    0.8288    0.8200


     NO SAMPLE DATA

        PARAMETER SET 1   0.8631    0.8020    0.7689    0.7487
        PAPAMETER SET 2   0.8333    0.7918    0.7747    0.7658
        PARAMETER SET 3   0.8788    0.8111    0.7691    0.7412


  PARAMETEP SET 1:     PARAMETER SET 2:     PARAMETER SET 3:

       V= 460.00            V= 250.00            V= 600.00

       U= 139.00            U=  50.00            U= 250.00

CORE USAGE       OBJECT CODE=   12064 BYTES,ARRAY AREA=   20856

DIAGNOSTICS        NUMBER OF ERRORS=       0, NUMBER OF WARNINGS=

COMPILE TIME=     0.90 SEC,EXECUTION TIME=   211.20 SEC,     11
```

Section 5.2 lists these additional criteria.

4.3.1 Time Horizon Variation

The variation in availability estimation methods due to time horizon is very slight. For Data Set 1, the average decrease in estimate from time T=100 to time T=400 for the maximum likelihood estimate (MLE) was 0.0481. The traditional Bayesian estimate's (TB) average decrease was 0.0576, and the decrease for Brender's Bayesian (BB) was 0.0541. The decreases for the Data Set 2 estimates are similarly alike in variation: MLE - 0.0699, TB - 0.0894, and BB - 0.0828. Thus, the estimation methods are equally variable with respect to time horizon, so no one method stands out as "best". To ease future comparisons, the availability estimate at time T=400 will be used, ignoring the others, since this is the value which, in theory, most approximates the estimate as $T \to \infty$, i.e., the steady state availability.

4.3.2 Variability Between Samples

Using the availability estimates at time T=400, the variabilities between samples for each method for each data set are outlined in Table 4-7 . The variability was calculated by simply subtracting the lowest value observed from the highest within each method for each data set. For the Bayesian estimators, since different prior parameter sets were used, the variability between samples was calculated for each parameter set, then all three were averaged to obtain the sample variability for each method. Note that the traditional Bayesian estimate exhibits the lowest variability between samples for both of the data sets, with the maximum likelihood estimate exhibiting the largest. This means the traditional Bayesian estimation method compensates for smaller and/or more biased samples than do the other methods.

Table 4-7: Variability Between Samples

		Data Set 1		Data Set 2
Maximum Likelihood Estimate		0.2761		0.4017
Traditional Bayesian Estimate		0.1739		0.2913
Parameter Set 1	0.1372		0.2148	
Parameter Set 2	0.2402		0.4600	
Parameter Set 3	0.1444		0.1991	
Brender's Bayesian Estimate		0.2056		0.3068
Parameter Set 1	0.2029		0.3013	
Parameter Set 2	0.2375		0.3396	
Parameter Set 3	0.1764		0.2794	

Parameter Set 1: V and U correspond exactly to the population mean on and off times

Parameter Set 2: V and U are too low

Parameter Set 3: V and U are too high

4.3.3 Closeness to Steady State Availability

Recall, the steady state availabilities are 0.9086 and 0.7682 for Data Sets 1 and 2, respectively. To evaluate the methods in terms of closeness, determine the percentages of estimates for each method that come within ±5% of the steady state availability. Thus, to be counted as "close" an estimate must fall between 0.8586 and 0.9586 for Data Set 1 and between 0.7282 and 0.8082 for Data Set 2. Table 4-8 lists the "close" estimates for each data set. Note that the traditional Bayesian estimation method provides the most "close" estimates, with Brender's Bayesian estimation a close second. The maximum likelihood method provides the least amount of "close" estimates

4.4 Sensitivity Analyses

The preceding results are useful when the three methods are being compared without regard to sample composition or prior parameter selection (for the Bayesian methods). But, often, it is desired to know what estimation method fares better given a certain sample size, or a certain sample type (i.e., biased or not), or a certain "adequacy" of prior parameters.

Again, using only the estimates at time T=400, the averages of the estimates according to the categories of sample size and sample type for each of the estimates in each of the data sets were calculated. The estimates according to sample size are given in Table 4-9 and the estimates according to sample type are listed in Table 4-10. An analysis of variance was run to determine the significance of and interaction between the two sample effects and the three availability estimation methods on the availability estimate. The analysis of variance showed no significant main or interaction effects. Therefore, in general, one cannot say a particular estimation method is better given a biased sample or given a certain sample size, based on this formal analysis.

Table 4-8: Number of Availability Estimates Close to the Steady-State Availability (± 5%)

	Data Set 1	Data Set 2	Overall
Maximum Likelihood Estimate	3 (50%)	0 (0%)	3 (25%)
Traditional Bayesian	7 (33%)	7 (33%)	14 (33%)
Brender's Bayesian	9 (43%)	4 (19%)	13 (31%)

Table 4-9: Availability Estimates by Sample Size

	Data Set 1	Data Set 2
Sample Size 3		
MLE	0.9151 *	0.7154
TB	0.7996	0.6342
BB	0.9102 *	0.7223
Sample Size 5		
MLE	0.8152	0.8281
TB	0.8072	0.6454
BB	0.8378	0.8142
Sample Size 8		
MLE	0.8501	0.8438
TB	0.8448	0.8110
BB	0.8547	0.8276
No Sample Data		
MLE	--	--
TB	0.8733 *	0.7389 *
BB	0.8738 *	0.7519 *

* "close" to steady state ($\pm$ 5%)

steady-state availabilities
Data Set 1: 0.9086
Data Set 2: 0.7682

Table 4-10: Availability Estimates by Sample Type (biasedness)

	Data Set 1	Data Set 2
Random Samples		
MLE	0.8992 *	0.8103
TB	0.8643 *	0.7178
BB	0.8950 *	0.7891 *
Biased Samples		
MLE	0.8210	0.7812 *
TB	0.7701	0.6759
BB	0.8402	0.7869 *

* "close" to steady state (± 5%)

steady-state availabilities

Data Set 1: 0.9086
Data Set 2: 0.7682

However, if an informal analysis is conducted (i.e., merely studying Tables 4- 9 and 4-10), the following observations are made:

(1) for the smallest sample size (N=3) Brender's Bayesian estimates are closest to steady-state

(2) the no-sample-data case provides, in general, closer estimates to steady-state than the larger sample size cases

(3) the random samples yield closer estimates to steady state than the biased samples (for Data Set 1)

(4) overall, traditional Bayesian estimation method yields the most estimates closest to steady state

(5) for the Bayesian methods, the no-data case yields estimates closer to steady state than the data cases.

Again, using an informal analysis on Tables 4-1 to 4-7, the following observations can be made with regard to "adequacy" of prior parameters:

(1) the variability between samples is less with Parameter Set 3 and greatest with Parameter Set 2.

(2) Parameter Set 3 provides the greatest amount of estimates closest to steady-state for the traditional Bayesian estimate and for Brender's Bayesian estimate.

In other words, if the experimenter cannot determine a good, close set of prior parameters, it is better to overestimate rather than underestimate them, given these two types of data sets.

One final comment: the traditional Bayesian estimates are almost always uniformly less than those of the other methods, which may be a fault in the method, but also this may be due to the numerical integration techniques. As stated before, these estimates may be slightly improved with a more refined numerical intergration technique.

Chapter 5 - EVALUATION AND SELECTION USING MULTIPLE ATTRIBUTE DECISION MAKING METHODS

Now that the estimates have been calculated using each of the three estimation methods: maximum likelihood, traditional Bayesian and Brender's Bayesian, under various conditions, the remaining task is to evaluate these results in order to select the best estimation method. But how is the choice to be made? Each method has its own unique advantages. Clearly, tradeoffs, along with their magnitudes, must be discovered and defined. But, again, how?

To begin the selection process, as with all problem solving processes, a set of goals (and/or objectives) must be established. These goals are more readily usable when translated into a set of attributes by which the alternatives are then judged. Many multiple attribute decision making (MADM) methods are available for this judging process [9]. For this selection problem, five methods: dominance, simple additive weighting, linear assignment, ELECTRE and TOPSIS, are used and the results will be compared, since different methods sometimes yield different choices. First, however, a brief introduction to the concepts of MADM are presented, along with the formulation of the goals and attributes of this particular problem.

5.1 An Introduction to MADM

In the study of decision making in complex environments, terms such as "multiple objectives," "multiple attributes," "multiple criteria," or "multi-dimensional" are used to describe decision situations. Often these terms are used interchangeably [13 a], and no universal definitions of these terms are available [11 a]. However, the term multiple criteria decision making (MCDM) has become the accepted designation for all methods dealing with multiple objective decision making (MODM) and/or multiple attribute decision making (MADM). MODM methods are used for an infinite set of alternatives implicity defined by a set of constraints, similar to linear programming, while MADM methods are used for a finite set of alternatives each with specified characteristics. In other words, MODM involves a design problem and MADM involves a selection problem. Note that the selection of the best estimation method problem falls into the MADM genre.

MADM methods, as inferred earlier, are management decision aids used in evaluating and selecting a desired alternative (here, estimation method) from several available [10]. They are used in decision analysis when two or more goals (objectives) are to be achieved and two or more alternatives are available. However, if one alternative is better for achieving one goal, another alternative for another goal, etc..., no single alternative dominates the others by being better than all other alternatives for all objectives, hence, a tradeoff of the achievement of one goal for the achievements of others must ultimately be made. This tradeoff is made by the decision maker (DM).

Many MADM methods are available. Different methods are used according to the type of information garnered from the DM and the salient feature of the information available. Because of the nature of the best estimation method problem, the MADM methods of dominance, linear assignment, simple additive weighting, ELECTRE, and TOPSIS will be used. For a complete review of all other MADM methods see Hwang and Yoon [9].

5.2 Solution of the Best Estimation Method Problem

In order to choose the best estimation method for system availability, the goals of estimation must first be specified, i.e., what is desirable for an estimation method.

Common sense would dictate that one goal should be closeness to the true availability. But, since the true availability is never known for a system, the steady state availability serves as the most logical approximation. Another goal for the estimation method would be for it to have a lesser dependence on the size and type of sample drawn. This would be convenient because if, perhaps, an extremely small or biased sample is the only one available, a fairly good estimate of the system availability could still be made, since the biasedness or small size would not affect the estimate much. Other important goals would be short computation time on a computer with the programming itself not too difficult; and, of course, the theory and methodology should not be too complex as to confuse the user.

These goals can easily be translated into a set of attributes by which the three estimation methods can be evaluated. These attributes are:

1. Closeness to steady state availability ($\pm 5\%$)
2. Variability between samples
3. Computer execution time
4. Ease of programming
5. Ease of understanding

Now, how can each estimation method be rated on each of the five attributes? The first three are specified through the results of the examples carried out in Chapter Four. The last two are essentially subjective, with the experimenter rating each of the methods on a scale from very easy, easy, moderate, difficult to very difficult. The methods, along with their ratings within the attributes are listed in Figure 5.1. This figure is then transformed into a decision matrix as seen below. The decision matrix is an important tool used in all MADM methods.

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\ \begin{bmatrix} 0.25 & 0.33 & 0.31 \\ 0.3389 & 0.2326 & 0.2562 \\ 0.58 & 143.28 & 2.99 \\ \text{very easy} & \text{difficult} & \text{easy} \\ \text{easy} & \text{moderate} & \text{difficult} \end{bmatrix} \end{array} .$$

Attributes / Methods	A_1 Closeness to Steady State (% of Est's $\pm$ 5%)	A_2* Variability Between Samples	A_3 Computer Execution Time (sec.)	A_4 Ease of Programming	A_5 Ease of Understanding
M_1 Maximum Likelihood	0.25	0.3389	0.58	very easy	easy
M_2 Traditional Bayesian	0.33	0.2326	143.28	difficult	moderate
M_3 Brender's Bayesian	0.31	0.2562	2.99	easy	difficult

* A_2 calculated from the average of the two data sets.

(very easy $\rightarrow$ very difficult)

Figure 5.1: Methods and their attributes

5.2.1 Transformation of the Attributes

Note that only three of the attributes have quantitative ratings. How can the qualitative attributes be evaluated to compare with the quantitative attributes? This is important because most MADM methods evaluate the alternatives according to the aggregate of the attribute values. One cannot add apples and oranges. Also, the decision matrix is most easily handled when all of its elements are numerical and on the same scale. So, all the quantitative values must also be modified so they are on a common scale.

To quantify the qualitative attributes, use a bipolar scale as shown in Figure 5.2.

Next, to scale the quantitative attributes, a range of values must first be picked. For this example values from 0.0 → 1.0 will be handy since A_4 and A_5 are already using this interval. To convert each attribute row, a linear transformation is made by using the formulae

$$r_{ij} = \frac{X_{ij}}{\max_i (X_{ij})} \quad , \text{ for benefit criterion} \tag{1}$$

$$r_{ij} = \frac{1/X_{ij}}{\max_i \left(\frac{1}{X_{ij}} \right)} = \frac{\min (X_{ij})}{X_{ij}} \quad , \text{ for cost criterion} \tag{2}$$

where X_{ij} are the original decision matrix values and r_{ij} are the transformed decision matrix values. Note that A_1, A_4 and A_5 are benefit criteria (i.e, higher values are best) while A_2 and A_3 are cost criteria (i.e., lower values are best).

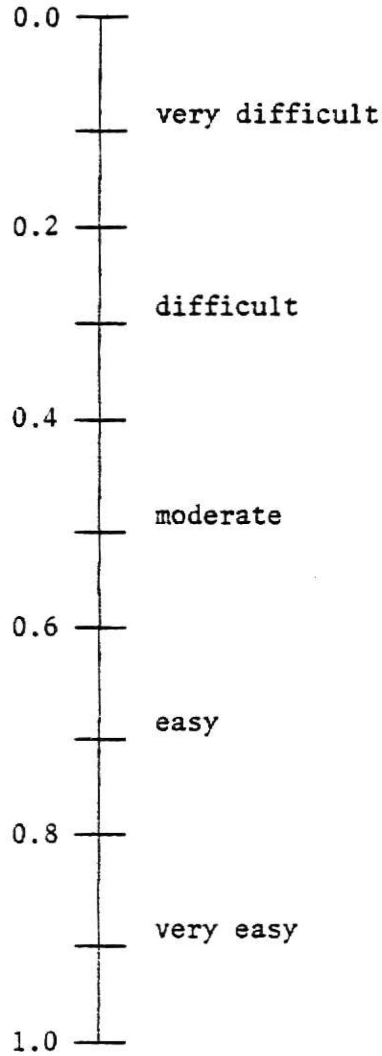


Figure 5-2: Bipolar scale for qualitative attributes A_4 and A_5.

The transformed decision matrix, using the bipolar scale and equations (1) and (2) is:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\ \begin{bmatrix} 0.7576 & 1.0 & 0.9394 \\ 0.6863 & 1.0 & 0.9079 \\ 1.0 & 0.0040 & 0.1940 \\ .9 & .3 & .7 \\ .7 & .5 & .3 \end{bmatrix} \end{array}$$

Note that <u>any</u> value closest to 1.0 is best, be it a benefit or a cost, due to the linear transformation used in eq. (2).

5.2.2 Weights of the Attributes

Some MADM methods such as simple additive weighting, the linear assignment method, ELECTRE, and TOPSIS require the decision maker (DM) to supply information about the relative importance of each of the attributes. This is usually stated by a set of weights normalized to sum to 1. In the case of n attributes, the weight set is defined as

$$\underline{w}^T = (w_1, w_2, \ldots w_n) \tag{3}$$

$$\sum_{i=1}^{n} w_i = 1 \tag{4}$$

Many methods can be used to determine this weight set using pairwise comparisons of the attributes, since the DM usually cannot state the weight set outright. For simplicity, however, in this problem assume the DM has supplied the following weights for each of the attributes:

$$\underline{w} = (0.3, 0.3, 0.1, 0.2, 0.1) \tag{5}$$

This weight vector states the DM feels attributes A_1 and A_2 are equally important with attributes A_3 and A_5 being only 1/3 as important. Attribute A_4 is twice as important as A_3 and A_5 but not quite as important as A_1 and A_2.

5.2.3 Solution by Dominance Method

Method

The MADM method of dominance does not require any transformation of attributes, so the original decision matrix can be used. The method of dominance creates a set of nondominated solutions (i.e. those which have no solutions better than this) through a simple screening process which eliminates all of the dominated alternatives from the decision process.

An alternative is said to be dominated if there is another alternative which excels it in one or more attributes and equals it in the remainder. It is desirable to uncover and eliminate these alternatives from further consideration so attention can be focused on the nondominated solutions.

The dominance method has the following steps:

(1) Compare the first alternative, attribute by attribute, with the second alternative. If one is dominated by the other, it is discarded.

(2) Compare the undominated alternative, attribute by attribute, with the third alternative. Again, discard the dominated one.

(3) Continue this process until all alternatives have been compared with each of the others.

The set of nondominated solutions remaining usually has multiple elements. Therefore, it is common to use this method as a first step in conjunction with other MADM methods, as a "weeding out" of clearly inferior alternatives.

Solution

Recall the original decision matrix.

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\ \left[\begin{array}{ccc} 0.25 & 0.33 & 0.31 \\ 0.3389 & 0.2326 & 0.2562 \\ 0.58 & 143.28 & 2.99 \\ \text{Very Easy} & \text{Difficult} & \text{Easy} \\ \text{Easy} & \text{Moderate} & \text{Difficult} \end{array}\right] \end{array}$$

Also remember that for attribute A_1, the larger the value, the better; while for attributes A_2 and A_3, the smaller the value, the better.

Comparing methods 1 and 2 (maximum likelihood and traditional Bayesian, respectively), note that M_2 is better than M_1 in terms of A_1 and A_2 while M_1 excels M_2 in terms of A_3, A_4 and A_5. Therefore neither dominates the other and both are nondominated.

Comparing methods M_1 and M_3, neither dominate the other; and when comparing M_2 and M_3, neither dominates. In this case, the dominance method was not of much help, for none of the alternatives was eliminated.

5.2.4 Solution by Simple Additive Weighting

Method

Probably the best known MADM method is simple additive weighting. This method selects as best, the alternative that has the maximum value of the weighted averages of the attributes. In other words, the alternative that satisfies the equation

$$M^* = \max_{\{M_j\}} \sum_{i=1}^{n} w_i x_{ij} / \sum_{i=1}^{n} w_i \qquad (6)$$

where

w_i = weight of the i^{th} attribute

x_{ij} = the value of the i^{th} attribute for the j^{th} alternative, using numerically comparable scales

Note: Usually the weights are normalized such that $\sum_{=1}^{n} w = 1$.

This approach is intuitively appealing and easy to execute, but has some drawbacks:

(1) This method assumes **independence** of attributes, and sometimes this is not easy to accomplish.

(2) Sometimes assessing the weights is difficult.

(3) How to change the x_{ij} into comparable values is sometimes a problem.

Solution

Using the weight vector specified by the DM in eq. (5) and the transformed decision matrix, the average weighted values for each method, using eq. (6) are:

$$M_1 = \sum_{i=1}^{5} w_i x_{ij}$$

$$= (.3)(.7576) + (.3)(.6863) + (.1)(1) + (.2)(.9) + (.1)(.7)$$

$$= 0.7832$$

M_2 = 0.7104

M_3 = 0.7436

Therefore method 1, the maximum likelihood method, is chosen as the best with methods 3 and 2 being the second and third best, respectively.

5.2.5 Solution by Linear Assignment Method

Method

The linear assignment method was developed by Bernardo and Blin [1b] and is based upon attributewise rankings with a set of attribute weights. This method, unlike simple additive weighting, features a linear compensatory process for attribute interactions and combinations. Another attractive feature is that qualitative attributes need not be scaled, nor the quantitative attributes need be put on a similar scale because the process uses ordinal rather than cardinal data.

The first task is to rank the methods by attributes and transform these attributewise ranks into overall ranks. A simple way to do this is to compute the sum of the ranks for each alternative, then rank the alternatives from the lowest sum to the highest sum.

For example, consider the following attributewise ranks assuming equal weights for the attributes.

		Attributes		
		X_1	X_2	X_3
	1st	A_1	A_1	A_2
Ranks	2nd	A_2	A_3	A_1
	3rd	A_3	A_2	A_3

Here, A_1 stands for alternative 1, and so on.

Adding to obtain the overall ranks:

$$\text{rank}\ (A_1) = 1 + 1 + 2 = 4$$
$$\text{rank}\ (A_2) = 6$$
$$\text{rank}\ (A_3) = 8$$

Therefore A_1 should be the best alternative.

However, this method is too simplistic because it fails to take into account all of the other alternative attributewise ranks at the same time, since a final rank is merely dependent upon its own summed attributewise rankings. The basic linear compensation requirement is violated. What is needed is a method which does take into account all the attributewise rankings at the same time, rather than using this information sequentially as in the sum-of-the-ranks method. Such a method is now outlined.

First, define a product-attribute matrix, called π, as a square (m x m) nonnegative matrix whose elements π_{ik} represent the frequency with which an A_i is ranked in the k^{th} places of the attributewise rankings. Using the previous example, the π matrix is (again, assuming equal attribute weights):

$$\pi = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} \text{1st} & \text{2nd} & \text{3rd} \end{array} \\ \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \end{array}$$

If equal attribute weights are not assumed, (i.e., for example, a weight vector, w = (.2, .3, .5) is used)the π matrix becomes

$$\pi = \begin{bmatrix} .2+.3 & .5 & .0 \\ .5 & .2 & .3 \\ 0 & .3 & .2+.5 \end{bmatrix} = \begin{bmatrix} .5 & .5 & 0 \\ .5 & .2 & .3 \\ 0 & .3 & .7 \end{bmatrix}$$

Each π_{ik} measures the contribution of A_i to the overall ranking if A_i is assigned to the k^{th} overall rank. The larger π_{ik} indicate more concordance in assigning A_i to the k^{th} overall rank, so the objective is to find the A_i for each k, k=1,2,...m, which maximizes $\sum_{k=1}^{m} \pi_{ik}$. Note this is an m! comparison problem, so a linear programming model is recommended for large m.

Define a permutation matrix P as a square (m x m) matrix whose element $P_{ik} = 1$ if A_i is assigned to the overall rank k and $P_{ik} = 0$ otherwise. The linear assignment method can be expressed in the following LP format:

$$\max \sum_{i=1}^{m} \sum_{k=1}^{m} \pi_{ik} P_{ik} \tag{7}$$

ST:

$$\sum_{i=1}^{m} P_{ik} = 1 \quad k=1, \ldots m \tag{8}$$

$$\sum_{=1}^{m} P_{ik} = 1 \quad i=1, \ldots m \tag{9}$$

$$P_{ik} \in (0,1)$$

Note that if any attribute is tied in the ranking, for example,

		$X_1(w_1)$
	1st	A_1, A_2
Ranks	2nd	-
	3rd	A_3

"split" the attribute $X_1(w_1)$ into two, X_{11} $(w_1/2)$ and X_{12} $(w_1/2)$ each with one-half the original weight value and reassign the alternative as follows:

		X_{11} $(w_1/2)$	$X_{12}(w_1/2)$
	1st	A_1	A_2
Ranks	2nd	A_2	A_1
	3rd	A_3	A_3

<u>Solution</u>

Using the original decision matrix and the same weight vector, eq. (5), as in the simple additive weighting method, the attributewise rankings of the three methods are:

$$\begin{array}{cc} \underline{w} & \\ (.3) & A_1 \\ (.3) & A_2 \\ (.1) & A_3 \\ (.2) & A_4 \\ (.1) & A_5 \end{array} \begin{array}{c} \begin{array}{ccc} \text{1st} & \text{2nd} & \text{3rd} \end{array} \\ \begin{bmatrix} M_2 & M_3 & M_1 \\ M_2 & M_3 & M_1 \\ M_1 & M_3 & M_2 \\ M_1 & M_3 & M_2 \\ M_1 & M_2 & M_3 \end{bmatrix} \end{array}$$

Therefore, the product attribute matrix becomes:

$$\begin{array}{c|ccc} & \text{1st} & \text{2nd} & \text{3rd} \\ \hline M_1 & .1+.2+.1 & 0 & .3+.3 \\ M_2 & .3+.3 & .1 & .1+.2 \\ M_3 & 0 & .3+.3+.1+.2 & .1 \end{array}$$

$$\pi = \begin{bmatrix} .4 & 0 & .6 \\ .6 & .1 & .3 \\ 0 & .9 & .1 \end{bmatrix}$$

The LP formulation with the above π matrix is

$$\max \sum_{i=1}^{3} \sum_{k=1}^{3} \pi_{ik} P_{ik} \tag{10}$$

ST:

$$\sum_{i=1}^{3} P_{ik} = 1 \qquad k = 1,2,3 \tag{11}$$

$$\sum_{k=1}^{3} P_{ik} = 1 \qquad i = 1,2,3 \tag{12}$$

$$P_{ik} \in (0,1)$$

The optimal permutation matrix, P* is

$$P^* = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

In other words, the final ranking of the methods is (M_2, M_3, M_1), or method 2 traditional Bayesian, is best.

5.2.6 Solution by ELECTRE

Method

The ELECTRE (Elimination et Choice Translating Reality) method was originally introduced by Benayoun, et al [1a]. Roy, Nijkamp, and van Delft, et al , then further developed the method into its present form [13b], [13c], [16a], [16b], [16c], [21a].

ELECTRE uses the concept of outranking relationships. An outranking relationship, $A_k \rightarrow A_\ell$, means that even though two alternatives k and ℓ are mathematically nondominated (see 5.2.3, Solution by Dominance), the decision maker accepts the risk of regarding A_k as almost surely better than A_ℓ. Through the successive assessments of outranking relationships of the other alternatives, the dominated alternatives generated by the outranking relationships can be eliminated. ELECTRE sets the criteria for the mechanical assessment of the outranking relationships, since the construction of them is not an unambiguous task for the DM.

This method consists of pairwise comparisons of alternatives which are based on the degree to which evaluations of the alternatives and their preference weights confirm or contradict the pairwise dominance relationships between the alternatives.

ELECTRE examines both the degree to which the preference weights are in agreement with the pairwise dominance relationships and the degree at which the weighted evaluations differ from each other. These stages are based on a "concordance and discordance" set, so this method is also known as concordance analysis. The method has nine steps:

Step 1 - Calculate the normalized decision matrix

This procedure transforms the various attribute dimensions into non-dimensional attributes, allowing comparison across the attributes. Each normalized value r_{ij} of the normalized decision matrix R is calculated as

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{j=1}^{n} x_{ij}^{2}}} \tag{13}$$

$$R = \begin{bmatrix} r_{11} & r_{12} \cdot \cdot \cdot \cdot & r_{1n} \\ & & \cdot \\ r_{21} & r_{22} \cdot \cdot \cdot \cdot & \cdot \\ & & \cdot \\ r_{m1} & r_{m2} \cdots \cdot \cdot & r_{mn} \end{bmatrix}$$

n = no. of alternatives
m = no. of attributes

Now, all attributes have the same vector unit length.

<u>Step 2</u> - <u>Calculate the weighted normalized decision matrix</u>

This matrix is calculated simply by multiplying each row of the R matrix with its associated weight w_i. Label this matrix V.

$$V = WR$$

$$= \begin{bmatrix} w_1 & 0 \; . \; . \; . & 0 \\ 0 & w_2 . \; . \; . & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & w_m \end{bmatrix} \quad X \quad \begin{bmatrix} r_{11} & r_{12} . \; . \; . \; . & r_{1n} \\ r_{21} & r_{22} . \; . \; . \; . & r_{2n} \\ & & \\ r_{m1} & r_{m2} . \; . \; . \; . & r_{mn} \end{bmatrix}$$

$$V = \begin{bmatrix} w_1 r_{11} & w_1 r_{11} \; . \; . \; . & w_1 r_{1n} \\ w_2 r_{21} & w_2 r_{22} \; . \; . \; . & w_2 r_{2n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ w_m r_{m1} & w_m r_{m2} \; . \; . \; . & w_m r_{mn} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \; . \; . \; . & v_{1n} \\ v_{21} & v_{22} \; . \; . \; . & v_{2n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ v_{m1} & v_{m2} \; . \; . \; . & v_{mn} \end{bmatrix}$$

Step 3 - Determine the concordance and discordance set

For each pair of alternatives k and ℓ ($k,\ell=1,2,\ldots n$, with $k \neq \ell$), the set of decision criteria $J = \{ i | i=1,\ldots m\}$ is divided into two distinct subsets. The concordance set $C_{k\ell}$ of A_k and A_ℓ is composed of all attributes for which A_k is preferred to A_ℓ. In other words,

$$C_{k\ell} = \{ i | x_{ik} \succ x_{i\ell} \} \tag{14}$$

the complementary subset is called the discordance set, which is

$$\begin{aligned} D_{k\ell} &= \{ i | x_{ik} \prec x_{i\ell} \} \\ &= J - C_{k\ell} \end{aligned} \tag{15}$$

Step 4 - Construct the concordance matrix

The concordance matrix is a composite of all the concordance indices $C_{k\ell}$. The concordance index $C_{k\ell}$ measures the relative value of the concordance set $C_{k\ell}$, is equal to the sum of the weights associated with those attributes in the $C_{k\ell}$. It is defined as

$$C_{k\ell} = \frac{\sum_{i \in C_{k\ell}} w_i}{\sum_{i=1}^{m} w_i} \tag{16}$$

But, when the weights are normalized (i.e., sum to one) the concordance index becomes simply

$$C_{k\ell} = \sum_{i=C_{k\ell}} w_i \tag{17}$$

The concordance index reflects the relative importance of A_k with respect to A_ℓ. Obviously, $0 \leq C_{k\ell} \leq \ell$, and a higher value of $C_{k\ell}$ indicates A_k is more preferred to A_ℓ as far as the concordance criteria are concerned.

The (nxn) concordance matrix is then formed by these concordance indices:

$$C = \begin{bmatrix} - & c_{12} & \cdots & c_{1(n-1)} & c_{1n} \\ c_{21} & - & \cdots & c_{2(n-1)} & c_{2n} \\ \vdots & & & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{n(n-1)} & - \end{bmatrix}$$

n= no. of alternatives

Note that , in general, the concordance matrix is not symmetric.

Step 5 - Construct the discordance matrix

The discordance matrix is formed from the discordance indices which represent the degree at which the evaluations of certain A_k are worse than the evaluations of competing A_ℓ. The discordance index $d_{k\ell}$ is defined as

$$d_{k\ell} = \frac{\max\limits_{i \varepsilon D_{k\ell}} \left| v_{ik} - v_{i\ell} \right|}{\max\limits_{i \varepsilon J} \left| v_{ik} - v_{i\ell} \right|} \tag{18}$$

where v_{ij} are the values from the weighted normalized decision matrix.

Note that $0 \leq d_{k\ell} \leq 1$ and a higher value of $d_{k\ell}$ implies A_k is *less favorable than* A_ℓ, according to the discordance criteria. The (nxn) discordance matrix is therefore

$$D_x = \begin{bmatrix} - & d_{12} & d_{13} & \cdots & d_{1n} \\ d_{21} & - & d_{23} & \cdots & d_{2n} \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ d_{n1} & d_{n2} & \cdots & d_{n(n-1)} & - \end{bmatrix}$$

Again, matrix D_x is generally not symmetric.

It is very important to note that the information contained in the concordance matrix differs significantly from that contained in the discordance matrix. Differences among weights are represented in the concordance matrix, while differences among attribute values are represented in the discordance matrix.

Step 6 - Determine the concordance dominance matrix

The concordance dominance matrix is determined with a threshold value of the concordance index. Define that A_k will dominate A_ℓ only if its corresponding concordance index $C_{k\ell}$ exceeds a certain threshold value $\bar{C}$, i.e., if $C_{k\ell} \geq \bar{C}$.

This threshold value can be determined many ways. For example, it could be the average concordance index:

$$\bar{C} = \sum_{\substack{k=1 \\ k \neq \ell}}^{n} \sum_{\substack{\ell=1 \\ \ell \neq k}}^{n} C_{k\ell} \,/\, n(n-1) \tag{19}$$

On the basis of this threshold value, a (nxn) Boolean matrix F (the concordance dominance matrix) can be constructed with the elements (0,1) where

$$f_{k\ell} = 1, \text{ if } C_{k\ell} \geq \bar{C}$$
$$f_{k\ell} = 0, \text{ if } C_{k\ell} < \bar{C} \tag{20}$$

Each element of 1 in the matrix F therefore represents a dominance of one alternative over another.

Step 7 - Determine the discordance dominance matrix

The discordance dominance matrix is constructed exactly like the concordance dominance matrix. A threshold value $\bar{d}$ is calculated as

$$\bar{d} = \sum_{\substack{k=1 \\ k \neq \ell}} \sum_{\substack{\ell=1 \\ \ell \neq k}} d_{k\ell} \,/\, n(n-1) \tag{21}$$

and the elements of the (nxn) discordance dominance matrix G are either (0,1) where

$$g_{k\ell} = 1, \text{ if } \quad d_{k\ell} \leq \bar{d}$$
$$g_{k\ell} = 0, \text{ if } \quad d_{k\ell} > \bar{d} \tag{22}$$

Each element of 1 in the matrix G also represents a dominance of one alternative over another.

Step 8 - Determine the aggregate dominance matrix

Next, the intersection of the concordance dominance matrix F and the discordance dominance matrix G is determined. The resulting matrix, called the (nxn) aggregate dominance matrix E, has its elements $e_{k\ell}$ defined as

$$\begin{aligned} e_{k\ell} &= 1, \text{ if } (f_{k\ell} = 1) \text{ and } (g_{k\ell} = 1) \\ e_{k\ell} &= 0, \text{ otherwise} \end{aligned} \tag{23}$$

Step 9- Eliminate the less favorable alternatives

The aggregate dominance matrix E gives the partial preference ordering of the alternatives. If $e_{k\ell} = 1$, then A_k is preferred to A_ℓ for both the concordance and discordance criteria, but A_k may still be dominated by other alternatives. Hence, the condition that A_k is not dominated, by the ELECTRE method, is

$$e_{k\ell} = 1, \text{for at least one } \ell,\ \ell = 1, \ldots n;\quad k \neq \ell$$

$$e_{ik} = 0, \text{for all } i,\ i = 1, 2, \ldots n;\quad i \neq k,\ i \neq \ell$$

This condition appears difficult to apply, but the dominated alternatives are easily identified in the E matrix. If any column of the E matrix has at least one element of 1, then this column is "ELECTREcally" dominated by the corresponding row(s). Hence, any columns which have an element of 1 are eliminated.

Solution

To begin, use the original decision matrix with the qualitative variables quantified using the bipolar scale in Fig. 5-2 .

Alternatives

$$\begin{array}{c c} \text{Attributes} & \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} & \begin{bmatrix} 0.25 & 0.33 & 0.31 \\ 0.3389 & 0.2326 & 0.2562 \\ 0.58 & 143.28 & 2.99 \\ .9 & .3 & .7 \\ .7 & .5 & .3 \end{bmatrix} \end{array}$$

The weight vector is again $\underline{w} = (.3, .3, .1, .2, .1)$

Step 1 - Calculate the normalized decision matrix.

$$R = \begin{bmatrix} 0.4834 & 0.6381 & 0.5994 \\ 0.6998 & 0.4803 & 0.5290 \\ 0.0040 & 0.9998 & 0.0209 \\ 0.7634 & 0.2545 & 0.5937 \\ 0.7684 & 0.5488 & 0.3293 \end{bmatrix}$$

Step 2 - Calculate the weighted normalized decision matrix.

$$V = \begin{bmatrix} 0.1450 & 0.1914 & 0.1798 \\ 0.2099 & 0.1441 & 0.1587 \\ 0.0004 & 0.1000 & 0.0021 \\ 0.1527 & 0.0509 & 0.1187 \\ 0.0768 & 0.0549 & 0.0329 \end{bmatrix}$$

Step 3 - Determine the concordance and discordance set

Remember that attributes A_1, A_4 and A_5 are benefit criteria (i.e., the higher the value, the better) while A_2 and A_3 are cost criteria (i.e., the lower the value, the better). Then $C_{12} = \{i | x_{i1} \succ x_{i2}\} = \{3,4,5\}$ and $D_{12} = \{1,2\}$.

The remaining combinations of $C_{k\ell}$ are:

$C_{13} = \{3,4,5\}$ $\quad$ $D_{13} = \{1,2\}$

$C_{21} = \{1,2\}$ $\quad$ $D_{21} = \{3,4,5\}$

$C_{23} = \{1,2,5\}$ $\quad$ $D_{23} = \{3,4\}$

$C_{31} = \{1,2\}$ $\quad$ $D_{31} = \{3,4,5\}$

$C_{32} = \{3,4\}$ $\quad$ $D_{32} = \{1,2,5\}$

Step 4 - Construct the concordance matrix

Recall, an element $C_{k\ell}$ is defined as

$$C_{k\ell} = \sum_{i \epsilon C_{k\ell}} w_i$$

so $c_{12} = \sum_{i=C_{12}} w_i = w_3 + w_4 + w_5 = .1 + .2 + .1 = .4$

The complete concordance matrix is then

$$C = \begin{bmatrix} - & 0.4 & 0.4 \\ 0.6 & - & 0.7 \\ 0.6 & 0.3 & - \end{bmatrix}$$

Step 5 - Construct the discordance matrix

Recall, an element $d_{k\ell}$ is defined as

$$d_{k\ell} = \frac{\max\limits_{i\epsilon D_{k\ell}} \left| v_{i1} - v_{i2} \right|}{\max\limits_{i\epsilon J} \left| v_{ik} - v_{i\ell} \right|}$$

so $$d_{12} = \frac{\max\limits_{i\epsilon D_{12}} \left| v_{i1} - v_{i2} \right|}{\max\limits_{i\epsilon J} \left| v_{i1} - v_{i2} \right|} = \frac{\max\{.0464,\ .0658\}}{\max\{.0464,\ .0658,\ .0996,\ .1018,\ .0219\}}$$

$$= \frac{.0658}{.1018} = 0.6464$$

The complete discordance matrix is then

$$D_x = \begin{bmatrix} - & 0.6464 & 1.0 \\ 1.0 & - & 1.0 \\ 0.8574 & 0.2247 & - \end{bmatrix}$$

Step 6 - Determine the concordance dominance matrix.

Taking the threshold value $\bar{c}$ to be the average of the $C_{k\ell}$, then

$$\bar{c} = \frac{.4 + .4 + .6 + .7 + .6 + .3}{6} = 0.5$$

The concordance dominance matrix is

$$F = \begin{bmatrix} - & 0 & 0 \\ 1 & - & 1 \\ I & 0 & - \end{bmatrix}$$

Step 7 - Determine the discordance dominance matrix

Taking the threshold value $\bar{d}$ to be the average of the $d_{K\ell}$, then

$$\bar{d} = \frac{.6464 + 1 + 1 + 1 + .8574 + .2247}{6} = 0.7881$$

The discordance dominance matrix is

$$G = \begin{bmatrix} - & 1 & 0 \\ 0 & - & 0 \\ 0 & 1 & - \end{bmatrix}$$

Step 8 - Determine the aggregate dominance matrix

Combining matrices F and G,

$$E = \begin{bmatrix} - & 0 & 0 \\ 0 & - & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 9 - Eliminate less favorable alternatives

From the E matrix, no dominance relationships evolve (since there are no 1's in the matrix). Therefore, in this case, the ELECTRE method yields no less favorable alternatives. Has the method been a waste of time? No. Consider what determines the concordance dominance and discordance dominance matrices: the threshold values of $\bar{c}$ and $\bar{d}$. This is one weak point in the ELECTRE method, since these values are rather arbitrary and yet still have a significant impact on the final solution.

Since the values are fairly arbitrary, as a remedy for this problem, relax the threshold values to permit more 1's in the F and G matrices which, hopefully, will leave some 1's in the E matrix with which to make some dominance conclusions.

To relax $\bar{c}$, lower it; and to relax $\bar{d}$, raise it. For this problem, relax $\bar{c}$ to 0.4 and d to 0.86. The F and G matrices then become

$$F = \begin{bmatrix} - & 1 & 1 \\ 1 & - & 1 \\ 1 & 0 & - \end{bmatrix} \qquad G = \begin{bmatrix} - & 1 & 0 \\ 0 & - & 0 \\ 1 & 1 & - \end{bmatrix}$$

The aggregate dominance matrix, E, is therefore

$$E = \begin{bmatrix} - & 1 & 0 \\ 0 & - & 0 \\ 1 & 0 & - \end{bmatrix}$$

Now, some dominance relationships can be recognized. It is clear that

$$M_1 \succ M_2$$
$$M_3 \succ M_1$$

so the following ranking of alternatives can be made:

$$M_3, M_1, M_2$$

Therefore, the best alternative is M_3, or Brender's Bayesian method.

5.2.7 Solution by TOPSIS

Method

TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) ranks the alternatives in a multi-attribute decision making problem according to their closeness to the "ideal" solution and their distance from the "negative-ideal" solution. Each attribute is assumed to have monotonically increasing (or decreasing) utility, (i.e., the larger the attribute outcome, the greater the preference for "benefit " criteria and the lesser the preference for "cost" criteria). It is then easy to locate both the "ideal" solution (i.e., the one composed of all the best attribute values attainable) and the "negative-ideal" solution, (i.e., the one composed of all the worst possible attributes values attainable). TOPSIS considers the alternatives' distances to both the ideal and negative-ideal solutions simultaneously by calculating the relative closeness to the ideal solution. The best alternative ends up as the point which is closest (in terms of Euclidean distance) to the ideal solution, but yet is far from the negative-ideal solution.

The TOPSIS method evaluates the following decision matrix containing m attributes and n alternatives.

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} X_1 & X_2 & \cdots & X_n \end{array} \\ \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x_{m1} & \cdots & \cdots & x_{mn} \end{bmatrix} \end{array}$$

where

A_i = the i^{th} attribute considered

X_j = the j^{th} alternative considered

x_{ij} = the numerical outcome of the j^{th} alternative with respect to the i^{th} attribute

Any outcome which is expressed in a nonnumerical way must be quantified, and each attribute must be weighted according to its importance in the final decision. The method can be summarized in the following steps.

Step 1 - Calculate the normalized decision matrix

As in the ELECTRE method, this procedure transforms the various attribute dimensions into nondimensional attributes, allowing comparison across the attributes. Each normalized value r_{ij} of the normalized decision matrix R is calculated as

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{j=1}^{n} x_{ij}^2}} \tag{24}$$

Each attribute now has the same vector unit length.

Step 2 - Calculate the weighted normalized decision matrix

Again, same as in the ELECTRE method, this matrix is calculated simply by multiplying each row of the R matrix with its associated weight w_i. Label the matrix V.

$$V = \begin{bmatrix} V_{11} & V_{12} & \cdots & V_{1n} \\ V_{21} & V_{22} & \cdots & V_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ V_{m1} & V_{m2} & \cdots & V_{mn} \end{bmatrix} = \begin{bmatrix} w_1 r_{11} & w_1 r_{12} & \cdots & w_1 r_{1n} \\ w_2 r_{21} & w_2 r_{22} & \cdots & w_2 r_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ w_m r_{m1} & w_m r_{m2} & \cdots & w_m r_{mn} \end{bmatrix}$$

Step 3 - Determine the ideal and negative-ideal solutions

Let the two artifical alternatives A^* and A^- (the ideal and negative-ideal solutions, respectively) be defined as

$$A^* = \{ (\max v_{ij} \mid i \varepsilon J), (\min v_{ij} \mid i \varepsilon J') \mid j = 1,2,\ldots,n\}$$

$$= \{ v_1^*, v_2^*, \ldots, v_m^*\} \tag{25}$$

where $J = \{ i = 1,2,\ldots,m \mid i$ associated with benefit criteria$\}$

$J' = \{ i = 1,2,\ldots,m \mid i$ associated with cost criteria$\}$

Step 4 - Calculate the separation measure

The separation between each alternative can be measured by the n-dimensional Euclidean distance. The separation of each alternative from the ideal is given as

$$S_{*j} = \sqrt{\sum_{i=1}^{m} (v_{ij} - v_i^*)^2}, \qquad j = 1,2,\ldots,n \tag{27}$$

Similarly, the separation from the negative-ideal is

$$S_{-j} = \sqrt{\sum_{i=1}^{m} (v_{ij} - v_i^-)^2}, \qquad j = 1,2,\ldots,n \tag{28}$$

Step 5 - Calculate the relative closeness to the ideal solution

The relative closeness of A_i with respect to the ideal A^* is defined as

$$C_{*j} = \frac{S_{-j}}{(S_{*j} + S_{-j})}, \qquad j = 1,\ldots,n, \tag{29}$$

Note that C_{*j} must be in the interval [0,1], and $C_{*j} = 1$ if $A_i = A^*$ and $C_{*j} = 0$ if $A_i = A^-$. Therefore, an alternative is closer to A^* as C_{*j} approaches 1.

Step 6 - Construct the preference order

Using the C_{*j}, the alternatives can now be ranked from most preferred to least preferred. The alternatives are ranked according to the descending order of the C_{*j}.

Solution

The first two steps of the solution are exactly the same as for the ELECTRE method.

Step 1 - Calculate the normalized decision matrix

$$R = \begin{bmatrix} 0.4834 & 0.6381 & 0.5994 \\ 0.6998 & 0.4803 & 0.5290 \\ 0.0040 & 0.9998 & 0.0209 \\ 0.7634 & 0.2545 & 0.5937 \\ 0.7684 & 0.5488 & 0.3293 \end{bmatrix}$$

Step 2 - Calculate the weighted normalized decision matrix

$$V = \begin{bmatrix} 0.1450 & 0.1914 & 0.1798 \\ 0.2099 & 0.1441 & 0.1587 \\ 0.0004 & 0.1000 & 0.0021 \\ 0.1527 & 0.0509 & 0.1187 \\ 0.0768 & 0.0549 & 0.0329 \end{bmatrix}$$

Step 3 - Determine ideal and negative-ideal solutions

$$A^* = (\max_j v_{1j}, \min_j v_{2j}, \min_j v_{3j}, \max_j v_{4j}, \max_j v_{5j}$$

$$= (0.1914, 0.1441, 0.0004, 0.1527, 0.0768)$$

$$A^- = (\min_j v_{1j}, \max_j v_{2j}, \max_j v_{3j}, \min_j v_{4j}, \min_j v_{5j})$$

$$= (0.1450, 0.2099, 0.1000, 0.0509, 0.0329)$$

Step 4 - Calculate the separation measures

$$S_{*j} = \sqrt{\sum_{i=1}^{5} (v_{ij} - v_i^*)^2}, \qquad j = 1,2,3$$

$$S_{*1} = 0.0805$$

$$S_{*2} = 0.1441$$

$$S_{*3} = 0.0586$$

$$S_{-j} = \sqrt{\sum_{i=1}^{5} (v_{ij} - v_i^-)^2}, \qquad j = 1,2,3$$

$$S_{-1} = 0.1490$$

$$S_{-2} = 0.0835$$

$$S_{-3} = 0.0918$$

Step 5 - Calculate the relative closeness to the ideal solution

$$C_{*1} = \frac{S_{-1}}{(S_{*1} + S_{-1})} = \frac{0.1490}{0.0805 + 0.1490} = 0.6492$$

$$C_{*2} = 0.3669$$

$$C_{*3} = 0.6104$$

Step 6 - Construct the preference order

The descending order of the C_{*i} gives the preference ranking of alternatives:

$$M_1, M_3, M_2$$

5.3 Overall Results and Selection

5.3.1 Using Original Weight Set

The results of the five MADM methods using the weight vector (.3, .3, .1, .2, .1) are presented in Table 5-1. Why are the results different? If one availability estimation method was truly best, it stands to reason that it should turn up best in all the MADM methods. This is not necessarily so, however, because recall that each method evaluates the alternatives using different principles and perspectives. The simple

Table 5-1:
MADM Results Using
$\underline{w}$ = (.3,.3,.1,.2,.1)

MADM Method	RANKING 1st	2nd	3rd
Dominance	M_1, M_2, M_3	M_1, M_2, M_3	M_1, M_2, M_3
Simple Additive Weighting	M_1	M_3	M_2
Linear Assignment	M_2	M_3	M_1
ELECTRE	M_3	M_1	M_2
TOPSIS	M_1	M_3	M_2

M_1 = Maximum Likelihood Estimate

M_2 = Traditional Bayesian Estimate

M_3 = Brender's Bayesian Estimate

additive weighting method constructs the preference order according to the alternatives' weighted average outcomes, while the linear assignment method uses ordinal attributewise rankings and then combines them into an overall preference ranking which is in closest agreement with the attributewise rankings. The ELECTRE method uses pairwise comparisons of alternatives and the preference order is based on the degree to which alternative outcomes and attribute weights confirm or contradict the pairwise dominance relationships between alternatives. The TOPSIS method ranks the alternatives according to the relative closeness to the ideal solution, along with the relative distance from the negative ideal solution. Hence, it is expected that even with the same data and the same weight set, the preference order obtained by each method could be different. Note that M_1 and M_3 most frequently turn up as the first or second choice, suggesting the best estimation method is either M_1 or M_3. But which? Only one must ultimately be chosen. Hwang and Yoon [9] suggest three ordering techniques to be used especially when there are conflicting results from the use of two or more MADM methods on the same data with the same weight set. Only one, the average ranking procedure, will be used here.

The average ranking procedure produces an aggregate rank order by calculating an average rank for each of the alternatives and uses these average ranks to determine the aggregate. If two or more alternatives are tied, then the one with the smallest standard deviation is ranked higher.

Using the average ranking procedure for this problem, alternative M_3 (Brender's Bayesian method) turns out to be the best. Table 5-2 lists the average ranks for each alternative along with their standard deviations. M_3 is the alternative with both the smallest average rank and smallest standard deviation.

5.3.2 Using Different Weight Sets

Up until now, the assumption has been that the DM feels attributes A_1 and A_2, the closeness to steady-state availability and the variability between

Table 5-2:
Preference Rankings by
the Four MADM Methods
$\underline{w}$ =((.3,.3,.1,.2,.1))

MADM Method	M_1 Maximum Likelihood	M_2 Traditional Bayesian	M_3 Brender's Bayesian
Simple Additive Weighting	1	3	2
Linear Assignment	3	1	2
ELECTRE	2	3	1
TOPSIS	1	3	2
Average Rank	1.75	2.5	1.75
Std. Dev.	0.9574	-	0.5

samples, respectively, are most important in determining the best estimation method of system availability. The normalized weights assigned (.3 for each) reflect the magnitude of this importance over the other three attributes A_3 (computer execution time), A_4 (ease of programming) and A_5 (ease of understanding) which were weighted as .1, .2 and .1, respectively.

But what if the DM considered the importance of each attribute differently? For example, perhaps the DM, due to a severe budgetary constraint, had to place a heavier importance on A_3, the computer execution time attribute. How would the results of the MADM analysis change? Would they change at all? In other words, how sensitive are the MADM methods to the data in conjunction with its attribute weight set?

Table 5-3 outlines the results of the last four MADM methods (the dominance method, since it is not dependent on any attribute weights, remains the same and therefore is not listed) when even more weight is given to the first two attributes (.4, .4, .05, .1, .05). Note, again, that more than one alternative is ranked first; but this time M_2 and M_3 are the contenders. Using the average ranking procedure, the average ranks being listed in Table 5-4, note M_2 becomes the best alternative with an average rank of 1.4, slightly edging out M_3 with its average rank of 1.6.

Assume, now, that the DM has no preconceived notion of what the attribute weights are, s/he just has the data of the decision matrix. One method of determining the weights using simply the data available is the entropy method [9], which is based on the amount of uncertainty present in the data. Ultimately, the attribute with the most uncertainty will exhibit the higher weight. The resultant weight set using this method, incorporating the original weight set of (.3, .3, .1, .2, .1) as the DM's a priori bias, is (.0159, .0316, .7732, .1356, .0437). Note that the heaviest weight is on A_3, the computer execution time, because this is the attribute that exhibits the largest range of data values (0.58 to 143.28), i.e., the largest uncertainty. The results

Table 5-3:
MADM Results Using
$\underline{w}$=(.4,.4,.05,.1,.05)

MADM Method	RANKINGS 1st	2nd	3rd
Simple Additive Weighting	M_2	M_3	M_1
Linear Assignment	M_2	M_3	M_1
ELECTRE	M_2,M_3	M_2,M_3	M_1
TOPSIS	M_3	M_2	M_1

M_1= Maximum Likelihood Estimate

M_2= Traditional Bayesian Estimate

M_3= Brender's Bayesian Estimate

Table 5-4:
Preference Rankings by
the Four MADM Methods
($\underline{w}$ = (.4,.4,.05,.1,.05))

MADM Method	M_1 Maximum Likelihood	M_2 Traditional Bayesian	M_3 Brender's Bayesian
Simple Additive Weighting	3	1	2
Linear Assignment	3	1	2
ELECTRE (1)	3	1	2
ELECTRE (2)	3	2	1
TOPSIS	3	2	1
Average Rank	3.0	1.4	1.6

of the four MADM methods using this weight set are found in Table 5-5. M_1, the maximum likelihood method, is the undisputed choice; meaning that, here, the savings in computer time greatly outweighs its system availability estimation inaccuracies. Realize, however, that this is the case simple because an enormous weight is given to the computer execution time attribute.

5.3.3 Discussion

Notice that with the given data in the matrix, the four MADM methods used yield a different aggregate "best" estimation method, when different weight sets are used. With the DM's original weight set, $\underline{w}$ = (.3, .3, .1, .2, .1); M_3, Brender's Bayesian method, is the best estimation method for system availability. When more weight is placed on the first two attributes of closeness to steady-state availability and variability between samples, $\underline{w}$ = (.4, .4, .05, .1, .05); M_2, the traditional Bayesian method is best. The reason for this stems from the fact that the traditional Bayesian method yields availability estimates closer to steady-state with smaller variability than Brender's Bayesian method. So when more emphasis is placed on these two attributes, the alternative that is best with regards to those attributes would be regarded as best overall if its deficiencies in the other attributes are not very great.

However, when more weight is placed on attribute A_3, computer execution time, as is the case when the weights are arrived at by the entropy method, alternative M_1, the maximum likelihood method, is the best method. But these weights do not truly reflect the DM's attribute weights, so any answer arrived at when using these entropy weights cannot be regarded as valid and in accordance with the DM's feelings about the attributes. It is extremely important to realize that in order to arrive at a correct solution using the four MADM methods of simple additve weighting, linear assignment, ELECTRE and TOPSIS, the DM's feelings about the attributes' importance on the final decision be correctly stated in the weight set. When this is not possible it would be better to use other MADM methods [9].

Table 5-5:

MADM Results Using

$\underline{w}$ = (.0159, .0316, .7732, .1356, .0437)

MADM Method	Rankings 1st	2nd	3rd
Simple Additive Weighting	M_1	M_3	M_2
Linear Assignment	M_1	M_3	M_2
ELECTRE	M_1	M_3	M_2
TOPSIS	M_1	M_3	M_2

M_1 = Maximum Likelihood Estimate

M_2 = Traditional Bayesian Estimate

M_3 = Brender's Bayesian Estimate

The most important conclusion that is wrought from the MADM analysis is that, when the DM's attribute weights are correctly established, the Bayesian methods of estimation are better than the classical maximum likelihood method. Both Bayesian methods prove better because they produce a larger proportion of estimates close ($\pm 5\%$) to the steady-state availability, along with dampening the sample-to-sample variability of availability estimates, which is extremely desirabl when large, unbiased samples are unavailable. And the Bayesian estimation methods are able to do this without much undue sacrifices pertaining to computer execution time, ease of computer programming and ease of understanding (theory-wise) of the methods.

More specifically, when comparing the two Bayesian methods, Brender's Bayesian method is considered better than the traditional Bayesian method proposed by Kuo [12] when the original weight set, implicitly assumed to be the correct assessment of the DM's feelings, is used. Even though M_3 has a lesser amount of estimates close to steady-state, a larger variability between samples, and is a bit harder, theoretically, to understand, the fact that is much easier to program and takes 45 times less computer time to calculate overrides the former deficiencies. So when an aggregate of the attributes is taken, as is done with the MADM methods, Brender's Bayesian method turns out to be superior.

Chapter 6 - CONCLUSION

The purpose of this study was to compare three methods of estimating system availability: the classical maximum likelihood method, the traditional Bayesian method with squared error loss function, and Brender's Bayesian method; with the objective of determining the best of the three. These three estimates were calculated for twelve samples, varying in size and type, drawn from two exponentially distributed sets of on and off time data. Using these numerical calculations and five multiple attribute decision making (MADM) techniques, the best method for estimating system availability was determined: Brender's Bayesian method.

6.1 Study Review

System availability was defined as the probability a system was operating satisfactorily at any point in time when used under stated conditions, where the [total] time considered was operating time and active repair time [22]. Availability, rather than reliability, was studied because of its increased usage as a measure of system effectiveness. System availability is most often estimated through the accumulation of data of a system's observed on and off times. However, problems can occur with the estimation when the amount of data is very small or nonexistant. When this is the case, Bayesian methods are most helpful.

The main reason for this study was to discover if, indeed, Bayesian methods were superior to a classical estimator. Two numerical examples incorporating small samples, biased samples, or no samples at all were explored. Also, since two Bayesian-type estimation methods were used, analyses were made to determine which, if either, was best. Further sensitivity analyses were made to discover, given certain combinations of sampling conditions, which estimation method was best. Not much work has been done, with the exception of Kuo [12], to make these determinations; hence, the impetus of this study.

System availability was represented using renewal theory; the states being the on and off states. An analytical expression was then derived for (1) a gamma distributed system, and (2) an exponentially distributed system. Analytical expressions for the maximum likelihood estimate and the traditional Bayesian estimate with squared error loss function for both system cases were derived with these system analytical expressions in mind. The analytical expression of Brender's Bayesian estimate was derived only for the exponentially distributed system case, and was derived separately, due to its unique theoretical background utilizing the Euler distribution.

To test and compare the three methods, numerical expressions of system availability were calculated, via computer, for six samples each drawn from two exponentially distributed systems. For comparative purposes, the samples were either biased or unbiased, and were comprised of either three, five or eight observations. Also, three different prior distributions were used for the Bayesian estimators.

The estimation methods were judged in terms of the five following criteria: number of values close (±.5%) to steady-state availability, variability between samples (determined from the samples themselves) , computer execution time, ease of programming and ease of understanding. The evaluation process was conducted using five MADM methods: dominance, simple additive weighting, linear assignment, ELECTRE, and TOPSIS. Also, further sensitivity analyses were conducted to determine, in terms of the first two criteria only, which method proved a better estimator given a certain bias of sample, or a certain size of sample.

6.2 Overall Results

The results of each of the five MADM methods were often conflicting, i.e., one MADM method chose one estimation method as best, while another MADM method chose a different estimation method as best. Initially, this seemed like something was wrong with either the alternatives or the MADM methods, but, as stated earlier this is a common occurrence, because each MADM method evaluates the alternatives according to different principles and perspectives. Because of these conflicting results, an ordering technique was used to determine the aggregate rank of each alternative based on each alternatives' rank in each of the MADM methods. Using the average ranking procedure as the ordering technique, Brender's Bayesian method was determined as best, with the maximum likelihood method as second best, and the traditional Bayesian method as the worst.

Recall, however, that these results were based on the assumption that the DM placed the following weights on the attributes: $\underline{w}$ = (.3, .3, .1, .2, .1). If the DM placed more weight on the first two criteria of closeness to steady-state and variability between samples with $\underline{w}$ = (.4, .4, .05, .1, .05); the traditional Bayesian method was considered best, with Brender's Bayesian method second best and the maximum likelihood method worst. When the attribute weight vector was determined through the entropy method, $\underline{w}$ = (.0159, .0316, .7732, .1356, .0437), the maximum likelihood estimate was considered best with Brender's Bayesian and traditional Bayesian following in that order. The noteworthy conclusion was that the preference order was highly dependent upon the DM's weighting scheme, so to obtain an accurate solution, the DM's attribute weights must have been correctly assessed, which, in any particular instance, may or may not be an easy task.

When analyzing the numerical example results in terms of the first two criteria only, the following observations were made:

(1) For the smallest sample size (n=3) Brender's Bayesian estimates were closest to steady-state.

(2) The no-sample-data case provided, in general, estimates closer to steady-state than the larger sample size cases.

(3) The random samples yielded estimates closer to steady-state than the biased samples.

(4) Overall, the traditional Bayesian method yielded the most estimates closest to steady state.

(5) If the experimenter could not determine a good close prior parameter set, it would be better, for these data sets, to overestimate rather than underestimate them.

(6) for the Bayesian methods, the no-data case yields estimates closer to steady state than the data cases.

Note that only the above informal observations could be made in terms of sample biasedness and size, because a formal analysis of variance between sample biasedness, size and estimation method showed no significant main or interaction effects between the three.

6.3 Ideas for Future Study

The conclusions from this study should not be construed as all encompassing for any system whose availability estimate is desired. Here, only two data sets, both exponentially distributed, were explored; only exponential priors were used; only three estimation methods were compared; only five MADM methods were used; only three weighting schemes were explored. Therefore, basically, the ideas for future study incorporate variations on the data sets and mathematical analyses used in this study. This is sorely needed, since not much other work has been done to explore Bayesian methods of estimating system availability. These ideas for future study include:

(1) Use the more general gamma distributed system.

(2) Use gamma distributed priors.

(3) Derive the availability expression, $\hat{g}_t$, other than via renewal theory (possibly through the use of a transition matrix).

(4) For the traditional Bayesian estimation method, experimentation can be done with different loss functions.

(5) Use other MADM methods that do not depend on a weight vector.

(6) The criteria for judging the alternatives could be expanded from the original five.

(7) The traditional Bayesian estimate could be refined through the use of better integral approximation techniques.

Hopefully, more studies will be conducted with respect to Bayesian estimation methods. This research is definitely worthy of attention because of its numerous practical applications to systems where no (or very little) sample data is available.

REFERENCES

1. Barlow, R. E., and F. Proschan, Mathematical Theory of Reliability, New York: John Wiley and Sons, 1965.

1a. Benayoun, R., B. Roy and N. Sussman, "Manual de Reference du Programme Electre," Note de Synthese et Formation, No. 25, Direction Scientifique SEMA, Paris, 1966.

1b. Bernardo, J. J., and J. M. Blin, "A Programming Model of Consumer Choice Among Multi-Attributed Brands," Journal of Consumer Research, Vol. 4, No. 2, pp. 111-118, 1977.

2. Brender, D. M., "The Bayesian Assessment of System Availability: Advanced Applications and Techniques," IEEE Trans. Reliability, Vol. R-17, No. 3, pp. 138-147, Sept. 1968.

3. Brender, D. M., "The Prediction and Measurement of System Availability: A Bayesian Treatment," IEEE Trans. Reliability, Vol. R-17, No. 3, pp. 127-138, Sept. 1968.

4. Crellin, G. L., "The philosophy and mathematics of Bayes' equation," IEEE Trans. Reliability, Vol. R-21, No. 3, pp. 131-135, Aug. 1972.

4a. Easterling, R. G., "A Personal View of the Bayesian Controversy in Reliability and Statistics," IEEE Trans. Reliability, Vol. R-21, No. 3, pp. 186-194, Aug. 1972

5. Epstein, B., "Tests for Validity of the Assumption that the Underlying Distribution of Life is Exponential: Part II," Technometrics, Vol. 2, No. 2, pp. 167-183, May 1960.

6. Evans, R. E., "Prior Knowledge, Engineers Versus Statisticians," IEEE Trans. Reliability, Vol. R-18, No. 4, p. 143, Nov. 1969.

7. Gaver, D. P. and M. Mazumdar, "Some Bayes' Estimates of Long-Run Availability in a Two-State System," IEEE Trans. Reliability, Vol. R-18, No. 4, pp. 184-189, Nov. 1969.

8. Grosh, D. L., Decision Theory, Lecture Notes, Dept. of I.E., Kansas State University, Manhattan, Kansas 1979.

9. Hwang, C. L., and K. Yoon, Methods and Applications of Multiple Attribute Decision Making, Lecture Notes, Dept. of I.E., Kansas State University, Manhattan, Kansas 1980.

10. Hwang, C. L., and K. Yoon, Principles for Evaluation of Air Conditioning System, Dept. of I.E., Kansas State University, Manhattan, Kansas 1980.

11. Johnson, N. L., and S. Kotz, Continuous Univariate Distributions, Boston: Houghton Mifflin Co., 1970.

11a. Keeny, R. L., and H. Raiffa, Decision with Multiple Objectives: Preferences and Value Tradeoffs, New York: John Wiley and Sons, 1976.

12. Kuo, W., "System Effectiveness Models via the Renewal Theory and Bayesian Inference," Kansas State University Ph.D. Dissertation, 1980.

13. Lie, C. H., C. L. Hwang and F. A. Tillman, "Availability of Maintained Systems: A State-of-the-Art Survey," AIIE Transactions, pp. 247-259, Sept. 1977.

13a. MacCrimmon, K. R., "Decision Making Among Multiple-Attribute Alternatives: A Survey and Consolidated Approach," RAND Memorandum, RM-4823-ARPA, 1968.

13b. Nijkamp, P., "A Multicriteria Analysis for Project Evaluation: Economic-Ecological Evaluation of a Land Reclamation Project," Papers of the Regional Science Association, Vol. 35, pp. 87-111, 1974.

13c. Nijkamp, P., "Stochastic Quantitative and Qualitative Multicriteria Analysis for Environmental Design," Papers of the Regional Science Association, Vol. 39, pp. 175-199, 1977.

14. Parzen, E., Stochastic Processes, San Francisco: Holden-Day, Inc., 1962.

15. Rainville, E. D., Special Functions, New York: Macmillan, 1960.

16. Rau, J. G., Optimization and Probability in Systems Engineering, New York: Van Nostrand Reinhold Co., 1970.

16a. Roy, B., "Problems and Methods with Multiple Objective Functions, Mathematical Programming, Vol. 1, No. 2, pp. 239-266, 1971.

16b. Roy, B., "How Outranking Relation Helps Multiple Criteria Decision Making," Cochrane, J. L. and M. Zeleny (eds.), Multiple Criteria Decision Making, Columbia, S.C.,: University of South Carolina Press, pp. 179-201, 1973.

16c. Roy, B., "Partial Preference Analysis and Decision-Aids: The Fuzzy Outranking Relationship Concept," Bell, D. E., R. L. Keeney and H. Raiffa (eds.), Conflicting Objectives in Decisions, New York: John Wiley and Sons, pp. 40-75, 1977.

17. Sandler, G. H., System Reliability Engineering, Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1963.

18. Schulhof, R. J., and D. L. Lindstrom, "Application of Bayesian Statistics in Reliability," 1966 Sysmposium on Reliability, pp. 684-695, 1966.

19. Snedecor, G. W., and W. G. Cochran, Statistical Methods, Sixth Edition, Ames, Iowa: The Iowa State University Press, 1967.

20. Thompson, W. E., and P. A. Palicio, "Bayesian Confidence Limits For the Availability of Systems," IEEE Trans. Reliability, Vol. R-24, No. 2, pp. 118-.20, June 1975.

21. Thompson, W. E., and M. D. Springer, "A Bayes' Analysis for a System Consisting of Several Independent Subsystems," IEEE Trans. Reliability, Vol. R-21, No. 4, pp. 212-214, Nov. 1972.

21a. van Delft, A., and P. Nijkamp, "A Multi-Objective Decision Model for Regional Development, Environmental Quality Control and Industrial Land Use," Papers of the Regional Science Association, Vol. 36, pp. 35-57, 1976.

22. Von Alven, W. H., (ed.), Reliability Engineering, ARINC Research Corp., Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1964.

23. Yoon,K., "Systems Selection by Multiple Attribute Decision Making," Kansas State University PhD Dissertation, 1980.

APPENDICES

APPENDIX A

Proof That The Renewal Counting Process for Gamma Distributed Inter-arrival Times is Poisson [12]

To prove the theorem, it suffices to show that for any $t \geq s \geq 0$, the increment $n(t)-n(s)$ is Poisson distributed with mean $\lambda(t-s)$ regardless what value $N(t')$ is for $t' \leq s$.

Let $\{n(t)-n(s),\ t \geq s\}$ be a renewal counting process, where $n(t) = k\ N(t)$ and $n(s) = k\ N(s)$, where $N(t)$ is given as the number of cycles finished at time t.

Let $\{N(t)-N(s),\ t \geq s\}$ be the renewal counting process corresponding to independent random variables T_1, T_2, ..., where T_1 is the time from s to the first event occurring after time s, and so on. Recall, the cycle times T_2, T_3, ... are gamma distributed with mean k/λ. Note, this is equivalent to the set of exponentially distributed random variables X_0, X_1, ... X_k, X_{k+1}, .., where X_j is the first subset of T_2 and each random variable has mean $\frac{1}{\lambda}$. This set of independent variables can also be described by the previously defined renewal counting process $\{n(t)-n(s),\ t>s\}$.

Similarly, since T_1 is gamma distributed with mean $\frac{k}{\lambda}$ also, X_0, X_1, ... X_{k-1} are also exponentially distributed with mean $\frac{1}{\lambda}$, no matter what the values of $n(t')$ for $t' \leq s$. Therefore, this fact allows the conditional distribution of $n(t)-n(s)$ (which describes the exponential probability of a cycles' endpoint occurring within a $s \rightarrow t$ interval) be equivalent to the unconditional distribution of $N(t-s)$ (which describes the gamma probability of the system having a cycles' endpoint outside the interval $s \rightarrow t$; i.e., the system being on)

The proof is completed by showing, for any $t > 0$, $n(t)$ is Poisson distributed with mean λt.

APPENDIX B

The Equivalence of the Gamma Function and the Cumulative Poisson Distribution [8]

Let x be a $G(\alpha,\beta)$ random variable so that $f(x) = \dfrac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$, $x > 0$.

Then $F(x;\ \alpha,\beta)$ is the cumulative distribution function defined as

$$F(x;\ \alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^x w^{\alpha-1} e^{-\beta w}\, dw$$

To solve, perform an integration by parts letting

$$u = e^{-\beta w} \qquad v = w^{\alpha}/\alpha$$

$$du = -\beta e^{-\beta w}\, dw \qquad dv = w^{\alpha-1}\, dw$$

so,

$$F(x;\ \alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \left[\left[\frac{w^{\alpha} e^{-\beta w}}{\alpha} \right] \Bigg|_0^x + \frac{\beta}{\alpha} \int_0^x w^{\alpha} e^{-\beta w}\, dw \right]$$

$$= \frac{\beta^{\alpha} x^{\alpha} e^{-\beta x}}{\alpha\Gamma(\alpha)} + \frac{\beta^{\alpha+1}}{\alpha\Gamma(\alpha)} \int_0^x w^{\alpha} e^{-\beta w}\, dw$$

$$F(x;\ \alpha,\beta) = \frac{(\beta x)^{\alpha} e^{-\beta x}}{\alpha!} + \frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^x w^{\alpha} e^{-\beta w}\, dw$$

Let $P_o(k;\lambda)$ indicate the Poisson probability $\dfrac{e^{-\lambda}\lambda^k}{k!}$

then

$$\begin{aligned} F(x;\ \alpha,\beta) &= P_o(\alpha;\ \beta x) + F(x;\ \alpha+1,\ \beta) \\ &= P_o(\alpha;\ \beta x) + P_o(\alpha+1,\ \beta x) + F(x;\ \alpha+2,\beta) \\ &= P_o(\alpha;\ \beta x) + P_o(\alpha+1,\ \beta x) + P_o(\alpha+2;\ \beta x) + \dots \\ &\quad + P_o(\ \alpha+r;\ \beta x) + \dots \end{aligned}$$

this is a convergent series since

$$F(x;\ \alpha+r,\ \beta) \to 0 \text{ as } r \to \infty$$

Thus,

$$F(x;\ \alpha,\beta) = \sum_{k=\alpha}^{\infty} P_o(k;\ \lambda) \text{ where } \lambda = \beta x \text{ and } \alpha \text{ must be an integer .}$$

(i.e., the left hand tail of a gamma distribution can be evaluated from the right hand tail of a suitably chosen Poisson distribution.)

APPENDIX C

Proof of the Theorem:

If $X \sim G(\alpha,\beta)$ then $2X\beta \sim \chi^2(2\alpha)$

$X \sim G(\alpha,\beta)$ means X has the density function

$$f(x) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \tag{1}$$

Make a transformation of variables. Let $u = 2x\beta$; then $x = \frac{u}{2\beta}$ and $\frac{dx}{du} = \frac{1}{2\beta}$.

Since $g(u) = f(x)\left|\frac{dx}{du}\right|$ due to the chain rule,

$$g(u) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{2\beta\ \Gamma(\alpha)}$$

$$= \frac{\beta^{\alpha} \left(\frac{u}{2\beta}\right)^{\alpha-1} e^{-\frac{\beta u}{2\beta}}}{2\beta\ \Gamma(\alpha)}$$

$$g(u) = \frac{u^{\alpha-1} e^{\frac{-u}{2}}}{\Gamma(\alpha)\ 2^{\alpha}} \tag{2}$$

Let $\alpha = \frac{\nu}{2}$ (so that $\nu = 2\alpha$). Thus,

$$g(u) = \frac{u^{\frac{u}{2}-1} e^{-\frac{u}{2}}}{\Gamma\left(\frac{\nu}{2}\right) 2^{\nu/2}} \tag{3}$$

Note that this density function denotes $u \sim \chi^2(\nu)$ or $u \sim \chi^2(2\alpha)$.

APPENDIX D

Proof of Convergence of

Eq. (61), Chapter 3 [12]

Eq. (61) in Chapter 3 states

$$B = \nu\gamma \int_0^\infty \int_\eta^\infty \eta\ e^{-\eta(\gamma-\nu)} \left[\frac{1}{\beta'} (1-e^{-\beta' t})\ e^{-\nu\beta'}\right] d\beta' d\eta \tag{D1}$$

Since the integrand in eq. (D1) is continuous, the following inequality holds true:

$$B \leq \nu\gamma \int_0^\infty \int_0^\infty \eta e^{-\eta(\gamma-\nu)} \left[\frac{1}{\beta'} (1-e^{-\beta' t})\ e^{-\nu\beta'}\right] d\beta' d\eta$$

$$= \nu\gamma \int_0^\infty \eta e^{-\eta(\gamma-\nu)}\ d\eta \int_0^\infty \frac{1}{\beta'} \left(1-e^{-\beta' t}\right) e^{-\nu\beta'}\ d\beta'$$

$$= \frac{\nu\gamma}{(\gamma-\nu)^2} \int_0^\infty \frac{1}{\beta'} \left(1-e^{-\beta' t}\right) e^{-\nu\beta'}\ d\beta' \tag{D2}$$

In order for D1 to converge, the integral

$$C = \int_0^\infty \frac{1}{\beta'} \left(1-e^{-\beta' t}\right) e^{-\nu\beta'}\ d\beta' \tag{D3}$$

must exist.

Since

$$\lim_{\beta' \to 0+} \frac{1 - e^{-\beta' t}}{\beta'}$$

$$= \lim_{\beta' \to 0+} t\ e^{-\beta' t}$$

$$= t$$

because of L'Hopital's Rule*, for each $\varepsilon>0$, there exists a $\delta>0$ such that

$$0 < \frac{1-e^{-\beta' t}}{\beta'} < t + \varepsilon$$

for all $\beta' \varepsilon (0,\delta)$.

Therefore,

$$C = \int_0^\delta \frac{1}{\beta'} \left(1-e^{-\beta' t}\right) e^{-\nu\beta'} \, d\beta' +$$

$$\int_\delta^1 \frac{1}{\beta'} \left(1-e^{-\beta' t}\right) e^{-\nu\beta'} \, d\beta' +$$

$$\int_1^\infty \frac{1}{\beta'} \left(1-e^{-\beta' t}\right) e^{-\nu\beta'} \, d\beta'$$

$$\leq \int_0^\delta e^{-\nu\beta'} \left(t+\varepsilon\right) d\beta' + \int_\delta^1 \frac{1}{\beta'} \left(1-e^{-\beta' t}\right) e^{-\nu\beta'} \, d\beta' + \int_1^\infty e^{-\nu\beta'} \, d\beta'$$

and since

$$\int_0^\delta e^{-\nu\beta'} \left(t+\varepsilon\right) d\beta' = \frac{t+\varepsilon}{\nu} \left(1-e^{-\nu\delta}\right)$$

$$\int_\delta \frac{}{\beta} \left(1-e^{-\beta' t}\right) e^{-\nu\beta'} \, d\beta' \text{ is finite}$$

and

$$\int_1^\infty e^{-\nu\beta'} \, d\beta' = \frac{1}{\nu} e^{-\nu},$$

then C is finite. Hence, C exists and the proof is completed.

---

* L'Hopital's Rule:

$$\text{The} \lim_{t\to 0} \frac{g(t)}{a(t)} = \lim_{t\to 0} \frac{g'(t)}{a'(t)}$$

APPENDIX E

SAS Generation of a Gamma Distributed Data Set [12]

To generate a gamma random variable X (exponential, if shape parameter =1) with parameters α,β and γ, and probability density function

$$f_x(x) = \frac{\beta^{\alpha}(x-\gamma)^{\alpha-1} e^{-\beta(x-\gamma)}}{\Gamma(\alpha)}$$

for $\alpha,\beta,\gamma > 0$ with $\mu = \alpha/\beta$, $\sigma^2 = \alpha/\beta^2$

where

α = integer shape parameter

β = scale parameter

γ = location parameter

the following procedure is used:

(1) Generate $\nu = 2\alpha$ independent unit normal random variables,

$N_1, N_2, \ldots N_2 \sim N(0,1)$

(2) The distribution of

$$y = \sum_{i=1}^{\nu} N_i^2$$

is then the chi-square distribution with ν degrees of freedom and $\chi^2(\nu) = \chi^2(2\alpha)$

(3) Use the transformation

$$X = \frac{y}{2\beta} + \gamma$$

to obtain the gamma random variable X with parameters (α,β,γ).

A computer program using SAS (Statistical Analysis System) used for generating two sets of gamma random variables (on and off times) is listed.

```
NOTE: THE JOB XPRS5306 HAS BEEN RUN UNDER RELEASE 79.3A OF SAS AT

NOTE: THIS IS THE MESSAGE BATCH USERS SEE
      WHEN OPTIONS NEWS IS SPECIFIED.
      THIS MESSAGE SHOULD BE REPLACED BY SOME
      MEANINGFULL TEXT.
      TO CHANGE THE MODULE RECOMPILE IT.
      IT IS: DSNAME=SAS.SOURCE(SASNEWS).
      SEE INSTALLATION INSTRUCTIONS PROGRAM LISTSRC.


          *     THIS SAS PROGRAM GENERATES GAMMA RANDOM
                VARIABLES FOR THE ON TIME AND OFF TIME.

                THE STEPS ARE:

                (1) GENERATE N(0,1) RANDOM VARIABLES, U
                (2) FIND THE SUM OF U**2 FROM 1 TO NU
                (3) DETERMINE THE CHI-SQUARE RANDOM
                    VARIABLE Y WITH NU DEGREES OF
                    FREEDOM.  Y=SUM(U**2) FOUND IN (2)
                (4) USE THE TRANSFORMATION X=Y/(BETA*2)
                    AND ALPHA=NU/2 TO OBTAIN A GAMMA
                    RANDOM VARIABLE X WITH PARAMETERS
                    ALPHA, BETA, AND GAMMA;

          *     NOTE:
                  WHEN X IS GAMMA DISTRIBUTED WITH
                  PARAMETERS ALPHA, BETA, AND GAMMA,
                  THE DENSITY FUNCTION IS (BETA**ALPHA)
                  *EXP(-BETA*(X-GAMMA))*(X-GAMMA)
                  **(ALPHA-1)/(ALPHA-1) ;

          *     HERE,
                  ALPHA=1 FOR BOTH TON AND TOFF (HENCE, EXPONENTIAL
                  (THEREFORE, NU=2 FOR BOTH TON AND TOFF)
                  BETA=1/4 FOR TON AND 1/2 FOR TOFF
                  GAMMA=0 FOR BOTH TON AND TOFF;

          *     NOTE:
                  TON IS ON TIME
                  TOFF IS OFF TIME
                  TO IS TOTAL CYCLE TIME;

          *       COL1 IS TON
                  COL2 IS TOFF
                  COL3 IS TO;

          *     THE GAMMA RANDOM VARIABLE TO IS GENERATED
                FROM THE SUM OF THE RANDOM VARIABLES
                TON AND TOFF;

          *     THE PROCEDURES TO GENERATE THE GAMMA
                RANDOM VARIABLES CAN BE REFERENCED TO
                JOHNSON AND KOTZ'S CONTINUOUS
                DISTRIBUTIONS;
```

```
S T A T I S T I C A L   A N A L Y S I S   S Y S T E M



        OPTIONS LS=64 NODATE PS=49 NONUMBER NOCENTER SKIP=3;

        DATA CSM;
             DO _N_ = 1 TO 80;
             E=NORMAL(57593);
             P=NORMAL(82981);
             OUTPUT;
             END;

NOTE: DATA SET WORK.CSM HAS 80 OBSERVATIONS AND 2 VARIABLES. 953
 CBS/TRK
NOTE: THE DATA STATEMENT USED 0.67 SECONDS AND 128K.

        DATA SQUARES;
             SET CSM;
             SQ=E*E;
             SP=P*P;
             KEEP SQ SP;

NOTE: DATA SET WORK.SQUARES HAS 80 OBSERVATIONS AND 2 VARIABLES.
 953 CBS/TRK
NOTE: THE DATA STATEMENT USED 0.23 SECONDS AND 128K.

        PROC MATRIX;
             Y=J(80,2,0);
             FETCH X DATA=SQUARES;
             DO I=1 TO 80 BY 2;
             II=I+1;
             Y(I,1)=X(I,1)+X(II,1);
             Y(I,2)=X(I,2)+X(II,2);
             END;
             A= 2 0 / 0 1;
             XX=Y*A;
             OUTPUT XX OUT=XEND;

NOTE: DATA SET WORK.XEND HAS 80 OBSERVATIONS AND 3 VARIABLES. 68
0 CBS/TRK
NOTE: THE PROCEDURE MATRIX USED 0.74 SECONDS AND 144K
      AND PRINTED PAGE 1.

        DATA MCCN;
```

```
S T A T I S T I C A L   A N A L Y S I S   S Y S T E M

                SET XEND;
                IF COL1¬=0.0 AND COL2¬=0.0;

NOTE: DATA SET WORK.MOON HAS 40 OBSERVATIONS AND 3 VARIABLES. 68
0 OBS/TRK
NOTE: THE DATA STATEMENT USED 0.26 SECONDS AND 123K.

          DATA GAMMA;
                SET MOON;
                COL3=COL1+COL2;
                T=_N_;
                KEEP T COL1 COL2 COL3;

NOTE: DATA SET WORK.GAMMA HAS 40 OBSERVATIONS AND 4 VARIABLES. 5
29 OBS/TRK
NOTE: THE DATA STATEMENT USED 0.25 SECONDS AND 128K.

          PROC PRINT DATA=GAMMA;
                TITLE A SET OF GAMMA DISTRIBUTED RANDOM VARIABLES
;

NOTE: THE PROCEDURE PRINT USED 0.50 SECONDS AND 134K
      AND PRINTED PAGE 2.

NOTE: SAS USED 144K MEMORY.

NOTE: SAS INSTITUTE INC.
      SAS CIRCLE
      BOX 8000
      CARY, N.C. 27511
```

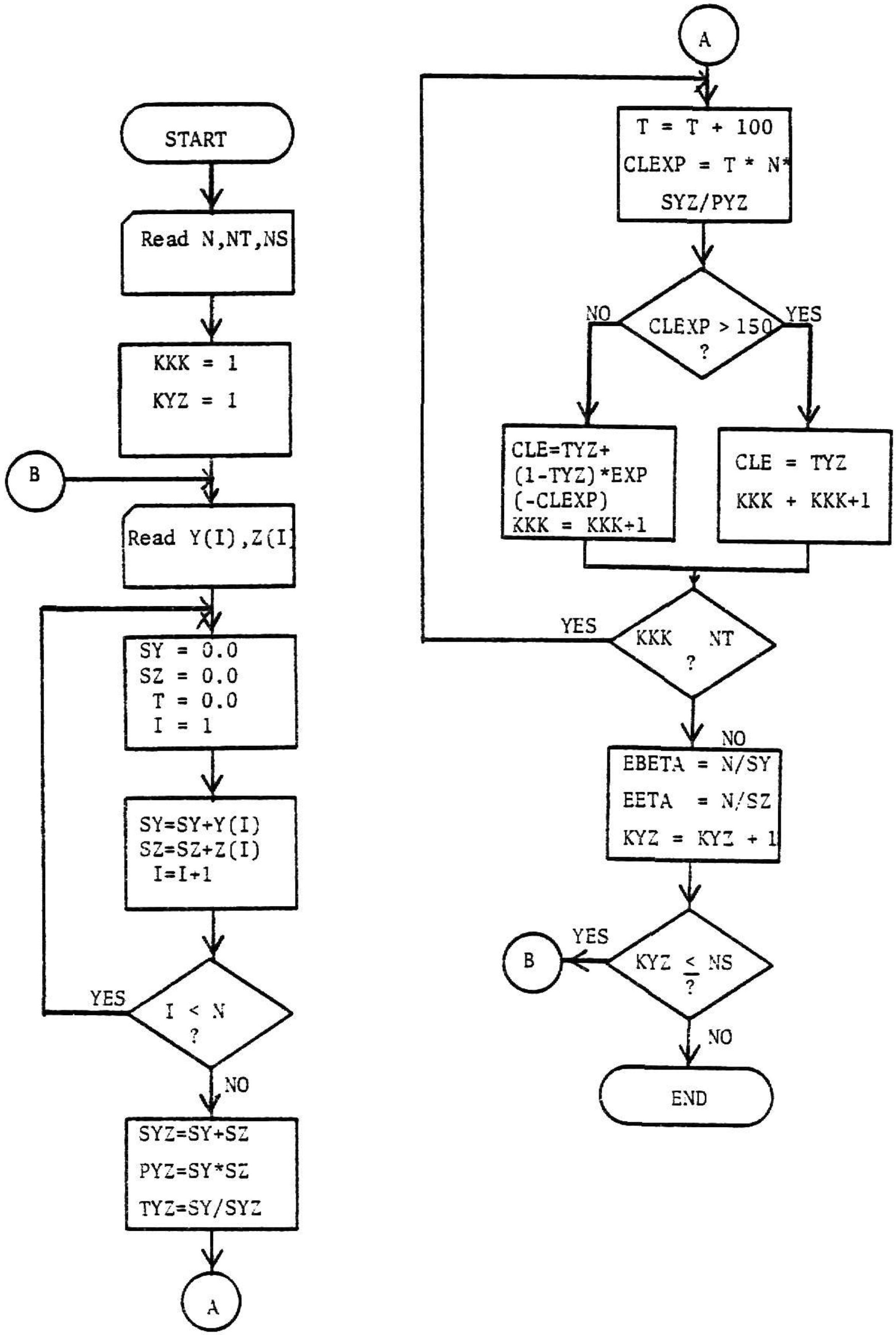
A SET OF GAMMA DISTRIBUTED RANDOM VARIABLES

OBS	COL1	COL2	COL3	T
1	0.6186	1.90787	2.5265	1
2	2.5610	0.86492	3.4259	2
3	4.5303	1.26596	5.7962	3
4	0.2069	0.86036	1.0672	4
5	0.0859	1.42172	1.5076	5
6	5.9580	0.30102	6.2590	6
7	0.9728	0.08195	1.0548	7
8	3.6668	1.83697	5.5038	8
9	1.4277	3.51557	4.9432	9
10	3.8742	2.12615	6.0003	10
11	7.7870	0.14148	7.9285	11
12	9.3676	0.17057	9.5382	12
13	1.3814	2.08018	3.4616	13
14	11.8043	0.90031	12.7046	14
15	14.1578	0.58616	14.7439	15
16	3.5732	8.16339	11.7366	16
17	0.0475	4.09503	4.1425	17
18	11.6031	3.73081	15.3340	18
19	0.2298	0.05406	0.2839	19
20	4.6266	0.12015	4.7467	20
21	0.3950	0.29294	0.6879	21
22	4.3218	0.06414	4.3859	22
23	9.6824	1.59550	11.2779	23
24	4.2697	0.23673	4.5064	24
25	0.9034	0.07164	0.9751	25
26	7.6568	0.12909	7.7859	26
27	9.2380	2.75301	11.9910	27
28	3.1117	0.03439	3.1461	28
29	7.3025	3.50084	10.8033	29
30	2.4932	0.47490	2.9681	30
31	4.5908	0.05483	4.6456	31
32	2.8995	0.10003	2.9995	32
33	0.6472	1.34214	1.9893	33
34	10.4190	0.13824	10.5573	34
35	2.7596	0.09740	2.8570	35
36	1.9168	0.34579	2.2626	36
37	2.9294	1.25289	4.1823	37
38	2.6000	1.21870	3.8187	38
39	1.7969	5.10970	6.9066	39
40	15.4431	2.43065	17.8737	40

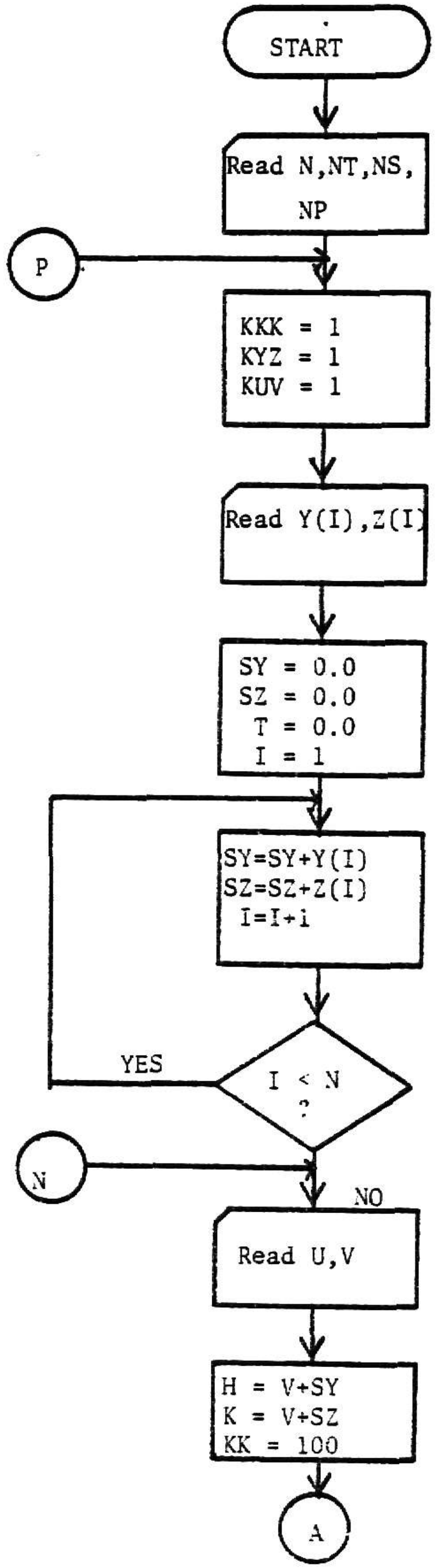
APPENDIX F

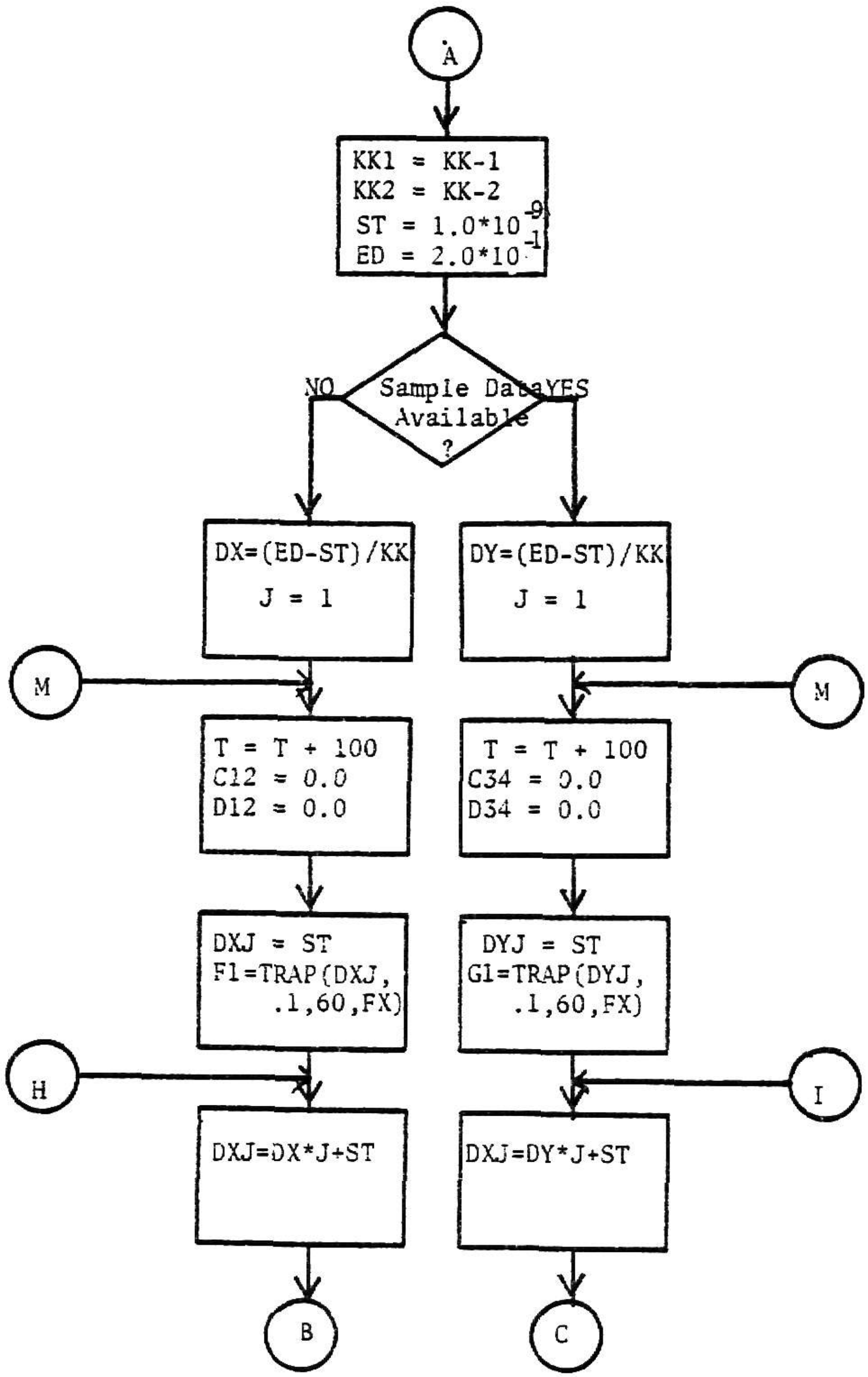
Flow Diagrams of Estimate Computations and Fortran Computer Routine

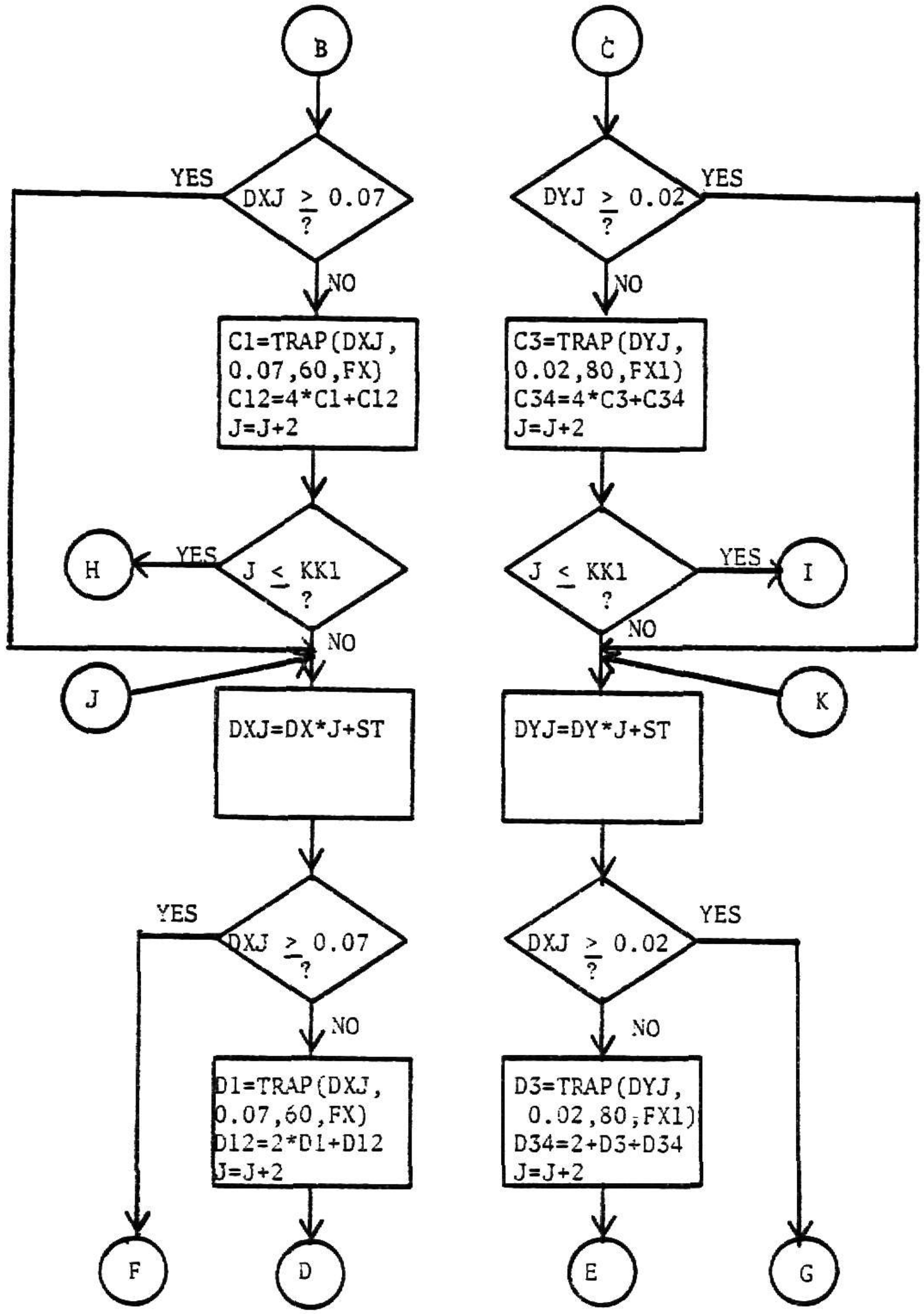
Flow Diagram of Maximum Likelihood Estimate Computation [Eq. (33) in Ch. 3]

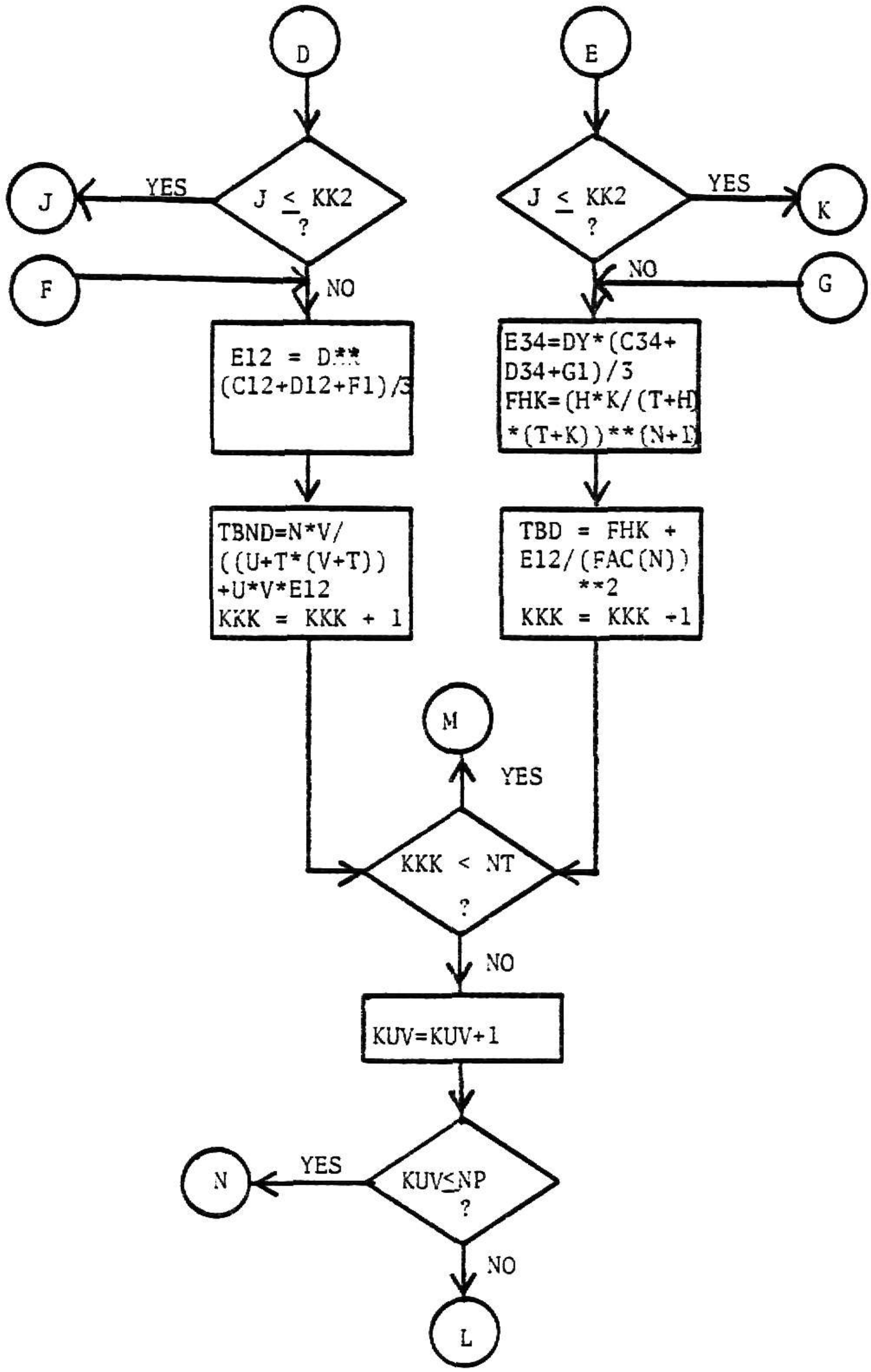


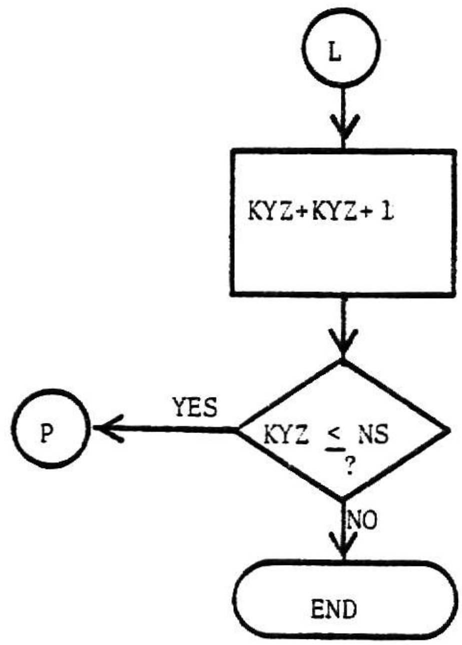
Flow Diagram of Traditional Bayesian Estimate Computations of [Eqs. (62) and (68) in Ch. 3]



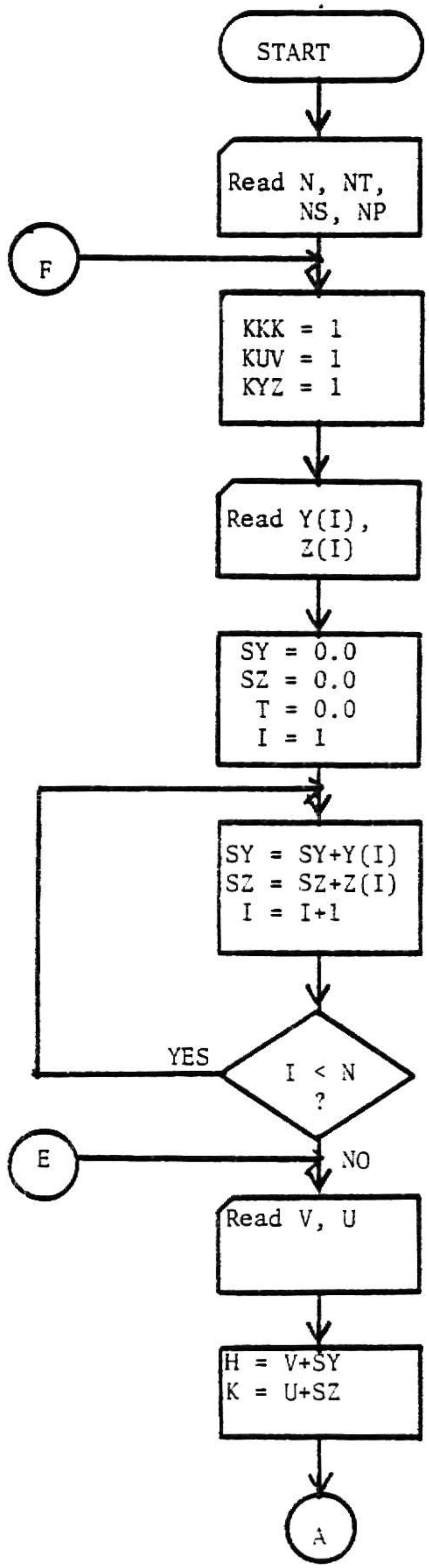

A
KK1 = KK-1
KK2 = KK-2
ST = 1.0*10^-9
ED = 2.0*10^-1
NO
Sample Data Available ?
YES
DX=(ED-ST)/KK
J = 1
DY=(ED-ST)/KK
J = 1
M
M
T = T + 100
C12 = 0.0
D12 = 0.0
T = T + 100
C34 = 0.0
D34 = 0.0
DXJ = ST
F1=TRAP(DXJ, .1,60,FX)
DYJ = ST
G1=TRAP(DYJ, .1,60,FX)
H
I
DXJ=DX*J+ST
DXJ=DY*J+ST
B
C

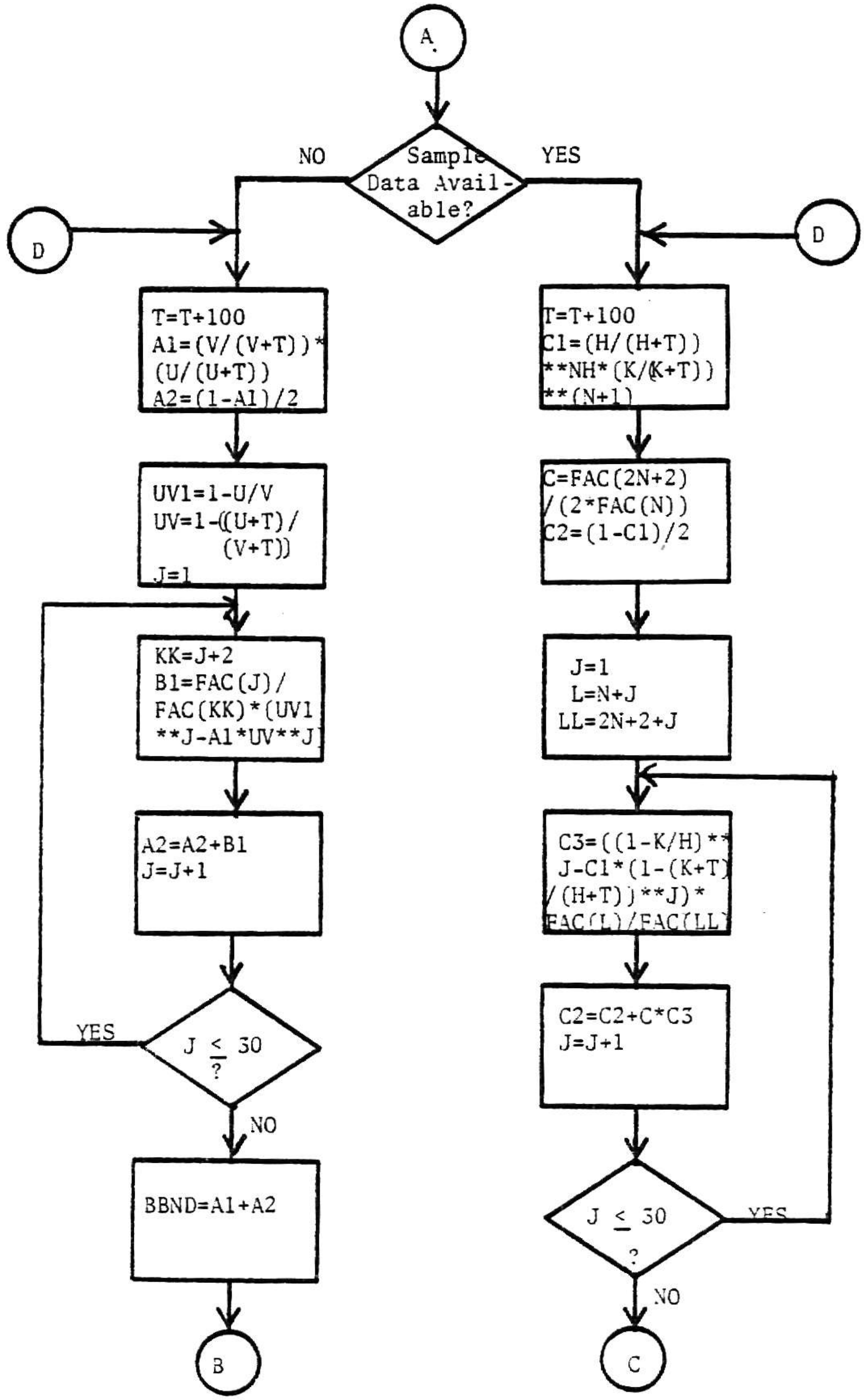

B
C
YES
DXJ ≥ 0.07 ?
DYJ ≥ 0.02 ?
YES
NO
NO
C1=TRAP(DXJ, 0.07,60,FX) C12=4*C1+C12 J=J+2
C3=TRAP(DYJ, 0.02,80,FX1) C34=4*C3+C34 J=J+2
H
YES
J ≤ KK1 ?
J ≤ KK1 ?
YES
I
NO
NO
J
K
DXJ=DX*J+ST
DYJ=DY*J+ST
YES
DXJ ≥ 0.07 ?
DXJ ≥ 0.02 ?
YES
NO
NO
D1=TRAP(DXJ, 0.07,60,FX) D12=2*D1+D12 J=J+2
D3=TRAP(DYJ, 0.02,80,FX1) D34=2+D3+D34 J=J+2
F
D
E
G

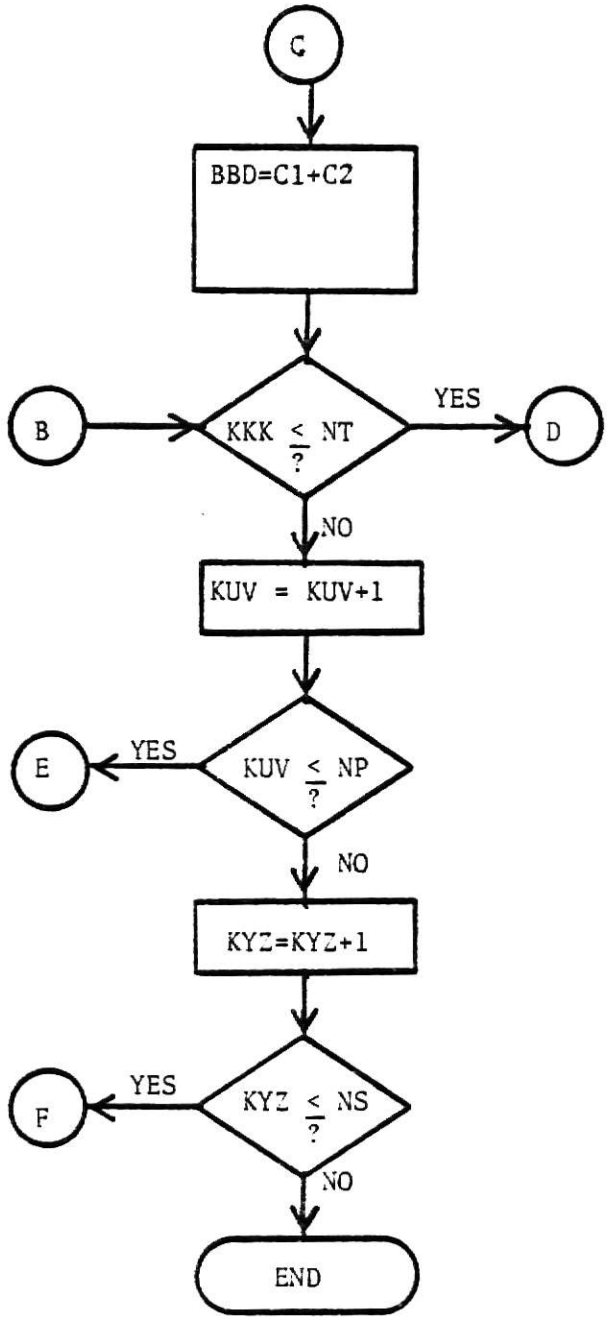

D
E
J ≤ KK2 ?
YES
J
J ≤ KK2 ?
YES
K
F
NO
NO
G
E12 = D**
(C12+D12+F1)/3
E34=DY*(C34+
D34+G1)/3
FHK=(H*K/(T+H)
*(T+K))**(N+1)
TBND=N*V/
((U+T*(V+T))
+U*V*E12
KKK = KKK + 1
TBD = FHK +
E12/(FAC(N))
**2
KKK = KKK +1
M
YES
KKK < NT
?
NO
KUV=KUV+1
N
YES
KUV≤NP
?
NO
L


L
KYZ+KYZ+1
YES
P
KYZ ≤ NS ?
NO
END

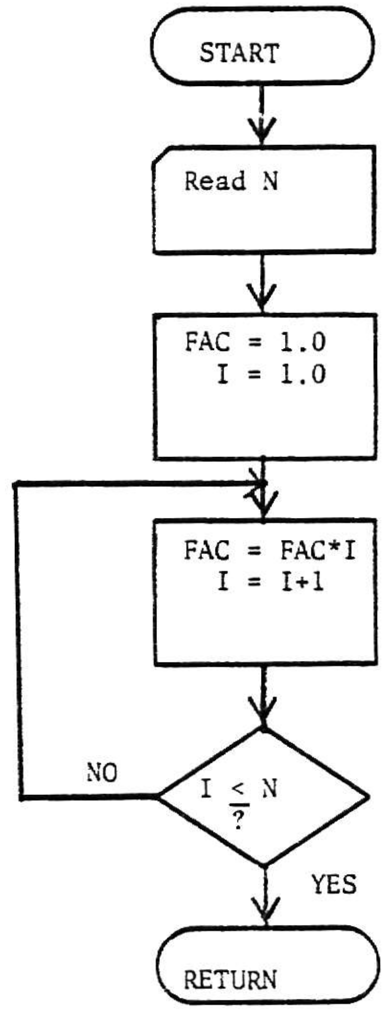
Flow Diagram of Brender's Bayesian Estimate Computations [Eqs. (119) and [122) in Ch. 3]



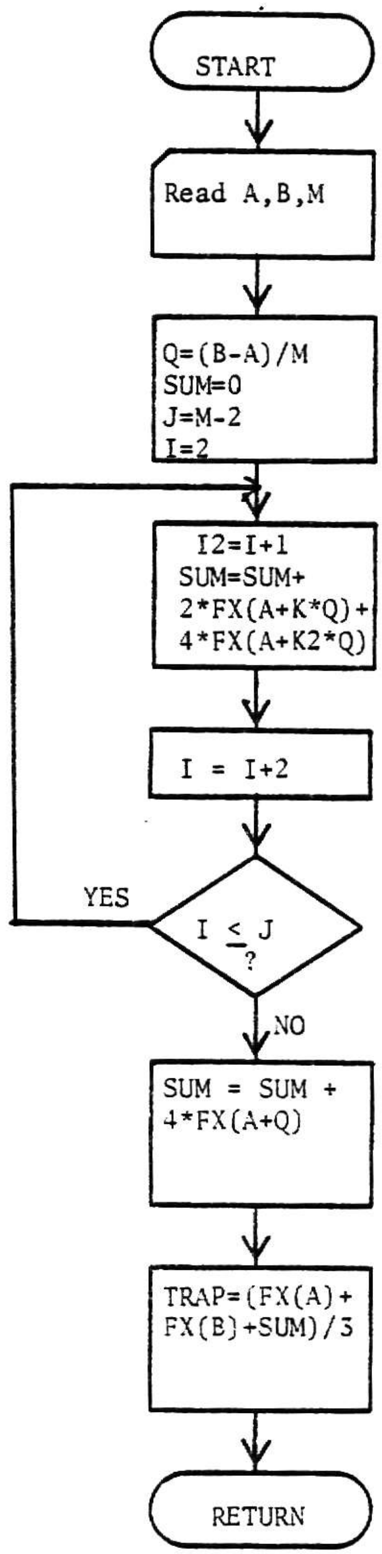

A
Sample Data Available?
NO
YES
D
D
T=T+100
A1=(V/(V+T))*(U/(U+T))
A2=(1-A1)/2
UV1=1-U/V
UV=1-((U+T)/(V+T))
J=1
KK=J+2
B1=FAC(J)/FAC(KK)*(UV1**J-A1*UV**J)
A2=A2+B1
J=J+1
J ≤ 30 ?
YES
NO
BBND=A1+A2
B
T=T+100
C1=(H/(H+T))**NH*(K/(K+T))**(N+1)
C=FAC(2N+2)/(2*FAC(N))
C2=(1-C1)/2
J=1
L=N+J
LL=2N+2+J
C3=((1-K/H)**J-C1*(1-(K+T)/(H+T))**J)*FAC(L)/FAC(LL)
C2=C2+C*C3
J=J+1
J ≤ 30 ?
YES
NO
C


C
BBD=C1+C2
B
KKK ≤ NT ?
YES
D
NO
KUV = KUV+1
YES
E
KUV ≤ NP ?
NO
KYZ=KYZ+1
YES
F
KYZ ≤ NS ?
NO
END

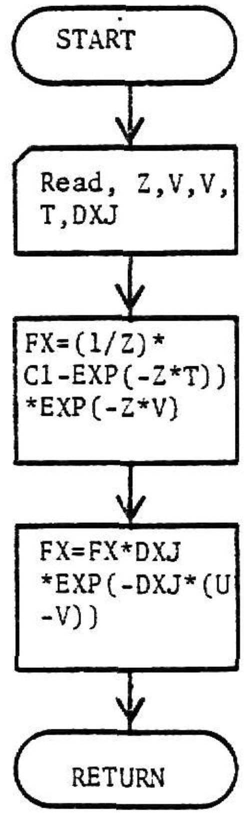
Flow Diagram of FAC Function



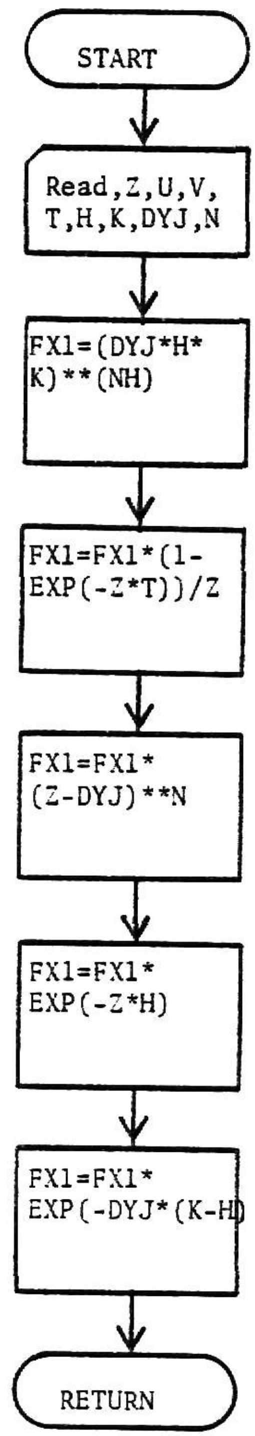
Flow Diagram of TRAP Function



Flow Diagram of FX Function



Flow Diagram of FX1 Function



```
$JOB            ,TIME=5,PAGES=20
C
C
C
C
C      THIS PROGRAM CALCULATES THE MAXIMUM LIKELIHOOD,
C      TRADITIONAL BAYESIAN, AND BRENDER'S BAYESIAN
C      ESTIMATES OF SYSTEM AVAILABILITY WHEN THE
C      SYSTEM IS COMPOSED OF NEGATIVE EXPONENTIALLY
C      DISTRIBUTED ON AND OFF TIMES.  THE ROUTINE,AS IS,
C      CAN ACCOMMODATE UP TO 10 SAMPLES AT A TIME WITH
C      UP TO 25 OBSERVATIONS EACH, AND WITH 10 DIFFERENT
C      PRIORS EACH.
C
C      NOTATION:
C          Y(I,J) = THE J TH ON TIME OBSERVATION FOR
C                   THE I TH SAMPLE
C          Z(I,J) = THE J TH OFF TIME OBSERVATION FOR
C                   THE I TH SAMPLE
C          EBETA(I) = THE MAXIMUM LIKELIHOOD ESTIMATE
C                     OF THE SYSTEM ON TIME PARAMETER
C                     BETA FOR THE I TH SAMPLE
C          EETA(I) = THE MAXIMUM LIKELIHOOD ESTIMATE OF
C                    THE SYSTEM OFF TIME PARAMETER
C                    ETA FOR THE I TH SAMPLE
C          CLE(I) = THE MAXIMUM LIKELIHOOD ESTIMATE
C                   OF THE SYSTEM AVAILABILITY FOR
C                   THE I TH SAMPLE
C          TBND(I,J) = THE TRADITIONAL BAYESIAN ESTIMATE
C                      OF SYSTEM AVAILABILITY FOR THE I TH
C                      PARAMETER SET AND THE J TH TIME
C                      WHEN SAMPLE DATA IS NOT AVAILABLE
C          TBD(I,J,K) = THE TRADITIONAL BAYESIAN ESTIMATE
C                       OF SYSTEM AVAILABILITY FOR THE I TH
C                       SAMPLE, THE J TH PARAMETER SET,
C                       AND THE K TH TIME, WHEN SAMPLE DATA
C                       IS AVAILABLE
C          BBND(I,J) = THE BRENDER'S   BAYESIAN ESTIMATE
C                      OF SYSTEM AVAILABILITY FOR THE I TH
C                      PARAMETER SET AND THE J TH TIME
C                      WHEN SAMPLE DATA IS NOT AVAILABLE
C          BBD(I,J,K) .= THE BRENDER'S   BAYESIAN ESTIMATE
C                       OF SYSTEM AVAILABILITY FOR THE I TH
C                       SAMPLE, THE J TH PARAMETER SET,
C                       AND THE K TH TIME, WHEN SAMPLE DATA
C                       IS AVAILABLE
C          N = THE NUMBER OF OBSERVATIONS PER SAMPLE
C          V = THE NEGATIVE EXPONENTIAL PRIOR PARAMETER
C              OF THE ON TIME DISTRIBUTION
C          U = THE NEGATIVE EXPONENTIAL PRIOR PARAMETER
C              OF THE OFF TIME DISTRIBUTION
C          NS = THE NUMBER OF SAMPLES   EXAMINED
C          NP = THE NUMBER OF PRIOR PARAMETER SETS
C               EXPLORED PER SAMPLE
C          NT = THE NUMBER OF TIME VALUES USED FOR
C               CALCULATING THE SYSTEM AVAILABILITY
C               ESTIMATES (IN INCREMENTS OF 100)
C      (NOTE:   THE NP AND NT MUST BE EQUAL FOR ALL
C               SAMPLES, ALTHOUGH THE PARAMETER SETS
C               THEMSELVES MAY BE DIFFERENT)
```

```
C
C
C
C
      DIMENSION Y(10,25),Z(10,25),NN(10)
      DIMENSION CLE(10,10),EBETA(10),EETA(10)
      DIMENSION TBND(10,10),TBD(10,10,10)
      DIMENSION BBND(10,10),BBD(10,10,10)
      DIMENSION VV(10),UU(10)
C
      DOUBLE PRECISION V,U,T,FAC,H,K
      DOUBLE PRECISION SYZ,PYZ,TYZ,CLE,CLEXP
      DOUBLE PRECISION EBETA,EETA
      DOUBLE PRECISION DX,DXJ,F1,C1,C12,D1,D12,E12
      DOUBLE PRECISION DY,DYJ,G1,C3,C34,D3,D34,E34,FHK
      DOUBLE PRECISION ST,ED,TBND,TBD
      DOUBLE PRECISION TRAP,FX,FX1
      DOUBLE PRECISION A1,A2,B1,BBND
      DOUBLE PRECISION CC,CC1,CC2,CC3,BBD
      DOUBLE PRECISION UV,UV1,VV,UU
C
      COMMON/CM1/DXJ
      COMMON/CM2/DYJ,K,H,N
      COMMON/CM3/T,V,U
      EXTERNAL FX,FX1
C
    4 FORMAT (I2)
    5 FORMAT (2F8.3)
    6 FORMAT (3I2)
   20 FORMAT ('1',///)
   21 FORMAT (18X,'TABLE 4-',I1,':  MAXIMUM LIKELIHOOD',
     1' ESTIMATES OF',/)
   22 FORMAT (18X,'TABLE 4-',I1,':  TRADITIONAL BAYESIAN',
     1' ESTIMATES OF'/)
   23 FORMAT (18X,'TABLE 4-',I1,':  BRENDERS BAYESIAN',
     1' ESTIMATES OF',/)
   24 FORMAT (30X,'SYSTEM AVAILABILITY FOR DATA SET ',I1,//)
   25 FORMAT (45X,'TIMES',/)
   26 FORMAT (30X,'T=100',5X,'T=200',5X,'T=300',5X,
     1'T=400',//)
   30 FORMAT (5X,'MAXIMUM LIKELIHOOD ESTIMATE',//)
   32 FORMAT (8X,'SAMPLE',I2,' (N=',I2,')',6X,F6.4,
     13(4X,F6.4),/)
   33 FORMAT (//,8X,'MAXIMUM LIKELIHOOD ESTIMATE OF BETA',
     1' IS ',F10.6)
   34 FORMAT ( 8X,'MAXIMUM LIKELIHOOD ESTIMATE OF  ETA',
     1' IS ',F10.6,//)
   40 FORMAT (5X,'TRADITIONAL BAYESIAN ESTIMATE')
   41 FORMAT (//,8X,'NO SAMPLE DATA',/)
   42 FORMAT (//,8X,'SAMPLE',I2,' (N=',I2,')',/)
   43 FORMAT (3(11X,'PARAMETER SET',I2,3X,F6.4,3(4X,F6.4),/))
   44 FORMAT (////,5X,'PARAMETER SET 1:',5X,'PARAMETER SET 2:',
     15X,'PARAMETER SET 3:',/)
   45 FORMAT (3(10X,'V=',F7.2),/)
   46 FORMAT (3(10X,'U=',F7.2))
   51 FORMAT (5X,'BRENDERS BAYESIAN ESTIMATE')
   60 FORMAT (30X,'DATA SET',I2,//)
   61 FORMAT (///,29X,'SAMPLE NO.',I2,///)
   62 FORMAT (11X,'OBSERVATION',10X,'TON',20X,'TOFF',//)
   63 FORMAT (15X,I2,10X,F10.2,13X,F10.2,/)
```

```
C
      READ (5,6) NS,NP,NT
      DO 500 KYZ=1,NS
        READ (5,4) N
        READ (5,5) (Y(KYZ,I),Z(KYZ,I),I=1,N)
        NN(KYZ)=N
      DO 400 KUV=1,NP
        READ (5,5) V,U
        VV(KUV)=V
        UU(KUV)=U
        SY=0.0
        SZ=0.0
        DO 50 I=1,N
          SY=SY+Y(KYZ,I)
   50     SZ=SZ+Z(KYZ,I)
        H=V+SY
        K=U+SZ
        SYZ=SY+SZ
        PYZ=SY*SZ
        TYZ=SY/SYZ
        ST=1.0D-09
        ED=2.0D-01
        KK=100
        DX=(ED-ST)/KK
        KK1=KK-1
        KK2=KK-2
        T=0.0D00
        DO 400 KKK=1,NT
          T=T+1.0D02
        CLEXP=T*DFLOAT(N)*SYZ/PYZ
C
C     CALCULATION OF TRADITIONAL BAYESIAN ESTIMATE
C     WHEN SAMPLE DATA IS NOT AVAILABLE
C
          C12=0.0D00
          D12=0.0D00
          DXJ=ST
          F1=TRAP(DXJ,1.0D-01,60,FX)
          DO 70 J=1,KK1,2
            DXJ=DX*J+ST
            IF(DXJ.GE.7.0D-02) GO TO 71
            C1=TRAP(DXJ,7.0D-02,60,FX)
   70       C12=4.0D00*C1+C12
   71     DO 72 J=2,KK2,2
            DXJ=DX*J+ST
            IF(DXJ.GE.7.0D-02) GO TO 73
            D1=TRAP(DXJ,7.0D-02,60,FX)
   72       D12=2.0D00*D1+D12
   73     E12=DX*(F1+C12+D12)/(3.0D00)
          TBND(KUV,KKK)=U*V/((U+T)*(V+T))+U*V*E12
C
C     CALCULATION OF BRENDERS BAYESIAN ESTIMATE
C     WHEN SAMPLE DATA IS NOT AVAILABLE
C
          A1=(V/(V+T))*(U/(U+T))
          A2=(1.0D00-A1)/2.0D00
          UV1=1.0D00-U/V
          UV=1.0D00-(U+T)/(V+T)
          DO 100 J=1,30
            JJ=J+2
```

```
               B1=FAC(J)/FAC(JJ)*(UV1**J-UV**J*A1)
   100         A2=A2+B1
              BBND(KUV,KKK)=A1+A2
C
C      CALCULATION OF TRADITIONAL BAYESIAN ESTIMATE
C      WHEN SAMPLE DATA IS AVAILABLE
C
              DY=(ED-ST)/KK
              C34=0.0D00
              D34=0.0D00
              DYJ=ST
              G1=TRAP(DYJ,1.0D-02,80,FX1)
              DO 90 J=1,KK1,2
                DYJ=DY*J+ST
                IF(DYJ.GE.2.0D-02) GO TO 91
                C3=TRAP(DYJ,2.0D-02,80,FX1)
    90          C34=4.0D00*C3+C34
    91        DO 92 J=2,KK2,2
                DYJ=DY*J+ST
                IF(DYJ.GE.2.0D-02) GO TO 93
                D3=TRAP(DYJ,2.0D-02,80,FX1)
    92          D34=2.0D00*D3+D34
    93        E34=DY*(G1+C34+D34)/(3.0D00)
              FHK=(H*K/((T+H)*(T+K)))**(N+1)
              TBD(KYZ,KUV,KKK)=FHK+E34/(FAC(N))**2
C
C      CALCULATION OF BRENDERS BAYESIAN ESTIMATE
C      WHEN SAMPLE DATA IS AVAILABLE
C
              CC=FAC(2*N+2)/(2.0*FAC(N))
              CC1=(H/(H+T))**(N+1)*(K/(K+T))**(N+1)
              CC2=(1.0D00-CC1)/2.0D00
              DO 150 JJ=1,30
                JJL=N+JJ
                JJLL=2*N+2+JJ
                CC3=((1.0D00-K/H)**JJ-CC1*(1.0D00-
     1             (K+T)/(H+T))**JJ)*FAC(JJL)/FAC(JJLL)
   150          CC2=CC2+CC*CC3
              BBD(KYZ,KUV,KKK)=CC1+CC2
C
C      CALCULATION OF MAXIMUM LIKELIHOOD ESTIMATE
C
              IF(CLEXP.GT.150) GO TO 175
              CLE(KYZ,KKK)=TYZ+(1.0D00-TYZ)*DEXP(-CLEXP)
              GO TO 400
   175        CLE(KYZ,KKK)=TYZ
   400        CONTINUE
C
C      CALCULATION OF MAXIMUM LIKELIHOOD ESTIMATES
C      OF THE SYSTEM PARAMETERS
C
            EBETA(KYZ)=DFLOAT(N)/SY
            EETA(KYZ)=DFLOAT(N)/SZ
   500 CONTINUE
            WRITE (6,20)
            WRITE (6,60) 1
          DO 700 IW1=1,NS
            WRITE (6,61) IW1
            WRITE (6,62)
            NN1=NN(IW1)
```

```
            WRITE (6,63) (I,Y(IW1,I),Z(IW1,I),I=1,NN1)
  700       CONTINUE
            WRITE (6,20)
            WRITE (6,21) 1
            WRITE (6,24) 1
            WRITE (6,25)
            WRITE (6,26)
            WRITE (6,30)
          DO 750 IW2=1,NS
            WRITE (6,32) IW2,NN(IW2),(CLE(IW2,LL),LL=1,NT)
            WRITE (6,33) EBETA(IW2)
  750       WRITE (6,34) EETA(IW2)
            WRITE (6,20)
            WRITE (6,22) 2
            WRITE (6,24) 1
            WRITE (6,25)
            WRITE (6,26)
            WRITE (6,40)
          DO 725 IW4=1,NS
            WRITE (6,42) IW4,NN(IW4)
            WRITE (6,43) (K1,(T2D(IW4,K1,L1),L1=1,NT),K1=1,NP)
  725       CONTINUE
            WRITE (6,41)
            WRITE (6,43) (K2,(TBND(K2,L2),L2=1,NT),K2=1,NP)
            WRITE (6,44)
            WRITE (6,45) (VV(I1),I1=1,NP)
            WRITE (6,46) (UU(I2),I2=1,NP)
            WRITE (6,20)
            WRITE (6,23) 3
            WRITE (6,24) 1
            WRITE (6,25)
            WRITE (6,26)
            WRITE (6,51)
          DO 800 IW3=1,NS
            WRITE (6,42) IW3,NN(IW3)
            WRITE (6,43) (K1,(B3D(IW3,K1,L1),L1=1,NT),K1=1,NP)
  800       CONTINUE
            WRITE (6,41)
            WRITE (6,43) (K2,(BBND(K2,L2),L2=1,NT),K2=1,NP)
            WRITE (6,44)
            WRITE (6,45) (VV(I1),I1=1,NP)
            WRITE (6,46) (UU(I2),I2=1,NP)
          STOP
          END
C
C
C
C
C         FAC IS A DOUBLE PRECISION FUNCTION WHICH DETERMINES
C         THE FACTORIAL OF ITS ARGUMENT.
C
C

          DOUBLE PRECISION FUNCTION FAC(N)
C
          FAC=1.0
          DO 5 I=1,N
        5 FAC=FAC*I
          RETURN
          END
```

```
C
C
C
C
C      TRAP IS A DOUBLE PRECISION FUNCTION WHICH USES
C      THE TRAPEZOIDAL APPROXIMATION (WITH SIMPSON'S
C      RULE) TO EVALUATE THE FINITE INTEGRAL,FX.
C
C
       DOUBLE PRECISION FUNCTION TRAP(A,B,M,FX)
       DOUBLE PRECISION A,B,Q,SUM,DXJ,DYJ
       DOUBLE PRECISION V,U,H,K,T,FX,FX1
       COMMON/CM1/DXJ
       COMMON/CM2/DYJ,K,H,N
       COMMON/CM3/T,V,U
C
       Q=(B-A)/M
       SUM=0.0D00
       J=M-2
       DO 10 I=2,J,2
         I2=I+1
    10   SUM=SUM+2.0D00*FX(A+I*Q)+4.0D00*FX(A+I2*Q)
         SUM=SUM+4.0D00*FX(A+Q)
       TRAP=(FX(A)+FX(B)+SUM)*(Q/(3.0D00))
       RETURN
       END
C
C
C
C
C      FX IS A DOUBLE PRECISION FUNCTION WHICH DEFINES
C      THE INTEGRAND USED IN THE FUNCTION TRAP IN THE
C      CASE WHERE SAMPLE DATA IS NOT AVAILABLE.
C
C
       DOUBLE PRECISION FUNCTION FX(Z)
       DOUBLE PRECISION Z,T,U,V,DXJ
       COMMON/CM1/DXJ
       COMMON/CM3/T,V,U
C
       FX=(1.0D00)/Z*((1.0D00)-DEXP(-Z*T))*
      1DEXP(-Z*V)
       FX=FX*DXJ*DEXP(-DXJ*(U-V))
       RETURN
       END
C
C
C
C
C      FX1 IS A DOUBLE PRECISION FUNCTION WHICH DEFINES
C      THE INTEGRAND USED IN THE FUNCTION TRAP IN THE
C      CASE WHERE SAMPLE DATA IS AVAILABLE.
C
C
       DOUBLE PRECISION FUNCTION FX1(Z)
       DOUBLE PRECISION H,K
       DOUBLE PRECISION Z,T,V,U,DYJ
```

```
      COMMON/CM2/DYJ,K,H,N
      COMMON/CM3/T,V,U
C
      FX1=(DYJ*H*K)**(N+1)
      FX1=FX1*(1.0D00-DEXP(-Z*T))/Z
      FX1=FX1*(Z-DYJ)**N
      FX1=FX1*DEXP(-Z*H/5.0D00)
      FX1=FX1*DEXP(-Z*H/5.0D00)
      FX1=FX1*DEXP(-Z*H/5.0D00)
      FX1=FX1*DEXP(-Z*H/5.0D00)
      FX1=FX1*DEXP(-Z*H/5.0D00)
      FX1=FX1*DEXP(-DYJ*(K-H))
      RETURN
      END
C
$ENTRY
```

EVALUATION OF ONE CLASSICAL AND TWO BAYESIAN ESTIMATORS
OF SYSTEM AVAILABILITY USING MULTIPLE ATTRIBUTE
DECISION MAKING TECHNIQUES

by

CYNTHIA S. MCCAHON

B.S.I.M., Industrial Management, Purdue University, West Lafayette, Indiana
1978

AN ABSTRACT OF A MASTER'S THESIS

submitted in partial fulfillment of the
requirements for the degree

MASTER OF SCIENCE

College of Engineering
Department of Industrial Engineering
Kansas State University
Manhattan, Kansas
1980

System availability is estimated for two systems whose on and off times are exponentially distributed. Three estimates, the classical maximum likelihood estimate, a traditional Bayesian estimate, and Brender's Bayesian estimate, are calculated numerous times using different sizes and types of samples. Prior distributions with different parameters are also investigated for the Bayesian estimators. From the three, a "best" system availability estimate is chosen given certain criteria via five multiple attribute decision making methods: dominance, simple additive weighting, linear assignment, ELECTRE and TOPSIS. In terms of the five criteria and their importance (weight) on the final decision, Brender's Bayesian estimation method was determined as superior over the other two methods.