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ON DIMENSION THEORY

by 45

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INTRODUCTION

In 1912, Poincare described intuitively the concept of dimension, stressing its inductive nature and the possibility of decomposing a space into subsets of lower dimension.

Before this time, mathematicians used "dimension" only in a vague sense: a configuration was said to be n-dimensional if n was the least number of real parameters required to describe its points. Fallacies of this approach were exhibited by Cantor, and, especially, Peano during the last part of the nineteenth century. Cantor disproved the notion that a plane is richer in points than the line, while Peano constructed what is now known as his space filling curve; this is a continuous mapping from the interval $I = [0,1]$ onto the square $I^2 = I \times I$. The latter demonstrated that the "dimension" could be raised and lowered by a continuous map.

Up to this time, it was not known whether or not the Euclidean n-space and the Euclidean m-space were topologically the same. In 1911, Brouwer answered this question negatively, and more important, he later introduced the notion of "Dimensiongrad", an integer valued function defined on the class of all spaces which turned out to be a topological invariant. Moreover, he showed that the "Dimensiongrad" of a Euclidean n-space is exactly n. The "Dimensiongrad" of a space is now known as the strong inductive dimension of that space, cf. [1.14].

While Brouwer was doing this important work, Lesbesque had taken another approach to the same problem. He observed that a square can be covered by arbitrarily small "bricks" in such a manner that no point of the square is contained in more than three bricks. He also noted that if the bricks were sufficiently small, at least three of them had a common point. Similarly, he observed that a cube in a Euclidean n-space could be decomposed into arbitrarily small bricks so that at most n+1 of these bricks met. Lesbesque conjectured that any decomposition of this cube into bricks would have at least n+1 of the bricks containing a common point. The conjecture was proved by Brouwer, and this gave another topological invariant. This invariant is now known as the covering (Lesbesque) dimension of a space, cf. [1.13].

We see that the new concepts of dimension gave a precise meaning to the statement that a Euclidean n-space has dimension n. Another important feature of the dimension was the wide class of spaces to which they could be applied. In 1941, W. Hurewicz and H. Wallman published their book, Dimension Theory, which gave a dimension theory for the class of all separable metric spaces. We note, that for this class of spaces, it was shown that the strong inductive dimension used by Brouwer is equivalent to the covering dimension of Lesbesque, and furthermore, that both were equivalent to the weak inductive dimension formulated by Menger and Urysohn.

In the last twenty-five years, advances in general topology have made possible the development of a dimension theory for general metric spaces. It is this theory which will be presented here. However, rather than giving a strictly traditional development, we shall use the approach that applies the notion of normal families, first developed by W. Hurewicz for separable metric spaces and later extended to general metric spaces by Morita. The purpose of this approach is to demonstrate the usefulness of normal families in the field of dimension theory.

In section I we shall state some notions from general topology which will be used throughout the paper. Also, certain notations will be introduced. We shall then develop some fundamental properties about normal families which we apply in III to prove some notions about 0-dimensional spaces. Repeating this process, we develop additional properties for normal families which we apply in V to obtain some characteristics about n-dimensional spaces, including the Sum Theorem, Product Theorem and Decomposition Theorem. With this background, we shall show that the notions of strong inductive dimension and covering dimension coincide on the class of metric spaces, and that they both coincide with the notion of weak inductive dimension of the class of separable metric spaces. Section VII gives an extended decomposition theorem, while the last section states some characterizations of dimension, specifically including an embedding theorem.

I. PRELIMINARIES

In this report, we assume a basic knowledge of general topology. However, this section contains a review of some important notions we will be using. Proofs which can be found in [1] will not be given.

Since every metric space is paracompact, and in particular, normal, we begin by discussing properties of these spaces.

1.1 Theorem A space Y is normal if and only if for each closed set F and each open set G containing F, there exists an open set V such that $F \subset V \subset \overline{V} \subset G$.

1.2 Definition A covering $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ of a space Y is said to be point-finite if for each point y in Y there are at most finitely many indices α in $\mathcal{A}$ such that y is in A_α.

1.3 Theorem (Shrinking Theorem) The following properties are equivalent:

(1). Y is normal.

(2). If $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ is any point-finite covering of Y by open sets, then there exists a covering $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ of Y by open sets such that $\overline{V}_\alpha \subset U_\alpha$ for each α in $\mathcal{A}$ and V_α is not empty whenever U_α is not.

Coverings are important in the development of a dimension theory for general metric spaces. Thus, we shall now exhibit some notions about coverings, especially those related to refinements.

1.4 Definition Let Y be a space and $\mathfrak{U} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$ be an open covering of Y. Then a covering $\mathfrak{D} = \{D_\beta \mid \beta \in \mathfrak{B}\}$ is said to be a refinement of $\mathfrak{U}$, denoted by $\mathfrak{B} < \mathfrak{U}$, if for every β in B there exists an α in $\mathcal{A}$ such that $D_\beta \subset U_\alpha$.

We say a covering $\mathfrak{A}$ is an open (closed) covering if each member of $\mathfrak{A}$ is open (closed). Similarly, we call a refinement $\mathcal{B}$ of a covering open (closed) if each member of $\mathcal{B}$ is open (closed).

A refinement of a covering may contain more sets than the given covering. We shall call a refinement $\{B_\beta \mid \beta \in \mathcal{B}\}$ of $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ precise if $\mathcal{B} = \mathcal{A}$ and $B_\alpha \subset A_\alpha$ for each α.

1.4.5 Definition Let Y be a space and $\mathfrak{A}$ an open covering of Y. Then $\mathfrak{A}$ is said to be a locally finite (= neighborhood finite) covering of Y if for each point $y \in Y$ there exists a neighborhood $\mathfrak{A}$ of X that meets only finitely many members of $\mathfrak{A}$.

1.5 Theorem If the covering $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ of a space Y has a point-finite (locally finite) refinement, $\{B_\beta \mid \beta \in \mathcal{B}\}$, it also has a precise point-finite (locally finite) refinement, $\{C_\alpha \mid \alpha \in \mathcal{A}\}$. Furthermore, if each B_β is an open set, then each C_α may be chosen to be open.

1.6 Definition Let $\mathfrak{A} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$ be a covering of a space Y. For any $B \subset Y$, the set $\bigcup\{U_\alpha \mid B \cap U_\alpha \neq \emptyset\}$ is called the star of B with respect to $\mathfrak{A}$, and is denoted by $S(B,\mathfrak{A})$.

1.7 Definition A covering $\mathcal{B}$ is called a barycentric refinement of a covering $\mathfrak{A}$ whenever the covering $\{S(y,\mathcal{B}) \mid y \in Y\}$ refines $\mathfrak{A}$.

1.8 Theorem Let Y be normal and $\mathfrak{A} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$ a locally finite open covering of Y. Then $\mathfrak{A}$ has an open barycentric refinement.

1.9 Theorem A T_1 space Y is paracompact if and only if each open covering of Y has a barycentric refinement.

1.10 Definition A covering $\mathfrak{B} = \{V_\beta \mid \beta \in \mathfrak{B}\}$ is called a star refinement of the covering $\mathfrak{A}$ whenever the covering $St\{V_\beta,\mathfrak{B}\}$ refines $\mathfrak{A}$.

1.11 Theorem A barycentric refinement $\mathfrak{C}$ of a barycentric refinement $\mathfrak{B}$ of a covering $\mathfrak{A}$ is a star refinement of $\mathfrak{A}$.

Thus, the above theorems show that every open covering of a metric space has a star refinement.

We now wish to fix various notations that will be used throughout the report. For any subset U of a space Y, we shall denote the boundary of U in X by B(U). If $U \subset A \subset X$, then the boundary of U with respect to the subspace A will be denoted by $B_A(U)$. Furthermore, if $\mathfrak{A} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$ is a collection of subsets of a space Y, we shall denote the collection $\{B(U_\alpha) \mid \alpha \in \mathcal{A}\}$ by $B(\mathfrak{A})$. For a family of collections $\{\mathfrak{A}_\alpha \mid \alpha \in \mathcal{A}\}$, we set

$$\wedge\{\mathfrak{A}_\alpha \mid \alpha \in \mathcal{A}\} = \{\cap\{U_\alpha \mid \alpha \in \mathcal{A}\} \mid U_\alpha \in \mathfrak{A}_\alpha \text{ for each } \alpha \in \mathcal{A}\}.$$

We note that $\wedge\{\mathfrak{A}_\alpha \mid \alpha \in \mathcal{A}\}$ is a covering of Y whenever each $\mathfrak{A}_\alpha$ is a covering of Y.

A collection $\mathfrak{A}$ of subsets of a space Y shall be called a σ-locally finite collection whenever it can be decomposed into an at most countable family of locally finite collections. It is well known that every metric space has a σ-locally finite basis. However, we desire to choose this basis so that it satisfies an additional property. To describe this property, we require the following terminology.

If $\mathfrak{A}$ is a collection of subsets of a metric space Y, we define the mesh of $\mathfrak{A}$, denoted by mesh $\mathfrak{A}$, to be the supremum of the set $\{\text{diameter } U \mid U \in \mathfrak{A}\}$.

1.12 Theorem Let Y be a metric space with metric d. Then there exists a

σ-locally finite basis $\mathfrak{U} = \bigcup_{i=1}^{\infty} \mathfrak{U}_i$ for the open sets of X such that $\lim_{i \to \infty} \text{mesh } \mathfrak{U}_i = 0$.

<u>Proof:</u> For each positive integer i, let

$$C_i = \{N_{1/i}(y) \mid y \in Y\},$$

where

$$N_{1/i}(y) = \{x \in Y \mid d(x,y) < 1/i\}.$$

Then, for each i, let $\mathfrak{U}_i$ be a locally finite refinement of C_i. By putting

$$\mathfrak{U} = \{U \mid U \in \mathfrak{U}_i,\ i = 1,2,\ldots\},$$

we have that $\mathfrak{U}$ is a σ-locally finite basis for Y having the desired property. Thus, the theorem is proved.

The basis constructed in the above proof shall henceforth be referred to as the canonical basis.

At this time, it is convenient to define two important dimension functions for general metric spaces, namely, covering dimension and strong inductive dimension.

1.13 <u>Definition</u> Let $\mathfrak{A}$ be a collection of subsets of a topological space X and p be a point in X. Then by the order of $\mathfrak{A}$ at p we mean the number of members of $\mathfrak{A}$ which contain p, and we denote it by $\text{ord}_p \mathfrak{A}$. If there exist infinitely many such members, then $\text{ord}_p \mathfrak{A} = +\infty$.

The order of $\mathfrak{U}$ will be the supremum of $\text{ord}_p \mathfrak{U}$ over $p \in X$ and will be denoted by $\text{ord } \mathfrak{U}$.

1.14 Definition If for any finite open covering $\mathfrak{A}$ of a topological space X there exists an open covering $\mathfrak{B}$ such that $\mathfrak{B} < \mathfrak{A}$ and ord $\mathfrak{B} \leq n+1$, then X has covering dimension $\leq n$, $\dim X \leq n$. The space X has covering dimension n, $\dim X = n$, if it is true that $\dim X \leq n$, and it is false that $\dim X \leq n-1$. If the statement, $\dim X \leq n$, is false for every integer n, then $\dim X = +\infty$. We define $\dim \emptyset = -1$.

1.15 Definition (a). A topological space X has strong inductive dimension -1, $\operatorname{Ind} X = -1$, if $X = \emptyset$.

(b). If for any two disjoint closed sets F and G of a topological space X there exists an open set U such that

$$F \subset U \subset X - G, \quad \operatorname{Ind} B(U) \leq n-1,$$

then X has strong inductive dimension $\leq n$, $\operatorname{Ind} X \leq n$. If it is true that $\operatorname{Ind} X \leq n$, and it is false that $\operatorname{Ind} X \leq n-1$, then $\operatorname{Ind} X = n$. If the statement, $\operatorname{Ind} X \leq n$, is false for every integer n, then $\operatorname{Ind} X = +\infty$.

We shall show that for any general metric space X, $\operatorname{Ind} X \leq n$ is equivalent to $\dim X \leq n$. There is a third notion of dimension which is called weak inductive dimension. However, P. Roy [5] proved that it is not equivalent to the two preceding notions of dimension for general metric spaces. However, all three agree for separable metric spaces.

1.16 Definition (a). A topological space X has weak inductive dimension -1, $\operatorname{ind} X = -1$, if $X = \emptyset$.

(b). If for every neighborhood U(p) of every point p in X there exists an open neighborhood V such that

$$p \in V \subset U(p)$$

and

$$\text{ind } B(V) \leq n-1,$$

then X has weak inductive dimension $\leq n$, ind $X \leq n$. If it is true that ind $X \leq n$, and it is false that ind $X \leq n-1$, then ind $X = n$. If the statement ind $X \leq n$ is false for every positive integer n, then ind $X = +\infty$.

1.17 <u>Proposition</u> Let $\mathfrak{G} = \{G_\alpha \mid \alpha \in \Omega\}$ be a locally finite system of open sets in a topological space R and let $\{F_\alpha \mid \alpha \in \Omega\}$ be a system of closed sets such that $F_\alpha \subset G_\alpha$ for each $\alpha \in \Omega$. Then

$$\mathfrak{G}' = \wedge\{G_\alpha, R - F_\alpha\}$$

is a locally finite open covering of R satisfying

$$S(F_\alpha, \mathfrak{G}') \subset G_\alpha.$$

<u>Proof:</u> Every non-empty set G' of $\mathfrak{G}'$ can be expressed in the form

$$(1) \qquad G' = \left[\bigcap_{\alpha \in \Gamma} G_\alpha\right] \cap \left[\bigcap_{\beta \notin \Gamma} R - F_\beta\right],$$

where Γ is a subset of Ω. Since $\mathfrak{G}$ is locally finite, Γ is always a finite set. We shall show that $\mathfrak{G}'$ is locally finite. Let p be a point in R. Then there exists a neighborhood U(p) such that the set $\Gamma_0(p)$ of indices α for which $U(p) \cap G_\alpha \neq \emptyset$ is a finite subset of Ω. If a set G' of $\mathfrak{G}'$, expressed in the form (1) intersects U(p), then $\Gamma \subset \Gamma_0(p)$. Hence the number of members of $\mathfrak{G}'$ meeting U(p) is finite. Therefore $\mathfrak{G}'$ is locally finite.

We shall now prove that $\mathfrak{G}'$ is an open covering. Let p be a point in $G' \in \mathfrak{G}$. If $\beta \notin \Gamma_{(p)}$, then $U_{(p)} \cap G_\beta = \emptyset$, and hence, $U_{(p)} \subset R - F_\beta$.

Therefore,

$$\left[\bigcap_{\alpha\in\Gamma} G_\alpha\right] \cap \left[\bigcap_{\beta\in\Gamma_0(p)-\Gamma}(R - F_\beta) \cap U(p)\right] = \bigcap_{\alpha\in\Gamma} G_\alpha \cap U(p) \subset G'$$

is a neighborhood of p. Thus G' is an open set. Finally if $G' \cap F_\alpha \neq \emptyset$, then $G' \subset G_\alpha$ which shows that $S(F_\alpha, \mathcal{G}') \subset G_\alpha$ for all F_α. This completes the proof.

1.18 <u>Definition</u> Let $\mathfrak{F} = \{F_\alpha \mid \alpha \in \mathcal{A}\}$ and $\mathcal{G} = \{G_\alpha \mid \alpha \in \mathcal{A}\}$ be families of subsets of a space X. Then we say $\mathfrak{F}$ is similar to $\mathcal{G}$ if for any subset B of $\mathcal{A}$, $\bigcap_{\beta\in B} F_\beta$ is not empty if and only if $\bigcap_{\beta\in B} G_\beta$ is not empty.

1.19 <u>Proposition</u> Let $\mathcal{G} = \{G_\alpha \mid \alpha \in \Omega_0\}$ be a locally finite system of open sets in a space R and $\mathfrak{F} = \{F_\alpha \mid \alpha \in \Omega\}$ a system of closed sets in R such that $F_\alpha \subset G_\alpha$ for each α in Ω. Then there exists a system of open sets $\{U_\alpha \mid \alpha \in \Omega\}$ such that

$$F_\alpha \subset U_\alpha \subset \overline{U}_\alpha \subset G_\alpha , \ \alpha \in \Omega$$

and

$$\{\overline{U}_\alpha \mid \alpha \in \Omega\} \text{ is similar to } \mathfrak{F}.$$

<u>Proof:</u> Let us assume that the set of indices α consists of all ordinal numbers which are less than a fixed ordinal ω_0. If we let φ_1 be the system of sets which are expressible as finite intersections $F_{\alpha_1} \cap \ldots \cap F_{\alpha_n}$ of sets of $\mathfrak{F}$ and are disjoint to F_1, then φ_1 is locally finite, since $\mathfrak{F}$ is locally finite. Hence if we denote the union of all members of φ_1 by S_1, we have that S_1 is closed in R and $F_1 \cap S_1 = \emptyset$. Therefore, there exists an open set U_1 such that

$$F_1 \subset U_1 \subset \overline{U}_1 \subset G_1$$

and

$$U_1 \subset R - S_1.$$

If we let

$$\mathfrak{A}_1 = \{\overline{U}_1, F_2, F_3, \ldots\}$$

then $\mathfrak{A}_1$ is similar to $\mathfrak{F}$.

We shall now complete the proof by transfinite induction. For this purpose, let us assume that for any β less than some fixed ordinal $\alpha < \omega_0$ there exists an open set U_β such that

$$F_\beta \subset U_\beta \subset \overline{U}_\beta \subset G_\beta$$

and

$$\mathfrak{A}_\beta = \{\overline{U}_\gamma \mid \gamma \leq \beta\} \cup \{F_\gamma \mid \beta < \gamma < \omega_0\} \text{ is similar to } \mathfrak{F}.$$

Now, if we let

$$\mathfrak{A}_\beta = \mathfrak{B} = \bigcup \{\mathfrak{A}_\beta \mid \beta < \alpha\} = \{\overline{U}_\gamma \mid \gamma < \alpha\} \cup \{F_\beta \mid \alpha \leq \beta < \omega_0\},$$

or

$$\mathfrak{B} = \{A_\gamma \mid \gamma < \omega_0\}.$$

It is clear from the induction hypothesis that $\mathfrak{B}$ is locally finite and similar to $\mathfrak{F}$. Hence, by the method described above, we can construct an open set U_α such that

$$F_\alpha \subset U_\alpha \subset \overline{U}_\alpha \subset G_\alpha$$

and

$$\mathfrak{A}_\alpha = \{\overline{U}_\gamma \mid \gamma \leq \alpha\} \cup \{F_\alpha \mid \alpha < \gamma < \omega_0\}$$

is similar to $\mathfrak{J}$.

The system

$$\mathfrak{U} = \{\overline{U}_\alpha \mid \alpha \in \Omega\}$$

of open sets has been constructed in such a way that it is similar to $\mathfrak{J}$. Therefore, the theorem is completely proved.

II. INTRODUCTION TO NORMAL FAMILIES

Normal families shall be consistently applied to develop a dimension theory for general metric spaces. Our immediate goal is to obtain those properties of normal families which will be useful in discussing 0-dimensional spaces.

Throughout the following chapters, all spaces are metric unless the contrary is explicitly stated.

By a locally countable covering $\mathcal{B}$ of a space Y, we mean a covering of Y such that for each y in Y there exists a neighborhood U of y that meets at most countably many members of $\mathcal{B}$.

2.1 <u>Definition</u> We shall say that a family $\mathfrak{M}$ of topological spaces is a normal family if the following two conditions are satisfied:

(a). If Y is a subspace of a space X, and X is an element of $\mathfrak{M}$, then Y is an element of $\mathfrak{M}$.

(b). If $\{A_\alpha\}$ is a locally countable closed covering of a space X, and each subspace A_α is an element of $\mathfrak{M}$, then X is an element of $\mathfrak{M}$.

The following proposition gives a more applicable form of the definition.

2.2 <u>Proposition</u> A family $\mathfrak{M}$ is a normal family if and only if it satisfies the following conditions:

(a). If Y is a subspace of a space X and X is an element of $\mathfrak{M}$, then Y is an element of $\mathfrak{M}$.

(b'). If $\{A_i\}$ is a countable closed covering of X, and each A_i is an element of $\mathfrak{M}$, then X is an element of $\mathfrak{M}$.

(b''). If $\{A_\alpha\}$ is a locally finite closed covering of X, and if each A_α

is an element of $\mathfrak{M}$, then X is an element of $\mathfrak{M}$.

Proof: Since (b) is a stronger condition than (b') and (b''), we need only show that (b') and (b'') imply (b).

Let $\mathfrak{M}$ be a family satisfying (a), (b') and (b'') of the proposition. If $\{F_\alpha \mid \alpha \in \mathcal{A}\}$ is a locally countable closed covering of a space X such that each F_α is an element of $\mathfrak{M}$, then for each point p in X we can find a neighborhood V(p) that meets at most countably many elements of $\{F_\alpha\}$. Since X is paracompact, and in particular, normal, 1.3 gives the existence of a locally finite open covering $\{U_\beta \mid \beta \in \mathcal{B}\}$ such that $\{\overline{U}_\beta \mid \beta \in \mathcal{B}\}$ refines $\{V(p) \mid p \in X\}$. Thus each $\overline{U}_\beta$ intersects at most countably many of the $\{F_\alpha\}$, which implies by (b') that each $\overline{U}_\beta$ is a member of $\mathfrak{M}$. But now $\{\overline{U}_\beta \mid \beta \in \mathcal{B}\}$ is a locally finite closed covering of X, each of whose members is an element of $\mathfrak{M}$ so that, by (b''), we have that X is a member of $\mathfrak{M}$.

2.3 Definition Let $\mathfrak{M}$ be a family of spaces. We define $\mathfrak{M}'$ to be a family of spaces such that $\mathfrak{M}' = \{X \mid$ for any pair of a closed set F and open set U of X with $F \subset U$, there exists an open set V of X such that $F \subset V \subset U$ and $B(V) \in \mathfrak{M}\}$.

2.4 Proposition Let X be a (metric) space and $\{\mathfrak{U}_i\}$ a countable collection of locally finite systems $\mathfrak{U}_i$ such that $\mathfrak{U} = \{U \mid U \in \mathfrak{U}_i,\ i = 1,2,\ldots\}$ is a basis for the open sets of X. Let $\mathfrak{M}$ be a normal family. If B(U) belongs to $\mathfrak{M}$ for each open set U of $\mathfrak{U}$, then X belongs to $\mathfrak{M}'$.

Proof: Let F and G be a pair of subsets of a space X with F closed, G open, and $F \subset G$. To prove the proposition, it suffices to find an open set V of X such that $F \subset V \subset G$ and $B(V) \in \mathfrak{M}$. By the normality of X, and in particular, 1.1, we can find an open set L such that $F \subset L \subset \bar{L} \subset G$. Let $\mathfrak{U}_i = \{U(i\alpha) \mid \alpha \in \mathcal{A}_i\}$. For each point p of X we can find an open set U of $\mathfrak{U}$ such that

$$p \in U,$$

(1). $\bar{U} \subset G$, in case $p \in \bar{L}$, and

(2). $\bar{U} \cap \bar{L} = \emptyset$ in case $p \notin \bar{L}$.

We denote U by $U(i(p), \alpha(p))$. Let H_j be the union of those $U(i(p), \alpha(p))$ for which $i(p) = j$ and p is in $\bar{L}$, and let K_j be the union of those $U(i(p), \alpha(p))$ for which $i(p) = j$ and p is not in $\bar{L}$. Then H_j and K_j are open sets, and, from the locally finiteness of the $\mathfrak{U}_j$, we have

$$B(H_j) \subset \bigcup \{B(U(i(p), \alpha(p))) \mid i(p) = j,\ p \in \bar{L}\}$$

and

$$B(K_j) \subset \bigcup \{B(U(i(p), \alpha(p))) \mid i(p) = j,\ p \notin \bar{L}\}.$$

It then follows from properties (a) and (b) of 2.1, that

(3). $B(H_i) \in \mathfrak{M}$, $B(K_i) \in \mathfrak{M}$.

Now let

$$P_1 = H_1,\ Q_1 = K_1 - \bar{H}_1$$

$$P_i = H_i - \bigcup_{j=1}^{i-1} \bar{K}_j,\ Q_i = K_i - \bigcup_{j=1}^{i} \bar{H}_j,\ i = (2,3,\dots)$$

$$P = \bigcup_{i=1}^{\infty} P_i,\ Q = \bigcup_{i=1}^{\infty} Q_i$$

Then

(4). $X = \bigcup_{j=1}^{\infty} \overline{P}_j \cup \left(\bigcup_{j=1}^{\infty} \overline{Q}_j\right)$,

(5). $P \cap Q = \emptyset$, $\overline{P}_j \subset G$, $Q \cap \overline{L} = \emptyset$,

(6). $B(P_j) \in \mathfrak{M}$, $B(Q_j) \in \mathfrak{M}$.

The statement in (4) follows immediately from the way the sets involved were defined, while the statements in (5) are obtained from (1) and (2). To see (6), we note that for each i, $B(P_j) \subset B(H_i)$, and since each $B(H_i)$ is a member of , we see $B(P_i)$ is in $\mathfrak{M}$. The remaining statement of (6) is obtained similarly.

Finally, let

$$V = X - \overline{Q}.$$

Since $Q \cap L = \emptyset$ and L is open, we have $\overline{Q} \cap L = \emptyset$, which yields $F \subset L \subset V$. On the other hand,

$$V = X - \overline{Q} \subset X - \bigcup_{i=1}^{\infty} \overline{Q}_i \subset \bigcup_{i=1}^{\infty} \overline{P}_i \subset G,$$

so that

(7). $F \subset V \subset G$.

Since $\overline{P}_i = P_i \cup B(P_i)$, $\overline{Q}_i = Q_i \cup B(Q_i)$, we obtain by (4) that

8. $X = P \cup Q \cup \left(\bigcup_{i=1}^{\infty} B(P_i)\right) \cup \left(\bigcup_{i=1}^{\infty} B(Q_i)\right)$

It follows from (5) and the openness of P that $P \cap \overline{Q} = \emptyset$. Hence

$$P \cap B(Q) = \emptyset.$$

By intersecting B(Q) with both sides of (8) we have

$$B(Q) \subset \bigcup_{i=1}^{\infty} B(P_i) \cup \left(\bigcup_{i=1}^{\infty} B(Q_i)\right).$$

Thus, by virtue of (a) and (b''), $B(Q) \in \mathfrak{m}$. Recalling the way V was defined, it is clear that $B(V) \subset B(Q)$, and hence

(9). $B(V) \in \mathfrak{m}$.

Consequently, V has the desired properties (7) and (9), so the proposition is now proved.

III. 0-DIMENSIONAL SPACES

The class of spaces having either strong inductive dimension equal to zero or covering dimension equal to zero will be referred to as 0-dimensional spaces. No ambiguity will arise from this, since the two notions coincide on this set of spaces, cf. [3.1]. In this section we wish to prove some fundamental properties of 0-dimensional spaces.

A necessary and sufficient condition for a space X to have strong inductive dimension less than or equal to zero is the existence of an open and closed set V corresponding to each pair of disjoint closed sets F and G such that $F \subset V \subset X - G$. We note that this is equivalent to showing the existence of an open and closed set V corresponding to each pair of an open set G and a closed set F such that $F \subset V \subset G$.

3.1 <u>Proposition</u> For any space X, $\dim X \leq 0$ if and only if $\operatorname{Ind} X \leq 0$.

<u>Proof:</u> Assume that $\dim X \leq 0$ and let F and G be disjoint closed subsets of X. Then $\{X - F, X - G\}$ is an open covering of X which has a precise open refinement of order 1. Let U be the member of the refinement contained in $X - G$. Then U is an open and closed set satisfying $F \subset U \subset X - G$ which implies, by definition, that $\operatorname{Ind} X \leq 0$.

Conversely, assume that $\operatorname{Ind} X \leq 0$, and let $\mathfrak{A} = \{U_1, \ldots, U_n\}$ be an open covering of X. Applying the Shrinking Theorem, 1.3, we obtain a closed refinement $\{F_1, F_2, \ldots, F_n\}$ of $\mathfrak{A}$ such that, for each $i = 1, 2, \ldots, n$, $F_i \subset U_i$. Since $\operatorname{Ind} X \leq 0$, we can find open and closed sets $\{V_1, \ldots, V_n\}$ such that, for each i, $F_i \subset V_i \subset U_i$. Let

$$G_1 = V_1$$

$$G_2 = V_2 - V_1$$
$$G_3 = V_3 - (V_1 \cup V_2)$$
$$\vdots$$
$$G_n = V_n - \bigcup_{i=1}^{n-1} V_i$$

Then $\{G_1, G_2, \ldots, G_n\}$ is an open refinement of $\mathfrak{A}$ of order 1. Hence dim $X \leq 0$ and the proof is complete.

3.2 <u>Proposition</u> Let A be a subset of a space X. If dim $A \leq 0$, then for any pair of a closed set F and an open set G with $F \subset G$, there exists an open set V such that

$$F \subset V \subset G, \text{ and}$$
$$B(V) \cap A = \emptyset.$$

<u>Proof:</u> By the normality of X, and in particular, 1.3, we may choose open sets L and M satisfying

$$F \subset L \subset \overline{L} \subset M \subset \overline{M} \subset G.$$

Also, since dim $A \leq 0$ implies Ind $A \leq 0$, there exists an open and closed set U_0 of the subspace A such that $\overline{L} \cap A \subset U_0 \subset M \cap A$. We note that $F \cup U_0$ and $(X - G) \cup (A - U_0)$ are separated sets. $(\overline{F \cup U_0} \cap [(X - G) \cup (A - U_0)] \subset (F \cup \overline{U_0}) \cap [(X - G) \cup (A - U_0)] = \emptyset)$. Thus, by the complete normality of X, we may find an open set V of X such that $F \cup U_0 \subset V$ and $\overline{V} \cap [(X - G) \cup (A - U_0)] = \emptyset$. Consequently, we have $F \subset V \subset G$ and $(\overline{V} - V) \cap A = \emptyset$. This completes the proof of the proposition.

3.3 <u>Proposition</u> Let $\mathfrak{A} = \bigcup_{i=1}^{\infty} \mathfrak{A}_i$ be a σ-locally finite open basis for the open sets of X such that each member U of $\mathfrak{U}$ is open and closed. Then dim X $\le$ 0.

Proof: Let $\mathcal{M}$ be the normal family consisting of the empty set. Then for each $U \in \mathfrak{U}$, $B(U) \in \mathcal{M}$; this gives by 2.4 that X is a member of $\mathcal{M}'$. The definition of $\mathcal{M}'$ shows that Ind X $\le$ 0, or equivalently, dim X $\le$ 0. This concludes the proof of the proposition.

3.4 Proposition Let A be a subset of a space X. If dim A $\le$ 0, then there exists a σ-locally finite open basis $\mathfrak{U}$ of X such that for each U in $\mathfrak{U}$, $(\overline{U} - U) \cap A = \emptyset$.

Proof: Let $\mathfrak{U}' = \bigcup_{i=1}^{\infty} \mathfrak{U}_i' = \{U_\alpha \mid U_\alpha \in \mathfrak{U}_i', i = 1,2,\ldots\}$ be the canonical basis for the open sets of X, where each $\mathfrak{U}_i' = \{U_{i\alpha} \mid \alpha \in \mathcal{A}_i'\}$ is locally finite. For each i, put

$$\mathfrak{U}_i'' = \{U_{i\alpha} \mid U_{i\alpha} \in \mathfrak{U}_i', U_{i\alpha} \cap A \neq \emptyset\} = \{U_{i\alpha} \mid \alpha \in \mathcal{A}_i''\}.$$

Also, let

$$\mathfrak{U}'' = \bigcup_{i=1}^{\infty} \mathfrak{U}_i''.$$

Then, by the normality of X, we may choose closed collections $\{F_{i\alpha} \mid \alpha \in \mathcal{A}_i''\}$ such that each $F_{i\alpha} \subset U_{i\alpha}$ and each $F_{i\alpha}$ is not empty. Now, since Ind A $\le$ 0, we can find collections $\{B_{i\alpha} \mid \alpha \in \mathcal{A}_i''\}$ consisting of open and closed sets of subspace A such that

$$F_{i\alpha} \cap A \subset B_{i\alpha} \subset U_{i\alpha}$$

for each $\alpha \in \mathcal{A}_i$, $i = 1,2,\ldots$.

By letting each $V_{i\alpha}$ be an open set in X, such that $V_{i\alpha} \cap A = B_{i\alpha}$, we see that

$$\{V_{i\alpha} \mid \alpha \in \mathcal{A}_i'', i = 1,2,\ldots\} \cup \{U_{i\alpha} \mid U_{i\alpha} \in \mathfrak{A}', U_{i\alpha} \notin \mathfrak{A}''\}$$

is the desired basis.

3.5 Proposition Let X be a space satisfying $\dim X \leq 0$. Then for any subset A of X, $\dim A \leq 0$.

Proof: By 3.4, $\dim X \leq 0$ implies the existence of an open and closed σ-locally finite basis $\mathfrak{A}$ for the topology on X. Thus $\{U \cap A \mid U \in \mathfrak{A}\}$ is a σ-locally finite basis for the space A, and this basis consists of open and closed sets of A. Therefore, it follows from 3.3 that $\dim A \leq 0$, as required.

3.6 Proposition Let $\{F_i \mid i = 1,2,\ldots\}$ be a closed covering of a space X such that for each i, $\dim F_i \leq 0$. Then $\operatorname{Ind} X \leq 0$.

Proof: Let G and H be disjoint closed subsets of X. To prove the proposition, it suffices to construct an open and closed set V of X such that $H \subset V \subset X - G$. Since $F_1 \cap G$ and $F_1 \cap H$ are disjoint closed subsets of F_1, we can find an open and closed set U_1 of F_1 satisfying

$$F_1 \cap H \subset U_1 \subset F_1 - G.$$

It follows from F_1 being closed in X that $H \cup U_1$ and $G \cup (F_1 - U_1)$ are closed in X. We also note that $H \cup U_1$ and $G \cup (F_1 - U_1)$ are disjoint. Thus, by the normality of X. there exist open sets V_1 and W_1 such that

$$H \cup U_1 \subset V_1, \; G \cup (F_1 \cup U_1) \subset W_1,$$

and

$$\overline{V}_1 \cap \overline{W}_1 = \emptyset.$$

Similarly, Ind $F_2 \leq 0$ implies there exists an open and closed set U_2 of F_2 satisfying

$$F_2 \cap H \subset F_2 \cap \bar{V}_1 \subset U_2 \subset F_2 - \bar{W}_1 \subset F_2 - G.$$

Again, since $\bar{V}_1 \cup U_2$ and $\bar{W}_1 \cup (F_2 - U_2)$ are disjoint closed sets in X, we can find open sets V_2 and W_2 of X such that

$$\bar{V}_1 \cup U_2 \subset V_2,\ \bar{W}_1 \cup (F_2 - U_2) \subset W_2,$$

and

$$\bar{V}_2 \cap \bar{W}_2 = \emptyset.$$

Inductively, we obtain the sequences $\{U_i \mid i = 1,2,\ldots\}$, $\{V_i \mid i = 1,2,\ldots\}$, and $\{W_i \mid i = 1,2,\ldots\}$, which satisfy

(1). $F_i \cap H \subset U_i \subset F_i - G$ $\quad (i = 1,2,\ldots)$,

(2). U_i is open and closed in F_i $\quad (i = 1,2,\ldots)$,

(3). $\bar{V}_{i-1} \cup U_i \subset V_i$, $\bar{W}_{i-1} \cup (F_i - U_i) \subset W_i$, $\bar{V}_i \cap \bar{W}_i = \emptyset$, and V_i, W_i are open in F_i $\quad (i = 2,\ldots)$.

Putting

$$V = \bigcup_{i=1}^{\infty} V_i, \quad W = \bigcup_{i=1}^{\infty} W_i,$$

we obtain by (3) and (0) that

$$V \supset H, \quad W \supset G.$$

Also, W and V are disjoint sets. (If not, there exist i and j such that $V_i \cap W_j \neq \emptyset$. If we assume i is greater than j, we have that $V_i \cap W_i \neq \emptyset$, which is impossible).

Again, by (3),

$$V_i \cup W_i \supset U_i \cup (F_i - U_i) = F_i$$

which implies

$$V \cup W \supset \bigcup_{i=1}^{\infty} F_i = X.$$

Thus V is the desired set, and the proof is complete.

3.7 Proposition Let $\{F_\alpha \mid \alpha \in \mathcal{A}\}$ be a locally finite closed covering of a space X such that $\dim F_\alpha \leq 0$ for every α in $\mathcal{A}$. Then $\dim X \leq 0$.

Proof: Since $\{F_\alpha \mid \alpha \in \mathcal{A}\}$ is a locally finite collection, there exists a collection $\{U(x) \mid x \in X\}$ of sets such that each member, $U(x)$, is a neighborhood of x which intersects only finitely many members of $\{F_\alpha \mid \alpha \in \ \}$. Let $\mathfrak{A} = \bigcup_{i=1}^{\infty} \mathfrak{A}_i$ be the canonical basis for X. For each i, let $\mathcal{C}_i = \{C_\beta \mid \beta \in \mathcal{B}_i\}$ be a locally finite open refinement of $\{U(x) \mid x \in X\} \wedge \mathfrak{A}_i$ such that $\lim_{i\to\infty} \text{mesh } \mathcal{C}_i = 0$, and each $\overline{C}_\beta$ meets only finitely many members of $\{F_\alpha \mid \alpha \in \mathcal{A}\}$. Then for every $\beta \in \bigcup_{i=1}^{\infty} \mathcal{B}_i$, there exist finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\overline{C}_\beta \subset \bigcup_{j=1}^{n} F_{\alpha_j}$. Thus, it follows from $\operatorname{Ind} F_\alpha \leq 0$ for each α in $\mathcal{A}$ and 3.6 that $\operatorname{Ind} \bigcup_{j=1}^{n} F_{\alpha_j} \leq 0$. Moreover, 3.5 then gives

(1). $\operatorname{Ind} \overline{C}_\beta \leq 0$.

By applying the normality of X we obtain an open covering $\mathcal{D} = \bigcup_{i=1}^{\infty} \mathcal{D}_i = \bigcup_{i=1}^{\infty} \{D_\beta \mid \beta \in \mathcal{B}_i\}$ of X such that for each i, and for each $\beta \in \mathcal{B}_i$, $\overline{D}_\beta \subset C_\beta$ and $\overline{D}_\beta \neq \emptyset$ if $C_\beta \neq \emptyset$. For each $\beta \in \bigcup_{i=1}^{\infty} \mathcal{B}_i$, it now follows from (1) that we can find an open and closed set W_β of $\overline{C}_\beta$ satisfying

$$\overline{D}_\beta \subset W_\beta \subset \overline{C}_\beta - (X - C_\beta).$$

Thus each W_β is contained in the open C_β, so that W_β is open and closed in X. Therefore, since $\{W_\beta \mid \beta \in \bigcup_{i=1}^{\infty} \mathcal{B}_i\}$ is a basis for the open sets of X which satisfies the condition of 3.3, we have Ind X $\leq$ 0. This completes the proof of the proposition.

3.8 Theorem The set $\mathfrak{M}_0$ of spaces for which dim X $\leq$ 0 forms a normal family.

Proof: The conditions of 2.2 are satisfied by 3.5, 3.6, and 3.7.

Example 1 Any finite nonempty metric space is an 0-dimensional space.

Example 2 Any countable metric space X has dim X $\leq$ 0. Since every point in X is closed, 3.8 proves the above statement.

Example 3 Let J be the space of irrational real numbers with the subspace topology of the Euclidean line, E^1. Then J is an 0-dimensional space. To see this, we first note that $\mathfrak{U} = \{N_\epsilon(r) \mid r$ is a rational number and ϵ is a positive rational number$\}$ is a countable basis for E^1. This implies $\mathfrak{U} \cap J = \{N_\epsilon(r) \cap J \mid N_\epsilon(r) \in \mathfrak{U}\}$ is a σ-locally finite basis for space J. However, since each member of $U \cap J$ is open and closed in J, it follows that Ind J $\leq$ 0. Therefore, J $\neq \emptyset$ gives Ind J = 0.

Example 4 The Euclidean line, E^1, is not 0-dimensional. This follows from E^1 being a connected space, or equivalently, that E^1 contains no proper, non-empty, open and closed subsets.

IV. MAIN THEOREMS ON NORMAL FAMILIES

In this section, we wish to relate the notions of 0-dimensional spaces and normal families. The following theorems, when combined with our results about 0-dimensional spaces, will indicate a natural method to study spaces of higher dimension.

4.1 <u>Proposition</u> Let $\mathfrak{M}$ be a normal family. If a space X belongs to $\mathfrak{M}'$, then there exist two subspaces A and B of X such that $X = A \cup B$, A is an element of $\mathfrak{M}$, and Ind $B \leq 0$.

<u>Proof:</u> Let $\mathfrak{U} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$ be the canonical basis of X. By applying the Shrinking Theorem, we can find a σ-locally finite open basis $\mathfrak{F} = \{F_\alpha \mid \alpha \in \mathcal{A}\}$ of X such that for each $\alpha \in \mathcal{A}$, $\overline{F}_\alpha \subset U_\alpha$. Since X is a member of $\mathfrak{M}'$, there exists a collection of open sets $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ such that for each α in $\mathcal{A}$,

$$\overline{F}_\alpha \subset V_\alpha \subset U_\alpha$$

and

$$B(V_\alpha) \in \mathfrak{M}.$$

Put

$$A = \bigcup \{B(V_\alpha) \mid \alpha \in \mathcal{A}\}$$

and

$$B = X - A.$$

Then $\{B(V_\alpha) \mid \alpha \in \mathcal{A}\}$ is a locally countable closed covering of A, which, by 2.2, yields that A is a member of $\mathfrak{M}$.

If we restrict $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ to B, we see that it becomes a σ-locally finite open basis of B consisting of open and closed subsets of B. Thus, by

3.3,

$$\text{Ind } B \leq 0.$$

The proposition is now completely proved.

4.2 Proposition If A and B are subspaces of a space X such that $X = A \cup B$, A is a member of $\mathfrak{M}$, and Ind $B \leq 0$, then X is a member of $\mathfrak{M}'$.

Proof: For any pair of a closed set F and open set G with $F \subset G$, there exists, by 3.2, an open set V such that $F \subset V \subset G$ and $B(V) \cap B = \emptyset$. Hence we have that the $B(V) \subset A$. By condition (a) of 2.1, we see that $B(V)$ is an element of $\mathfrak{M}$, which shows that X is an element of $\mathfrak{M}'$. This concludes the proof.

The following theorem is the combined result of 4.1 and 4.2.

4.3 Theorem Let $\mathfrak{M}$ be a normal family. Then a necessary and sufficient condition that a space X belong to $\mathfrak{M}'$ is the existence of subspaces A and B satisfying $X = A \cup B$, $A \in \mathfrak{M}$, and dim $B \leq 0$.

4.4 Theorem If $\mathfrak{M}$ is a normal family, then $\mathfrak{M}'$ is also a normal family.

Proof: Let $S \subset X$, where X is an element of $\mathfrak{M}'$. It then follows from 4.3 that we can find subspaces A and B of X satisfying $X = A \cup B$, with A in $\mathfrak{M}$ and Ind $B \leq 0$. Since $A \in \mathfrak{M}$, it follows from the normality of the family $\mathfrak{M}$ that $A \cap S$ is a member of $\mathfrak{M}$. Moreover, 3.5 gives that Ind $(B \cap S) \leq 0$. Thus, by again applying 4.3, we obtain that $S = (A \cap S) \cup (B \cap S)$ belongs to $\mathfrak{M}'$, or equivalently, $\mathfrak{M}'$ satisfies (a) of 2.1.

To show (b'') holds for $\mathfrak{M}'$, we let $\{F_\nu \mid \nu < \omega\}$ be a locally finite closed covering of a space X such that

(1). $F_\nu \in \mathfrak{M}'$, for every $\nu < \omega$.

Then each

(2). $G_\nu = F_\nu - \bigcup\{F_\mu \mid \mu < \nu\}$

is open in F_ν, which implies it is an F_σ set in F_ν. But F_ν is an F_σ set in X, so that G_ν is F_σ in X.

Since (1) implies that G_ν is in $\mathfrak{M}'$, we can decompose G_ν as

$$G_\nu = A_\nu \cup B_\nu$$

where

$$A_\nu \in \mathfrak{M}, \text{ and}$$

$$\operatorname{Ind} B_\nu \leq 0.$$

By (2) we have that G_ν and G_μ are disjoint whenever ν and μ are different. Hence each A_ν is an F_σ set in $A = \bigcup\{A_\nu \mid \nu < \omega\}$. Similarly B_ν is F_σ in B. Thus $\{A_\nu \mid \nu < \omega\}$ is a locally finite covering of A consisting of F_σ sets, each of whose members is an element of $\mathfrak{M}$, which means that A is the union of a locally countable closed collection, each member of which belongs to $\mathfrak{M}$.

Since $\mathfrak{M}$ is a normal family, it follows that

$$A \in \mathfrak{M}.$$

Moreover, we have from 3.8 that

$$\operatorname{Ind} B \leq 0.$$

Since

$$X = A \cup B,$$

4.3 shows that

$$X \in \mathfrak{M}.$$

Therefore $\mathfrak{M}'$ satisfies (b'') of 2.1. Similarly, we can show that $\mathfrak{M}'$ satisfies (b'). Thus $\mathfrak{M}'$ is a normal family, and the proof is complete.

V. GENERAL PROPERTIES OF $\mathfrak{M}$-DIMENSIONAL SPACES

We now wish to obtain some properties one expects a dimension theory to have. Our development will begin by combining previous results about normal families and 0-dimensional spaces to obtain some properties of higher dimensional spaces. Many of the results are immediate; all that is required is the following inductive interpretation of normal families.

Denote by $\mathfrak{M}^{(-1)}$ the normal family consisting of the empty set. Now, by letting $\mathfrak{M}^{(h+1)} = \left(\mathfrak{M}^{(h)}\right)'$, $h = -1,0,1,\ldots$, we see that for each n, $\mathfrak{M}^{(n)}$ is the normal family consisting of all spaces X with Ind X $\leq$ n.

5.1 <u>Theorem</u> (The Subspace Theorem) If Y is a subspace of a space X, and if the Ind X $\leq$ n, then Ind Y $\leq$ n.

5.2 <u>Theorem</u> (The Sum Theorem) If $\{A_\alpha\}$ is a locally countable closed covering of a space X, where Ind $A_\alpha \leq$ n for each member of A_α, then Ind X $\leq$ n.

5.3 <u>Theorem</u> (The Decomposition Theorem) Let X be a space. Then Ind X $\leq$ n if and only if X can be decomposed into the sum of n+1 subspaces, each having strong inductive dimension $\leq$ 0.

<u>Example 1</u> The strong inductive dimension of the Euclidean line, E^1, is exactly 1. To see this, we first note that $E^1 = Q \cup J$, where Q is the set of rational numbers in E^1 and J is the set of irrational numbers in E^1. However, since Ind Q = 0, cf. [II, Example 2] and Ind J = 0, cf. [III, Example 3], the Decomposition Theorem yields that Ind $E^1 \leq 1$. However, we have already shown that Ind $E^1 > 0$, cf. [III, Example 4], so we have Ind $E^1 = 1$.

5.4 Theorem Let $X = Y \cup Z$. If $\operatorname{Ind} Y \leq m$ and $\operatorname{Ind} Z \leq n$, then $\operatorname{Ind} X \leq m+n+1$.

Proof: By 5.3, we can decompose Y and Z such that

$$Y = \bigcup_{i=1}^{m+1} Y_i \ , \quad \text{where } \operatorname{Ind} Y_i \leq 0 \text{ for } i = 1,2,\ldots,m+1$$

and

$$Z = \bigcup_{i=1}^{n+1} Z_j \ , \quad \text{where } \operatorname{Ind} Z_j \leq 0 \text{ for } j = 1,2,\ldots,n+1.$$

Since

$$X = \left(\bigcup_{i=1}^{m+1} Y_i\right) \cup \left(\bigcup_{j=1}^{n+1} Z_j\right) \ ,$$

it follows by the Decomposition Theorem that

$$\operatorname{Ind} X \leq (m+1) + (n+1) - 1 = m+n+1$$

as desired.

5.5 Theorem Let A be a subset of a space X. If $\operatorname{Ind} A \leq n$, then for every pair of a closed set F and an open set G satisfying $F \subset G$, there exists an open set V such that

$$F \subset V \subset G \ , \text{ and}$$

$$\operatorname{Ind} B(V) \cap A \leq n - 1.$$

Proof: The proof shall be by induction.

For the case when $n = 0$, the theorem reduces to 3.2.

Let $n > 0$. Then by 5.3 there exist subsets P and Q of A such that $\operatorname{Ind} P \leq$

$n - 1$, $\text{Ind } Q \leq 0$, and $A = P \cup Q$. From 3.2, it follows that there exists an open set V such that $F \subset V \subset G$ and $B(V) \cap Q = \emptyset$. Hence, we have $\text{Ind } B(V) \cap A \leq \text{Ind } B(V) \cap P \leq \text{Ind } P \leq n - 1$, which completes the proof.

5.6 Theorem Let A be a subset of a space X and let $\text{Ind } A \leq n$. Then there exists a G_δ-set H such that

$$A \subset H \quad \text{and}$$

$$\text{Ind } A = \text{Ind } H.$$

Proof: In case $n = 0$, there exists, by 3.4, a σ-locally finite open basis $\mathfrak{U} = \bigcup_{i=1}^{\infty} \mathfrak{U}_i$ of X such that each $\mathfrak{U}_i$ is a cover of X and such that for every $U \in \mathfrak{U}$, $B(U) \cap A = \emptyset$. If we put

$$H = X - \bigcup_{U \in \mathfrak{U}'} B(U),$$

then H is a G_δ-set. Now, by letting

$$\mathfrak{B} = \{U \cap H \mid U \in \mathfrak{U}\},$$

we have that $\mathfrak{B}$ is a σ-locally finite basis for the open sets of H such that for $U \cap H$ in $\mathfrak{B}$, $B_H(U \cap H) \subset B(U) \cap H = \emptyset$. Hence it follows from 3.3 that $\text{Ind } H \leq 0$. Also, if $\text{Ind } A = 0$, we have that $A \neq \emptyset$, and thus, $\text{Ind } H = 0$.

If $n > 0$, 5.3 gives the existence of $n+1$ subsets $A_0, A_1, \ldots, A_n$ of A satisfying

$$A = \bigcup_{i=0}^{n} A_i$$

and

$$\text{Ind } A_i \leq 0 \quad (i = 0,1,2,\ldots,n).$$

Thus the case for $n = 0$ implies that for each i there exists a G_δ-set H_i such that $A_i \subset H_i$ and $\text{Ind } H_i = 0$. If we put $H = \bigcup_{i=0}^{n} H_i$, then H is a G_δ-set of X such that $A \subset H$ and $\text{Ind } H \leq n$, but $\text{Ind } H \geq \text{Ind } A = n$ gives that $\text{Ind } H = n$, and the theorem is completely proved.

5.7 <u>Proposition</u> Let A be a subset of a space X such that $\text{Ind } A \leq n$. Then there exists a countable collection of locally finite open coverings, $\mathfrak{U}_i$, such that $\mathfrak{U} = \{U \mid U \in \mathfrak{U}_i,\ i = 1,2,\ldots\}$ is a basis for the open sets of X and $\text{Ind } B(U) \cap A \leq n - 1$ for each $U \in \mathfrak{U}$.

<u>Proof:</u> Let $\{\mathcal{B}_i \mid i = 1,2,\ldots\}$ be a countable collection of locally finite open coverings of X such that $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ is the canonical basis for the open sets of X. Then, for each i, let $\mathcal{D}_i$ be a barycentric refinement of $\mathcal{B}_i$. It follows that for each x in X, $\{St(x,\mathcal{D}_i) \mid i = 1,2,\ldots\}$ is a basis for the neighborhoods of x. Let $\mathcal{D}_i = \{D_{i\alpha} \mid \alpha \in \mathcal{A}_i\}$. For each $\mathcal{D}_i$, we now apply the Shrinking Theorem to obtain a closed covering $\{C_{i\alpha} \mid \alpha \in \mathcal{A}_i\}$ such that each $C_{i\alpha} \subset D_{i\alpha}$. By 5.5, it now follows that there exist open sets $U_{i\alpha}$ such that

$$C_{i\alpha} \subset U_{i\alpha} \subset D_{i\alpha} \quad \text{for } \alpha \in \mathcal{A}_i,\ i = 1,2,\ldots$$

and

$$(1). \quad \text{Ind } B(U_{i\alpha}) \cap A \leq n - 1 \quad \text{for } \alpha \in \mathcal{A}_i,\ i = 1,2,\ldots$$

Since $\mathfrak{U}_i$ is a refinement of $\mathcal{B}_i$, and hence $\{St(x,\mathfrak{U}_i) \mid i = 1,2,\ldots\}$ is a basis of neighborhoods at each point x of X, it follows that $\mathfrak{U} = \bigcup_{i=1}^{\infty} \mathfrak{U}_i$ is a σ-locally finite basis of X satisfying (1). This completes the proof of the proposition.

5.8 Proposition Let X be a space. If there exists a σ-locally finite basis $\mathfrak{B}$ of X such that

$$\operatorname{Ind} B(V) \leq n - 1 \quad \text{for every } V \in \mathfrak{B},$$

then $\operatorname{Ind} X \leq n$.

Proof: To prove this we first put

$$A = \bigcup \{B(V) \mid V \in \mathfrak{B}\}$$

and

$$B = R - A.$$

Then, since $\{B(V) \mid V \in \mathfrak{B}\}$ is a σ-locally finite closed covering of A and since $\operatorname{Ind} B(V) \leq n - 1$ for each $V \in \mathfrak{B}$, it follows from 5.2 that

$$\operatorname{Ind} A \leq n - 1.$$

Also, since $\mathfrak{B}$ restricted to B is an open basis of B satisfying the condition of 3.3, we have

$$\operatorname{Ind} B \leq 0.$$

Therefore, by 5.4,

$$\operatorname{Ind} X = \operatorname{Ind}(A \cup B) \leq \operatorname{Ind} A + \operatorname{Ind} B + 1 \leq n,$$

and the proposition is proved.

5.9 Theorem (The Product Theorem) Let X and Y be spaces. Then

$$\operatorname{Ind}(X \times Y) \leq \operatorname{Ind} X + \operatorname{Ind} Y.$$

Proof: The proof shall be by induction on $K = \operatorname{Ind} X + \operatorname{Ind} Y$.

If $K = -1$, we have that either $\operatorname{Ind} X = -1$ or $\operatorname{Ind} Y = -1$. Thus, $X = \emptyset$ or $Y = \emptyset$, which implies $X \times Y = \emptyset$ and the theorem holds.

We now assume the theorem holds whenever $\operatorname{Ind} X + \operatorname{Ind} Y < K$. Let X and Y be spaces such that $\operatorname{Ind} X \leq n$ and $\operatorname{Ind} Y \leq m$, where $m + n = K$. Then, by 5.7, we can find countable collections of locally finite coverings $\mathfrak{U}_i$ and $\mathfrak{D}_i$ such that $\mathfrak{U} = \{ U \mid U \in \mathfrak{U}_i, i = 1,2,\ldots\}$ and $\mathfrak{D} = \{D \mid D \in \mathfrak{D}_i, i = 1,2,\ldots\}$ are bases for the open sets of X and Y, respectively. Furthermore, these bases may be chosen in a manner such that

$$\operatorname{Ind} B(U) \leq n - 1, \text{ for } U \in \mathfrak{U},$$

and

$$\operatorname{Ind} B(D) \leq m - 1, \text{ for } D \in \mathfrak{D}.$$

If we put

$$\mathfrak{R} = \{U \times D \mid U \in \mathfrak{U}_i, D \in \mathfrak{D}_j, i = 1,2,\ldots, \quad j = 1,2,\ldots\},$$

then $\mathfrak{R}$ is a σ-locally finite open basis for the product space $X \times Y$.

Now, since

$$B_{X\times Y}(U \times V) = \left(B_X(U) \times \overline{V}\right) \cup \left(\overline{U} \times B_Y(V)\right), \quad \text{for } U \times V \in \mathfrak{R},$$

the induction assumption implies each summand is a closed set having strong inductive dimension $\leq n + m - 1$, and, by 5.2, we have that

$$\operatorname{Ind} B_{X\times Y}(U \times V) \leq m + n - 1.$$

Therefore, by 5.8, we have that $\operatorname{Ind}(X \times Y) \leq m + n$. This concludes the proof of the theorem.

After developing more material, we shall prove that Ind $R^{\aleph_0} = 1$, where $R^{\aleph_0}$ is the set of points in the Hilbert space all of whose coordinates are rational, cf. [VI, Example 3]. If we assume this result, the following example demonstrates that the inequality derived in the last theorem cannot, in general, be changed to equality.

Example 2 Since $R^{\aleph_0} = R^{\aleph_0} \times R^{\aleph_0}$, Ind $(R^{\aleph_0} \times R^{\aleph_0}) <$ Ind $R^{\aleph_0}$ + Ind $R^{\aleph_0}$.

5.10 Theorem Let $\{X_i \mid i = 1,2,\ldots\}$ be a collection of 0-dimensional spaces. Then Ind $\Pi\{X_i \mid i = 1,2,\ldots\} = 0$.

Proof: For each X_i there exists a countable collection $\{\mathfrak{U}_{ij} \mid j = 1,2,\ldots\}$ of locally finite open systems such that $\mathfrak{U}_i = \{U \mid U \in \mathfrak{U}_{ij},\ i = 1,2,\ldots\}$ is a basis for the open sets of X_i. Then $\mathfrak{U} = \{U \mid U \in \mathfrak{U}_{ij},\ i = 1,2,\ldots,\ j = 1, 2,\ldots\}$ is a σ-locally finite collection. If P_i is the projection from ΠX_k onto X_i, we have that $\{P_k^{-1}(U) \mid U \in \mathfrak{U}\}$ is a subbasis for the product space ΠX_k. Thus the basis determined by $\mathfrak{U}$ is σ-locally finite. Furthermore, since each element of this basis has empty boundary, 3.2 implies Ind $\Pi X_i \leq 0$. This proves the theorem.

VI. EQUIVALENCE OF STRONG INDUCTIVE DIMENSION, COVERING DIMENSION AND WEAK INDUCTIVE DIMENSION

In this section, we wish to prove that for every metric space X and for every positive integer n, dim X $\leq$ n is equivalent to Ind X $\leq$ n. We will also show that in the class of all separable metric spaces, the notion of weak inductive dimension coincides with those of strong inductive dimension and covering dimension.

6.1 Proposition If F is a closed set in a space X, then dim F $\leq$ dim X.

Proof: Let $C = \{C_\alpha \mid \alpha \in \mathcal{A}\}$ be any finite open covering of subspace F. Then for each $\alpha \in \mathcal{A}$, there exists an open U_α in X such that $U_\alpha \cap F = C_\alpha$. Since $\{U_\alpha \mid \alpha \in \mathcal{A}\} \cup \{CF\}$ is a finite open covering of X, dim X $\leq$ n implies there exists an open refinement $\{V_\beta \mid \beta \in \mathcal{B}\}$ of $\{U_\alpha \mid \alpha \in \mathcal{A}\} \cup \{CF\}$ satisfying

$$\text{order } \{V_\beta \mid \beta \in \mathcal{B}\} \leq n + 1.$$

Then $\{V_\beta \cap F \mid \beta \in \mathcal{B}\}$ is a refinement of $\{C_\alpha \mid \alpha \in \mathcal{A}\}$ such that

$$\text{ord } \{V_\beta \cap F\} \leq n + 1.$$

Thus, by definition, dim F $\leq$ n, and the proposition is proved.

6.2 Theorem Let R be a normal space, such that dim R $\leq$ n. If $\mathfrak{G} = \{G_\alpha \mid \alpha \in \Omega\}$ is a locally finite covering of R, then there exists an open covering $\mathfrak{U} = \{U_\alpha \mid \alpha \in \Omega\}$ of order $\leq$ n+1 such that $U_\alpha \subset G_\alpha$ for α in Ω.

Proof: Since the theorem reduces to the definition of covering dimension in case Ω is finite, we assume $\aleph$ = card Ω to be infinite and complete the proof by transfinite induction.

Suppose the theorem holds in case card $\Omega < \aleph$. Without loss of generality we may assume $\mathfrak{G} = \{G_\alpha \mid \alpha < \aleph\}$.

Since $\mathfrak{G}$ is locally finite there exists a closed refinement $\{F_\alpha \mid \alpha < \aleph\}$ of $\mathfrak{G}$. Also, for each $\alpha < \Omega$, we can find an open set L_α satisfying $F_\alpha \subset L_\alpha \subset \bar{L}_\alpha \subset G_\alpha$. We shall now construct an open covering $\{\mathfrak{U}_\alpha\}$ by transfinite induction.

Assume that for any ordinal β less than some fixed $\alpha < \aleph$ we have constructed open sets $U_{\beta\gamma} (\gamma < \aleph)$ such that

(1). $$U_{\beta\gamma} = \emptyset \quad \text{for } \beta < \gamma < \aleph$$

(2). $$\bar{U}_{\beta\gamma} \subset L_\beta \cap L_\gamma \quad \text{for } \gamma \leq \beta$$

(3). $$F_\beta \subset \bigcup_{\gamma \leq \beta} U_{\beta\gamma}$$

and

(4). $$\text{ord} \bigcup_{\sigma \leq \beta} \{\bar{U}_{\sigma\gamma} \mid \gamma \leq \beta\} \leq n + 1$$

Now we shall show the existence of open sets $U_{\alpha\gamma} (\gamma < \aleph)$ satisfying (1), (2), (3), and (4). If we put

(5). $$U_{\gamma}' = \bigcup_{\beta < \alpha} U_{\beta\gamma} \quad \text{for } \gamma < \alpha,$$

we have

(6). $$\bar{U}_{\gamma}' = \bigcup_{\beta < \alpha} \bar{U}_{\beta\gamma} \subset L_\gamma,$$

since $U_{\beta\gamma} \subset L_\beta$ and hence the system $\{\bar{U}_{\beta\gamma} \mid \beta < \alpha\}$ is locally finite. We shall now prove that

(7). $$\text{order} \{\bar{U}_{\gamma}' \mid \gamma < \alpha\} \leq n + 1.$$

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To see this we note that if

$$p \in \bigcap_{i=1}^{n+2} U'_{\gamma_i}$$

then there exist $\beta_i < \alpha$, $i = 1,2,\ldots,n+2$ such that

$$p \in \bigcap_{i=1}^{n+2} U_{\beta_i \gamma_i} ,$$

and since there exists a β such that $\beta_i \leq \beta < \alpha$, with $\gamma_i \leq \beta$ for $i = 1,2,\ldots,$ $n+2$, we have

$$p \in \bigcap_{i=1}^{n+2} \left[\bigcup_{\sigma \leq \beta} U_{\sigma \gamma_i} \right]$$

This contradicts (4).

Now, by 1.18 we know that there exist open sets V_γ for $\gamma < \alpha$ such that

(8). $$U'_\gamma \subset V_\gamma \subset \overline{V}_\gamma \subset L_\gamma$$

and

(9). $\{V_\gamma \mid \gamma < \alpha\}$ is similar to $\{U'_\gamma \mid \gamma < \alpha\}$.

Also, by applying 1.17, we have that $\mathcal{B} = \wedge\{V_\alpha, R - \overline{U}_\alpha\}$ is a locally finite open covering of R which satisfies

(10). $$S(U'_\gamma, \mathcal{B}) \subset V_\gamma \quad \text{for } \gamma < \alpha.$$

Since we are dealing only with transfinite numbers, we know that the cardinal number of the family of sets $\mathcal{B}$ is not greater than the cardinal number of the set $\{\gamma \mid \gamma < \alpha\}$. Therefore, since 6.1 yields that dim $F_\alpha \leq$

dim $R \leq n+1$, the induction assumption implies the existence of a system of open sets $\{H_\lambda \mid \lambda \in \Lambda\}$ satisfying

$$F_\alpha \subset \bigcup_\lambda H_\lambda \ , \ \bar{H}_\lambda \subset L_\alpha,$$

(11). $$\{\bar{H}_\lambda \mid \lambda \in \Lambda\} \text{ refines } \mathcal{R}$$

$$\text{order } \{H_\lambda \mid \lambda \in \Lambda\} \leq n + 1.$$

We may assume that $\{H_\lambda \mid \lambda \in \Lambda\}$ is locally finite because it refines the locally finite collection $\mathcal{R}$.

For all $j < \alpha$, define

$$U_{\alpha j} = \bigcup \{H_\lambda \mid \bar{H}_\lambda \cap U'_j \neq \emptyset\}$$

Also, put

$$U_{\alpha\alpha} = \bigcup \{H_\lambda \mid \bar{H}_\lambda \cap U'_\gamma = \emptyset \text{ for } \gamma < \alpha\}.$$

For $\gamma > \alpha$, put

$$U_{\alpha\gamma} = \emptyset.$$

Then the open sets $U_{\alpha\gamma} (\gamma < \aleph)$ satisfy (1), (2), (3), and (4). To prove this, we need only to show

(12). $$\text{order } \left[\{U'_\alpha \cup U_{\alpha\gamma} \mid \gamma < \alpha\} \cup \{U_{\alpha\alpha}\} < n + 1\right]$$

and

(13). $$U_{\alpha\gamma} \subset L_\alpha \cap L_\gamma \quad \text{for } \gamma < \alpha.$$

By (10), (11), and the construction of the H_λ we have for $\gamma < \alpha$ that

Finally, if we put

$$U_\alpha = \bigcup_\sigma U_{\sigma\alpha} \quad \text{for } \alpha < \aleph$$

we have from $\{F_\alpha \mid \alpha \in \aleph\}$ forming a covering and (1), (2), (3), (4) that

$$U_\alpha \subset L_\alpha \subset G_\alpha$$

and

$$R = \bigcup_\alpha U_\alpha.$$

Hence $\mathfrak{U} = \{U_\alpha \mid \alpha < \aleph\}$ is an open covering of R whose order $\leq n+1$. To see this, we note that if

$$p \in \bigcup_{i=1}^{n+2} U_{\alpha_i}$$

then $p \in U_{\sigma_i\alpha_i}$, for $\sigma_i < \aleph$, $i = 1,2,\ldots,n+2$. Hence

$$p \in \bigcup_{\sigma>\beta} U_{\sigma\alpha_i}, \quad i = 1,2,\ldots,n+2$$

for some β such that for $i = 1,2,\ldots,n+2$

$$\sigma_i \leq \beta$$

and

$$\alpha_i \leq \beta \leq \aleph .$$

This contradicts (4), and so the theorem is completely proved.

6.3 <u>Corollary</u> Let X be a space. Then dim X $\leq$ n if and only if for any open covering $\mathfrak{U}$ of X there exists a locally finite open refinement $\mathfrak{B}$ of $\mathfrak{U}$ with

$$(14). \qquad U_{\alpha\gamma} \subset S(U'_\gamma, e) \subset V_\gamma \subset \overline{V}_\gamma \subset L_\gamma$$

and hence (13).

To prove (12), suppose

$$X = \bigcap_{i=1}^{r} \{(U'_{\gamma_i} \cup U_{\alpha\gamma_i}) \cap U_{\alpha\alpha}\} \neq \emptyset$$

where $\gamma_1 < \gamma_2 < \dots < \gamma < \alpha$.

Then, since $U'_{\gamma_i} \cap U_{\alpha\alpha} = \emptyset$ for $i = 1,2,\dots,r$, we have

$$X = \bigcap_{i=1}^{r} (U_{\alpha\gamma_i} \cap U_{\alpha\alpha}) \neq \emptyset ,$$

so that (11) gives that $r + 1 \leq n + 1$.

If

$$X = \bigcap_{i=1}^{r} (U'_{\gamma_i} \cup \overline{U}'_{\alpha\gamma_i}) \neq \emptyset$$

then by (7) and (8) we have

$$X \subset \bigcap_{i=1}^{r} V_{\gamma_i}$$

and hence

$$r \leq n + 1$$

is established by (7) and (9).

Thus, for any ordinal $\beta < \aleph$, we have proved the existence of open sets $U_{\beta\gamma}$ satisfying (1), (2), (3) and (4).

order $\mathfrak{B} \leq n+1$.

<u>Proof:</u> Assume dim $X \leq n$, and let $\mathfrak{U}$ be any open covering of X. Then, by the paracompactness of X, there exists a locally finite open refinement of $\mathfrak{U}$. The corollary now follows from 6.2.

Conversely, we see that the given condition is stronger than required by 6.2, and the corollary is completely proved.

6.4 <u>Proposition</u> Let $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ be a locally finite open collection in a space X, where dim $X \leq n$, and let $\{F_\alpha \mid \alpha \in \mathcal{A}\}$ be a closed collection in X so that $F_\alpha \subset U_\alpha$ for every α in $\mathcal{A}$. Then there exist open collections $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ and $\{W_\alpha \mid \alpha \in \mathcal{A}\}$ such that

$$F_\alpha \subset V_\alpha \subset \overline{V}_\alpha \subset W_\alpha \subset U_\alpha$$

and

$$\text{order } \{W_\alpha - \overline{V}_\alpha \mid \alpha \in \mathcal{A}\} \leq n.$$

<u>Proof:</u> For each $\alpha \in \mathcal{A}$, let $\mathfrak{U}_\alpha = \{U_\alpha, X - F_\alpha\}$. Then each $\mathfrak{U}_\alpha$ is an open covering of X. Thus, we have that $\wedge\{\mathfrak{U}_\alpha \mid \alpha \in \mathcal{A}\}$ is a locally finite open covering of X.

For each point x in X, let $N(x)$ be a neighborhood of x meeting only finitely many of the $\{F_\alpha\}$. If we take a common locally finite refinement of $\wedge\{\mathfrak{U}_\alpha \mid \alpha \in \mathcal{A}\}$ $\wedge\{N(x) \mid x \in X\}$, it follows by 6.2 that there exists a locally finite open refinement $\{N_\delta \mid \delta \in \Delta\}$ of this refinement satisfying

$$\text{order } \{N_\delta \mid \delta \in \Delta\} \leq n+1.$$

Now let $\{K_\delta \mid \delta \in \Delta\}$ be a precise closed refinement of $\{N_\delta \mid \delta \in \Delta\}$. For

THE FOLLOWING PAGES CONTAIN CROOKED TYPING AND IS THE BEST POSSIBLE IMAGE AVAILABLE each N_δ, and for each F_α intersecting N_δ, we define an open set $N_\delta(F_\alpha)$ such that

$$\text{(1).} \qquad K_\delta \subset N_\delta(F_\alpha) \subset \overline{N_\delta(F_\alpha)} \subset N_\delta$$

and such that

$$\text{(2).} \quad N_\delta(F_\alpha) \subset N_\delta(F_\alpha) \subset N_\delta(F_{\alpha'}) \text{ or } N_\delta(F_{\alpha'}) \subset N_\delta(F_{\alpha'}) \subset N_\delta(F_\alpha) \text{ for } \alpha \neq \alpha'.$$

To do this, fix δ and let $F_{\alpha_1}, F_{\alpha_2}, \ldots, F_{\alpha_n}$ be the only members of $\{F_\alpha \mid \alpha \in \mathcal{A}\}$ which meet N_δ. By the normality of X, there exist open sets $M_{\delta_1}, \ldots, M_{\delta_n}$ such that

$$K_\delta \subset M_{\delta_1} \subset \overline{M_{\delta_1}} \subset M_{\delta_2} \subset \overline{M_{\delta_2}} \subset \ldots \subset M_{\delta_n} \subset \overline{M_{\delta_n}} \subset N_\delta.$$

Let

$$N_\delta(F_{\alpha_1}) = M_{\delta_1}$$

$$N_\delta(F_{\alpha_2}) = M'_{\delta_2}$$

$$\vdots$$

$$N_\delta(F_{\alpha_n}) = M_{\delta_n}.$$

Thus, we obtain the $N_\delta(F_{\alpha_i})$ with the properties (1) and (2). Now let

$$V_\alpha = \bigcup \{N_\delta(F_\alpha) \mid \delta \in \Delta\}.$$

Then, because $\{N_\delta \mid \delta \in \Delta\}$ refines $\mathfrak{U}_\alpha$ and $N_\delta(F_\alpha) \subset N_\delta \subset U_\alpha$ whenever $N_\delta \cap F_\alpha \neq \emptyset$, we have $F_\alpha \subset V_\alpha \subset \overline{V}_\alpha \subset U_\alpha$.

We shall now show that order $\{\overline{V}_\alpha - V_\alpha \mid \alpha \in \mathcal{A}\} \leq n$. Suppose not. Then there exist $V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_{n+1}}$ such that

(3). $\bigcap_{i=1}^{n+1} (\overline{V}_{\alpha_i} - V_{\alpha_i}) \neq \emptyset$, for distinct α_i, $i = 1,2,\ldots,n+1$. Thus, by the local finiteness of $\{N_\delta \mid \delta \in \Delta\}$, we know that each V_α is only a finite union, and there exist $N_{\delta_i}(F_{\alpha_i})$, $i = 1,2,\ldots,n+1$ such that

(4). $$\bigcap_{i=1}^{n+1} \left(\overline{N_{\delta_i}(F_{\alpha_i})} - N_{\delta_i}(F_{\alpha_i})\right) \neq \emptyset .$$

If $\delta_i = \delta_j$ and $i \neq j$, then by (2), we have

$$\left[\overline{N_{\delta_i}(F_{\alpha_i})} - N_{\delta_i}(F_{\alpha_i})\right] \cap \left[\overline{N_{\delta_j}(F_{\alpha_j})} - N_{\delta_j}(F_{\alpha_j})\right] = \emptyset,$$

which is a contradiction to (3). Therefore, the members of $\{\delta_i \mid i = 1,2,\ldots, n+1\}$ are distinct. Let

$$p \in \bigcap_{i=1}^{n+1} \left[\overline{N_{\delta_i}(F_{\alpha_i})} - N_{\delta_i}(F_{\alpha_i})\right].$$

Then by (1),

(5). $$p \notin \bigcup_{i=1}^{n+1} K_{\delta_i} .$$

However, since $\{K_\delta \mid \delta \in \Delta\}$ is a covering of X, it follows that there exists a K_δ that contains p, which implies by (5), that $\delta \neq \delta_i$, for $i = 1,2,\ldots,n+1$.

On the other hand, by (1) and (4), we have

$$p \in \bigcap_{i=1}^{n+1} N_{\delta_i} .$$

Thus

$$p \in K_\delta \cap \left[\bigcap_{i=1}^{n+1} N_{\delta_i}\right] \subset N_\delta \cap \bigcap_{i=1}^{n+1} N_{\delta_i} .$$

But this contradicts order $\{N_\delta \mid \delta \in \Delta\} \le n+1$, so we have

(6). $$\text{order } \{\overline{V}_\alpha - V_\alpha \mid \alpha \in \mathcal{A}\} \le n.$$

For each $\alpha \in \mathcal{A}$, let

$$\mathfrak{A}'_\alpha = \{U_\alpha, X - B(V_\alpha)\}.$$

Since the local finiteness of $\{U_\alpha\}$ implies $\{\overline{V}_\alpha - V_\alpha \mid \alpha \in \mathcal{A}\}$ is a locally finite collection, we can find, for each point x in X, a neighborhood, $G'(x)$, of x which meets only finitely many members of $\{\overline{V}_\alpha - V_\alpha \mid \alpha \in \mathcal{A}\}$. For each $G'(x)$, let

$$G(x) = G'(x) - \bigcup \{B(V_\alpha) \mid x \notin B(V_\alpha)\}.$$

Then $G(x)$ is open. By (6), it follows that each $G(x)$ meets at most n members of $\{\overline{V}_\alpha - V_\alpha \mid \alpha \in \mathcal{A}\}$. Now, let $\mathcal{C}$ be a barycentric refinement of

$$\wedge\{\mathfrak{A}'_\alpha \mid \alpha \in \mathcal{A}\} \wedge \{G(x) \mid x \in X\}.$$

Then for each point x in X, $St(x,\mathcal{C})$ meets at most n members of $\{\overline{V}_\alpha - V_\alpha \mid \alpha \in \mathcal{A}\}$.

Now, if we put

$$W_\alpha = \overline{V}_\alpha \cup S(B(V_\alpha,\mathcal{C}) \quad , \text{ for each } \alpha \in \mathcal{A},$$

then

$$\text{order } \{W_\alpha - \overline{V}_\alpha \mid \alpha \in \mathcal{A}\} \le n.$$

Since, if not, there exists an x and $W_{\alpha_1}, \ldots, W_{\alpha_{n+1}}$ such that

$$x \in \bigcap_{i=1}^{n+1} W_{\alpha_1}$$

which implies

$$x \in \bigcap_{i=1}^{n+1} \left[St(V_{\alpha_i}) - \overline{V}_{\alpha_i} \right]$$

where the V_α are distinct and each V_{α_i} meets some $C_i \in C$. But then $St(x,C)$ meets n+1 members of $\{\overline{V}_\alpha - V_\alpha \mid \alpha \in \mathcal{A}\}$, which is a contradiction, and we have that $\{W_\alpha \mid \alpha \in \mathcal{A}\}$ and $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ are the desired covers. Thus the theorem is proved.

6.5 <u>Proposition</u> Let $\{U_\alpha \mid \alpha \in \mathcal{A}_i\}$, $i = 1,2,\ldots$ be locally finite open collections in a space X with $\dim X \le n$, and $\{F_\alpha \mid \alpha \in \mathcal{A}_i\}$, $i = 1,2,\ldots$ be closed collections in X such that $F_\alpha \subset U_\alpha$ for $\alpha \in \bigcup_{i=1}^{\infty} \mathcal{A}_i$. Then there exist open sets V_α for $\alpha \in \bigcup_{i=1}^{\infty} \mathcal{A}_i$ such that

$$F_\alpha \subset V_\alpha \subset \overline{V}_\alpha' \subset U_\alpha$$

and

$$\text{order } \{\overline{V}_\alpha - V_\alpha \mid \alpha \in \bigcup_{i=1}^{\infty} \mathcal{A}_i\} \le n.$$

<u>Proof:</u> By applying 6.4, we obtain open sets

V_α^1, W_α^1 for every $\alpha \in \mathcal{A}_1$,

V_α^2, W_α^2 for every $\alpha \in \mathcal{A}_1 \cup \mathcal{A}_2$

. . . .

V_α^j, W_α^j for every $\alpha \in \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_j$,

. . . .

such that

$$F_\alpha \subset \overline{V}^j_\alpha \subset W^j_\alpha \subset U_\alpha,$$

$$\overline{V}^j_\alpha \subset V^{j+1}_\alpha \subset \overline{W}^{j+1}_\alpha \subset W^j_\alpha,$$

and

(1). $$\text{order}\,\{W^j_\alpha - \overline{V}^j_\alpha \mid \alpha \in \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots \cup \mathcal{A}_j\} \leq n.$$

Put

$$V_\alpha = \bigcup_j V^j_\alpha, \quad \text{for } \alpha \in \bigcup_{i=1}^{\infty} \mathcal{A}_i$$

where the union is over the j such that $\alpha \in \mathcal{A}_j$. Then for each α in $\bigcup_{i=1}^{\infty} \mathcal{A}_i$, let i_α be the smallest integer such that $\alpha \in \mathcal{A}_{i_\alpha}$. Then

(2). $$F_\alpha \subset V^{i_\alpha}_\alpha \subset V_\alpha \subset \overline{V}_\alpha \subset \overline{W}^{i_\alpha+1}_\alpha \subset U_\alpha,$$

Thus,

(3). $$F_\alpha \subset V_\alpha \subset \overline{V}_\alpha \subset U_\alpha$$

and

$$\overline{V}_\alpha - V_\alpha \subset W^{i_\alpha}_\alpha - \overline{V}^{i_\alpha}_\alpha.$$

But, then, by (1), we have

(4). $$\text{order}\,\{\overline{V}_\alpha - V_\alpha \mid \alpha \in \bigcup_{j=1}^{\infty} \mathcal{A}_i\} \leq n.$$

Therefore (2), (3), and (4) show that $\{V_\alpha \mid \alpha \in \bigcup_{i=1}^{\infty} \mathcal{A}_i\}$ is the desired collection of open sets, and the proposition is proved.

6.6 <u>Proposition</u> If there exists a σ-locally finite open basis $\mathcal{B}$ of a space X such that order $B(\mathcal{B}) \leq n$, then Ind $X \leq n$.

<u>Proof:</u> Let $\mathcal{B} = \{V_\alpha \mid \alpha \in \mathcal{A}\}$. The proof shall be by induction. We note that the case for $n = 0$ is simply 3.3. To show the proposition holds for $n > 0$, we note that 2.4 indicates that it suffices to show that for each V_α in $\mathcal{B}$, Ind $B(V_\alpha) \leq n - 1$.

To prove this, we first note that if we put

$$\mathcal{B}_\alpha = \{B(V_\alpha) \cap V_{\alpha'} \mid \alpha' \in \mathcal{A}\},$$

then $\mathcal{B}_\alpha$ is a σ-locally finite basis of $B(V_\alpha)$. Also, since

$$B(B(V_\alpha) \cap V_{\alpha'}) \subset B(V_\alpha)$$

we have that

$$\text{order}\,\{B_{B(V_\alpha)}(V) \mid V \in \mathcal{B}_\alpha\} \leq \text{order}\,\{B(V_{\alpha'}) \mid \alpha' \in \mathcal{A}\} \leq n.$$

But since $B(B(V_\alpha) \cap V_\alpha) = B(\emptyset) = \emptyset$, it follows that

$$\text{order}\,\{B_{B(V_\alpha)}(V) \mid V \in \mathcal{B}_\alpha\} \leq n - 1.$$

Thus, by the induction hypothesis,

$$\text{Ind}\, B(V_\alpha) \leq n - 1,$$

which completes the proof of the proposition.

6.7 <u>Theorem</u> For any space X, dim $R \leq n$ if and only if Ind $X \leq n$.

Proof: Assume Ind $X \leq n$. Then, by the Decomposition Theorem, we can express X as

$$X = \bigcup_{i=1}^{n+1} A_i$$

with Ind $A_i \leq 0$ for $i = 1,2,\ldots,n+1$.

Let $\mathfrak{U} = \{U_j \mid j = 1,\ldots,h\}$ be an open covering of X.

For each A_i, we shall now construct an open covering $\{V_j^i \mid j = 1,2,\ldots,h\}$ such that

$$\text{(1).} \qquad \text{order } \{V_j^i \mid j = 1,2,\ldots,h\} \leq 1$$

and

$$\text{(2).} \qquad V_j^i \subset U_j \text{ for each } j = 1,2,\ldots,h.$$

We first choose a covering $\{D_j \mid j = 1,\ldots,h\}$ of A_i which consists of open and closed sets of A_i such that $D_j \subset U_j$ for $j = 1,2,\ldots,h$. Putting

$$V_j = D_j - \bigcup_{k=1}^{j-1} D_k,$$

we obtain the desired covering of A_i.

For every point x in V_j, we choose $\varepsilon(x) > 0$ such that

$$N_{\varepsilon(x)}(x) \cap A_i \subset V_j$$

and

$$N_{\varepsilon(x)}(x) \subset U_j,$$

where

$$N_{\varepsilon(x)}(x) = \{y \in X \mid d(x,y) < \varepsilon(x)\}.$$

Let

$$W_j = \bigcup \{N_{\varepsilon(x)/2}(x) \mid x \in V_j\}.$$

Then

$$\mathcal{W}_i = \{W_j \mid j = 1,2,\ldots,h\}$$

is an open collection of sets which cover A_i. We also note that $W_j \subset U_j$ for $j = 1,2,\ldots,h$ and thus, by (1), we have

$$\text{order } \mathcal{W}_i \leq 1.$$

Hence

$$\mathcal{W} = \bigcup_{i=1}^{n+1} \mathcal{W}_i$$

is an open refinement of $\mathfrak{U}$ and it has order $\leq n+1$. Therefore $\dim R \leq n$.

Conversely, we assume $\dim X \leq n$. Let $\mathfrak{U}_i = \{U_\alpha \mid \alpha \in \mathcal{A}_i\}$, $i = 1,2,\ldots$ be a sequence of locally finite open coverings of X such that

$$\text{mesh } \mathfrak{U}_i < \frac{1}{2^i}$$

Then, by the normality of X, we obtain a sequence of closed coverings $\mathfrak{F}_i = \{F_\alpha \mid \alpha \in \mathcal{A}_i\}$, $i = 1,2,\ldots$ such that $F_\alpha \subset U_\alpha$ for each α. Now applying 6.5, we obtain open sets V_α for α in $\bigcup_{i=1}^{\infty} \mathcal{A}_i$ such that

$$F_\alpha \subset V_\alpha \subset \overline{V}_\alpha \subset U_\alpha$$

and

$$\text{order } \{\overline{V}_\alpha - V_\alpha \mid \alpha \in \bigcup_{i=1}^{\infty} \mathcal{A}_i\} \leq n.$$

By the choice of the $\mathfrak{A}_i$, in particular, mesh $\mathfrak{A}_i < \frac{1}{2^i}$, we have that

$$\mathcal{B} = \{V_\alpha \mid \alpha \in \bigcup_{i=1}^{\infty} \mathcal{A}_i\}$$

is a σ-locally finite basis of X, and thus, by 6.6, we have Ind R $\leq$ n, and the theorem is completely proved.

When working with separable metric spaces, it is often convenient to use the weak inductive dimension, cf. [1.16] Therefore, we now prove its equivalence to strong inductive dimension. First, however, we require the Subspace Theorem for weak inductive dimension.

6.8 Theorem Let X be a separable metric space such that ind X $\leq$ k. Then for any $A \subset X$, ind A $\leq$ k.

Proof: The proof will be by induction on k. We note that the theorem is trivially true for k = -1. Assume it holds for k = n - 1. Let $x \in X$ and V a neighborhood of x in A. Then there exists an open set V' of X such that $V' \cap A = V$. Since ind X $\leq$ n, we can find an open set U' in X satisfying

$$x \in U' \subset V'$$

and

$$\text{ind } B(U') \leq n - 1.$$

Then $U = U' \cap A$ is an open set of the space A such that

$$x \in U \subset V \subset A.$$

Furthermore, since $B(U) \cap A$ is contained in $B(U')$ in X, it follows from the induction hypothesis that

$$\text{ind } B(U) \cap A \leq n - 1.$$

Therefore

$$\text{ind } A \leq n$$

and the theorem is proved.

6.9 Theorem Let X be a separable metric space. Then Ind $X \leq n$ if and only if ind $X \leq n$.

Proof: We note that since each point in a metric space is a closed set, the proof of the "only if" part of the theorem is immediate. The proof of the "if" part of the theorem shall be by induction.

It is clear that the theorem holds for $n = -1$. Assume it is true for any space whose strong inductive dimension is n-1. Let X be a space with ind $X \leq n$. Then, by 1.17, there exists a basis $\mathfrak{A}'$ of X such that for every U in $\mathfrak{A}$

(1). $$\text{ind } B(U) \leq n\text{-}1.$$

Since X is a separable metric space, and thus second countable, there exists a countable basis $\{W_i \mid i = 1,2,\ldots\}$ for the open sets of X. For each W_i, and for each x in W_i, there exists $U_i(x) \in \mathfrak{A}$ such that

$$x \in U_i(x) \subset W_i.$$

If, for each i, we put

$$\mathfrak{A}_i = \{U_i(x) \mid x \in W_i\},$$

then $\mathfrak{A}_i$ is an open covering of W_i and thus has a countable subcovering C_i. Now, by putting

$$C = \{U_i(x) \mid U_i(x) \in C_i,\ i = 1,2,\ldots\}$$

we have that C is a countable basis of X and, in particular, is σ-locally finite. Also, from (1) and the induction hypothesis, it follows that for each $U_i(x) \in C$, $\operatorname{Ind} B(U_i(x)) \leq n - 1$. Therefore, the theorem is proved by 5.8.

Example 1 The set $R'_{\aleph_0}$ of points in the Hilbert cube $I_{\aleph_0}$ all of whose coordinates are rational has weak inductive dimension = 0. Suppose

$$a = (a_1, a_2, \ldots)$$

is an arbitrary point in $I_{\aleph_0}$ and U is a neighborhood of a in $I_{\aleph_0}$. By taking n large enough and p_i, q_i sufficiently close to a_i, $p_i < a_i < q_i$, $i = 1,2,\ldots,n$, we obtain a neighborhood of a contained in U consisting of the points

$$x = (x_1, x_2, \ldots)$$

of $I_{\aleph_0}$ whose first n coordinates are restricted by

$$(1). \qquad p_i < x_i < q_i$$

and whose other coordinates are restricted by

$$|x_i| \leq 1/i.$$

Now, let $a \in R'_{\aleph_0}$. By taking p_i and q_i to be irrational, we have a neighborhood V of a, each of whose boundary points has at least one irrational coordinate. Hence V has empty boundary in $R'_{\aleph_0}$ and we have $\operatorname{ind} R'_{\aleph_0} = 0$.

Example 2 The set $R_{\aleph_0}$ of points in the Hilbert space all of whose coordinates are rational has $\operatorname{Ind} R_{\aleph_0} > 0$.

To show this, we will prove ind $R_{\aleph_0} > 0$. Therefore, it suffices to prove that any bounded neighborhood U in $R_{\aleph_0}$ of the origin has non-empty boundary. Consider the line, $-\infty < x_1 < \infty$, $0 = x_2 = x_3 = \ldots$. This contains points in U and points in $R_{\aleph_0} - U$. Therefore, we can find a rational number a_1 such that the point

$$p_1 = (a_1, 0, 0, \ldots)$$

is contained in U and has distance less than 1 from $R_{\aleph_0} - U$. Similarly, by considering the straight line, $x_1 = a_1$, $-\infty < x_2 < +\infty, 0, 0, \ldots$, we determine a point

$$p_\gamma = (a_1, a_2, 0, 0, \ldots)$$

in U such that its distance from $R_{\aleph_0} - U$ is less than 1/2

Inductively, we determine a sequence $\{p_n\}$ such that

$$p_n = (a_1, a_2, \ldots, a_n, 0, 0, \ldots),$$

where each a_n is rational, each p_n is in U and

$$d(p_n, R_{\aleph_0} - U) < 1/n.$$

Then

$$p = (a_1, a_2, \ldots, a_h, a_{h+1}, \ldots)$$

is a boundary point of U.

We note that $\sum_{i=1}^{n} a_i^2 < d^2$, $n = 1, 2, \ldots$, where d is the diameter of U, hence $\sum_{i=1}^{\infty} a_i^2 < \infty$, and p is a point of the Hilbert space.

<u>Example 3</u> The space, $R_{\aleph_0}$, of points in the Hilbert space, all of whose coordinates are rational has $\dim R_{\aleph_0} = 1$.

To see this, we recall that we have just shown $\dim R_{\aleph_0} \geq 1$. We now shall show that $\dim R_{\aleph_0} \leq 1$ by proving that each point in $R_{\aleph_0}$ can be surrounded by arbitrarily small neighborhoods whose boundaries are 0-dimensional. Let $S = \{x \mid d(x,0) < 1\}$. It suffices to prove $\dim S \cap R_{\aleph_0} \leq 1$. For each point x in S where

$$x = (x_1, x_2, \ldots)$$

associate the point

$$x' = \left(x_1, \frac{x_2}{2}, \ldots, \frac{x_i}{i}, \ldots\right).$$

We note that x' is an element of the Hilbert cube. Also, S has the property that a sequence $\{p_n\}$ of points in S converges to a point p in S if and only if the sequence of the <u>i</u>th coordinate of the $\{p_n\}$ converges the <u>i</u>th coordinate of p for each i. Since $I_{\aleph_0}$ has the same property, we see that the map from S to $I_{\aleph_0}$ defined by $x \to x'$ is a homeomorphism onto a subset of $I_{\aleph_0}$. But this homeomorphism sends $S \cap R_{\aleph_0}$ onto a subset of $R'_{\aleph_0}$. Thus, since $\dim R'_{\aleph_0} \leq 0$, cf.(Example 1) it follows that $\dim (S \cap R_{\aleph_0}) \leq 0$.

VII. GENERALIZATIONS OF THE DECOMPOSITION AND COVERING THEOREMS

7.1 Theorem A space X has dimension $\leq n$ if and only if there exists a σ-locally finite open basis $\mathfrak{B}$ such that

$$\text{order } B(\mathfrak{B}) \leq n.$$

Proof: The "if" part of the theorem follows from 6.6. To prove the "only if" part, we first let

$$\mathfrak{A} = \bigcup_{i=1}^{\infty} \mathfrak{A}_i$$

be the canonical basis of X. Then, since X is normal, there exists a precise σ-locally finite open refinement

$$\mathcal{W} = \{W_\gamma \mid \gamma \in \bigcup_{i=1}^{\infty} \quad {}_i\}$$

such that

$$\overline{W}_\alpha \subset U_\alpha$$

for each $\alpha \in \bigcup_{i=1}^{\infty} \mathcal{A}_i$. Now applying 6.5, we construct open sets V_α for α in $\bigcup_{i=1}^{\infty} \mathcal{A}_i$ satisfying

$$\overline{W}_\alpha \subset V_\alpha \subset U_\alpha$$

and

$$\text{order } \{B(V_\alpha) \mid \alpha \in \bigcup_{i=1}^{\infty} \mathcal{A}_i\} \leq n.$$

That $\mathcal{B}$ is a σ-locally finite basis of X follows as in the proof of 6.7. Thus $\mathcal{B}$ is the desired collection, and the theorem is proved.

7.2 Theorem Let $\{G_\alpha \mid \alpha \in \mathcal{A}\}$ be a locally finite system of open sets in a space X and $\{F_\alpha \mid \alpha \in \mathcal{A}\}$ be a system of closed sets such that $F_\alpha \subset G_\alpha$, for $\alpha \in \mathcal{A}$. If A is a closed subset of X and $\dim A \leq n$, then there exist two systems $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ and $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ of open sets of X such that

(1). $F_\alpha \subset V_\alpha \subset \overline{V}_\alpha \subset U_\alpha \subset G_\alpha$, for $\alpha \in \mathcal{A}$,

(2). order $\{(\overline{U}_\alpha - V_\alpha) \cap A \mid \alpha \in \mathcal{A}\} \leq n$.

Proof: By 6.4, we obtain open collections $\{U'_\alpha \mid \alpha \in \mathcal{A}\}$ and $\{W_\alpha \mid \alpha \in \mathcal{A}\}$ such that

$$F_\alpha \subset W_\alpha \subset \overline{W}_\alpha \subset U'_\alpha \subset G_\alpha$$

and

$$\text{order } \{(U'_\alpha - \overline{W}_\alpha) \cap A \mid \alpha \in \mathcal{A}\} \leq n.$$

Then, by the normality of X, we obtain the open collections $\{V_\alpha \mid \alpha \in \mathcal{A}\}$ and $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ satisfying

$$F_\alpha \subset W_\alpha \subset \overline{W}_\alpha \subset V_\alpha \subset \overline{V}_\alpha \subset U_\alpha \subset \overline{U}_\alpha \subset U'_\alpha \subset G_\alpha$$

Then, for each α in $\mathcal{A}$, we have

$$\overline{U}_\alpha - V_\alpha \subset U'_\alpha - \overline{W}_\alpha,$$

which gives

$$\text{order } \{(\overline{U}_\alpha - V_\alpha) \cap A \mid \alpha \in \mathcal{A}\} \leq n.$$

7.3 Proposition Let X be a space such that $\dim X \leq n$. If $\mathfrak{U} = \bigcup_{i=1}^{\infty} \mathfrak{U}_i$ is a countable collection of open coverings such that $\mathfrak{U}$ is a basis for the open sets of X, with order $\{B(U) \mid U \in \mathfrak{U}\} \leq n$, then $\dim \left[\bigcap_{i=1}^{r} B(U_j)\right] \leq n-r$ for each finite system of distinct sets $U_1, \ldots, U_r$ of $\mathfrak{U}$.

Proof: Let $A = \bigcap_{j=1}^{r} B(U_j)$. Then $\mathcal{W}_i = \{U' \cap A \mid U' \in \mathfrak{U}\}$ is a locally finite open covering of A where we assume each member of the collection is not empty. Then $\mathcal{W} = \{W \mid W \in \mathcal{W}_i,\ i = 1,2,\ldots\}$ is an open basis of A. Also, since $B_A(U' \cap A) \subset B(U') \cap A \subset \bigcap_{j=1}^{r} B(U_j) \cap B(U')$, we have order $\{B_A(W) \mid W \in \mathcal{W}\} \leq n-r$, which proves $\dim (A) \leq n-r$, by 7.1, as desired.

7.4 Theorem Let $\{A_i \mid i = 1,2,\ldots\}$ be a countable collection of closed sets in a space X. Let $\{G_{i\alpha} \mid \alpha \in \mathcal{A}_i\}$, $i = 1,2,\ldots$ be a countable number of locally finite open collections and $\{F_{i\alpha} \mid \alpha \in \mathcal{A}_i\}$, $i = 1,2,\ldots$ be a countable number of systems of closed collections such that $F_{i\alpha} \subset G_{i\alpha}$. Then there exists a system of open sets $H_{i\alpha}$, $\alpha \in \mathcal{A}_i$, $i = 1,2,\ldots$ such that

(1). $F_{i\alpha} \subset H_{i\alpha} \subset G_{i\alpha}$, $\alpha \in \mathcal{A}_i$

(2). $\dim \left[\bigcap_{\nu=1}^{r} \mathcal{B}(H_{i_\nu \alpha_\nu}) \cap A_j\right] \leq \dim A_j - r$ for any system of r distinct pairs (i_ν, α_ν), $\nu = 1,2,\ldots,r$.

Proof: Let $\mathfrak{B}_i$, $i = 1,2,\ldots$ be a countable collection of locally finite families of open sets such that mesh $\mathcal{D}_i \leq \frac{1}{i}$ and $\{D \mid D \in \mathcal{D}_i, i = 1,2,\ldots\}$ is a basis for the open sets of X. Now, for each i, let $\{\mathcal{W}_{i\beta} \mid \beta \in \mathcal{B}_i\}$ be a barycentric refinement of $\mathcal{D}_i$. Thus, $\{S(p, \mathcal{W}_i) \mid i = 1,2,\ldots\}$ is a basis for the neighborhoods at the point p of X. Applying the normality of X, we now obtain, for each i, a closed collection $\{C_{i\alpha} \mid \alpha \in \mathcal{A}_i\}$ satisfying $C_{i\alpha} \subset W_{i\alpha}$.

Since $\{G_{1\alpha} \mid \alpha \in \mathcal{A}_1\} \cup \{W_{1\beta} \mid \beta \in \mathcal{B}_1\}$ and $\{F_{1\alpha} \mid \alpha \in \mathcal{A}_1\} \cup \{C_{1\beta} \mid \beta \in \mathcal{B}_1\}$ are locally finite families with

$$F_{1\alpha} \subset G_{1\alpha}, \text{ for } \alpha \in \mathcal{A}_1$$

and

$$C_{1\beta} \subset W_{1\beta}, \text{ for } \beta \in \mathcal{B}_1,$$

we can apply 7.2 and obtain open collections $\{H^1_{1\alpha} \mid \alpha \in \mathcal{A}_1\}$, $\{L^1_{1\alpha} \mid \alpha \in \mathcal{A}_1\}$, $\{U^1_{1\beta} \mid \beta \in \mathcal{B}_1\}$, and $\{V^1_{1\beta} \mid \beta \in \mathcal{B}_1\}$ such that

$$F_{1\alpha} \subset H^1_{1\alpha} \subset \overline{H^1_{1\alpha}} \subset L^1_{1\alpha} \subset G_{1\alpha}, \text{ for each } \alpha \in \mathcal{A}_1,$$

$$C_{1\beta} \subset U^1_{1\beta} \subset \overline{U^1_{1\beta}} \subset V^1_{1\beta} \subset W_{1\beta}, \text{ for each } \beta \in \mathcal{B}_1,$$

and

$$\text{order}\left[\{(\overline{L^1_{1\alpha}} - H^1_{1\alpha}) \cap A_1 \mid \alpha \in \mathcal{A}_1\} \cup \{(\overline{V^1_{1\beta}} - U^1_{1\beta}) \cap A_1 \mid \beta \in \mathcal{B}_1\}\right] \leq \dim A_1.$$

Inductively, we obtain

$$F_{i\alpha} \subset H^i_{i\alpha} \subset \overline{H^i_{i\alpha}} \subset L^i_{i\alpha} \subset G_{i\alpha}, \text{ for every } \alpha \in \mathcal{A}_i,$$

$$C_{i\beta} \subset U^i_{i\beta} \subset \overline{U^i_{i\beta}} \subset V^i_{i\beta} \subset W_{i\beta}, \text{ for every } \beta \in \mathcal{B}_i,$$

$$\overline{H^{i-1}_{k\alpha}} \subset H^{i}_{k\alpha} \subset \overline{H^{i}_{k\alpha}} \subset L^{i}_{k\alpha} \subset L^{i-1}_{k\alpha}, \text{ for every } \alpha \in \mathcal{A}_k,\ k = 1,2,\ldots,i,$$

$$U^{i-1}_{k\beta} \subset U^{i}_{k\beta} \subset \overline{U^{i}_{k\beta}} \subset V^{i}_{k\beta} \subset \overline{V^{i-1}_{k\beta}}, \text{ for every } \beta \in \mathcal{B}_k,\ k = 1,2,\ldots,i,$$

and

(1). order $\{(\overline{L^{i}_{k\alpha}} - H^{i}_{k\alpha}) \cap A_j,\ (\overline{V^{i}_{k\beta}} - U^{i}_{k\beta}) \cap A_j \mid \alpha \in \mathcal{A}_k,\ \beta \in \mathcal{B}_k,\ k = 1,$ $2,\ldots,i\} \le \dim A_j,\ j = 1,2,\ldots,i.$

Let

$$H_{k\alpha} = \bigcup_{i=k}^{\infty} H^{i}_{k\alpha},\ \alpha \in \mathcal{A}_k$$

and

$$U_{k\beta} = \bigcup_{i=k}^{\infty} U^{i}_{k\beta},\ \beta \in \mathcal{B}_k.$$

We have

$$B(H_{k\alpha}) = \overline{\bigcup_{i=k}^{\infty} H^{i}_{k\alpha}} - \bigcup_{i=k}^{\infty} H^{i}_{k\alpha} \subset \overline{L^{k}_{k\alpha}} - H^{k}_{k\alpha}$$

and, similarly,

$$B(U_{k\beta}) \subset V^{k}_{k\beta} - U^{k}_{k\beta}.$$

Therefore, by (1), it follows that for each $j = 1,2,\ldots$ the system

$$\{B(H_{i\alpha}) \cap A_j,\ B(U_{i\beta}) \cap A_j \mid \alpha \in \mathcal{A}_i,\ \beta \in \mathcal{B}_i,\ i = 1,2,\ldots\}$$

has order $\le \dim A_j$. If we now note that $\{U_{i\beta} \mid \beta \in \mathcal{B}_i,\ i = 1,2,\ldots\}$ is a basis, we see that if we restrict this basis to A_j, we may apply 7.3 and obtain

$$\dim\left[\bigcap_{\nu=1}^{r} B(H_{i_\nu \alpha_\nu}) \cap A_j\right] \leq \dim A_j - r$$

for any r distinct pairs (i_ν, α_ν), $\nu = 1,2,\ldots,r$. Thus, the theorem is now proved.

The proof of the following theorem is included in the above proof.

7.5 <u>Theorem</u> Let A_i, $i = 1,2,\ldots$ be a countable number of finite dimensional closed sets of a space X. Then there exists a countable collection of locally finite open coverings $\mathfrak{U}_i$ of X such that $\mathfrak{U} = \{U \mid U \in \mathfrak{U}_i, i = 1, 2,\ldots\}$ is a basis for the open sets of X and

$$\dim\left[\bigcap_{j=1}^{r} B(U_j) \cap A_i\right] \leq \dim A_i - r$$

for distinct sets $U_1,\ldots,U_r$ of $\mathfrak{U}$.

We now have sufficient material with which we can prove a generalization of the Decomposition Theorem.

7.6 <u>Theorem</u> Let A_i, $i = 1,2,\ldots$ be a countable number of finite dimensional closed sets of a space X. Then there exists a countable number of sets P_i, $i = 0,1,2,\ldots$ of dimension ≤ 0 and a set P_∞ such that

(1). $$X = \bigcup_{i=0}^{\infty} P_i \cup P_\infty.$$

(2). $$A_j = \bigcup_{i=0}^{n_j} (A_j \cap P_i), \text{ where } n_j = \dim A_j.$$

Proof: Let $\mathfrak{U} = \bigcup_{i=1}^{\infty} \mathfrak{U}_i$ be a σ-locally finite basis for the open sets of X, with each $\mathfrak{U}_i$ a locally finite open covering of X. Also, by 7.5, we may choose $\mathfrak{U}$ so that

$$\dim \left[\bigcap_{j=1}^{r} B(U_j) \cap A_i\right] \leq \dim A_i - r$$

for any r distinct sets $U_1, \ldots, U_n$ of $\mathfrak{U}$.

Put

$$K_r = \bigcup\left(\bigcap_{i=1}^{r} B(U_i)\right), \quad r = 1, 2, \ldots,$$

$$K_0 = X,$$

$$P_r = K_r - K_{r+1},$$

and

$$P_\infty = \bigcap_{r=1}^{\infty} K_r$$

where the union is taken over all systems of r distinct sets $U_1, \ldots, U_r$ of $\mathfrak{U}$. Since $\mathfrak{U}$ restricted to $\bigcap_{j=1}^{r} B(U_j) - K_{r+1}$ is a basis for its open sets, and since for any $U \in \mathfrak{U}$, $B(U) \cap \left[\bigcap_{j=1}^{r} B(U_j) - K_{r+1}\right] = B(U) \cap \left[\bigcap_{j=1}^{r} B(U_j)\right] - K_{r+1} = \emptyset$, we have

$$\dim \left[\bigcap_{i=1}^{r} B(U_i) - K_{r+1}\right] \leq 0.$$

But, since $\mathfrak{U}$ is a σ-locally finite open basis of X, we know that $\{B(U) \mid U \in \mathfrak{U}\}$ is a locally countable closed covering of P_r which implies $\mathfrak{J} = \{\bigcap_{i=1}^{r} B(U_i) - K_{r+1} \mid U_1,\ldots,U_r \in \mathfrak{U}\}$ is also. Since each member of $\mathfrak{J}$ is 0-dimensional, we have, by 3.8, that

$$\dim P_r \leq 0$$

for $r = 0,1,\ldots$.

Since (1) is obvious, we need only to show (2) to prove the theorem. To see (2), we note that if $\dim A_i = n$, it then follows from $\dim \left[\bigcap_{j=1}^{r} (B(U_j) \cap A_i)\right] \leq \dim A_i$ that

$$P_n = K_n - K_{n+1} = \bigcup\left[\bigcap_{j=1}^{n} B(U_j) \cap A_i\right] - \bigcup\left[\bigcap_{j=1}^{n+1} B(U_j) \cap A_i\right]$$

$$= \bigcup\left[\bigcap_{j=1}^{n} B\, U_j \;\cap A_i\right] - \emptyset.$$

We obtain the following theorem as a direct consequence of 7.4.

7.7 <u>Theorem</u> Let A_i, $i = 1,2,\ldots$ be a countable number of finite dimensional closed sets of a space X and $\{G_\alpha \mid \alpha \in \mathcal{A}\}$ be a locally finite open covering of X. Then there exists a closed covering $\{\overline{H_\alpha} \mid \alpha \in \mathcal{A}\}$ which consists of the closures of open sets H_α and satisfies the conditions:

(1). $$\overline{H_\alpha} \subset G_\alpha,\ \alpha \in \mathcal{A}$$

(2). $\dim \left[\bigcap_{i=0}^{r} \overline{H_{\alpha_i}} \;\cap A_j\right] \leq \dim A_j - r$ for any r+1 distinct indices

indices $\alpha_0,\ldots,\alpha_n$ of $\mathcal{A}$.

VIII. CHARACTERIZATIONS OF DIMENSION

We have already seen some characterizations of dimension, cf. [5.6], [6.2], [6.3], [7.1]. However, we will restate 7.1.

7.1 Theorem A space X has dimension $\leq n$ if and only if there exists a σ-locally finite basis $\mathcal{B}$ of X such that order $B(\mathcal{B}) \leq n$.

8.1 Theorem A space X has dimension $\leq n$ if and only if for every open collection $\{U_i \mid i = 1,2,\ldots,k\}$ and closed collection $\{F_i \mid i = 1,2,\ldots,k\}$ with $F_i \subset U_i$, there exists an open collection $\{V_i \mid i = 1,2,\ldots,k\}$ such that

$$F_i \subset V_i \subset U_i$$

and

$$\text{order } \{B(V_i) \mid i = 1,\ldots,k\} \leq n.$$

Proof: Since the "only if" part of the theorem is a special case of 6.5, we shall prove the "if" part only. Let $\{U_i \mid i = 1,2,\ldots,k\}$ be any finite open covering of X, and $\{F_i \mid i = 1,2,\ldots,k\}$ be any closed covering of X such that $F_i \subset U_i$ for $i = 1,2,\ldots,k$. Then, by the normality of X and the condition of the theorem, we know that there exists an open collection $\{V_i \mid i = 1,2,\ldots,k\}$ satisfying

$$F_i \subset V_i \subset \overline{V}_i \subset U_i, \text{ for } i = 1,2,\ldots,k$$

and

(1). $$\text{order } \{B(V_i) \mid i = 1,2,\ldots,k\} \leq n.$$

Let

$$G_i = \overline{V}_i - \bigcup_{j=1}^{i-1} V_j,$$

then

$$\text{order } \{G_i \mid i = 1,2,\ldots,k\} \leq n+1.$$

For, if p were a point such that

$$p \in G_{i_1} \cap G_{i_2} \cap \ldots \cap G_{i_{n+2}},$$

where

$$i_1 < i_2 < \ldots < i_{n+2},$$

then we have

$$p \in (\overline{V}_{i_1} - V_{i_1}) \cap (\overline{V}_{i_2} - V_{i_2}) \cap \ldots \cap (\overline{V}_{i_{n+1}} - V_{i_{n+1}}),$$

which contradicts (1).

Hence $\{G_i \mid i = 1,2,\ldots,k\}$ is a closed covering with order $\leq n+1$, and $G_i \subset U_i$ for $i = 1,2,\ldots,k$.

For each $i = 1,2,\ldots,k$, let $\mathfrak{U}_i = \{U_i, X - G_i\}$. Also, for each point p in X define $C_p = X - \bigcup\{G_i \mid p \notin G_i\}$. Then each C_p meets at most n+1 members of $\{G\ ,\ldots,G_k\}$, since ord $\{G_i \mid i = 1,2,\ldots,k\} \leq n+1$. Now, if we let $\mathfrak{D}$ be a barycentric refinement of $\{C_p \mid p \in X\}$, and then choose $\mathfrak{U}$ such that

$$\text{(2).} \qquad \mathfrak{U} < \wedge\, \mathfrak{U}_i \wedge \mathfrak{D},$$

we see that for each point p of X

$$S(p,\mathfrak{U}) \subset St\ (p,\mathfrak{D}) \subset C_p,,$$

for some $p' \in X$ and thus, for each p in X, $S(p,\mathfrak{A})$ meets at most n+1 members of $\{G_i \mid i = 1,2,\ldots,k\}$.

By putting,

$$(3). \qquad W_i = S(G_i,\mathfrak{A}),$$

it follows from $\mathfrak{A} < \wedge \mathfrak{A}_i$ that

$$W_i \subset U_i, \text{ for } i = 1,2,\ldots,k.$$

Now, if we let $\mathcal{W} = \{W_i \mid i = 1,2,\ldots,k\}$ we have order $\mathcal{W} \leq n+1$, for otherwise there exist $i_1 < i_2 < \ldots < i_{n+2}$ such that $W_{i_1} \cap W_{i_2} \cap \ldots \cap W_{i_{n+2}} \neq \emptyset$, which in turn implies by (3) that there exist $A_{i_j} \in \mathfrak{A}$, $j = 1,2,\ldots,n+2$, such that $A_{i_j} \cap G_{i_j} \neq \emptyset$ and $A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_{n+2}} \neq \emptyset$. If $p \in \bigcap_{j=1}^{n+2} A_{i_j}$, then $S(p,\mathfrak{A})$ meets n+2 members of $\{G_1,\ldots,G_k\}$, which is a contradiction. Thus, $\mathcal{W}$ is a refinement of $\{U_i \mid i = 1,2,\ldots,k\}$ of order $\leq n+1$, so dim $X \leq n$, and the theorem is completely proved.

8.2 Corollary A space X has dimension $\leq n$ if and only if for every open collection $\{U_i \mid i = 1,\ldots,n+1\}$ and every closed collection $\{F_i \mid i = 1,\ldots, n+1\}$, there exists an open collection $\{V_i \mid i = 1,\ldots,n+1\}$ such that

$$F_i \subset V_i \subset U_i, \text{ for } i = 1,2,\ldots,n+1$$

and

$$\bigcap_{i=1}^{n+1} B(V_i) = \emptyset.$$

Proof: Since the "only if" part is a direct consequence of the theorem, we shall prove only the "if" part. Let $\{U_i \mid i = 1,2,\ldots,k\}$ be a given finite open covering of X and $\{F_i \mid i = 1,2,\ldots,k\}$ be a given finite closed covering of X such that $F_i \subset U_i$ for each $i = 1,2,\ldots,k$. Then we number all the combinations

$$C = \{i_1,\ldots,i_{n+1}\}$$

of n+1 numbers from $\{1,2,\ldots,k\}$ as

$$C_1,\ C_2,\ \ldots\ ,\ C_m,$$

where $m = \binom{k}{n+1}$. By the condition and the normality of X, we can define open sets V_i^1 and W_i^1 for i in C_1 such that

$$F_i \subset V_i^1 \subset \overline{V}_i^1 \subset W_i^1 \subset U_i$$

and

$$\bigcap \{B(V_i^1) \mid i \in C_1'\} = \emptyset.$$

Also, we may define the V_i^1 and W_i^1 such that

$$\bigcap \{W_i^1 - \overline{V}_i^1 \mid i \in C_1\} = \emptyset.$$

Assume that open sets V_i^j and W_i^j, for $i \in C_j$, have been defined for $j < \ell$. Then, as above, we can define open sets V_i^ℓ and W_i^ℓ for i in C_ℓ so that if

$$i \notin \bigcup_{k=1}^{\ell-1} C_k,$$

then

(1). $$F_i \subset V_i^\ell \subset \overline{V}_i^\ell \subset W_i^\ell \subset U_i,$$

if

$$i \in \bigcup_{k=1}^{\ell-1} C_k,$$

then

(2). $$F_i \subset V_i^j \subset \overline{V}_i^j \subset V_i^\ell \subset \overline{V}_i^\ell \subset W_i^\ell \subset W_i^j \subset U_i$$

for every j for which $1 \leq j \leq \ell - 1$, $i \in C_j$, and so that

(3). $$\bigcap \{W_i^\ell - \overline{V}_i^\ell \mid i \in C_\ell \} = \emptyset.$$

Let i_0 be fixed and let ℓ_1 be the largest index such that $i_0 \in C_{\ell_1}$. Then if we put

$$V_{i_0} = \bigcup \{V_{i_0}^\ell \mid i_0 \in C_\ell \ , \ \ell = 1,2,\ldots,m\},$$

then

$$V_{i_0} = V_1^{\ell_1},$$

and

$$F_{i_0} \subset V_{i_0} \subset U_{i_0}.$$

Now, for each $\ell = 1,\ldots,m$, let

$$C_{\ell'} = C_\ell$$

Then, by the process just completed, we can construct $V_i^{\ell'}$ and $W_i^{\ell'}$ such that both collections have the properties, (1), (2), (3) and also satisfy

$$F_i \subset V_i^{\ell_i} \subset \overline{V}_i^{\ell_i} \subset V_i^{\ell'} \subset \overline{V}_i^{\ell'} \subset W_i^\ell \subset U_i.$$

Then, for each $i = 1,2,\ldots,k$, put

$$V_i = \bigcup \{V_i^{\ell'} \mid i \in C_\ell \ , \ \ell = 1,2,\ldots\}.$$

Then

$$F_i \subset V_i \subset U_i$$

and

$$B(V_i) \subset W_i^{\ell} - \overline{V}_i^{\ell}$$

for every ℓ satisfying $i \in C_{\ell}$.

Thus, for every combination C_{ℓ} of n+1 numbers from $\{1,2,\ldots,k\}$,

$$\bigcap \{B(V_i) \mid i \in C_{\ell}\} \subset \bigcap \{W_i^{\ell} - \overline{V}_i^{\ell} \mid i \in C_{\ell}\} = \emptyset.$$

Therefore, by the theorem, we conclude $\dim X \leq n$, and the corollary is proved.

8.3 <u>Theorem</u> Let R be a topological space. Then with R as a metric space, dim R = 0 if and only if R is a dense subset of an inverse limit space of a sequence of discrete spaces.

<u>Proof:</u> Assume dim R = 0. Then there exists a sequence, $\{\mathfrak{U}_i\}$, of open coverings of R such that for each positive integer, i, order $\mathfrak{U}_i \leq 1$ and mesh $\mathfrak{U}_i < 2^{-i}$. Since $R \neq \emptyset$, we may assume that each element of $\mathfrak{U} = \bigcup_{i=1}^{\infty} \mathfrak{U}_i$ is nonempty. For each i, put $\mathfrak{R}_i = \wedge_{j \leq i} \mathfrak{U}_j$; then $\mathfrak{R}_i = \{V_\alpha \mid \alpha \in \mathcal{A}_i\}$ is an open covering of R satisfying

(1). order $\mathfrak{R}_i \leq 1$.

(2). mesh $\mathfrak{R}_i < 2^{-i}$.

(3). $\mathfrak{R}_{i+1}$ refines $\mathfrak{R}_i$.

(4). $\bigcup_{i=1}^{\infty} \mathfrak{R}_i$ is a basis for the open sets of R.

Define a map $f_{i+1,i}$ from $\mathcal{A}_{i+1}$ to $\mathcal{A}_i$ by letting $f_{i+1,i}(\alpha) = \beta$ if and only if $V_\alpha \subset V_\beta$. This is well-defined because of (1) and (3).

If we consider the $\mathcal{A}_i$ as topological spaces, each having the discrete topology, it is clear that $\{\mathcal{A}_i, f_{i+1,i}\}$ is an inverse spectrum.

Define

$$A_0 = \{a = \{\alpha_1, \alpha_2, \dots\} \in A \mid \bigcap_{i=1}^{\infty} V_{\alpha_i} \neq \emptyset\}.$$

We note that since $\operatorname{mesh}_{i \to \infty} \mathcal{B}_i = 0$, it follows that whenever $\bigcap_{i=1}^{\infty} V_{\alpha_i} \neq \emptyset$, it must be a single point.

We will now show that A_0 is homeomorphic to R. Define φ to be a mapping from A_0 into R such that for each a in A_0

$$\varphi(a) = \bigcap_{i=1}^{\infty} V_{\Pi_i(a)}$$

where $\Pi_i = p_i \mid A_0$, where p_i is the canonical projection of A_0 into $\mathcal{A}_i$. That φ is well-defined follows from the above remarks.

Let x be an element of R. By (1) we have that for each i, there exists exactl one element V_{α_i} of $\mathcal{B}_i$ satisfying $x \in V_{\alpha_i}$. Since

$$V_{\alpha_{i+1}} \subset V_{\alpha_i}, \text{ for every } i = 1,2,\dots,$$

we have that

$$a = \{\alpha_1, \alpha_2, \dots\} \in A_0$$

and

$$\varphi(a) = x.$$

Hence φ is a surjection.

Let a and b be distinct points of A_0. Then there exists an i such that $\Pi_i(a) \neq \Pi_i(b)$. Since $V_{\Pi_i(a)} \cap V_{\Pi_i(b)} = \emptyset$, by (1), we have $\varphi(a) \neq \varphi(b)$. Therefore φ is an injection.

To see that φ is an open continuous map, we note that

$$(5). \qquad \varphi\left[\Pi_i^{-1}(\Pi_i(a))\right] = V_{\Pi_i(a)}$$

for every a in A_0 and for every $i = 1,2,\ldots$. Since φ is a bijective, it follows from (5) that

$$(6). \qquad \Pi_i^{-1}(\Pi_i(a)) = \varphi^{-1}\left[V_{\Pi_i(a)}\right],$$

for every $a \in A_0$ and for every $i = 1,2,\ldots$. But

$$\bigcup_{i=1}^{\infty} \mathcal{B}_i = \{V_{\Pi_i(a)} \mid a \in A_0,\ i = 1,2,\ldots\}$$

is a basis for the open sets of R and $\{\Pi_i^{-1}\Pi(a) \mid a \in A_0 \text{ and } i = 1,2,\ldots\}$ is a basis for the open sets of A_0; so that φ is open by (5) and that φ is continuous by (6). Thus φ is a homeomorphism.

To complete the "only if" part, we only need to show that A_0 is a dense subset of A. Since the set

$$\{p_i^{-1}(\alpha) \mid \alpha \in \mathcal{A}_i \text{ and } i = 1,2,\ldots\}$$

forms a basis for the open sets of A, it suffices to show that each set of the form $p_i^{-1}(\alpha_i)$, $\alpha_i \in \mathcal{A}_i$, contains a point of A_0. Since $V_{\alpha_i} \neq \emptyset$, there exists a sequence $\{V_\beta\}$ such that $\beta_i = \alpha_i$ and $\bigcap_j V_{\beta_j} \neq \emptyset$; hence $p_i^{-1}(\alpha_i)$ contains the point $\varphi^{-1}[\bigcap_j V_{\beta_j}]$ of A_0. Thus A_0 is dense in A.

Conversely, assume that R is homeomorphic to a dense subset of an inverse limit space of a sequence of discrete spaces, A_i. Since discrete spaces are metrizable and 0-dimensional, ΠA_i is metrizable and 0-dimensional, where the latter follows from 5.10. Consequently, any subspace is metrizable and 0-dimensional. Since R is homeomorphic to a subspace of ΠA_i, R is metrizable and 0-dimensional.

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ON DIMENSION THEORY

by

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AN ABSTRACT OF A REPORT

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A dimension theory for general metric spaces had been established. This paper will offer a different approach to the development of this dimension theory by applying the notion of normal families. A family $\mathfrak{M}$ of metric spaces is said to be a normal family if it satisfies

(i) if $S \subset R$ and $R \in \mathfrak{M}$, then $S \in \mathfrak{M}$

(ii) if $\{F_\gamma \mid \gamma \in \Gamma\}$ is a locally countable closed covering of a space R such that $F_\gamma \in \mathfrak{M}$ for each $\gamma \in \Gamma$, then $R \in \mathfrak{M}$.

We begin by proving that for each integer n, the class of n-dimensional spaces is a normal family. In section V, we apply this result to obtain many theorems of the dimension theory, including the Sum Theorem, Product Theorem and the Decomposition Theorem.

The approach of normal families leads to the proof that covering dimension and strong inductive dimension agree on the class of all metric spaces. In section VI we prove this result and also show that the above two notions of dimension also coincide with the notion of weak inductive dimension on the class of all separable metric spaces.

We next give extensions of the Decomposition Theorem and the Sum Theorem. Finally, some characterizations of dimension are given, including an embedding theorem.