APPLICATION OF THE MAXIMUM PRINCIPLE TO THE

OPTIMAL CONTROL OF LIFE SUPPORT SYSTEMS

by

GOVINDASWAMY NADIMUTHU

B.E. (Mechanical)
Madras University, India, 1968

D.I.I.T. (Industrial Engineering)
Indian Institute of Technology, Madras, India, 1969

A MASTER'S REPORT

submitted in partial fulfillment of the

requirement for the degree

MASTER OF SCIENCE

Department of Industrial Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1971

Approved by:

Ching-Lai Hwang
Major Professor

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

This report deals with the application of the maximum principle to the optimal temperature control of life support systems. A life support system is a system for creating, maintaining, and controlling an environment so as to permit personnel to function efficiently. The control of temperature is probably the most important role of the life support system.

The need for providing an automatic control system to an air-conditioning system has long been recognized [24, 42]. It is also a well-known fact that use of automatic control is necessary for the life support system of a space cabin or submarine or underground shelter [43, 44]. It appears that analysis and synthesis of the control systems for the air-conditioning and life support systems have so far been carried out by the classical approach [1,6,7,13,22,24,26,42,43,44,45] which is essentially a trial-and-error procedure or a disturbance-response (or input-output) approach in which extensive use is made of the transform methods such as the Laplace transform (s-domain), Fourier transform (ω-domain), and z-transform (discrete time-domain). Inspite of the extensive use of mathematics, the classical approach is essentially an empirical one [41].

An approach known as the modern (optimal) control theory, distinctly different from the classical one, to the analysis and synthesis of a control system has recently been developed [3,14,32,34,35,36,41]. It is based on the state-space characterization of a system. The state-space is the abstract space whose coordinates are the state properties of the system or the variables which define the characteristics of the system [41]. This approach involves mainly maximization or minimization of an objective function (functional) which is a function of state and control variables which are in turn functions of time and/or distance coordinate. The objective function is specified, constraints are imposed on the state and decision variables, and an optimal control policy is determined by extremizing the objective function by means of mathematical techniques such as the calculus of variations, maximum principle, and dynamic programming [2,3,41]. This modern approach is entirely theoretical in the sense that no trial-and-error is involved in'adjusting the controller'.

A series of five papers containing the results of an original investigation of the temperature control of confined spaces such as those in any building and life support systems by means of the modern optimal control theory have recently been published in Building Science [17,18,19,20,21]. All these five papers are concerned only with open-loop control. Also, these papers deal only with problems of controlling systems subjected to an impulse heat disturbance with the minimization of the objective function, $S = \int_0^T dt$. The first of this series of articles, contains the derivation of the mathematical models of several different systems and the simulations of their behaviour and characteristics. In the second of the series, the most basic form of Pontryagin's maximum principle, which together with dynamic programming constitutes the bulk of the modern control theory, is outlined and its use is demonstrated by some numerical examples. The optimal control of a system with equality state variable constraints imposed at the end of the control action is considered in the third part. The fourth part deals with some realistic problems of controlling systems with inequality constraints imposed on the state variable, namely the temperature of the cabin. In the final part of the series, some aspects of sensitivity analysis are presented and discussed by fully exploiting the results obtained in the four preceding parts.

Bhandiwad [5] has dealt with open-loop control of systems subjected to both impulse and step heat disturbances. In problems dealing with an impulse heat disturbance he considers minimization of the objective function, $S = \int_0^T (a + b\theta^2)dt$. He limits himself to problems having specified final control time and free right end in the case of systems subjected to a step heat disturbance with the minimization of the objective function,

$$S = \int_0^T [b\theta^2 + c(x_1 - x_{1d})^2]dt.$$

The aim of this study is two-fold (i) to deal with some of the unexplored problems of controlling systems subjected to a step heat disturbance in the case of open-loop control and thus acting as a complement to the two earlier works; (ii) to make an original investigation of the closed-loop temperature control of systems subjected to both an impulse and a step heat disturbances separately and to compare the results with those of open-loop control.

The open-loop system considered in this report consists of a confined space (1 CST model) subjected to a heat disturbance and a heat exchanger (of negligible time constant) while the closed-loop system considered here has an additional element - the feedback element - namely the thermostat. In Chapter 3, the performance equations which represent the dynamic characteristics of the system for a step heat disturbance as well as an impulse heat disturbance are considered [17]. In Chapter 4, the basic form of

Pontryagin's maximum principle is stated and it is then applied to obtain the open-loop optimal control policy for a simple system with a linear performance equation and a quadratic objective function. In Chapter 5, the basic form of the linear regulator problem (the first solution of which was due to Kalman) is outlined and it is then used to obtain the closed-loop optimal control law for the same simple example of Chapter 4, thereby showing that the open-loop control policy and the closed-loop control law are one and the same for a system with linear performance equation and quadratic objective function. Chapter 6 deals with the open-loop control of a heating system, subjected to a step heat disturbance, having the initial and final values of the state variable - namely, the temperature - fixed but the final time T to be determined and with the following objective functions to be minimized:

(i) $$S = \int_0^T dt$$

(ii) $$S = \int_0^T (a+b\theta^2)dt$$

(iii) $$S = \int_0^T [b\theta^2 + c(x_1-x_{1d})^2]dt$$

(iv) $$S = \int_0^T [a + b\theta^2 + c(x_1-x_{1d})^2]dt$$

Chapter 7 contains the open-loop as well as the closed-loop control of two examples - one dealing with a cooling system subjected to an impulse heat input and the other dealing with a heating system subjected to a step heat input. In each of these examples the performance equation is of linear form and the objective function is of quadratic form. Also, each example has the right-end free and deals with both the cases of specified and unspecified final time. It is found, in each example, that the open-loop control policy and the closed-loop control law are exactly the same whether optimal T is specified or not, as long as the linear system has a quadratic functional.

CHAPTER 2

AUTOMATIC CONTROL SYSTEMS

2.1 The Control Systems

Nowadays automatic control systems play an important role in the development of both civilization and technology. Domestically, automatic thermostats in furnaces and air conditioners regulate the temperature and humidity of modern houses for comfortable living. Industrially, automatic control systems are employed to improve both the quantity and the quality of manufactured products. In modern weapons systems, the applications of control systems have become overwhelmingly important.

The basic control system may be described by the simple block diagram shown in Fig. 2.1. The output variable c is controlled by the input variable r through the elements of the control system. For instance, the angular position of the steering wheel of an automobile controls the direction of the front wheels. In this case, the position of the steering wheel is the input, and the direction of the front wheels is the output; the control system elements are composed of the steering mechanisms [33].

Now there are two main classifications of the control systems, namely open-loop and closed-loop (feedback).

An open-loop system is a system in which the output has no effect upon the input signal while a closed-loop control system is a system in which the output has an effect upon the input quantity in such a manner as to maintain the desired output value [12].

It was as long ago as 1788 that James Watt decided that a man controlling the opening and closing of steam valves was not the best way of keeping the speed of his steam engines constant. So the Watt governor, which used the 'lift' of rotating balls as a speed monitor, automatically shutting off the steam as the speed tended to increase and vice versa, was the first feedback control system brought to prominence, although undoubtedly not the first to be applied [27].

2.2 Definition of Feedback Control System

American Institute of Electrical Engineers (A.I.E.E.) defines a feedback control system as follows [37]: "A feedback control system is a control system which tends to maintain a prescribed relationship of one system variable to another by comparing functions of these variables and using the difference as means of control".

According to the Institution of Radio Engineers (I.R.E.) [39], "A feedback control system is a control system comprising one or more feedback control loops, which combines functions of the controlled signals with functions of the commands to tend to maintain prescribed relationships between the commands and the controlled signals.

According to Hammond [25], "A feedback system comprises one or more distinguishable elements which react on each other in a predetermined manner and are arranged so that a closed ring or loop of dependencies is formed.

Kuo's definition of feedback control systems runs as [33] "systems comprising one or more feedback loops which compare the controlled signal c with the command signal r; the difference ($e = r - c$) is used to drive c into correspondence with r.

2.3 The Principle of Closed-loop Control [27]

A closed-loop system is actuated by a signal dependent upon the difference between the output and the input, commonly termed 'error actuation'. The output is monitored, the signal from the monitor being in the same form as the input signal so that the monitored output can be subtracted from the input; the resulting 'actuating signal' (loosely called the 'error signal') only exists when the monitored output differs from the input. This signal is used to control the power supplied to the output in such a sense that when the output is not aligned to the input, the resulting actuating signal drives the output so as to reduce this signal to zero (negative feedback) in which state no further power is fed to the output and it is correctly aligned and maintained. Any changes in the input will cause the output to follow.

Figure 2.2 shows a simple schematic diagram of an open-loop and a closed-loop system.

2.4 Closed-loop Versus Open-loop System [27]

The two important features of the closed loop as compared to the open loop system are:

(i) Since the output power is only controlled by, and not supplied by, the actuating signal, high power outputs can be accurately controlled from low power inputs.

(ii) Closed-loop systems have a self-regulating property. If disturbances on the output tend to change the output from its controlled value then, since the input is unaltered, an actuating signal will develop which produces a reaction to the disturbances, tending to maintain the output.

2.5 Elements of a Closed-loop Control System [8]

The elements in the control system may be classified into the following three categories:

(i) The controlled system, which may also be called the plant, the process or the fixed component.

(ii) The sensors or measuring devices.

(iii) The system controller.

2.6 Types of Feedback Control Systems [12, 33]

The feedback action which is the fundamental difference between the open-and closed-loop systems may be continuous or discontinuous. The discontinuous control may be a relay type or a sampling type [see Fig. 2.3].

Continuous control implies that the output is continuously being fed back, in time, and compared with the reference output. The continuous data may either be modulated, in which case the system is referred to as an a-c carrier servo system, or unmodulated, when the system is called a d-c system. For instance, a closed-loop system used for positioning a load is a typical continuous data d-c system.

The relay-type discontinuous control system is one in which the actuating signal must reach a prescribed value before the dynamic unit will react to it; that is, the control action is discontinuous in amplitude rather than in time. For example, in the thermostatic control of a furnace, the furnace is turned either 'on' or 'off', depending upon whether the room temperature is below or above the preset reference.

In the sampling-data discontinuous control system, the input and output quantities are periodically sampled and compared; that is, the control action is discontinuous in time. A radar tracking system is an example of a sampled data system.

Closed-loop control systems can be categorized [27] as electrical, kinetic or process controls as shown in Fig. 2.4.

Some examples of the application of closed-loop control are [27]:

1. Aircraft flight control.
2. Automatic landing of aircraft.
3. Missile control.
4. Radar and gun control.

5. Ship steering and roll stabilization.
6. Machine-tool control.
7. Remote position control.
8. Nuclear power control.
9. Speed control.
10. High-speed mechanical systems.
11. Voltage and current stabilizers.
12. Process controls.

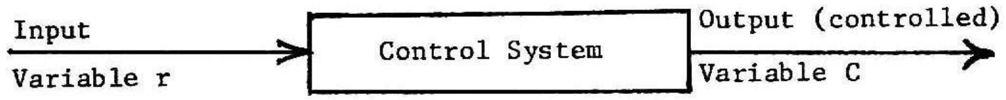

Fig. 2.1 The basic control system [33]

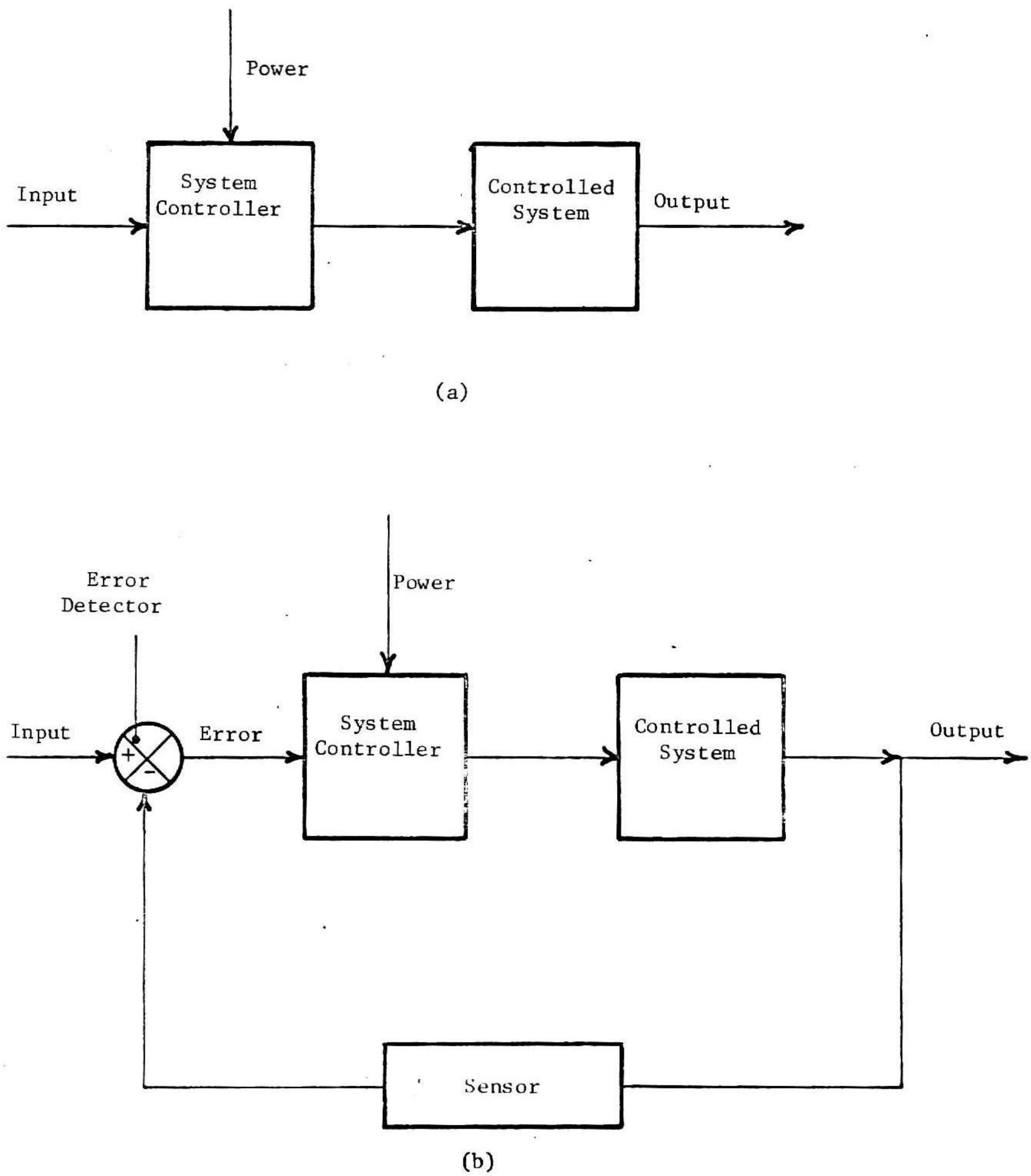

Fig. 2.2 (a) An open-loop, and
(b) a closed-loop system.

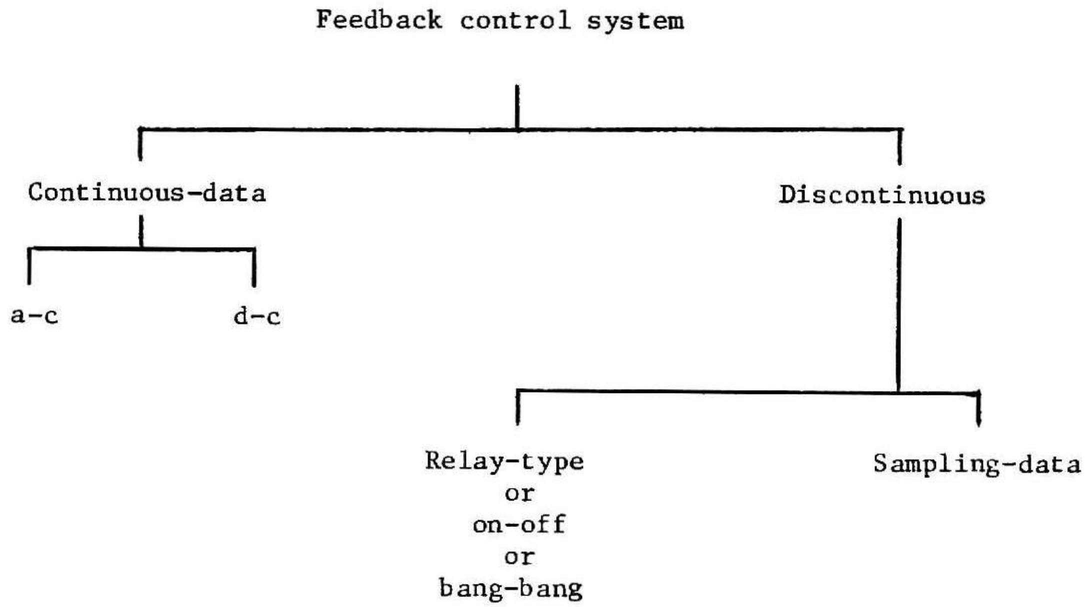

Fig. 2.3 Types of feedback control system

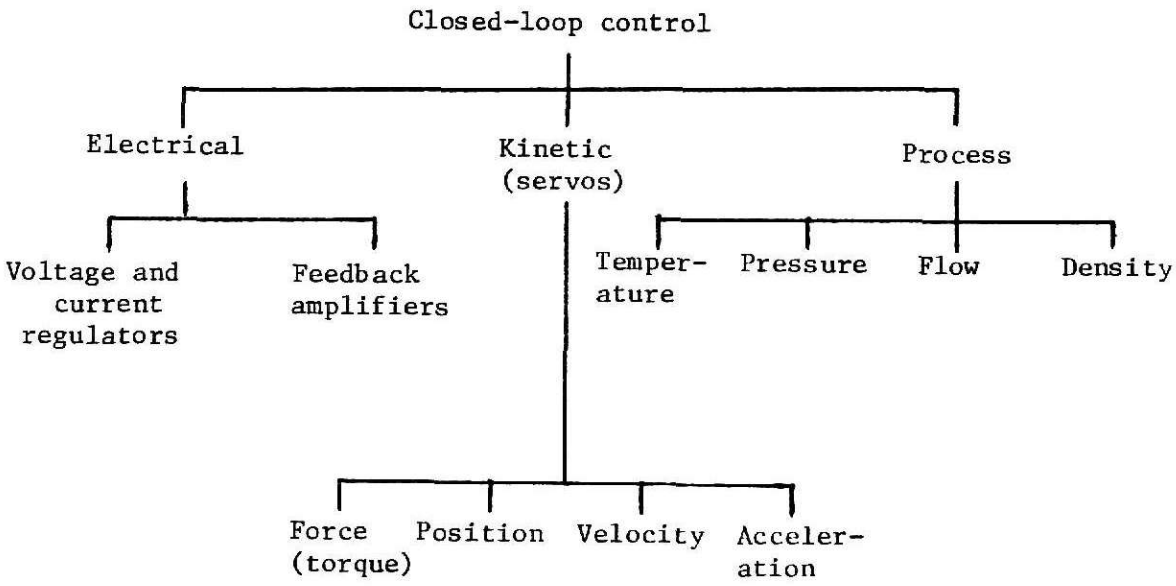

Fig. 2.4 Classification of feedback control systems [27]

CHAPTER 3

ENVIRONMENTAL SYSTEM OF CONFINED SPACES

A control system usually consists of three elements: the feedback element, the control element, and the system proper [11]. The feedback element in a life support system or an environmental control system may be composed of a thermostat, humidistat and pressure regulator, or any combination of these depending on the purpose of control. The control element may include a heat exchanger, humidifier, dehumidifier, blower, portable air-conditioner, or any combination of these, depending on the objective of control. For instance, both the thermostat and heat exchanger are often used to control the air temperature inside a building. The system proper may be a confined space, e.g., an underground shelter, a space vehicle, a space suit, a submarine or a building.

The system considered here is shown schematically in Fig. 3.1. The confined space may be a typical office located in a multi-story building or the cabin of a spaceship. Air or oxygen or a mixture of oxygen and nitrogen is circulated through the room or confined space via an air duct by mechanical means, e.g., a blower or a fan. Control of air temperature in the system is accomplished with a duct system. The thermostat in the system adjusts the position of the control valve of the heat exchanger in order to provide the desired temperature [17].

The performance equations of the system, which represent the dynamic characteristics of the system and system components (see Fig. 3.2 and Fig. 3.3) are derived in [5, 17].

3.1 The System Proper

The following three main assumptions are made concerning the system proper:

(i) Room or cabin air is well mixed, i.e., air temperature within the system proper is uniform throughout at any instant in time.

(ii) The thermal capacitance of room walls, floors, ceiling and windows is neglected, as well as that of any furniture within the system proper.

(iii) Heat loss through the walls and windows is negligible.

Using the continuity law or heat balance, the performance equation of the system proper subjected to a step heat input is derived in dimensionless form as

$$\frac{dx_1}{dt} + x_1 = r_1 x_2 + r_2 K_\alpha + \sigma_s U_0(t) \qquad x_1 = \alpha_0 \text{ at } t = 0 \tag{1}$$

where

t = dimensionless time.

x_1 = dimensionless room temperature.

x_2 = dimensionless temperature of the circulation air.

σ_s = dimensionless disturbance temperature.

$U_0(t)$ = step heat disturbance function.

$\left.\begin{matrix} r_1 \\ r_2 \\ K_\alpha \end{matrix}\right\}$ = system constants.

3.2 The Control Element

The heat exchanger which is the control element of the system under consideration, can perform its control function in various ways, for example, by changing the temperature or flow rate of the heat transfer medium, or changing both.

Again using the continuity law or heat balance, the performance equation of the control element of the system is derived in dimensionless form as

$$\frac{\tau_2}{\tau_1} \frac{dx_2}{dt} + x_2 = x_1 - (K_\beta \theta + K_\gamma) \tag{2}$$

where

θ = control variable constrained between -1 corresponding to maximum heating and +1 corresponding to maximum cooling.

τ_1 = time constant of the system proper.

τ_2 = time constant of the heat exchanger.

$\left.\begin{matrix} K_\beta \\ K_\gamma \end{matrix}\right\}$ = system constants.

3.3 The Feedback Element - Thermostat

Here it is assumed that the sensing element measures the room temperature instantaneously and that there is no accumulation of heat in the element, or for simplicity, it will be assumed that the sensing element is the zero order element with its time constant, τ_3, equal to zero. A detailed discussion of the response of the thermostat can be found in [11].

For simplicity, in this report a life support system consisting of an air-conditioned room or cabin and a heat exchanger of negligibly small time constant ($\tau_2 = 0$) subjected to a step heat disturbance is considered.

The performance equation of such a system is obtained by combining equations (1) and (2) and setting $\tau_2 = 0$ as

$$\frac{dx_1}{dt} + r_2x_1 = r_2K_\alpha - r_1K_\beta\theta - r_1K_\gamma + \sigma_s \tag{3}$$

$$x_1 = \alpha_0 \quad \text{at} \quad t = 0$$

Now the performance equations of the system subjected to an impulse heat disturbance are also derived in [5].

The performance equation of the system proper subjected to an impulse heat disturbance which is taken into account in the initial condition immediately after the onset of the process, is derived in dimensionless form as

$$\begin{aligned} &\frac{dx_1}{dt} + x_1 = \frac{r_1K_1x_2}{K_4} + r_2K_1 \\ &x_1 = 1 \quad \text{at} \quad t = 0^+ \end{aligned} \tag{4}$$

where

$$\left.\begin{matrix} K_1 \\ K_4 \end{matrix}\right\} = \text{system constants.}$$

The performance equation of the control element of the system is also derived in dimensionless form as

$$\tau_2\frac{dx_2}{dt} + \tau_1x_2 = \tau_1(x_1 - K_2K_4\theta - K_3K_4) \tag{5}$$

where

$$\left.\begin{matrix} K_2 \\ K_3 \end{matrix}\right\} = \text{system constants.}$$

Again for simplicity, in this report a life support system consisting of an air-conditioned room or cabin and a heat exchanger of negligibly small time constant ($\tau_2 = 0$) subjected to an impulse heat disturbance is considered.

The performance equation of such a system is obtained by combining equations (4) and (5) and setting $\tau_2 = 0$ as

$$\begin{aligned} &\frac{dx_1}{dt} + r_2 x_1 = r_2 K_1 - r_1 K_1 K_2 \theta - r_1 K_1 K_3 \\ &x_1 = 1 \quad \text{at} \quad t = 0^+ \end{aligned} \tag{6}$$

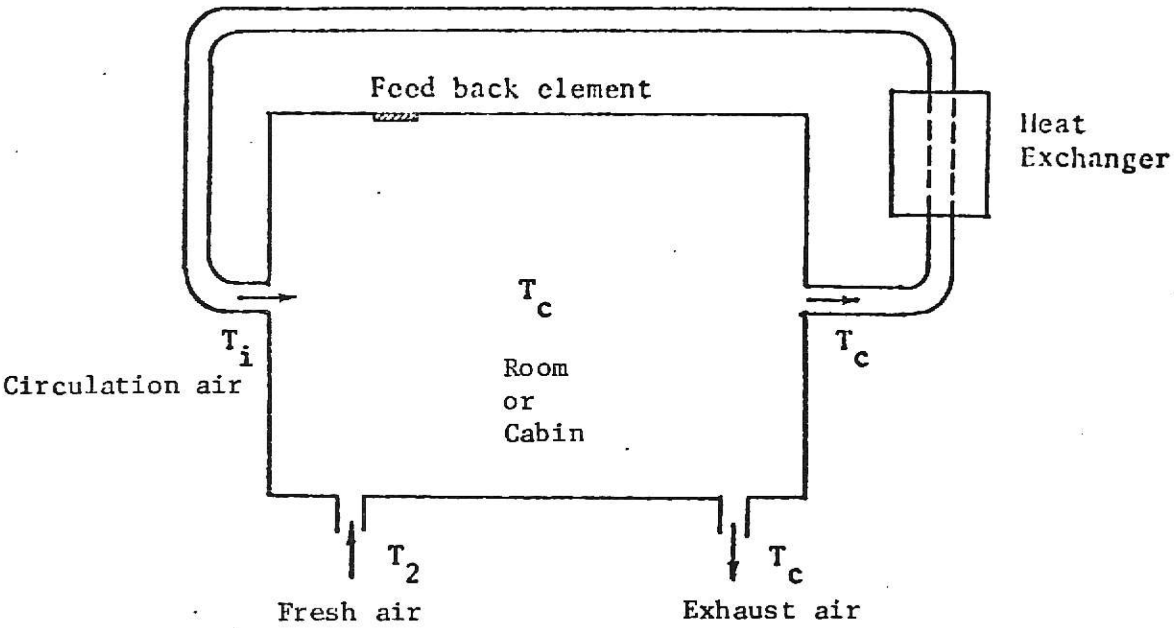

Fig. 3.1 The system, an air-conditioned room

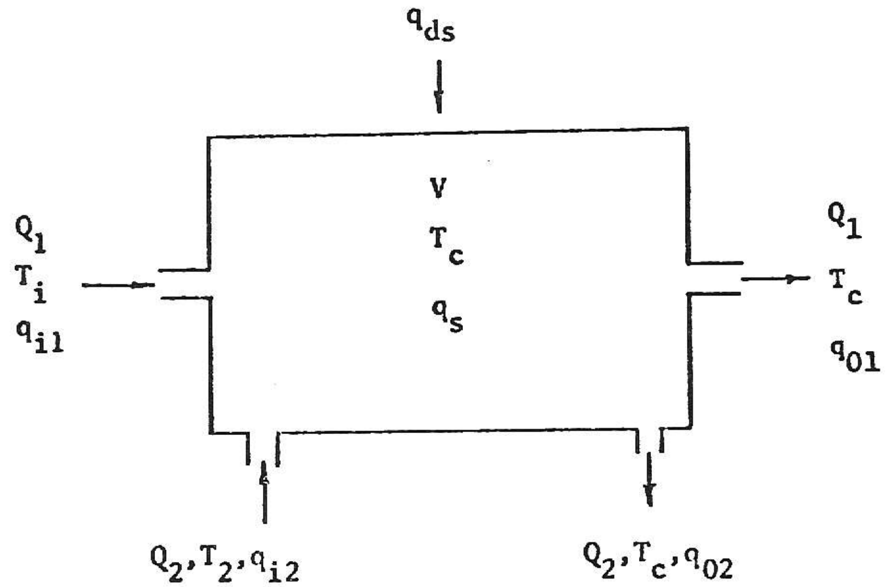

Fig. 3.2 The system proper: heat flow rates

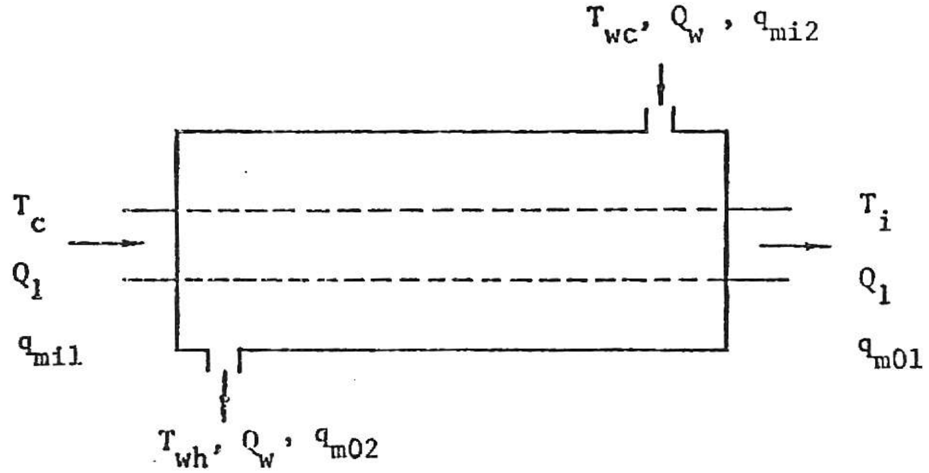

Fig. 3.3 Schematic diagram of the heat exchanger

CHAPTER 4

OPEN-LOOP CONTROL VIA THE MAXIMUM PRINCIPLE

4.1 Statement of the Algorithm

The basic notion of the original version of Pontryagin's maximum principle [15, 36] is introduced. It can be used to treat a wide variety of optimization problems associated with simple continuous processes.

Consider that the dynamic behaviour of a controlled system can be represented by a set of differential equations

$$\frac{dx_i}{dt} = f_i[x_1(t),\ x_2(t), \ldots\ldots, x_s(t);\ \theta_1(t),\ \theta_2(t), \ldots\ldots, \theta_r(t)],$$

$$i = 1,\ 2, \ldots\ldots,\ s; \qquad t_o \leq t \leq T \tag{i}$$

or in vector form

$$\frac{dx}{dt} = f[x(t),\ \theta(t)], \qquad t_o \leq t \leq T \tag{ia}$$

where $x(t)$ is an s-dimensional vector function representing the state of the process at time t and $\theta(t)$ is an r-dimensional vector function representing the decision at time t [15, 36]. The functions f_i, $i = 1,2, \ldots\ldots,s$, are single valued, bounded, differentiable with respect to the x's with bounded first partial derivatives, and are continuous in the θ's on a product region $x\theta$, where x and θ are closed regions in the s-dimensional x-space and r-dimensional θ-space respectively [9]. Note that we are dealing with the autonomous systems in which the right-hand side of the performance equation, equation (i), depends implicitly on time t. The non-autonomous systems are those in which the right-hand side of the performance equation, equation (i), depends explicitly on time t.

A typical optimization problem associated with such a process is to find a piecewise continuous decision vector function, $\theta(t)$, subject to the p-dimensional constraints

$$h_i[\theta(t)] \leq 0, \qquad i = 1, 2, \ldots\ldots, p \qquad \text{(ii)}$$

such that the performance index

$$S = \sum_{i=1}^{s} c_i \, x_i \, (T), \qquad c_i = \text{constant} \qquad \text{(iii)}$$

is minimum (or maximum) when the initial conditions

$$x_i(t_o) = x_{io}, \qquad i = 1, 2, \ldots\ldots, s \qquad \text{(iv)}$$

are given. The duration of control, T, is specified and the final conditions of state variables are unfixed. This type of problem is often called the free right-end problem (with fixed T). The decision vector (or a collection of control variables) so chosen is called an optimal decision vector (or optimal control variables) and is denoted by $\overline{\theta}$ (t).

The procedure for solving the problem is to introduce an s-dimensional adjoint vector z(t) and a Hamiltonian function H which satisfy the following relations

$$H[x(t), \theta(t), z(t)] = \sum_{i=1}^{s} z_i(t) \quad f_i[x(t), \theta(t)] \qquad \text{(v)}$$

$$\frac{dz_i}{dt} = - \frac{\partial H}{\partial x_i} = - \sum_{j=1}^{s} z_j \frac{\partial f_j}{\partial x_i}, \qquad i = 1, 2, \ldots\ldots, s \qquad \text{(vi)}$$

$$z_i(T) = c_i, \qquad i = 1, 2, \ldots\ldots, s \qquad \text{(vii)}$$

The set of equations, equations (i), (iv), (vi) and (vii), constitutes a two-point split boundary value problem, whose solution depends on θ(t). The optimal decision vector $\overline{\theta}$(t) which makes S an extremum also makes the Hamiltonian an extremum for all t, i.e., $t_o \leq t \leq T$ [15, 16, 23, 36, 40].

A necessary condition for S to be an extremum with respect to $\theta(t)$ is

$$\frac{\partial H}{\partial \theta_i} = 0, \qquad i = 1, 2, \ldots\ldots, r \qquad \text{(viii)}$$

if the optimal decision vector is interior to the set of admissible decisions $\theta(t)$ [the set given by equation (ii)]. If $\theta(t)$ is constrained, the optimal decision vector $\overline{\theta}(t)$ is determined either by solving equation (viii) or by searching the boundary of the set. More specifically, the extremum value of Hamiltonian is maximum (or minimum) when the control variables are on the constraint boundary. Furthermore, the extremum value of the Hamiltonian is constant at every point of the time under the optimal condition. It is worth noting that the final conditions of the adjoint variables, $z_i(T)$, are often given as $-c_i$ instead of c_i as shown in equation (vii), in employing the maximum principle of Pontryagin. The use of such final conditions of $z_i(t)$ gives rise to the condition that the Hamiltonian is maximum when the objective function is minimized, and minimum when the objective function is maximized as stated in the original version of the maximum principle of Pontryagin [15, 36, 38].

If both the initial and final conditions of state variables are given, the problem is said to be a boundary value problem. The basic algorithm presented except the condition given by equation (vii) is still applicable.

If optimization (usually minimization) of time t is involved in the objective function in a problem with an unfixed duration of control, T, the problem is then called a time optimal problem. In this case, the basic algorithm presented is still applicable with an additional condition that the extremal value of the Hamiltonian is not only a constant but also identical to zero. The simplest example of the time optimal control problem is one in which the performance index is of the form

$$S = \int_0^T dt$$

Such a problem is often called a minimum time problem.

4.2 An Example

The use of the maximum principle to obtain open-loop control is demonstrated here by considering a simple example in detail. Let the performance equation of a simple process be

$$\frac{dx_1}{dt} = x_1 + \theta \tag{1}$$

with the initial condition

$$x_1(0) = 1$$

The objective function to be minimized is the sum of the integrated control effort to maintain the state of the system in the desired state and the integrated deviation from the desired state over a specified control time and is given by

$$S = \frac{1}{2}\int_0^T (\theta^2 + x_1^2)dt \tag{2}$$

This problem as stated has a fixed time interval with free right-end.

Introducing another state variable $x_2(t)$ such that

$$x_2(t) = \frac{1}{2}\int_0^t (\theta^2 + x_1^2)dt$$

it follows that

$$\frac{dx_2}{dt} = \frac{1}{2}\theta^2 + \frac{1}{2}x_1^2, \qquad x_2(0) = 0 \tag{3}$$

The problem is thus transformed into that of minimizing $x_2(T)$.

According to Pontryagin's maximum principle the Hamiltonian is

$$H = z_1(\theta + x_1) + \frac{1}{2} z_2(\theta^2 + x_1^2) \tag{4}$$

The adjoint variables are defined by

$$\frac{dz_1}{dt} = - \frac{\partial H}{\partial x_1} = - z_1 - z_2 x_1 , \qquad z_1(T) = 0 \tag{5}$$

$$\frac{dz_2}{dt} = - \frac{\partial H}{\partial x_2} = 0 , \qquad z_2(T) = 1 \tag{6}$$

Solving equation (6) for z_2 gives

$$z_2(t) = 1, \qquad 0 \leq t \leq T \tag{7}$$

Hence the Hamiltonian can be rewritten as

$$H = z_1(\theta + x_1) + \frac{1}{2} (\theta^2 + x_1^2) \tag{8}$$

According to the maximum principle, H must be a minimum in θ with the values of x and z considered as fixed. Putting

$$\frac{\partial H}{\partial \theta} = 0$$

we have

$$\frac{\partial H}{\partial \theta} = z_1 + \theta = 0 \tag{9}$$

or

$$\theta(t) = - z_1(t) \tag{10}$$

Substitution of equations (7) and (10) into equations (1) and (5) respectively, gives

$$\frac{dx_1}{dt} = x_1 - z_1 \tag{11}$$

and

$$\frac{dz_1}{dt} = -x_1 - z_1 \tag{12}$$

The system of differential equations, equations (11) and (12), is solved simultaneously. From equation (11) we have

$$z_1 = x_1 - \frac{dx_1}{dt} \tag{13}$$

Differentiation of equation (13) with respect to t yeilds

$$\frac{dz_1}{dt} = \frac{dx_1}{dt} - \frac{d^2x_1}{dt^2} \tag{14}$$

By substituting equations (13) and (14) into equation (12) we obtain

$$\frac{d^2x_1}{dt^2} - 2x_1 = 0 \tag{15}$$

The solution of equation (15) is

$$x_1(t) = A_1e^{\lambda t} + A_2e^{-\lambda t} \tag{16}$$

where

$$\lambda = \sqrt{2}$$

By differentiating equation (16) with respect to t and substituting the result together with equation (16) into equation (13), we obtain

$$z_1(t) = -A_1(\lambda-1)e^{\lambda t} + A_2(\lambda+1)e^{-\lambda t} \tag{17}$$

Application of the boundary conditions

$$x_1(0) = 1$$

and

$$z_1(T) = 0$$

to equations (16) and (17), respectively, gives

$$x_1(0) = A_1 + A_2 = 1 \tag{18}$$

$$z_1(T) = -A_1(\lambda-1)e^{\lambda T} + A_2(\lambda+1)e^{-\lambda T} = 0 \tag{19}$$

The solution of equations (18) and (19) for A_1 and A_2 is

$$A_1 = \frac{(\lambda+1)e^{-\lambda T}}{(\lambda-1)e^{\lambda T} + (\lambda+1)e^{-\lambda T}} \tag{20}$$

$$A_2 = \frac{(\lambda-1)e^{+\lambda T}}{(\lambda-1)e^{\lambda T} + (\lambda+1)e^{-\lambda T}} \tag{21}$$

The optimal control $\overline{\theta}(t)$, which may be obtained by the substitution of equations (10), (20), and (21) and the relation $\lambda^2 = 2$ into equation (17), is

$$\overline{\theta}(t) = \frac{e^{-\lambda(T-t)} - e^{\lambda(T-t)}}{(\lambda-1)e^{\lambda T} + (\lambda+1)e^{-\lambda T}} \tag{22}$$

The objective function $S = x_2(T)$ becomes

$$\begin{aligned} S = x_2(T) &= \frac{1}{2}\int_0^T (\theta^2 + x_1^2)dt \\ &= \frac{1}{2}\int_0^T \left\{[(\lambda-1)A_1e^{\lambda t} - (\lambda+1)A_2e^{-\lambda t}]^2 + [A_1e^{\lambda t} + A_2e^{-\lambda t}]^2\right\}dt \\ &= \frac{1}{2}[A_1^2(\lambda-1)\ (e^{2\lambda T}-1) - A_2^2(\lambda+1)\ (e^{-2\lambda T}-1)] \end{aligned} \tag{23}$$

Further simplification of equation (23) gives

$$x_2(T) = \frac{1}{2}[A_2(\lambda+1) - A_1(\lambda-1)]$$

$$= \frac{1}{2}\,\frac{(e^{\lambda T} - e^{-\lambda T})}{(\lambda-1)e^{\lambda T} + (\lambda+1)e^{-\lambda T}} \tag{24}$$

If the final time T is left unspecified, the additional condition that the minimum of the Hamiltonian is zero can be employed as follows.

$$\text{Min } H = 0 = -\theta^2 - \theta x_1 + \frac{1}{2}\theta^2 + \frac{1}{2}x_1^2$$

$$= \theta^2 + 2\theta x_1 - x_1^2 \tag{25}$$

Application of the boundary condition at $t = 0$, namely, $x_1(0) = 1$, to equation (25) gives

$$[\theta(0)]^2 + 2\theta(0) - 1 = 0 \tag{26}$$

The roots of this quadratic equation are $\theta(0) = -1 - \sqrt{2}$ and $\theta(0) = -1 + \sqrt{2}$. For

$$\theta(0) = -1 - \sqrt{2}$$

combination of equations (10) and (17) yields

$$\theta(0) = -z_1(0) = A_1(\lambda-1) - A_2(\lambda+1) = -1 - \sqrt{2} \tag{27}$$

The solution of equations (18) and (27) for A_1 and A_2 is

$$A_1 = \frac{\lambda - \sqrt{2}}{2\lambda} = 0 \tag{28}$$

$$A_2 = \frac{\lambda + \sqrt{2}}{2\lambda} = 1 \tag{29}$$

By substituting equations (28) and (29) into equation (19), the final time T is obtained as

$$T = +\infty$$

Instead using $\theta(0) = -1 + \sqrt{2}$ in the above procedure produces $T = -\infty$ which is not physically feasible.

The objective function S becomes

$$S = x_2(T) = 1/2 \int_0^T (\theta^2 + x_1^2)\, dt$$

$$= 1/2 \int_0^T \{[(\lambda-1)A_1 e^{\lambda t} - (\lambda+1)A_2 e^{-\lambda t}]^2 + [A_1 e^{\lambda t} + A_2 e^{-\lambda t}]^2\}\, dt$$

$$= 1/2[A_1^2(\lambda-1)(e^{2\lambda T}-1) - A_2^2(\lambda+1)(e^{-2\lambda T}-1)] \tag{31}$$

Further simplification of equation (31) gives

$$x_2(T) = \frac{1+\sqrt{2}}{2}\ (1 - e^{-2\sqrt{2}\,T}) \tag{32}$$

The results of this simple example of open-loop control via the maximum principle for both the T-specified and the T-not specified cases are shown in Figs. 4.1 and 4.2.

These results are later compared with those of the simple example of closed-loop control via the maximum principle (Chapter 5).

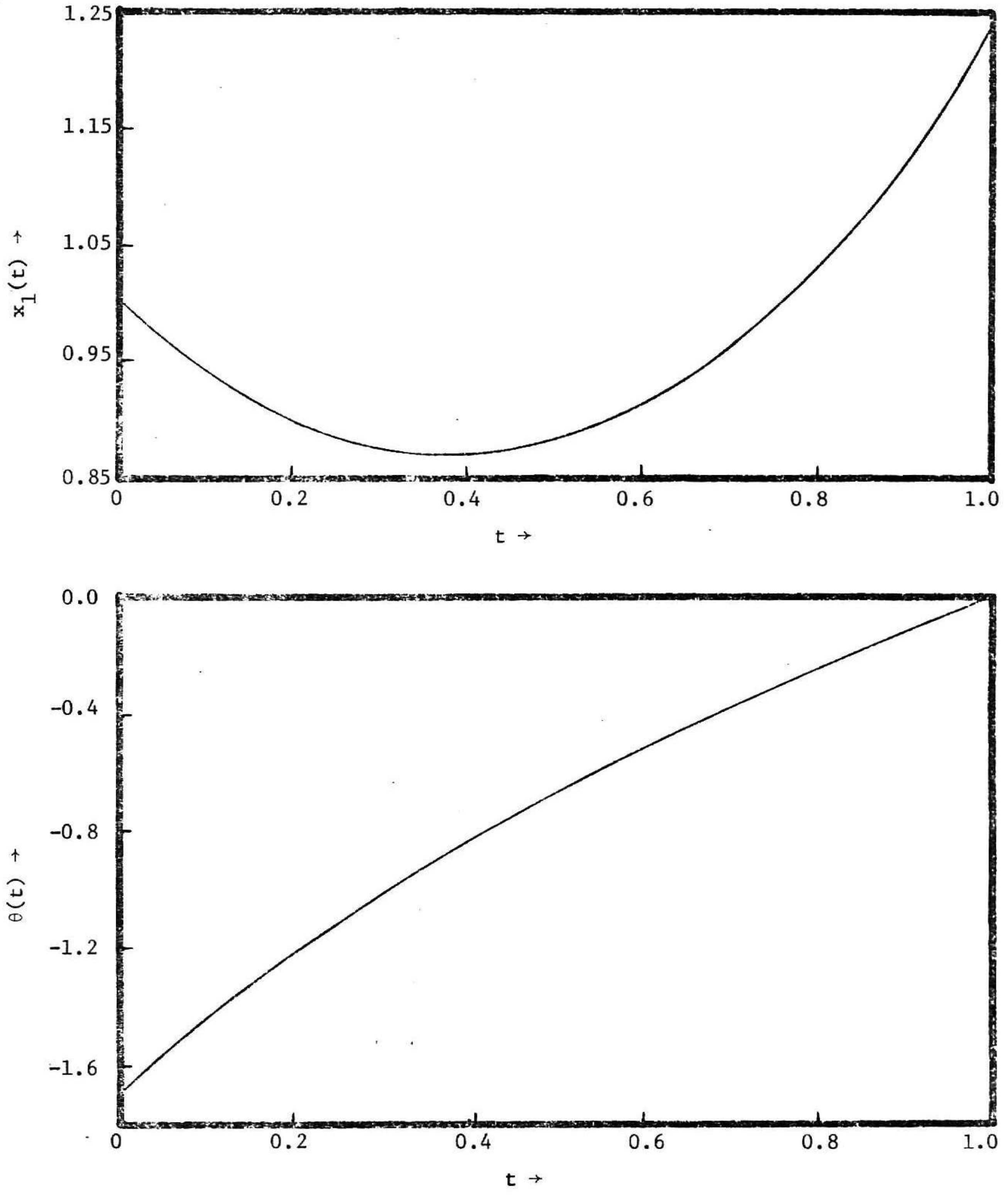

Fig. 4.1 Optimal trajectory and optimal control policy of the simple example (T - specified) T = 1.0

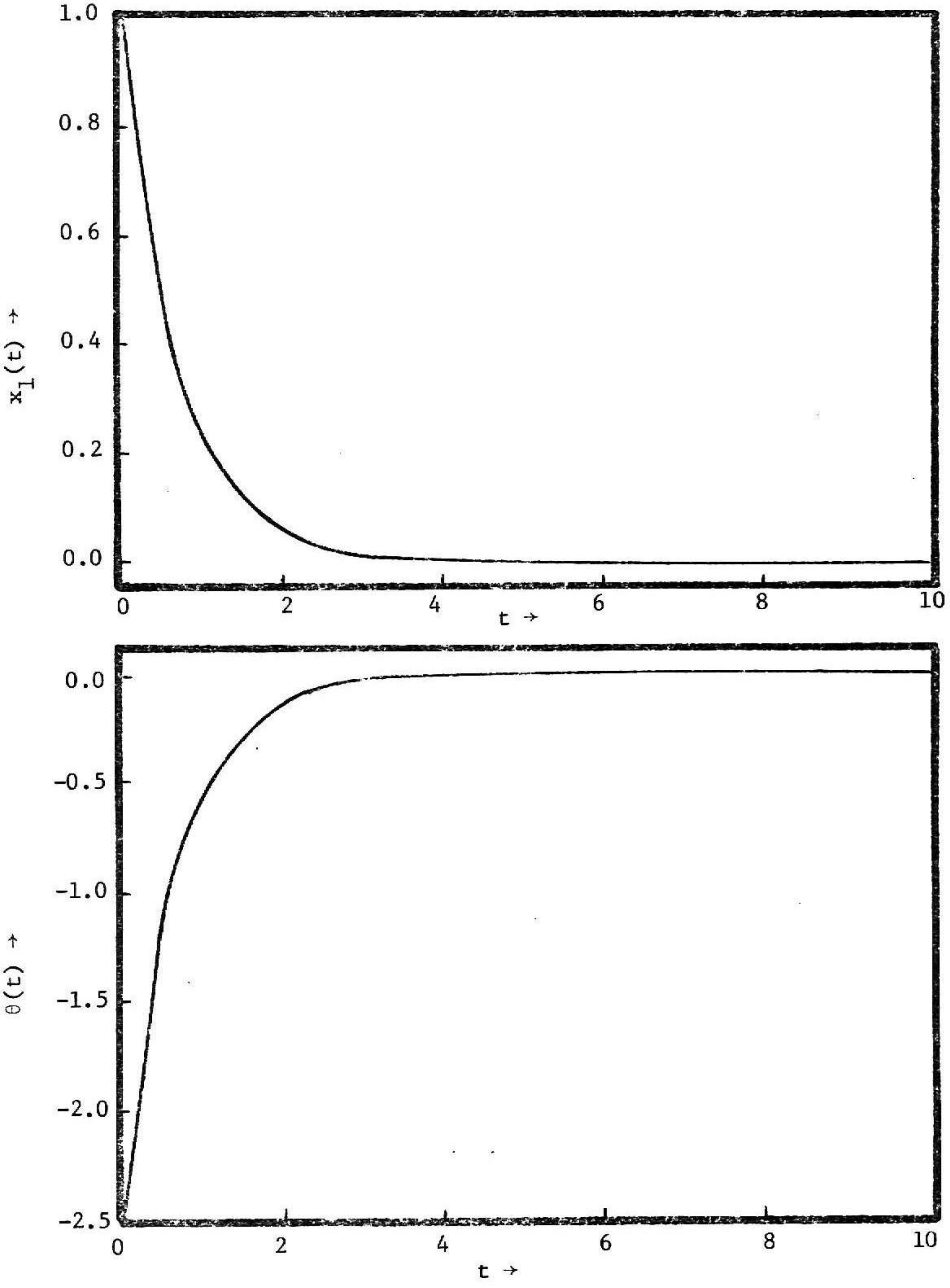

Fig. 4.2 Optimal trajectory and optimal control policy of the simple example (T - not specified).

CHAPTER 5

CLOSED-LOOP CONTROL VIA THE MAXIMUM PRINCIPLE

5.1 Introduction

For the continuous-time system described by a set of ordinary or partial differential equations, the problem of optimal control is that of determining the "control function" $\theta(t)$ from among the admissible set which brings the state of the system from an initial condition $x(0)$ to a final condition $x(T)$ such that a suitable functional, objective function or performance index is minimized.

Now there are problems of engineering interest, for example, regulator problems, for which the possible range of initial conditions is very large. To provide optimal system response over the set of initial conditions that might be encountered using the control function solution would require determining $\theta(t)$ for each possible initial condition. This obviously can be impractical. By contrast, if the "control law" $\theta(x, t)$ is known, the optimal control is determined at any time by knowledge of the current state $x(t)$. When the optimal control function is obtained, the system is said to be operating in an "open-loop" manner; that is, $\theta(t)$ is a function of time. In the same sense, determining an optimal control law implies operating the system in a "closed-loop" manner; that is, $\theta(x, t)$ is a function of both the state and time.

At the other extreme there is a problem in which only one initial state is to be expected. Here the control function solution $\theta(t)$ would appear to have more meaning. However, even in this case feedback information is required to overcome the effects of errors and disturbances which change the system response from the nominal trajectory. If $\theta(t)$ is followed without correction in the presence of these disturbances, the resulting trajectory cannot be expected to be either optimal or to satisfy terminal conditions. Again, if the control law $\theta(x, t)$ is known, this difficulty would not occur. Knowledge of the current state $x(t)$ and the time t would suffice to determine the optimal control without reference to whether the state $x(t)$ resulted from a disturbance from the open-loop optimal trajectory [10].

It has often been said that the optimal control policy as obtained by the maximum principle is open-loop. This is not always true as will be shown here in obtaining the feedback control law for the linear regulator problem, the first solution of which was due to Kalman [28, 29, 30, 31].

5.2 Statement of the Methodology

Let a linear differential system be described by

$$\dot{\underline{x}} = A\underline{x} + B\underline{\theta} \tag{i}$$

with the inital condition as

$$\underline{x}(0) = \underline{x}_0$$

where

$\underline{x}$ = a vector of s-dimensional state variables

$\underline{\theta}$ = a vector of r-dimensional control variables

A = s x s matrix

B = s x r matrix

It is required to find the optimal control $\underline{\theta}$ which minimizes the functional (for T fixed)

$$S = \frac{1}{2}\underline{x}^T(T)N\underline{x}(T) + \frac{1}{2}\int_0^T [\underline{x}^T L\underline{x} + \underline{\theta}^T M\underline{\theta}]dt \qquad \text{(ii)}$$

where the matrices L(s x s), M(r x r) and N(s x s) are assumed to be symmetric with no loss of generality. Considering $\underline{x}$ as the deviation of the system state from the desired state, minimization of equation (ii) can be interpreted as minimizing the deviation at the final time T and also minimizing the deviation and control effort during the transient from 0 to T.

The Hamiltonian is

$$H = \frac{1}{2}\underline{x}^T L\underline{x} + \frac{1}{2}\underline{\theta}^T M\underline{\theta} + \underline{z}^T A\underline{x} + \underline{z}^T B\underline{\theta} \qquad \text{(iii)}$$

where

$\underline{z}$ = a vector of s-dimensional adjoint variables

The adjoint variables are defined by

$$\frac{\partial H}{\partial \underline{x}} = -\dot{\underline{z}} = L\underline{x} + A^T\underline{z} \qquad \text{(iv)}$$

with the end condition

$$\underline{z}(T) = N\underline{x}(T) \qquad \text{(v)}$$

The optimal control is obtained from the following necessary condition for optimality.

$$\frac{\partial H}{\partial \underline{\theta}} = 0 = M\underline{\theta} + B^T\underline{z} \qquad \text{(vi)}$$

which gives

$$\underline{\theta} = -M^{-1}B^T\underline{z} \qquad \text{(vii)}$$

Now we inquire whether this can be converted to a closed-loop control by assuming that the solution for $\underline{z}$ is similar to equation (v), i.e.,

$$\underline{z}(t) = P(t)\underline{x}(t) \tag{vii}$$

where

$P(t)$ = s x s matrix of functions of time

Employing equation (viii) into equations (i) and (vii) yields

$$\dot{\underline{x}} = A\underline{x} - BM^{-1}B^{T}P\underline{x} \tag{ix}$$

Also from equations (iv) and (viii), we require

$$\dot{\underline{z}} = \dot{P}\underline{x} + P\dot{\underline{x}} = -L\underline{x} - A^{T}P\underline{x} \tag{x}$$

Combining equations (ix) and (x) gives

$$[\dot{P} + PA + A^{T}P - PBM^{-1}B^{T}P + L]\underline{x} = 0 \tag{xi}$$

Since equation (xi) must hold for all nonzero $\underline{x}$, the term premultiplying $\underline{x}$ must be zero. Thus the s x s symmetric matrix P having $n(n + 1)/2$ different terms, must satisfy the matrix Riccati equation

$$\dot{P} = - PA - A^{T}P + PBM^{-1}B^{T}P - L \tag{xii}$$

with an end condition given by equations (v) and (viii)

$$P(T) = N \tag{xiii}$$

Thus we solve equation (xii) backward in time from T to 0, store the matrix

$$K(t) = - M^{-1}B^{T}P \tag{xiv}$$

and then obtain the closed-loop control from

$$\underline{\theta}(\underline{x}, t) = K(t)\underline{x}(t) \tag{xv}$$

A block diagram for accomplishing this solution is shown in Figure 5.1.

From the consideration of second variation it has been shown that [38] L, M and N must be at least positive semidefinite in order to establish the sufficient condition for a minimum. In addition, from equation (vii) M must have an inverse; therefore, it is sufficient that M be positive definite.

In problems where the final time T is not specified, the additional condition that the minimum Hamiltonian is zero is used. Combining equations (iii), (vii) and (viii) gives

$$\text{Min } H = \frac{1}{2}\underline{x}^T L\underline{x} + \frac{1}{2}\underline{x}^T PBM^{-1}B^T P\underline{x} + \underline{x}^T PA\underline{x} - \underline{x}^T PBM^{-1}B^T P\underline{x}$$

$$= \frac{1}{2}\underline{x}^T L\underline{x} - \frac{1}{2}\underline{x}^T PBM^{-1}B^T P\underline{x} + \frac{1}{2}\underline{x}^T PA\underline{x} + \frac{1}{2}\underline{x}^T A^T P\underline{x} \qquad \text{(xvi)}$$

(using the identity $PA = \frac{1}{2}PA + \frac{1}{2}A^T P$)

Using Min H = 0, equation (16) becomes

$$PA + A^T P + L - PBM^{-1}B^T P = 0 \qquad \text{(xvii)}$$

which is the right hand side of equation (xii) with sign changed. Equation (xvii) is a set of $n(n + 1)/2$ algebraic equations whose solution yields a constant matrix P. Thus in case of problems with final time unspecified, the solution of the $n(n + 1)/2$ nonlinear equations is far simpler, since these are algebraic rather than differential equations.

5.3 An Example

The use of the maximum principle to obtain closed-loop control is illustrated here by considering the same simple example of chapter 4. Let the performance equation of a simple process be

$$\dot{x}_1 = x_1 + \theta \qquad (1)$$

with the initial condition

$$x_1(0) = 1$$

The objective function to be minimized is the sum of the integrated control effort to maintain the state of the system in the desired state and the integrated deviation from the desired state over a specified control time and is given by

$$S = \frac{1}{2}\int_0^T (\theta^2 + x_1^2)dt \tag{2}$$

This problem as stated has a fixed time interval with free right-end.

The Hamiltonian is

$$H = \frac{1}{2}\theta^2 + \frac{1}{2}x_1^2 + z_1(x_1 + \theta) \tag{3}$$

The adjoint variable is defined by

$$\frac{\partial H}{\partial x_1} = -\dot{z}_1 = x_1 + z_1 \tag{4}$$

with the end condition

$$z_1(T) = 0 \tag{5}$$

The optimal control is obtained from the following necessary condition for optimality

$$\frac{\partial H}{\partial \theta} = 0 = \theta + z_1 \tag{6}$$

which gives

$$\theta = -z_1 \tag{7}$$

Now we inquire whether this can be converted to a closed-loop control by assuming the solution for z_1 as

$$z_1 = p_{11}x_1 \tag{8}$$

Employing equation (8) into equations (1) and (7) yields

$$\dot{x}_1 = x_1 - p_{11}x_1 \tag{9}$$

Also from equations (4) and (8) we require

$$\dot{z}_1 = \dot{p}_{11}x_1 + p_{11}\dot{x}_1 = - x_1 - p_{11}x_1 \tag{10}$$

Combining equations (9) and (10) gives

$$(\dot{p}_{11} + 2p_{11} - p_{11}^2 + 1)x_1 = 0 \tag{11}$$

Since equation (11) must hold for all nonzero x_1, the term premultiplying x_1 must be zero. Thus we have

$$\dot{p}_{11} = - 2p_{11} + p_{11}^2 - 1 \tag{12}$$

with an end condition given by equations (5) and (8)

$$p_{11}(T) = 0 \tag{13}$$

From equation (9) we have

$$p_{11} = \frac{x_1 - \dot{x}_1}{x_1} \tag{14}$$

Differentiation of equation (14) with respect to t yields

$$\dot{p}_{11} = \frac{- x_1\ddot{x}_1 + \dot{x}_1^2}{x_1^2} \tag{15}$$

By substituting equations (14) and (15) into equation (12) we obtain

$$\ddot{x}_1 - 2x_1 = 0 \tag{16}$$

The solution of equation (16) is

$$x_1(t) = A_1e^{\lambda t} + A_2e^{-\lambda t} \tag{17}$$

where

$$\lambda = \sqrt{2}$$

By differentiating equation (17) with respect to t and substituting the result together with equation (17) into equation (14), we obtain

$$p_{11}(t) = \frac{-A_1(\lambda-1)e^{\lambda t} + A_2(\lambda+1)e^{-\lambda t}}{A_1 e^{\lambda t} + A_2 e^{-\lambda t}} \tag{18}$$

Application of the boundary conditions

$$x_1(0) = 1$$

and

$$p_{11}(T) = 0$$

to equations (17) and (18), respectively, gives

$$x_1(0) = A_1 + A_2 = 1 \tag{19}$$

$$p_{11}(T) = -A_1(\lambda-1)e^{\lambda T} + A_2(\lambda+1)e^{-\lambda T} = 0 \tag{20}$$

The solution of equations (19) and (20) for A_1 and A_2 is

$$A_1 = \frac{(\lambda+1)e^{-\lambda T}}{(\lambda-1)e^{\lambda T} + (\lambda+1)e^{-\lambda T}} \tag{21}$$

and

$$A_2 = \frac{(\lambda-1)e^{\lambda T}}{(\lambda-1)e^{\lambda T} + (\lambda+1)e^{-\lambda T}} \tag{22}$$

The feedback gain $k_{11}(t)$ is obtained as

$$k_{11}(t) = -p_{11}(t) = \frac{A_1(\lambda-1)e^{\lambda t} - A_2(\lambda+1)e^{-\lambda t}}{A_1 e^{\lambda t} + A_2 e^{-\lambda t}} . \tag{23}$$

Then the optimal control policy $\bar{\theta}(t)$ may be obtained as

$$\bar{\theta}(t) = k_{11}(t)x_1(t) = \frac{A_1(\lambda-1)e^{\lambda t} - A_2(\lambda+1)e^{-\lambda t}}{A_1 e^{\lambda t} + A_2 e^{-\lambda t}} x_1(t)$$

$$= A_1(\lambda-1)e^{\lambda t} - A_2(\lambda+1)e^{-\lambda t} \tag{24}$$

Using equations (21) and (22) and the relation $\lambda^2 = 2$ into equation (24) yields

$$\bar{\theta}(t) = \frac{e^{-\lambda(T-t)} - e^{\lambda(T-t)}}{(\lambda-1)e^{\lambda T} + (\lambda+1)e^{-\lambda T}} \tag{25}$$

The objective function S becomes

$$S = x_2(T) = \frac{1}{2}\int_0^T (\theta^2 + x_1^2)dt$$

$$= \frac{1}{2}\int_0^T \left\{[(\lambda-1)A_1 e^{\lambda t} - (\lambda+1)A_2 e^{-\lambda t}]^2 + [A_1 e^{\lambda t} + A_2 e^{-\lambda t}]^2\right\}dt$$

$$= \frac{1}{2}[A_1^2(\lambda-1)(e^{2\lambda T}-1) - A_2^2(\lambda+1)(e^{-2\lambda T}-1)] \tag{26}$$

Further simplification of equation (26) gives

$$S = \frac{1}{2}[A_2(\lambda+1) - A_1(\lambda-1)]$$

$$= \frac{1}{2}\frac{(e^{\lambda T} - e^{-\lambda T})}{(\lambda-1)e^{\lambda T} + (\lambda+1)e^{-\lambda T}} \tag{27}$$

In the above simple example if the final time T is left unspecified (which is the same as saying that the process is to be operated for a semi-infinite period of time), equation (12) becomes

$$p_{11}^2 - 2p_{11} - 1 = 0 \tag{28}$$

The solution of equation (28) gives a positive constant value for p_{11} as

$$p_{11} = 1 + \sqrt{2} \tag{29}$$

The constant feedback gain k_{11} is obtained as

$$k_{11} = - p_{11} = - (1 + \sqrt{2}) \tag{30}$$

Now the solution of equation (9) can be obtained as

$$x_1(t) = A\, e^{(1-p_{11})t} \tag{31}$$

Application of the boundary condition

$$x_1(0) = 1$$

to equation (31) gives

$$A = 1 \tag{32}$$

Then the optimal control policy $\bar{\theta}(t)$ may be obtained as

$$\begin{aligned} \bar{\theta}(t) &= k_{11}x_1(t) \\ &= - (1 + \sqrt{2})e^{(1-p_{11})t} \end{aligned} \tag{33}$$

The objective function S becomes

$$\begin{aligned} S &= \frac{1}{2}\int_0^T (\theta^2 + x_1^2)dt \\ &= \frac{1}{2}\int_0^T \left\{p_{11}^2 e^{2(1-p_{11})t} + e^{2(1-p_{11})t}\right\}dt \\ &= \frac{1 + p_{11}^2}{4(1 - p_{11})}\left[e^{2(1-p_{11})T} - 1\right] \end{aligned} \tag{34}$$

Further simplification of equation (34) gives

$$S = \frac{1 + \sqrt{2}}{2} (1 - e^{-2\sqrt{2}\, T}) \tag{35}$$

5.4 Conclusion

The results of this simple example of closed-loop control via the maximum principle for both the T-specified and the T-not specified cases are shown in Figures 5.2 and 5.3.

The optimal trajectory and the optimal control policy are exactly the same as those of the simple example of open-loop control via maximum principle (Chapter 4). Thus both the open-loop as well as the closed-loop control via the maximum principle give the same results for a system with linear performance equation and quadratic objective function.

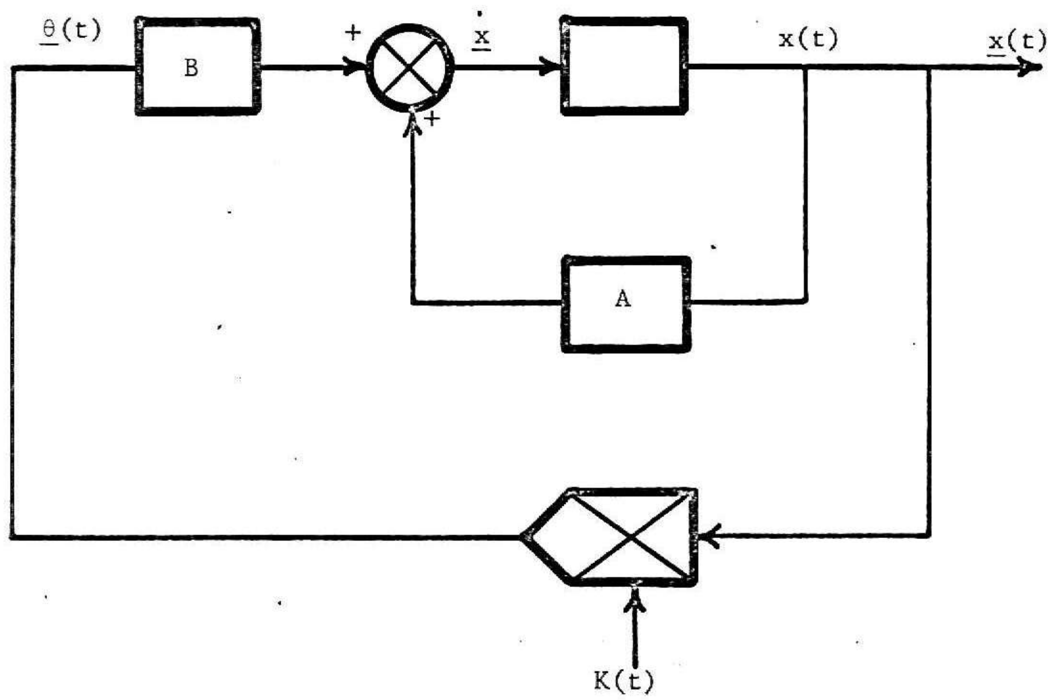

Fig. 5.1 Block diagram for the optimum linear closed-loop regulator.

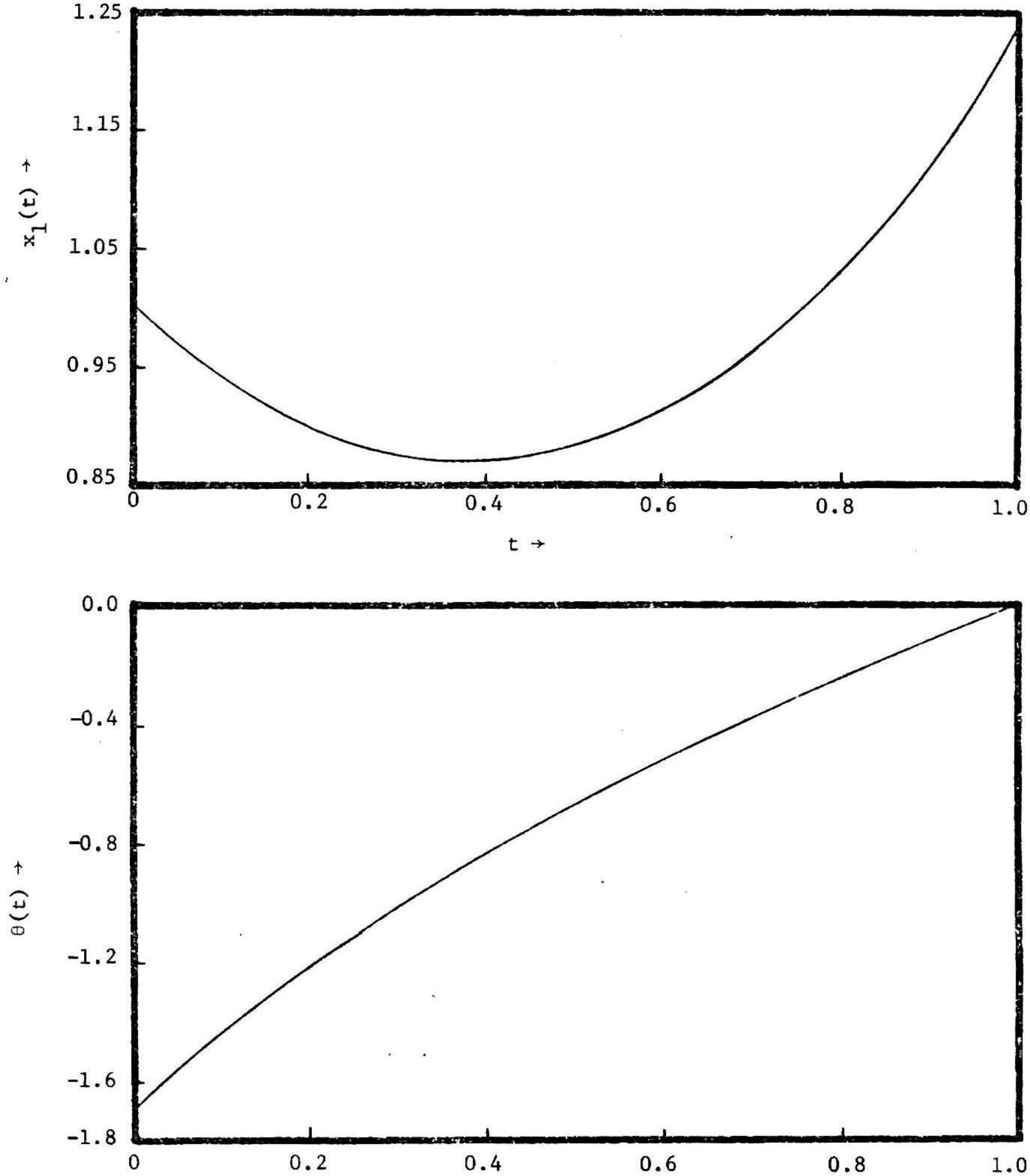

Fig. 5.2 Optimal trajectory and optimal control policy of the simple example (T - specified) T = 1.0

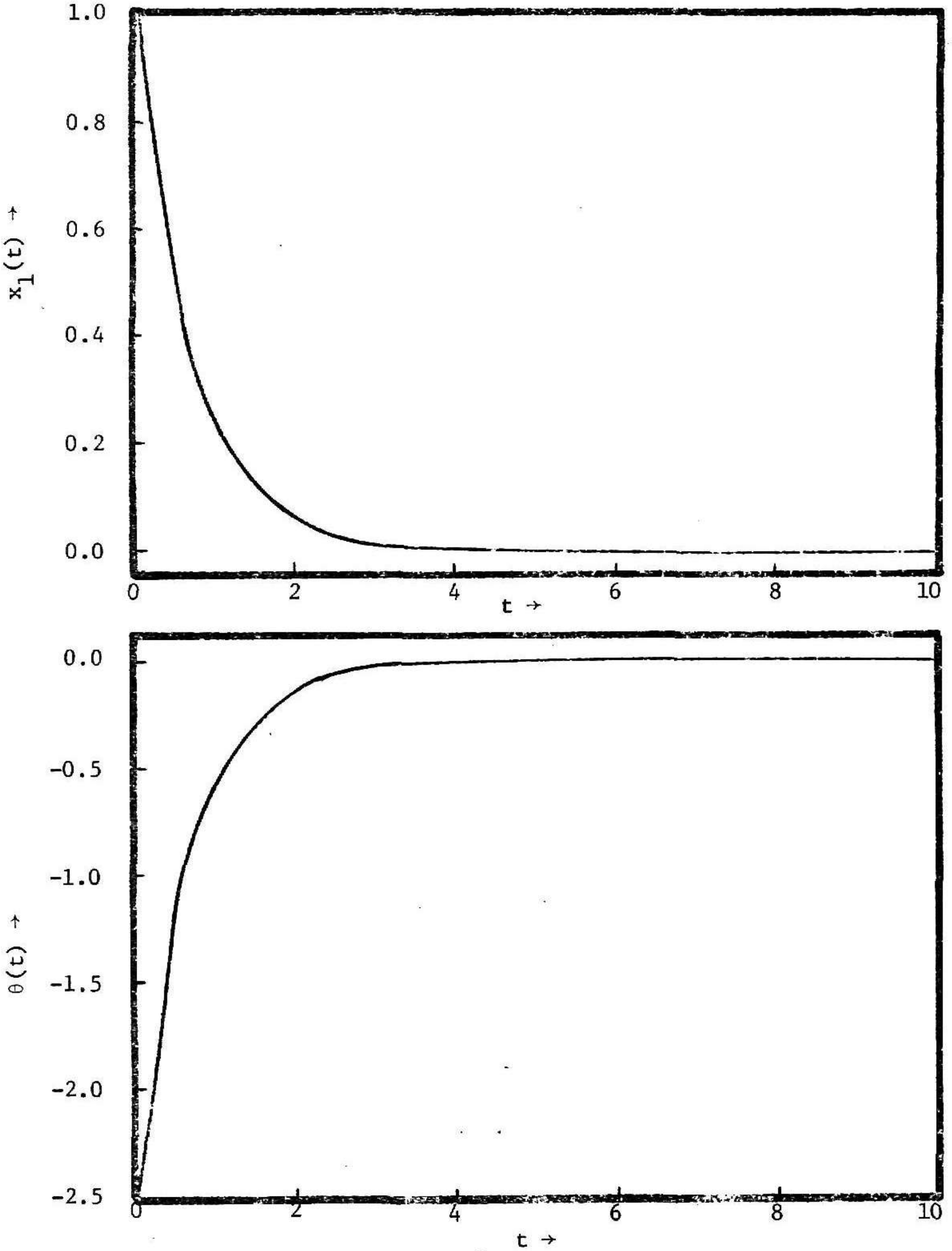

Fig. 5.3 Optimal trajectory and optimal control policy of the simple example (T - not specified).

CHAPTER 6

OPEN-LOOP CONTROL OF LIFE SUPPORT SYSTEMS SUBJECTED TO A STEP HEAT DISTURBANCE

6.1 Introduction

This chapter deals with the open-loop control of a heating system consisting of an air-conditioned room or cabin and a heat exchanger of negligibly small *time constant* ($\tau_2 = 0$) subjected to a step heat disturbance, having the initial and final values of the state variable-namely, the temperature fixed but the final time T to be determined.

Pontryagin's maximum principle is used in determining the optimal control policies for each of the following examples with the corresponding objective functions to be minimized:

Example 1: $S = \int_0^T dt$

Example 2: $S = \int_0^T (a + b\theta^2)\, dt$

Example 3: $S = \int_0^T [b\theta^2 + c(x_1 - x_{1d})^2]\, dt$

Example 4: $S = \int_0^T [a + b\theta^2 + c(x_1 - x_{1d})^2]\, dt$

where a, b and c are weighting factors and x_{1d} is the desired state of the system.

It is worth noting here that the objective functions of the first three examples are particular cases of the objective function of the fourth example.

6.2 Example 1: Minimal Time Control

Let a life support system consisting of an air-conditioned room or cabin and a heat exchanger of negligibly small time constant ($\tau_2 = 0$) be subjected to a step heat disturbance.

The performance equation of such a system is written as

$$\frac{dx_1}{dt} + r_2 x_1 = r_2 K_\alpha - r_1 K_\beta \theta - r_1 K_\gamma + \sigma_s \tag{1.1}$$

The initial condition is

$$x_1(0) = \alpha_0$$

It is required to find an optimal control that will bring the system to the desired condition in an unspecified time interval T. That is, the final (desired) condition is

$$x_1(T) = 1, \quad T \text{ - not specified.}$$

At the same time it is also required to minimize the functional (objective function)

$$S = \int_0^T dt \tag{1.2}$$

The objective function consists of the minimal time control function alone.

The control variable is constrained as

$$-1 \leq \theta \leq +1$$

This problem as stated has the initial and final values of the state vector function $x_1(0)$ and $x_1(T)$ fixed but the final time T to be determined.

Introducing another state variable $x_2(t)$ such that

$$x_2(t) = \int_0^t dt$$

it follows that

$$\frac{dx_2}{dt} = 1, \qquad x_2(0) = 0 \tag{1.3}$$

The problem is thus transformed into that of minimizing $x_2(T)$.

According to Pontryagin's maximum principle, the Hamiltonian is

$$H = z_1(-r_2x_1+r_2K_\alpha-r_1K_\beta\theta-r_1K_\gamma+\sigma_s) + z_2 \tag{1.4}$$

The adjoint variables are defined by

$$\frac{dz_1}{dt} = -\frac{\partial H}{\partial x_1} = r_2z_1 \tag{1.5}$$

$$\frac{dz_2}{dt} = -\frac{\partial H}{\partial x_2} = 0, \qquad z_2(T) = 1 \tag{1.6}$$

Solutions of equations (1.5) and (1.6) are

$$z_1(t) = Ae^{r_2t} \tag{1.7}$$

$$z_2(t) = 1, \qquad 0 \le t \le T \tag{1.8}$$

where A is the integration constant to be determined later.

Using equation (1.8) into equation (1.4) yields

$$H = z_1(-r_2x_1+r_2K_\alpha-r_1K_\beta\theta-r_1K_\gamma+\sigma_s) + 1 \tag{1.9}$$

The switching function H*, the portion of H which depends on θ, is

$$H^* = - r_1K_\beta z_1\theta \tag{1.10}$$

Inspection of H* shows the basic structure of the time optimal control policy is of the bang-bang type. The conditions for which the Hamiltonian be minimum are

$$\left.\begin{array}{llll} \theta = \theta_{min} = -1 & \text{if} & -r_1K_\beta z_1 > 0 \\ \theta = \theta_{max} = +1 & \text{if} & -r_1K_\beta z_1 < 0 \end{array}\right\} \tag{1.11}$$

provided that the controller shifts from θ_{min} to θ_{max} instantaneously and inertialessly, or vice versa.

Now, the maximum principle requires that the system equations (1.1) and (1.3) and the adjoint equation (1.5) be integrated simultaneously in such a way that the two-point boundary conditions

$$\begin{array}{ll} x_1(0) = \alpha_0, & x_1(T) = 1 \\ x_2(0) = 0, & x_2(T) = \text{undetermined} \\ z_1(0) = \text{undetermined}, & z_1(T) = \text{undetermined} \end{array}$$

be satisfied. Meanwhile the Hamiltonian must remain at zero at every point of its response under the optimal condition.

In order to bring the initial deviated state $[x_1(0) = \alpha_0]$ to the final desired state $[x_1(T) = 1]$, we intuitively reject the control $\theta = \theta_{max} = +1$

(which corresponds to the minimum heating action).

Equation (1) can be integrated with the conditions

$$\theta = \theta_{min} = -1 \tag{1.12}$$

and

$$x_1(0) = \alpha_0 \tag{1.13}$$

as

$$x_1(t) = K'(1-e^{-r_2 t}) + \alpha_0 e^{-r_2 t} \tag{1.14}$$

where

$$K' = (r_2 K_\alpha + r_1 K_\beta - r_1 K_\gamma + \sigma_s)/r_2$$

The integration constant A in equation (1.7) can be determined by using the condition that minimum H is zero for all the process time in time optimal control. At $t = 0$, we have from equations (1.7), (1.9), (1.12) and (1.13)

$$A = z_1(0) = \frac{-1}{r_2(K'-\alpha_0)}$$

and

$$z_1(t) = \frac{-1}{r_2(K'-\alpha_0)} e^{r_2 t} \tag{1.15}$$

Equation (1.15) implies that $z_1(t)$ will not change sign since $z_1(t) \to 0$ only when t approaches negative infinity, or in other words, control will not shift from θ_{min} to θ_{max} (or from θ_{max} to θ_{min}). Therefore, this problem is a particular case of bang-bang control which has the bang part only.

The final control time can be obtained from equations (1.9) and (1.12) together with the final condition

$$x_1(T) = 1$$

as follows

$$H = z_1(T)[-r_2x_1(T)+K'r_2] + 1 = 0$$

or solving for $z_1(T)$

$$z_1(T) = \frac{-1}{r_2(K'-1)} \tag{1.16}$$

Also we have from equation (1.15) at $t = T$

$$z_1(T) = \frac{-1}{r_2(K'-\alpha_0)} e^{r_2T} \tag{1.17}$$

Solving for T from equations (1.16) and (1.17), we have

$$T = \frac{1}{r_2} \ln \left\{\frac{K'-\alpha_0}{K'-1}\right\} \tag{1.18}$$

This solution may be verified by inserting it into equation (1.14) as

$$x_1(T) = K'(1-e^{-r_2T}) + \alpha_0 e^{-r_2T}$$

$$= (\alpha_0 - K') e^{-r_2T} + K'$$

$$= (\alpha_0 - K')e^{-\ln\left\{\frac{K'-\alpha_0}{K'-1}\right\}} + K'$$

$$= (\alpha_0-K') \left\{\frac{K'-1}{K'-\alpha_0}\right\} + K'$$

This indicates that the Hamiltonian is kept at zero at every point of its response in this minimum time problem.

The objective function becomes

$$S = x_2(T) = \int_0^T dt$$

$$= T \tag{1.19}$$

The following two cases are considered in this example:

Case number	Value of the dimensionless disturbance temperature (σ_s)
1	0.75
2	1.20

The values of the various constants used are:

$r_1 = 0.8$ $\quad\quad$ $K_\alpha = 1.5$

$r_2 = 0.2$ $\quad\quad$ $K_\beta = K_\gamma = 0.625$

The results of this example are shown in Table 6.1.1 and Figs. 6.1.1 and 6.1.2.

Since this is a heating system, the greater the value of σ_s, the lesser is the control time required to reach the final desired state.

The state variable x_1 approaches asymptotically the final state, the control variable θ remains at its minimum namely -1 until the final state is reached, and the adjoint vector increases asymptotically.

Table 6.1.1

Optimal Solutions of Example 1:

Case Number	Value of σ_s	Final Time T
1	0.75	1.0565
2	1.20	0.7155

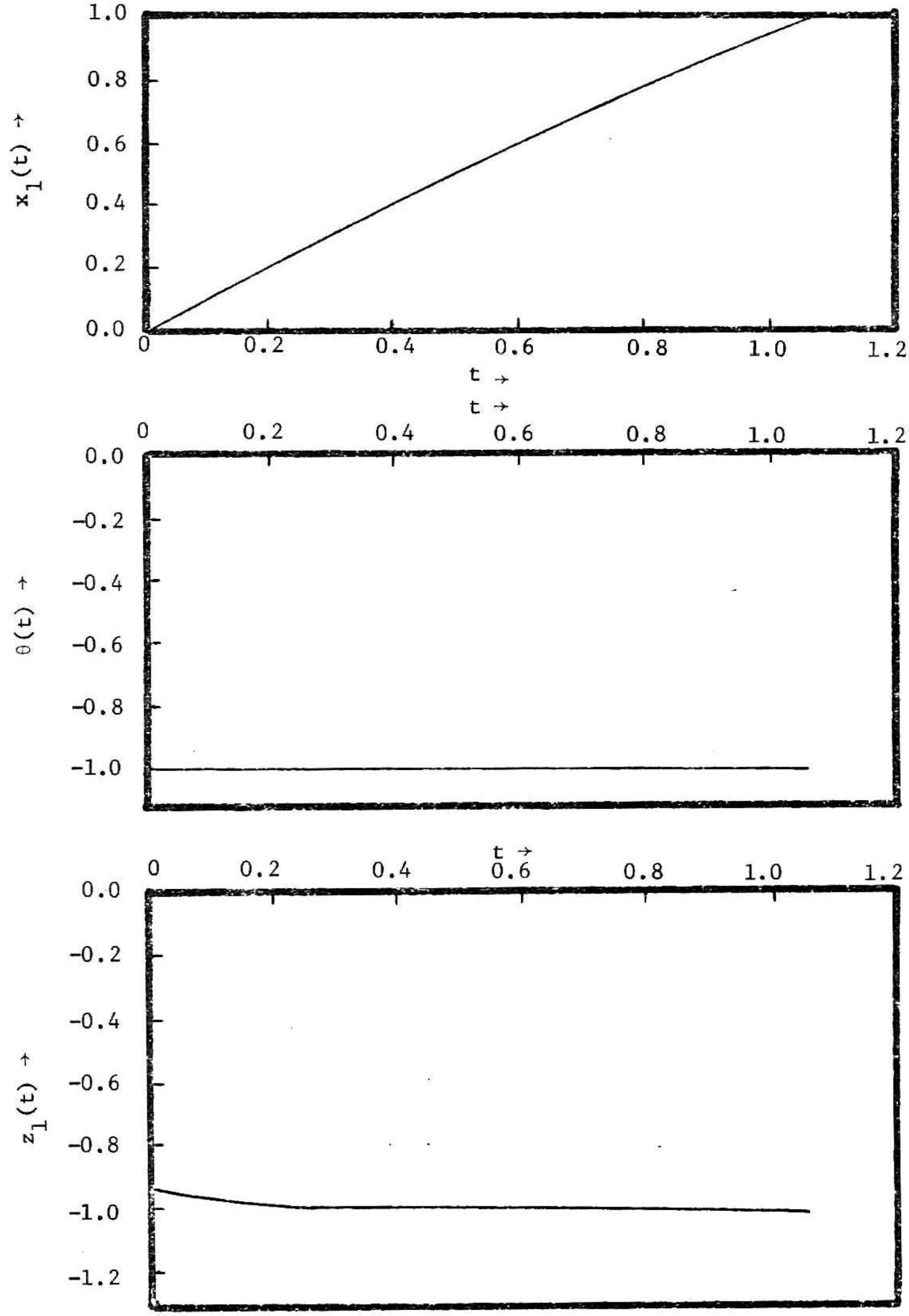

Fig. 6.1.1 Optimal control policy and system response of Example 1 ($\sigma_s = 0.75$).

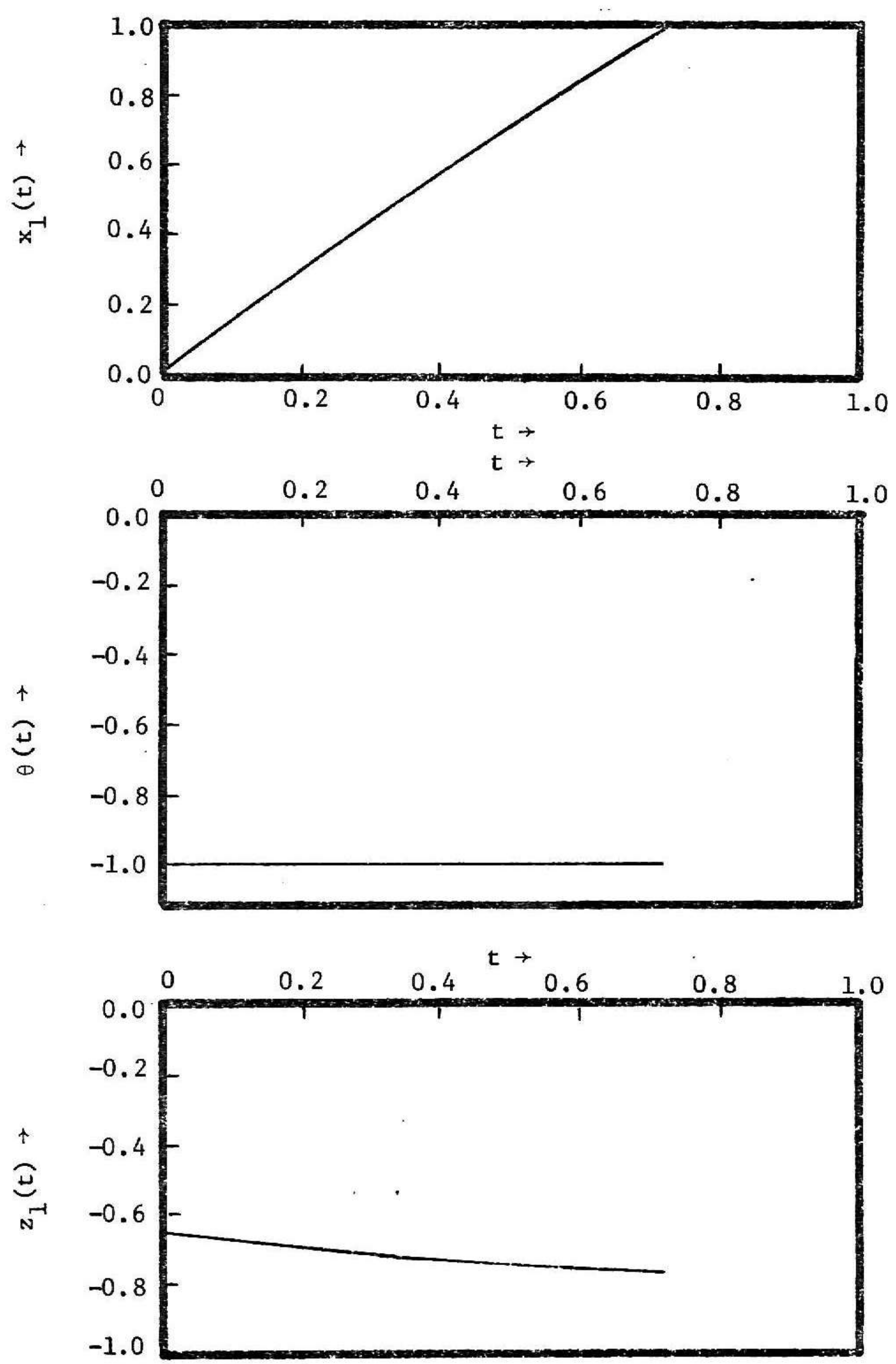

Fig. 6.1.2 Optimal control policy and system response of Example 2 ($\sigma_s = 1.2$).

6.3 Example 2: Minimal Time and Integrated Effort Control

Let a life support system consisting of an air-conditioned room or cabin and a heat exchanger of negligibly small time constant ($\tau_2 = 0$) be subjected to a step heat disturbance.

The performance equation of such a system is written as

$$\frac{dx_1}{dt} + r_2 x_1 = r_2 K_\alpha - r_1 K_\beta \theta - r_1 K_\gamma + \sigma_s \tag{2.1}$$

The initial condition is

$$x_1(0) = \alpha_0$$

It is required to find an optimal control that will bring the system to the desired condition in an unspecified time interval T. That is, the final (desired) condition is,

$$x_1(T) = 1 \quad , \qquad T \text{ not specified.}$$

At the same time it is also required to minimize the functional (objective function)

$$S = \int_0^T (a+b\theta^2)dt \tag{2.2}$$

where a and b are weighting factors.

The objective function is obtained by combining the unspecified control time and the integrated control effort to bring the state of the system to the desired state over the unspecified control time.

The control variable is constrained as

$$-1 \leq \theta \leq +1$$

This problem as stated has the initial and final values of the state vector function $x_1(0)$ and $x_1(T)$ fixed but the final time T to be determined.

Introducing another state variable $x_2(t)$ such that

$$x_2(t) = \int_0^t (a+b\theta^2)dt$$

it follows that

$$\frac{dx_2}{dt} = a + b\theta^2 \quad , \qquad\qquad x_2(0) = 0 \tag{2.3}$$

The problem is thus transformed into that of minimizing $x_2(T)$.

According to Pontryagin's maximum principle, the Hamiltonian is

$$H = z_1(-r_2x_1+r_2K_\alpha-r_1K_\beta\theta-r_1K_\gamma+\sigma_s) + z_2(a+b\theta^2) \tag{2.4}$$

The adjoint variables are defined by

$$\frac{dz_1}{dt} = -\frac{\partial H}{\partial x_1} = r_2z_1 \tag{2.5}$$

$$\frac{dz_2}{dt} = -\frac{\partial H}{\partial x_2} = 0 \quad , \qquad\qquad z_2(T) = 1 \tag{2.6}$$

Solutions of equations (2.5) and (2.6) are

$$z_1(t) = Ae^{r_2t} \tag{2.7}$$

$$z_2(t) = 1, \qquad\qquad 0 \leq t \leq T \tag{2.8}$$

where A is the integration constant to be determined later.

Using equation (2.8) into equation (2.4) yields

$$H = z_1(-r_2x_1-r_2K_\alpha-r_1K_\beta\theta-r_1K_\gamma+\sigma_s) + a + b\theta^2 \tag{2.9}$$

The variable portion of H that depends on θ, H^*, is

$$H^* = -r_1 K_\beta z_1 \theta + b\theta^2$$

Inspection of H* shows that the optimal control is of continuous type and is obtained from the following necessary condition for optimality.

$$\frac{\partial H^*}{\partial \theta} = 0 = -r_1 K_\beta z_1 + 2b\theta \tag{2.10}$$

which gives

$$\theta = \frac{r_1 K_\beta z_1}{2b} \tag{2.11}$$

The integration constant A in equation (2.7) can be determined using the condition that minimum H is zero at all the process time. At $t = 0$, from equations (2.7), (2.9), (2.11) and the initial condition

$$x_1(0) = \alpha_0$$

we have

$$\frac{(r_1 K_\beta)^2}{4b} A^2 + (r_2\alpha_0 - r_2 K_\alpha + r_1 K_\gamma - \sigma_s) - a = 0 \tag{2.12}$$

The solution of equation (2.12) gives

$$A = 2b\left\{-(r_2\alpha_0 - r_2K_\alpha + r_1K_\gamma - \sigma_s) - \sqrt{(r_2\alpha_0 - r_2K_\alpha + r_1K_\gamma - \sigma_s)^2 + \frac{a}{b}(r_1K_\beta)^2}\right\} / (r_1K_\beta)$$

Using equations (2.7) and (2.11), equation (2.1) can be integrated as

$$x_1(t) = A_1 e^{-r_2 t} + \frac{r_2 K_\alpha - r_1 K_\gamma + \sigma_s}{r_2} - \frac{(r_1 K_\beta)^2}{4br_2} A e^{r_2 t} \tag{2.13}$$

The constant of integration A_1 can be determined using the initial condition that $x_1(0) = \alpha_0$ as

$$A_1 = \alpha_0 - \frac{1}{r_2}(r_2K_\alpha - r_1K_\gamma + \sigma_s) + \frac{(r_1K_\beta)^2}{4br_2} A$$

The final time T is found by employing the final condition that $x_1(T) = 1$ in equation (2.13) as

$$A_1e^{-r_2T} + \frac{1}{r_2}(r_2K_\alpha - r_1K_\gamma + \sigma_s) - \frac{(r_1K_\beta)^2}{4br_2} Ae^{r_2T} = 1$$

which gives

$$T = \left\{\frac{1}{r_2}\right\} \ln \left[(2br_2)\left\{\frac{1}{r_2}(r_2K_\alpha - r_1K_\gamma + \sigma_s - r_2) - \sqrt{\frac{1}{r_2^2}(r_1K_\alpha - r_1K_\gamma + \sigma_s - r_2)^2 + \frac{(r_1K_\beta)^2}{br_2} A_1A}\right\} / \left\{(r_1K_\beta)^2 A\right\}\right] \tag{2.14}$$

The optimal control policy can now be determined from equations (2.7) and (2.11) as

$$\theta(t) = \frac{r_1K_\beta}{2b} Ae^{r_2t} \tag{2.15}$$

The objective function becomes

$$S = x_2(T) = \int_0^T (a+b\theta^2)dt = aT + \frac{(r_1K_\beta A)^2}{8br_2}[e^{2r_2T} - 1] \tag{2.16}$$

In cases where the weighting factor, a, is appreciably larger compared to the weighting factor b, the optimal control given by equation (2.15) will violate the constraint $|\theta| \leq 1$. Therefore the condition

$$\frac{\partial H^*}{\partial \theta} = 0$$

does not yield the admissible control action.

In order to bring the initial deviated state $[x_1(0) = \alpha_0]$ to the final desired state $[x_1(T) = 1]$, we intuitively reject the control $\theta = +1$ (which corresponds to minimum heating action). With $\theta = -1$, equation (1) can be integrated as

$$x_1(t) = B_1 e^{-r_2 t} + K' \tag{2.17}$$

where

$$K' = \frac{r_2 K_\alpha + r_1 K_\beta - r_1 K_\gamma + \sigma_s}{r_2}$$

The constant of integration, B_1, can be determined by employing the initial condition in equation (2.17) which yields

$$B_1 = \alpha_0 - K'$$

The solution of equation (5) is

$$z_1(t) = A' e^{r_2 t} \tag{2.18}$$

The integration constant A' in equation (2.18) is determined by using the condition that minimum H is zero for all the process time.

Now the minimum value of H is

$$H = z_1(-r_2 x_1 + r_2 K_\alpha + r_1 K_\beta - r_1 K_\gamma + \sigma_s) + a + b = 0 \quad , \qquad 0 \leq t \leq T$$

or

$$H = z_1(-r_2 x_1 + K' r_2) + a + b = 0 \quad , \qquad 0 \leq t \leq T \tag{2.19}$$

Application of the boundary condition at $t = 0$, that is, $x_1(0) = \alpha_0$ to equation (2.19) gives

$$z_1(0) = -\frac{(a + b)}{r_2(K' - \alpha_0)} \tag{2.20}$$

Also from equation (2.18) at time $t = 0$, we have

$$z_1(0) = A' \tag{2.21}$$

Combining equations (2.20) and (2.21) yields

$$A' = -\frac{(a + b)}{r_2(K' - \alpha_0)}$$

Now equation (2.18) implies that $z_1(t)$ will not change sign since $z_1(t) \to 0$ only when t approaches negative infinity.

The final control time can be obtained from the minimum H equation, namely equation (2.19) together with the condition

$$x_1(T) = 1$$

as follows

$$H = z_1(T)[- r_2x_1(T) + K'r_2] + a + b = 0$$

or solving for $z_1(T)$

$$z_1(T) = -\frac{(a + b)}{r_2(K' - 1)} \tag{2.22}$$

Also we have from equation (2.18) at $t = T$

$$z_1(T) = A'e^{r_2 T} \tag{2.23}$$

Solving for T from equations (2.22) and (2.23), we have

$$T = \frac{1}{r_2} \ln \left\{\frac{K' - \alpha_0}{K' - 1}\right\} \tag{2.24}$$

This solution may be verified by inserting it into equation (2.17) as

$$\begin{aligned} x_1(T) &= B_1 e^{-r_2 T} + K' \\ &= (\alpha_0 - K')e^{-r_2 T} + K' \\ &= (\alpha_0 - K')e^{-\ln\left\{\frac{K' - \alpha_0}{K' - 1}\right\}} + K' \\ &= (\alpha_0 - K')\left\{\frac{K' - 1}{K' - \alpha_0}\right\} + K' \end{aligned}$$

This indicates that the Hamiltonian is kept at zero at every point of its response.

Then the optimal policy and response are summarized as follows:

$$\theta = -1 \quad , \qquad 0 \le t \le T \tag{2.25a}$$

$$x_1 = B_1 e^{-r_2 t} + K' \quad , \qquad 0 \le t \le T \tag{2.25b}$$

Now the objective function becomes

$$\begin{aligned} S = x_2(T) &= \int_0^T (a + b\theta^2)dt \\ &= (a+b)T \end{aligned} \tag{2.26}$$

The following four cases are considered in this example:

case 1: $a = 1$, $b = 0.1$

case 2: $a = 1$, $b = 1$

case 3: $a = 1$, $b = 10$

case 4: $a = 1$, $b = 100$

The values of the various constants used are:

$r_1 = 0.8$ $\qquad K_\alpha = 1.5$

$r_2 = 0.2$ $\qquad K_\beta = K_\gamma = 0.625$

$\alpha_0 = 0$

The results of this example for $\sigma_s = 0.75$ and $\sigma_s = 1.2$ are shown in Table 6.2.1 and Figures 6.2.1 and 6.2.2.

The control time, required to reach the final desired state, increases with increase in the weighting factor b on the control effort. Also since this is a heating system, the greater the value of σ_s, the lesser is the control time required to reach the final desired state.

The value of the objective function is minimum for the maximum value of σ_s and the minimum value of the weighting factor b on the control effort.

In case 1, where more weight is given to the control time than to the control effort, the optimal control is of bang-bang type which has the bang part alone. In case 4, where more weight is given to the control effort, the optimal control has a very small negative value. Cases 2 and 3 are intermediate between the two extreme cases 1 and 4.

Table 6.2.1

Optimal Solutions of Example 2:

σ_s	Case Number	Weighting Factors	Final Time T	Values of the Objective Function
0.75	1	a = 1, b = 0.1	1.05654	1.16220
	2	a = 1, b = 1	1.48840	1.79244
	3	a = 1, b = 10	2.12263	2.18899
	4	a = 1, b = 100	2.24484	2.25233
1.20	1	a = 1, b = 0.1	0.71550	0.78705
	2	a = 1, b = 1	0.97325	1.03955
	3	a = 1, b = 10	1.09850	1.10701
	4	a = 1, b = 100	1.11339	1.11426

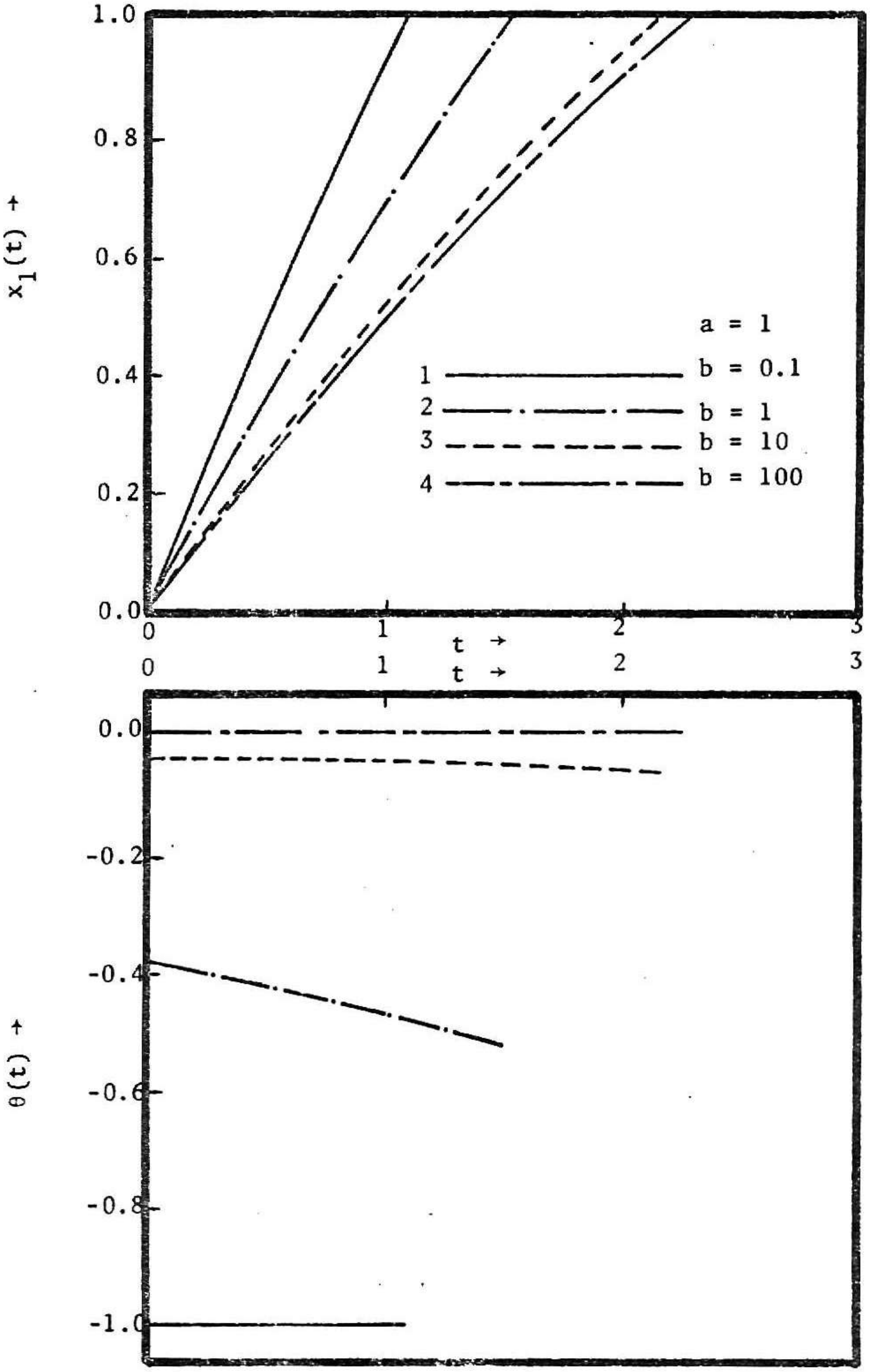

Fig. 6.2.1 Optimal control policy and system response of Example 2 (σ_s = 0.75)

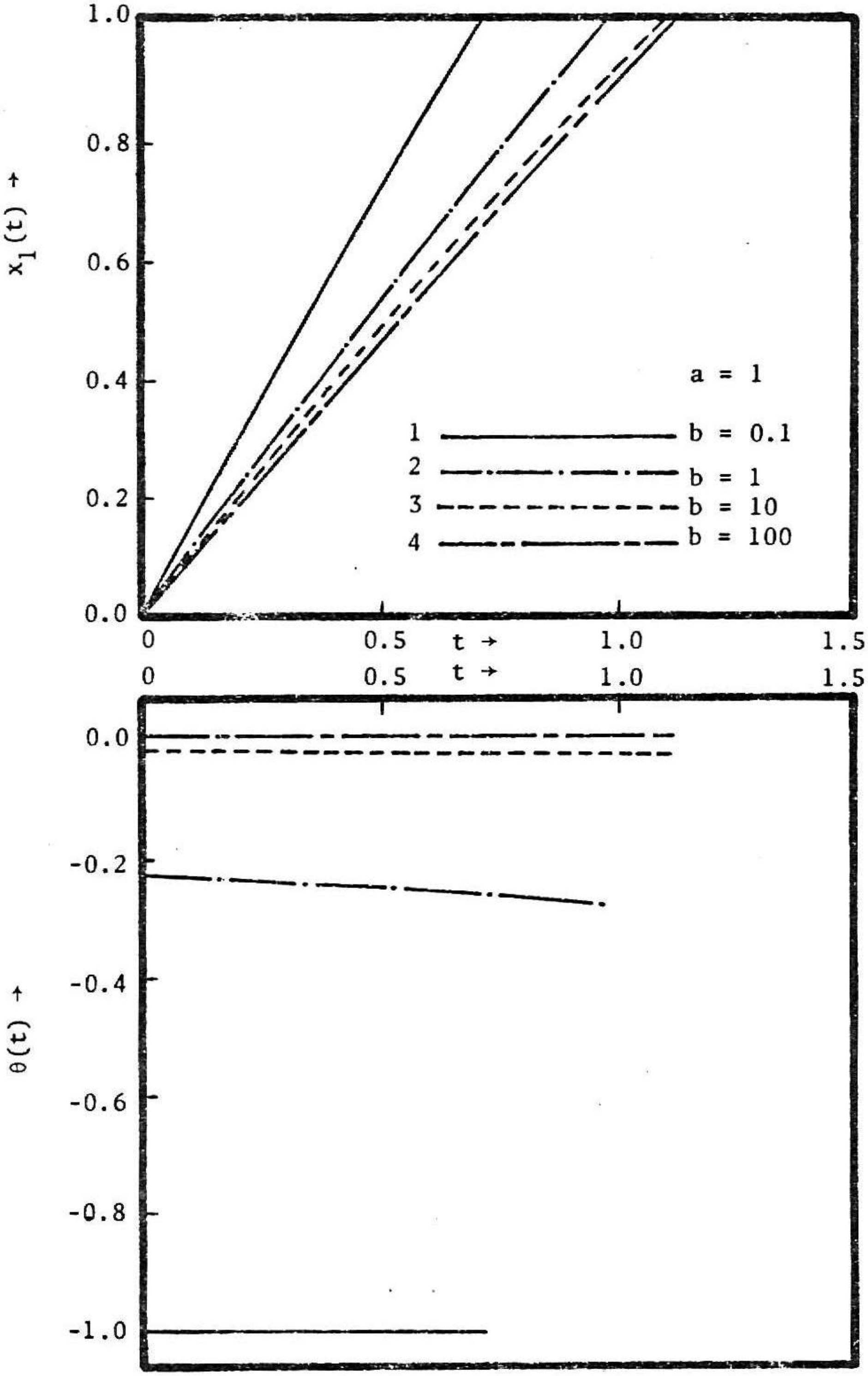

Fig. 6.2.2 Optimal control policy and system response of Example 2 (σ_s = 1.2)

6.4 EXAMPLE 3: Minimal Integrated Effort and Integrated Error Control

Let a life support system consisting of an air-conditioned room or cabin and a heat exchanger of negligibly small time constant ($\tau_2 = 0$) be subjected to a step heat disturbance.

The performance equation of such a system is written as

$$\frac{dx_1}{dt} + r_2 x_1 = r_2 K_\alpha - r_1 K_\beta \theta - r_1 K_\gamma + \sigma_s \tag{3.1}$$

The initial condition is

$$x_1(0) = \alpha_0$$

It is required to find an optimal control that will bring the system to the desired condition in an unspecified time interval T. That is, the final (desired) condition is

$$x_1(T) = 1 \ , \qquad T \text{ - not specified.}$$

At the same time it is also required to minimize the functional (objective function)

$$S = \int_0^T [b\theta^2 + c(x_1 - x_{1d})^2]dt \tag{3.2}$$

where b and c are weighting factors. The desired state, x_{1d}, is equal to one.

The objective function is obtained by combining the integrated control effort to bring the state of the system to the desired state and the integrated deviation from the desired state over an unspecified control time.

The control variable is constrained as

$$-1 \leq \theta \leq +1$$

This problem as stated has the initial and final values of the state vector function $x_1(0)$ and $x_1(T)$ fixed but the final time T to be determined.

Introducing another state variable $x_2(t)$ such that

$$x_2(t) = \int_0^t [b\theta^2 + c(x_1 - 1)^2]dt$$

it follows that

$$\frac{dx_2}{dt} = b\theta^2 + c(x_1 - 1)^2 , \qquad x_2(0) = 0 \tag{3.3}$$

The problem is thus transformed into that of minimizing $x_2(T)$.

According to Pontryagin's maximum principle, the Hamiltonian is

$$H = z_1(-r_2x_1 + r_2K_\alpha - r_1K_\beta\theta - r_1K_\gamma + \sigma_s) + z_2[b\theta^2 + c(x_1 - 1)^2] \tag{3.4}$$

The adjoint variables are defined by

$$\frac{dz_1}{dt} = -\frac{\partial H}{\partial x_1} = r_2z_1 - 2z_2c(x_1 - 1) \tag{3.5}$$

$$\frac{dz_2}{dt} = -\frac{\partial H}{\partial x_2} = 0 , \qquad z_2(T) = 1 \tag{3.6}$$

From equation (3.6), the solution of z_2 is obtained as

$$z_2(t) = 1 , \qquad 0 \leq t \leq T \tag{3.7}$$

Hence the Hamiltonian can now be written as

$$H = z_1(-r_2x_1 + r_2K_\alpha - r_1K_\beta\theta - r_1K_\gamma + \sigma_s) + b\theta^2 + c(x_1 - 1)^2 \tag{3.8}$$

The variable portion of H that depends on θ, H^*, is

$$H^* = -r_1K_\beta z_1\theta + b\theta^2$$

Inspection of H* shows that the optimal control is of continuous type and is obtained from the following necessary condition for optimality.

$$\frac{\partial H^*}{\partial \theta} = 0 = - r_1 K_\beta z_1 + 2b\theta \tag{3.9}$$

which gives

$$\theta = \frac{r_1 K_\beta}{2b} z_1 \tag{3.10}$$

Using this relationship in equation (3.1) and eliminating x_1 from equations (3.1) and (3.5) yields

$$\frac{d^2 z_1}{dt} - [r_2^2 + \frac{c}{b} (r_1 K_\beta)^2] z_1 = 2c(r_2 - r_2 K_\alpha + r_1 K_\gamma - \sigma_s)$$

The solution of this equation is

$$z_1 = A_1 e^{\lambda t} + A_2 e^{-\lambda t} + K \tag{3.11}$$

where

$$\lambda = \sqrt{r_2^2 + \frac{c}{b}(r_1 K_\beta)^2}$$

$$K = \frac{2c(r_2 K_\alpha - r_1 K_\gamma + \sigma_s - r_2)}{r_2^2 + \frac{c}{b}(r_1 K_\beta)^2}$$

and A_1 and A_2 are constants of integration.

The solution of x_1 can be obtained from equations (3.5) and (3.11) as

$$x_1(t) = \frac{1}{2c}[(r_2 - \lambda)A_1 e^{\lambda t} + (r_2 + \lambda)A_2 e^{-\lambda t} + r_2 K] + 1 \tag{3.12}$$

Now employing the initial condition, $x_1(0) = \alpha_0$ and the final condition, $x_1(T) = 1$ in equation (3.12) gives

$$(r_2 - \lambda)A_1 + (r_2 + \lambda)A_2 + r_2K = 2c(\alpha_0 - 1) \tag{3.13}$$

and

$$(r_2 - \lambda)A_1e^{\lambda T} + (r_2 + \lambda)A_2e^{-\lambda T} + r_2K = 0 \tag{3.14}$$

respectively.

Now the minimum value of H is,

$$H = \frac{2b\theta}{r_1K_\beta}(-r_2x_1 + r_2K_\alpha - r_1K_\beta\theta - r_1K_\gamma + \sigma_s) + b\theta^2 + c(x_1 - 1)^2 = 0 \ ,$$

$$0 \leq t \leq T$$

That is,

$$H = -b\theta^2 + \frac{2b}{r_1K_\beta}(-r_2x_1 + r_2K_\alpha - r_1K_\gamma + \sigma_s)\theta + c(x_1 - 1)^2 = 0 \ ,$$

$$0 \leq t \leq T \tag{3.15}$$

Application of the boundary condition at t = 0, that is, $x_1(0) = \alpha_0$ to equation (3.15) gives

$$\theta(0) = \left[- \left\{\frac{2b}{r_1K_\beta}\right\} (-r_2\alpha_0 + r_2K_\alpha - r_1K_\gamma + \sigma_s) + \sqrt{\frac{4b^2}{r_1^2K_\beta^2} (-r_2\alpha_0 + r_2K_\alpha - r_1K_\gamma + \sigma_s)^2 + 4bc(\alpha_0 - 1)^2}\right] / (-2b) \tag{3.16}$$

Using equation (3.11) in equation (3.10) for time t = 0, yields

$$\theta(0) = \frac{r_1K_\beta}{2b} (A_1 + A_2 + K) \tag{3.17}$$

Combining equations (3.16) and (3.17) gives

$$A_1 + A_2 + K = \left[-\left\{\frac{2b}{r_1 K_\beta}\right\} (-r_2\alpha_0 + r_2K_\alpha - r_1K_\gamma + \sigma_s) + \sqrt{\frac{4b^2}{r_1^2K_\beta^2} (-r_2\alpha_0 + r_2K_\alpha - r_1K_\gamma + \sigma_s)^2 + 4bc(\alpha_0 - 1)^2}\right] / (-r_1K_\beta) \quad (3.18)$$

The constants A_1 and A_2 can be determined from equations (3.13) and (3.18) as

$$A_1 = \left\{\frac{1}{2\lambda}\right\} \left\{ r_2K - 2c(\alpha_0 - 1) - K(r_2 + \lambda) + (r_2 + \lambda)\left[-\left(\frac{2b}{r_1K_\beta}\right)(-r_2\alpha_0 + r_2K_\alpha - r_1K_\gamma + \sigma_s) + \sqrt{\frac{4b^2}{r_1^2K_\beta^2} (-r_2\alpha_0 + r_2K_\alpha - r_1K_\gamma + \sigma_s)^2 + 4bc(\alpha_0 - 1)^2}\right] / (-r_1K_\beta) \right\}$$

$$A_2 = \left\{\frac{1}{2\lambda}\right\} \left\{ -r_2K + 2c(\alpha_0 - 1) + K(r_2 - \lambda) - (r_2 - \lambda)\left[-\left(\frac{2b}{r_1K_\beta}\right)(-r_2\alpha_0 + r_2K_\alpha - r_1K_\gamma + \sigma_s) + \sqrt{\frac{4b^2}{r_1^2K_\beta^2} (-r_2\alpha_0 + r_2K_\alpha - r_1K_\gamma + \sigma_s)^2 + 4bc(\alpha_0 - 1)^2}\right] / (-r_1K_\beta) \right\}$$

The optimal control policy can now be determined from equations (3.10) and (3.11) as

$$\theta(t) = \frac{r_1K_\beta}{2b} (A_1e^{\lambda t} + A_2e^{-\lambda t} + K) \quad (3.19)$$

The final time is found from equation (3.14) as

$$T = \frac{1}{\lambda}\,\ell n \left\{ \frac{-r_2K + \sqrt{(r_2K)^2 - 4(r_2^2-\lambda^2)A_1A_2}}{2(r_2-\lambda)A_1} \right\} \tag{3.20}$$

The objective function becomes

$$S = x_2(T) = \int_0^T [b\theta^2 + c(x_1 - 1)^2]dt$$

$$= \int_0^T \left[b \left\{ \frac{r_1K_\beta}{2b} (A_1e^{\lambda t} + A_2e^{-\lambda t} + K) \right\}^2 \right.$$

$$\left. + c \left\{ \frac{1}{2c}[(r_2 - \lambda)A_1e^{\lambda t} + (r_2 + \lambda)A_2e^{-\lambda t} + r_2K] \right\}^2 \right] dt$$

$$= \frac{r_1^2K_\beta^2}{4b} \left\{ \frac{A_1^2}{2\lambda} (e^{2\lambda T} - 1) - \frac{A_2^2}{2\lambda} (e^{-2\lambda T} - 1) + K^2T \right.$$

$$\left. + 2A_1A_2T - \frac{2KA_2}{\lambda}(e^{-\lambda T} - 1) + \frac{2KA_1}{\lambda}(e^{\lambda T} - 1) \right\}$$

$$+ \frac{1}{4c} \left\{ \frac{(r_2-\lambda)^2A_1^2}{2\lambda}(e^{2\lambda T} - 1) - \frac{(r_2+\lambda)^2A_2^2}{2\lambda}(e^{-2\lambda T} - 1) \right.$$

$$+ r_2^2K^2T + 2(r_2^2 - \lambda^2)A_1A_2T$$

$$\left. - \frac{2(r_2-\lambda)r_2KA_2}{\lambda}(e^{-\lambda T} - 1) + \frac{2(r_2-\lambda)r_2KA_1}{\lambda}(e^{\lambda T} - 1) \right\} \tag{3.20a}$$

In cases where the weighting factor c is appreciably larger compared to the weighting factor b, the optimal control given by equation (3.19) will violate the constraint $|\theta| \leq 1$. Therefore the condition

$$\frac{\partial H^*}{\partial \theta} = 0$$

does not yield the admissible control action. The optimal control for $0 \leq t \leq t_s$ is

$$\theta = -1$$

where t_s is the time when the saturation period ends, which is to be determined. With $\theta = -1$, equation (3.1) can be integrated as

$$x_1(t) = B_1 e^{-r_2 t} + K' , \qquad 0 \leq t \leq t_s \tag{3.21}$$

where

$$K' = \frac{r_2 K_\alpha + r_1 K_\beta - r_1 K_\gamma + \sigma_s}{r_2}$$

The constant of integration, B_1, can be determined by employing the initial condition in equation (3.21) which yields

$$B_1 = \alpha_0 - K'$$

With the solution of $x_1(t)$ given by equation (3.21), equation (3.5) can be integrated as

$$z_1(t) = B_2 e^{r_2 t} + \frac{cB_1}{r_2} e^{-r_2 t} + \frac{2c(K'-1)}{r_2} , \qquad 0 \leq t \leq t_s \tag{3.22}$$

where B_2 is the integration constant to be determined later.

After time t_s the control action will no longer be saturated and thus the condition

$$\frac{\partial H^*}{\partial \theta} = 0$$

can be used to determine the optimal condition. Thus the optimal control, optimal state, and adjoint variable given by equations (3.19), (3.12) and (3.11) can be used for $t_s \leq t \leq T$. The constants A_1 and A_2 in equations (3.11) and (3.12) can be determined by using the fact that x_1 and z_1 are continuous with respect to t. Hence at $t = t_s$.

$$z_1(t_s) = A_1 e^{\lambda t_s} + A_2 e^{-\lambda t_s} + K$$

$$= B_2 e^{r_2 t_s} + \frac{cB_1}{r_2} e^{-r_2 t_s} + \frac{2c(K'-1)}{r_2} \tag{3.23}$$

$$x_1(t_s) = \frac{1}{2c}[(r_2 - \lambda)A_1 e^{\lambda t_s} + (r_2 + \lambda)A_2 e^{-\lambda t_s} + r_2 K] + 1$$

$$= B_1 e^{-r_2 t_s} + K' \tag{3.24}$$

Also at $t = t_s$, $\theta = -1$. Hence from equation (3.19), we have

$$\frac{r_1 K_\beta}{2b} (A_1 e^{\lambda t_s} + A_2 e^{-\lambda t_s} + K) = -1 \tag{3.25}$$

Application of the boundary condition at $t = T$, that is $x_1(T) = 1$ to the minimum H equation, namely equation (3.15) gives

$$\left.\begin{aligned} &\theta(T) = 0 \\ \text{or} \\ &\theta(T) = \left\{\frac{2}{r_1 K_\beta}\right\} (-r_2 + r_2 K_\alpha - r_1 K_\gamma + \sigma_s) \end{aligned}\right\} \tag{3.26}$$

Using equation (3.11) in equation (3.12) for time $t = T$ yields

$$\theta(T) = \frac{r_1 K}{2b} (A_1 e^{\lambda T} + A_2 e^{-\lambda T} + K) \tag{3.27}$$

Combining equations (3.26) and (3.27) gives

$$A_1 e^{\lambda T} + A_2 e^{-\lambda T} + K = \frac{4b}{(r_1 K_\beta)^2} (-r_2 + r_2 K_\alpha - r_1 K_\gamma + \sigma_s) \tag{3.28}$$

There are five unknowns A_1, A_2, B_2, t_s, and T in equations (3.14), (3.23), (3.24), (3.25), and (3.28). These equations can be solved simultaneously using a search technique to determine these constants.

Then the optimal policy and response are summarized as follows:

$$\theta = -1, \qquad 0 \leq t \leq t_s \tag{3.29a}$$

$$x_1 = B_1 e^{-r_2 t} + K', \qquad 0 \leq t \leq t_s \tag{3.29b}$$

$$\theta = \frac{r_1 K_\beta}{2b} (A_1 e^{\lambda t} + A_2 e^{-\lambda t} + K), \qquad t_s \leq t \leq T \tag{3.29c}$$

$$x_1 = \frac{1}{2c} \{(r_2 - \lambda) A_1 e^{\lambda t} + (r_2 + \lambda) A_2 e^{-\lambda t} + r_2 K\} + 1,$$

$$t_s \leq t \leq T \tag{3.29d}$$

Now the objective function becomes

$$S = x_2(T) = \int_0^T [b\theta^2 + c(x_1 - 1)^2]\, dt$$

$$= \int_0^{t_s} \{b + c[B_1 e^{-r_2 t} + (K' - 1)]^2\}\, dt$$

$$+ \int_{t_s}^T [b\{\frac{r_1 K_\beta}{2b} (A_1 e^{\lambda t} + A_2 e^{-\lambda t} + K)\}^2$$

$$+ c\, \{\frac{1}{2c} [(r_2 - \lambda) A_1 e^{\lambda t} + (r_2 + \lambda) A_2 e^{-\lambda t} + r_2 K]\}^2]\, dt$$

$$= bt_s + c\{\frac{B_1^2}{2r_2} (1 - e^{-2r_2 t_s}) + \frac{2B_1}{r_2} (K'-1)(1 - e^{-r_2 t_s}) + (K'-1)^2 t_s\}$$

$$+ \frac{r_1^2 K_\beta^2}{4b} \{\frac{A_1^2}{2\lambda} (e^{2\lambda T} - e^{2\lambda t_s}) - \frac{A_2^2}{2\lambda} (e^{-2\lambda T} - e^{-2\lambda t_s}) + K^2 (T - t_s)$$

$$+ 2A_1A_2(T-t_s) - \frac{2A_2K}{\lambda}(e^{-\lambda T} - e^{-\lambda t_s})$$

$$+ \frac{2A_1K}{\lambda}(e^{\lambda T} - e^{\lambda t_s})\}$$

$$+ \frac{1}{4c}\{\frac{(r_2-\lambda)^2A_1^2}{2\lambda}(e^{2\lambda T} - e^{2\lambda t_s}) - \frac{(r_2+\lambda)^2A_2^2}{2\lambda}(e^{-2\lambda T} - e^{-2\lambda t_s})$$

$$+ r_2^2K^2\ (T-t_s) + 2(r_2^2-\lambda^2)A_1A_2(T-t_s)$$

$$- \frac{2(r_2+\lambda)r_2KA_2}{\lambda}(e^{-\lambda T} - e^{-\lambda t_s})$$

$$+ \frac{2(r_2-\lambda)r_2KA_1}{\lambda}(e^{\lambda T} - e^{\lambda T_s})\} \tag{3.30}$$

The following four cases are considered in this numerical example:

case 1: $b = 1, \quad c = 0.1$

case 2: $b = 1, \quad c = 1$

case 3: $b = 1, \quad c = 10$

case 4: $b = 1, \quad c = 100$

The values of the various constants used are:

$r_1 = 0.8 \qquad K_\alpha = 1.5$

$r_2 = 0.2 \qquad K_\beta = K_\gamma = 0.625$

$\alpha_0 = 0$

The results of this example for $\sigma_s = 0.75$ and $\sigma_s = 1.2$ are shown in Table 6.3.1 and Figs. 6.3.1, 6.3.2.

The control time, required to reach the final desired state, decreases with increase in the weighting factor c on the deviation. Also when the weighting factor c takes the values as 10 and 100 which are appreciably larger compared to the weighting factor b, there results a switching time t_s upto which point the heat exchanger operates at its limiting capacity (here $\theta = -1$). Since this is a heating system, the greater the value of σ_s, the lesser is the control time required to reach the final desired state.

The value of the objective function is minimum for the lowest value of the weighting factor c on the deviation and the highest value of σ_s.

In case 1, where more weight is given to the control effort, θ, the optimal control policy has a very small negative value. In cases 3 and 4, where more weight is given to the deviation of x_1 from the desired value, the optimal control θ has an approximately constant value that is required to maintain the system in the desired state, i.e., $x_1 = 1$. Case 2 is an intermediate one.

Table 6.3.1

Optimal Solutions of Example 3:

σ_s	Case Number	Weighting Factors	Switching Time t_s	Final Time T	Value of the Objective Function
0.75	1	b = 1, c = 0.1	-	2.22645	0.06620
	2	b = 1, c = 1	-	2.00340	0.59922
	3	b = 1, c = 10	0.26897	1.23861	3.13482
	4	b = 1, c = 100	0.34994	0.65998	24.53221
1.20	1	b = 1, c = 0.1	-	1.11085	0.03501
	2	b = 1, c = 1	-	1.07151	0.33871
	3	b = 1, c = 10	0.21812	0.89242	2.80417
	4	b = 1, c = 100	0.28000	0.50000	18.67151

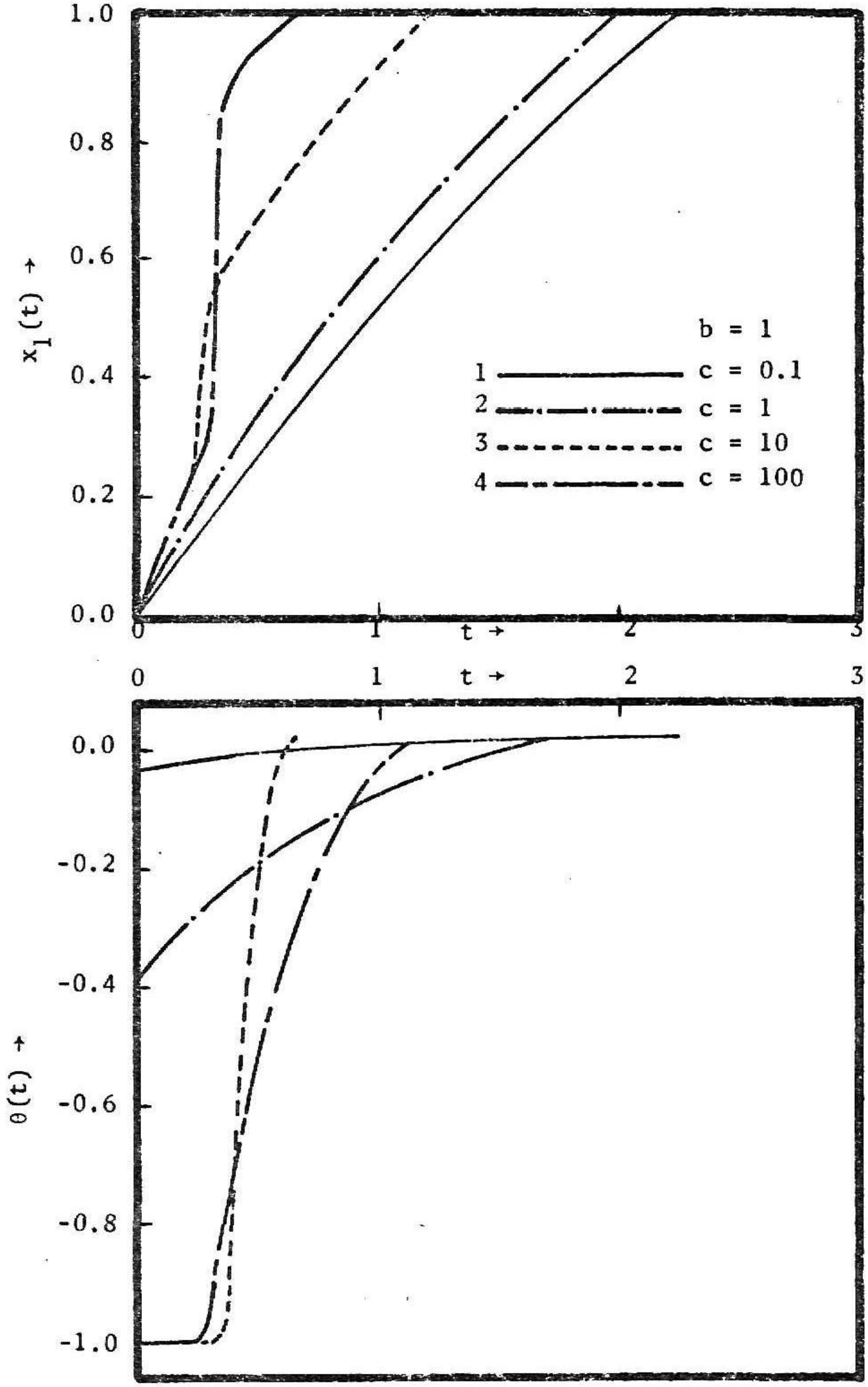

Fig. 6.3.1 Optional control policy and system response of Example 3 (σ_s = 0.75)

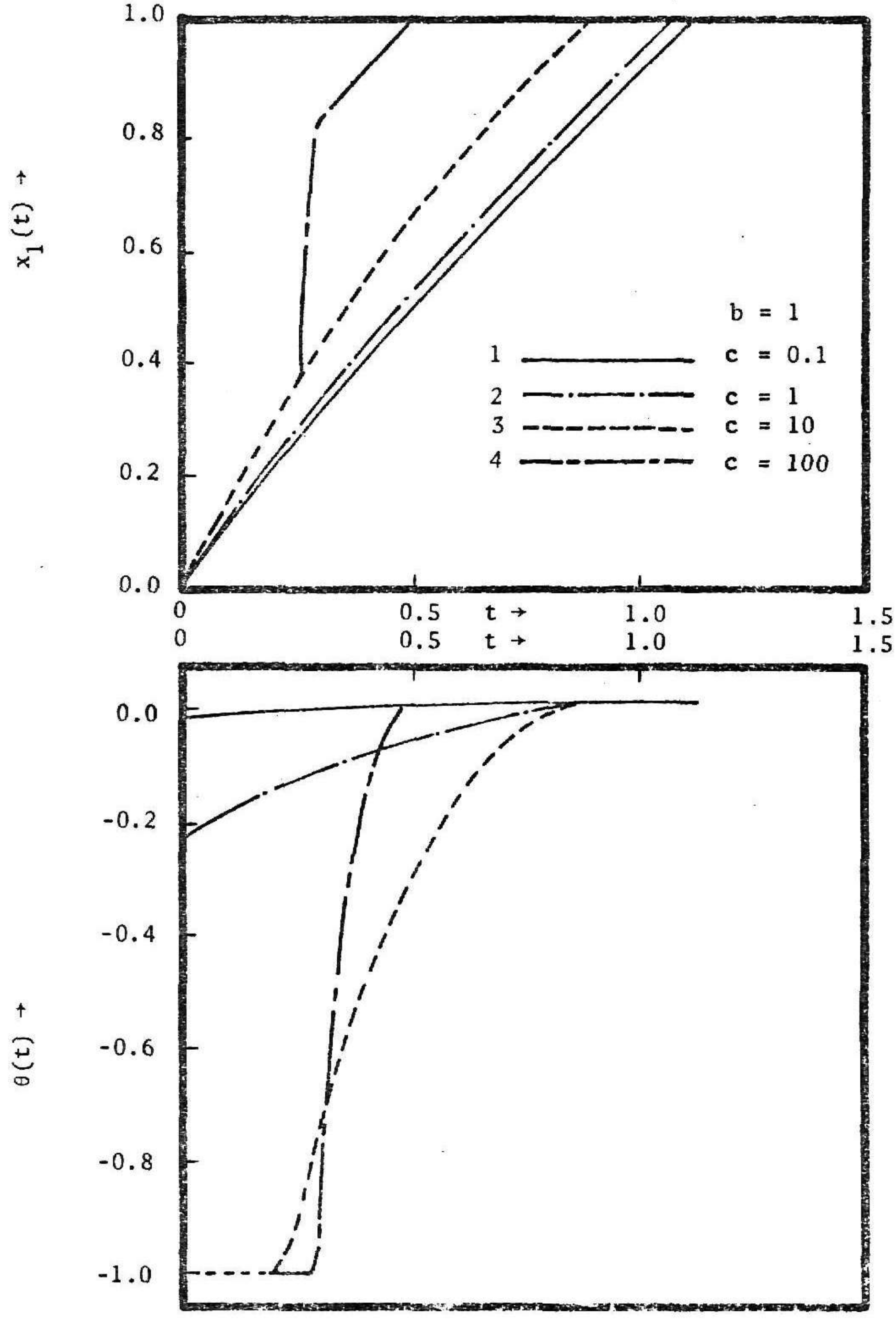

Fig. 6.3.2 Optimal control policy and system response of Example 3 (σ_s = 1.2)

6.5 Example 4: Minimal Time, Integrated Effort and Integrated Error Control

Let a life support system consisting of an air-conditioned room or cabin and a heat exchanger of negligibly small time constant ($\tau_2 = 0$) be subjected to a step heat disturbance.

The performance equation of such a system is written as

$$\frac{dx_1}{dt} + r_2 x_1 = r_2 K_\alpha - r_1 K_\beta \theta - r_1 K_\gamma + \sigma_s \tag{4.1}$$

The initial condition is

$$x_1(0) = \alpha_0$$

It is required to find an optimal control that will bring the system to the desired condition in an unspecified time interval T. That is, the final (desired) condition is

$$x_1(T) = 1 \quad , \qquad\qquad T \text{ - not specified.}$$

At the same time it is also required to minimize the functional (objective function)

$$S = \int_0^T [a + b\theta^2 + c(x_1 - x_{1d})^2]dt \tag{4.2}$$

where a, b, and c are weighting factors. The desired state, x_{1d}, is equal to one.

The objective function is obtained by combining the unspecified control time, the integrated control effort to bring the state of the system to the desired state and the integrated deviation from the desired state over the unspecified control time.

The control variable is constrained as

$$-1 \leq \theta \leq +1$$

This problem as stated has the initial and final values of the state vector function $x_1(0)$ and $x_1(T)$ fixed but the final time T to be determined.

Introducing another state variable $x_2(t)$ such that

$$x_2(t) = \int_0^t [a + b\theta^2 + c(x_1-1)^2]dt$$

it follows that

$$\frac{dx_2}{dt} = a + b\theta^2 + c(x_1-1)^2 \quad , \qquad x_2(0) = 0 \tag{4.3}$$

The problem is thus transformed into that of minimizing $x_2(T)$.

According to Pontryagin's maximum principle, the Hamiltonian is

$$H = z_1(-r_2x_1+r_2K_\alpha-r_1K_\beta\theta-r_1K_\gamma+\sigma_s) + z_2[a + b\theta^2 + c(x_1-1)^2] \tag{4.4}$$

The adjoint variables are defined by

$$\frac{dz_1}{dt} = -\frac{\partial H}{\partial x_1} = r_2z_1 - 2z_2c(x_1-1) \tag{4.5}$$

$$\frac{dz_2}{dt} = -\frac{\partial H}{\partial x_2} = 0 \quad , \qquad z_2(T) = 1 \tag{4.5}$$

From equation (4.6), the solution of z_2 is obtained as

$$z_2(t) = 1 \quad , \qquad 0 \leq t \leq T \tag{4.7}$$

Hence the Hamiltonian can now be written as

$$H = z_1(-r_2x_1+r_2K_\alpha-r_1K_\beta\theta-r_1K_\gamma+\sigma_s) + a + b\theta^2 + c(x_1-1)^2 \tag{4.8}$$

The variable portion of H that depends on θ, H*, is

$$H^* = -r_1K_\beta z_1\theta + b\theta^2$$

Inspection of H* shows that the optimal control is of continuous type and is obtained from the following necessary condition for optimality.

$$\frac{\partial H^*}{\partial\theta} = 0 = -r_1K_\beta z_1 + 2b\theta \tag{4.9}$$

which gives

$$\theta = \frac{r_1K_\beta}{2b} z_1 \tag{4.10}$$

Using this relationship in equation (4.1) and eliminating x_1 from equations (4.1) and (4.5) yields

$$\frac{d^2z_1}{dt^2} - [r_2^2 + \frac{c}{b}(r_1K_\beta)^2]z_1 = 2c(r_2-r_2K_\alpha+r_1K_\gamma-\sigma_s)$$

The solution of this equation is

$$z_1 = A_1e^{\lambda t} + A_2e^{-\lambda t} + K \tag{4.11}$$

where

$$\lambda = \sqrt{r_2^2 + \frac{c}{b}(r_1K_\beta)^2}$$

$$K = \frac{2c(r_2K_\alpha-r_1K_\gamma+\sigma_s-r_2)}{r_2^2 + \frac{c}{b}(r_1K_\beta)^2}$$

and A_1 and A_2 are constants of integration.

The solution of x_1 can be obtained from equations (4.5) and (4.11) as

$$x_1(t) = \frac{1}{2c}[(r_2-\lambda)A_1e^{\lambda t} + (r_2+\lambda)A_2e^{-\lambda t} + r_2K] + 1 \tag{4.12}$$

Now employing the initial condition, $x_1(0) = \alpha_0$ and the final condition, $x_1(T) = 1$ in equation (4.12) gives

$$(r_2-\lambda)A_1 + (r_2+\lambda)A^2 + r_2K = 2c(\alpha_0-1) \tag{4.13}$$

and

$$(r_2-\lambda)A_1e^{\lambda T} + (r_2+\lambda)A_2e^{-\lambda T} + r_2K = 0 \tag{4.14}$$

respectively.

Now the minimum value of H is,

$$H = \frac{2b\theta}{r_1K_\beta}(-r_2x_1+r_2K_\alpha-r_1K_\beta\theta-r_1K_\gamma+\sigma_s) + a + b\theta^2 + c(x_1-1)^2 = 0,$$
$$0 \leq t \leq T$$

That is,

$$H = -b\theta^2 + \frac{2b}{r_1K_\beta}(-r_2x_1+r_2K_\alpha-r_1K_\gamma+\sigma_s)\theta + c(x_1-1)^2 + a = 0 \quad ,$$
$$0 \leq t \leq T \tag{4.15}$$

Application of the boundary condition at t = 0, that is, $x_1(0) = \alpha_0$ to equation (4.15) gives

$$\theta(0) = \left[-\left\{\frac{2b}{r_1K_\beta}\right\}(-r_2\alpha_0+r_2K_\alpha-r_1K_\gamma+\sigma_s) + \right.$$

$$\left. \sqrt{\frac{4b^2}{r_1^2K_\beta^2}(-r_2\alpha_0+r_2K_\alpha-r_1K_\gamma+\sigma_s)^2 + 4b\{c(\alpha_0-1)^2 + a\}} \right] \Big/ (-2b) \quad (4.$$

Using equation (4.11) in equation (4.10) for time t = 0 yields

$$\theta(0) = \frac{r_1K_\beta}{2b}(A_1+A_2+K) \quad (4.$$

Combining equations (4.16) and (4.17) gives

$$A_1 + A_2 + k = \left[-\left\{\frac{2b}{r_1K_\beta}\right\}(-r_2\alpha_0+r_2K_\alpha-r_1K_\gamma+\sigma_s) + \right.$$

$$\left. \sqrt{\frac{4b^2}{r_1^2K_\beta^2}(-r_2\alpha_0+r_2K_\alpha-r_1K_\gamma+\sigma_s)^2 + 4b[c(\alpha_0-1)^2 + a]} \right] \Big/ (-r_1K_\beta) \quad (4.$$

The constants A_1 and A_2 can be determined from equations (4.13) and (4.18) as

$$A_1 = \left\{\frac{1}{2\lambda}\right\}\left\{ r_2K - 2c(\alpha_0-1) - K(r_2+\lambda) + \right.$$

$$(r_2+\lambda)\left[-\left\{\frac{2b}{r_1K_\beta}\right\}(-r_2\alpha_0+r_2K_\alpha-r_1K_\gamma+\sigma_s) + \right.$$

$$\left.\left. \sqrt{\frac{4b^2}{r_1^2K_\beta^2}(-r_2\alpha_0+r_2K_\alpha-r_1K_\gamma+\sigma_s)^2 + 4b[c(\alpha_0-1)^2 + a]} \right] \Big/ (-r_1K) \right\}$$

$$A_2 = \left\{\frac{1}{2\lambda}\right\}\left\{-r_2 K + 2c(\alpha_0 - 1) + K(r_2 - \lambda) - \right.$$

$$(r_2 - \lambda)\left[- \left\{\frac{2b}{r_1 K_\beta}\right\}(-r_2\alpha_0 + r_2 K_\alpha - r_1 K_\gamma + \sigma_s) + \right.$$

$$\left.\left.\sqrt{\frac{4b^2}{r_1^2 K_\beta^2}(-r_2\alpha_0 + r_2 K_\alpha - r_1 K_\gamma + \sigma_s)^2 + 4b[c(\alpha_0 - 1)^2 + a]}\right] \Big/ (-r_1 K_\beta)\right\}$$

The optimal control policy can now be determined from equations (4.10) and (4.11) as

$$\theta(t) = \frac{r_1 K_\beta}{2b}(A_1 e^{\lambda t} + A_2 e^{-\lambda t} + K) \tag{4.19}$$

The final time is found from equation (4.14) as

$$T = \frac{1}{\lambda} \ln \left\{\frac{-r_2 K + \sqrt{(r_2 K)^2 - 4(r_2^2 - \lambda^2)A_1 A_2}}{2(r_2 - \lambda)A_1}\right\} \tag{4.20}$$

The objective function becomes

$$S = x_2(T) = \int_0^T [a + b\theta^2 + c(x_1 - 1)^2]dt$$

$$= \int_0^T \left\{ a + b\left\{\frac{r_1 K_\beta}{2b}(A_1 e^{\lambda t} + A_2 e^{-\lambda t} + K)\right\}^2 \right.$$

$$\left. + c\left\{\frac{1}{2c}[(r_2 - \lambda)A_1 e^{\lambda t} + (r_2 + \lambda)A_2 e^{-\lambda t} + r_2 K]\right\}^2 \right\} dt$$

$$= aT + \frac{r_1^2 K_\beta^2}{4b}\left\{\frac{A_1^2}{2\lambda}(e^{2\lambda T}-1) - \frac{A_2^2}{2\lambda}(e^{-2\lambda T}-1) + K^2 T\right.$$

$$\left. + 2A_1A_2T - \frac{2KA_2}{\lambda}(e^{-\lambda T}-1) + \frac{2KA_1}{\lambda}(e^{\lambda T}-1)\right\}$$

$$+ \frac{1}{4c}\left\{\frac{(r_2-\lambda)^2A_1^2}{2\lambda}(e^{2\lambda T}-1) - \frac{(r_2+\lambda)^2A_2^2}{2\lambda}(e^{-2\lambda T}-1)\right.$$

$$+ r_2^2K^2T + 2(r_2^2-\lambda^2)A_1A_2T$$

$$- \frac{2(r_2-\lambda)r_2KA_2}{\lambda}(e^{-\lambda T}-1)$$

$$\left. + \frac{2(r_2-\lambda)r_2KA_1}{\lambda}(e^{\lambda T}-1)\right\} \qquad (4.20a)$$

In cases where the weighting factor c is appreciably larger compared to the weighting factor b, the optimal control given by eqution (4.19) will violate the constraint $|\theta| \leq 1$. Therefore the condition

$$\frac{\partial H^*}{\partial \theta} = 0$$

does not yield the admissible control action. The optimal control for $0 \leq t \leq t_s$ is

$$\theta = -1$$

where t_s is the time when the saturation period ends, which is to be determined. With $\theta = -1$, equation (4.1) can be integrated as

$$x_1(t) = B_1 e^{-r_2 t} + K' \quad , \qquad 0 \leq t \leq t_s \qquad (4.21)$$

where

$$K' = \frac{r_2 K_\alpha + r_1 K_\beta - r_1 K_\gamma + \sigma_s}{r_2}$$

The constant of integration, B_1, can be determined by employing the initial condition in equation (4.21) which yields

$$B_1 = \alpha_0 - K'$$

With the solution of $x_1(t)$ given by equation (4.21), equation (4.5) can be integrated as

$$z_1(t) = B_2 e^{r_2 t} + \frac{cB_1}{r_2} e^{-r_2 t} + \frac{2c(K'-1)}{r_2} \quad , \qquad 0 \leq t \leq t_s \quad (4.22)$$

where B_2 is the integration constant to be determined later.

After time t_s the control action will no longer be saturated and thus the condition

$$\frac{\partial H^*}{\partial \theta} = 0$$

can be used to determine the optimal condition. Thus the optimal control, optimal state, and adjoint variable given by equations (4.19), (4.12), and (4.11) can be used for $t_s \leq t \leq T$. The constants A_1 and A_2 in equations (4.11) and (4.12) can be determined by using the fact that x_1 and z_2 are continuous with respect to t. Hence at $t = t_s$

$$\begin{aligned} z_1(t_s) &= A_1 e^{\lambda t_s} + A_2 e^{-\lambda t_s} + K \\ &= B_2 e^{r_2 t_s} + \frac{cB_1}{r_2} e^{-r_2 t_s} + \frac{2c(K'-1)}{r_2} \end{aligned} \quad (4.23)$$

$$x_1(t_s) = \frac{1}{2c}[(r_2-\lambda)A_1e^{\lambda t_s} + (r_2+\lambda)A_2e^{-\lambda t_s} + r_2K] + 1$$

$$= B_1e^{-r_2t_s} + K' \tag{4.24}$$

Also at $t = t_s$, $\theta = -1$. Hence from equation (4.19), we have

$$\frac{r_1K_\beta}{2b}(A_1e^{\lambda t_s}+A_2e^{-\lambda t_s}+K) = -1 \tag{4.25}$$

Application of the boundary condition at $t = T$, that is, $x_1(T) = 1$ to the minimum H equation, namely equation (4.15) gives

$$\theta(T) = \left[-\left\{\frac{2b}{r_1K_\beta}\right\}(-r_2+r_2K_\alpha-r_1K_\gamma+\sigma_s) + \sqrt{\frac{4b^2}{r_1^2K_\beta^2}(-r_2+r_2K_\alpha-r_1K_\gamma+\sigma_s)^2 + 4ba}\right] / (-2b) \tag{4.26}$$

Using equation (4.11) in equation (4.12) for time $t = T$ yields

$$\theta(T) = \frac{r_1K_\beta}{2b}(A_1e^{\lambda T} + A_2e^{-\lambda T} + K) \tag{4.27}$$

Combining equations (4.26) and (4.27) gives

$$A_1e^{\lambda T} + A_2e^{-\lambda T} + K = \left[-\left\{\frac{2b}{r_1K_\beta}\right\}(-r_2+r_2K_\alpha-r_1K_\gamma+\sigma_s) + \sqrt{\frac{4b^2}{r_1^2K_\beta^2}(-r_2+r_2K_\alpha-r_1K_\gamma+\sigma_s)^2 + 4ba}\right] / (-r_1K_\beta) \tag{4.28}$$

There are five unknowns A_1, A_2, B_2, t_s, and T in equations (4.14), (4.23), (4.24) (4.25), and (4.28). These equations can be solved simultaneously using a search technique to determine these constants.

Then the optimal policy and response are summarized as follows:

$$\theta = -1 \quad , \qquad 0 \le t \le t_s \qquad (4.29a)$$

$$x_1 = B_1 e^{-r_2 t} + K' \quad , \qquad 0 \le t \le t_s \qquad (4.29b)$$

$$\theta = \frac{r_1 K_\beta}{2b}(A_1 e^{\lambda t} + A_2 e^{-\lambda t} + K) \quad , \qquad t_s \le t \le T \qquad (4.29c)$$

$$x_1 = \frac{1}{2c}\left\{(r_2-\lambda)A_1 e^{\lambda t} + (r_2+\lambda)A_2 e^{-\lambda t} + r_2 K\right\} + 1 \quad ,$$

$$t_s \le t \le T \qquad (4.29d)$$

Now the objective function becomes

$$S = x_2(T) = \int_0^T [a + b\theta^2 + c(x_1-1)^2]dt$$

$$= \int_0^{t_s} \left\{a + b + c[B_1 e^{-r_2 t} + (K'-1)]^2\right\} dt$$

$$+ \int_{t_s}^{T} \left(a + b\left\{\frac{r_1 K_\beta}{2b} (A_1 e^{\lambda t} + A_2 e^{-\lambda t} + K)\right\}^2\right.$$

$$\left. + c\left\{\frac{1}{2c}[(r_2-\lambda)A_1 e^{\lambda t} + (r_2+\lambda)A_2 e^{-\lambda t} + r_2 K]\right\}^2\right) dt$$

$$= (a+b)t_s + c\left\{\frac{B_1^2}{2r_2}(1-e^{-2r_2t_s}) - \frac{2B_1}{r_2}(K'-1)(1-e^{-r_2t_s}) + (K'-1)^2t_s\right\}$$

$$+ a(T-t_s) + \frac{r_1^2K_\beta^2}{4b}\left\{\frac{A_1^2}{2}(e^{2\lambda T}-e^{2\lambda t_s}) - \frac{A_2^2}{2\lambda}(e^{-2\lambda T}-e^{-2\lambda t_s})\right.$$

$$+ K^2(T-t_s) + 2A_1A_2(T-t_s) - \frac{2A_2K}{\lambda}(e^{-\lambda T}-e^{-\lambda t_s})$$

$$\left. + \frac{2A_1K}{\lambda}(e^{\lambda T}-e^{\lambda t_s})\right\}$$

$$+ \frac{1}{4c}\left\{\frac{(r_2-\lambda)^2A_1^2}{2\lambda}(e^{2\lambda T}-e^{2\lambda t_s}) - \frac{(r_2+\lambda)^2A_2^2}{2\lambda}(e^{-2\lambda T}-e^{-2\lambda t_s})\right.$$

$$+ r_2^2K^2(T-t_s) + 2(r_2^2-\lambda^2)A_1A_2(T-t_s)$$

$$- \frac{2(r_2+\lambda)r_2KA_2}{\lambda}(e^{-\lambda T}-e^{-\lambda t_s})$$

$$\left. + \frac{2(r_2-\lambda)r_2KA_1}{\lambda}(e^{\lambda T}-e^{\lambda t_s})\right\} \tag{4.29e}$$

In cases where the weighting factor a is appreciably larger compared to the weighting factor b, the optimal control given by equation (4.19) will violate the constraint $|\theta| \leq 1$. Therefore the condition

$$\frac{\partial H^*}{\partial \theta} = 0$$

does not yield the admissible control action. The optimal control for $0 \leq t \leq T$ is

$$\theta = -1$$

where T is the final time to be determined. With $\theta = -1$, equation (4.1) can be integrated as

$$x_1(t) = B_1 e^{-r_2 t} + K' \quad , \qquad 0 \leq t \leq T \qquad (4.30)$$

where

$$K' = \frac{r_2 K_\alpha + r_1 K_\beta - r_1 K_\gamma + \sigma_s}{r_2}$$

The constant of integration, B_1, can be determined by employing the initial condition in equation (4.30) which yields

$$B_1 = \alpha_0 - K'$$

With the solution of $x_1(t)$ given by equation (4.30), equation (4.5) can be integrated as,

$$z_1(t) = B_2 e^{r_2 t} + \frac{cB_1}{r_2} e^{-r_2 t} + \frac{2c(K'-1)}{r_2} \quad , \qquad 0 \leq t \leq T \qquad (4.31)$$

The integration constant B_2 in equation (4.31) is determined by using the condition that minimum H is zero for all the process time.

Now the minimum value of H is,

$$H = z_1(-r_1 x_1 + r_2 K_\alpha + r_1 K_\beta - r_1 K_\gamma + \sigma_s) + a + b + c(x_1 - 1)^2 = 0 \quad , \quad 0 \leq t \leq T$$

That is

$$H = z_1(-r_2 x_1 + K' r_2) + a + b + c(x_1 - 1)^2 = 0 \quad , \qquad 0 \leq t \leq T \qquad (4.32)$$

Application of the boundary condition at $t = 0$, that is, $x_1(0) = \alpha_0$ to equation (4.32) gives

$$z_1(0) = -\frac{[a + b + c(\alpha_0 - 1)^2]}{r_2(K' - \alpha_0)} \tag{4.33}$$

Also from equation (4.31) at time $t = 0$, we have

$$z_1(0) = B_2 + \frac{cB_1}{r_2} + \frac{2c(K'-1)}{r_2} \tag{4.34}$$

Combining equations (4.33) and (4.34) yields

$$B_2 = -\frac{[a + b + c(\alpha_0 - 1)^2]}{r_2(K' - \alpha_0)} - \frac{cB_1}{r_2} - \frac{2c(K'-1)}{r_2}$$

The final control time can be obtained from the minimum H equation, namely equation (4.32) together with the condition

$$x_1(T) = 1$$

as follows:

$$H = z_1(T)\ [-r_2 x_1(T) + K' r_2] + a + b + c[x_1(T) - 1]^2 = 0$$

or solving for $z_1(T)$

$$z_1(T) = -\frac{(a+b)}{r_2(K'-1)} \tag{4.35}$$

Also we have from equation (4.31) at $t = T$

$$z_1(T) = B_2 e^{r_2 T} + \frac{cB_1}{r_2} e^{-r_2 T} + \frac{2c(K'-1)}{r_2} \tag{4.36}$$

Solving for T from equations (4.35) and (4.36), we have

$$T = \frac{1}{r_2} \ln \left\{ \frac{-\left(\frac{2c(K'-1)^2 + a + b}{r_2(K'-1)}\right) - \sqrt{\left(\frac{2c(K'-1)^2 + a + b}{r_2(K'-1)}\right)^2 - \frac{4cB_1B_2}{r_2}}}{2B_2} \right\} \quad (4.3$$

Then the optimal policy and response are summarized as follows:

$$\theta = -1 \quad , \qquad 0 \leq t \leq T \qquad (4.38a)$$

$$x_1 = B_1 e^{-r_2 t} + K' \quad , \qquad 0 \leq t \leq T \qquad (4.38b)$$

Now the objective function becomes

$$S = x_2(T) = \int_0^T [a + b\theta^2 + c(x_1-1)^2]dt$$

$$= \int_0^T [a + b + c(B_1 e^{-r_2 t} + K'-1)^2]dt$$

$$= (a+b)T + c\left\{\frac{B_1^2}{2r_2}(1-e^{-2r_2 T}) + \frac{2B_1}{r_2}(K'-1)(1-e^{-r_2 T}) + (K'-1)^2 T\right\} \qquad (4.39)$$

The following twelve cases are considered in this example:

Case Number	Weighting factors a	b	c	Case Number	Weighting factors a	b	c
1	0.1	1	1	7	1	10	1
2	1	1	1	8	1	100	1
3	10	1	1	9	1	1	0.1
4	100	1	1	10	1	1	1
5	1	0.1	1	11	1	1	10
6	1	1	1	12	1	1	100

Cases 2, 6 and 10 are one and the same; however they are repeated for convenience in the comparison of results.

The values of the various constnats used are:

$r_1 = 0.8$ $\qquad K_\alpha = 1.5$

$r_2 = 0.2$ $\qquad K_\beta = K_\gamma = 0.625$

$\alpha_0 = 0$

The results of this example for $\sigma_\beta = 0.75$ and $\sigma_\beta = 1.2$ are shown in Table 6.4.1 and Figures 6.4.1, 6.4.2, 6.4.3, 6.4.4, 6.4.5, 6.4.6.

The control time, required to reach the final desired state, decreases with increase in the weighting factor a on the control time while b and c are kept constant; increases with increase in the weighting factor b on the control effort while a and c are kept constant; and decreases with increase in the weighting factor c on the state deviation while a and b are kept constant. Also when the weighting factor c takes the values as 10 and 100 which are appreciably larger compared to the weighting factor b, there results a switching time t_s until when the heat exchanger operates at its limiting capacity (here $\theta = -1$). Again the greater the value of σ_s, the lesser is the control time required to reach the final desired state, this being the heating system.

The value of the objective function increases with the increase in the magnitude of the weighting factor under consideration while all other weighting factors are kept constant. With the highest value of σ_s and the lowest magnitude of the weighting factor a on the control time (b and c being kept constant), the objective function attains its minimum value.

In cases 3, 4 and 5, where more weight is given to the control time than to the control effort, the optimal control is of bang-bang type having only the bangpart. In case 8 where more weight is given to the control effort, θ, than to the state deviation, the optimal control has a very small negative value. In cases 11 and 12, where more weight is given to the deviation of x_1 from the desired value, the optimal control θ has an approximately constant value that is required to maintain the system in the desired state, i.e., $x_1 = 1$. All other cases are intermediate ones.

Table 6.4.1

Optimal Souutions of Example 4:

σ_s	Case Number	Weighting Factors a	b	c	Switching Time t_s	Final Time T	Value of the Objective Function
0.75	1	0.1	1	1		1.89786	0.79412
	2	1	1	1		1.38925	2.23503
	3	10	1	1		1.05653	11.95565
	4	100	1	1		1.05653	107.04370
	5	1	0.1	1		1.05633	1.79584
	6	1	1	1		1.38925	2.23503
	7	1	10	1		2.09380	2.81999
	8	1	100	1		2.24150	2.91905
	9	1	1	0.1		1.47677	1.83927
	10	1	1	1		1.38925	2.23503
	11	1	1	10	0.51316	1.08976	5.08517
	12	1	1	100	0.37997	0.55558	26.21930
1.20	1	0.1	1	1		1.05604	0.44508
	2	1	1	1		0.94215	1.34126
	3	10	1	1		0.71551	8.10081
	4	100	1	1		0.71551	72.49586
	5	1	0.1	1		0.71545	1.01710
	6	1	1	1		0.94215	1.34126
	7	1	10	1		1.09388	1.45237
	8	1	100	1		1.11344	1.46562
	9	1	1	0.1		0.96989	1.07054
	10	1	1	1		0.94215	1.34126
	11	1	1	10	0.25078	0.79997	3.51820
	12	1	1	100	0.29996	0.46450	19.67662

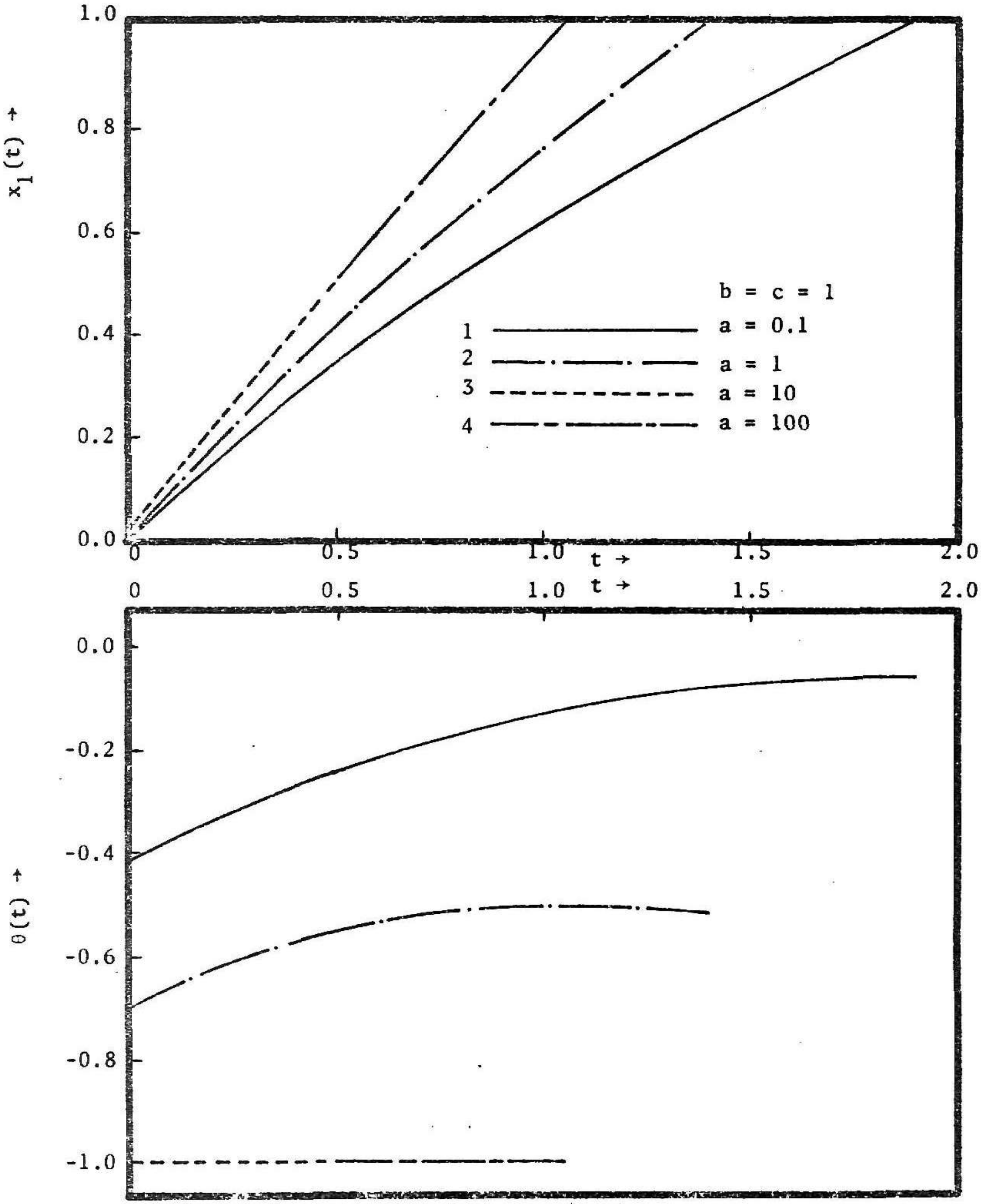

Fig. 6.4.1 Optimal control policy and system response of Example 4 (σ_s = 0.75)

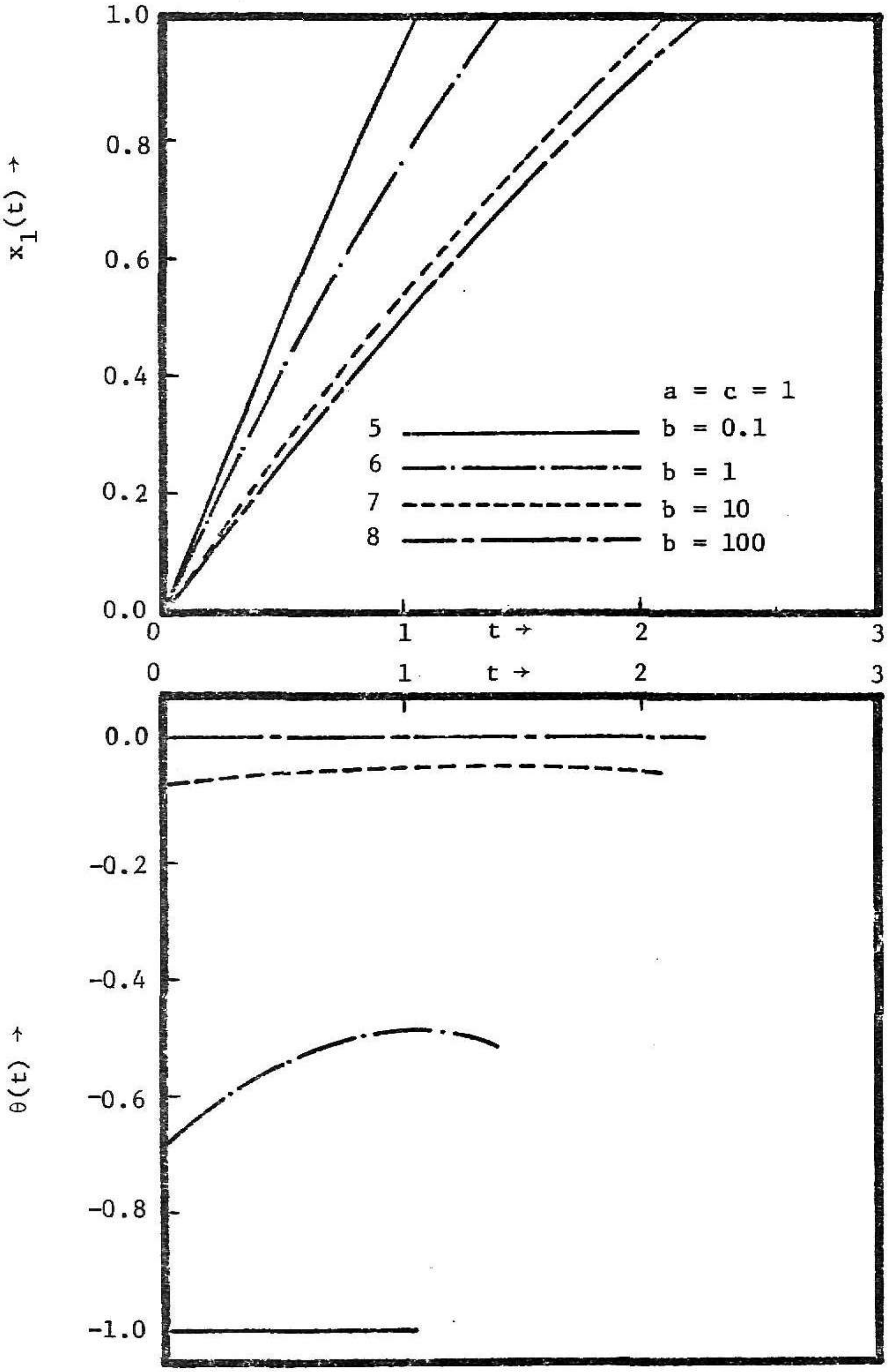

Fig. 6.4.2 Optimal control policy and system response of Example 4 (σ_s = 0.75)

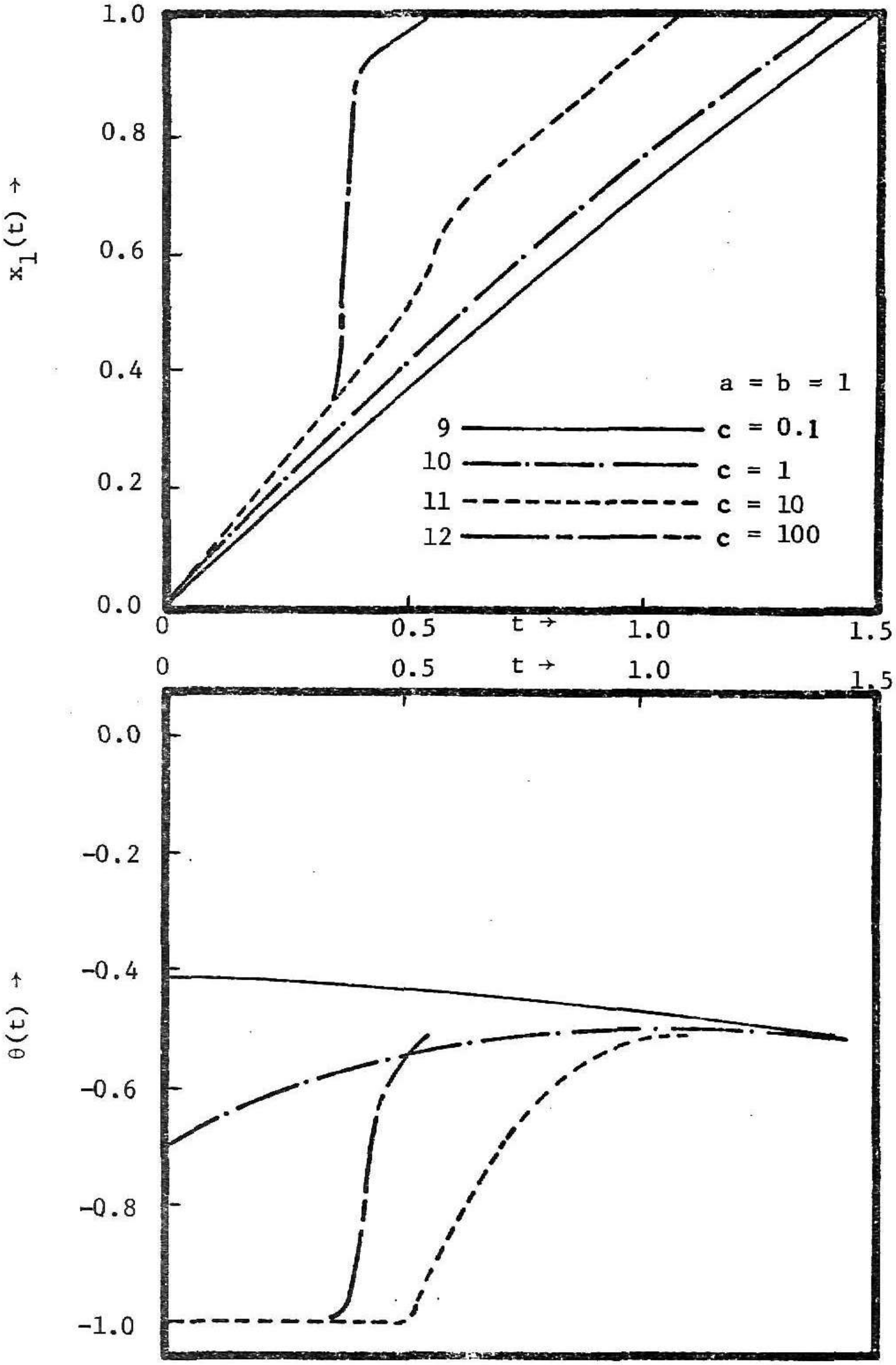

Fig. 6.4.3 Optimal control policy and system response of Example 4 (σ_s = 0.75)

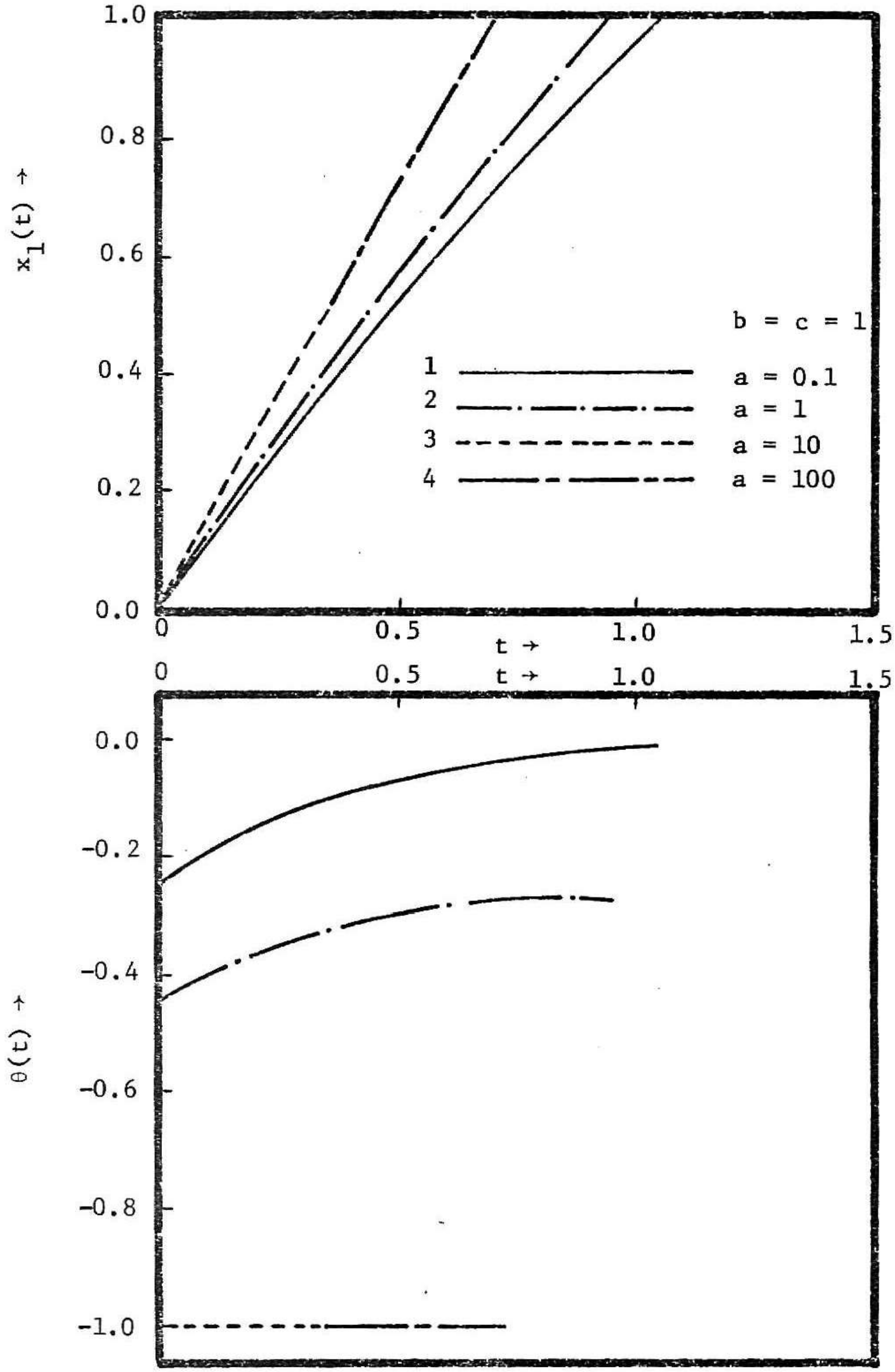

Fig. 6.4.4 Optimal control policy and system response of Example 4 ($\sigma_s = 1.2$)

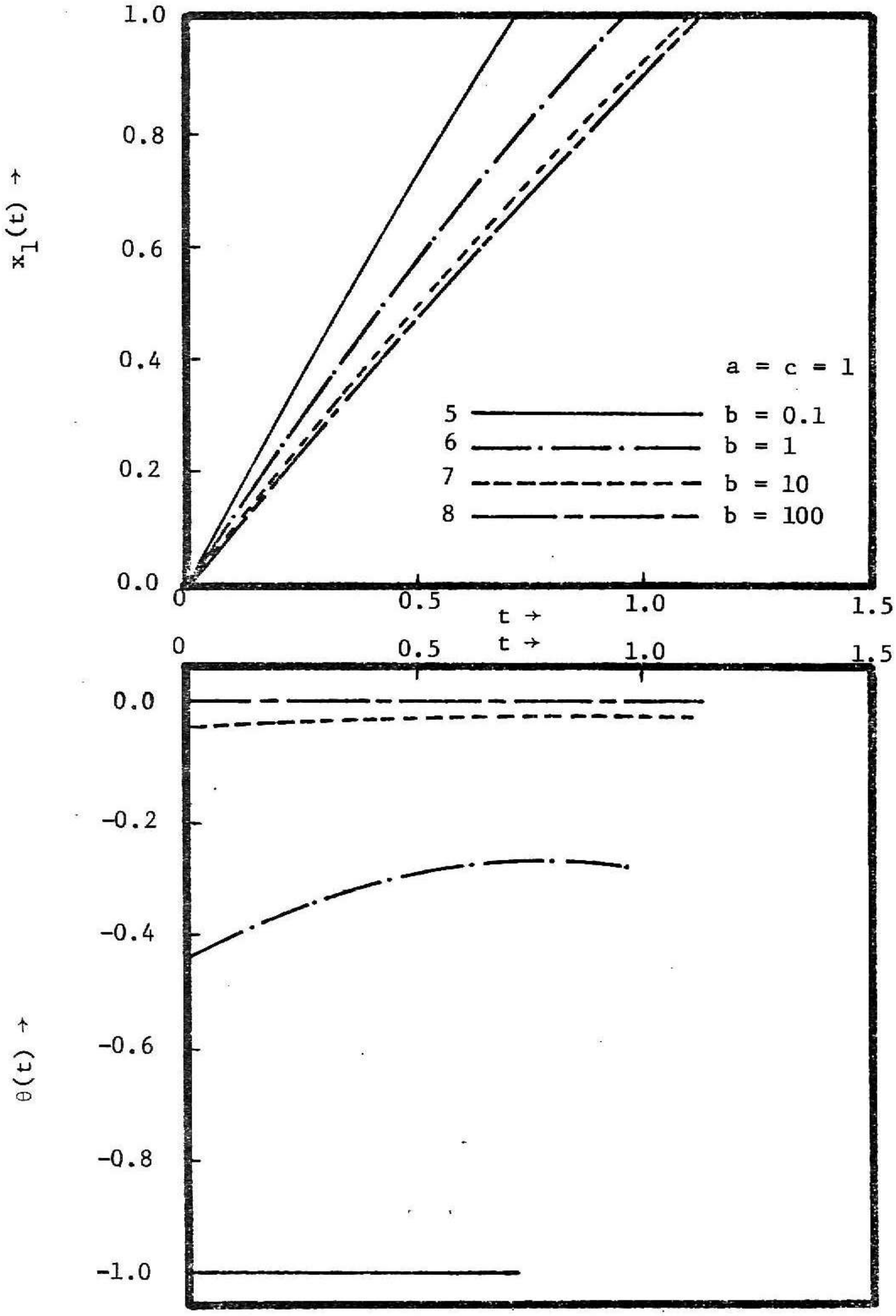

Fig. 6.4.5 Optimal control policy and system response of Example 4 (σ_s = 1.2)

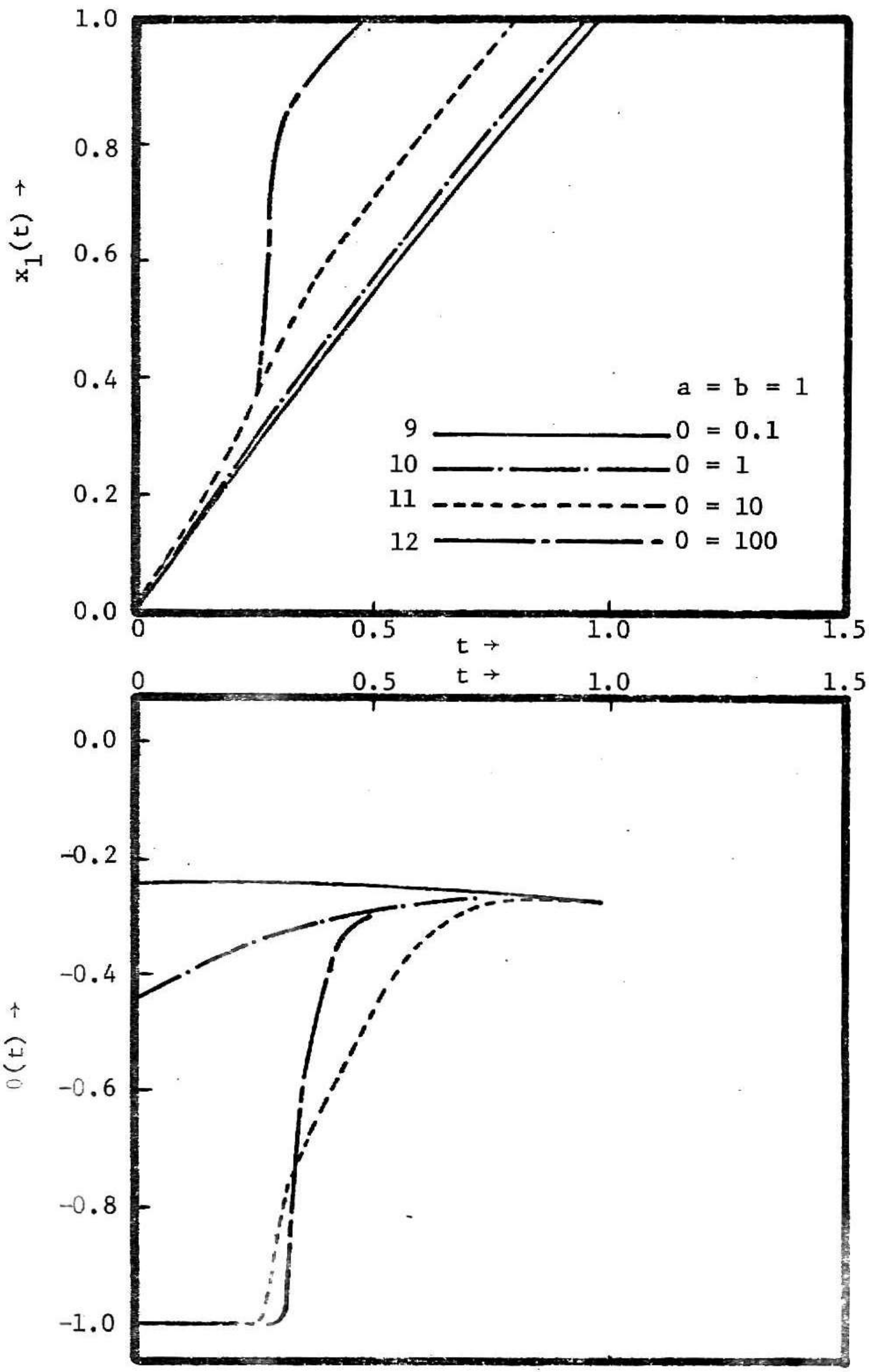

Fig. 6.4.6 Optimal control policy and system response of Example 4 (σ_s = 1.2)

CHAPTER 7

COMPARISON OF CLOSED-LOOP AND OPEN-LOOP CONTROL OF LIFE SUPPORT SYSTEMS WITH LINEAR PERFORMANCE EQUATION AND QUADRATIC OBJECTIVE FUNCTION

7.1 Introduction

This chapter deals with the closed-loop as well as the open-loop control of two examples-one dealing with a cooling system subjected to an impulse heat input and the other dealing with a heating system subjected to a step heat input. In each of these examples the performance equation is of linear form and the objective function is of quadratic form. Also each example has the right end free and deals with both the cases of specified and unspecified final time.

Kalman's linear regulator methodology is used in determining the optimal control law for the first example subjected to an impulse heat disturbance. However, since the performance equation of the second example subjected to a step heat disturbance depends explicitly on disturbance, a slightly modified version of the linear regulator methodology is used in this case.

It is found, in each example, that the open-loop control policy and the closed-loop control law are exactly same whether optimal T is specified or not, as long as the linear system has a quadratic functional.

7.2 EXAMPLE 1: Impulse Heat Disturbance

Let a life support system consisting of an air-conditioned room or cabin and a heat exchanger of negligibly small time constant ($\tau_2 = 0$) be subjected to an impulse heat disturbance.

The performance equation of such a system is written as

$$\frac{dx_1}{dt} = - r_2x_1 + r_2K_1 - r_1K_1K_2\theta - r_1K_1K_3 \qquad \text{(i)}$$

The initial condition is

$$x_1(0^+) = 1$$

It is required to find an optimal control that will bring the system to the desired condition in a time interval T, which may or may not be specified.

The objective function to be minimized is the sum of the integrated control effort to maintain the state of the system in the desired state and the integrated deviation from the desired state over a specified/unspecified control period and is given by

$$S = \int_0^T (b\theta^2 + cx_1^2)dt \qquad \text{(ii)}$$

where b and c are weighting factors.

It is required to compare the open-loop as well as the closed-loop solutions of the above problem for both the specified and unspecified control periods.

Since the closed-loop algorithm cannot handle constraints, it is assumed that the control variable is unconstrained, that is, two or more heat exchangers are operated in series if required.

a. Closed-loop control:

First considering the closed-loop case with a specified control period, the problem as stated has a fixed time interval with free right-end.

Introducing a new control variable Φ, which is a function of θ alone (so that minimization of Φ is equivalent to that of θ), as

$$\Phi = r_2K_1 - r_1K_1K_2\theta - r_1K_1K_3$$

in order to put the above problem in the standard form suitable for the application of the linear regulator algorithm, the original problem is rewritten as follows:

Performance equation:

$$\frac{dx_1}{dt} = - r_2x_1 + \Phi \tag{1.1}$$

Initial condition:

$$x_1(0^+) = 1$$

Objective function:

Minimize

$$S = \int_0^T (r\phi^2 + qx_1^2)dt \tag{1.2}$$

where r and q are new weighting factors.

The Hamiltonian is

$$H = r\phi^2 + qx_1^2 - z_1r_2x_1 + z_1\phi \qquad (1.3)$$

The adjoint variable is defined by

$$\frac{\partial H}{\partial x_1} = -\dot{z}_1 = 2qx_1 - z_1r_2 \qquad (1.4)$$

with the end condition

$$z_1(T) = 0 \qquad (1.5)$$

The optimal control is obtained from the following necessary condition for optimality

$$\frac{\partial H}{\partial \phi} = 0 = 2r\phi + z_1 \qquad (1.6)$$

which gives

$$\phi = -\frac{z_1}{2r} \qquad (1.7)$$

Now we inquire whether this can be converted to a closed-loop control by assuming the solution for z_1 as

$$z_1 = p_{11}\, x_1 \qquad (1.8)$$

Employing equation (1.8) into equations (1.1) and (1.7) yields

$$\dot{x}_1 = -r_2x_1 - \frac{p_{11}x_1}{2r} \qquad (1.9)$$

Also from equations (1.4) and (1.8) we require

$$\dot{z}_1 = \dot{p}_{11}x_1 + p_{11}\dot{x}_1 = -2qx_1 + p_{11}x_1r_2 \tag{1.10}$$

combining equations (1.9) and (1.10) gives

$$(\dot{p}_{11} - 2p_{11}r_2 - \frac{p_{11}^2}{2r} + 2q)x_1 = 0 \tag{1.11}$$

Since equation (1.11) must hold for all nonzero x_1, the term premultiplying x_1 must be zero. Thus we have

$$\dot{p}_{11} = 2p_{11}r_2 + \frac{p_{11}^2}{2r} - 2q \tag{1.12}$$

with an end condition given by equations (1.5) and (1.8)

$$p_{11}(T) = 0 \tag{1.13}$$

From equation (1.9) we have

$$p_{11} = -2r\left\{\frac{r_2x_1 + \dot{x}_1}{x_1}\right\} \tag{1.14}$$

Differentiation of equation (1.14) with respect to t yields

$$\dot{p}_{11} = -2r\left\{\frac{x_1\ddot{x}_1 - \dot{x}_1^2}{x_1^2}\right\} \tag{1.15}$$

By substituting equations (1.14) and (1.15) into equation (1.12) we obtain

$$\ddot{x}_1 - (r_2^2 + \frac{q}{r})x_1 = 0 \tag{1.16}$$

The solution of equation (1.16)

$$x_1(t) = A_1e^{\lambda t} + A_2e^{-\lambda t} \tag{1.17}$$

where

$$\lambda = \sqrt{r_2^2 + \frac{q}{r}}$$

By differentiating equation (1.17) with respect to t and substituting the result together with equation (1.17) into equation (1.14), we obtain

$$p_{11}(t) = -2r\left\{\frac{A_1(\lambda+r_2)e^{\lambda t} - A_2(\lambda-r_2)e^{-\lambda t}}{A_1e^{\lambda t} + A_2e^{-\lambda t}}\right\} \tag{1.18}$$

Application of the boundary conditions

$$x_1(0^+) = 1$$

and

$$p_{11}(T) = 0$$

to equations (1.17) and (1.18), respectively, gives

$$x_1(0^+) = A_1 + A_2 = 1 \tag{1.19}$$

$$p_{11}(T) = -A_1(\lambda+r_2)e^{\lambda T} + A_2(\lambda-r_2)e^{-\lambda T} = 0 \tag{1.20}$$

The solution of equations (1.19) and (1.20) for A_1 and A_2 is

$$A_1 = \frac{(\lambda-r_2)e^{-\lambda T}}{(\lambda+r_2)e^{\lambda T} + (\lambda-r_2)e^{-\lambda T}} \tag{1.21}$$

and

$$A_2 = \frac{(\lambda+r_2)e^{\lambda T}}{(\lambda+r_2)e^{\lambda T} + (\lambda-r_2)e^{-\lambda T}} \tag{1.22}$$

The feedback gain k(t) is obtained as

$$k(t) = -\frac{p_{11}(t)}{2r} = \frac{A_1(\lambda+r_2)e^{\lambda t} - A_2(\lambda-r_2)e^{-\lambda t}}{A_1e^{\lambda t} + A_2e^{-\lambda t}} \tag{1.23}$$

Then the optimal control policy $\bar{\phi}(t)$ may be obtained as

$$\bar{\phi}(t) = k(t)\; x_1(t)$$

$$= \frac{A_1(\lambda+r_2)e^{\lambda t} - A_2(\lambda-r_2)e^{-\lambda t}}{A_1e^{\lambda t} + A_2e^{-\lambda t}}\; x_1(t)$$

$$= A_1(\lambda+r_2)e^{\lambda t} - A_2(\lambda-r_2)e^{-\lambda t} \tag{1.24}$$

Using equations (1.21) and (1.22) and the relation $\lambda^2 - r_2^2 = \frac{q}{r}$ into equation (1.24) yields

$$\bar{\phi}(t) = \frac{\{\frac{q}{r}\}[e^{-\lambda(T-t)} - e^{\lambda(T-t)}]}{(\lambda+r_2)e^{\lambda T} + (\lambda-r_2)e^{-\lambda T}} \tag{1.25}$$

The objective function S becomes

$$S = x_2(T) = \int_0^T (r\phi^2 + qx_1^2)dt$$

$$= \int_0^T [r\{(\lambda+r_2)A_1e^{\lambda t} - (\lambda-r_2)A_2e^{-\lambda t}\}^2 + q\{A_1e^{\lambda t} + A_2e^{-\lambda t}\}^2]dt$$

$$= \frac{1}{2\lambda} [A_1^2(e^{2\lambda T}-1)\{q+r(\lambda+r_2)^2\} - A_2^2(e^{-2\lambda T}-1)\{q+r(\lambda-r_2)^2\}] \tag{1.26}$$

Next considering the same closed-loop case but with unspecified control period, equation (1.12) becomes

$$p_{11}^2 + 4rr_2p_{11} - 4qr = 0 \tag{1.27}$$

The solution of equation (1.27) gives a positive constant value for p_{11} as

$$p_{11} = -2r\left\{r_2 - \sqrt{r_2^2 + \frac{q}{r}}\right\} \tag{1.28}$$

The constant feedback gain k is obtained as

$$k = -\frac{p_{11}}{2r} = r_2 - \sqrt{r_2^2 + \frac{q}{r}} \tag{1.29}$$

Now the solution for equation (1.9) can be obtained as

$$x_1(t) = Ae^{-(r_2 + \frac{p_{11}}{2r})t} \tag{1.30}$$

Application of the boundary condition

$$x_1(0^+) = 1$$

to equation (1.30) gives

$$A = 1 \tag{1.31}$$

Then the optimal control policy $\bar{\phi}(t)$ may be obtained as

$$\bar{\phi}(t) = kx_1$$

$$= \left\{r_2 - \sqrt{r_2^2 + \frac{q}{r}}\right\} e^{-(r_2 + \frac{p_{11}}{2r})t} \tag{1.32}$$

The objective function S becomes

$$S = \int_0^T (r\phi^2 + qx_1^2)dt$$

$$= \int_0^T \left[r\left\{ (r_2 - \sqrt{r_2^2 + \frac{q}{r}}) e^{-(r_2 + \frac{p_{11}}{2r})t} \right\}^2 + q\left\{ e^{-(r_2 + \frac{p_{11}}{2r})t} \right\}^2 \right] dt$$

$$= \frac{p_{11}^2 + 4qr}{-4r(2r_2 + \frac{p_{11}}{r})} \left[e^{-(2r_2 + \frac{p_{11}}{r})T} - 1 \right] \tag{1.33}$$

b. Open-loop control:

Now the same problem is considered for open-loop control. Although Pontryagin's maximum principle can handle constraints, it is assumed that the control variable is unconstrained for the purpose of comparison of the results of the open-loop control with those of the feed-back control.

First considering the open-loop case with a specified control period, we proceed as follows:

Introducing another state variable $x_2(t)$ such that

$$x_2(t) = \int_0^t (r\phi^2 + qx_1^2)dt$$

it follows that

$$\frac{dx_2}{dt} = r\phi^2 + qx_1^2, \quad x_2(0) = 0 \tag{1.34}$$

The problem is thus transformed into that of minimizing $x_2(T)$.

According to Pontryagin's maximum principle the Hamiltonian is

$$H = - z_1 r_2 x_1 + z_1\phi + z_2 r\phi^2 + z_2 q x_1^2 \tag{1.35}$$

The adjoint variables are defined by

$$\frac{dz_1}{dt} = - \frac{\partial H}{\partial x_1} = z_1 r_2 - 2z_2 q x_1, \; z_1(T) = 0 \tag{1.36}$$

$$\frac{dz_2}{dt} = - \frac{\partial H}{\partial x_2} = 0, \qquad z_2(T) = 1 \tag{1.37}$$

Solving equation (1.37) for z_2 gives

$$z_2(t) = 1, \qquad 0 \leq t \leq T \tag{1.38}$$

Hence the Hamiltonian can be rewritten as

$$H = -z_1 r_2 x_1 + z_1\phi + r\phi^2 + qx_1^2 \tag{1.39}$$

According to the maximum principle, H must be a minimum in ϕ with the values of x_1 and z_1 considered as fixed. Putting

$$\frac{\partial H}{\partial \phi} = 0$$

we have

$$\frac{\partial H}{\partial \phi} = z_1 + 2r\phi = 0 \tag{1.40}$$

or

$$\phi = - \frac{z_1}{2r} \tag{1.41}$$

Substitution of equations (1.38) and (1.41) into equations (1.1) and (1.36) respectively, gives

$$\frac{dx_1}{dt} = -r_2x_1 - \frac{z_1}{2r} \tag{1.42}$$

and

$$\frac{dz_1}{dt} = r_2z_1 - 2qx_1 \tag{1.43}$$

The system of differential equations, equations (1.42) and (1.43), is solved simultaneously. From equation (1.42) we have

$$z_1 = -2r \left(r_2x_1 + \frac{dx_1}{dt}\right) \tag{1.44}$$

Differentiation of equation (1.44) with respect to t yields

$$\frac{dz_1}{dt} = - 2r \left(r_2 \frac{dx_1}{dt} + \frac{d^2x_1}{dt^2}\right) \tag{1.45}$$

By substituting equations (1.44) and (1.45) into equation (1.43) we obtain

$$\frac{d^2x_1}{dt^2} - x_1\left(r_2^2 + \frac{q}{r}\right) = 0 \tag{1.46}$$

The solution of equation (1.46) is,

$$x_1(t) = A_1e^{\lambda t} + A_2e^{-\lambda t} \tag{1.47}$$

where

$$\lambda = \sqrt{r_2^2 + \frac{q}{r}}$$

and A_1 and A_2 are constants of integration. By differentiating equation (1.47) with respect to t and substituting the result together with equation (1.47) and equation (1.44), we obtain

$$z_1(t) = 2r[-A_1(\lambda+r_2)e^{\lambda t} + A_2(\lambda-r_2)e^{-\lambda t}] \tag{1.48}$$

Application of the boundary conditions

$$x_1(0^+) = 1$$

and

$$z_1(T) = 0$$

to equations (1.47) and (1.48), respectively, gives

$$x_1(0^+) = A_1 + A_2 = 1 \tag{1.49}$$

$$z_1(T) = 2r[-A_1(\lambda+r_2)e^{\lambda T} + A_2(\lambda-r_2)e^{-\lambda T}] = 0 \tag{1.50}$$

The solution of equations (1.49) and (1.50) for A_1 and A_2 is

$$A_1 = \frac{(\lambda-r_2)e^{-\lambda T}}{(\lambda+r_2)e^{\lambda T} + (\lambda-r_2)e^{-\lambda T}} \tag{1.51}$$

$$A_2 = \frac{(\lambda+r_2)e^{\lambda T}}{(\lambda+r_2)e^{\lambda T} + (\lambda-r_2)e^{-\lambda T}} \tag{1.52}$$

The optimal control $\bar{\phi}(t)$, which may be obtained by the substitution of equations (1.41), (1.51), (1.52), and the relation $\lambda^2 - r_2^2 = \frac{q}{r}$ into equation (1.48) is

$$\bar{\phi} = \frac{(\frac{q}{r})[e^{-\lambda(T-t)} - e^{\lambda(T-t)}]}{(\lambda+r_2)e^{\lambda T} + (\lambda-r_2)e^{-\lambda T}} \tag{1.53}$$

The objective function $S = x_2(T)$ becomes

$$S = x_2(T) = \int_0^T (r\phi^2 + qx_1^2)dt$$

$$= \int_0^T [r\{(\lambda+r_2)A_1e^{\lambda t} - (\lambda-r_2)A_2e^{-\lambda t}\}^2 + q\{A_1e^{\lambda t}+A_2e^{-\lambda t}\}^2]dt$$

$$= \frac{1}{2\lambda}\{A_1^2(e^{2\lambda T}-1)[q+r(\lambda+r_2)^2] - A_2^2(e^{-2\lambda T}-1)[q+r(\lambda-r_2)^2]\} \tag{1.54}$$

Next considering the same open-loop case but with unspecified control period, we equate the minimum value of Hamiltonian to zero.

$$\text{Min } H = 0 = -r_2(-2r\phi)x_1 + qx_1^2 + (-2r\phi)\phi + r\phi^2$$

$$= -r\phi^2 + 2rr_2x_1\phi + qx_1^2 \tag{1.55}$$

Application of the boundary condition at $t = 0$, namely, $x_1(0^+) = 1$ to equation (1.55) gives

$$-r[\phi(0)]^2 + 2rr_2\phi(0) + q = 0 \tag{1.56}$$

The roots of this quadratic equation are

$$\phi(0) = r_2 \pm \sqrt{r_2^2 + \frac{q}{r}}$$

For $\phi(0) = r_2 - \sqrt{r_2^2 + \frac{q}{r}}$

combination of equations (1.41) and (1.48) yields

$$\phi(0) = -\frac{z_1(0)}{2r} = A_1(\lambda+r_2) - A_2(\lambda-r_2)$$

$$= r_2 - \sqrt{r_2^2 + \frac{q}{r}} \tag{1.57}$$

The solution of equations (1.49) and (1.57) for A_1 and A_2 is

$$A_1 = 0 \tag{1.58}$$

$$A_2 = 1 \tag{1.59}$$

The final time T is found from equation (1.50) as follows

$$-A_1(\lambda+r_2)e^{2\lambda T} + A_2(\lambda-r_2) = 0$$

or

$$e^{2\lambda T} = \frac{A_2(\lambda-r_2)}{A_1(\lambda+r_2)}$$

This gives

$$T = \frac{1}{2\lambda}\,\ell n\left\{\frac{A_2(\lambda-r_2)}{A_1(\lambda+r_2)}\right\} \tag{1.60}$$

Further simplification of equation (1.60) gives

$$T = +\infty \tag{1.61}$$

Instead using $\phi(0) = r_2 + \sqrt{r_2^2 + \frac{q}{r}}$ in the above procedure produces $T = -\infty$ which is not physically feasible.

The objective function S becomes

$$S = x_2(T) = \int_0^T (r\phi^2 + qx_1^2)\, dt$$

$$= \int_0^T [r\{(\lambda+r_2)A_1e^{\lambda t} - (\lambda-r_2)A_2e^{-\lambda t}\}^2 + q\{A_1e^{\lambda t} + A_2e^{-\lambda t}\}^2]dt$$

$$= \frac{1}{2\lambda}\{A_1^2(e^{2\lambda T}-1)[q+r(\lambda+r_2)^2] - A_2^2(e^{-2\lambda T}-1)[q+r(\lambda-r_2)^2]\} \tag{1.62}$$

Further simplification of equation (1.62) gives

$$S = x_2(T) = \frac{[q + r(\lambda - r_2)^2]}{2\lambda} (1 - e^{-2\lambda T}) \tag{1.63}$$

The following four combinations of weighting factors are considered in this example.

Case 1: $r = 1$, $q = 0.1$

Case 2: $r = 1$, $q = 1$

Case 3: $r = 1$, $q = 10$

Case 4: $r = 1$, $q = 100$

The values of the various constants used are:

$r_1 = 0.8$ $r_2 = 0.2$

$K_1 = 0.5$ $K_2 = 1.5$

$K_3 = 1.5$ $\sigma = 2$

The results of this example for both the open-loop as well as the closed-loop cases, which are exactly same (because of linear performance equation and quadratic objective function) are shown in Table 1.1 and Figs. 1.1, 1.2 and 1.3.

The value of the objective function increases with increase in the weighting factor q on the deviation. Also the longer the control period T, the greater is the value of the objective function. However when the weighting factor q takes appreciably larger values such as 100 compared to the weighting factor r (which is 1), the objective function is affected in the least by the length of the control period T.

In case 1, where more weight is given to the control effort, ϕ, the optimal control policy has a very small negative value. Since the control effort ϕ is unconstrained it takes high negative values for larger values of q (cases 2, 3 and 4) and thus requiring two or more heat exchangers to be operated in series. However, the control ϕ reaches zero in all the cases at the end of the control period (T=1 or 5). Also in the case of unspecified control period, the control ϕ attains zero in all the cases at $T \to \infty$.

Table 1.1

Values of the Objective Function of Example 1:

Specified Time	Case Number	Weighting Factors	Value of the Objective Function
1.0	1	$r = 1, \quad q = 0.1$	0.08024
	2	$r = 1, \quad q = 1$	0.65583
	3	$r = 1, \quad q = 10$	2.95872
	4	$r = 1, \quad q = 100$	9.80200
5.0	1	$r = 1, \quad q = 0.1$	0.16882
	2	$r = 1, \quad q = 1$	0.81975
	3	$r = 1, \quad q = 10$	2.96859
	4	$r = 1, \quad q = 100$	9.80200

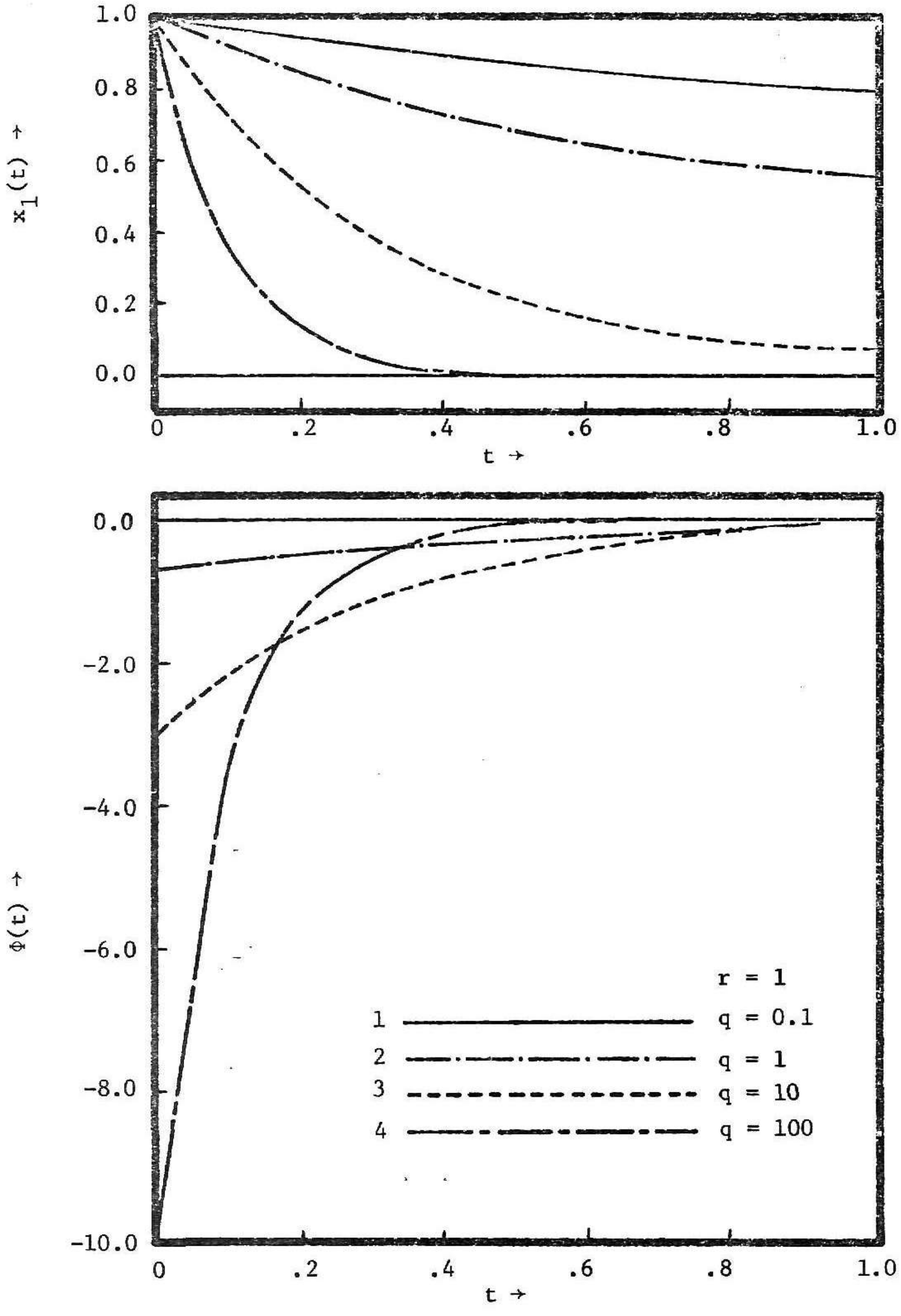

Fig. 7.11 Optimal control policy and system response of Example 1 (T - specified) T = 1.0

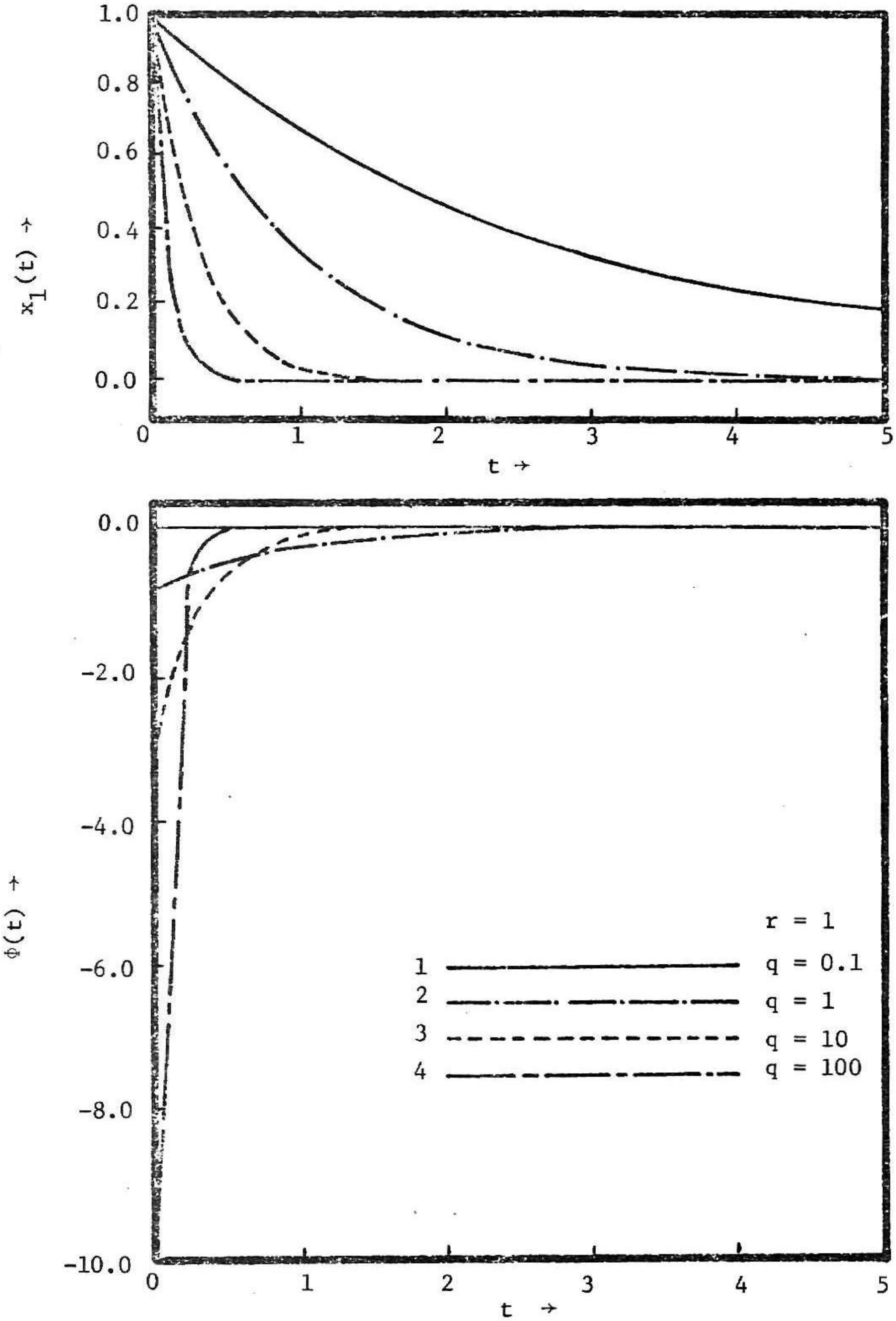

Fig. 7.1.2 Optimal control policy and system response of Example 1 (T - specified) T = 5.0

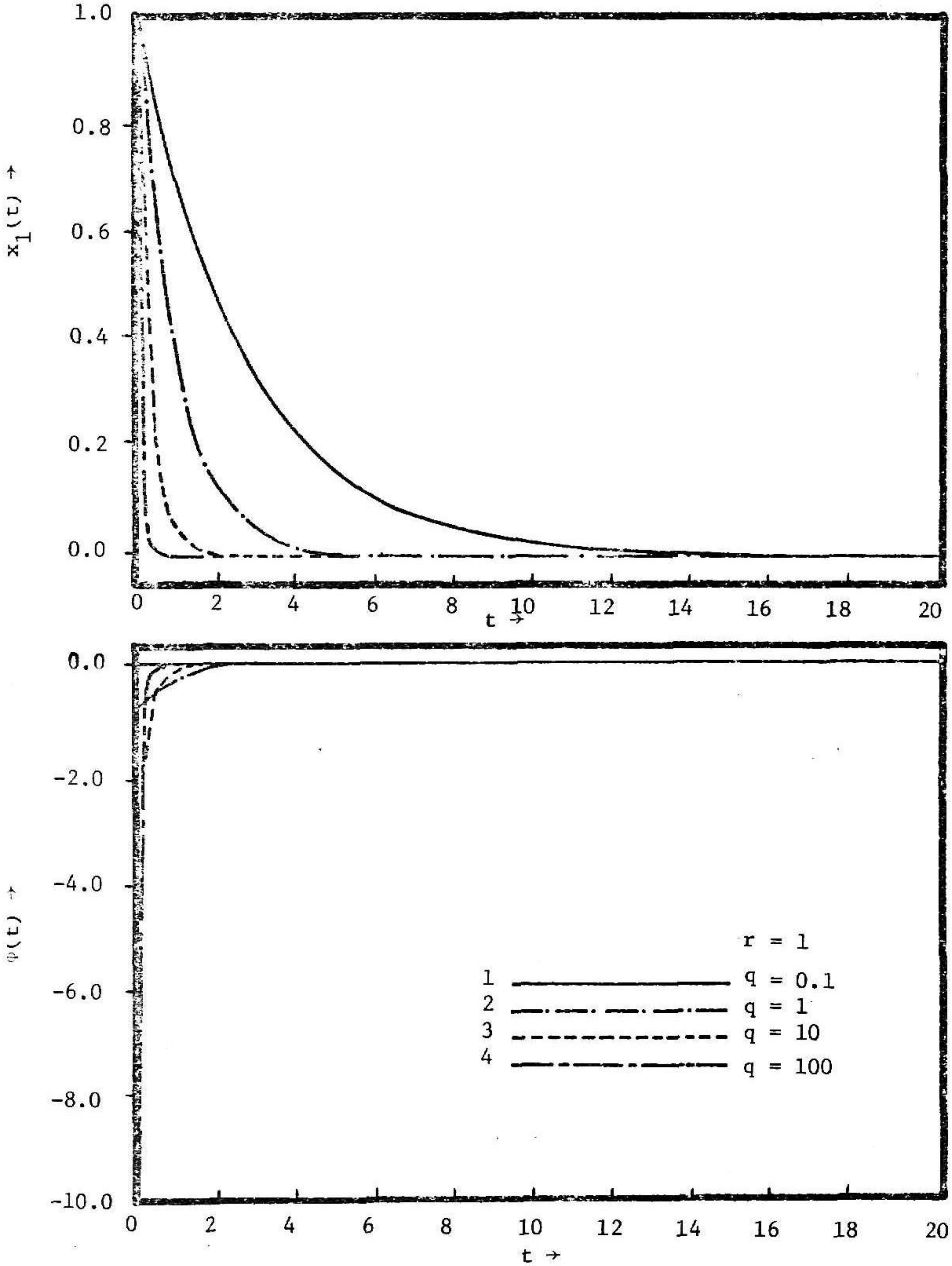

Fig. 7.1.3 Optimal trajectory and optimal control policy of Example 1. (T - not specified).

7.3 EXAMPLE 2: Step Heat Disturbance

Let a life support system consisting of an air-conditioned room or cabin and a heat exchanger of negligibly small time constant ($\tau_2 = 0$) be subjected to a step heat disturbance.

The performance equation of such a system is written as

$$\frac{dx_1}{dt} + r_2 x_1 = r_2 K_\alpha - r_1 K_\beta \theta - r_1 K_\gamma + \sigma_s \tag{i}$$

The initial condition is

$$x_1(0) = 0$$

It is required to find an optimal control that will bring the system to the desired condition in a time interval T which may or may not be specified.

The objective function to be minimized is the sum of the integrated control effort to maintain the state of the system in the desired state and the integrated deviation from the desired state over a specified/un-specified control period and is given by

$$S = \int_0^T [b\theta^2 + c\,(x_1 - x_{1d})^2]dt \tag{ii}$$

where b and c are weighting factors. The desired state x_{1d} is equal to one.

It is required to compare the open-loop as well as the closed-loop solutions of the above problem for both the specified and unspecified control periods.

Since the closed-loop algorithm cannot handle constraints, it is assumed that the control variable is unconstrained, that is, two or more heat exchangers are operated in series if required.

a. Closed-loop control:

First considering the closed-loop case with a specified control period, the problem as stated has a fixed time interval with free right-end.

Since the performance equation of this problem depends explicitly on disturbance, a slightly modified version of the linear regulator methodology is used here.

Introducing a new state variable y_1, to represent the deviation of the system state from the desired state, as

$$y_1 = x_1 - 1$$

and a new control variable Φ, which is a function of θ alone (so that minimization of Φ is equivalent to that of θ), as

$$\Phi = -r_2 + r_2 K_\alpha - r_1 K_\beta \theta - r_1 K_\gamma$$

in order to put the above problem in the standard form suitable for the application of the algorithm, the original problem is rewritten as follows:
Performance equation:

$$\frac{dy_1}{dt} = -r_2 y_1 + \Phi + \sigma_s \tag{2.1}$$

Initial condition:

$$y_1(0) = -1$$

Objective function:

Minimize

$$S = \int_0^T (r\phi^2 + qy_1^2)\, dt \tag{2.2}$$

where r and q are new weighting factors.

The Hamiltonian is

$$H = r\phi^2 + qy_1^2 - z_1 r_2 y_1 + z_1\phi + z_1\sigma_s \tag{2.3}$$

The adjoint variable is defined by

$$\frac{\partial H}{\partial y_1} = -\dot{z}_1 = 2qy_1 - z_1 r_2 \tag{2.4}$$

with the end condition

$$z_1(T) = 0 \tag{2.5}$$

The optimal control is obtained from the following necessary condition for optimality

$$\frac{\partial H}{\partial \phi} = 0 = 2r\phi + z_1 \tag{2.6}$$

which gives

$$\phi = -\frac{z_1}{2r} \tag{2.7}$$

Now we inquire whether this can be converted to a closed-loop control by assuming the solution for z_1 as

$$z_1 = -2r(p_{11}y_1 + n_{11}) \tag{2.8}$$

Employing equation (2.8) into equations (2.1) and (2.7) yields

$$\dot{y}_1 = - r_2 y_1 + p_{11} y_1 + n_{11} + \sigma_s \tag{2.9}$$

Also from equations (2.4) and (2.8) we require

$$\dot{z}_1 = - 2r\ (\dot{p}_{11} y_1 + p_{11}\dot{y}_1 + \dot{n}_{11}) = - 2qy_1 - 2rr_2\ (p_{11} y_1 + n_{11}) \tag{2.10}$$

Combining equations (2.9) and (2.10) gives

$$y_1(\dot{p}_{11} - r_2 p_{11} + p_{11}^2) + p_{11} n_{11} + p_{11}\sigma_s + \dot{n}_{11}$$

$$= y_1(r_2 p_{11}) + r_2 n_{11} \tag{2.11}$$

Equating coefficients of y_1 in equation (2.11), we obtain an equation for $p_{11}(t)$

$$\dot{p}_{11} - 2r_2 p_{11} + p_{11}^2 - \frac{q}{r} = 0 \tag{2.12}$$

and therefore $n_{11}(t)$ must satisfy

$$\dot{n}_{11} + n_{11}(p_{11} - r_2) + p_{11}\ \sigma_s = 0 \tag{2.13}$$

To establish boundary conditions for equations (2.12) and (2.13), it is assumed that at some time $t < T$ the disturbance vanishes and remains identically zero. Equation (2.12) is the usual Riccati ordinary differential equation with the end condition

$$p_{11}(T) = 0 \tag{2.14}$$

Hence it follows from equations (2.5) and (2.8), that the suitable boundary condition for equation (2.13) is

$$n_{11}(T) = 0 \tag{2.15}$$

Thus the procedure is to

(a) solve equation (2.12) for $p_{11}(t)$ as

$$p_{11}(t) = \frac{C + Be^{(t+A)(B-C)}}{1 + e^{(t+A)(B-C)}} \tag{2.16}$$

where B and C are the roots of the quadratic algebraic equation

$$p_{11}^2 - 2r_2 p_{11} - \frac{q}{r} = 0,$$

that is,

$$B = r_2 + \sqrt{r_2^2 + \frac{q}{r}}$$

and

$$C = r_2 - \sqrt{r_2^2 + \frac{q}{r}}$$

and

$$A = \frac{1}{B-C} \ln\{-\frac{C}{B}\} - T$$

(b) solve for $n_{11}(t)$ from equation (2.13), which is a linear nonhomogeneous differential equation, backward in time from T to 0.

(c) solve equation (2.9) for y_1, forward in time from 0 to T.

(d) obtain the closed-loop control from

$$\Phi(y_1, t) = p_{11}(t)\, y_1(t) + n_{11}(t) \tag{2.17}$$

Next considering the same closed-loop case but with unspecified control period, that is, $T \to \infty$, the solution of the Riccati equation (2.12) is

$$p_{11} = r_2 - \sqrt{r_2^2 + \frac{q}{r}} \tag{2.18}$$

and equation (2.13) is

$$\dot{n}_{11} = \sqrt{r_2^2 + \frac{q}{r}}\, n_{11} = \left(- r_2 + \sqrt{r_2^2 + \frac{q}{r}}\right) \sigma_s, \qquad n_{11}(\infty) = 0 \tag{2.19}$$

which has the solution

$$n_{11}(t) = - \left\{- r_2 + \sqrt{r_2^2 + \frac{q}{r}}\right\} \int_t^{\infty} e^{-\sqrt{r_2^2 + \frac{q}{r}}\,(\tau - t)} \sigma_s \, d\tau \tag{2.20}$$

that is

$$n_{11}(t) \simeq \left\{ \frac{r_2}{\sqrt{r_2^2 + \frac{q}{r}}} - 1 \right\} \sigma_s \tag{2.21}$$

Hence the solution of equation (2.9) for y_1 yields

$$y_1 = De^{-(r_2 - p_{11})t} + \frac{n_{11} + \sigma_s}{r_2 - p_{11}} \tag{2.22}$$

where

$$D = - \left\{1 + \frac{n_{11} + \sigma_s}{r_2 - p_{11}}\right\}$$

and p_{11} and n_{11} are given by equations (2.18) and (2.21).

Then the optimal closed-loop control becomes

$$\Phi(y_1,t) = p_{11}\, y_1(t) + n_{11} \tag{2.23}$$

where p_{11} and n_{11} are constants.

(b) Open-loop control:

Now the same above problem is considered for open-loop control. Although Pontryagin's maximum principle can handle constraints, it is assumed that the control variable is unconstrained for the purpose of comparison of the results of the open-loop control with those of the feed-back control.

First considering the open-loop case with a specified control period, we proceed as follows:

Introducing another state variable $y_2(t)$ such that

$$y_2(t) = \int_0^t (r\phi^2 + qy_1^2)dt$$

it follows that

$$\frac{dy_2}{dt} = r\phi^2 + qy_1^2, \qquad y_2(0) = 0 \tag{2.24}$$

The problem is thus transformed into that of minimizing $y_2(T)$.

According to Pontryagin's maximum principle the Hamiltonian is

$$H = z_1(-r_2y_1 + \phi + \sigma_s) + z_2(qy_1^2 + r\phi^2) \tag{2.25}$$

The adjoint variables are defined by

$$\frac{dz_1}{dt} = -\frac{\partial H}{\partial y_1} = z_1 r_2 - 2z_2 q y_1, \qquad z_1(T) = 0 \tag{2.26}$$

$$\frac{dz_2}{dt} = -\frac{\partial H}{\partial y_2} = 0 , \qquad z_2(T) = 1 \tag{2.27}$$

Solving equation (2.27) for z_2 gives

$$z_2(t) = 1 , \qquad 0 \leq t \leq T \tag{2.28}$$

Hence the Hamiltonian can be rewritten as

$$H = z_1(-r_2 y_1 + \phi + \sigma_s) + q y_1^2 + r\phi^2 \tag{2.29}$$

According to Pontryagin's maximum principle, H must be a minimum in ϕ with the values of y_1 and z_1 considered as fixed. Putting

$$\frac{\partial H}{\partial \phi} = 0$$

we have

$$\frac{\partial H}{\partial \phi} = z_1 + 2r\phi = 0 \tag{2.30}$$

or

$$\phi = -\frac{z_1}{2r} \tag{2.31}$$

Substitution of equations (2.28) and (2.31) into equations (2.1) and (2.26) respectively, gives

$$\frac{dy_1}{dt} = -r_2 y_1 - \frac{z_1}{2r} + \sigma_s \tag{2.32}$$

and

$$\frac{dz_1}{dt} = r_2 z_1 - 2q y_1 \tag{2.33}$$

The system of differential equations, equations (2.32) and (2.33), is solved simultaneously. From equation (2.32) we have

$$z_1 = -2r(r_2 y_1 + \frac{dy_1}{dt} - \sigma_s) \tag{2.34}$$

Differentiation of equation (2.34) with respect to t yields

$$\frac{dz_1}{dt} = -2r\left(r_2 \frac{dy_1}{dt} + \frac{d^2 y_1}{dt^2}\right) \tag{2.35}$$

By substituting equations (2.34) and (2.35) into equation (2.33) we obtain

$$\frac{d^2 y_1}{dt^2} - y_1(r_2^2 + \frac{q}{r}) + r_2 \sigma_s = 0 \tag{2.36}$$

The solution of equation (2.36) is,

$$y_1 = A_1 e^{\lambda t} + A_2 e^{-\lambda t} + K \tag{2.37}$$

where

$$\lambda = \sqrt{r_2^2 + \frac{q}{r}}$$

$$K = \frac{r_2 \sigma_s}{r_2^2 + \frac{q}{r}}$$

and A_1 and A_2 are constants of integration.

By differentiating equation (2.37) with respect to t and substituting the result together with equation (2.37) into equation (2.34), we obtain

$$z_1(t) = -2r[A_1(r_2+\lambda)e^{\lambda t} + A_2(r_2-\lambda)e^{-\lambda t} + (r_2 K-\sigma_s)] \tag{2.38}$$

Application of the boundary conditions

$$y_1(0) = -1$$

and

$$z_1(T) = 0$$

to equations (37) and (38), respectively, gives

$$y_1(0) = A_1 + A_2 + K = -1 \tag{2.39}$$

$$z_1(T) = -2r[A_1(r_2+\lambda)e^{\lambda T} + A_2(r_2-\lambda)e^{-\lambda T} + r_2K - \sigma_s] = 0 \tag{2.40}$$

The solution of equations (2.39) and (2.40) for A_1 and A_2 is

$$A_1 = \frac{-(r_2-\lambda)(K+1)e^{-\lambda T} + r_2K - \sigma_s}{(r_2-\lambda)e^{-\lambda T} - (r_2+\lambda)e^{\lambda T}} \tag{2.41}$$

$$A_2 = \frac{(r_2+\lambda)(K+1)e^{\lambda T} - r_2K + \sigma_s}{(r_2-\lambda)e^{-\lambda T} - (r_2+\lambda)e^{\lambda T}} \tag{2.42}$$

The optimal control $\overline{\phi}(t)$, which may be obtained by the substitution of equation (2.31) into equation (2.38) is

$$\overline{\phi}(t) = A_1(r_2+\lambda)e^{\lambda T} + A_2(r_2-\lambda)e^{-\lambda T} + r_2K - \sigma_s \tag{2.43}$$

The objective function $S = x_2(T)$ becomes

$$S = x_2(T) = \int_0^T (r\phi^2 + qy_1^2)dt$$

$$= \int_0^T \Big\{ r[A_1(r_2+\lambda)e^{\lambda t} + A_2(r_2-\lambda)e^{-\lambda t} + r_2K - \sigma_s]^2$$

$$+ q[A_1e^{\lambda t} + A_2e^{-\lambda t} + K]^2 \Big\} dt$$

$$= \frac{A_1^2}{2\lambda}\,[r(r_2+\lambda)^2 + q](e^{2\lambda T}-1)$$

$$- \frac{A_2^2}{2\lambda}\,[r(r_2-\lambda)^2 + q](e^{-2\lambda T}-1)$$

$$+ \frac{2A_1}{\lambda}[r(r_2+\lambda)(r_2K-\sigma_s) + qK](e^{\lambda T}-1)$$

$$- \frac{2A_2}{\lambda}\,[r(r_2-\lambda)(r_2K-\sigma_s) + qK](e^{-\lambda T}-1)$$

$$+ T\left\{2A_1A_2[r(r_2^2-\lambda^2) + q] + r(r_2K-\sigma_s)^2 + qK^2\right\} \qquad (2.44)$$

Next considering the same open-loop case but with unspecified control period, we equate the minimum value of Hamiltonian to zero.

$$\text{Min } H = 0 = (-2r\phi)(-r_2y_1 + \phi + \sigma_s) + qy_1^2 + r\phi^2$$

$$= -r\phi^2 + 2r\phi(r_2y_1 - \sigma_s) + qy_1^2 \qquad (2.45)$$

Application of the boundary condition at t = 0, namely, $y_1(0) = -1$ to equation (2.45) gives

$$r[\phi(0)]^2 + 2r(r_2 + \sigma_s)\phi(0) - q = 0 \qquad (2.46)$$

The roots of this quadratic equation are

$$\phi(0) = -(r_2 + \sigma_s) \pm \sqrt{(r_2 + \sigma_s)^2 + \frac{q}{r}}$$

For $\phi(0) = -(r_2 + \sigma_s) + \sqrt{(r_2 + \sigma_s)^2 + \frac{q}{r}}$ combination of equations (2.31) and (2.38) yields

$$\phi(0) = A_1(r_2 + \lambda) + A_2(r_2 - \lambda) + r_2K - \sigma_s$$

$$= -(r_2 + \sigma_s) + \sqrt{(r_2 + \sigma_s)^2 + \frac{q}{r}} \tag{2.47}$$

The solution of equations (2.39) and (2.47) for A_1 and A_2 is

$$A_1 = \frac{\sqrt{(r_2 + \sigma_s)^2 + \frac{q}{r}} - \lambda(K+1)}{2\lambda} \tag{2.48}$$

$$A_2 = \frac{-\sqrt{(r_2 + \sigma_s)^2 + \frac{q}{r}} - \lambda(K + 1)}{2\lambda} \tag{2.49}$$

The final time T is found from equation (2.40) as follows

$$A_1(r_2 + \lambda)e^{2\lambda T} + (r_2K - \sigma_s)e^{\lambda T} + A_2(r_2 - \lambda) = 0$$

or

$$e^{\lambda T} = \frac{-(r_2K - \sigma_s) \pm \sqrt{(r_2K - \sigma_s)^2 - 4A_1A_2(r_2^2 - \lambda^2)}}{2A_1(r_2 + \lambda)}$$

This gives

$$T = \frac{1}{\lambda} \ln \left\{ \frac{-(r_2K - \sigma_s) \pm \sqrt{(r_2K - \sigma_s)^2 - 4A_1A_2(r_2^2 - \lambda^2)}}{2A_1(r_2 + \lambda)} \right\} \tag{2.50}$$

Instead using $\phi(0) = -(r_2 + \sigma_s) - \sqrt{(r_2 + \sigma_s)^2 + \frac{q}{r}}$ in the above procedure produces negative final time which is not physically feasible.

Here, when the final time T is not specified, the open-loop solution gives a finite value for optimal final time, that is, $T \nrightarrow \infty$. However the closed-loop solution is obtained for $T \to \infty$, when the final time is not specified. Hence for the purpose of comparison of the open-loop control policy when the unspecified final time has a finite value T, the closed-loop control law is also obtained for that optimal finite final T (as if the final time T is specified).

The following four combinations of weighting factors are considered in this example.

case 1:	$r = 1$,	$q = 0.1$
case 2:	$r = 1$,	$q = 1$
case 3:	$r = 1$,	$q = 10$
case 4:	$r = 1$,	$q = 100$

The values of the various constants used are:

$$r_1 = 0.8 \qquad r_2 = 0.2$$

$$K_\alpha = 1.5 \qquad K_\beta = K_\gamma = 0.625$$

The results of this example for $\sigma_s = 0.75$ and $\sigma_s = 1.2$ are shown in Table 1.1 and Figures 2.1 through 2.8.

The results of open-loop and closed-loop control are exactly same for both the final time specified as well as the final time (optimal) not specified cases.

When the value of the final time T is specified, the objective function increases with increase in the weighting factor q on the deviation. Also the longer the control period T, the greater is the value of the objective function, for the same value of the step heat disturbance.

In case 1, where more weight is given to the control effort, ϕ, the optimal control policy has a very small negative value. Since the control effort ϕ is unconstrained it takes high positive values for larger values of q (cases 2, 3, and 4) and thus requiring two or more heat exchangers to be operated in series. However, the control effort ϕ reaches zero in all the cases at the end of the control period (T = 1 or 2).

When the (optimal) value of the final time T is not specified, the objective function increases with increase in the weighting factor q on the deviation. Since this is a heating system, the greater the value of σ_s, the lesser is the value of the objective function, for the same values of the weighting factors.

Also in the case of unspecified (optimal) control period, the control ϕ attains zero at the end of the optimal period, in all the four different combinations of weighting factors.

Again in the case of unspecified control period with $T \to \infty$ (for closed-loop control alone), as the weighting factor q on the deviation increases, the control effort ϕ takes higher positive values (since ϕ is unconstrained) and thus requiring two or more heat exchangers to be operated in series. However, the control effort ϕ as well as the deviation y_1 attain their respective steady-state values at different times in all the four different combinations of weighting factors - with the case 4 [r = 1 and q = 100] reaching the steady state at the earliest and the case 1 [r = 1 and q = 0.1] attaining the steady state at the latest.

Table 2.1

Values of the Objective Function of

Example 2: (Final time T specified).

σ_s	Specified Time	Case Number	Weighting Factors	Value of the Objective Function
0.75	1.0	1	r = 1, q = 0.1	0.03640
		2	r = 1, q = 1	0.32335
		3	r = 1, q = 10	2.04640
		4	r = 1, q = 100	8.83707
	2.0	1	r = 1, q = 0.1	0.04566
		2	r = 1, q = 1	0.41521
		3	r = 1, q = 10	2.50150
		4	r = 1, q = 100	9.39923
1.20	1.0	1	r = 1, q = 0.1	0.02512
		2	r = 1, q = 1	0.23981
		3	r = 1, q = 10	1.85926
		4	r = 1, q = 100	8.74289
	2.0	1	r = 1, q = 0.1	0.09365
		2	r = 1, q = 1	0.68598
		3	r = 1, q = 10	3.11722
		4	r = 1, q = 100	10.18210

Table 2.2

Values of the Objective Function of

Example 2: (Final time T not specified).

σ_s	Case Number	Weighting Factors	Optimal Time	Value of the Objective Function
0.75	1	r = 1, q = 0.1	1.15992	0.03648
	2	r = 1, q = 1	1.02294	0.32335
	3	r = 1, q = 10	0.66307	2.01150
	4	r = 1, q = 100	0.32660	8.54257
1.20	1	r = 1, q = 0.1	0.76410	0.02452
	2	r = 1, q = 1	0.71489	0.23042
	3	r = 1, q = 10	0.52356	1.66323
	4	r = 1, q = 100	0.27991	7.92176

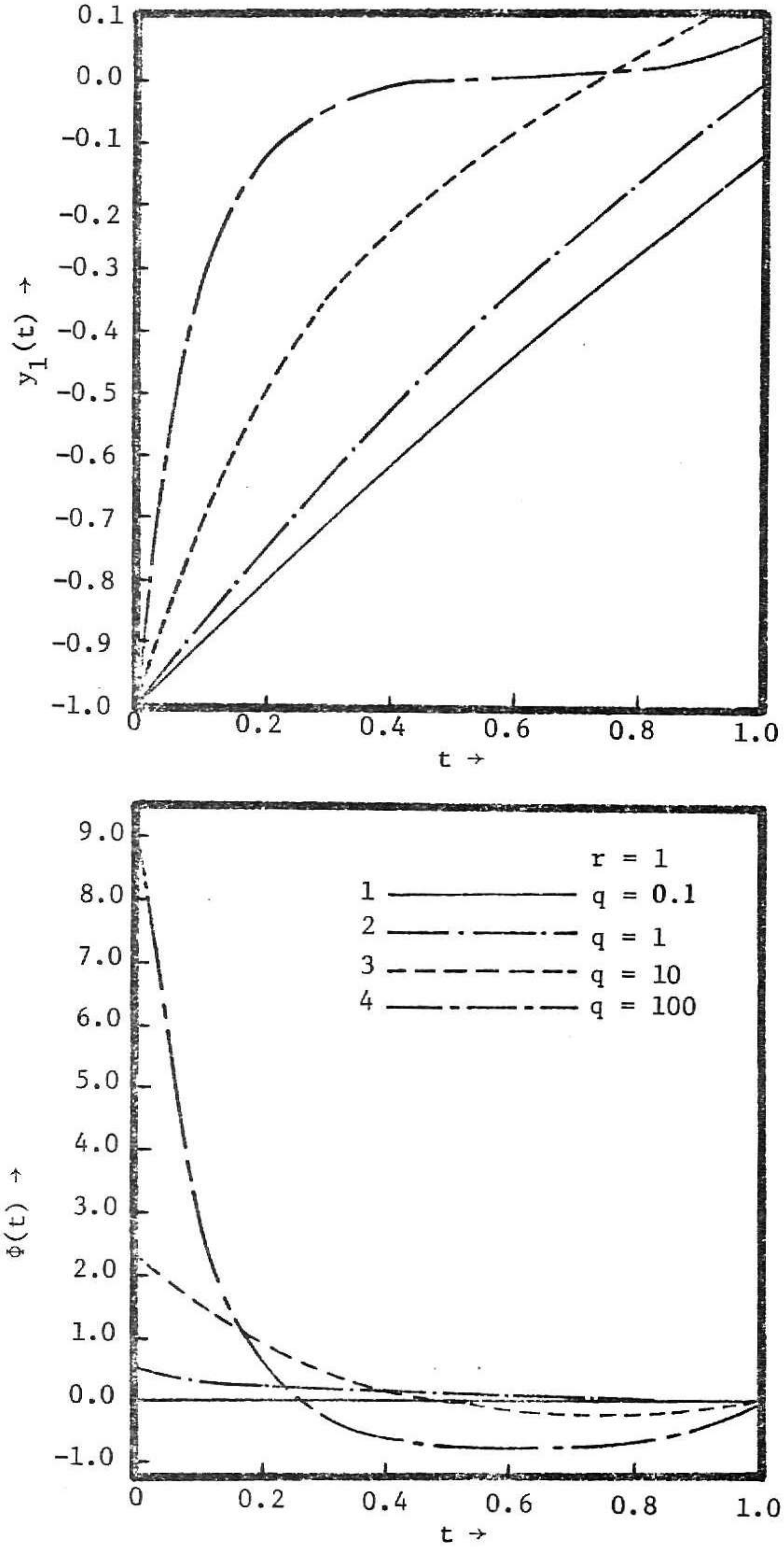

Fig. 7.2.1 Optimal control policy and system response of Example 2 (T specified) T = 1.0; $\sigma_s = 0.75$

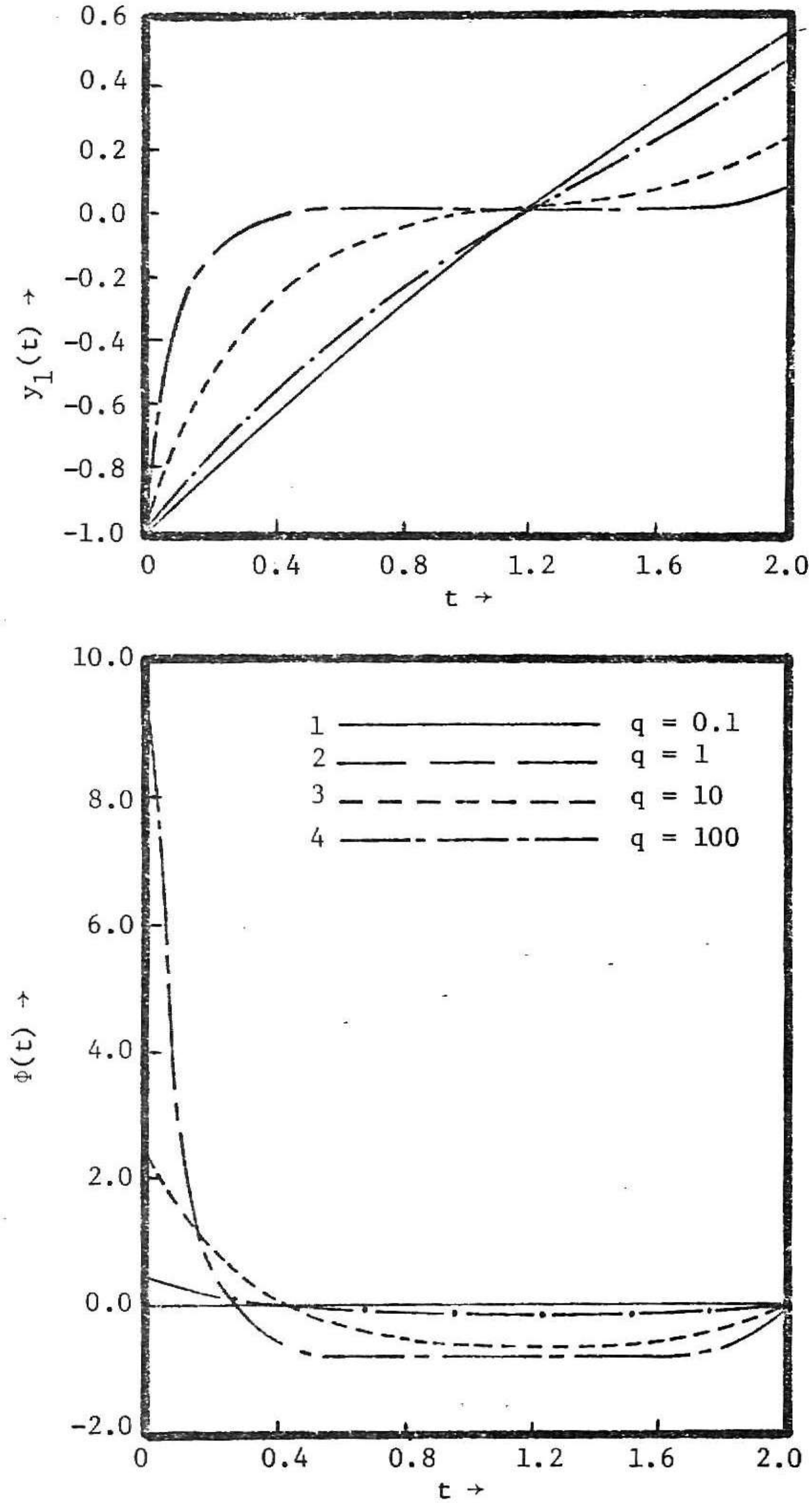

Fig. 7.2.2 Optimal control policy and system response of Example 2 (T specified) T =2.0; σ_s = 0.75

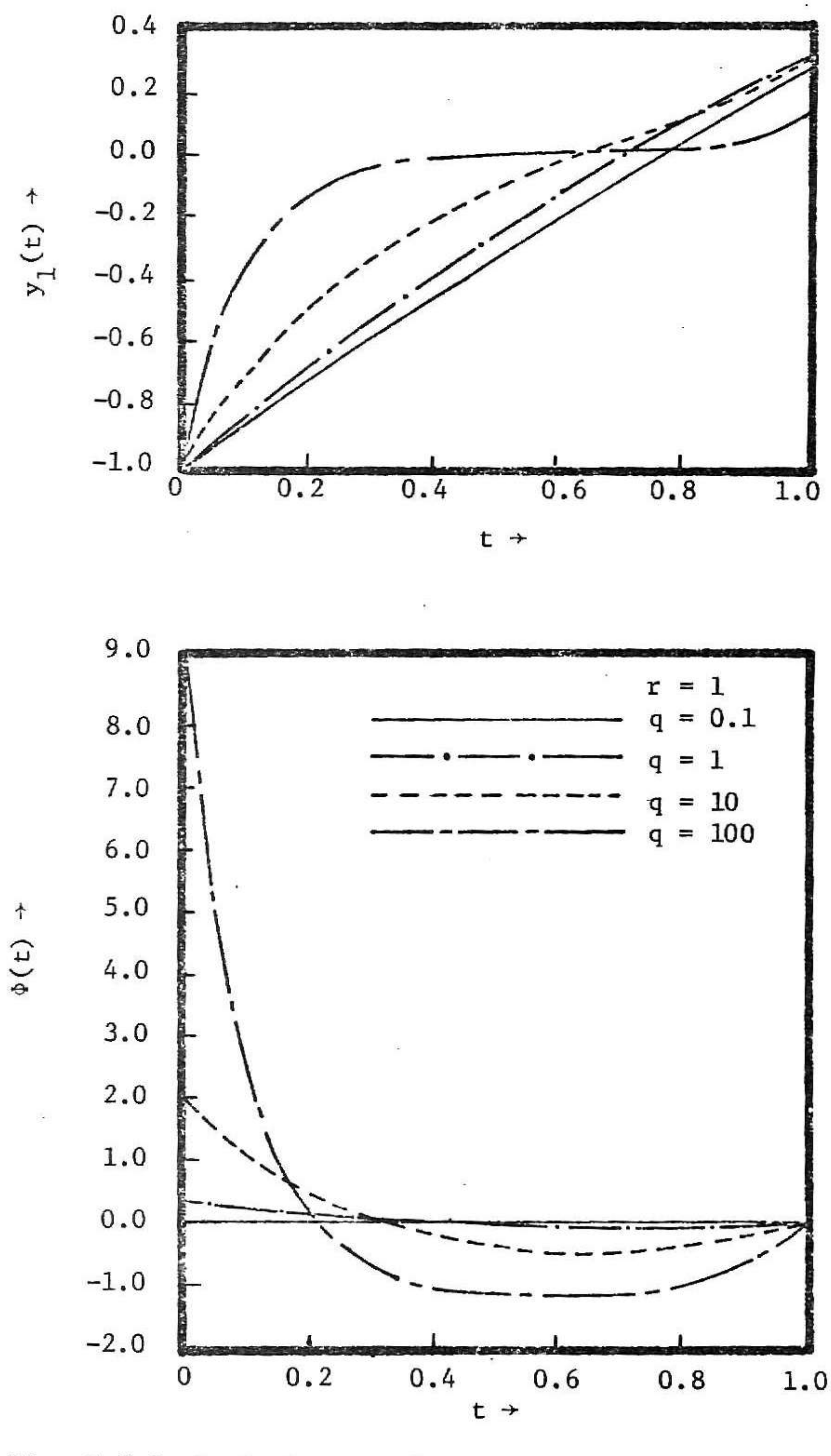

Fig. 7.2.3 Optimal control policy and system response of Example 2 (T specified) $T = 2.0$; $\sigma_s = 1.2$

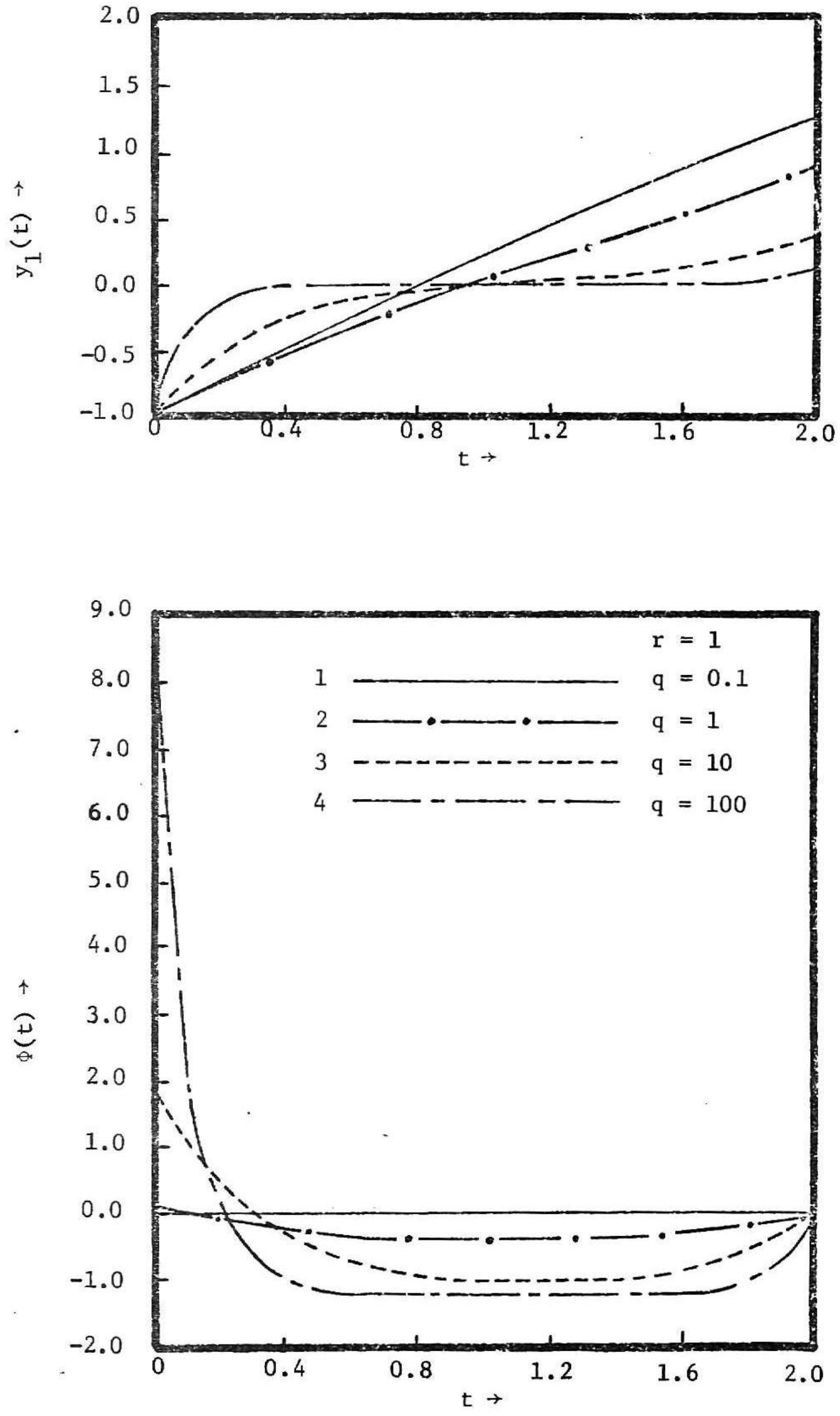

Fig. 7.2.4 Optimal control policy and system response of Example 2 (T specified) T = 2.0; $\sigma_s = 1.2$

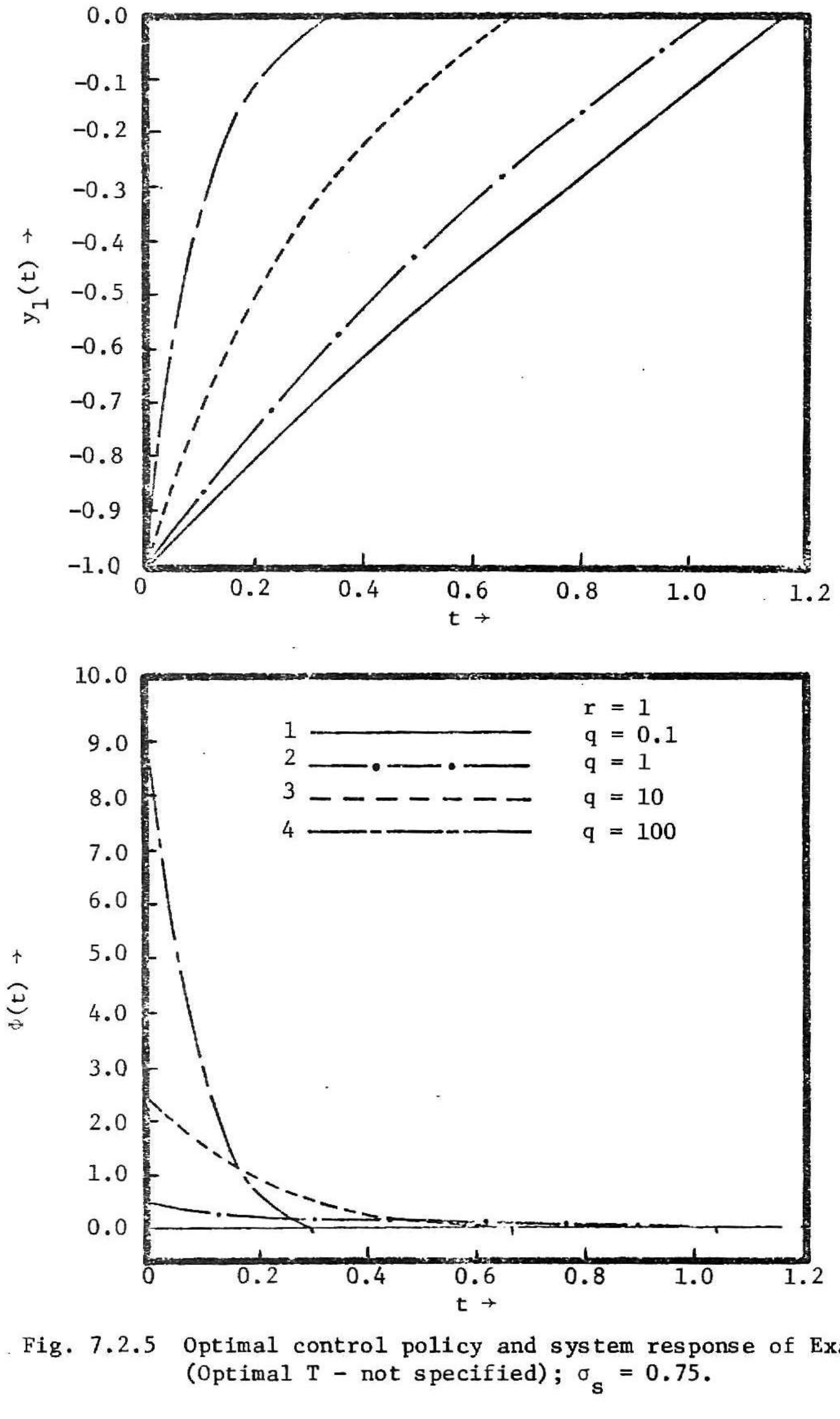

Fig. 7.2.5 Optimal control policy and system response of Example 2 (Optimal T - not specified); $\sigma_s = 0.75$.

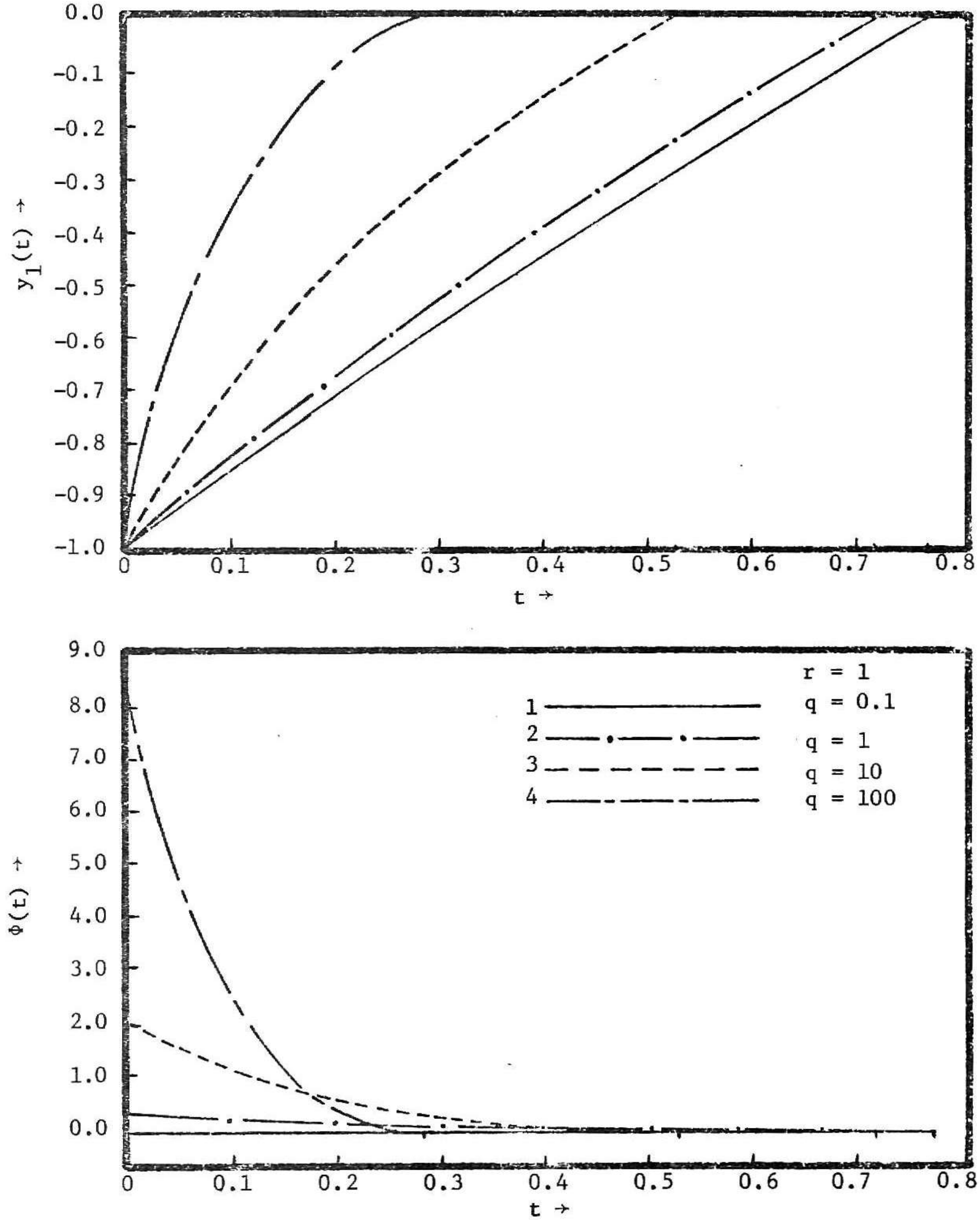

Fig. 7.2.6 Optimal control policy and system response of Example 2 (Optimal T - not specified); $\sigma_s = 1.2$

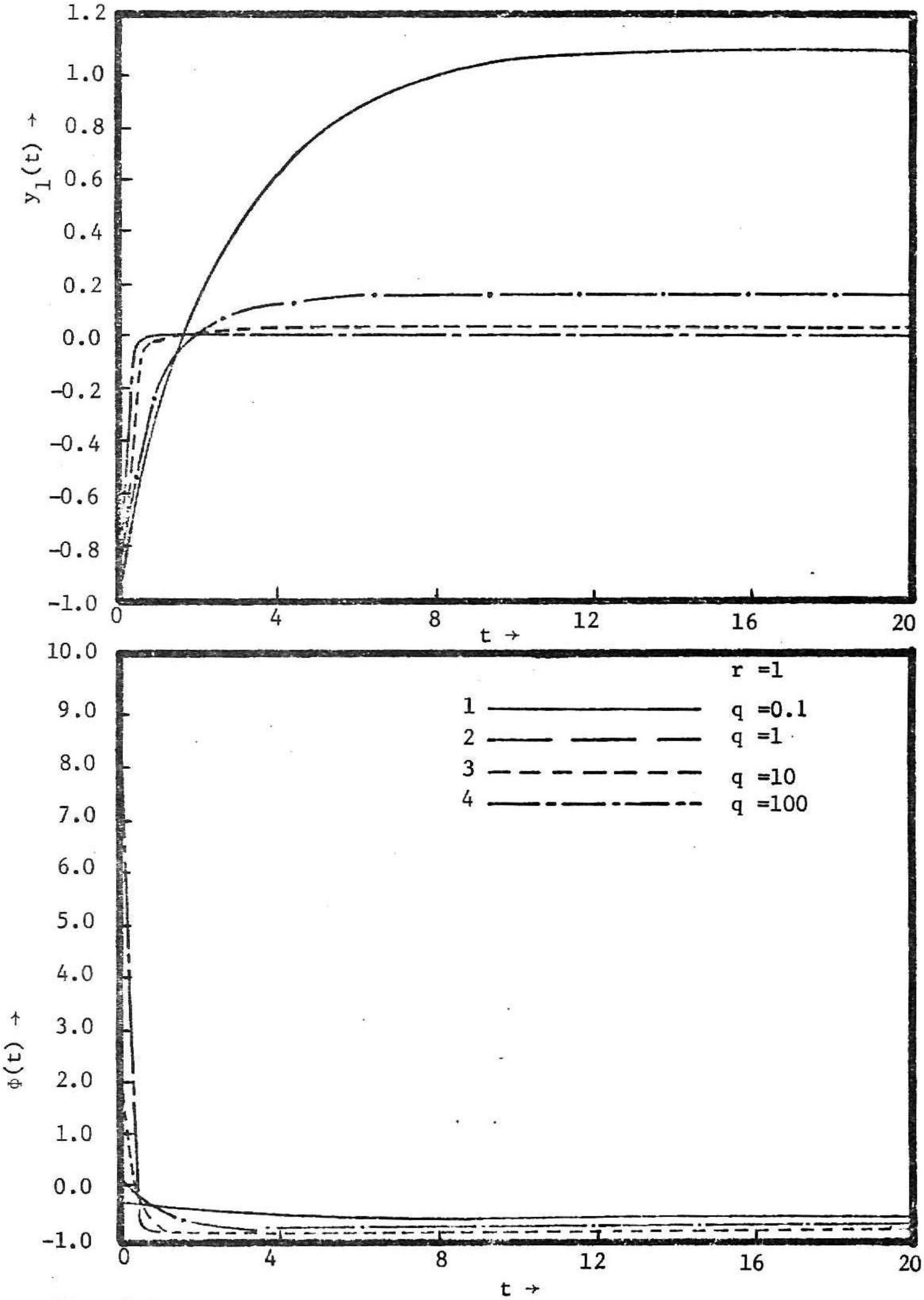

Fig. 7.2.7 Optimal control policy and system response of Example 2 (T - not specified) $T \rightarrow \infty$; $\sigma_s = 0.75$

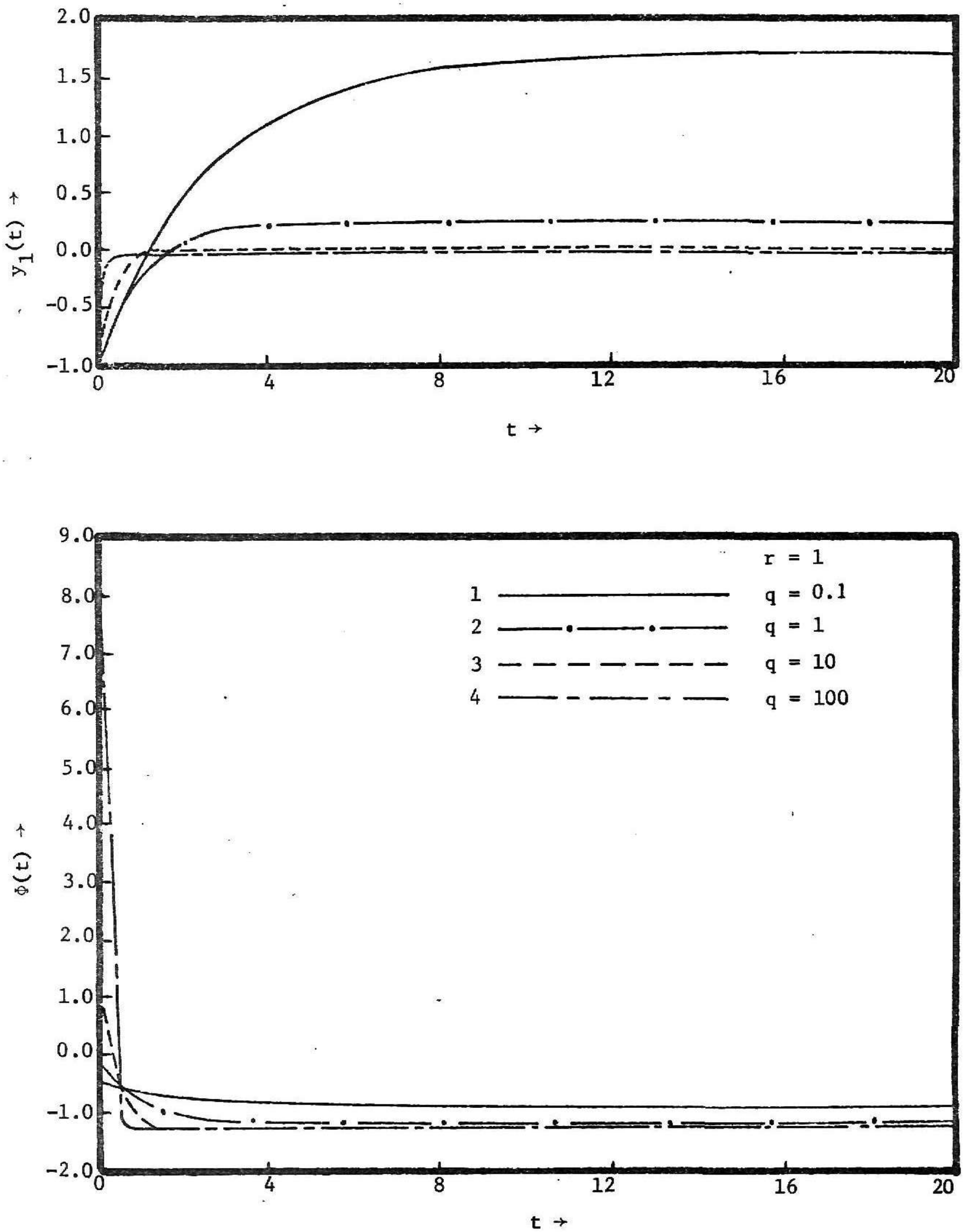

Fig. 7.2.8 Optimal control policy and system response of Example 2 (T - not specified) $T \rightarrow \infty$; $\sigma_s = 1.2$

CHAPTER 8

CONCLUSION

The modern optimal control theory can be employed for models of systems in which mass and momentum transfer take place in addition to heat transfer. It can also be applied to establish optimal control policy for systems for control of humidity, purity and noise.

The maximum principle has a certain advantage over other modern optimal control techniques in that it can be applied not only to the system with linear performance equations but also to those with non-linear performance equations. The maximum principle can handle constraints on state variables. Thus, any environmental control problem in which the temperature of the confined space has to be higher than a certain temperature - for example, a biomedical process - can be solved by means of the maximum principle [18]. The maximum principle can also be used to evaluate the number of switching points of the bang-bang control policy via the switching function and adjoint vectors. Bellman [4] has proven theoretically the number of switching points is one less than the dimension of the problem for linear systems. But this theory cannot be applied to non-linear systems.

It has often been said that the optimal control policy as obtained by the maximum principle is open-loop. This is not always true as shown through the discussion of the linear regulator problem in chapter 5 and through its application to real-life problems in chapter 7.

In summary, this report is a complement to the earlier work [17, 18, 19, 20, 21]. The models considered throughout this report can be made more realistic, though not perfect, by considering multiple CST's-in-series (instead of only one CST model). In this case, the time constants of the sensing element and the heat exchanger are no longer negligible. Computation of the optimal trajectory will be very difficult, if not impossible. Also, solution of the performance equations which consists of combination of the ordinary differential equation representing the dynamic behaviour of the system element, and the partial differential equation representing the dynamic behaviour of the heat exchanger will be more realistic, though the proceudre of solving this problem is quite sophisticated.

ACKNOWLEDGEMENT

The author thanks Dr. C. L. Hwang and Dr. L. T. Fan for their inspiring guidance, valuable criticisms and constant encouragement in bringing this work to a successful completion.

The author also thanks Dr. A. Ichikawa for his timely discussions, criticisms and suggestions.

The author expresses his gratitude to Dr. F. A. Tillman for his constant guidance and serving on his committee.

The author also thanks Mr. T. Natarajan for his checking the calculations and proof-reading.

Appreciation is expressed to Mrs. Marie Jirak for her efficient and excellent typing.

This study was supported in part by NASA Grant No. NGR-17-001-034 and by The Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under Contract F44620-68-0020 (Themis Project).

NOMENCLATURE

$a_1 = r\ K_1/K_4$

$a_2 = r_2\ K_1$

$a_3 = K_1\sigma$

$a_4 = r\ K_4/K_1$

$a_5 = r\ K_2\ K_4$ (Impulse heat disturbance)

$a_5 = r\ K_\beta$ (Step heat disturbance)

$a_5' = K_2\ K_4$

$a_6 = r\ K_3\ K_4$ (Impulse heat disturbance)

$a_6 = r\ K_\gamma$ (Step heat disturbance)

$a_6' = K_3\ K_4$

c_p = Specific heat of air in Kcal/Kg ^oC

c_{pw} = Specific heat of coolant in Kcal/Kg ^oC

$$K_\alpha = \frac{T_2}{T_{c0}}$$

$$K_\beta = \frac{1}{2T_{c0}}\ [T_{r\ max} - T_{r\ min}]$$

$$K_\gamma = \frac{1}{2T_{c0}}\ [T_{r\ max} - T_{r\ min}]$$

$$K_1 = \frac{T_2}{T_{c0}}$$

$$K_2 = \frac{1}{2T_2}\ (T_{r\ max} - T_{r\ min})$$

$$K_3 = \frac{1}{2T_2}\ (T_{r\ max} + T_{r\ min})$$

$K_4 \quad = \dfrac{T_2}{T_{i0}}$

q_{ds} = Step heat disturbance in Kcal/sec

q_{i1} = Heat flow into the system proper by circulation air in Kcal/sec

q_{i2} = Heat flow into the system proper by fresh air in Kcal/sec

q_{mi1} = Heat flow into the heat exchanger by circulation air in Kcal/sec

q_{mi2} = Heat flow into the heat exchanger by cooling water in Kcal/sec

q_{m01} = Heat flow out of the heat exchanger by circulation air in Kcal/sec

q_{m02} = Heat flow out of the heat exchanger by cooling water in Kcal/sec

q_{ms} = Heat stored in the heat exchanger in Kcal/sec

q_{01} = Heat flow out of the system proper by circulation air in Kcal/sec

q_{02} = Heat flow out of the system proper by fresh air in Kcal/sec

q_s = Rate of heat accumulation in the system proper

Q = $Q_1 + Q_2$, flow rate of air in the system proper in m^3/sec

Q_1 = Air flow rate by circulation air in m^3/sec

Q_2 = Flow rate of fresh air in m^3/sec

Q_w = Flow rate of coolant in m^3/sec

r = $\dfrac{\tau_1}{\tau_2}$, the ratio of time constant of system proper to that of heat exchanger

r_1 = $\dfrac{Q_1}{Q_1 + Q_2}$, the fraction of circulation air

$r_2 \quad = \dfrac{Q_2}{Q_1 + Q_2}$, the fraction of fresh air

$t \quad = \dfrac{\alpha}{\tau_1}$, dimensionless time

t_a = Reference temperature in ^oC

t_c = Room temperature in ^oC

t_d = Disturbance temperature in ^oC

t_i = Temperature of incoming circulation air in ^oC

t_o = Initial time

t_s = Switching time

t_{wc} = Inlet temperature of coolant in ^oC

t_{wh} = Outlet temperature of coolant in ^oC

t_2 = Outside air temperature in ^oC

T = Final time, dimensionless

T_c = $(t_c - t_a)$, room temperature in ^oC

T_{c0} = Room temperature at $\alpha = 0^+$ in ^oC

T_d = $(t_d - t_a)$, disturbance temperature in ^oC

T_i = $(t_i - t_a)$, temperature of the circulation air into the system proper, in ^oC

T_{i0} = Temperature of the circulation air into the system proper at $\alpha = 0^+$ in ^oC

$T_r \quad = \dfrac{Q_w \, \rho_w \, c_{pw}(T_{wh} - T_{wc})}{Q_1 \, \rho \, c_p}$, hypothetical temperature

T_{rf} = Final steady state value of T_r

$T_{r\,max}$ = Upper bound of T_r in ^oC

$T_{r\,min}$ = Lower bound of T_r in ^oC

T_{r0} = Value of T_r at $\alpha = 0$

T_{wc} = $t_{wc} - t_c$ in ^oC

T_{wh} = $t_{wh} - t_a$ in ^oC

T_2 = $(t_2 - t_a)$, outside air temperature

$U_0(t)$ = Step heat disturbance function

V_1 = Volume of room in m^3

V_2 = Volume of heat exchanger in m^3

$x_1(t)$ = $\frac{T_c}{T_{c0}}$, dimensionless room temperature

x_{1d} = Desired value of x_1

$x_2(t)$ = $\frac{T_i}{T_{i0}}$, dimensionless temperature of the circulation air

GREEK LETTERS

α = Time in sec.

α_f = Final time in sec.

$\delta(\alpha)$ = Impulse heat disturbance function, sec^{-1}

ρ = Air density in Kg/m^3

ρ_w = Density of coolant in Kg/m^3

σ = $\frac{T_d}{T_2}$, dimensionless disturbance temperature (Impulse)

σ_s = $\frac{T_d}{T_{c0}}$, dimensionless disturbance temperature (Step)

τ_1 = $\frac{V_1}{Q_1 + Q_2}$, time constant of the system proper in sec.

τ_2 = $\frac{V_2}{Q_1}$, time constant of heat exchanger in sec.

$$\theta = \frac{T_r - 1/2\ (T_{r\ max} + T_{r\ min})}{T_{r\ max} - 1/2\ (T_{r\ max} + T_{r\ min})}, \text{ control variable}$$

$$= \begin{cases} +1 & \text{at} \quad T_r = T_{r\ max} \\ -1 & \text{at} \quad T_r = T_{r\ min} \end{cases}$$

$\bar{\theta}(t)$ = Optimum value of $\theta(t)$

REFERENCES

1. Allen, R., "Time Modulations Control Increases Comfort," Part I and II, American Artisan, July (1950).
2. Bellman, R., Dynamic Programming, Princeton University Press, Princeton, New Jersey, 1957.
3. Bellman, R., Some Vistas of Modern Mathematics, University of Kentucky Press, 1968.
4. Bellman, R., I. Glicksberg and O. Gross, "On the Bang-bang Control Problems," Quarterly Appl. Math., 14, 11 (1955).
5. Bhandiwad, M. A., A Master's Report, Kansas State University (1970).
6. Buchberg, H., "Cooling Load from Thermal Network Solutions," Heating, Piping and Air Conditioning, October (1957).
7. Buchberg, H., "Cooling Load from Pretabulated Impedances," Heating, Piping and Air Conditioning, February 1958.
8. Chang, S. S. L., Synthesis of Optimum Control Systems, McGraw-Hill, New York, 1961.
9. Chang, S. S. L., "A Modified Maximum Principle for Optimum Control of a System with Bounded Phase Space Coordinates," Automatica, 1, 55 (1962).
10. Citron, S. J., Elements of Optimal Control, Holt, Rinehart and Winston, Inc., (1969).
11. Coughanour, D. R., and L. B. Koppel, Process Systems Analysis and Control, McGraw-Hill, New York, 1965.
12. D'Azzo, J. J. and C. H. Houpis, Feedback Control System Analysis and Synthesis, McGraw-Hill, New York, 1960.

13. Drake, W. B., "Transfer Admittance Functions," ASHRAE (1959).

14. Eltimsahy, A. H., "The Optimization of Domestic Heating Systems," Ph.D. Dissertation, 1967, University of Michigan.

15. Fan, L. T., The Continuous Maximum Principle: A Study of Complex Systems Optimization, John Wiley and Sons, New York, 1966.

16. Fan, L. T. and C. S. Wang, The Discrete Maximum Principle: A Study of Multistage Systems Optimization, John Wiley and Sons, New York, 1964.

17. Fan, L. T., Y. S. Hwang, and C. L. Hwang, "Applications of Modern Optimal Control Theory to Environmental Control of Confined Spaces and Life Support Systems, Part 1. Modeling and Simulation," Build. Sci. Vol. 5, pp. 57-71 (1970).

18. Fan, L. T., Y. S. Hwang, and C. L. Hwang, "Applications of Modern Optimal Control Theory to Environmental Control of Confined Spaces and Life Support Systems, Part 2. Basic Computational Algorithm of Pontryagin's Maximum Principle and its Applications," Build. Sci. Vol. 5, pp. 81-94 (1970).

19. Fan, L. T., Y. S. Hwang, and C. L. Hwang, "Applications of Modern Optimal Control Theory to Environmental Control of Confined Spaces and Life Support Systems, Part 3. Optimal Control of Systems in which State Variables have Equality Constraints at the Final Process Time," Build. Sci. Vol. 5, pp. 125-136 (1970).

20. Fan, L. T., Y. S. Hwang, and C. L. Hwang, "Applications of Modern Optimal Control Theory to Environmental Control of Confined Spaces and Life Support Systems, Part 4. Control of Systems with Inequality Constraints Imposed on State Variables," Build. Sci. Vol. 5, pp. 137-147 (1970).

21. Fan, L. T., Y. S. Hwang, and C. L. Hwang, "Applications of Modern Optimal Control Theory to Environmental Control of Confined Spaces and Life Support Systems, Part 5. Optimality and Sensitivity Analysis," Build. Sci. Vol. 5, pp. 149-152 (1970).

22. Gartner, J. R., H. L. Harrison, "Dynamic Characteristics of Water to Air Cross Flow Heat Exchangers," ASHRAE (1965).

23. Gurel, O. and L. Lapidus, "The Maximum Principle and Discrete Systems," I&EC Fundamentals, 7, 617 (1968).

24. Haines, J. E., Automatic Control of Heating and Air Conditioning, pp. v-vi, McGraw-Hill, New York, 1953.

25. Hammond, P. H., Feedback Theory and its Applications, The MacMillan Company, New York, 1958.

26. Harrison, H. L., W. S. Hansen and R. E. Zelenski, "Development of a Room Transfer Function Model for Use in the Study of Short-term Transient Response," ASHRAE J., 5, 1968.

27. Healey, M., Principles of Automatic Control, The English Universities Press Ltd., St. Paul House Warwick Lane, London EC4., 1967.

28. Kalman, R. E., "Contributions to the Theory of Optimal Control," Bol. Soc. Mat. Mex., Vol. 5, pp. 102-119, 1960.

29. Kalman, R. E., "The Theory of Optimal Control and the Calculus of Variations," Mathematical Optimization Techniques, R. Bellman, ed., University of California Press, Berkeley, Calif., 1963.

30. Kalman, R. E., "Mathematical Description of Linear Dynamical Systems," J. SIAM Control, Ser. A., Vol. 1, pp. 152-192, 1963.

31. Kalman, R. E., et al., "Fundamental Study of Adaptive Control Systems," Wright-Patterson Air Force Base Tech. Rept., ASD-TR-61-27, Vol. 1, April, 1962.

32. Kelley, H. J., "Methods of Gradients," G. Leitman, ed., Optimization Techniques, Chapter 6, Academic Press, New York, 1962.

33. Kuo, B. C., Automatic Control Systems, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1962.

34. Lapidus, L. and R. Luus, Optimal Control of Engineering Processes, Blaisdell, Walthem, Massachusetts, 1967.

35. Leondes, C. T., Modern Control Systems Theory, McGraw-Hill, New York, 1965.

36. Pontryagin, L. S., V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenke, The Mathematical Theory of Optimal Process, (English Translation by K. N. Trirogoff), Interscience, New York, 1962.

37. Proposed Symbols and Terms for Feedback Control Systems, AIIE Committee Reports, Elec. Eng., vol. 70, pp. 905-909, 1951.

38. Sage, A. P., Optimum Systems Control , Prentice-Hall, (1968).

39. Standards on Terminology for Feedback Control Systems, Institution of Radio Engineers (U.S.A), Proceedings I.R.E., January 1956.

40. Takahashi, Y., "The Maximum Principle and its Applications," ASME paper 63-WA-333 (1963).

41. Tou, J. C., Modern Control Theory, pp. 5-12, McGraw-Hill, New York, 1954.

42. Uchida, H., et al., Automatic Control of Air Conditioning (in Japanese), pp. 7-44, Scientific Technology Center, Tokyo, 1963.

43. Vaughan, R. L., H. M. Stephenes, and R. S. Barker, "Generalized Environmental Control and Life Support System; Fortran Program - Vol. 1," Douglas Report SM-49403, Santa Monica, Calif., May 1966.

44. Webb, P., J. F. Annis, and S. J. Troutman, "Automatic Control of Water Cooling in Space Suits," NASA Report CR-1085, National Aeronautics and Space Administration, Washington, D.C., 1968.

45. Zermuehlen, R. O. and H. L. Harrison, "Room Temperature Response to A Sudden Heat Disturbance Input," ASHRAE Trans., 71, Part 1, 206 (1965).

APPLICATION OF THE MAXIMUM PRINCIPLE TO THE OPTIMAL CONTROL OF LIFE SUPPORT SYSTEMS

by

GOVINDASWAMY NADIMUTHU

B.E. (Mechanical)
Madras University, India, 1968

D.I.I.T. (Industrial Engineering)
Indian Institute of Technology, Madras, India, 1969

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirement for the degree

MASTER OF SCIENCE

Department of Industrial Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1971

ABSTRACT

The temperature control of an environmental control system is studied. The system consists of a confined space or cabin subjected to a heat disturbance, a heat exchanger of negligible time constant and a feedback element such as a thermostat. Both a step heat disturbance and an impulse heat disturbance are considered.

The basic forms of Pontryagin's maximum principle for open-loop control and Kalman's linear regulator problem for closed-loop control are outlined and their application is illustrated through a simple example with a linear performance equation and a quadratic objective function. The environmental control system considered in this report also has a linear system equation and a quadratic objective function.

The optimal open-loop control policies of a heating system - subjected to a step heat disturbance, having the initial and final values of the state variable, namely the temperature, fixed but the final time T to be determined - are established for the minimization of the following objective functions.

(i) $$S = \int_0^T dt$$

(ii) $$S = \int_0^T (a + b\theta^2)\,dt$$

(iii) $$S = \int_0^T [b\theta^2 + c(x_1 - x_{1d})^2]\,dt$$

(iv) $$S = \int_0^T [a + b\theta^2 + c(x_1 - x_{1d})^2]dt$$

The optimal open-loop control policies and the optimal closed-loop control laws are determined for two examples - one dealing with a cooling system subjected to an impulse heat input and the other dealing with a heating system subjected to a step heat input. In each of these examples the performance equation is of linear form and the objective function is of quadratic form. Also, each example has the right-end free and deals with both the cases of specified and unspecified final time. It is found in each example that the results obtained by both the open-loop control policy and the closed-loop control law are exactly same whether optimal T is specified or not, as long as the linear system has a quadratic functional.