

Nanoscale problems for an elastic semi-space subjected to an axisymmetric
loading

by

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B. S., California State University - Northridge, 2015

AN ABSTRACT OF A DISSERTATION

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Department of Mathematics
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Abstract

Due to technological advancements, problems at nanoscale have gained significant interest in recent years. Material properties at nanoscale exhibit size-dependent behavior which can be attributed to the influence of surfaces and interfaces. Correspondingly, problems at nanoscale have to take into account surface energy. Here, we consider two problems involving an isotropic half-space subjected to either nanoscale contact or nanoscale patch loading.

In the first problem, we consider an isotropic half-space subjected to nanoscale contact with a rigid punch. The surface energy in the Steigmann-Ogden form is used to model the half-space while linear elasticity is used to model the bulk of the material. The nanoindentation problem is solved using Boussinesq's displacement potentials and Hankel integral transforms. The problem is reduced to a single integral equation, the character of which is studied, and a numerical method of solution to the corresponding integral equation using Gauss-Chebyshev quadrature is presented.

In the second problem, we consider another problem at nanoscale in which an isotropic half-space is subjected to a nanoscale patch load. The surface effects are considered on a circular subset of the surface by assuming the Steigmann-Ogden model on the subset, while the remainder of the surface is assumed to not have surface effects. Using techniques presented previously in the work, the problem is reduced to a system of singular integro-differential equations with boundary conditions. Solutions corresponding to the dominant systems with Cauchy kernels are explored. A numerical method of solution of the system of integral equations using approximation by Chebyshev polynomials and Gauss-Chebyshev quadrature for computation of the integrals is presented and results are compared to those available the literature.

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Dedication

I dedicate this work to my sister Hannah, my friend Vanessa, and lastly to my partner Nicholas, for supporting and encouraging me.

Chapter 1

Introduction

In recent years, technological advancements have allowed scientists the ability to study materials at nanoscale, discovering that the behavior of materials exhibit size dependence at these smaller scales. Properties such as melting point, electrical conductivity, magnetic permeability, and many other observable effects change as a function of the size of the material at this scale [15, 53, 65]. Even the movement of electrons changes at this scale compared to larger scales, making light react differently and changing the observable color of a material. It has been shown that the size dependency of material properties at nanoscale are attributed to quantum effects and physical effects such as expanded surface area. The study of materials at this scale is necessitated by industrial efforts that utilize the special properties of materials at nanoscale. The ability for an observable property of a material to be ‘fine tuned’ allows for medical applications where precision tests are developed to find bio-markers for diseases that were previously undetectable [61]. The large surface area of nanomaterials also has applications in renewable energy as well as water purification.

At these small scales, it has been shown that the behavior of materials is influenced by surface effects [1, 6, 7, 19, 20, 43, 49, 58, 77, 78]. For a nanomaterial, the surface area to volume ratio is far greater than for a larger scale object of similar proportions. The effect of the surface has been observed to be in the inverse relationship between material hardness and the indenter size.

The study of the influence of surface mechanics on the behavior of solid materials has its beginnings over a century ago, in the works of Gibbs [21]. A systematic derivation of the theory of surface elasticity, the study of how the surface of a material will react to force by bending or deforming, using a variational formulation of the problem was first presented by Gurtin and Murdoch [26, 27]. The theory, referred to as the Gurtin-Murdoch theory of surface elasticity, treats the surface of the material as an infinitely thin membrane which resists stretching and may have an applied prestress. Steigmann and Ogden [59, 60] pointed out that the Gurtin-Murdoch surface elasticity model does not support compressive surface loads. To avoid the contradiction this creates, Steigmann and Ogden developed a model for the surface by augmenting the Gurtin-Murdoch model with the ability of the surface to resist to bending. The mathematical significance of the extension from the Gurtin-Murdoch model to the Steigmann-Ogden model is that the boundary conditions on the surface contain a combination of tractions and higher order derivatives of the displacements. The Steigmann-Ogden model of surface elasticity has been shown to play an important role at the smallest scales, observed in the context of fracture problems [71].

The study of contact problems, where two bodies come into contact with one another to cause a deformation, and patch loading problems, where the surface has an applied external load, with the presence of surface effects has been considered by several authors, especially in recent years. Problems in both two- and three-dimensional settings have been studied [18, 79–81]. The effect of the surface’s resistance to bending on a patch-loaded semi-space using the Steigmann-Ogden model of surface elasticity has been studied [37, 44]. The Flamant solution for a semi-plane with the Steigmann-Ogden model of surface elasticity has been studied by Li et al. [39]. A Hertzian contact problem with the Gurtin-Murdoch model has been studied for both semi-plane and semi-space [40, 41]. The problems have been reduced to integral equations of the first kind with additional conditions. The contact problem for a semi-plane with the Gurtin-Murdoch model has been explored and the problem reduced to an integral equation of the second kind on a semi-infinite interval [50, 51]. Solutions for frictionless and adhesive contact problems of a rigid stamp indentation into a semi-plane with the Steigmann-Ogden model on the surface have been derived [69, 70]. The nanoindentation of a cylindrical

roller onto a semi-plane, assuming the Steigmann-Ogden model of surface elasticity, has also been explored [38]. Axisymmetric problems for transversely isotropic materials with the Gurtin-Murdoch model of surface elasticity have been investigated [55, 64]. Contact problems with curvature-dependent surface tension have also been studied [63].

The study of aspects and applications of the Gurtin-Murdoch and Steigmann-Ogden models of surface elasticity is an active field. The existence and uniqueness of weak solutions has been studied from a variational perspective in [13, 14]. Problems involving fracture mechanics with the surface modeled using the Steigmann-Ogden model of surface elasticity have been studied in [69, 71, 72]. The influence of surface elasticity on fibers and spherical inclusions has been studied [2, 28, 46, 73, 74]. Antiplane and plane problems for a thin inclusion where the Gurtin-Murdoch model of surface elasticity is used have been studied in [3, 47]. A penny-shaped crack problem has been studied using the Gurtin-Murdoch model of surface elasticity in [29] and using the Steigmann-Ogden model in [76]. The interaction between a line edge dislocation using the Steigmann-Ogden model of surface elasticity to model the two elastic half-planes has been studied in [10]. Plane and anti-plane fractures using the Gurtin-Murdoch model of surface elasticity have been studied in [22, 30–35]. A model with a different type of curvature dependence of the surface energy has been investigated in [16, 54, 62, 66, 67, 75].

A particular case of a problem of axisymmetric contact of a rigid punch and a semi-space, modeled with the Steigmann-Ogden model of surface elasticity was studied in [36]. The problem presented within Chapter 3 of the current work differs from the literature in several ways. First, the problem presented in the current work assumes the punch to be of an arbitrary smooth axisymmetric shape whereas the work by Li and Mi [36] assumes a punch of approximately spherical shape. The most important distinction in our work is that the kernel of the integral equation is written as a single integral instead of a double integral. This allows for more computational efficiency since both kernels involve oscillatory integrals on a semi-infinite interval which are notoriously difficult to compute. The reason for this is simple to understand. When computing an integral numerically, it is approximated using a finite set of data points and because of the oscillatory behavior, whole intervals of either

positive or negative values of the function may not be taken into account. Fortunately in the problem presented in Chapter 3, the type of kernel can be computed accurately and efficiently using the available IIPBF algorithm [42, 52]. Finally, the form of the kernel presented in the current work allows for a study of the type of equation that is obtained. The work contained in this chapter is published, see reference [76].

The patch loading problem on the semi-space has been solved by Mi [44] where the surface is modeled using the Steigmann-Ogden model of surface elasticity. The work contained in Chapter 4 differs from the available literature in that surface effects are considered on only a subset of the surface. The Steigmann-Ogden model of surface elasticity is assumed on this subset, while the remainder of the surface is assumed to have no surface effects present. Because the surface has two regions with differing assumptions on the influence of surface effects, care is needed at the shared boundary of these two subsets. The problem reduces to either one singular integral equation or a system of two singular integral equations depending on the surface effects, with additional tip conditions. The equations may be transformed into integral equations of Fredholm type of the second kind. In both the Steigmann-Ogden model and Gurtin-Murdoch model of surface elasticity, the prestress plays a large role in the behavior of the solution near the boundary of the surface subset. This can be seen in the numerical exploration done in this chapter. The numerical solution using the boundary conditions assumed in Mi's paper [44] and the boundary conditions assumed in this work are compared numerically in order to explore the influence of the prestress on the behavior of the solution.

For the remainder of this work, the material is assumed to be linearly elastic. This means that the deformations are assumed to be small, and the material obeys Hooke's law. The material is also assumed to be isotropic, meaning it has the same measurable properties in all directions. Some examples of isotropic materials are metals and glass.

The remaining chapters are outlined as follows. The model of linear isotropic elasticity, used to model the bulk of the material, and the Steigmann-Ogden model of surface elasticity, used to model the surface, are detailed in Chapter 2. Chapter 3 describes a contact problem between a rigid stamp and a material given by the upper semi-space in three dimensions at

nanoscale. In Chapter 4, a patch-loading problem at nanoscale on a semi-space is presented, where only a subset of the surface is modeled using the Steigmann-Ogden model of surface elasticity. Finally, conclusions and future directions of work are presented in Chapter 5.

Chapter 2

Models for the bulk and the surface of the material

In this chapter, the models used for the bulk and surface of the solid material are introduced. For the problems discussed in this work, the conditions that allow for modeling by linear elasticity are assumed for the bulk of the material. The material is also assumed to be isotropic, meaning it has the same measurable properties in all directions. The solid body is assumed to occupy the upper semi-space with surface boundary $z = 0$. An external force or displacement is applied on a circular region of the surface of the material.

For the two problems discussed in later chapters, it will be helpful to rewrite the equations for the bulk and surface elasticity in cylindrical coordinates. The cylindrical coordinate system generalizes the polar coordinate system in two dimensions by defining a point in three dimensional space by (r, θ, z) where r is the radial coordinate, θ is the azimuthal coordinate, and z is the vertical coordinate. The unit coordinate vectors are given by

$$\mathbf{e}_r = \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix}, \quad \mathbf{e}_\theta = \begin{bmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{bmatrix}, \quad \mathbf{e}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The origin of the coordinate system is taken to be at the center of the circular region on the

surface where the external force or displacement is applied.

The remainder of this chapter is as follows: Section 2.1 is devoted to discussion of the model of linear elasticity for an isotropic material in cylindrical coordinates. Boussinesq potentials are used to represent the displacements and stresses in terms of integrals involving Bessel functions. It is assumed that the displacements and external forces acting on the material are axisymmetric. Specifically, these functions are rotationally symmetric about the z -axis and independent of the value of θ . Section 2.2 outlines the Steigmann-Ogden model of surface elasticity, which is derived using variational methods. The equations are then specified for cylindrical coordinates under the axisymmetric assumption.

2.1 Linear Elasticity in the bulk and Boussinesq displacement potentials

By assuming that the displacements of the points of the material are small, the solid object may be modeled using a linear relationship between stress, the force acting in the material per unit area, and strain, the relative change in the shape of the material. When body forces are neglected, Navier's equations of equilibrium for an isotropic bulk material can be expressed in a vector form by [4]

$$\nabla(\nabla \cdot \mathbf{u}) + (1 - 2\nu)\nabla^2 \mathbf{u} = 0,$$

where ν is Poisson's ratio, ∇^2 is the vector Laplacian differential operator, and $\mathbf{u}$ is the vector of displacements. In terms of cylindrical displacement components (u_r, u_θ, u_z) the equations may be written as:

$$\begin{aligned} \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} + \frac{1}{1 - 2\nu} \frac{\partial}{\partial r} \left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \right) &= 0, \\ \nabla^2 u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} + \frac{1}{r(1 - 2\nu)} \frac{\partial}{\partial \theta} \left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \right) &= 0, \end{aligned}$$

$$\nabla^2 u_z + \frac{1}{1-2\nu} \frac{\partial}{\partial z} \left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \right) = 0.$$

The strategy to obtain solutions of Navier's equations given in the literature finds its origins in potential theory and is used to solve coupled differential equations similar to the ones above. The method involves writing the unknown functions in terms of a scalar and a vector potential [4, 24]. This method simplifies the equations by reducing the number of unknowns. For problems with axial symmetry, we only need two of these displacement potentials. The following Boussinesq [5] representations for the displacements can be used

$$2\mu u_r(r, z) = \frac{\partial \Phi_0(r, z)}{\partial r} + z \frac{\partial \Phi_3(r, z)}{\partial r}, \quad (2.1)$$

$$2\mu u_z(r, z) = \frac{\partial \Phi_0(r, z)}{\partial z} - (3-4\nu)\Phi_3(r, z) + z \frac{\partial \Phi_3(r, z)}{\partial z},$$

where μ is the shear modulus of the material and Φ_0 , Φ_3 are both harmonic functions. This representation is a special case of the more complete Papkovitch-Neuber displacement potentials. Using the strain-displacement relation and the generalized Hooke's law, the corresponding stresses can be expressed as:

$$\sigma_{rz}(r, z) = \frac{\partial^2 \Phi_0(r, z)}{\partial r \partial z} + z \frac{\partial^2 \Phi_3(r, z)}{\partial r \partial z} - (1-2\nu) \frac{\partial \Phi_3(r, z)}{\partial r}, \quad (2.2)$$

$$\sigma_{zz}(r, z) = \frac{\partial^2 \Phi_0(r, z)}{\partial z^2} + z \frac{\partial^2 \Phi_3(r, z)}{\partial z^2} - 2(1-\nu) \frac{\partial \Phi_3(r, z)}{\partial z}.$$

A harmonic function in cylindrical coordinates Φ is a twice-differentiable function which satisfies $\Delta \Phi = 0$, where Δ is the Laplace operator in cylindrical coordinates given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

For the semi-space $z \geq 0$, the solution to the cylindrical Laplacian can be found by separation of variables. This method of solution is commonplace in solving partial differential equations

and assumes there is a solution of the type

$$V(r, \theta, z) = R(r)P(\theta)Z(z).$$

Plugging this in to the Laplacian and dividing by V gives

$$\frac{1}{R(r)} \frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{rR(r)} \frac{\partial R(r)}{\partial r} + \frac{1}{r^2 P(\theta)} \frac{\partial^2 P(\theta)}{\partial \theta^2} + \frac{1}{Z(z)} \frac{\partial^2 Z(z)}{\partial z^2} = 0$$

which can be rewritten as three equations

$$\begin{aligned} \frac{\partial^2 Z(z)}{\partial z^2} - \lambda^2 Z(z) &= 0, \\ \frac{\partial^2 P(\theta)}{\partial \theta^2} + n^2 P(\theta) &= 0, \\ r^2 \frac{\partial^2 R(r)}{\partial r^2} + r \frac{\partial R(r)}{\partial r} + (\lambda^2 r^2 - n^2) R(r) &= 0. \end{aligned}$$

The first two equations have exponential type solutions and the last equation is a scaled Bessel equation of order n , resulting in solutions given by

$$Z(z) = e^{\pm \lambda z}, \quad P(\theta) = e^{\pm i n \theta}, \quad R(r) = J_n(\lambda r) \text{ and/or } Y_n(\lambda r),$$

where the Bessel function of the first kind of order n , J_n , is regular at $r = 0$ and the Bessel function of the second kind of order n , Y_n , has a singularity at $r = 0$. By asserting there are no singularities at $r = 0$, only the Bessel function J_n is used in the final solution. The solution is also assumed to approach zero as $z \rightarrow \infty$, forcing the input λ to be restricted to the set of positive reals. Noting that the values of λ are taken on a continuum, we can write the solution as a linear combination of the terms given by

$$V(r, \theta, z) = \sum_n \int_0^\infty (v_n(\lambda) \cos(n\theta) + w_n(\lambda) \sin(n\theta)) J_n(\lambda r) e^{-\lambda z} d\lambda.$$

For the axisymmetric case, only the first term in the summation over n is nonzero, resulting

in the solution to the Laplacian in cylindrical coordinates for axisymmetric functions:

$$V(r, z) = \int_0^\infty v(\lambda) J_0(\lambda r) e^{-\lambda z} d\lambda.$$

For the semi-space $z \geq 0$, the functions Φ_0 and Φ_3 appearing in (2.1), (2.2) may be written as

$$\Phi_0(r, z) = \int_0^\infty A(\lambda) J_0(\lambda r) e^{-\lambda z} d\lambda, \quad \Phi_3(r, z) = \int_0^\infty B(\lambda) J_0(\lambda r) e^{-\lambda z} d\lambda,$$

where $A(\lambda)$ and $B(\lambda)$ are unknown functions to be determined by satisfying the boundary conditions at the plane boundary. The displacement components may be written as:

$$2\mu u_r(r, z) = \int_0^\infty \lambda(A(\lambda) + zB(\lambda)) J'_0(\lambda r) e^{-\lambda z} d\lambda, \quad (2.3)$$

$$2\mu u_z(r, z) = - \int_0^\infty (\lambda A(\lambda) + (3 - 4\nu + \lambda z)B(\lambda)) J_0(\lambda r) e^{-\lambda z} d\lambda.$$

The corresponding stresses are:

$$\sigma_{zz}(r, z) = \int_0^\infty \lambda(\lambda A(\lambda) + (2 - 2\nu + \lambda z)B(\lambda)) J_0(\lambda r) e^{-\lambda z} d\lambda, \quad (2.4)$$

$$\sigma_{rz}(r, z) = - \int_0^\infty \lambda(\lambda A(\lambda) + (1 - 2\nu + \lambda z)B(\lambda)) J'_0(\lambda r) e^{-\lambda z} d\lambda.$$

Note since our problem is axisymmetric, u_θ or any stress component with θ equals zero. This concludes the information on the model for the bulk of the material. Next, the model used to represent surface elasticity is presented.

2.2 The Steigmann-Ogden model of surface elasticity

For the problems discussed in later chapters, the surface of the material is modeled by the Steigmann-Ogden model of surface elasticity [59, 60]. The Steigmann-Ogden model of surface elasticity can be obtained using a variational formulation [13, 68, 73]. Below is a brief derivation of the equations. This derivation relies on the principle of virtual work, which

asserts that the total virtual work done by the applied forces and internal forces must be zero. This may be written as:

$$\partial E = \partial P$$

where E is the functional of strain energy and P is the functional of work of external forces. The strain energy is defined as the sum of volumetric and surface terms

$$E = \int \int \int_V \mathcal{W} dv + \int \int_{A_s} \mathcal{U} da,$$

where $\mathcal{W}$ and $\mathcal{U}$ are the bulk and surface strain energy densities, respectively. The set A_s is the subset of the surface where surface energy is applied.

For an isotropic bulk, the following constitutive equations hold:

$$\mathcal{W} = \mu \mathbf{e} : \mathbf{e} + \frac{1}{2} \lambda tr^2 \mathbf{e}, \quad \sigma = 2\mu \mathbf{e} + \lambda I tr \mathbf{e}, \quad \mathbf{e} = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T),$$

where $\mathbf{e}$ is the strain tensor, λ is Lamé's first constant, μ is the shear modulus of the bulk, the function denoted by the symbol $:$ is the scalar double-dot product of two second-order tensors, $\mathbf{u}$ is the displacement vector, and ∇ is the gradient operator.

For the surface strain energy, first define the function $\mathbf{v} = \mathbf{u}|_{A_s}$. The surface strain energy density is given by:

$$\mathcal{U} = \sigma_0(1 + tr \boldsymbol{\varepsilon}) + \frac{1}{2}(\lambda_0 + \sigma_0)tr^2 \boldsymbol{\varepsilon} + (\mu_0 - \sigma_0)\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} + \chi \boldsymbol{\kappa} : \boldsymbol{\kappa} + \frac{1}{2}\zeta tr^2 \boldsymbol{\kappa}, \quad (2.5)$$

$$\boldsymbol{\varepsilon} = \frac{1}{2} (\nabla_s \mathbf{v} \cdot \mathbf{I}_s + \mathbf{I}_s \cdot (\nabla_s \mathbf{v})^T), \quad (2.6)$$

$$\boldsymbol{\kappa} = -\frac{1}{2} (\nabla_s \vartheta \cdot \mathbf{I}_s + \mathbf{I}_s \cdot (\nabla_s \vartheta)^T), \quad \vartheta = \nabla_s (\mathbf{n} \cdot \mathbf{v}) - \nabla_s \mathbf{n} \cdot \mathbf{v}, \quad (2.7)$$

where the physical parameters on the surface for the Steigmann-Ogden model are the surface Lamé constant λ_0 , shear modulus μ_0 , residual surface stress σ_0 , and the flexural rigidity parameters described by the constants ζ, χ . When the last two terms accounting for flexural rigidity are set to zero, the Steigmann-Ogden model is equivalent to the Gurtin-Murdoch

model. Also appearing in the equations is the surface identity tensor $\mathbf{I}_s = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}$ and surface gradient operator $\nabla_s = \mathbf{I}_s \cdot \nabla$. The quantity $\boldsymbol{\varepsilon}$ describes the measure of surface strain and $\boldsymbol{\kappa}$ is the bending strain measure, called the tensor of curvature.

Consider the variation of the surface energy:

$$\partial \mathcal{U} = \frac{d}{dt} \int \int_{A_s} [\mathcal{U}(\mathbf{v} + t\partial \mathbf{v})] \big|_{t=0} da = \int \int_{A_s} [\boldsymbol{\Sigma} : (\nabla_s \delta \mathbf{v})^T + \mathbf{M} : \delta \boldsymbol{\kappa}] da$$

where

$$\boldsymbol{\Sigma} = \sigma_0 \mathbf{I}_s + (\lambda_0 + \sigma_0)(\text{tr} \boldsymbol{\varepsilon}) \mathbf{I}_s + 2(\mu_0 - \sigma_0) \boldsymbol{\varepsilon} + \sigma_0 \nabla_s \mathbf{v},$$

$$\mathbf{M} = \zeta(\text{tr} \boldsymbol{\kappa}) \mathbf{I}_s + 2\chi \boldsymbol{\kappa}.$$

The tensors $\boldsymbol{\Sigma}$ and $\mathbf{M}$ are the surface stress and bending moment tensors, respectively. Simplifying the variational equation results in the boundary condition

$$\mathbf{n} \cdot \boldsymbol{\sigma} = \nabla_s \cdot \boldsymbol{\Sigma} + \nabla_s \cdot (\nabla_s \cdot \mathbf{M}) \mathbf{n} - (\nabla_s \cdot \mathbf{n}) \mathbf{n} \cdot (\nabla_s \cdot \mathbf{M}) \mathbf{n} + \mathbf{T}^{(\mathbf{n})}. \quad (2.8)$$

where $\mathbf{T}^{(\mathbf{n})}$ the vector of external stresses applied to the material surface. This equation directly modifies the conventional boundary condition across the material surface.

Also the tip conditions on the free boundary of the surface A_s can be written as [14, 74]:

$$\boldsymbol{\nu} \cdot \boldsymbol{\Sigma} + \boldsymbol{\nu} \cdot (\nabla_s \cdot \mathbf{M}) \mathbf{n} + \frac{\partial}{\partial s} [(\boldsymbol{\nu} \cdot \mathbf{M} \cdot \boldsymbol{\tau}) \mathbf{n}] = 0, \quad (2.9)$$

$$\boldsymbol{\nu} \cdot \mathbf{M} \cdot \boldsymbol{\nu} = 0, \quad \partial A_s, \quad (2.10)$$

where $\boldsymbol{\tau}$ is the unit tangent to the edge of the surface modeled by the Steigmann-Ogden boundary and $\boldsymbol{\nu}$ is the unit normal to the edge such that $\boldsymbol{\nu} \cdot \mathbf{n} = 0$.

2.2.1 Steigmann-Ogden boundary conditions in cylindrical coordinates

For the semi-space $z \geq 0$ the outward unit normal to the plane boundary, surface projection tensor, and surface gradient differential operator can be represented as follows:

$$\mathbf{n} = -\mathbf{e}_z, \quad \mathbf{I}_s = \mathbf{e}_r \otimes \mathbf{e}_r + \mathbf{e}_\theta \otimes \mathbf{e}_\theta, \quad \nabla_s = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta}.$$

Using the differential identities $\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta$ and $\frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r$ and denoting the vector of displacements on the surface $\mathbf{v}(r, \theta, 0) = (v_r, 0, v_z)$ we find that

$$\nabla_s \mathbf{v} = \frac{\partial v_r}{\partial r} \mathbf{e}_r \otimes \mathbf{e}_r + \frac{\partial v_z}{\partial r} \mathbf{e}_r \otimes \mathbf{e}_z + \frac{1}{r} v_r \mathbf{e}_\theta \otimes \mathbf{e}_\theta,$$

$$\boldsymbol{\varepsilon} = \frac{\partial v_r}{\partial r} \mathbf{e}_r \otimes \mathbf{e}_r + \frac{v_r}{r} \mathbf{e}_\theta \otimes \mathbf{e}_\theta,$$

$$\vartheta = -\frac{\partial v_z}{\partial r} \mathbf{e}_r, \quad \kappa = \frac{\partial^2 v_z}{\partial r^2} \mathbf{e}_r \otimes \mathbf{e}_r + \frac{1}{r} \frac{\partial v_z}{\partial r} \mathbf{e}_\theta \otimes \mathbf{e}_\theta,$$

$$\begin{aligned} \boldsymbol{\Sigma} = & \left\{ \lambda_0 \left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r} \right) + 2\mu_0 \frac{\partial v_r}{\partial r} + \sigma_0 \left(1 + \frac{v_r}{r} \right) \right\} \mathbf{e}_r \otimes \mathbf{e}_r + \sigma_0 \frac{\partial v_z}{\partial r} \mathbf{e}_r \otimes \mathbf{e}_z \\ & + \left\{ \lambda_0 \left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r} \right) + 2\mu_0 \frac{v_r}{r} + \sigma_0 \left(1 + \frac{v_r}{r} \right) \right\} \mathbf{e}_\theta \otimes \mathbf{e}_\theta, \end{aligned}$$

$$\mathbf{M} = \left\{ (\zeta + 2\chi) \frac{\partial^2 v_z}{\partial r^2} + \zeta \frac{1}{r} \frac{\partial v_z}{\partial r} \right\} \mathbf{e}_r \otimes \mathbf{e}_r + \left\{ \zeta \frac{\partial^2 v_z}{\partial r^2} + (\zeta + 2\chi) \frac{1}{r} \frac{\partial v_z}{\partial r} \right\} \mathbf{e}_\theta \otimes \mathbf{e}_\theta.$$

By substitution and simplification it can be seen that

$$\nabla_s \cdot \boldsymbol{\Sigma} = (\lambda_0 + 2\mu_0) \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} \right) \mathbf{e}_r + \sigma_0 \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} \right) \mathbf{e}_z,$$

$$\nabla_s \cdot (\nabla_s \cdot \mathbf{M}) \mathbf{n} = -(\zeta + 2\chi) \left(\frac{\partial^4 v_z}{\partial r^4} + \frac{2}{r} \frac{\partial^3 v_z}{\partial r^3} - \frac{1}{r^2} \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r^3} \frac{\partial v_z}{\partial r} \right) \mathbf{e}_z.$$

Due to the infinite radii of the curvature of the plane boundary, only the displacement component normal to the the surface enters into the final expression. This also causes the third term in the force-balance condition (2.8) to play no role in the surface elasticity of the boundary, i.e. $(\nabla_s \cdot \mathbf{n})\mathbf{n} \cdot (\nabla_s \cdot \mathbf{M})\mathbf{n} = 0$.

In the second problem, discussed in Chapter 4, the Steigmann-Ogden model of surface elasticity is only assumed on a subset of the surface, unlike the problem in Chapter 3 where surface effects are assumed on the entire surface. The region where surface effects are assumed is a circular membrane. For this case, the boundary conditions on the edge of the disk where the Steigmann-Ogden model is assumed are given by (2.9), (2.10). Using cylindrical coordinates, the boundary of the surface model interface is given by $r = a$, resulting in the unit tangent to the boundary $\boldsymbol{\tau} = \mathbf{e}_\theta$ and the inward unit normal vector $\boldsymbol{\nu} = -\mathbf{e}_r$. By substitution the boundary conditions on the edge ∂A_s may be written as:

$$\left[\sigma_0 + (2\mu_0 + \lambda_0)v_{r,r} + (\lambda_0 + \sigma_0)\frac{1}{r}v_r \right] \bigg|_{r=a} = 0, \quad (2.11)$$

$$\left[-\sigma_0 v_{z,r} + (2\chi + \zeta) \frac{\partial}{\partial r} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) v_z \right] \bigg|_{r=a} = 0, \quad (2.12)$$

$$\left[(2\chi + \zeta)v_{z,rr} + \zeta \frac{1}{r}v_{z,r} \right] \bigg|_{r=a} = 0, \quad (2.13)$$

where (2.11), (2.12) come from equating the components of the vectors $\mathbf{e}_r$ and $\mathbf{e}_z$ in the first equation in the boundary condition (2.9), and (2.13) from the second equation (2.10).

Chapter 3

A frictionless indentation of a semi-space by a rigid stamp

In this chapter, we introduce a problem in which an axisymmetric rigid stamp is pressed into the surface of a semi-space. Steigmann-Ogden surface elasticity is used to model the surface, while linear isotropic elasticity is used to model the bulk. The chapter is organized in the following way. First, in Section 3.1 the boundary conditions on the surface are given for the problem using the axisymmetric representation of the Steigmann-Ogden model of surface elasticity outlined in Chapter 2.2. An auxiliary problem is introduced, and the problem is rewritten in terms of a function that represents the unknown pressure in the normal direction. In Section 3.2, the displacements and stresses are expressed using the representations by Boussinesq potentials in the form of Hankel transforms of unknown functions. Inverting the Hankel transforms results in reframing the problem as an integral equation of the first kind with a condition on the total applied force. Section 3.3 covers the solvability of the integral equation by studying the type of integral kernel, depending on the values of the surface parameters. Finally, a numerical scheme is presented and the results are given in Section 3.4.

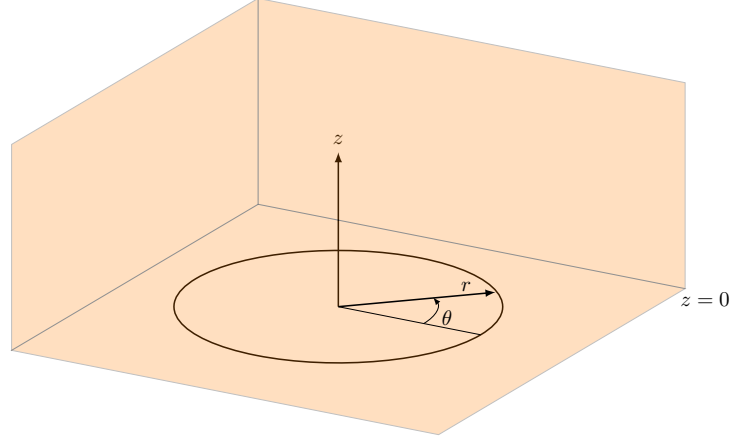


Figure 3.1: An upper semi-space in three dimensions with cylindrical coordinates.

3.1 Boundary conditions for a frictionless contact problem in the presence of the Steigmann-Ogden surface elasticity

Consider an elastic upper semi-space with surface boundary $z = 0$ in three-dimensional space subjected to an indentation by a rigid stamp. The region of contact is given by the circle $r < a$ on the surface. Using the axisymmetric representation of the Steigmann-Ogden model of surface elasticity given in Chapter 2.2, the boundary conditions for the problem are given by

$$2\mu u_z(r, 0) = g(r), \quad 0 \leq r < a, \quad (3.1)$$

$$\sigma_{zr}(r, 0) = -(\lambda_0 + 2\mu_0) \frac{\partial}{\partial r} \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) u_r(r, 0), \quad 0 \leq r < a,$$

$$\sigma_{zr}(r, 0) = -(\lambda_0 + 2\mu_0) \frac{\partial}{\partial r} \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) u_r(r, 0), \quad r > a,$$

$$\sigma_{zz}(r, 0) = -\sigma_0 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) u_z(r, 0) + (\zeta + 2\chi) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right)^2 u_z(r, 0), \quad r > a,$$

where $g(r)$ is a known function.

We reformulate the problem by introducing a function $p(r)$ which represents the external

normal stress $\sigma_{zz}(r, 0)$ inside the region of contact needed in order to generate the known displacement on the surface $g(r)$. This function $p(r)$ is initially unknown and needs to be determined by solving the problem. By introducing this pressure function, it is possible to substitute the first equation (3.1) with the following equation

$$\sigma_{zz}(r, 0) = -\sigma_0 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) u_z(r, 0) + (\zeta + 2\chi) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right)^2 u_z(r, 0) - p(r), \quad 0 \leq r < a. \quad (3.2)$$

The advantage to reframing the problem using this unknown function $p(r)$ is that we can write the functions $A(\lambda)$ and $B(\lambda)$ in terms of the unknown function $p(r)$ by inverting Hankel transforms.

3.2 Reformulating the problem as a singular integral equation

Substituting the representations (2.3), (2.4) into the boundary conditions results in two equations

$$\int_0^\infty \lambda \left((\gamma_1 \lambda^2 + \lambda) A(\lambda) + (1 - 2\nu) B(\lambda) \right) J'_0(\lambda r) d\lambda = 0 \quad (3.3)$$

$$\begin{aligned} \int_0^\infty \lambda \left((\gamma_3 \lambda^4 + \gamma_2 \lambda^2 + \lambda) A(\lambda) + ((3 - 4\nu)(\gamma_3 \lambda^3 + \gamma_2 \lambda) + (2 - 2\nu)) B(\lambda) \right) J_0(\lambda r) d\lambda \\ = \begin{cases} -p(r), & 0 \leq r \leq a \\ 0, & r > a \end{cases} \end{aligned} \quad (3.4)$$

where the constants γ_1 , γ_2 , γ_3 describe the properties of the surface by $\gamma_1 = (\lambda_0 + 2\mu_0)/2\mu$, $\gamma_2 = \sigma_0/2\mu$, $\gamma_3 = (\zeta + 2\chi)/2\mu$. The constant γ_1 represents the ability of the surface to resist stretching, the parameter γ_2 represents the surface tension, also referred to as surface prestress, and the parameter γ_3 represents the surface's resistance to bending. The equations

(3.3), (3.4) can be thought of as Hankel transforms of order one and zero, noting that $J'_0(x) = -J_1(x)$. Apply the inverse Hankel transform to each equation to obtain the system of linear equations of $A(\lambda)$, $B(\lambda)$:

$$\begin{bmatrix} \gamma_1\lambda^2 + \lambda & 1 - 2\nu \\ \gamma_3\lambda^4 + \gamma_2\lambda^2 + \lambda & (3 - 4\nu)(\gamma_3\lambda^3 + \gamma_2\lambda) + 2 - 2\nu \end{bmatrix} \begin{bmatrix} A(\lambda) \\ B(\lambda) \end{bmatrix} = \begin{bmatrix} 0 \\ -\int_0^a tp(t)J_0(\lambda t)dt \end{bmatrix}$$

resulting in the functions $A(\lambda)$, $B(\lambda)$ represented by the formulas

$$A(\lambda) = \frac{1 - 2\nu}{\lambda q(\lambda)} \int_0^a rp(r)J_0(\lambda r)dr, \quad B(\lambda) = -\frac{\gamma_1\lambda + 1}{q(\lambda)} \int_0^a rp(r)J_0(\lambda r)dr, \quad (3.5)$$

with

$$q(\lambda) = \gamma_1\gamma_3(3 - 4\nu)\lambda^4 + \gamma_3(2 - 2\nu)\lambda^3 + \gamma_1\gamma_2(3 - 4\nu)\lambda^2 + (\gamma_1 + \gamma_2)(2 - 2\nu)\lambda + 1. \quad (3.6)$$

Equation (3.1) can be written in terms of these functions $A(\lambda)$, $B(\lambda)$ resulting in the equation

$$\int_0^\infty (\lambda A(\lambda) + (3 - 4\nu)B(\lambda))J_0(\lambda r)d\lambda = g(r). \quad (3.7)$$

We manipulate this equation by first taking a derivative with respect to r and then substitute the representations of $A(\lambda)$ and $B(\lambda)$ in terms of the unknown function $p(r)$. This results in the equation

$$\int_0^\infty R(\lambda)J'_0(\lambda r) \int_0^a J_0(\lambda t)tp(t)dt d\lambda = g'(r),$$

where $R(\lambda) = \frac{\lambda(\gamma_1(3-4\nu)\lambda+2-2\nu)}{q(\lambda)}$ with the function $q(\lambda)$ in the denominator given by (3.6).

An additional condition given by $\int_0^a tp(t)dt = \frac{Y}{2\pi}$ is necessary to enforce the depth of the stamp penetration where Y is the total force applied to the stamp. Switching the order of integration, results in

$$\int_0^a K(t, r)tp(t)dt = g'(r),$$

where $K(t, r) = \int_0^\infty R(\lambda) J'_0(\lambda r) J_0(\lambda t) d\lambda$. Rescaling from $r \in [0, a]$ to $r \in [0, 1]$ and new notations: $ap(at) = P(t)$ and $ag'(ar) = G(r)$, the integral equation and additional condition become

$$\int_0^1 K(t, r) t P(t) dt = G(r), \quad (3.8)$$

$$\int_0^1 t P(t) dt = \frac{Y}{2\pi a}. \quad (3.9)$$

Note that the equation (3.8) is an integral equation of the first kind with an additional condition given by (3.9). The following section will explore this equation for various cases of surface parameters $\gamma_1, \gamma_2, \gamma_3$.

3.3 Solvability of the singular integral equation for various values of surface coefficients

3.3.1 The classical case

In the absence of all surface effects, $\gamma_1 = \gamma_2 = \gamma_3 = 0$, the integral equation is given by

$$\int_0^1 t K(t, r) P(t) dt = G(r),$$

where $K(t, r) = (2 - 2\nu) \int_0^\infty \lambda J'_0(\lambda r) J_0(\lambda t) d\lambda$ with the condition $\int_0^1 t P(t) dt = \frac{Y}{2\pi a}$. The integral kernel can be computed explicitly [23]:

$$K(t, r) = \frac{4(1 - \nu)}{\pi} \begin{cases} \frac{t}{r} \frac{E(r/t)}{(t^2 - r^2)} - \frac{1}{tr} K(r/t), & r < t, \\ \frac{E(t/r)}{(t^2 - r^2)}, & t < r. \end{cases} \quad (3.10)$$

The equations (3.8), (3.9) can be extended to the segment $[-1, 1]$ by defining [12]

$$tK(t, r) = \frac{2 - 2\nu}{\pi} \left(-\frac{1}{t + r} + \frac{1}{t - r} \right) H(t, r),$$

where

$$H(t, r) = \begin{cases} \frac{t^2}{r^2}E(r/t) - \frac{t^2-r^2}{r^2}K(r/t), & r < t, \\ \frac{t}{r}E(t/r), & t < r. \end{cases}$$

Note that the function $H(t, r)$ is Hölder continuous on the set $(0, 1) \times (0, 1)$. Using the symmetric continuations of the functions on the segment $(-1, 0]$ defined by

$$P(t) = P(-t), \quad G(r) = -G(-r), \quad H(-t, -r) = H(-t, r) = H(t, -r) = H(t, r),$$

the integral equation may be written as

$$\frac{2-2\nu}{\pi} \int_{-1}^1 \frac{1}{t-r} H(t, r) P(t) dt = G(r), \quad |r| < 1, \quad (3.11)$$

with condition

$$\int_{-1}^1 |t| P(t) dt = \frac{Y}{\pi a}. \quad (3.12)$$

The equation (3.11) is a singular integral equation with index zero [48], and can be regularized by inverting the Cauchy kernel with the assumption that the solution $P(r)$ has square root singularities at the boundary points $r = \pm 1$. The method involves writing the problem as a Riemann-Hilbert problem using the Sokhotski-Plemelj formula. This produces a regularized equation with one real constant which can be used to satisfy the condition (3.12). The solution of a frictionless contact problem is well known in the literature and can be found in Green and Zerna's 'Theoretical Elasticity' [24].

3.3.2 The Gurtin-Murdoch model without prestress

In this case the surface of the semi-space resists stretching but does not have prestress, and does not resist bending. This corresponds to the surface coefficients taken as: $\gamma_1 \neq 0$, $\gamma_2 = \gamma_3 = 0$. The integral equation (3.8) has the kernel $K(t, r) = \int_0^\infty R(\lambda) J'_0(\lambda r) J_0(\lambda t) d\lambda$

with $R(\lambda) = \frac{\lambda(\gamma_1/a(3-4\nu)\lambda+2-2\nu)}{\gamma_1/a(2-2\nu)\lambda+1}$. We can write the kernel as

$$K(t, r) = \frac{3-4\nu}{2-2\nu} \int_0^\infty \lambda J'_0(\lambda r) J_0(\lambda t) d\lambda + \frac{a(1-2\nu)^2}{\gamma_1(2-2\nu)^2} \int_0^\infty J'_0(\lambda r) J_0(\lambda t) d\lambda \\ - \frac{a(1-2\nu)^2}{\gamma_1(2-2\nu)^2} \int_0^\infty \frac{1}{\gamma_1/a(2-2\nu)\lambda+1} J'_0(\lambda r) J_0(\lambda t) d\lambda.$$

The first integral in the sum will be the same as in the classical case given by (3.10). The second integral can be computed explicitly [23], given by

$$\int_0^\infty J'_0(\lambda r) J_0(\lambda t) d\lambda = \begin{cases} -1/r, & t < r, \\ 0, & r < t. \end{cases} \quad (3.13)$$

The third integral in the sum is regular since the integrand converges with order λ^{-2} as $\lambda \rightarrow \infty$ for all values of $(t, r) \in (0, 1) \times (0, 1)$. In this case, equation (3.8) with the condition (3.9) is the same type of equation as the classical case. This means that the problem can be reduced to a Cauchy singular equation on $(-1, 1)$ with index zero. The equation can be regularized similar to the classical case and the solution is expected to have square root singularities at the endpoints $r = \pm 1$. There will be one free constant, making it possible to enforce the additional condition (3.9). This result shows that the case where the surface is modeled by the Gurtin-Murdoch model of surface elasticity without prestress is similar to the classical case in the type of solution that arises and the solvability of the problem.

3.3.3 The Gurtin-Murdoch model with non-zero prestress

For the case of the complete Gurtin-Murdoch model where both the surface resistance to stretching and surface prestress are taken into account, the values of the surface parameters satisfy $\gamma_1, \gamma_2 \neq 0$ and $\gamma_3 = 0$. The rational function has asymptotic $R(\lambda) \sim O(1)$ as $\lambda \rightarrow \infty$, and the kernel $K(t, r)$ has the form

$$K(t, r) = \frac{a}{\gamma_2} \int_0^\infty J'_0(\lambda r) J_0(\lambda t) d\lambda - \frac{a}{\gamma_2} \int_0^\infty \frac{(\gamma_1/a(2-2\nu)\lambda+1) J'_0(\lambda r) J_0(\lambda t) d\lambda}{\gamma_1 \gamma_2 / a^2 (3-4\nu) \lambda^2 + (\gamma_1 + \gamma_2) / a (2-2\nu) \lambda + 1}.$$

The first integral in the kernel can be computed explicitly using (3.13) so that the kernel can be written as

$$tK(t, r) = -\frac{a}{\gamma_2} \frac{t}{r} \chi_{[0, r]}(t) + L(t, r),$$

where $\chi_{[0, r]}(t)$ is the characteristic function of the interval $[0, r]$ and $L(t, r)$ is continuous in both r and t . The integral equation for $P(t)$ is

$$-\frac{a}{\gamma_2} \int_0^r \frac{t}{r} P(t) dt + \int_0^1 P(t) L(t, r) dt = G(r),$$

which can be reduced to a Fredholm integral equation of the second kind using differentiation with respect to r . The resulting Fredholm equation of the second kind has at most a logarithmic singularity at $t = r$ in the kernel. Using Fredholm theory, the integral equation (3.8) can be solved uniquely as long as the corresponding adjoint homogeneous problem has only trivial solution. While the integral equation has a unique solution, we cannot enforce the condition (3.9) in general.

3.3.4 The Steigmann-Ogden model

Finally, consider the case where the full Steigmann-Ogden model is assumed, with surface bending rigidity taken into account. This case is characterized by the surface parameter $\gamma_3 \neq 0$. The rational function $R(\lambda)$ decays as $R(\lambda) \sim O(\lambda^{-2})$ at infinity and the integral equation (3.8) is a Fredholm integral equation of the first kind. In general, the Fredholm equation of the first kind does not have solutions unless the right hand side of the equation satisfies additional conditions (see, for instance [25, 26]). This means that the problem (3.8), (3.9) with the assumption that the surface is modeled using the complete Steigmann-Ogden model represents an ill-posed problem.

In the case a solution does exist, the work below justifies the manipulations done in order to obtain the Fredholm equation (3.8). Consider the displacement function $u_z(r, z)$ given by

(2.3) and differentiate by the following operator

$$2\mu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right)^2 u_z(r, z) = - \int_0^\infty \lambda^4 (\lambda A(\lambda) + (3 - 4\nu)B(\lambda)) J_0(\lambda r) e^{-\lambda z} d\lambda. \quad (3.14)$$

From the solution of the problem we obtain the representations for $A(\lambda)$, $B(\lambda)$ in terms of an integral function with the traction function $p(r)$ given in (3.5). By substitution, the function inside the integral representation of $u_z(r, z)$ is given by

$$\lambda A(\lambda) + (3 - 4\nu)B(\lambda) = - \frac{2 - 2\nu + (3 - 4\nu)\gamma_1 \lambda}{q(\lambda)} \int_0^a r p(r) J_0(\lambda r) dr.$$

The convergence of integrals correspondingly depends on the asymptotic behavior of $\int_0^a r p(r) J_0(\lambda r) dr$ for λ near infinity. Note that for λr large we have

$$|J_0(\lambda r)| < \frac{c}{\sqrt{\lambda r}}$$

for a constant c . Then

$$\left| \int_0^a r p(r) J_0(\lambda r) dr \right| \leq \frac{c}{\sqrt{\lambda}} \int_0^a |\sqrt{r} p(r)| dr$$

so assuming $\sqrt{r} p(r)$ is integrable on $[0, a]$ we obtain at least $O(\lambda^{-1/2})$ asymptotics for the integral. Then the following approximation of the asymptotic behaviors hold:

$$\lambda A(\lambda) + (3 - 4\nu)B(\lambda) \approx O(\lambda^{-7/2}),$$

$$\lambda^3 (\lambda A(\lambda) + (3 - 4\nu)B(\lambda)) \approx O(\lambda^{-1/2}).$$

The problem (3.8), (3.9) may be rewritten equivalently as

$$\int_0^1 t P(t) K_0(t, r) dt = G(r), \quad 0 \leq r \leq 1,$$

with $K_0(t, r) = \int_0^\infty \frac{\gamma_1/a(3-4\nu)\lambda+2-2\nu}{q(\lambda/a)} J_0(\lambda t) J_0(\lambda r) d\lambda$. Observe that the kernel $K_0(t, r)$ of the

integral equation is continuously differentiable in both variables twice and the partial derivatives exist up to second order. Observe also that the kernel $K_0(t, r)$ is symmetric so it is self-adjoint. All eigenvalues are real and the eigenfunctions may be chosen so that they are real and orthogonal. There is at most a countable number of eigenvalues and the only limit point is zero.

Consider the eigenfunctions $\varphi_i(r)$ with corresponding eigenvalues λ_i . Then

$$\lambda_i \varphi_i(r) = \int_0^1 \varphi_i(t) K(t, r) dt.$$

Observe that the right hand side is twice continuously differentiable. Correspondingly, so should the left-hand side. The eigenfunctions corresponding to $\lambda = 0$ constitute the solution to the homogeneous equation

$$\int_0^1 \varphi(t) K(t, r) dt = 0.$$

Continuity for these functions cannot be guaranteed. The solution to the equation (3.8) exists as long as $G(r) \in \text{Range}(K)$. Observe that $\{\varphi_i\}$ is a complete set over $\text{Range}(K)$. Then $G(r)$ can be expanded in a series of eigenfunctions

$$G(r) = \sum_{i=1}^{\infty} \langle G, \varphi_i \rangle \varphi_i$$

where none of the eigenfunctions in the series correspond to $\lambda = 0$. Expand $tP(t)$ into this series as well

$$tP(t) = \sum_{i=1}^{\infty} A_i \varphi_i$$

and assume there are no eigenfunctions corresponding to the eigenvalue $\lambda = 0$ in this series either. Substitute into the equation and obtain

$$\sum_{i=1}^{\infty} A_i \int_0^{\infty} K(t, r) \varphi_i(t) dt = \sum_{i=1}^{\infty} \langle G, \varphi_i \rangle \varphi_i$$

or by using the eigenfunction property,

$$\sum_{i=1}^{\infty} A_i \lambda_i \varphi_i(t) = \sum_{i=1}^{\infty} \langle G, \varphi_i \rangle \varphi_i$$

then using the orthogonality of φ_i ,

$$A_i = \frac{1}{\lambda_i} \langle G, \varphi_i \rangle .$$

Correspondingly, if the solution $tP(t)$ exists, it is a twice differentiable function as long as the homogeneous equation has only trivial solution. From [57], p.3, formula (3.2), we obtain the following predicted asymptotics of the integral

$$\int_0^a rp(r)J_0(\lambda r)dr = O(\lambda^{-3/2}).$$

The integral representation of the derivative operator acting on $u_z(r, 0)$ given by (3.14) is ensured through the boundary conditions (3.1).

3.4 Numerical approximations and results

In this section, a numerical scheme for equations (3.8), (3.9) is developed. In the cases where the Steigmann-Ogden model or the Gurtin-Murdoch model with non-zero prestress are assumed the solution is understood in the approximate sense. To obtain numerical approximations for the solution of (3.8), (3.9) the computation of the integral over t is done using the method of Gauss-Chebyshev quadrature.

The Chebyshev nodes of the first and second kind are used to evaluate the integrals:

$$r_k = \cos \left(\frac{k\pi}{2(N+1)} \right), \quad t_j = \cos \left(\frac{(2j-1)\pi}{4(N+1)} \right).$$

By evaluating the integrals at these nodes, the equations can be written as

$$\frac{\pi}{2(N+1)} \sum_{j=1}^{N+1} t_j P(t_j) K(t_j, r_k) \sqrt{1 - t_k^2} = G(r_k),$$

$$\frac{\pi}{2(N+1)} \sum_{j=1}^{N+1} t_j P(t_j) \sqrt{1 - t_j^2} = \frac{Y}{2\pi a}.$$

These equations form a system with $N + 1$ equations and $N + 1$ unknowns. Note that the kernel $K(t, r)$ is an oscillatory integral so care must be taken in order to evaluate it. These types of integrals are notoriously difficult to approximate numerically, but for the specific integrand in $K(t, r)$ this computation is accomplished using the IIPBF algorithm [42, 52]. The IIPBF algorithm uses extrapolation on a sequence of partial sums and requires writing the product of Bessel functions as the sum of two more well behaved functions. Once $K(t, r)$ is computed numerically at the values selected, the system is solved for values of the function $P(r)$ at the nodes r_k . Using the values found for $P(r)$, the values of the displacements and stresses are computed using the integral representation formulas (2.3), (2.4).

In order to test the accuracy and efficiency of the algorithm we compare solutions to the known classical solutions by setting all surface parameters equal to zero. These classical solutions are well known in the literature [24]. The material we model is Si[100] which correspondingly has the values $\mu = 40.23$ GPa for the shear modulus and $\nu = 0.33$ Poisson's ratio. The comparison to the classical solution and the numerical solutions obtained by varying the number of evaluation points is presented in fig. 3.2. The graphs are plotted for the straight stamp $G(r) = 0$ and the total force $Y/a^2 = 5 \cdot 10^9$ Pa. It can be seen that even at $N = 30$ evaluation points the solution is in close agreement to the classical solution. In the remaining part of this section the number of evaluation points is set to $N = 50$.

Further testing the algorithm, we compare the contact pressure $p(r)$ to results in given by Pinyochotiwong [51] in the case of the complete Gurtin-Murdoch model. The following parameters were used in the rest of this exploration, $\mu = 26.13$ GPa, $\nu = 0.345$, $\lambda_0 = 6.8511$ N/m, $\mu_0 = -0.376$ N/m, $\sigma_0 = 1$ N/m, and $\gamma_1 = 0.116707$. In order to compare with the

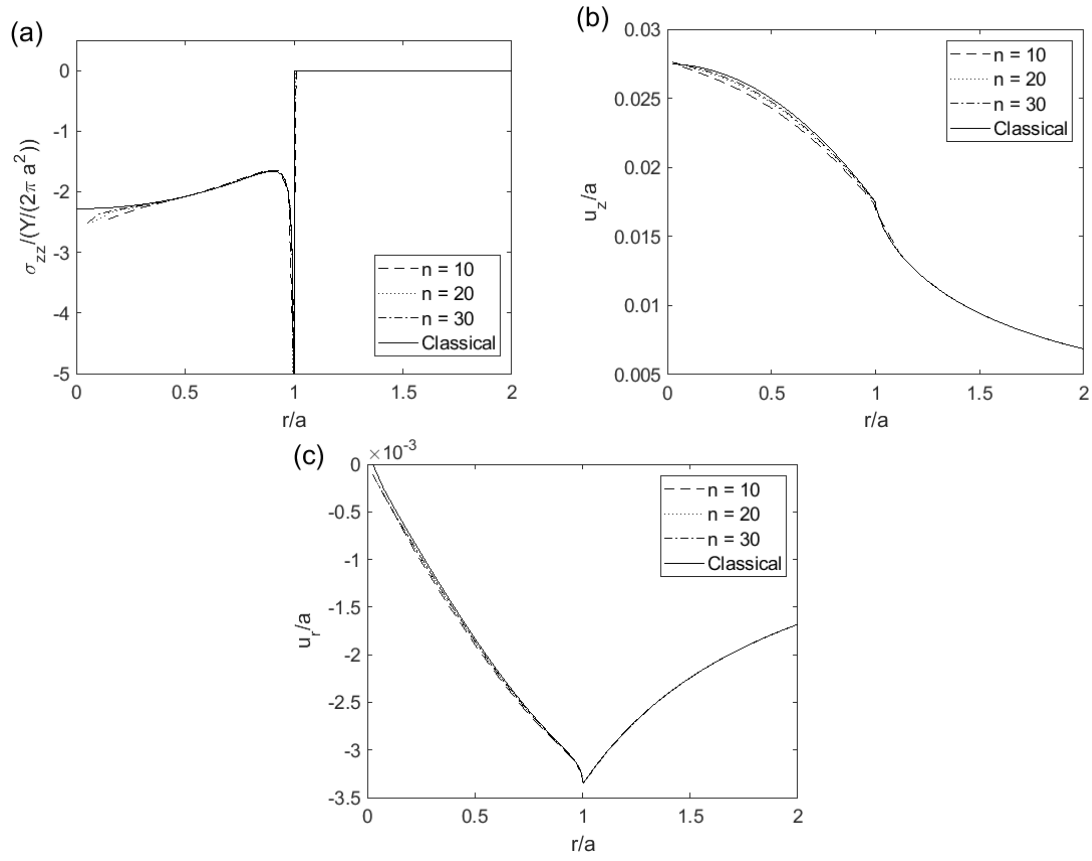


Figure 3.2: Comparison of the results with the exact solution in the case of no surface effects.

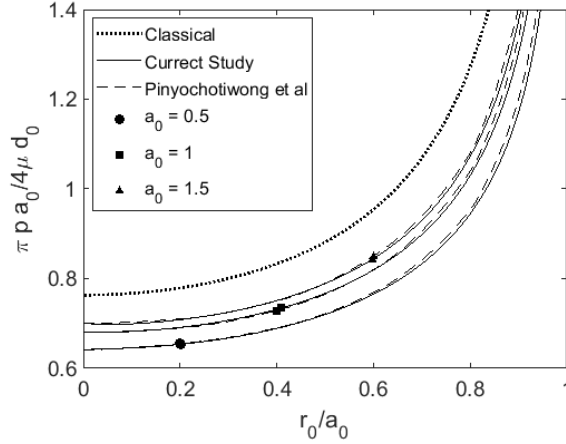


Figure 3.3: Comparison of the results for the contact pressure to known results.

results by Pinyochotiwong [51] all the geometric parameters were normalized as $a_0 = a/\gamma_1$, $r_0 = r/\gamma_1$, $d_0 = d/\gamma_1$ where $d = u_z(0,0)$ is the penetration depth of the stamp at the center of the contact radius. The solid lines of the figure denote the results from the current study while the dashed lines correspond to results by Pinyochotiwong [51]. Finally, the dot results correspond to the classical solution. The rescaled values of the contact radius a_0 that are considered are $a_0 = 0.5, 1$, and 1.5 . It can be seen from fig. 3.3 that the results are very close to those in Pinyochotiwong's [51] which were obtained using different theoretical and computational methods. Unless stated otherwise, the following parameters are used in the remainder of this chapter: $\mu = 40.23$ GPa, $\nu = 0.33$, $\lambda_0 = 4.4939$ N/m, $\mu_0 = 2.7779$ N/m, $\sigma_0 = 0.6506$ N/m, and $\chi/(2\mu) = \zeta/(2\mu) = 1.9915 \cdot 10^{-30} \gamma \text{ m}^3$ where the values of γ are $\gamma = 0, 1, 10, 100$. These parameter values are based on atomistic simulations [8, 9, 11, 45, 56]. The value of $a = 10$ nm is assumed unless stated otherwise and Y is taken to be $Y/a^2 = 5 \cdot 10^9$ Pa.

Fig. 3.4 shows the values of the stresses $\sigma_{rz}(r, 0)$, $\sigma_{zz}(r, 0)$ and displacements $u_r(r, 0)$, $u_z(r, 0)$ on the semi-space boundary $z = 0$ for various values of the parameter γ . The numerical results presented in fig. 3.4 are derived for a flat stamp profile $G(r) = 0$. The plots for the classical results with no surface effects are solid black lines, the plots for the complete Gurtin-Murdoch model in the absence of surface resistance ($\gamma = 0$) are dashed lines, plots for the Steigmann-Ogden model with $\gamma = 1$ are shown with dot lines, plots

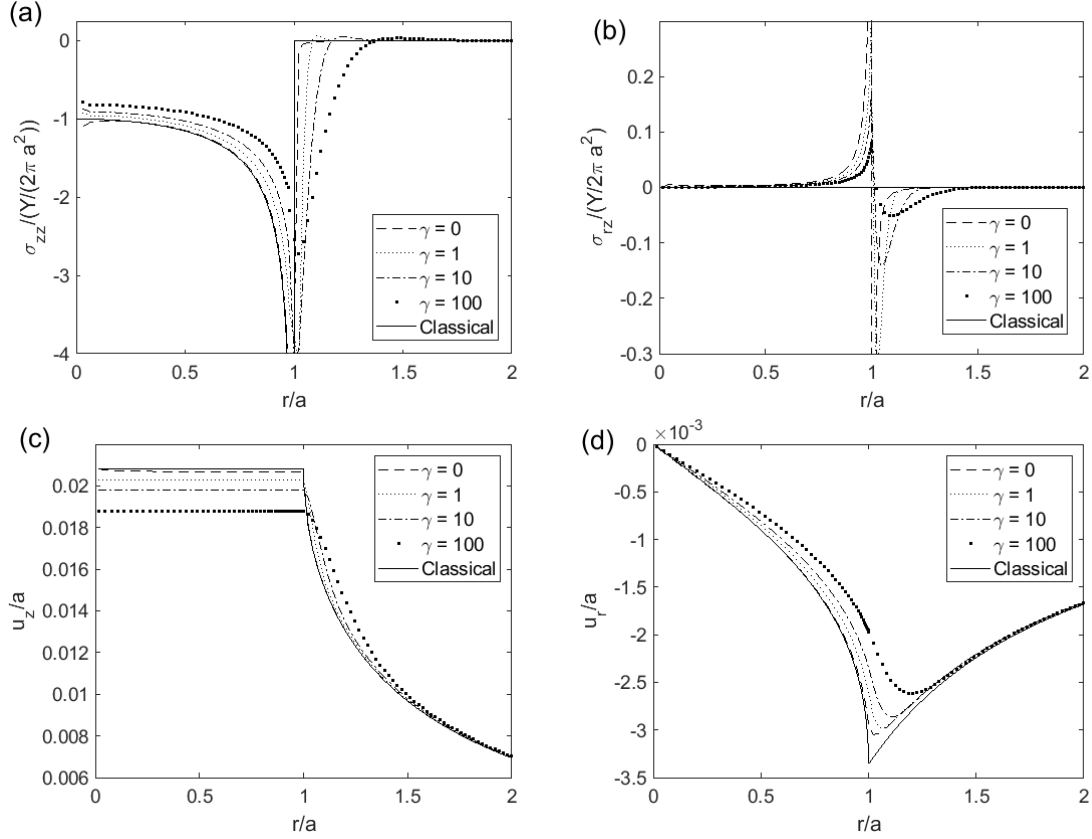


Figure 3.4: Comparison of the results for different values of the surface bending resistance parameter γ for a flat stamp profile.

for the Steigmann-Ogden model with $\gamma = 10$ are shown with dash-dot lines, and plots for the Steigmann-Ogden model with $\gamma = 100$ are shown with dots. The stresses $\sigma_{zz}(r, 0)$ for the parameter $\gamma \neq 0$ may exhibit oscillations in a neighborhood close to the tip of the contact region. The condition numbers for the corresponding systems of linear equations with $N = 50$ terms are approximately within the range of $10^4 - 10^7$.

Figure 3.4 shows that the presence of surface effects causes a smoother behavior near the edge of the contact radius. The stresses can be seen to decrease near the end of the contact interval. It can also be seen from fig. 3.4c that the presence of surface effects corresponds to a reduction in the indentation depth of the stamp into the semi-space and correspondingly, an increase in the hardness of the material. This effect depends strongly on the value of the bending resistance of the surface.

It can be seen in equations (3.8), (3.9) that the contact radius a has a proportional

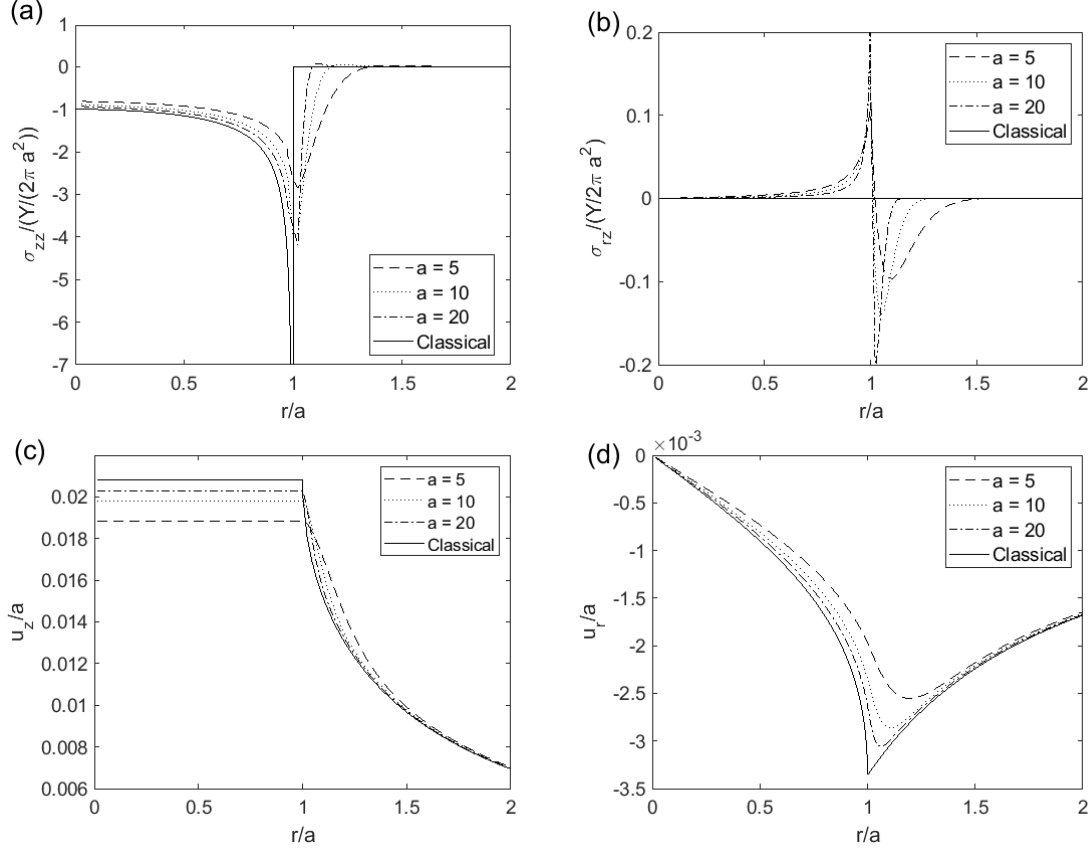


Figure 3.5: Comparison of the results for different values of the contact radius a for a flat stamp.

relationship with the coefficients involving the surface parameters. These relationships γ_1/a , γ_2/a , and γ_3/a^3 are explored numerically in fig. 3.5 by keeping all parameters constant, with $\gamma = 10$, and varying values of a where $a = 5, 10, 20$ nm. Fig. 3.5 shows the displacements and stresses for a flat profile stamp at the surface $z = 0$ for $a = 5, 10, 20$ nm. The classical solution represented as a solid black line, results for $a = 5$ nm are dashed lines, $a = 10$ nm are dot lines, and $a = 20$ nm are dot-dashed lines. As the value of the contact radius a gets larger, the results converge toward the classical solutions while for smaller values of a we see that solutions appear to be smoother. For larger values of a the scaled surface parameters proportionally get smaller, and thus approximate the classical solution in the absence of surface effects. Similarly, for smaller values of a the surface effects become more pronounced and the smoothing that was seen in fig. 3.4 is also seen in fig. 3.5.

The hardness of a material, its resistance to being deformed, can be represented by the

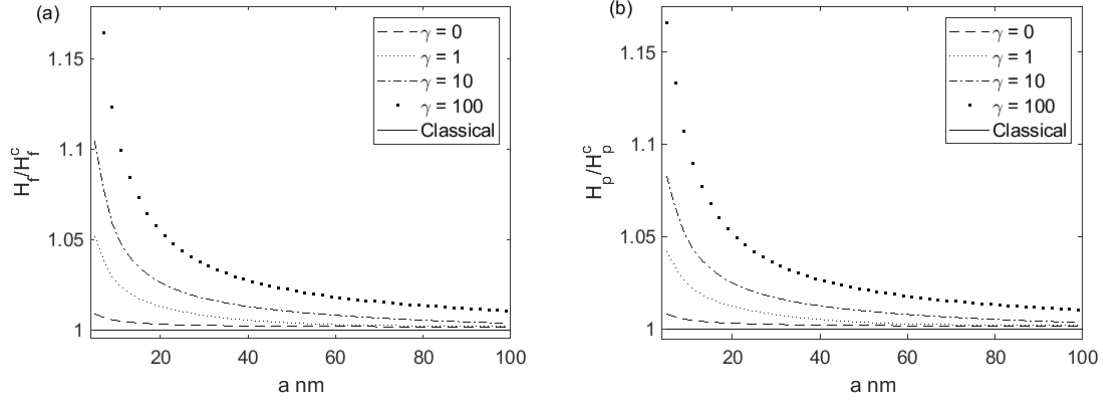


Figure 3.6: Indentation hardness for various parameters γ and a flat stamp profile (a) or a parabolic stamp profile (b).

ratio of the total force applied to the depth of penetration,

$$H = \frac{Y}{u_z(0,0)}.$$

Fig. 3.6 shows the ratio of the indentation hardness of the results with surface effects to the hardness of the classical result H^c . The total pressure Y is set to $Y/a^2 = 5 \cdot 10^9$ Pa for the straight stamp (fig. 3.6a) and $Y/a^2 = 50 \cdot 10^9$ for the paraboloidal stamp (fig. 3.6b). The parameter γ assumes the values 0, 1, 10, and 100. All other parameters are fixed and set to the values given above. For larger values of γ the material with assumed surface effects has a greater resistance to deformation compared to the classical solution. The surface parameters also have a much greater contribution for smaller values of the contact radius a but as the values of a are taken to be larger, we see that surface parameters effect on the hardness diminish.

Chapter 4

A patch-loaded circular membrane attached to a surface of a semi-space

The problem presented in the following chapter is motivated by the solvability of the full Steigmann-Ogden case of Chapter 3. In Section 3.3 it is discovered that unique solvability for a frictionless contact problem is not guaranteed in the most general case. This result motivates an exploration of differing models for surface elasticity of an object with infinite surface area. In this chapter, we consider a circular membrane attached to a surface of a semi-space loaded with an axisymmetric normal stress. The membrane is modeled by the Steigmann-Ogden model of surface elasticity and surface effects are assumed to be absent on the rest of the semi-space.

The chapter is organized as follows. First, Section 4.1 covers the boundary conditions for the membrane on the surface of the semi-space, modeled by the Steigmann-Ogden surface elasticity model. These conditions are derived from the variational formulation in Section 2.2. The displacements and stresses are represented through Boussinesq potentials in the form of Hankel transforms of unknown functions in Section 4.2. In Section 4.3, a re-framing of the problem is used to write the unknown functions in terms of new unknowns given by differential equations of the surface displacement functions. The unknowns with the boundary conditions form a system of singular integral equations whose character depends

on the value of the surface parameters. The solvability of the system of equations with boundary conditions for each case is covered in Section 4.4 by studying the kernels of the equations. Finally, a numerical method using Gauss-Chebyshev quadrature is presented in Section 4.5.

4.1 Boundary conditions for a patch-loaded membrane with the Steigmann-Ogden surface elasticity

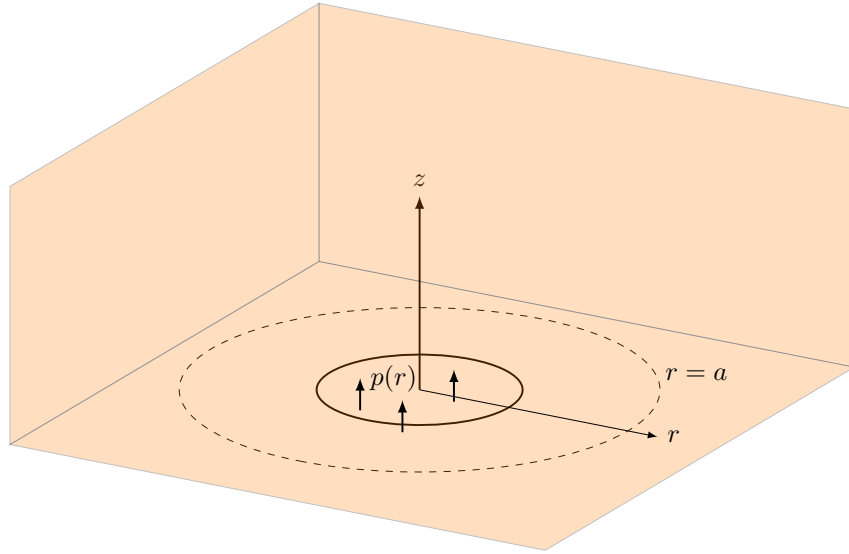


Figure 4.1: Material with surface $z = 0$ with a circular membrane attached, modeled by Steigmann-Ogden surface elasticity on the region $r < a$. The material is subjected to an axisymmetric pressure $p(r)$ in a circular region.

Consider a material occupying the upper semi-space in three dimensions with surface boundary given by $z = 0$ subjected to a nanoscale force in a circular region. We will model the bulk of the material using linear isotropic elasticity with shear modulus μ and Poisson's ratio ν . The boundary of the material given by the plane $z = 0$ will be thought of as the union of two sets, one of which is a circular patch-loaded region with radius $r = a$, while the other region on the surface is the complement of this circle on the plane. The circular region on the surface will be modeled by the Steigmann-Ogden model of surface elasticity.

The axisymmetric representation of the Steigmann-Ogden model of surface elasticity, as

well as the boundary conditions where surface effects are assumed inside the circle $r = a$ have been derived in Chapter 2.2. The conditions on the surface of the semi-space in the axisymmetric case with vector displacement function $\mathbf{u}(r, z) = (u_r(r, z), 0, u_z(r, z))$ can be written as:

$$\begin{aligned}\sigma_{zr}(r, 0) &= -(\lambda_0 + 2\mu_0) \frac{\partial}{\partial r} \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) u_r(r, 0), \quad 0 \leq r < a, \\ \sigma_{zz}(r, 0) &= -\sigma_0 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) u_z(r, 0) + (2\zeta + \chi) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right)^2 u_z(r, 0) - p(r), \quad 0 \leq r < a, \\ \sigma_{rz}(r, 0) &= 0, \quad \sigma_{zz}(r, 0) = 0, \quad r > a.\end{aligned}\tag{4.1}$$

The function $p(r)$ represents the applied external traction in the direction normal to the surface. Using the cylindrical coordinate representation of the Steigmann-Ogden model the boundary conditions on the membrane edge are given by (2.11-2.13), as shown in Chapter 2.2.

4.2 Boussinesq potentials for the stresses and strains

Using the theory of linear elasticity for the bulk, the displacements and stresses may be represented in terms of integral equations with unknown functions $A(\lambda)$ and $B(\lambda)$ as detailed in Section 2.1. For the axisymmetric case this results in the representations (2.3), (2.4) on the surfaces.

The conditions (4.1) with (2.11-2.13) form a mixed boundary problem, where there are different conditions on the part of the surface assuming the Steigmann-Ogden model and the remainder of the surface. This means that we cannot solve the problem directly by inverting Hankel transforms.

4.3 A system of singular integral equations for the solution of the problem

Consider the auxiliary problem

$$\sigma_{rz}(r, 0) = \begin{cases} -\gamma_1 g(r), & 0 \leq r < a, \\ 0, & r > a, \end{cases} \quad \sigma_{zz}(r, 0) = \begin{cases} h_0(r) - p(r), & 0 \leq r < a, \\ 0, & r > a, \end{cases}$$

where the functions $g(r)$ and $h_0(r)$ are initially unknown. Substituting into the traction balance condition for the Steigmann-Ogden surface model, produces two integral equations

$$\int_0^\infty \lambda (\lambda A(\lambda) + (1 - 2\nu)B(\lambda)) J'_0(\lambda r) d\lambda = \begin{cases} \gamma_1 g(r), & 0 \leq r < a \\ 0, & r > a \end{cases} \quad (4.2)$$

$$\int_0^\infty \lambda (\lambda A(\lambda) + (2 - 2\nu)B(\lambda)) J_0(\lambda r) d\lambda = \begin{cases} h_0(r) - p(r), & 0 \leq r \leq a \\ 0, & r > a. \end{cases} \quad (4.3)$$

These integrals with Bessel functions can be seen as a Hankel transforms of order one and order zero respectively, noting that $J'_0(x) = -J_1(x)$. Applying the inverse Hankel transform to each equation results in a system of equations:

$$\begin{bmatrix} \lambda & 1 - 2\nu \\ \lambda & 2 - 2\nu \end{bmatrix} \begin{bmatrix} A(\lambda) \\ B(\lambda) \end{bmatrix} = \begin{bmatrix} \gamma_1 \int_0^a t g(t) J'_0(\lambda t) dt \\ \int_0^a t (h_0(t) - p(t)) J_0(\lambda t) dt \end{bmatrix}.$$

The functions $A(\lambda)$, $B(\lambda)$ can be written as:

$$\lambda A(\lambda) = (2 - 2\nu)\gamma_1 \int_0^a t g(t) J'_0(\lambda t) dt - (1 - 2\nu) \int_0^a t (h_0(t) - p(t)) J_0(\lambda t) dt,$$

$$B(\lambda) = -\gamma_1 \int_0^a t g(t) J'_0(\lambda t) dt + \int_0^a t (h_0(t) - p(t)) J_0(\lambda t) dt,$$

where the surface parameters are defined by $\gamma_1 = (\lambda_0 + 2\mu_0)/2\mu$, $\gamma_2 = \sigma_0/2\mu$, and $\gamma_3 = (2\zeta + \chi)/2\mu$. Note these are the same definitions used in Chapter 3.

If $g(r)$ and $h_0(r)$ were known, then these equations would satisfy the boundary conditions (4.1) identically. The goal is to find the functions $g(r)$ and $h_0(r)$ that also satisfy (2.11-2.13). These functions can be written in terms of the displacements by noting that

$$h_0(r) = \gamma_3 \left(h''(r) + \frac{1}{r} h'(r) \right) - \gamma_2 h(r), \quad (4.4)$$

with

$$g(r) = 2\mu \frac{\partial}{\partial r} \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) u_r(r, 0), \quad h(r) = 2\mu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) u_z(r, 0).$$

The last two equations may also be rewritten as

$$\int_0^r g(t) dt + c_1 = 2\mu \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) u_r(r, 0), \quad \frac{1}{r} \int_0^r t h(t) dt + \frac{c_2}{r} = 2\mu \frac{\partial}{\partial r} u_z(r, 0),$$

where c_1 is a constant whose character will be discussed later. Assuming that the derivative of $u_z(r, 0)$ at $r = 0$ is smooth, forces $c_2 = 0$. The character of the function $h_0(r)$ changes depending on the values of the surface parameters γ_2, γ_3 . By the orthogonality property of Bessel functions, the equations simplify to

$$\int_0^r g(t) dt + c_1 = (1 - 2\nu)(h_0(r) - p(r)) - \gamma_1 \frac{4(1 - \nu)}{\pi} \int_0^a K(r, t) t g(t) dt, \quad (4.5)$$

$$\frac{1}{r} \int_0^r t h(t) dt = (1 - 2\nu) \gamma_1 g(r) - \frac{4(1 - \nu)}{\pi} \int_0^a K(t, r) t (h_0(t) - p(t)) dt, \quad (4.6)$$

where the kernel $K(t, r)$ is a constant multiple of the kernel from the last chapter given by (3.10).

The next step is to substitute $h(r)$ for its representation in terms of $h_0(r)$ in order to write the two equations in terms of the two unknown functions $g(r)$ and $h_0(r)$ as well as the known external pressure $p(r)$. The following section will isolate the different cases and discuss the analysis of solvability for each case.

4.4 The integral equations for various assumptions on the surface effects

4.4.1 The complete Steigmann-Ogden model

This section covers the case where the membrane is modeled using the Steigmann-Ogden model with all surface parameters taken into account, represented by the surface parameters $\gamma_1, \gamma_2, \gamma_3 \neq 0$. The relationship between $h_0(r)$ and $h(r)$ given by (4.4) is a second order ODE. The solution is found by the method of variation of parameters, where the fundamental system of solutions for the homogeneous equation consists of the modified Bessel functions of order zero. The general solution is given by

$$h(r) = \left(c_3 + \frac{1}{\gamma_3} \int_0^r t K_0(kt) h_0(t) dt \right) I_0(kr) + \left(c_4 - \frac{1}{\gamma_3} \int_0^r t I_0(kt) h_0(t) dt \right) K_0(kr)$$

where $k = \sqrt{\gamma_2/\gamma_3}$ and $I_\alpha(x)$, $K_\alpha(x)$ denote the modified Bessel functions of order α of the first and second kind, respectively. This representation for $h(r)$ allows for the integral of $h(r)$ on the left side to be written as

$$\frac{1}{r} \int_0^r t h(t) dt = \frac{c_3}{k} I_1(kr) - \frac{c_4}{k} (K_1(kr) - 1/kr) + \frac{1}{k\gamma_3} \int_0^r B(t, r) h_0(t) dt \quad (4.7)$$

where

$$B(t, r) = t K_0(kt) I_1(kr) + t I_0(kt) K_1(kr) - \frac{t}{rk}. \quad (4.8)$$

Substitute (4.7) into (4.5), (4.6), and rescale from $(r, t) \in (0, a) \times (0, a)$ to $(r, t) \in (0, 1) \times (0, 1)$ using the new functions:

$$ap(ar) = P(r), \quad ah_0(ar) = H_0(r), \quad a^2 \frac{\gamma_1}{a} g(ar) = G(r),$$

$$ac_1 = C_1, \quad a^2 c_3 = C_3, \quad a^2 c_4 = C_4.$$

Continue the equations by symmetry onto $(r, t) \in (-1, 1) \times (-1, 1)$ by introducing the

functions

$$\begin{aligned}
H_1(t, r) &= \begin{cases} E(r/t), & r < t, \\ \frac{r}{t} E(t/r) - \frac{r^2 - t^2}{rt} K(t/r), & t < r, \end{cases} \\
H_2(t, r) &= \begin{cases} \frac{t}{r} E(t/r), & t < r, \\ \frac{t^2}{r^2} E(r/t) - \frac{t^2 - r^2}{r^2} K(r/t), & r < t, \end{cases}
\end{aligned} \tag{4.9}$$

and note that $tK(r, t) = \left(\frac{-1}{t+r} + \frac{1}{r-t}\right) H_1(t, r)$ and $tK(t, r) = \left(-\frac{1}{t+r} + \frac{1}{t-r}\right) H_2(t, r)$. Also, define the following symmetric continuations on $(-1, 0]$:

$$G(-r) = -G(r), \quad H_0(-r) = H_0(r), \quad P(-r) = P(r),$$

$$B(t, -r) = -B(t, |r|), \quad B(-t, r) = -B(|t|, r),$$

$$H_1(-t, -r) = H_1(t, r) = H_1(t, -r) = H_1(-t, r),$$

$$H_2(-t, -r) = H_2(t, r) = H_2(t, -r) = H_2(-t, r).$$

Introducing the functions

$$Q_1(t, r) = \frac{2 - 2\nu}{\pi} \left(\frac{H_1(t, r) - 1}{t - r} \right), \quad Q_2(t, r) = \frac{2 - 2\nu}{\pi} \left(\frac{H_2(t, r) - 1}{t - r} \right), \tag{4.10}$$

the integral equations may be written as

$$\begin{aligned}
\frac{a}{\gamma_1} \int_0^{|r|} G(t) dt + C_1 &= (1 - 2\nu)(H_0(r) - P(r)) + \frac{2 - 2\nu}{\pi} \int_{-1}^1 G(t) \frac{dt}{t - r} + \int_{-1}^1 G(t) Q_1(t, r) dt, \\
\frac{C_3}{ak} \operatorname{sgn}(r) I_1(ka|r|) - \frac{C_4}{ak} \left(\operatorname{sgn}(r) K_1(ka|r|) - \frac{1}{akr} \right) &+ \frac{a^3}{\gamma_3} \int_0^r B(akt, akr) H_0(t) dt \\
&= (1 - 2\nu)G(r) - \frac{2 - 2\nu}{\pi} \int_{-1}^1 \frac{(H_0(t) - P(t))dt}{t - r} - \int_{-1}^1 Q_2(t, r)(H_0(t) - P(t))dt.
\end{aligned}$$

Following the method outlined by Erdogan [12], first write the equations so that the kernel is a sum of a Cauchy singular kernel and the regular part of the kernel. This results

in the equations:

$$(1 - 2\nu)H_0(r) + \frac{2 - 2\nu}{\pi} \int_{-1}^1 \frac{G(t)dt}{t - r} - \int_{-1}^1 R_1(t, r)G(t)dt = p_1(r), \quad (4.11)$$

$$(1 - 2\nu)G(r) - \frac{2 - 2\nu}{\pi} \int_{-1}^1 \frac{H_0(t)dt}{t - r} - \int_{-1}^1 R_2(t, r)H_0(t)dt = p_2(r), \quad (4.12)$$

where

$$p_1(r) = (1 - 2\nu)P(r) + C_1,$$

$$\begin{aligned} p_2(r) = & \frac{C_3}{ak} \operatorname{sgn}(r) I_1(ka|r|) - \frac{C_4}{ak} \left(\operatorname{sgn}(r) K_1(ka|r|) - \frac{1}{akr} \right) \\ & - \frac{2 - 2\nu}{\pi} \int_{-1}^1 P(t) \frac{dt}{t - r} - \int_{-1}^1 Q_2(t, r)P(t)dt, \end{aligned}$$

$$R_1(t, r) = \frac{a}{\gamma_1} \chi_{[0, |r|]}(t) - Q_1(t, r), \quad R_2(t, r) = \frac{a^2}{k\gamma_3} \chi_{[0, r]}(t) B(akt, akr) + Q_2(t, r).$$

Assuming, for now, that the regular part of the integral equation is known, the system of dominant equations has the form

$$(1 - 2\nu)H_0(r) + \frac{2 - 2\nu}{\pi} \int_{-1}^1 \frac{G(t)dt}{t - r} = P_1(r), \quad (4.13)$$

$$(1 - 2\nu)G(r) - \frac{2 - 2\nu}{\pi} \int_{-1}^1 \frac{H_0(t)dt}{t - r} = P_2(r), \quad (4.14)$$

where

$$P_1(r) = p_1(r) + \int_{-1}^1 R_1(t, r)G(t)dt, \quad P_2(r) = p_2(r) + \int_{-1}^1 R_2(t, r)H_0(t)dt.$$

Dividing (4.13) by i and adding that to (4.14) gives

$$(1 - 2\nu)\varphi(r) + \frac{2 - 2\nu}{i\pi} \int_{-1}^1 \frac{\varphi(t)dt}{t - r} = T(r), \quad (4.15)$$

where $T(r) = P_2(r) - iP_1(r)$, $\varphi(r) = G(r) - iH_0(r)$. The known function $T(r)$ is assumed to be Hölder continuous on $[-1, 1]$ including the endpoints. The solution is sought to be in the class $h(-1, 1)$ which is defined as the collection of functions that are Hölder continuous on the interior of the domain but may have integrable singularities at the ends. To match formulae in [17], divide (4.15) by $\sqrt{4\nu - 3}$.

A Cauchy singular integral equation (4.15) may be written as a Riemann-Hilbert problem by defining

$$\Phi(r) = \frac{1}{2\pi i} \int_{-1}^1 \frac{\varphi(t)dt}{t - r}.$$

Then using the Sokhotski–Plemelj formula [17]

$$\begin{aligned}\Phi^+(r) - \Phi^-(r) &= \varphi(r) \\ \Phi^+(r) + \Phi^-(r) &= \frac{1}{\pi i} \int_{-1}^1 \frac{\varphi(t)dt}{t - r}\end{aligned}$$

and rewriting (4.15) in terms of Φ^+ and Φ^- gives the Riemann-Hilbert problem,

$$\Phi^+(r) = \frac{-1}{3 - 4\nu} \Phi^-(r) + \frac{1}{3 - 4\nu} T(r), \quad r \in (-1, 1). \quad (4.16)$$

Solving the homogeneous problem first, call $\gamma = \frac{-1}{(3-4\nu)}$ so that the homogeneous problem may be written as

$$\chi^+(r) = \gamma \chi^-(r), \quad r \in (-1, 1).$$

The solution of a homogeneous Riemann-Hilbert problem is presented in Muskhelishvili's ‘Singular Integral Equations’ [48] as well as Gakhov’s ‘Boundary Value Problems’ [17]. There are no special ends, meaning that the function $\mathbf{G}(r) = \gamma$ is not a real positive quantity on the endpoints. The solution to the homogeneous problem is given by

$$\Phi(r) = \chi(r)C_0,$$

where

$$\chi^+(r) = \sqrt{\gamma} \left(\frac{r+1}{r-1} \right)^{i\omega} (r^2 - 1)^{-1/2}, \quad \chi^-(r) = \frac{1}{\sqrt{\gamma}} \left(\frac{r+1}{r-1} \right)^{i\omega} (r^2 - 1)^{-1/2},$$

with $\omega = \log(\gamma)/2\pi$ and a complex constant C_0 . Note that the exponent $i\omega$ has an imaginary component, which causes oscillatory behavior to appear in the solution. The solution to the inhomogeneous Riemann-Hilbert problem (4.16), is then given by

$$\Phi(r) = \frac{\chi(r)}{2\pi i} \int_{-1}^1 \frac{-\gamma T(t)}{\chi^+(t)(t-r)} dt + C_0 \chi(r).$$

Using the Sokhotski-Plemelj formulas, the equation may be written in terms of the unknown function $\varphi(r)$:

$$\varphi(r) = -\frac{1-2\nu}{3-4\nu} T(r) + \frac{2-2\nu}{3-4\nu} \frac{Z(r)}{\pi i} \int_{-1}^1 \frac{T(t)}{Z(t)} \frac{dt}{t-r} + \sqrt{\gamma}(2-2\nu)C_0 Z(r), \quad (4.17)$$

where $Z(r) = \frac{1}{\sqrt{\gamma}} \chi^+(r)$. Observe that the constant C_0 is restricted to real values to preserve the parity of the functions $G(r)$ and $H_0(r)$. Note that the function $Z(r)$ has oscillatory singularities at $r = \pm 1$. Since singular integrals preserve oscillatory singularities, the second term in the sum for the solution $\varphi(r)$ given by (4.17) does not have oscillatory singularities. The only term that contributes to possible oscillatory behavior at the boundary of the membrane is the term with C_0 . In order to avoid this behavior, the constant C_0 is taken to equal zero, allowing the solution to be written as:

$$\varphi(r) = -\frac{1-2\nu}{3-4\nu} T(r) + \frac{2-2\nu}{3-4\nu} \frac{Z(r)}{\pi i} \int_{-1}^1 \frac{T(t)}{Z(t)} \frac{dt}{t-r}. \quad (4.18)$$

Name the real and complex components of $Z(r)$: $\text{Re}Z(r) = Z_R(r)$, $\text{Im}Z(r) = Z_I(r)$. By substituting the function $T(r)$ into the solution (4.18), the original system of two singular integral equations (4.11), (4.12) may be reformed into a system of two Fredholm equations

in terms of $G(r)$, $H_0(r)$:

$$G(r) + \int_{-1}^1 \mathbf{K}_1(\tau, r) G(\tau) d\tau + \int_{-1}^1 \mathbf{K}_2(\tau, r) H_0(\tau) d\tau = \mathbf{q}_1(r), \quad (4.19)$$

$$H_0(r) + \int_{-1}^1 \mathbf{K}_3(\tau, r) G(\tau) d\tau + \int_{-1}^1 \mathbf{K}_4(\tau, r) H_0(\tau) d\tau = \mathbf{q}_2(r),$$

with the functions

$$\mathbf{q}_1(r) = -\frac{1-2\nu}{3-4\nu} p_2(r) - \frac{1}{\pi} \frac{2-2\nu}{3-4\nu} \int_{-1}^1 [p_1(t) Z_1(t, r) + p_2(t) Z_2(t, r)] \frac{dt}{t-r},$$

$$\mathbf{q}_2(r) = -\frac{1-2\nu}{3-4\nu} p_1(r) - \frac{1}{\pi} \frac{2-2\nu}{3-4\nu} \int_{-1}^1 [p_1(t) Z_2(t, r) - p_2(t) Z_1(t, r)] \frac{dt}{t-r},$$

and the kernels

$$\mathbf{K}_1(\tau, r) = \frac{1}{\pi} \frac{2-2\nu}{3-4\nu} \int_{-1}^1 R_1(\tau, t) Z_1(t, r) \frac{dt}{t-r},$$

$$\mathbf{K}_2(\tau, r) = -\frac{1}{\pi} \frac{2-2\nu}{3-4\nu} \int_{-1}^1 R_2(\tau, t) Z_2(t, r) \frac{dt}{t-r} + \frac{1-2\nu}{3-4\nu} R_2(\tau, r),$$

$$\mathbf{K}_3(\tau, r) = \frac{1}{\pi} \frac{2-2\nu}{3-4\nu} \int_{-1}^1 R_1(\tau, t) Z_2(t, r) \frac{dt}{t-r} + \frac{1-2\nu}{3-4\nu} R_1(\tau, r),$$

$$\mathbf{K}_4(\tau, r) = -\frac{1}{\pi} \frac{2-2\nu}{3-4\nu} \int_{-1}^1 R_2(\tau, t) Z_1(t, r) \frac{dt}{t-r},$$

where

$$Z_1(t, r) = \frac{Z_R(r) Z_R(t) + Z_I(r) Z_I(t)}{Z_R(t)^2 + Z_I(t)^2}, \quad Z_2(t, r) = \frac{Z_R(r) Z_I(t) - Z_I(r) Z_R(t)}{Z_R(t)^2 + Z_I(t)^2}.$$

The kernels and known functions on the right side of each equation will have integrable singularities at the endpoints of the interval $r = \pm 1$. As they are, the known functions and kernels of the two equations do not satisfy the conditions to use Fredholm theory. The infinities at the endpoints can be eliminated by solving instead for the functions $\mathbf{H}_0(r) =$

$H_0(r)\sqrt{1-r^2}$, $\mathbf{G}(r) = G(r)\sqrt{1-r^2}$ and performing a change of variables. The kernels will have at most logarithmic singularities at $\tau = r$ and satisfy the conditions necessary to use Fredholm theory. The system of integral equations can be solved uniquely as long as the corresponding homogeneous adjoint problem has only trivial solutions.

For the boundary conditions given by (2.11-2.13), the equations can be rescaled similarly to the integral equations. The displacements (2.3) may be substituted in with the definitions for $A(\lambda)$ and $B(\lambda)$. This results in:

$$\begin{aligned} \sigma_0 + \int_0^1 G(t)dt + \frac{\gamma_1}{a}C_1 - \frac{\sigma_0 - 2\mu_0}{\pi\mu a}(2 - 2\nu) \int_0^1 G(t)(E(t^2) - K(t^2))dt \\ - \frac{(\sigma_0 - 2\mu_0)}{2\mu a}(1 - 2\nu) \int_{-1}^1 tP(t)dt = 0, \\ M_a \left(C_3 + \frac{a^3}{\gamma_3} \int_0^1 tK_0(kat)H_0(t)dt \right) + N_a C_4 - \left(N_a + \frac{1}{k^2 a^2} \right) \frac{a^3}{\gamma_3} \int_0^1 tI_0(kat)H_0(t)dt = 0, \\ \int_0^1 tH_0(t)dt = 0, \end{aligned}$$

where $M_a = \frac{(2\chi+\zeta)}{2\chi}I_0(ka) - \frac{I_1(ka)}{ka}$ and $N_a = \frac{(2\chi+\zeta)}{2\chi}K_0(ka) + \frac{K_1(ka)-1/ka}{ka}$. These conditions given by the three equations above allow for solving for the constants C_1 , C_3 , C_4 uniquely.

4.4.2 The Steigmann-Ogden model in the absence of prestress

Consider the case where the Steigmann-Ogden model of surface elasticity is assumed on the membrane with resistance to surface bending and stretching, but without prestress. This case is characterized by the surface parameters $\gamma_3, \gamma_1 \neq 0$ and $\gamma_2 = 0$. The relationship between $h_0(r)$ and $h(r)$ given by (4.4) is a second order ODE. We may solve the differential equation by the method of variation of parameters. This results in the function

$$h(r) = c_3 \log(r) + c_4 + \frac{1}{\gamma_3} \int_0^r (\log(r) - \log(t))th_0(t)dt.$$

The integral of $h(r)$ appearing in equation (4.6) can then be written as

$$\frac{1}{r} \int_0^r th(t)dt = c_3 \frac{r(2 \log(r) - 1)}{4} + c_4 \frac{r}{2} + \frac{1}{\gamma_3} \int_0^r \left[\frac{tr(2 \log(r/t) - 1)}{4} + \frac{t^3}{4r} \right] h_0(t)dt. \quad (4.20)$$

First substitute the representation of the integral of $h(r)$ into (4.6) and rescale (r, t) from $(0, a) \times (0, a)$ to $(0, 1) \times (0, 1)$ with

$$ah_0(ar) = H_0(r), \quad ap(ar) = P(r), \quad a^2 \frac{\gamma_1}{a} g(ar) = G(r),$$

$$ac_1 = C_1, \quad a^2 c_3 = C_3, \quad a^2 c_4 = C_4,$$

Extend the equations to $(r, t) \in (-1, 1) \times (-1, 1)$, using the equations $Q_1(t, r)$, $Q_2(t, r)$ defined by (4.10) and the symmetric continuations on $(-1, 0]$:

$$G(-r) = -G(r), \quad H_0(-r) = H_0(r), \quad P(-r) = P(r).$$

The two integral equations extended to $r \in (-1, 1)$ are given by

$$\begin{aligned} \frac{a}{\gamma_1} \int_0^{|r|} G(t)dt + C_1 &= (1 - 2\nu)(H_0(r) - P(r)) + \frac{2 - 2\nu}{\pi} \int_{-1}^1 G(t) \frac{dt}{t - r} + \int_{-1}^1 Q_1(t, r) G(t)dt, \\ C_3 \frac{r(2 \log |ar| - 1)}{4} + C_4 \frac{r}{2} + \frac{a^3}{\gamma_3} \int_0^r L(t, r) H_0(t)dt &= (1 - 2\nu)G(r) \\ &\quad - \frac{2 - 2\nu}{\pi} \int_{-1}^1 \frac{(H_0(t) - P(t))dt}{t - r} - \int_{-1}^1 Q_2(t, r)(H_0(t) - P(t))dt, \end{aligned}$$

where

$$L(t, r) = \frac{tr(2 \log |r/t| - 1)}{4} + \frac{t^3}{4r}. \quad (4.21)$$

The solution is found by following the method used in Section 4.4.1, with the functions $p_1(r)$, $p_2(r)$ and the kernels $R_1(t, r)$, $R_2(t, r)$ in this case defined as:

$$p_1(r) = (1 - 2\nu)P(r) + C_1,$$

$$p_2(r) = C_3 \frac{r(2 \log |ar| - 1)}{4} + C_4 \frac{r}{2} - \frac{2 - 2\nu}{\pi} \int_{-1}^1 P(t) \frac{dt}{t - r} - \int_{-1}^1 Q_2(t, r) P(t) dt,$$

$$R_1(t, r) = \frac{a}{\gamma_1} \chi_{[0, |r|]}(t) - Q_1(t, r), \quad R_2(t, r) = \frac{a^3}{\gamma_3} \chi_{[0, r]}(t) L(t, r) + Q_2(t, r).$$

Solutions $G(r)$, $H_0(r)$ are sought in the class $h(-1, 1)$. As shown in Section 4.4.1, the system may be rewritten as two Fredholm equations, given by (4.19) with the functions $p_1(r)$, $p_2(r)$ and the kernels $R_1(t, r)$, $R_2(t, r)$ for this case substituted in. The kernels appearing in this system of Fredholm equations will have at most logarithmic singularities at $t = r$ and square root singularities at the endpoints $r = \pm 1$. Transforming the two Fredholm equations further, by substitutions and relabeling, the singularities at $r = \pm 1$ may be eliminated, leaving the logarithmic singularities at $t = r$. By Fredholm theory, a unique solution exists so long as the homogeneous adjoint equation as a solution.

The boundary conditions, rescaled to $r = 1$, in this case are derived from (2.11-2.13) given by

$$\begin{aligned} \int_0^1 \frac{a}{\gamma_1} G(t) dt + C_1 - \frac{\mu_0}{\mu a} (2 - 2\nu) \int_0^1 G(t) (E(t^2) - K(t^2)) dt &= -\frac{\mu_0}{\mu a} (1 - 2\nu) \int_0^1 t P(t) dt, \\ \int_0^1 t H_0(t) dt &= 0, \\ C_3 (\log(a) + 1/2) + C_4 - \frac{a^3}{\gamma_3} \int_0^1 (\log(a) - \log(t) + 1/2 - t^2/2) t H_0(t) dt &= 0. \end{aligned}$$

These conditions given by the three equations above allow us to solve for the constants C_1 , C_3 , C_4 uniquely.

4.4.3 The Steigmann-Ogden model with prestress and without resistance to stretching

This section will cover the case where the membrane is modeled using the Steigmann-Ogden model with resistance to bending and a nonzero prestress, but without resistance to stretching. The parameters are characterized by $\gamma_2, \gamma_3 \neq 0$ and $\gamma_1 = 0$. In this case, the problem reduces to solving only one integral equation, given by (4.6). The function $h(r)$ is the same

as in the case of complete Steigmann-Ogden in Section 4.4.1. The integral of $h(r)$ on the left side of equation (4.6) is given by (4.7). The integral equation is rescaled and extended so that $r \in (-1, 1)$. To do so, first define the functions

$$ah_0(ar) = H_0(r), \quad ap(ar) = P(r), \quad a^2c_3 = C_3, \quad a^2c_4 = C_4.$$

Extend the equation to $(r, t) \in (-1, 1) \times (-1, 1)$, using the equation $Q_2(t, r)$ defined by (4.10) and the symmetric continuations on $(-1, 0]$:

$$H_0(-r) = H_0(r), \quad P(-r) = P(r).$$

The equation (4.6) can be written as

$$\frac{1}{\pi} \int_{-1}^1 \frac{H_0(t)dt}{t-r} = \mathbf{F}(r) - \int_{-1}^1 k(t, r)H_0(t)dt,$$

with

$$\begin{aligned} \mathbf{F}(r) = & \frac{1}{\pi} \int_{-1}^1 P(t) \frac{dt}{t-r} + \frac{1}{2-2\nu} \int_{-1}^1 Q_2(t, r)P(t)dt - \frac{C_3}{ak(2-2\nu)} \text{sgn}(r)I_1(kar) \\ & + \frac{C_4}{ak(2-2\nu)} \left(\text{sgn}(r)K_1(kar) - \frac{1}{akr} \right). \end{aligned}$$

The kernel is given by $k(t, r) = \frac{1}{2-2\nu} \left(\frac{a^3}{\gamma_3} \chi_{[0,r]}(t)B(akt, akr) + Q_2(t, r) \right)$, with $B(t, r)$ defined by (4.8) and $Q_2(t, r)$ defined by (4.10). Using regularization by solving the dominant equation, we assume first that the function on the right side of the equation is known. The solution $H_0(r)$ will be sought for in the class $h(-1, 1)$. The singular integral equation may be written as a Fredholm equation of the second kind given by [17]

$$H_0(r) + \int_{-1}^1 \mathbf{K}(t, r)H_0(t)dt = f(r) \tag{4.22}$$

with

$$\mathbf{K}(t, r) = -\frac{1}{\pi\sqrt{1-r^2}} \int_{-1}^1 \frac{k(\tau, t)\sqrt{1-\tau^2}d\tau}{\tau-r},$$

$$f(r) = -\frac{1}{\pi\sqrt{1-r^2}} \int_{-1}^1 \frac{\mathbf{F}(t)\sqrt{1-t^2}dt}{t-r} + \frac{C_0}{\sqrt{1-r^2}}.$$

The one free constant C_0 may be used to obtain a bounded solution at $r = 1$. The kernel $\mathbf{K}(t, r)$ and free term $f(r)$ will have square root singularities at the endpoints $r = \pm 1$. As they are now, they do not satisfy the conditions of Fredholm theory. A change of variables for $H_0(r)$ and the variable of integration eliminates these singularities at the ends.

The kernel of the Fredholm equation after eliminating the square root singularities is given by

$$\mathbf{K}(t, r) = -\frac{a^3}{\gamma_3\pi(2-2\nu)} \int_0^t \frac{B(ak\tau, akt)\sqrt{1-\tau^2}d\tau}{\tau-r} - \frac{1}{\pi(2-2\nu)} \int_{-1}^1 \frac{Q_2(t, \tau)\sqrt{1-\tau^2}d\tau}{\tau-r}.$$

The first integral in the kernel equation does not have any singularities. The kernel $Q_2(t, \tau)$ has a logarithmic singularity at $t = \tau$, meaning the second integral in the kernel has a logarithmic-squared singularity at $t = r$. The kernel $\mathbf{K}(t, r)$ has a logarithmic-squared singularity at $t = r$. Then by Fredholm theory, the equation will be solvable as long as the homogeneous adjoint equation has only trivial solution.

The boundary conditions given by (2.11-2.13) rescaled and evaluated on the edge $r = 1$ can be written as:

$$\int_0^1 tP(t)dt = \frac{2\mu}{1-2\nu},$$

$$M_a \left(C_3 + \frac{a^3}{\gamma_3} \int_0^1 tK_0(kat)H_0(t)dt \right) + N_a C_4 - \left(N_a + \frac{1}{k^2 a^2} \right) \frac{a^3}{\gamma_3} \int_0^1 tI_0(kat)H_0(t)dt = 0,$$

$$\int_0^1 tH_0(t)dt = 0,$$

where $M_a = \frac{(2\chi+\zeta)}{2\chi} I_0(ka) - \frac{I_1(ka)}{ka}$ and $N_a = \frac{(2\chi+\zeta)}{2\chi} K_0(ka) + \frac{K_1(ka)-1/ka}{ka}$. The first condition restricts the value of the total force applied to the semi-space. The second two conditions will allow us to solve for the constants C_3, C_4 uniquely.

4.4.4 The Steigmann-Ogden model without resistance to stretching or prestress

This section will cover the case where the membrane is modeled using the Steigmann-Ogden model of surface elasticity with resistance to bending and without resistance to stretching or prestress. The parameters are characterized by $\gamma_3 \neq 0$ and $\gamma_1, \gamma_2 = 0$. In this case, only one integral equation is needed to solve. The function $h(r)$ is the same as in the case where the Steigmann-Ogden model of surface elasticity is assumed without any prestress, covered in Section 4.4.2. The integral of $h(r)$ on the left side of the equation (4.6) is given by (4.20). The equation (4.6) can be rescaled and extended to $(r, t) \in (-1, 1) \times (-1, 1)$ by defining the scaled functions

$$ah_0(ar) = H_0(r), \quad ap(ar) = P(r), \quad a^2c_3 = C_3, \quad a^2c_4 = C_4,$$

and symmetric continuations

$$H_0(-r) = H_0(r), \quad P(-r) = P(r).$$

The solution $H_0(r)$ is sought for in the class $h(-1, 1)$. The solution is found by method of regularization by solving the dominant equation as presented in Section 4.4.3, the equation (4.6) can be written as a Fredholm integral equation of the second kind given by (4.22) with

$$\mathbf{F}(r) = -C_3 \frac{r(2 \log |ar| - 1)}{4(2 - 2\nu)} - C_4 \frac{r}{2(2 - 2\nu)} + \frac{1}{\pi} \int_{-1}^1 P(t) \frac{dt}{t - r} + \frac{1}{2 - 2\nu} \int_{-1}^1 Q_2(t, r) P(t) dt,$$

$$k(t, r) = \frac{1}{2 - 2\nu} \left(\frac{a^2}{k\gamma_3} \chi_{[0, r]}(t) L(t, r) + Q_2(t, r) \right),$$

with $L(t, r)$ defined by (4.21) and $Q_2(t, r)$ defined by (4.10). The known function $\mathbf{F}(r)$ belongs to the class of functions that are Hölder continuous. The one free constant C_0 may be used to obtain a bounded solution at $r = 1$. The kernel $\mathbf{K}(t, r)$ and free term $f(r)$ will have square root singularities at the endpoints $r = \pm 1$. After a change of variables for $H_0(t)$

and the argument of integration, the square root singularities at $r = \pm 1$ can be eliminated.

The kernel of the Fredholm equation, after eliminating the square root singularities, is given by

$$\mathbf{K}(t, r) = -\frac{a^2}{k\gamma_3\pi(2-2\nu)} \int_0^t \frac{B(\tau, t)\sqrt{1-\tau^2}d\tau}{\tau-r} - \frac{1}{\pi(2-2\nu)} \int_{-1}^1 \frac{Q_2(t, \tau)\sqrt{1-\tau^2}d\tau}{\tau-r}.$$

The kernel $\mathbf{K}(t, r)$ has a logarithmic singularity at $t = r$. Then by Fredholm theory, the equation will be solvable as long as the homogeneous adjoint equation has only trivial solution.

The boundary conditions given by (2.11-2.13) rescaled and evaluated on the edge $r = 1$ can be written as:

$$\begin{aligned} \int_0^1 tP(t)dt &= \frac{2\mu}{1-2\nu}, \\ \int_0^1 tH_0(t)dt &= 0, \\ C_3(\log(a) + 1/2) + C_4 - \frac{a^3}{\gamma_3} \int_0^1 (\log(a) - \log(t) + 1/2 - t^2/2) tH_0(t)dt &= 0. \end{aligned}$$

The first condition restricts the value of the total force applied to the semi-space. The second two conditions will allow us to solve for the constants C_3, C_4 uniquely.

4.4.5 The complete Gurtin-Murdoch model

This section covers the case where the membrane is modeled by the Gurtin-Murdoch model of surface elasticity with nonzero prestress and resistance to surface stretching. This is characterized by the surface parameters: $\gamma_3 = 0$ and $\gamma_1, \gamma_2 \neq 0$. The function $h_0(r) = -\gamma_2 h(r)$, so all that is needed to write the integral of $h(r)$ in (4.6) in terms of $h_0(r)$ is a direct substitution. Rescale and extend the integrals (4.5), (4.6) to $(r, t) \in (-1, 1) \times (-1, 1)$ by using the functions $Q_1(t, r), Q_2(t, r)$ defined in (4.10), relabeling:

$$ah_0(ar) = H_0(r), \quad ap(ar) = P(r), \quad a^2 \frac{\gamma_1}{a} g(ar) = G(r), \quad ac_1 = C_1,$$

and using the symmetric continuations:

$$G(-r) = -G(r), \quad H_0(-r) = H_0(r), \quad P(-r) = P(r).$$

The two integral equations are rewritten as a system of Fredholm equations of the second kind, following the method outlined in Section 4.4.1. The functions labeled $p_1(r)$, $p_2(r)$ and kernels $R_1(t, r)$, $R_2(t, r)$ in the equations appearing in Section 4.4.1 for this case are given by:

$$p_1(r) = (1 - 2\nu)P(r) + C_1, \quad p_2(r) = -(2 - 2\nu) \int_{-1}^1 P(t) \frac{dt}{t - r} - \int_{-1}^1 Q_2(t, r)P(t)dt,$$

$$R_1(t, r) = \frac{a}{\gamma_1} \chi_{[0, |r|]}(t) - Q_1(t, r), \quad R_2(t, r) = -\frac{a}{\gamma_2} \frac{t}{r} \chi_{[0, r]}(t) + Q_2(t, r).$$

The resulting two Fredholm equations are given by (4.19) with the functions $p_1(r)$, $p_2(r)$ and the kernels $R_1(t, r)$, $R_2(t, r)$ for this particular case substituted in. The kernels in the system of Fredholm equations have logarithmic singularities at $t = r$ and square root singularities at the edge of the circular boundary $r = \pm 1$. As seen in Section 4.4.1, the square root singularities at the endpoints can be eliminated by performing a change of variables and relabeling. By Fredholm theory, the solution will exist as long as the homogeneous adjoint equation has only trivial solution.

For this case, the boundary conditions (2.11-2.13) reduce to

$$\begin{aligned} \int_0^1 G(t)dt + \frac{\gamma_1}{a} a C_1 - \frac{\mu_0}{\mu a} (2 - 2\nu) \int_0^1 G(t)(E(t^2) - K(t^2))dt &= -\frac{\mu_0}{\mu a} (1 - 2\nu) \int_0^1 t P(t)dt, \\ \int_0^1 t H_0(t)dt &= 0. \end{aligned}$$

The first condition allows us to solve for the constant C_1 . For the second boundary condition, the constant C_0 in (4.17) that was previously assumed to be zero can be taken to be real and found by using this condition. The solutions in this case will have oscillatory behavior near the membrane boundary.

4.4.6 The Gurtin-Murdoch model without surface resistance to stretching

Consider the case where the Gurtin-Murdoch model of surface elasticity is assumed on the membrane with an applied prestress but no resistance to surface stretching, characterized by $\gamma_3, \gamma_1 = 0$ and $\gamma_2 \neq 0$. In this case, the function $h_0(r) = -\gamma_2 h(r)$ and the problem involves only one integral equation, given by (4.6). First, rescale equation (4.6) so that $(r, t) \in (0, 1) \times (0, 1)$ with the functions $ah_0(ar) = H_0(r)$ and $ap(ar) = P(r)$. The equation can be extended to $(r, t) \in (-1, 1) \times (-1, 1)$ using the kernel $Q_2(t, r)$ defined in (4.10) and the following symmetric continuations on $(-1, 0]$: $P(-t) = P(t)$, $H_0(-t) = H_0(t)$.

The method of regularization by solution of the dominant equation is used where the solution $H_0(r)$ is sought in the class $h(-1, 1)$. Following the work in Section 4.4.3, the equation may be written as a Fredholm integral equation of the second kind given by (4.22) with

$$\mathbf{F}(r) = \frac{1}{\pi} \int_{-1}^1 \frac{P(t)}{t-r} dt + \frac{1}{2-2\nu} \int_{-1}^1 P(t) Q_2(t, r) dt,$$

$$k(t, r) = \frac{1}{2-2\nu} \left(-\frac{a}{\gamma_2} \frac{t}{r} \chi_{[0,r]}(t) + Q_2(t, r) \right).$$

Performing a change of variables, and solving for the function $\mathbf{H}_0(r) = H_0(r)\sqrt{1-r^2}$ leads to a Fredholm equation that will satisfy the conditions of Fredholm theory.

The kernel $\mathbf{K}(t, r)$ without square root singularities is given by

$$\mathbf{K}(t, r) = \frac{a}{\gamma_2 \pi (2-2\nu)} \frac{1}{t} \int_0^t \frac{\tau \sqrt{1-\tau^2} d\tau}{\tau-r} - \frac{1}{\pi (2-2\nu)} \int_{-1}^1 \frac{Q_2(t, \tau) \sqrt{1-\tau^2} d\tau}{\tau-r}.$$

The first term in the sum has a logarithmic singularity at $t = r$. The second term has an order logarithmic-squared singularity for $t = r$, meaning the kernel as a whole has a logarithmic singularity at $t = r$. By Fredholm theory, the solution exists and is unique as long as the homogeneous adjoint problem only has trivial solution.

The boundary conditions (2.11-2.13) in this case are given by the boundary conditions

may be written as

$$\int_0^1 tP(t)dt = \frac{2\mu}{1-2\nu}, \quad \int_0^1 tH_0(t)dt = 0.$$

Note that the first boundary condition forces a condition on the value of the total external force applied. In this case, this condition must be satisfied to have a solution. The constant C_0 can be found by using the second boundary condition. The equation $H_0(r)$ is not guaranteed to be bounded on the ends in this case.

4.4.7 The Gurtin-Murdoch model without prestress

This section covers the case where the Gurtin-Murdoch model of surface elasticity is assumed on the membrane, with resistance to surface stretching and no applied prestress. The surface parameters are characterized by $\gamma_3, \gamma_2 = 0$ and $\gamma_1 \neq 0$. The function $h_0(r) = 0$, so there is only one unknown function, $g(r)$. This reduces the problem to the equation (4.5) with $h_0(r) = 0$. Rescale (r, t) from $(0, a) \times (0, a)$ to $(0, 1) \times (0, 1)$ by writing the equation in terms of the scaled functions:

$$ap(at) = P(t), \quad a^2 \frac{\gamma_1}{a} g(at) = G(t), \quad ac_1 = C_1.$$

The equation can be extended to $(r, t) \in (-1, 1) \times (-1, 1)$ by using the function $Q_1(t, r)$ defined in (4.10) and the symmetric continuations on $(-1, 0]$ given by: $P(t) = P(-t)$, $G(t) = -G(-t)$.

The solution is sought in the class $h(-1, 1)$. Following the work in Section 4.4.3, the singular equation can be rewritten as a Fredholm equation. The functions

$$\mathbf{F}(r) = \frac{(1-2\nu)P(r) + C_1}{2-2\nu}, \quad k(t, r) = \frac{1}{2-2\nu} \left(-\frac{a}{\gamma_1} \chi_{[0, |r|]}(t) + Q_1(t, r) \right),$$

in this case may be substituted into (4.22) with the desired function $G(r)$ instead of $H_0(r)$. Performing a change of variables, and instead solving for the function $\mathbf{G}(r)$ where $G(r) = \mathbf{G}(r)/\sqrt{1-r^2}$ leads to a Fredholm equation that will satisfy the conditions of Fredholm

theory. The free constant, C_0 , generated by solving must be taken equal to zero in order to preserve the parity of the function $G(r)$. The solution $G(r)$ is allowed to have integrable singularities at the endpoints.

Now to investigate the character of the kernel of the Fredholm equation,

$$\mathbf{K}(t, r) = \frac{1}{\pi(2-2\nu)} \int_0^{|t|} \frac{\sqrt{1-\tau^2} d\tau}{\tau-r} - \frac{1}{\pi(2-2\nu)} \int_{-1}^1 \frac{Q_1(\tau, t) \sqrt{1-\tau^2} d\tau}{\tau-r}.$$

The first term in the sum will have a logarithmic singularity at $t = r$. The second term in the sum has a logarithmic-squared singularity for $t = r$. The kernel $\mathbf{K}(t, r)$ has at most a logarithmic singularity at $t = r$. Using Fredholm theory, the integral equation can be solved uniquely as long as the corresponding homogeneous adjoint problem has only trivial solution.

In this case, the boundary conditions (2.11-2.13) reduce to one equation which, by substitution, gives

$$\begin{aligned} \int_0^1 \frac{a}{\gamma_1} G(t) dt + C_1 - \frac{\mu_0}{\mu a} (2-2\nu) \int_0^1 G(t) (E(t^2) - K(t^2)) dt \\ + \frac{\mu_0}{\mu a} (1-2\nu) \int_0^1 t P(t) dt = 0. \end{aligned}$$

The constant C_1 will be uniquely determined using this condition.

4.4.8 The classical case

In the classical case, we do not have the issue of mixed boundary conditions and can therefore use the classical method of inverting Hankel transforms in (2.4) and substituting in the explicit results for $A(\lambda)$ and $B(\lambda)$ into the equations (2.3) to write the displacement functions as a function of the external force $p(r)$. Inverting the tangential stress gives the following relationship between $A(\lambda)$ and $B(\lambda)$:

$$\lambda A(\lambda) = -(1-2\nu) B(\lambda).$$

By substituting this into the normal stress equation and inverting the Hankel transform we have

$$-\int_0^a tp(r)J_0(\lambda t)dt = \lambda A(\lambda) + (2 - 2\nu)B(\lambda)$$

which results in

$$\lambda A(\lambda) = (1 - 2\nu) \int_0^a tp(t)J_0(\lambda t)dt, \quad B(\lambda) = - \int_0^a tp(t)J_0(\lambda t)dt.$$

The solution in this case are well known in the literature, and can be found in Green and Zerna's 'Theoretical Elasticity' [24].

4.5 Numerical approximations and results

This section presents a numerical scheme used to find solutions for the set of integral equations (4.5), (4.6) with the boundary conditions given by (2.11-2.13). The method used involves approximating the unknown functions by a summation of Chebyshev polynomials. The scheme makes use of the orthogonality of Chebyshev polynomials to form a matrix equation where the unknown coefficients are solved for by matrix inversion.

The unknown functions $H_0(r)$ and $G(r)$, taking into account evenness/oddness, are represented as

$$H_0(r) = \sum_{n=1}^N \frac{A_n T_{2n-2}(r)}{\sqrt{1-r^2}}, \quad G(r) = \sum_{m=1}^N \frac{B_m T_{2m-1}(r)}{\sqrt{1-r^2}},$$

where T_j is a Chebyshev polynomial of the first kind. These representations are substituted into the two integral equations (4.5), (4.6) as well as the conditions on the membrane boundary given by (2.11-2.13). To approximate the integrals over t we use Gauss-Chebyshev quadrature where the integrals are evaluated at Chebyshev nodes of the first kind t_k and Chebyshev nodes of the second kind r_j :

$$t_k = \cos\left(\frac{(2k-1)\pi}{2n}\right), \quad r_j = \cos\left(\frac{j\pi}{n}\right),$$

with $k = 1, \dots, n$ and $j = 1, \dots, n - 1$. For evaluation of an integral using nodes of the first kind the following formula is used

$$\int_{-1}^1 f(t) \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{n} \sum_{k=1}^n f(t_k).$$

For evaluation at nodes of the second kind the following formula is used

$$\int_{-1}^1 f(r) dr = \frac{\pi}{n} \sum_{k=1}^{n-1} f(r_j) \sqrt{1-r_j^2}.$$

After substitution of $G(r)$ and $H_0(r)$, the integral equations (4.5), (4.6) are multiplied by the Chebyshev polynomial $T_k(r)$ and integrated with respect to r over $(-1, 1)$. Utilizing the orthogonality property

$$\int_{-1}^1 T_m(r) T_n(r) \frac{dr}{\sqrt{1-r^2}} = \begin{cases} 0 & m \neq n, \\ \pi & m = n = 0, \\ \pi/2 & m = n \neq 0 \end{cases}$$

simplifies the equations. The Chebyshev polynomials of the first and second kind appearing in the equations are evaluated using the formulas

$$T_n(x) = \cos(n\theta), \quad U_n(x) = \frac{\sin((n+1)\theta)}{\sin(\theta)},$$

for $\theta = \arccos(x)$. The trigonometric representations allow for faster results and maintains the accuracy of the numerical scheme. This process allows for construction of a matrix M such that $MC = D$ where

$$C = \begin{bmatrix} A_1 & \cdots & A_N & B_1 & \cdots & B_N & C_1 & C_3 & C_4 \end{bmatrix}^T$$

which has $2N + 3$ unknowns and D is a vector with values from integrals involving the

known function $P(r)$ and constants to be determined by the equations. This will give us the coefficients of the polynomial expansion for $H_0(r)$ and $G(r)$.

To compare the outputs to previous work in the literature [44], the semi-space is treated as an isotropic aluminium material, with shear modulus $\mu = 26.13$ GPa and Poisson's ratio $\nu = 0.345$. The radius of the circular membrane where surface effects are assumed is set to $a = 10$ nm. The radius of loaded region, s , will be set such that $s = a/4$. These parameters will be used henceforth unless otherwise specified. The function for external pressure applied to the surface will be set to the Hertzian contact pressure

$$p(r) = \frac{3P}{2\pi s^3} \sqrt{s^2 - r^2}, \quad r \leq s$$

with average pressure $p = P/(\pi s^2)$.

First, we test the numerical convergence of the algorithm by comparing the solution outputs from the scheme to the classical solution. These classical solutions are well known and can be found in the literature [24]. The classical tangential stress on the surface $\sigma_{rz}(r, 0)$ is assumed to equal zero for all values. We do not take the values of the surface parameters to equal zero since there is division by the parameters in many of the equations. For simplicity, we set $\mu_0, \lambda_0, \sigma_0 = 10^{-4}$ so that $\gamma_1/2\mu a$ and $\gamma_2/2\mu a$ are of the order 10^{-7} and $\gamma_3/2\mu a^3$ is of the order 10^{-10} . In the numerical model, we are able to set both the number of evaluation points and the number of terms in the Chebyshev polynomial sum approximation. The next figures explore the convergence of the numerical approximation with either the number of polynomials, denoted N , or the number of evaluation points, denoted n , fixed while the other varies.

In fig. 4.2, the number of polynomials in the summation approximation for both $H_0(r)$ and $G(r)$ is set to $N = 50$ and the number of evaluation points is varied. The functions $\sigma_{zz}(r, 0)$ and $u_z(r, 0)$ are very close to the classical solution even for smaller numbers of evaluation points. The convergence for $\sigma_{rz}(r, 0)$ and $u_r(r, 0)$ is slow, but at $n = 2N$ where n is the number of evaluation points and N is the number of polynomials is very close to the classical solution. In fig. 4.3 the number of evaluation points is set to $n = 200$ and the

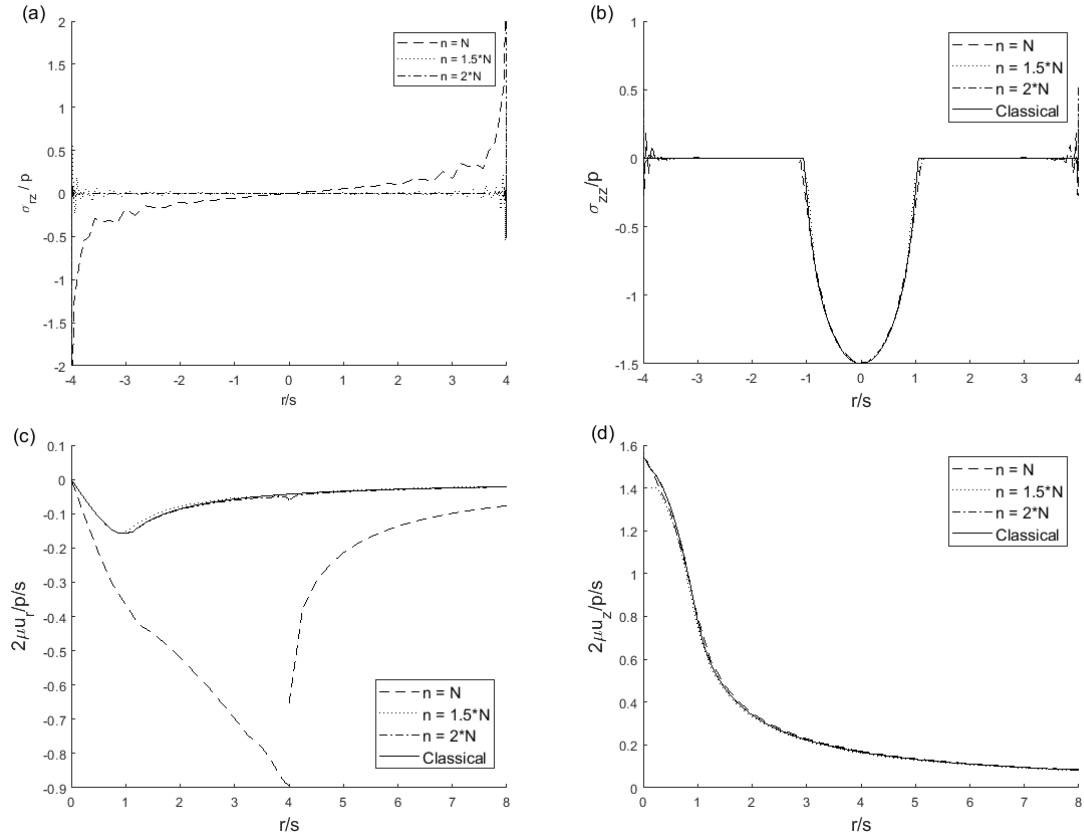


Figure 4.2: Comparison of the fixed polynomial approximation results with the exact solution in the case of no surface effects. The number of polynomials in the summation approximation of $G(r)$ and $H_0(r)$ is given by N and the total number of evaluation points is given by n .

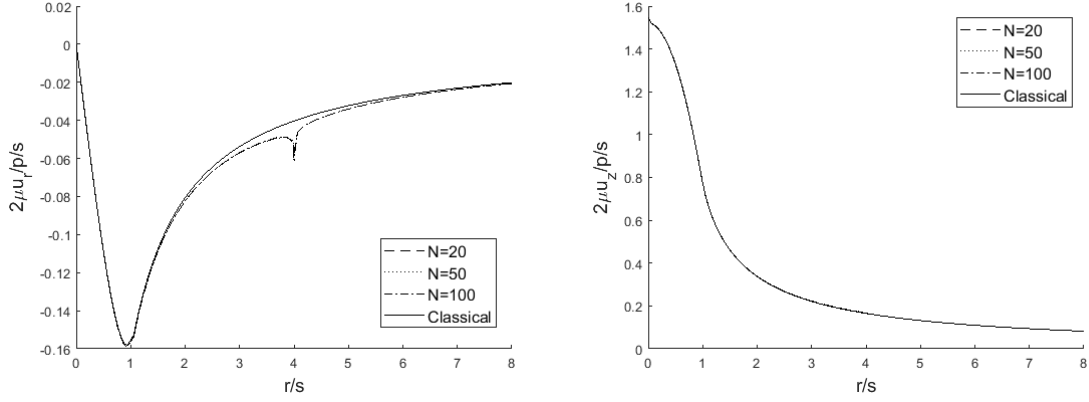


Figure 4.3: Comparison of the fixed evaluation points approximation results with the exact solution in the case of no surface effects. The number of polynomials in the summation approximation of $H_0(r)$ and $G(r)$ is given by N .

number of polynomials in the summation approximation for both $H_0(r)$ and $G(r)$ is varied. The number of polynomials N varies in each plot such that $N = 20, 50, 100$. For a large number of evaluation points $n \geq 2N$ the solutions tend to be very close. This is true even for polynomial approximations with small upper summation bounds. For the remainder of the results we set the number of polynomials in the summation approximations $N = 100$ and the number of evaluation points $n = 2N$.

It is prudent to observe that the boundary conditions derived through variational methods given by (2.11-2.13), has a term in the first equation where σ_0 appears as a constant. This causes the behavior of the model at the boundary of the membrane to be dominated by the value of the prestress. In fig. 4.4 the surface prestress parameter σ_0 was set so that $\sigma_0 = \gamma \cdot 10^{-4}$ where $\gamma = 1, 10, 100$. The surface Lamé parameters were set to $\mu_0 = -0.376$ and $\lambda_0 = 6.8511$ and the bending resistance parameters were set so that $\zeta + 2\chi = 1$ nN·nm to match the values used by Mi [44]. It can be seen in these plots that for larger values of σ_0 the displacement $u_r(r, 0)$ seems to have a singularity at the boundary of the membrane, here $r/s = 4$ since the inner radius s is scaled such that $s = a/4$ and r is scaled to r/a . In comparison, fig. 4.5 shows the results for varying the bending resistance, γ_3 , with boundary conditions given by (2.11-2.13). The value of γ_3 was set so that $\gamma_3 = \gamma \cdot 10^{-19}$ where $\gamma = 1, 10, 100, 10^4$. As can be seen, the results barely change for the different values of γ ,

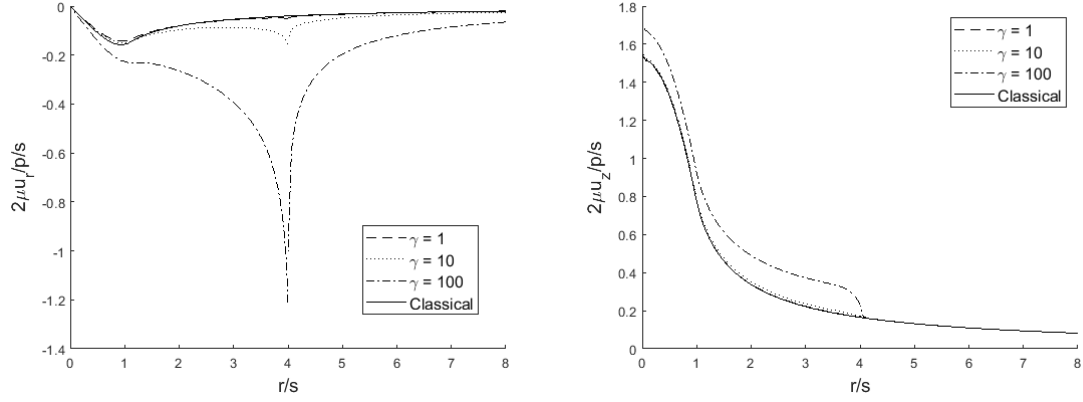


Figure 4.4: Comparison of the results obtained varying the prestress parameter γ_2 so that $\gamma_2 = \gamma \cdot C$ for a constant C .

with the most notable change at the largest value for γ_3 .

The results of this work differ from the literature in that surface elasticity is assumed on a finite subset of the surface. The results with boundary conditions (2.11-2.13) and the results given by Mi [44] with the “fixed tip” boundary conditions are compared in fig. 4.6. The “fixed tip” boundary conditions are given by

$$\begin{aligned} \left[\left(\frac{\partial}{\partial r} + \frac{1}{r} \right) u_r(r, 0) \right] \Big|_{r=a} &= 0, \\ \left[-\sigma_0 u_{z,r}(r, 0) + (2\chi + \zeta) \frac{\partial}{\partial r} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) u_z(r, 0) \right] \Big|_{r=a} &= 0, \\ \left[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) u_z(r, 0) \right] \Big|_{r=a} &= 0, \end{aligned}$$

which satisfies continuity for the auxiliary problem presented in Section 4.3. To compare results to Mi, the following parameters are used: $\mu = 26.13$, $\nu = 0.345$, $\lambda_0 = 6.8511$, $\mu_0 = -0.376$, $\sigma_0 = 1$, and $(\zeta + 2\chi) = 10^{-18}$. For numerical stability of the system, the number of polynomials in the approximations of $G(r)$ and $H_0(r)$ is taken to be $N = 8$ and the number of evaluation points is taken to be $n = 160$.

In fig. 4.6, the results for the model with boundary conditions given by (2.11-2.13) are plotted with small dots and labelled ‘BC 1’, results with the numerical method presented in this section but boundary conditions given by the “fixed tip” boundary are plotted with

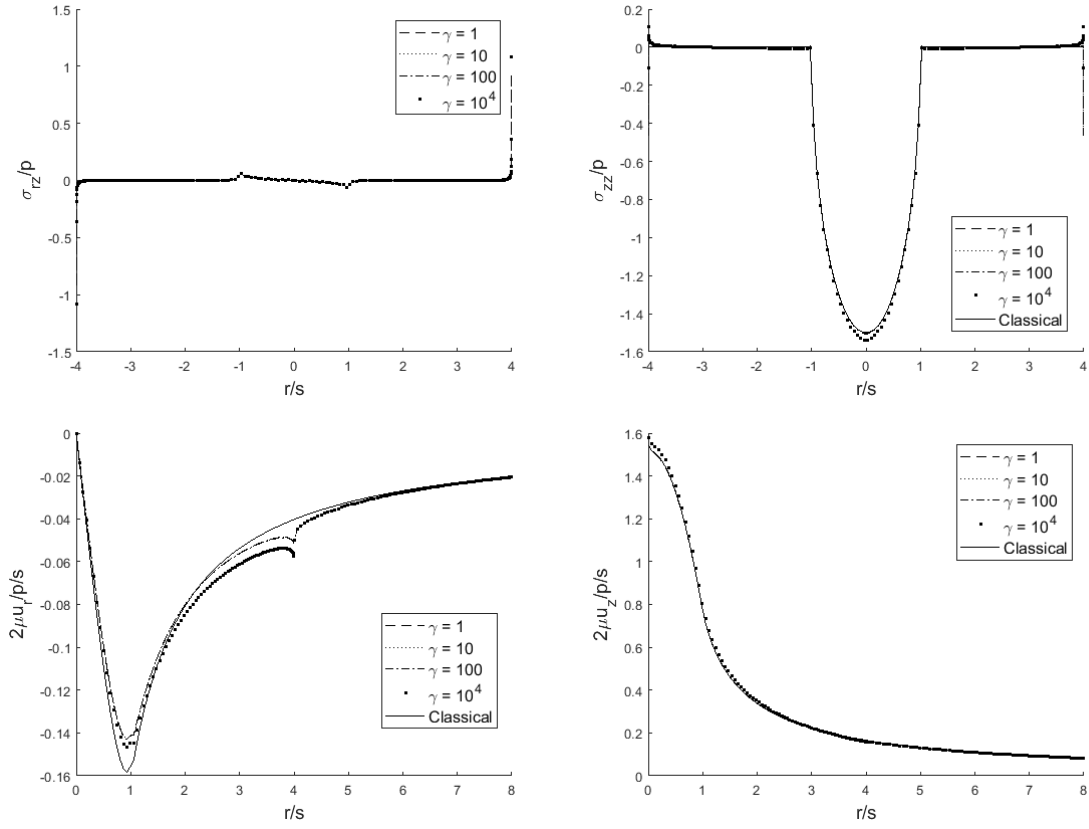


Figure 4.5: Comparison of the results when the surface parameter corresponding to the bending rigidity is varied as $\gamma_3 = \gamma \cdot C$ for constant C .

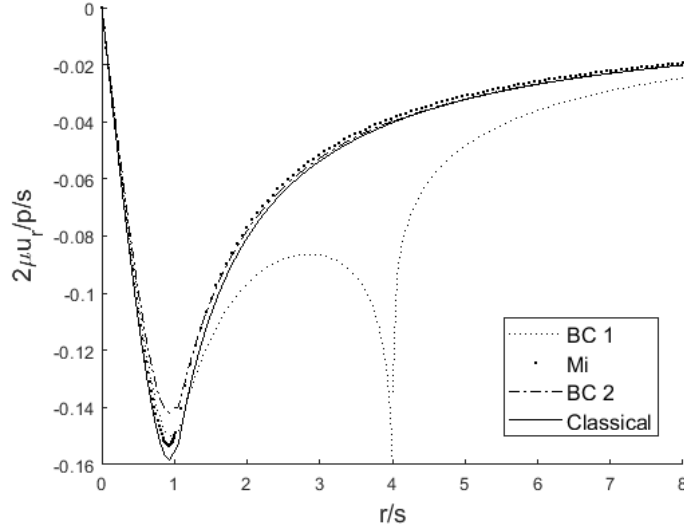


Figure 4.6: Comparison of the results to known results in the literature where the whole surface is modeled with Steigmann-Ogden.

dashed dots and labeled ‘BC 2’, and results using Mi’s numerical methods are plotted in large dots. It can be seen that for the chosen parameters the results for the “fixed tip” boundary conditions match more closely to Mi’s results. This is to be expected since the problem presented by Mi assumes surface effects on the entire infinite surface, which is exactly what is defined as the “fixed tip” boundary conditions.

Continuing to use the “fixed tip” boundary conditions, in fig. 4.7 the results for varying the prestress parameter γ_2 such that $\gamma_2 = \gamma \cdot C$ for a constant C and $\gamma = 1, 10, 100$. The other surface parameters are given by $\mu_0 = -0.376$ and $\lambda_0 = 6.8511$ and the bending resistance parameters were set so that $\zeta + 2\chi = 1 \text{ nN}\cdot\text{nm}$.

In fig. 4.8 the results for varying the bending rigidity parameter γ_3 such that $\gamma_3 = \gamma \cdot C$ for a constant C and $\gamma = 1, 10, 100$. The other surface parameters are given by: surface Lamé parameters $\mu_0 = -0.376$, $\lambda_0 = 6.8511$, and prestress parameter $\sigma_0 = 1$. It can be seen that the results using the “fixed tip” boundary conditions are almost identical, except at the membrane boundary where there is some deviation from the behavior of the classical solution. The results given by Mi [44] do not show this behavior since the entire plane is modeled by the Steigmann-Ogden model of surface elasticity, so this behavior is pushed out to infinity.

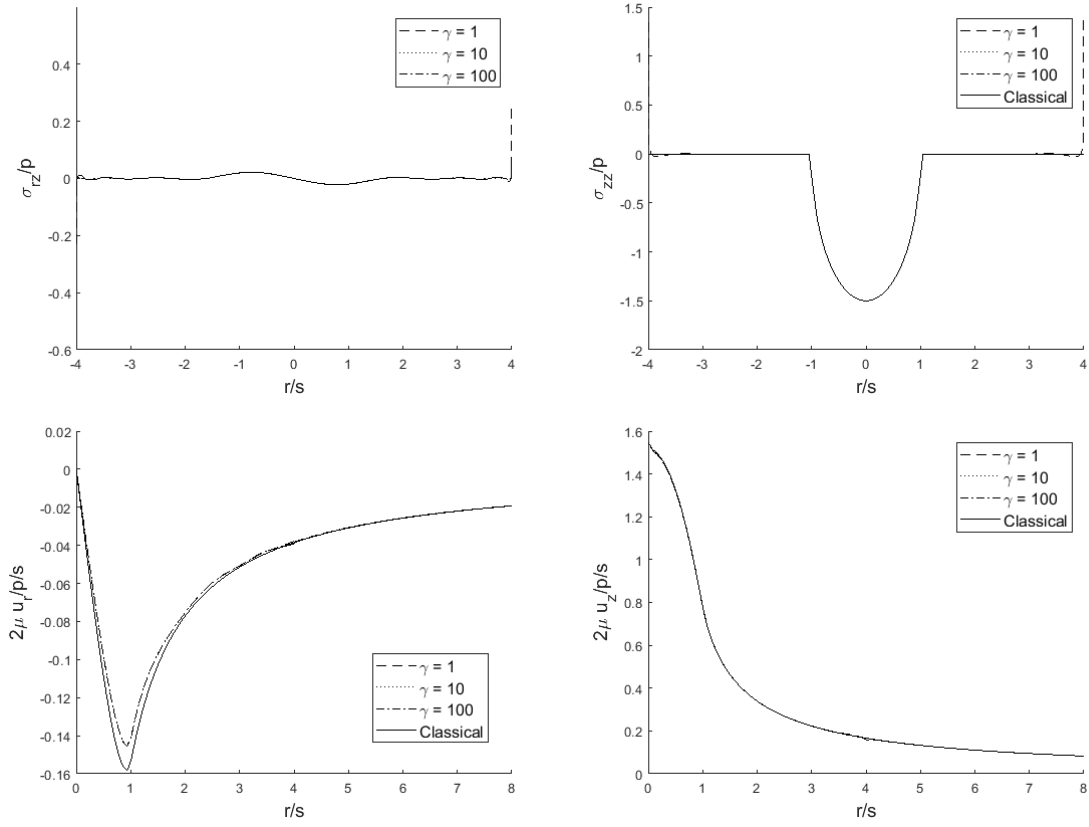


Figure 4.7: Comparison of the results obtained using the “fixed tip” boundary conditions varying the prestress parameter γ_2 so that $\gamma_2 = \gamma \cdot C$ for a constant C .

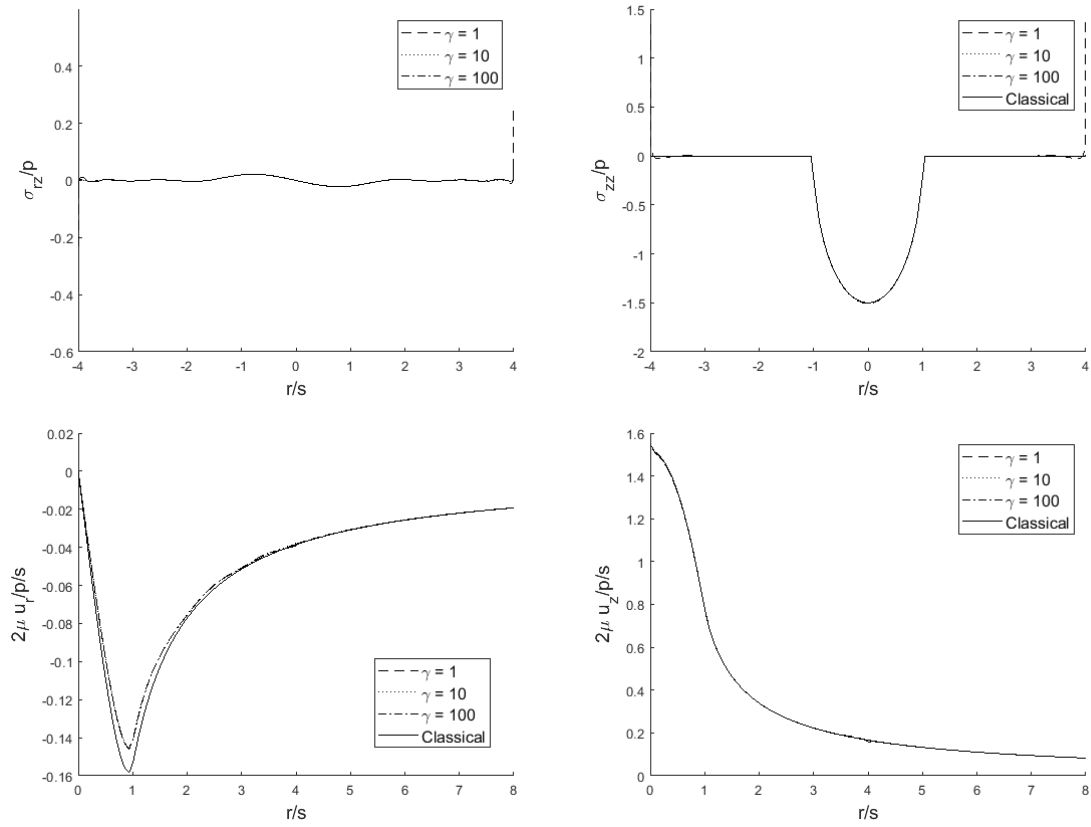


Figure 4.8: Comparison of the results using the “fixed tip” boundary conditions when the surface parameter corresponding to the bending rigidity is varied as $\gamma_3 = \gamma \cdot C$ for constant C .

Chapter 5

Conclusions

This work covers two problems involving deformation of a material at the nanoscale. The surface plays a large role at this scale, where materials have been observed to have size-dependencies of measurable properties. In each problem, the Steigmann-Ogden model of surface elasticity is chosen to model the effect of the surface while linear isotropic elasticity is used to model the bulk. Three parameters are used to describe the surface properties $\gamma_1 = (\lambda_0 + 2\mu_0)/2\mu$, $\gamma_2 = \sigma_0/2\mu$, $\gamma_3 = (2\chi + \zeta)/2\mu$. The parameter γ_1 represents the surface's resistance to stretching, γ_2 is the value of the prestress, and the parameter γ_3 represents the resistance of the surface to bending.

Chapter 3 studies an axisymmetric problem of contact between a rigid stamp and an elastic semi-space. In this problem, the infinite plane boundary of the semi-space is modeled using the Steigmann-Ogden model. The problem is reduced to a singular integral equation where the character of the kernel is dependent on the values of the surface parameters. It is shown that the problem without surface effects matches the classical solution. In the cases of the Gurtin-Murdoch model with prestress and the Steigmann-Ogden model, the resulting equation is, in general, ill-posed, and can be solved only in the least squares sense.

The numerical scheme of the solution of the resulting integral equation of the first kind is based on using the Gauss-Chebyshev quadrature. It is shown that the surface parameters play a significant role in the behavior of the solution for contact radii in the range of 1 – 100

nm. The influence of the surface parameters diminishes and eventually becomes negligible for large values of the contact radius. The presence of the surface energy generally has the effect of smoothing the stresses and displacements near the end of the contact interval, and results in significant size-dependency. The material hardness increases due to the presence of surface effects, an effect that has been observed experimentally [43, 49]. Similar theoretical observations have been made before for the the Gurtin-Murdoch model [51].

In Chapter 4, a problem where an elastic semi-space is subjected to an axisymmetric nanoscale circular patch loading is studied. A circular membrane attached to the boundary of the semi-space is modeled using the Steigmann-Ogden model of surface elasticity. This membrane is assumed to share its center with the loaded circular region. Outside of the circular membrane the surface is assumed to have no surface effects. The problem is reduced to a system of two singular integral equations with conditions on the boundary of the membrane. The character of these equations is shown to be dependent on the values of the surface parameters assumed on the membrane. In the absence of all surface effects, the solution is solved by inverting Hankel transforms. One of the integral equations in the system only appears when the surface is assumed to resist stretching, $\gamma_1 \neq 0$. When the surface resists stretching and either has nonzero prestress or resists bending, there are two singular integral equations that can be rewritten as two Fredholm equations of the second kind. In the cases where the problem reduces to a single equation with singular kernel, this equation can also be rewritten as a Fredholm equation of the second kind. The kernels in each respective case are shown to have logarithmic singularities along the diagonal. In the most general case of the Steigmann-Ogden model of surface elasticity, it is shown the solution can be found uniquely provided the corresponding adjoint homogeneous equation has only trivial solution.

The numerical method of Gaussian quadrature and approximation by Chebyshev polynomials are used to obtain the approximate solutions to the problems. Values of prestress play a dominant role in the solution. It was also shown that varying γ_3 did not change the solution in a significant way, as opposed to results from other works [44] where the Steigmann-Ogden model is taken on the whole infinite plane boundary. Assuming the fixed

tip boundary conditions for the membrane leads to solutions that are smooth at the membrane boundary. The solutions do not change for varying values of the prestress or bending resistance parameters. This behavior differs from solutions given by assuming the boundary conditions given in Chapter 4 or solutions for the problem discussed in Chapter 3.

The frictionless contact problem with the Steigmann-Ogden model of surface elasticity in general, is not solvable for a stamp of arbitrary shape. The patch loading problem for a finite-area membrane that lies on the surface of a semi-space can be solved uniquely as long as the homogeneous adjoint problem has only trivial solution. Assuming surface effects on a membrane attached to the surface with finite area is promising for investigations of solvability for future problems. It is important to note the choice of conditions on the membrane has a significant effect on numerical solutions. Further exploration of these differing choices of boundary conditions is important for the validity of numerical solutions.

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