EFFECT OF VISCOELASTIC FOUNDATION ON THE STABILITY OF A TANGENTIALLY LOADED CANTILEVER COLUMN

by

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I. INTRODUCTION

The vibration and stability of elastic systems involving follower forces has attracted much attention during the last few decades. The need for such a study arises from the fact that many practical engineering problems can be classified under this category. For example, follower forces play an important role in the study of pod-mounted jet engines [1], vertical take-off and landing aircrafts [2,3,4], aerodynamic flutter of panels [5], thermally loaded spacecraft antennas [6,7], and cantilever pipes conveying fluid [8]. Under certain conditions, chemical or electromagnetic energy can also induce follower type forces into a system [9].

All these systems are subjected to forces which follow the motion of the system in some prescribed manner. Since the work done on the system by these forces is dependent on the loading path, the presence of follower forces causes the system to be nonconservative.

In general, a nonconservative system has two modes of instability, static and dynamic, in contrast to conservative systems which only have static instabilities. Static instability in elastic systems (also called buckling or divergence) occurs when the system assumes a new equilibrium position close to the original equilibrium configuration of the system. Dynamic instability, or flutter, results when a small disturbance about an equilibrium position causes oscillations which increase without bound. With either type of instability, obviously a system fails to operate in its designed configuration. Although the static method yields stability conditions for conservative systems, it fails to predict the dynamic instabilities of nonconservative systems. Therefore, nonconservative elastic systems, in general, require a dynamic stability analysis.

In order to understand the fundamental characteristics of these engineering problems, investigators have suggested various models which involve follower forces. Beck [10] was the first to obtain the correct solution of the linear elastic cantilevered column under the action of a concentrated tangential force. The stability investigation of this special nonconservative system is now referred to as Beck's problem. He used the dynamic approach to solve the equation of motion and found the flutter load and frequency. This basic solution generated much interest among scientists and engineers working in the field of dynamic stability.

Beck's result was verified by many authors and extended to explore the effects of shear and rotary inertia, internal and external viscous damping, different boundary conditions imposed on the cantilever's free end, as well as elastic and viscoelastic support. Excellent reviews of vibration and stability studies related to elastic systems subjected to follower forces have been presented by Herrmann [11] and Sundararajan [12]. Nemat-Nasser [13] used the Timoshenko beam theory to develop the equation of motion. He found the resulting critical load to be less than that based on the Euler-Bernoulli beam theory. Using a cantilever column made of a viscoelastic material, Ziegler [14] found the unexpected result that under certain conditions small internal viscous damping destabilized the otherwise stable system. Nemat-Nasser and Herrmann [15], Prasad and Herrmann [16], and Bolotin and Zhinzher [17] have shown that the flutter load found for an undamped elastic system subjected to follower forces is the upper bound for systems with slight internal damping. Other dissipative mechanisms besides viscous damping have also been considered. Huang and Shieh [18] found thermal-mechanical coupling to have a pronounced destabilizing effect on Beck's column while Jong [19] showed that bilinear hysteretic damping can also cause destabilization.

Studies considering the effect of external viscous damping also yielded rather unusual results. Plaut and Infante [20], Anderson [21], and Pedersen [22] found that external viscous damping increases the flutter load but only to an asymptotic value. As a result of these studies and those mentioned previously by Ziegler and others, damping plays an uncertain role in the stability of nonconservative systems. Hence, each system must be considered independently with no general statement available to describe the effect of damping.

Besides these studies, some investigators have considered the effect of boundary conditions imposed on the free end. Barta [23] and Sundararajan [24] found that rigid or elastic end supports can have a destabilizing effect. Pedersen [22] extended the problem to include a concentrated tip mass, a linear elastic spring, and a partial follower force. He described the effect of these boundary parameters on the flutter load and the lowest natural frequency.

Designing adequate support for elastic systems in order to prevent failures in flutter is a current engineering challenge. Some progress has been made to this end by authors who studied the stability characteristics of Beck's column supported with elastic and viscoelastic foundations. Peterson [25], Smith and Herrmann [26], and Sundararajan [27] found that adding a continuous elastic support does not change the flutter load, but only increases the flutter frequency. Anderson [28] considered the effect of an elastic foundation on the stability of cantilever columns subjected to uniformly and linearly distributed tangential forces. Wahed [29] considered supporting the cantilever with an elastic foundation in the presence of external viscous damping and found that the flutter load depends on the damping coefficient as well as the foundation stiffness. Kar [30] considered a linearly tapered cantilever of rectangular cross section made of viscoelastic material and supported by a Kelvin-Voigt viscoelastic foundation. He formulated an appropriate variational principle to determine the approximate critical flutter load for various combinations of taper and internal damping parameters of the beam and viscous damping and stiffness parameters of the foundation.

Although these studies which consider Beck's problem with a foundation have given some insight, they possess several limitations. In most cases, either an approximate analytical technique, such as the Galerkin's method, or a numerical scheme was used in the analysis. These techniques are limited in that it is usually difficult to determine general behavior patterns and the exact dependence of the solution on the system parameters. In particular, for the case of numerical computation it is necessary to obtain data for a large number of cases and even then it may be difficult to ascertain whether some phenomenon or characteristic is being overlooked or concealed. For these reasons, most of the previous studies do not include results for a complete range of system parameters. In addition, the viscoelastic foundation models are primarily restricted to the Kelvin-Voigt type. This foundation is probably selected since it can be shown that a Maxwell type viscoelastic foundation has no effect on the critical load for conservative systems. The important question regarding the effect of Maxwell foundations on the stability of nonconservative systems, therefore, remains unanswered.

The purpose of the present study is to eliminate these limitations through an exact analysis of this special nonconservative elastic system. The system consists of a slender elastic column, fixed at one end and axially loaded with a constant force at its free end, supported with a Standard Linear viscoelastic foundation. This foundation has as special cases the well known Kelvin-Voigt model and the Maxwell, or relaxation model. This system is analyzed using the dynamic theory of stability.

The equation of motion for each foundation model is developed in Chapter II and written in terms of the appropriate nondimensional constants. A separable solution is then assumed and the associated eigenvalue problem is formulated. A numerical scheme is devised to obtain the real and complex eigenvalues of the characteristic equation. In Chapter III, the effect of the eigenvalues on the time dependent part of the solution is considered. To investigate the stability characteristics, a modified Routh-Hurwitz criteria is developed. Then, for each foundation model, the stability constraints are derived for the full range of foundation parameters. The primary results and recommendations for further study are given in Chapter IV.

II. EQUATION OF MOTION AND GENERAL SOLUTION

2.1 Equation of Motion

Consider small transverse motion about the undisturbed equilibrium position of a uniform elastic column, of length ℓ, supported with a visco-elastic foundation and subjected to a constant compressive axial load p as shown in figure 2.1. Let y be the lateral displacement, x the distance along the beam, and q(x,t) the total reaction pressure from the foundation. Note that the supporting foundation is assumed to be continuously connected to the beam so it opposes motion regardless of the direction. The flexural rigidity EI and the density per unit length ρ are assumed to be constant.

Using Euler-Bernoulli beam theory and Hamilton's principle [31], the dynamic equation of motion for the system is

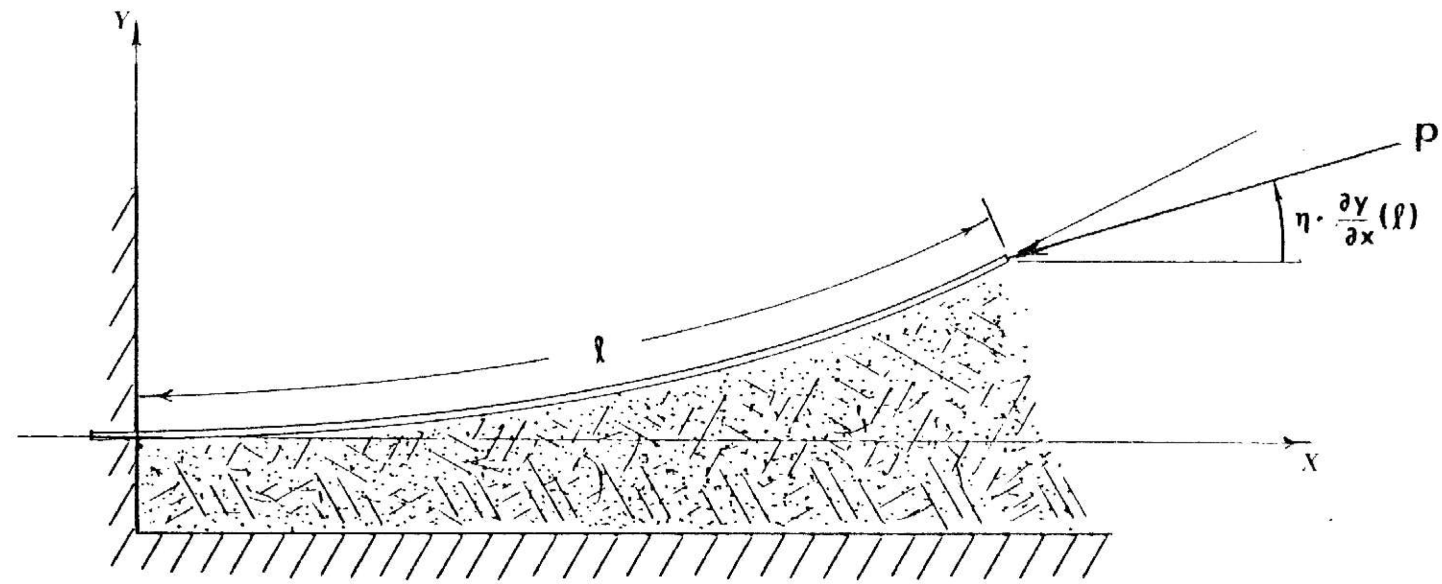
$$EI\,\frac{\partial^4 y}{\partial x^4} + p\,\frac{\partial^2 y}{\partial x^2} + \rho\,\frac{\partial^2 y}{\partial t^2} = -q(x,t). \qquad (2.1)$$

To obtain a realistic effect of the foundation on the motion, its "in-phase" mass M^* must be included in the inertia term of equation (2.1). Following the suggestion of Veletsos [32], the total foundation reaction q^* is

$$q^* = q - M^*\,\frac{\partial^2 y}{\partial t^2}\ . \qquad (2.2)$$

Consequently, equation (2.1) becomes

$$EI\,\frac{\partial^4 y}{\partial x^4} + p\,\frac{\partial^2 y}{\partial x^2} + (\rho + M^*)\frac{\partial^2 y}{\partial t^2} = -q^*\ . \qquad (2.3)$$



<u>Figure 2.1</u> Cantilever Column Supported with Viscoelastic Foundation Subjected to Follower Force

The right hand side of equation (2.3) depends on the type of model used for the foundation. In the following, various viscoelastic models are discussed and the corresponding equations of motion are derived.

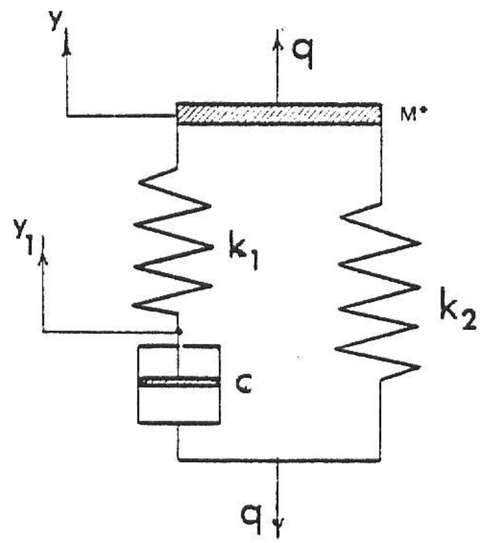
2.1.1 Standard Linear Solid Foundation

Freudenthal and Lorsch [33] reported that a viscoelastic medium reproduces actual foundation behavior better than an elastic medium since its force-deformation relations are time dependent. Following this suggestion, the foundation is represented by a Standard Linear Solid, which has the capacity to both store and dissipate energy. For analysis, the continuum is represented by independent viscoelastic elements, each supporting a differential beam element. The Standard Linear model, shown in figure 2.2(a), consists of an effective mass M^* supported by a series combination of an elastic spring k_1 and viscous dashpot c connected in parallel with another elastic spring k_2. Special cases of this general model can be used to represent two other models, viz., the Kelvin-Voigt and the Maxwell foundation models shown in figures 2.2(b) and (c), respectively. The first results when k_1 approaches infinity and the second when k_2 is identically zero. These models are further discussed later. The equation of motion for the Standard Linear model is developed in the following.

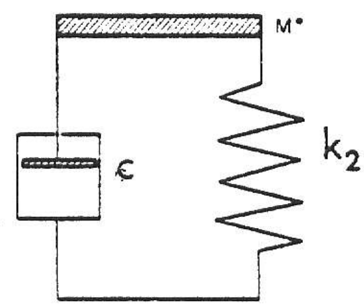
Following Lin [34], let q_1 and q_2 be the pressure acting on springs k_1 and k_2 such that

$$q^* = q_1 + q_2 \; . \qquad (2.4)$$

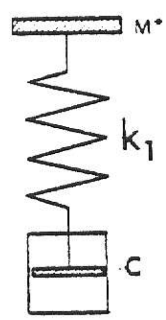
Referring to figure 2.2(a), let y_1 be the displacement of the node connecting k_1 and c. The force q_1 is transmitted through both the spring



(a) Standard Linear Model



(b) Kelvin-Voigt Model



(c) Maxwell Model

Figure 2.2 Viscoelastic Models

and the dashpot so the following relationships hold. First, note that

$$q_1 = k_1(y - y_1) \; . \tag{2.5}$$

Dividing by k_1 and Δt, an increment of time, yields

$$\frac{1}{\Delta t}\left(\frac{q_1}{k_1}\right) = \frac{1}{\Delta t}\,(y - y_1) \; . \tag{2.6}$$

In the limit Δt approaches zero and equation (2.6) takes the form

$$\frac{1}{k_1}\left(\frac{\partial q_1}{\partial t}\right) = \frac{\partial y}{\partial t} - \frac{\partial y_1}{\partial t} \; . \tag{2.7}$$

From the force in the dashpot,

$$\frac{1}{c}\,q_1 = \frac{\partial y_1}{\partial t} \; . \tag{2.8}$$

Using equations (2.7) and (2.8),

$$\frac{\partial y}{\partial t} = \frac{1}{k_1}\frac{\partial q_1}{\partial t} + \frac{\partial y_1}{\partial t} = \frac{1}{k_1}\frac{\partial q_1}{\partial t} + \frac{1}{c}\,q_1 \; . \tag{2.9}$$

The force in spring k_2 is

$$q_2 = k_2 y \tag{2.10}$$

Combining equations (2.4) and (2.9) and also noting that

$$\frac{\partial q^*}{\partial t} = \frac{\partial q_1}{\partial t} + \frac{\partial q_2}{\partial t} \; , \tag{2.11}$$

q_1 can be eliminated to obtain

$$\frac{\partial y}{\partial t} = \frac{1}{k_1}\left(\frac{\partial q^*}{\partial t} - \frac{\partial q_2}{\partial t}\right) + \frac{1}{c}\,(q^* - q_2) \; . \tag{2.12}$$

Substituting from equation (2.10) for q_2 in equation (2.12) yields,

$$\frac{\partial y}{\partial t} = \frac{1}{k_1}\frac{\partial q^*}{\partial t} - \frac{k_2}{k_1}\frac{\partial y}{\partial t} + \frac{1}{c}q^* - \frac{k_2 y}{c} \tag{2.13}$$

Differentiating with respect to t, equation (2.3) becomes

$$\frac{\partial q^*}{\partial t} = -\left[EI\frac{\partial^5 y}{\partial t\partial x^4} + p\frac{\partial^3 y}{\partial t\partial x^2} + (\rho + M^*)\frac{\partial^3 y}{\partial t^3}\right] . \tag{2.14}$$

Substituting equations (2.3) and (2.14) into equation (2.13) yields the following partial differential equation of motion for the Standard Linear foundation.

$$\frac{EI}{k_1}\frac{\partial^5 y}{\partial t\partial x^4} + \frac{EI}{c}\frac{\partial^4 y}{\partial x^4} + \frac{p}{k_1}\frac{\partial^3 y}{\partial t\partial x^2} + \left(\frac{\rho + M^*}{k_1}\right)\frac{\partial^3 y}{\partial t^3} + \frac{p}{c}\frac{\partial^2 y}{\partial x^2} + \left(\frac{\rho + M^*}{c}\right)\frac{\partial^2 y}{\partial t^2} + \left(1 + \frac{k_2}{k_1}\right)\frac{\partial y}{\partial t} + \frac{k_2}{c}y = 0 . \tag{2.15}$$

Introducing nondimensional parameters

$$w = \frac{y}{\ell} , \quad \xi = \frac{x}{\ell} , \quad \tau = t\left[\frac{(\rho + M^*)\ell^4}{EI}\right]^{-\frac{1}{2}} ,$$

$$K_1 = \frac{k_1\ell^4}{EI} , \quad K_2 = \frac{k_2\ell^4}{EI} , \tag{2.16}$$

$$P = \frac{p\ell^2}{EI} , \quad C = c\left[\frac{\ell^4}{(\rho + M^*)EI}\right]^{\frac{1}{2}} ,$$

the above equation can be rewritten in the nondimensional form

$$\frac{C}{K_1}\frac{\partial^5 w}{\partial\tau\partial\xi^4} + \frac{\partial^4 w}{\partial\xi^4} + \frac{C}{K_1}P\frac{\partial^3 w}{\partial\tau\partial\xi^2} + \frac{C}{K_1}\frac{\partial^3 w}{\partial\tau^3} + P\frac{\partial^2 w}{\partial\xi^2} + \frac{\partial^2 w}{\partial\tau^2} + C\left(1 + \frac{K_2}{K_1}\right)\frac{\partial w}{\partial\tau} + K_2 w = 0 . \tag{2.17}$$

Notice that if C = 0, the equation of motion for a column supported by an elastic foundation is recovered as

$$\frac{\partial^4 w}{\partial \xi^4} + P\,\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \tau^2} + K_2 w = 0, \tag{2.18}$$

which is the same as obtained by Smith and Herrmann [26].

2.1.2 Kelvin-Voigt Foundation

In the limit when k_1 approaches infinity, the Standard Linear model becomes the well-known Kelvin-Voigt model (figure 2.2(b)). This model consists of an elastic spring and viscous dashpot combined in parallel. It has been shown that this foundation model represents delayed elasticity or after effect. Upon removal of a force on the element the deformation is gradually recovered.

The equation of motion for the Kelvin-Voigt model can be obtained by taking the limit of equation (2.15) as k_1 approaches infinity. This yields

$$EI\,\frac{\partial^4 y}{\partial x^4} + p\,\frac{\partial^2 y}{\partial x^2} + (\rho + M^*)\frac{\partial^2 y}{\partial t^2} + c\,\frac{\partial y}{\partial t} + k_2 y = 0. \tag{2.19}$$

Using equation (2.16), the nondimensional form becomes

$$\frac{\partial^4 w}{\partial \xi^4} + P\,\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \tau^2} + C\,\frac{\partial w}{\partial \tau} + K_2 w = 0. \tag{2.20}$$

This equation also describes the motion of a column when supported by an elastic foundation in the presence of external damping [29]. Also observe that viscous damping is a special case of the Kelvin-Voigt model. If K_2 is identically zero, then the nondimensional equation of motion for a column in the presence of viscous damping is

$$\frac{\partial^4 w}{\partial \xi^4} + P\,\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \tau^2} + C\,\frac{\partial w}{\partial \tau} = 0. \qquad (2.21)$$

This is identical to the equation reported by Plaut and Infante [20].

2.1.3 Maxwell Foundation

If spring k_2 is absent from the Standard Linear model, it takes the form of the Maxwell model as shown in figure 2.2(c). This model consists of an elastic spring and viscous dashpot combined in series, and represents the characteristics of creep and relaxation. For a constant force applied to the element, the deformation increases linearly with time. The equation of motion is easily found by letting $k_2 = 0$ in equation (2.15) to yield

$$\begin{aligned} &\frac{EI}{k_1}\frac{\partial^5 y}{\partial t\partial x^4} + \frac{EI}{c}\frac{\partial^4 y}{\partial x^4} + \frac{p}{k_1}\frac{\partial^3 y}{\partial t\partial x^2} + \left(\frac{\rho+M^*}{k_1}\right)\frac{\partial^3 y}{\partial t^3} + \\ &\qquad + \frac{p}{c}\frac{\partial^2 y}{\partial x^2} + \left(\frac{\rho+M^*}{c}\right)\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial t} = 0. \end{aligned} \qquad (2.22)$$

Consequently, the nondimensional equation becomes

$$\begin{aligned} &\frac{C}{K_1}\frac{\partial^5 w}{\partial \tau\partial \xi^4} + \frac{\partial^4 w}{\partial \xi^4} + \frac{C}{K_1}P\frac{\partial^3 w}{\partial \tau\partial \xi^2} + \frac{C}{K_1}\frac{\partial^3 w}{\partial \tau^3} \\ &\qquad + P\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \tau^2} + C\frac{\partial w}{\partial \tau} = 0\,, \end{aligned} \qquad (2.23)$$

with the constants defined by equation (2.16).

2.2 Separable Solution

As shown in figure 2.1, the column is fixed at one end and subjected to a constant follower force p on the free end. The entire length is supported by a viscoelastic foundation. The loading is a function of the follower parameter η to allow for a subtangential force. When η is zero, the force is horizontal and corresponds to Euler-type conservative loading. When η is one, the force is totally tangential. The complete boundary-value problem is given by the equation of motion (2.15) about the undisturbed equilibrium position and the following boundary conditions.

$$y(0,t) = 0, \quad y'(0,t) = 0,$$
$$y''(\ell,t) = 0, \quad y'''(\ell,t) + p(1-\eta)y'(\ell,t) = 0. \tag{2.24}$$

These can be rewritten in terms of the nondimensional variables as

$$w(0,\tau) = 0, \quad w'(0,\tau) = 0,$$
$$w''(1,\tau) = 0, \quad w'''(1,\tau) + P(1-\eta)w'(1,\tau) = 0. \tag{2.25}$$

Consider a solution of the form

$$w(\xi,\tau) = \Phi(\xi)T(\tau). \tag{2.26}$$

Substituting this into equation (2.17) yields

$$(C\dot{T} + K_1T)\Phi'''' + P(C\dot{T} + K_1T)\Phi'' +$$
$$+(C\dddot{T} + K_1\ddot{T} + C(K_1 + K_2)\dot{T} + K_1K_2T)\Phi = 0. \tag{2.27}$$

Dividing by $\Phi\cdot(C\dot{T} + K_1T)$, equation (2.27) can be rearranged as

$$\frac{\Phi'''' + P\Phi''}{\Phi} = \frac{-(C\dddot{T} + K_1\ddot{T} + C(K_1 + K_2)\dot{T} + K_1K_2T)}{C\dot{T} + K_1T} \tag{2.28}$$

If equation (2.28) is to hold for all values of ξ and τ, both sides must equal a constant. Let this constant be denoted by λ^2. Equation (2.28) now separates into the two ordinary differential equations.

$$\Phi'''' + P\Phi'' - \lambda^2\Phi = 0 \tag{2.29}$$

$$C\dddot{T} + K_1\ddot{T} + C(\lambda^2 + K_1 + K_2)\dot{T} + K_1(\lambda^2 + K_2)T = 0. \tag{2.30}$$

The boundary conditions in terms of Φ become

$$\Phi(0) = 0, \qquad \Phi'(0) = 0,$$

$$\Phi''(1) = 0, \qquad \Phi'''(1) + (1-\eta)P\Phi'(1) = 0. \tag{2.31}$$

It is observed that the boundary value problem defined by equations (2.29) and (2.31) must be solved for λ^2 in order to define equation (2.30), which determines the temporal part of the solution. The linear differential operator in equation (2.29), in general, is not self-adjoint due to the boundary conditions given by equation (2.31) and therefore, the eigenvalue λ^2 is not always real. As a result, the constant λ^2 is complex and can be defined as

$$\lambda^2 = \alpha \pm i\beta, \quad i = (-1)^{\frac{1}{2}}. \tag{2.32}$$

Let the solution of equation (2.30) be of the form

$$T(\tau) = Ae^{s\tau}, \tag{2.33}$$

where

$$s = \sigma + i\omega . \tag{2.34}$$

Substituting equation (2.33) into equation (2.30) yields

$$Cs^3 + K_1 s^2 + C(\lambda^2 + K_1 + K_2)s + K_1(\lambda^2 + K_2) = 0. \tag{2.35}$$

Since, in general, λ^2 is complex, the above polynomial has complex coefficients. From equation (2.33), it is seen that the roots of this complex polynomial play the key role in determining the stability criteria for the system. From equation (2.34), it is obvious that the necessary and sufficient condition for the solution to remain bounded is $\sigma \leq 0$. If $\sigma > 0$, the solution is unbounded and instability prevails. Thus, it is clear that the stability conditions for the system are functions of the foundation parameters K_1, K_2, C, and λ^2 which depends on the loading parameters P and η.

2.3 Eigenvalue Problem

First, the eigenvalue problem given by equations (2.29) and (2.31) is considered. It is known that the general solution of equation (2.29) can be expressed as

$$\Phi(\xi) = C_1 \sin a\xi + C_2 \cos a\xi + C_3 \sinh b\xi + C_4 \cosh b\xi. \tag{2.36}$$

By substituting this general solution in equation (2.29), one obtains

$$(a^4 - Pa^2 - \lambda^2)(C_1 \sin a\xi + C_2 \cos a\xi) +$$

$$+ (b^4 + Pb^2 - \lambda^2)(C_3 \sinh b\xi + C_4 \cosh b\xi) = 0. \tag{2.37}$$

For this to be zero for all ξ, since the C's are arbitrary, the following conditions must hold.

$$a^4 - Pa^2 - \lambda^2 = 0, \quad b^4 + Pb^2 - \lambda^2 = 0. \tag{2.38}$$

These produce the following relationships among a, b, P, and λ^2,

$$a^2 = \frac{P}{2} + \left(\frac{P^2}{4} + \lambda^2\right)^{\frac{1}{2}}, \quad b^2 = -\frac{P}{2} + \left(\frac{P^2}{4} + \lambda^2\right)^{\frac{1}{2}},$$

$$P = a^2 - b^2, \ \lambda = ab. \tag{2.39}$$

Imposing the boundary conditions (2.31) on the general solution (2.36) creates a set of four linear algebraic equations in the four constants C_1 through C_4. For a nontrivial solution, the determinant of the coefficients of the C's must vanish, i.e.,

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ a & 0 & b & 0 \\ -a^2 \sin a & -a^2 \cos a & b^2 \sinh b & b^2 \cosh b \\ \left[(1-\eta)P - a^2\right] a \cos a, & \left[a^2 - (1-\eta)P\right] a \sin a, & \left[(1-\eta)P + b^2\right] b \cosh b, & \left[b^2 + (1-\eta)P\right] b \sinh b \end{vmatrix} = 0. \tag{2.40}$$

Evaluating the determinant leads to the transcendental equation

$$\begin{aligned} &ab\left[2(1-\eta)P + b^2 - a^2\right]\sin a \sinh b - (1-\eta)P(b^2 - a^2) + \\ &+\left[(1-\eta)P(b^2 - a^2) - 2a^2b^2\right]\cos a \cosh b - (a^4 + b^4) = 0. \end{aligned} \tag{2.41}$$

Substituting equations (2.38) and (2.39) into equation (2.41) simplifies the transcendental equation to the form

$$2\lambda^2 + \eta P^2 + P\lambda(2\eta - 1)\sin a \sinh b + \left[2\lambda^2 + P^2(1-\eta)\right]\cos a \cosh b = 0. \tag{2.42}$$

2.4 Instability Mechanisms

For a given set of loading parameters P and η, equation (2.42) can be solved numerically to obtain the roots λ_i^2. These roots may be real or complex. Before discussing the solution of this transcendental equation, first, the instability mechanism and its dependence on the eigenvalues λ^2 must be clearly understood. A discussion of this nature is also necessary in order to realize the need for viscoelastic support rather than an elastic support. Therefore, a review of the instability phenomena under various conditions, as reported by past investigators, is presented in the following.

2.4.1 Divergence

First, consider the motion of the column about the equilibrium position under the action of a conservative force ($\eta = 0$) with no supporting foundation. By setting the foundation parameters C and K_2 and the load parameter η equal to zero in equation (2.17), the nondimensional equation of motion is easily obtained as

$$\frac{\partial^4 w}{\partial \xi^4} + P \frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \tau^2} = 0. \tag{2.43}$$

Assuming a separable solution in the form of equation (2.26) leads to the differential equation (2.29) with the transformed boundary conditions

$$\begin{aligned} &\Phi(0) = 0, \qquad \Phi'(0) = 0, \\ &\Phi''(1) = 0, \qquad \Phi'''(1) + P\Phi'(1) = 0, \end{aligned} \tag{2.44}$$

and the temporal equation

$$\ddot{T} + \lambda^2 T = 0. \tag{2.45}$$

The eigenvalue problem given by equations (2.29) and (2.44) yields the transcendental equation

$$(2\lambda^2 + P^2)\cos a \cosh b + 2\lambda^2 - P\lambda \sin a \sinh b = 0, \tag{2.46}$$

where a and b are defined by equation (2.39).

Static instability, or divergence, occurs when $\lambda^2 = 0$, since the temporal equation (2.45) would become

$$\ddot{T} = 0, \tag{2.47}$$

which has an unbounded solution. By letting $\lambda^2 = 0$ in equation (2.46), it is seen that

$$P^2 \cos(P^{\frac{1}{2}}) = 0 \tag{2.48}$$

gives possible static solutions. For nonzero divergence loads, equation (2.48) requires that

$$\cos(P^{\frac{1}{2}}) = 0, \tag{2.49}$$

which gives the condition

$$P = (2n - 1)^2 \frac{\pi^2}{4} \quad . \tag{2.50}$$

The lowest nondimensional load P, corresponding to n = 1, is the familiar Euler buckling load for a cantilever column,

$$P = \frac{\pi^2}{4} = 2.47 \quad . \tag{2.51}$$

Now, consider the effect of an elastic foundation on the static buckling load. THe equation of motion was obtained in equation (2.18). A separable solution leads to equation (2.29) and

$$\ddot{T} + (\lambda^2 + K_2)T = 0, \tag{2.52}$$

with the boundary conditions given again by equation (2.44). Notice from equation (2.52) that the stiffness parameter shifts the characteristic exponent by an amount K_2. Because of the elastic foundation, possible budkling loads now occur when the quantity $K_2 + \lambda^2$ becomes zero, leading to an unbounded solution. It can be shown that these buckling loads are higher than the corresponding buckling loads for an unsupported column. Thus, an elastic foundation does stabilize the conservative system which fails in divergence. Note that since the load is conservative, a static approach would also yield the same result.

At this point, consider the motion of the column in the presence of external viscous damping, described by equation (2.21). A separable solution once again leads to the spatial equation (2.29) and the equation

$$\ddot{T} + C\dot{T} + \lambda^2 T = 0. \tag{2.53}$$

For stability of the second order differential equatiom (2.53), it is necessary and sufficient that all the coefficients remain positive. Since only positive demping is considered, this stability condition restricts λ^2 to be greater than zero. Thus, the stability boundary occurs when $\lambda^2 = 0$, which defines the critical buckling load. From this it is seen that damping has no effect on the buckling load of this conservative system.

2.4.2 Flutter

First, consider the motion of a cantilever column under the action of a nonconservative ($\eta = 1$) force, commonly referred to as Beck's column. The motion is again described by equation (2.43) from which a separable solution leads to equations (2.29) and (2.45). The boundary conditions are now given as

$$\Phi(0) = \Phi'(0) = \Phi''(1) = \Phi'''(1) = 0, \tag{2.54}$$

since the tangential end load has no shear component. Substituting these boundary conditions and the general solution (2.36) in equation (2.29) leads to the transcendental equation

$$P^2 + P\lambda \sin a \sinh b + 2\lambda^2(1 + \cos a \cosh b) = 0, \tag{2.55}$$

as reported by Beck [10], Bolotin [35] and others. Equation (2.55) has an infinite set of eigenvalues λ_i^2 for each value of P. These eigenvalues occur in pairs such that as load P increases from zero, each pair of eigenvalues approach each other. After merging, they become a pair of complex conjugates. It can be seen from equation (2.55) that no possible static solutions ($\lambda^2 = 0$) exist, in contrast to the conservative problem when $\eta = 0$. The stability of the system is now determined from the solution of the temporal equation (2.45), which is affected by the eigenvalues of the transcendental equation (2.55). To observe their effect on stability, let equation (2.45) have a solution of the form (2.33), which leads to

$$s^2 + \lambda^2 = 0, \tag{2.56}$$

or

$$s = \pm i\lambda \ . \tag{2.57}$$

As seen from the assumed exponential solution, for stability, it is necessary and sufficient that the s roots (2.57) have zero or negative real parts. This requires that the imaginary part of the complex λ be non-negative. However, if λ is complex, a conjugate pair would satisfy equation (2.55) with one having a negative imaginary part. Realizing this, the critical load occurs when λ becomes complex since higher loads would cause one s root to have a positive real part, inducing flutter instability. Beck [10] found that the two lowest eigenvalues become equal at the critical load of 20.05. The higher pairs of eigenvalues demonstrate this same trend with successively higher critical load values. The flutter load corresponding to the lowest eigenvalues is of primary engineering interest and will hereafter be denoted by P_f. Beck's column is only stable for real eigenvalues corresponding to loads less than $P_f = 20.05$.

If Beck's column was supported by an elastic foundation, the separated solution consists of the eigenvalue problem defined by equation (2.29) with the boundary conditions (2.54) and the temporal equation (2.52). By assuming an exponential solution of the form (2.33), equation (2.52) becomes

$$s^2 + (\lambda^2 + K_2) = 0 \ . \tag{2.58}$$

Using an argument similar to that given after equation (2.57) the quantity $\lambda^2 + K_2$ must remain real for stability. This restricts λ^2 to be real, which leads to the critical flutter load of P_f. As shown by Smith and Herrmann [26] and others, an elastic foundation has no effect on the critical flutter load, and instead only increases the flutter frequency. Flutter instability still occurs if $P > P_f$.

Now, consider the effect of external viscous damping on Beck's column. Again a separable solution leads to the spatial equation (2.29) with boundary conditions (2.54) and the equation for T given by (2.53). Letting the solution for T have the form of equation (2.33), yields the quadratic in s

$$s^2 + Cs + \lambda^2 = 0 \ , \tag{2.59}$$

which has the roots

$$s_{1,2} = -\frac{C}{2} \pm \left\{\frac{C^2}{4} - \lambda^2\right\}^{\frac{1}{2}} . \tag{2.60}$$

Expressing λ^2 as the complex constant (2.32), yields

$$s_{1,2} = -\frac{C}{2} \pm \left(\frac{C^2}{4} - (\alpha \pm i\beta)\right)^{\frac{1}{2}} . \tag{2.61}$$

Leipholz [36] has shown that the real part of each s root remains negative as long as the inequality

$$\alpha > \frac{\beta^2}{C^2} \tag{2.62}$$

is satisfied. Since the smallest value of β^2/C^2 is zero, α must remain positive for stability. Rearranging inequality (2.62), the stability boundary becomes

$$C^2 = \frac{\beta^2}{\alpha} \ . \tag{2.63}$$

When λ^2 is real ($\beta = 0$) zero damping is required, but when λ^2 becomes complex ($\beta > 0$) damping must be present which satisfies inequality (2.62) The limiting value of $\alpha = 0$ requires infinite damping. Plaut and

Infante [20] and others have shown that the critical load corresponding to $\alpha = 0$ is 37.7. By adding external viscous damping to Beck's column, the critical load increases with increasing damping from P_f at zero damping to the limiting value of 37.7 for very large damping. At loads higher than 37.7 the system fails in flutter, regardless of the damping value.

In summary, the conservative problem ($\eta = 0$) fails in divergence at a nondimensional buckling load of 2.47. The addition of a continuous elastic support increases this critical buckling load, but external viscous damping has no effect. The nonconservative problem ($\eta = 1$), with or without an elastic foundation, fails in flutter when the eigenvalues of equation (2.55) become complex. This corresponds to a load of 20.05. External viscous damping stabilizes the system up to the limiting case when the real part becomes zero. This occurs for the load 37.7.

It is anticipated that a viscoelastic foundation can further increase the flutter load. This suggests that eigenvalues for loads greater than 37.7 will be necessary for a complete stability analysis. With an understanding of the effect of the eigenvalues on the stability of the system and the range of values needed, a numerical procedure for solving the general transcendental equation (2.42) with respect to the real and complex eigenvalues λ^2 is presented in the following section.

2.5 Numerical Solution of Eigenvalues

Because of the complicated nature of transcendental equation (2.42), few techniques are available to solve for the exact eigenvalues. As a result, several approximate methods have been suggested. Plaut and Infante [20] used a Galerkin approach to develop an approximate frequency equation and found expressions for α and β as functions of load P.

Herrmann and others [37] discretized the continuous nonconservative problem to a two degrees of freedom model. Routh-Hurwitz criteria gave the necessary and sufficient conditions for stability of this lumped parameter system.

The most direct method for solving the exact eigenvalue problem substitutes the complex form of λ^2 in equation (2.42) and separates the real and the imaginary parts to form two simultaneous transcendental equations in α and β [20]. Although this technique gives implicit expressions for the exact solution, the form is very complicated and extremely difficult to solve.

As an alternative to this approach, Pedersen [22] offers a more tractable method of solution. By transforming the transcendental equation into an easily differentiable form, the Newton-Raphson method [38] is used to solve for the real or complex eigenvalues corresponding to a particular load. Since this approach is most suitable for the exact analysis, it is used to proceed with the solution of the eigenvalue problem.

The characteristic equation (2.42) can be rewritten as

$$D(P,\lambda) = c_1 f_1 + c_2 f_2 \quad , \tag{2.64}$$

where

$$c_1 = 2\lambda^2 + (1-\eta)P^2 \; ,$$
$$c_2 = (2\eta-1)P, \tag{2.65}$$

and

$$f_1(P,\lambda) = f_1(a,b) = 1 + \cos a \cosh b \quad ,$$

$$f_2(P,\lambda) = f_2(a,b) = (a^2 - b^2) + ab \sin a \sinh b \quad , \tag{2.66}$$

with a and b given by equation (2.39). Now the following complex quantities can be defined.

$$g = a + ib, \quad g^2 = P + i2\lambda \,,$$
$$\bar{g} = a - ib, \quad \bar{g}^2 = P - i2\lambda \,. \tag{2.67}$$

Then, f_1 and f_2 of equation (2.66) become the complex functions

$$f_1 = 1 + \frac{1}{2}[\cos(g) + \cos(\bar{g})] \,,$$
$$f_2 = P + i\frac{\lambda}{2}[\cos(g) - \cos(\bar{g})] \,. \tag{2.68}$$

Using these complex functions, equation (2.64) can be easily differentiated to yield

$$\frac{\partial D}{\partial \lambda} = 4\lambda f_1 + c_1 \frac{\partial f_1}{\partial \lambda} + c_2 \frac{\partial f_2}{\partial \lambda} \,, \tag{2.69}$$

where

$$\frac{\partial f_1}{\partial \lambda} = -\, i\, \frac{1}{2}\left[\frac{\sin(g)}{g} - \frac{\sin(\bar{g})}{\bar{g}}\right] \,,$$
$$\frac{\partial f_2}{\partial \lambda} = i\, \frac{1}{2}\, [\cos(g) - \cos(\bar{g})] + \frac{1}{2}\lambda\left[\frac{\sin(g)}{g} + \frac{\sin(\bar{g})}{\bar{g}}\right] \,. \tag{2.70}$$

The Newton-Raphson method is used to solve equation (2.64) to yield real or complex values of λ for a given set of parameters η and load P. The iteration procedure is given by

$$\lambda_{n+1} = \lambda_n + \Delta\lambda_n \,, \tag{2.71}$$

where

$$\Delta\lambda_n = \frac{-D(P,\lambda)}{\left(\frac{\partial D}{\partial \lambda}\right)_{P,\lambda_n}} \,. \tag{2.72}$$

A FORTRAN program in complex variables, which uses this procedure to solve for the eigenvalues is given in Appendix A. Because of the availability of a Hewlett-Packard 9845 desktop computer, a Muller subroutine in BASIC language (given in Appendix B) was used to verify the real eigenvalues obtained from the FORTRAN program. The complex eigenvalues for $\eta = 1$ corresponding to load values ranging from 20.1 to 100 are included in Appendix C. These were compared with the partial listing given by Plaut and Infante [20]. Interactive graphics capabilities using an HP9872A plotter produced the results.

The real eigenvalues for several values of follower parameter η are plotted in figure 2.3. Observe that each η curve intersects the λ axis at the first two natural frequencies of the unsupported column ($\lambda_1 = 3.516$, $\lambda_2 = 22.034$). These follow from equation (2.42) with P = 0, i.e.,

$$2\lambda^2\left[1 + \cos a \cosh b\right] = 0, \tag{2.73}$$

which is independent of η. As P increases, these frequencies approach each other, up to a maximum value of P_f where they become equal. For P greater than this value, the frequencies become a complex conjugate pair and flutter occurs for the unsupported column.

This system also has possible divergence configurations, where the lowest frequency becomes zero. Static failure conditions can be found from (2.42) by setting $\lambda = 0$ as

$$P^2\left[\eta + (1-\eta)\cos a \cosh b\right] = 0. \tag{2.74}$$

Recall that, from equation (2.39) when $\lambda = 0$,

$$a^2 = P \quad \text{and} \quad b^2 = 0. \tag{2.75}$$

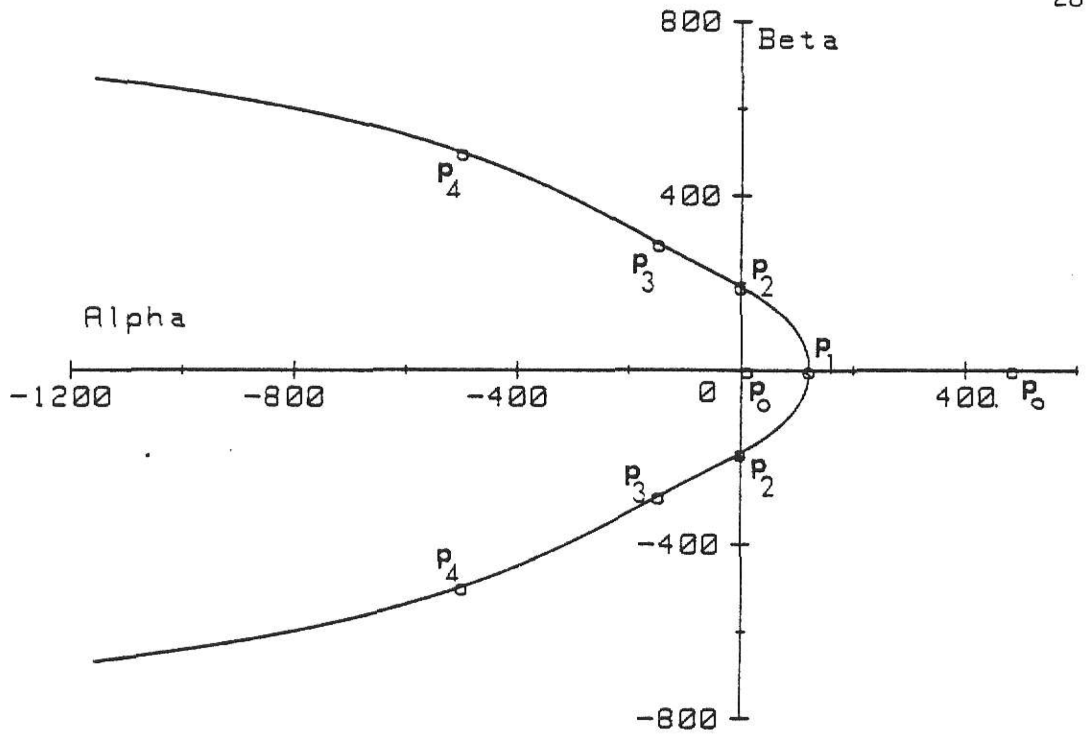


Figure 2.4 Complex Eigenvalues of Transcendental Equation (2.42)

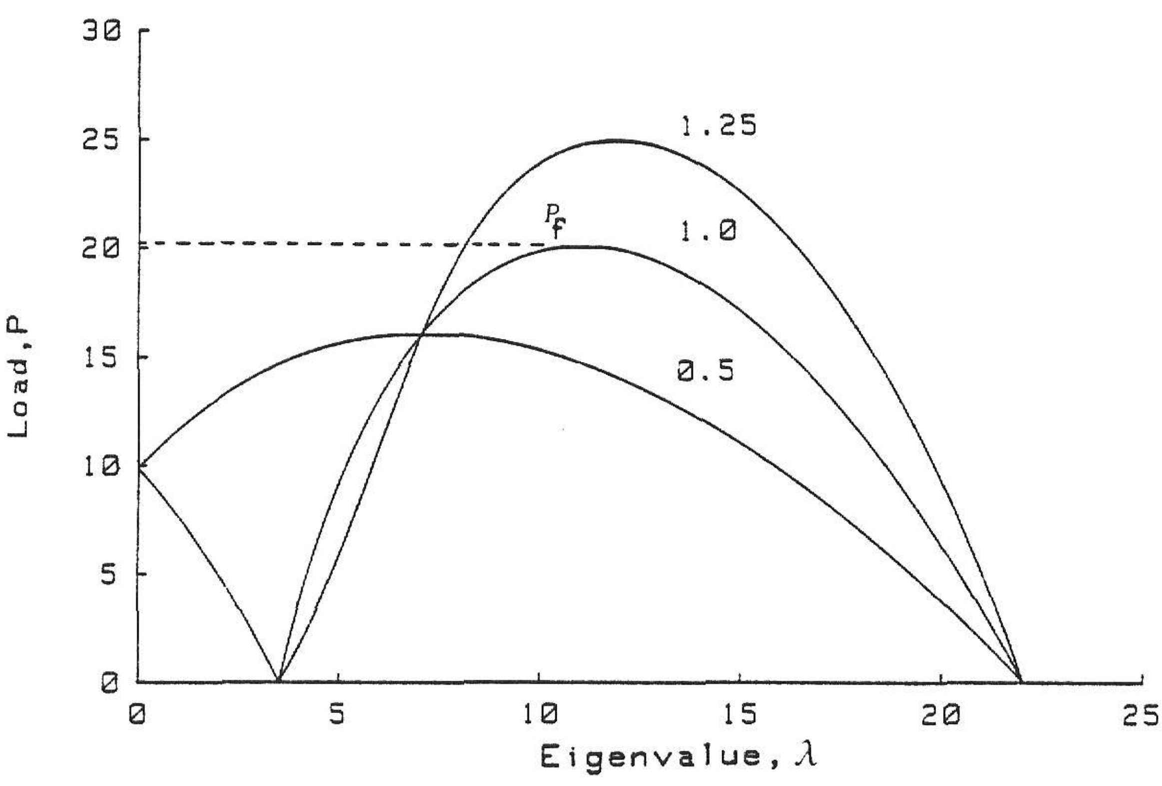


Figure 2.3 Real Eigenvalues of Transcendental Equation (2.42)

This reduces equation (2.74) to

$$P^2\left[\eta + (1-\eta)\cos(P^{\frac{1}{2}})\right] = 0. \tag{2.76}$$

Then for a nonzero P, possible critical static loads are given by

$$P = \left[\cos^{-1}\left(\frac{\eta}{\eta-1}\right)\right]^2 \tag{2.77}$$

For example, substituting $\eta = 0$ yields $P = \frac{\pi^2}{4}$, the familiar Euler buckling load.

It also follows from equation (2.77) that divergence instability is not possible with positive P for $\eta > 0.5$. Since the argument of $\cos^{-1}(\ )$ is bounded such that

$$-1 < \left(\frac{\eta}{\eta-1}\right) < 1\ , \tag{2.78}$$

the two relationships

$$-(\eta-1) > \eta \quad \text{and} \quad \eta > \eta - 1, \tag{2.79}$$

yield the necessary condition for divergence as

$$\eta < 0.5\ . \tag{2.80}$$

Thus, $\eta < 0.5$ is the constraint on η for possible divergence configurations.

Notice that for $\eta = 0.5$ in figure 2.3, the lowest eigenvalue decreases with increasing load to zero and then increases with a further increase of load. Divergence occurs when λ becomes zero ($P = \pi^2$, from equation (2.77)), then with increasing load the first and second roots merge and become complex causing flutter. Pedersen [22] has shown that the possibility of both divergence and flutter failures exists for η between 0.354 and 0.5.

Strictly divergence failure occurs for $0 < \eta < 0.354$.

The complex eigenvalues for $\eta = 1$ are plotted in figure 2.4. Note that α and β are the real and the imaginary parts of λ^2, respectively. Points p_0 through p_4 mark the complex roots corresponding to loads of special interest. The first two real natural frequencies corresponding to $P = 0$ in figure 2.3, are shown at points p_0. The roots move toward each other on the real axis, with increasing P, until the load becomes P_f at point p_1. Complex roots exist for loads higher than P_f. At point p_2 the real parts become zero, which corresponds to a load $P = 37.7$. This is the allowable load for the system in the presence of infinite external viscous damping, as shown by several investigators. [20,21,22] Higher loads yield eigenvalues with negative real parts. This is demonstrated with points p_3 and p_4 which represent loads of 50 and 75, respectively.

In the next chapter, the stability criteria for various foundation models are developed. The discussion is restricted to the case of $\eta = 1$ which corresponds to a totally tangential follower force.

III. STABILITY ANALYSIS AND RESULTS

The time dependent part of the assumed solution governed by equation (2.30) determines the stability criteria of the system. If following a small disturbance, the motion about the equilibrium configuration remains bounded, then this equilibrium position is considered stable; otherwise it is unstable. Furthermore, if the steady state motion returns to the original equilibrium configuration, this equilibrium position is considered asymptotically stable.

By assuming a solution in the form of equation (2.33), the temporal equation becomes a third order characteristic equation in s with complex coefficients, as seen from equation (2.35). Because of the assumed exponential solution, the condition that the real parts of the roots of this polynomial remain negative, insures asymptotic stability. The problem then reduces to the analysis of a characteristic equation with complex coefficients.

3.1 Characteristic Equations with Complex Coefficients

A review of past investigations shows that only a few techniques exist for such analyses. If the numerical values of the characteristic roots are desired then the complex form of the roots can be substituted into the n^{th} order characteristic polynomial and the equation separated into its real and imaginary parts. Setting both parts equal to zero yields two n^{th} order polynomials with real coefficients which must be solved simultaneously for the real and the imaginary parts of the roots.

By requiring the real parts of the roots to remain negative, stability conditions can be derived. However, this approach may become very tedious for higher order systems. It also fails to give closed form expressions relating the effect of the various system parameters on the stability of the system. Another method, which does not require explicit solutions for the roots, has been suggested by Chebotarev and Meiman, as indicated by Bolotin [35]. In this approach the coefficients of the complex polynomial are arranged in an array similar to the well-known Routh-Hurwitz matrix. The stability criteria are then determined from the requirement that the principle minors remain positive. The resulting conditions show what effect the system's parameters have on the stability of the system. Although this seems to be a reasonably good approach, yet another technique can be developed by realizing the fact that the n^{th} order complex polynomial can always be transformed into a polynomial of order 2n with real coefficients [39]. The traditional Routh-Hurwitz or similar criteria can then be used to provide the stability conditions involving the system parameters.

First, a description of this method for an arbitrary polynomial with complex coefficients is presented. Then this technique is applied to the characteristic equation (2.35).

Consider an n^{th} order characteristic equation represented by

$$P(s) = c_0 s^n + c_1 s^{n-1} + \ldots + c_n = 0, \tag{3.1}$$

where the coefficients $c_0, c_1, \ldots, c_n$ are, in general, complex. Let the polynomial R(s) be defined by

$$R(s) = \bar{c}_0 s^n + \bar{c}_1 s^{n-1} + \ldots + \bar{c}_n = 0, \tag{3.2}$$

where the bars denote the complex conjugates of the quantities. Since both P(s) and R(s) are independently zero their product will also be zero. Multiplying equations (3.1) and (3.2) results in a polynomial

$$Q(s) = P(s)R(s) = a_0 s^{2n} + a_1 s^{2n-1} + \ldots + a_n = 0, \qquad (3.3)$$

where the real coefficients $a_0, a_1, \ldots, a_n$ are given by

$$\begin{aligned} a_0 &= c_0\bar{c}_0, \\ a_1 &= c_0\bar{c}_1 + \bar{c}_0 c_1, \\ a_2 &= c_0\bar{c}_2 + c_1\bar{c}_1 + c_2\bar{c}_0 , \\ &\;\;\vdots \\ a_n &= c_n\bar{c}_n . \end{aligned} \qquad (3.4)$$

The 2n roots of the equation (3.3) are the n roots of polynomial P(s) and the n roots of polynomial R(s). If the roots of P(s) are real or complex conjugate pairs then the 2n roots will occur as n roots of multiplicity two [39].

Now, the above procedure is applied to the characteristic equation (2.35). Since the eigenvalue λ^2 can be expressed as $\lambda^2 = \alpha \pm i\beta$, the equations P(s) and R(s) corresponding to equation (2.35) take the forms

$$P(s) = Cs^3 + K_1 s^2 + C[(\alpha+K_1+K_2) + i\beta]s + K_1[(\alpha+K_2) + i\beta] = 0, \quad (3.5)$$

$$R(s) = Cs^3 + K_1 s^2 + C[(\alpha+K_1+K_2) - i\beta]s + K_1[(\alpha+K_2) - i\beta] = 0. \quad (3.6)$$

Since equations (3.5) and (3.9) are independently zero, their product is also zero. After multiplying these and combining like terms the following sixth order equation with real coefficients is obtained

$$
\begin{aligned}
&C^2s^6 + 2K_1Cs^5 + \left[K_1^2 + 2C^2(\alpha + K_1 + K_2)\right]s^4 + \\
&+ C\left[4(\alpha+K_2)K_1 + 2K_1^2\right]s^3 + \left[2K_1^2(\alpha+K_2) + C^2(\alpha+K_1+K_2)^2 + C^2\beta^2\right]s^2 + \\
&+ 2K_1C\left[(\alpha+K_2)(\alpha+K_1+K_2) + \beta^2\right]s + \\
&+ K_1^2\left[(\alpha+K_2)^2 + \beta^2\right] = 0 .
\end{aligned}
\tag{3.7}
$$

The Hurwitz criteria may now be used to analyze the roots of this equivalent sixth order system.

Before establishing the stability conditions for each foundation model, it is helpful to review the Hurwitz criteria for an n^{th} order system and present an extension of this method proposed by Mikhailov [40].

3.2 Routh-Hurwitz-Mikhailov (RHM) Criteria

Independent of the work done by Routh [41], Hurwitz [42] developed an algebraic criteria in 1895 to determine the stability of systems described by an ordinary differential equation of arbitrary order. The characteristic equation for such a system is given by

$$
Q(s) = a_0s^n + a_1s^{n-1} + \ldots + a_{n-1}s + a_n = 0, \tag{3.8}
$$

where a_0, a_1, . . . , a_n are real constants. For the system to be asymptotically stable, the n roots must have negative real parts. Hurwitz showed that for this to be true, it is necessary and sufficient that the principle minors of the square matrix

$$\begin{bmatrix} a_1 & a_3 & a_5 & a_7 & \cdots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & a_6 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & \cdots & 0 & 0 & 0 \\ 0 & a_0 & a_2 & a_4 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & a_{n-4} & a_{n-2} & a_n \end{bmatrix} \tag{3.9}$$

involving the coefficients of equation (3.8), be positive. The conditions can be written as

$$\Delta_1 = a_1 > 0, \tag{3.10}$$

$$\Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} > 0, \tag{3.11}$$

$$\Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0, \tag{3.12}$$

and so on up to Δ_n, which is simply the determinant of the entire matrix (3.9). If expanded about the last column, Δ_n can also be expressed as

$$\Delta_n = a_n \Delta_{n-1} \ , \tag{3.13}$$

which reduces the positive definiteness condition on Δ_n to

$$a_n > 0 \ . \tag{3.14}$$

The "stability limit" is determined by

$$\Delta_{n-1} = 0, \tag{3.15}$$

with the appearance of a pair of purely imaginary roots of equation (3.8), or by

$$a_n = 0, \tag{3.16}$$

with the appearance of a zero root, provided all the remaining determinants are positive.

An equivalent but useful technique for determining the conditions for the stability limit for linear systems of finite order has also been developed by Aleksandr Vasil'evich Mikhailov [40]. Since the criteria suggested by Mikhailov is more convenient for the present investigation, a brief description of his method follows. A more detailed discussion is given by Popov [43].

Consider the n^{th} order characteristic equation of a linear system given by equation (3.8). For the real roots to be negative it is necessary and sufficient that all the coefficients be positive, but this alone does not insure that the roots have negative real parts if they are complex. To guarantee this, additional conditions, analogous to the Hurwitz determinants, must be satisfied. Substituting $s = i\omega$ in equation (3.8) and separating the real and imaginary parts, yields

$$Q(i\omega) = X(\omega) + iY(\omega), \tag{3.17}$$

where

$$X(\omega) = a_n - a_{n-2}\omega^2 + a_{n-4}\omega^4 - \ . \ . \ . \tag{3.18}$$

and

$$Y(\omega) = a_{n-1}\omega - a_{n-3}\omega^3 + a_{n-5}\omega^5 - \ . \ . \ . \tag{3.19}$$

Following Popov [43], $Q(i\omega)$ can be represented in the complex plane (X,iY) as a vector for a given value of ω. As ω varies between zero and infinity, the tip of the vector traces a curve in the complex plane. Mikhailov found that this configuration reveals the number of roots having negative (or positive) real parts. For all roots to have negative real parts, it is necessary and sufficient that the vector $Q(i\omega)$ representing the n^{th} order linear system rotate through an angle of

$$\phi = n\frac{\pi}{2} \tag{3.20}$$

radians as ω varies from zero to infinity. This is equivalent to requiring the curve to pass through n quadrants of the complex plane in succession. The limit of stability occurs when the expression for the Mikhailov curve, equation (3.17), becomes zero. Graphically, this happens when the curve passes through the origin for some value of ω. Analytically, the stability boundary is represented by

$$X(\omega) = 0 \tag{3.21}$$

and

$$Y(\omega) = 0, \tag{3.22}$$

which become simultaneous equations in ω. A solution of equations (3.21) and (3.22) yields the required stability condition in terms of the coefficients of the characteristic equation (3.8).

For higher order systems this method becomes more tedious and the Hurwitz criteria would give cleaner results. Comparing these methods, Popov [43] found that they produce equivalent conditions for the stability limit. For the stability of a finite order system the Hurwitz criteria require that all the principle minors of matrix (3.9) be positive with the determinant Δ_{n-1} giving the stability boundary. This is exactly equivalent to the Mikhailov requirement that all the coefficients be positive and that the Mikhailov curve rotate through $n \frac{\pi}{2}$ radians with the stability boundary occurring when the curve passes through the origin.

For systems of order less than seven, Mikhailov's criteria provide stability conditions without the tedious evaluation of the determinants required by the Hurwitz criteria. Since the present study deals with systems of at most sixth order, this criteria is used for convenience.

Hereafter, this criteria will be referred to as the Routh-Hurwitz-Mikhailov (RHM) criteria. In the following, the stability conditions for the sixth order characteristic equation (3.7) are derived through an application of the RHM criteria. Recall that equation (3.7) represents the Standard Linear foundation from which the special results for Kelvin-Voigt and Maxwell foundations can also be obtained. First, the Kelvin-Voigt model is considered.

3.3 Kelvin-Voigt Foundation

The motion of the column supported by a Kelvin-Voigt foundation (figure 2.2(b)) is governed by equation (2.20). A separable solution leads to equation (2.29) in the space variable ξ and the equation for T as

$$\ddot{T} + C\dot{T} + (K_2 + \lambda^2) = 0. \tag{3.23}$$

Assuming a solution in the form of equation (2.33), the following characteristic equation is obtained.

$$s^2 + Cs + (K_2 + \lambda^2) = 0, \tag{3.24}$$

where λ^2 is complex as described by equation (2.32). As shown in section

(3.1), this second order polynomial with complex coefficients can be transformed into a fourth order polynomial with real coefficients

$$Q(s) = a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0, \qquad (3.25)$$

where

$$\begin{aligned} a_0 &= 1, \\ a_1 &= 2C, \\ a_2 &= 2(\alpha + K_2) + C^2, \\ a_3 &= 2C(\alpha + K_2), \\ a_4 &= (\alpha + K_2)^2 + \beta^2. \end{aligned} \qquad (3.26)$$

Notice that this same result could have been obtained from equation (3.7) in the limit as K_1 approaches infinity.

The RHM criteria developed in the last section yield conditions which insure negative real parts of the roots of equation (3.25). Following the development of Popov [43] for fourth order systems, the polynomial $Q(i\omega)$ separates into a real part $X(\omega)$ and an imaginary part $Y(\omega)$ such that

$$X(\omega) = a_0\omega^4 - a_2\omega^2 + a_4, \quad Y(\omega) = -a_1\omega^3 + a_3\omega. \qquad (3.27)$$

The value of ω necessary for stability is found by setting $Y(\omega) = 0$ as

$$\omega_s^2 = \frac{a_3}{a_1} \qquad (3.28)$$

Substituting for ω_s in $X(\omega)$ and requiring the expression to be less than zero yields the condition for stability

$$X(\omega_s) = a_0 \frac{a_3^2}{a_1^2} - a_2 \frac{a_3}{a_1} + a_4 = \frac{-a_3(a_1 a_2 - a_0 a_3) + a_4 a_1^2}{a_1^2} < 0 \qquad (3.29)$$

Then for necessary and sufficient conditions for stability, the RHM criteria requires that all coefficients of equation (3.25) be positive and that the inequality

$$a_3(a_1a_2 - a_0a_3) - a_4a_1^2 > 0 \tag{3.30}$$

be satisfied. The stability limit occurs when inequality (3.30) becomes an equality with all the coefficients remaining positive.

To find the limiting values of the system parameters for stability, first consider the condition that all coefficients be positive. From the expressions (3.26), it is seen that a_0 and a_4 are always positive, a_1 is positive for positive damping, and that a_2 and a_3 are positive if the inequalities

$$\alpha + K_2 > -\frac{C^2}{2} \tag{3.31}$$

and

$$\alpha + K_2 > 0 \tag{3.32}$$

are satisfied. It is clear that if inequality (3.32) is satisfied then inequality (3.31) also holds. Therefore, with positive damping, all the coefficients are positive if $\alpha + K_2 > 0$. Now, consider inequality (3.30). Substituting for the coefficients from expressions (3.26) yields

$$\alpha + K_2 > \frac{\beta^2}{C^2} . \tag{3.33}$$

Since inequality (3.33) is more restrictive than (3.32), the required condition for stability is given by (3.33). Notice that if $K_2 = 0$, the Kelvin-Voigt model reduces to a viscous damper. Setting K_2 to zero in inequality (3.33), the stability condition for this case can be recovered as

$$\alpha > \frac{\beta^2}{C^2} , \qquad (3.34)$$

which was reported by Leipholz [36], Plaut and Infante [20], Pedersen [22] and others. These studies found that external viscous damping increases the flutter load only to a limiting value, which can be concluded from inequality (3.34). If the damping constant C was to approach infinity, then the critical load would only increase to the load corresponding to $\alpha = 0$. This α value is represented on figure 2.4 by point p_2, which corresponds to a load of 37.7. The effect of external viscous damping is shown in figure 3.1 when $K_2 = 0$.

With the presence of stiffness K_2 it is seen from inequality (3.33) that the critical load increases to the limiting value corresponding to $\alpha + K_2 = 0$, as C approaches infinity. The stability limit occurs when

$$\alpha + K_2 = \frac{\beta^2}{C^2} . \qquad (3.35)$$

Thus, loads corresponding to eigenvalues with negative real parts up to the magnitude of stiffness K_2 can be allowed for a stable configuration. The results are shown in figure 3.1 for several values of K_2. For a given stiffness parameter K_2, the critical load increases with increasing damping to the limiting value where the quantity $\alpha + K_2$ becomes zero. Wahed [29] and Kar [30] have reported the same trend. However, since their studies were limited to a much smaller range of damping values, the results failed to show that for a given K_2, the critical load increases only to a limiting value, even if the damping is very large.

Recall from Smith and Herrmann [26] that an elastic foundation alone has no effect on the critical load for Beck's column. However, as seen from figure 3.1, the combined influence of elasticity and damping,

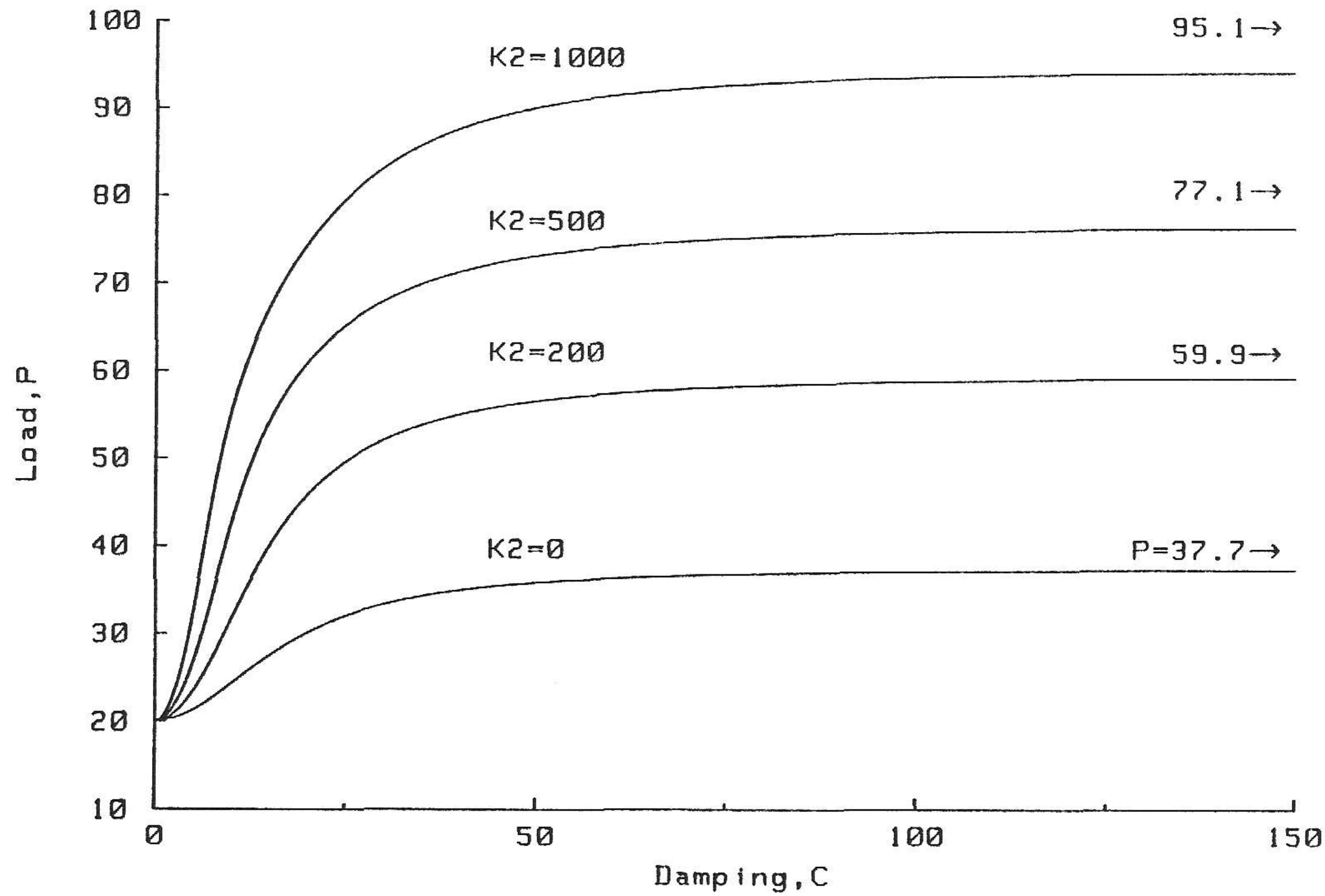


Figure 3.1 Effect of Kelvin-Voigt Foundation on Flutter Load

provided by a Kelvin-Voigt foundation, does result in higher critical loads.

3,4 Maxwell Foundation

The nondimensional equation (2.23) describes the motion of Beck's column when supported by a Maxwell type foundation (figure 2.2(c)). As before, a separable solution leads to the equation (2.29) and the temporal equation

$$C\dddot{T} + K_1\ddot{T} + C(K_1 + \lambda^2)\dot{T} + K_1\lambda^2 T = 0, \qquad (3.36)$$

which yields the characteristic equation

$$Cs^3 + K_1 s^2 + C(K_1 + \lambda^2)s + K_1\lambda^2 = 0. \qquad (3.37)$$

The roots of this equation govern the stability of the motion about the equilibrium position. Since λ^2 is complex, the method presented in Section 3.1 can be used to form an equivalent sixth order charcteristic equation

$$Q(s) = a_0 s^6 + a_1 s^5 + a_2 s^4 + \ldots + a_5 s + a_6 = 0 \qquad (3.38)$$

with real coefficients

$$\begin{aligned}
a_0 &= C^2, \\
a_1 &= 2K_1 C, \\
a_2 &= K_1^2 + 2C^2(\alpha + K_1), \\
a_3 &= 2CK_1(2\alpha + K_1), \\
a_4 &= 2\alpha K_1^2 + C^2[(\alpha + K_1)^2 + \beta^2], \\
a_5 &= 2K_1 C[(\alpha + K_1)\alpha + \beta^2], \\
a_6 &= K_1^2(\alpha^2 + \beta^2).
\end{aligned} \qquad (3.39)$$

Since the Maxwell model is a special case of the Standard Linear model, this result could have been obtained from (3.7) by setting K_2 equal to zero.

The RHM criteria, given in Section 3.2, is used to determine the conditions for a bounded solution. Following the development of Popov [43] for sixth order systems, substituting $i\omega$ for s in equation (3.38) describes the Mikhailov curve. For stability, the trajectory must pass through six successive quadrants of the complex plane. The real and imaginary parts of $Q(i\omega)$ are

$$\begin{aligned} X(\omega) &= a_6 - a_4\omega^2 + a_2\omega^4 - a_0\omega^6, \\ Y(\omega) &= (a_5 - a_3\omega^2 + a_1\omega^4)\omega . \end{aligned} \tag{3.40}$$

To insure that the curve passes through six quadrants as ω varies from zero to infinity, it is necessary to observe the values of $X(\omega)$ where the trajectory crosses the X axis. Setting $Y(\omega)$ equal to zero gives the required values of ω for these critical X values.

$$(\omega_s^2)_{1,2} = \frac{1}{2a_1}\left[a_3 \pm (a_3^2 - 4a_1a_5)^{\frac{1}{2}}\right] \tag{3.41}$$

Substituting these values of ω_s into the $X(\omega)$ expression and combining like terms yields the following expressions for X.

$$\begin{aligned} X_1 = \frac{1}{8a_1^3}\{&2a_1^3a_6 + (a_3^2 - 2a_1a_5)(a_1a_2 - a_0a_3) - a_1a_3(a_1a_4 - a_0a_5) - \\ &- \left[a_3(a_1a_2 - a_0a_3) - a_1(a_1a_4 - a_0a_5)\right](a_3^2 - 4a_1a_5)^{\frac{1}{2}}\} \end{aligned} \tag{3.42}$$

and

$$X_2 = \frac{1}{8a_1^3}\{2a_1^3a_6 + (a_3^2 - 2a_1a_5)(a_1a_2 - a_0a_3) - a_1a_3(a_1a_4 - a_0a_5) +$$
$$+ [a_3(a_1a_2 - a_0a_3) - a_1(a_1a_4 - a_0a_5)](a_3^2 - 4a_1a_5)^{1/2}\}. \tag{3.43}$$

Therefore, to insure stability it is required that X_1 be negative and X_2 be positive. This requirement leads to the inequalities

$$a_3(a_1a_2 - a_0a_3) - a_1(a_1a_4 - a_0a_5) > 0 \tag{3.44}$$

and

$$2a_1^3 a_6 + (a_3^2 - 2a_1a_5)(a_1a_2 - a_0a_3) - a_1a_3(a_1a_4 - a_0a_5]$$
$$< [a_3(a_1a_2 - a_0a_3) - a_1(a_1a_4 - a_0a_5)](a_3^2 - 4a_1a_5)^{1/2} . \tag{3.45}$$

The last inequality can be rewritten as [43]

$$(a_1a_2 - a_0a_3)[a_5(a_4a_3 - a_2a_5) + a_6(2a_1a_5 - a_3^2)] +$$
$$+ (a_1a_4 - a_0a_5)[a_1a_3a_6 - a_5(a_1a_4 - a_0a_5)] \tag{3.46}$$
$$- a_1^3 a_6^2 > 0 .$$

Thus for asymptotic stability, it is necessary and sufficient that all coefficients of equation (3.38) be positive and that inequalities (3.44) and (3.46) be satisfied simultaneously. At the stability boundary, inequality (3.46) becomes an equality [44]. Substituting for the coefficients in terms of the system parameters from (3.39) and performing some simple but rather lengthy manipulations yield the following expressions.

$$(a_1a_2 - a_0a_3) = 2K_1^2 C(K_1 + C^2) ,$$

$$(a_1a_4 - a_0a_5) = 2K_1^2 C(2\alpha K_1 + \alpha C^2 + K_1C^2),$$

$$a_5(a_4a_3 - a_2a_5) + a_6(2a_1a_5 - a_3^2) = 4C^2K_1^3[(K_1 + C^2)\alpha^4 + (3C^2K_1 + 2K_1^2)\alpha^3 +$$

$$+ (3K_1^2C^2 + 2K_1\beta^2)\alpha^2 + (K_1C^2\beta^2 + K_1^3 C^2 - 2K_1^2\beta^2)\alpha +$$

$$+ (K_1 - C^2)\beta^4 + K_1^2(C^2 - K_1)\beta^2], \qquad (3.47)$$

$$a_1a_3a_6 - a_5(a_1a_4 - a_0a_5) =$$

$$-4C^2K_1^3 [C^2\alpha^3 + K_1(K_1 + 2C^2)\alpha^2 + C^2(K_1^2 + \beta^2)\alpha - K_1(K_1 - C^2)\beta^2],$$

$$a_1^3 a_6^2 = 8C^3K_1^7(\alpha^2 + \beta^2)^2.$$

With further calculations the stability boundary is then obtained from (3.46) as

$$d_1C^4 + d_2C^2 + d_3 = 0, \qquad (3.48)$$

where

$$d_1 = \alpha\beta^2(\alpha + K_1) + \beta^4,$$

$$d_2 = \alpha K_1^2(2\beta^2 - K_1^2) - K_1^3\beta^2 , \qquad (3.49)$$

$$d_3 = K_1^4\beta^2.$$

Equation (3.48) is quadratic in C^2 with the roots

$$C_{1,2}^2 = \frac{d_2}{2d_1} \pm \frac{1}{2d_1}(d_2^2 - 4d_1d_3)^{\frac{1}{2}}. \qquad (3.50)$$

For real damping constant C, the discriminant

$$(d_2 - 4d_1d_3)^{\frac{1}{2}} \tag{3.51}$$

as well as the entire right hand side of equation (3.50) must be non-negative. In terms of the system parameters given by definitions (3.49), the condition on the discriminant (3.51) becomes

$$\left(\alpha + \frac{\beta^2}{K_1}\right) (K_1^2 - 4\beta^2)^{\frac{1}{2}} \geq 0. \tag{3.52}$$

From equation (3.50), if the inequality

$$4d_1d_3 \geq 0 \tag{3.53}$$

is satisfied, then the entire right hand side is non-negative, provided the discriminant (3.51) remains non-negative. In terms of the system parameters, this requires that

$$\alpha^2 + \alpha K_1 + \beta^2 \geq 0. \tag{3.54}$$

Therefore, to insure real damping C, inequalities (3.52) and (3.54) place bounds on K_1 as

$$2\beta \leq K_1 \leq \frac{\beta^2}{-\alpha} \tag{3.55}$$

and

$$K_1 \leq \frac{\alpha^2 + \beta^2}{-\alpha}, \tag{3.56}$$

respectively.

A simple computer program was written in BASIC language in order to find the roots of equation (3.48) for a given set of α and β corresponding to a unique load P (Appendix D). The condition that the coefficients defined in equation (3.39) remain positive and the inequality (3.44) hold was also incorporated in the computer program to determine the stability boundary. The two roots of equation (3.48) satisfying these conditions, were then obtained for sets of α and β corresponding to increasing values of load P. This computation was performed to generate the stability boundaries for various values of the stiffness parameter K_1. An HP9845 desktop computer and a HP9872 plotter system was used to produce the graphical results shown in figures 3.2 through 3.11.

For a fixed K_1, the two roots of the damping constant C approach each other as the load is increased to a maximum value where the two roots become equal. As seen from equation (3.50), the roots become equal when the discriminant (3.52) is zero. For the values of K_1 shown in figure 3.2, the discriminant is zero when

$$K_1 = 2\beta. \tag{3.57}$$

Thus the maximum load occurs when β equals $K_1/2$. At this maximum load value, equation (3.50) reduces to

$$C^2 = -\frac{d_2}{2d_1}, \tag{3.58}$$

which, in terms of the system parameters, defines the damping value at the peak load as

$$C^2 = \frac{4\beta^2}{\alpha + \beta}. \tag{3.59}$$

Beyond this value of load, the roots of equation (3.48) become complex.

However, this does not happen for stiffness values greater than $K_1 = 579.58$, which will be designated as K^*. For K_1 values greater than K^*, the discriminant again becomes zero if

$$\alpha + \frac{\beta^2}{K_1} = 0. \tag{3.60}$$

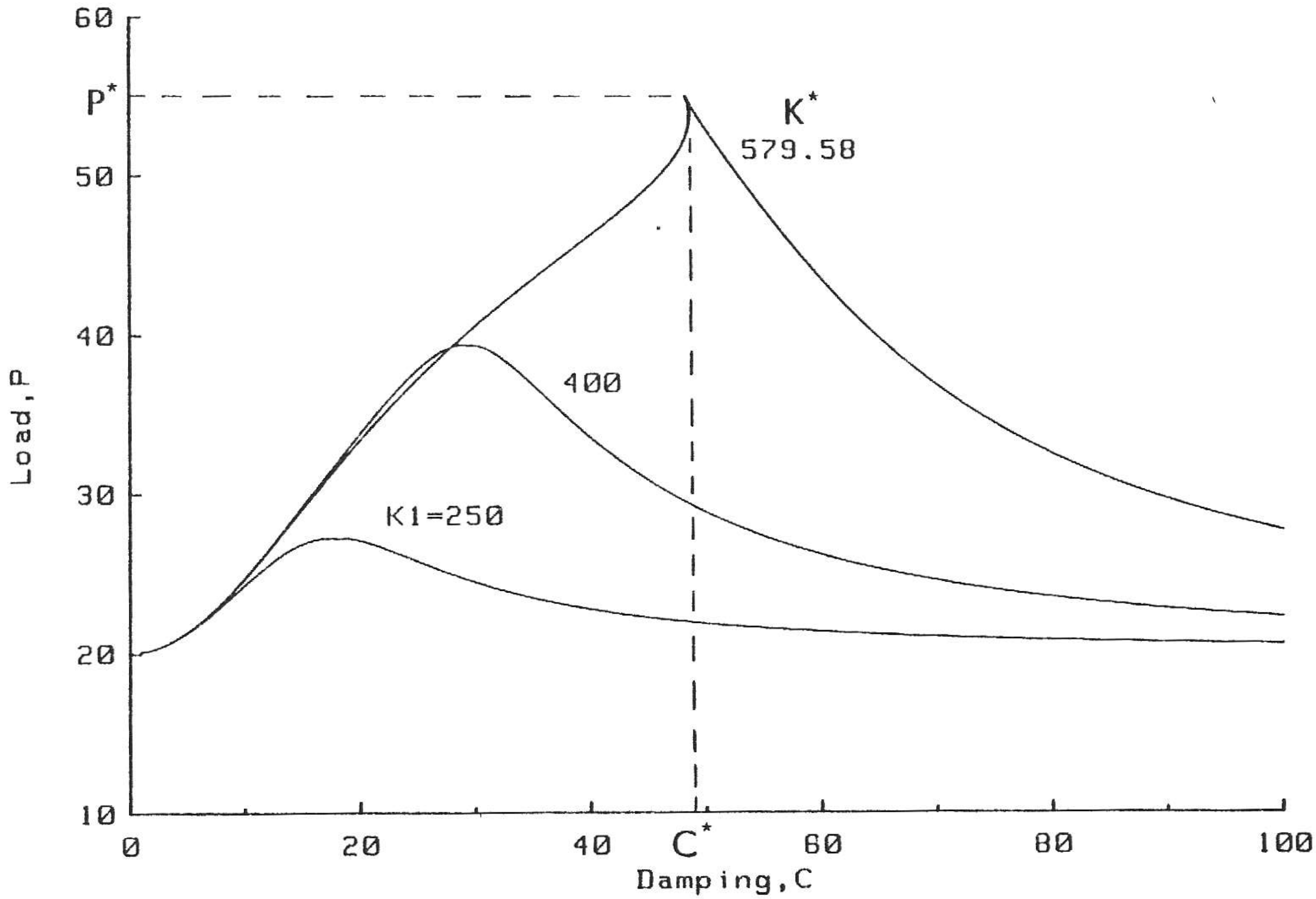


Figure 3.2 Effect of Maxwell Foundation on Flutter Load ($K_1 \leq K^*$)

At this point the roots are again equal. With increasing loads the damping values remain real since inequality (3.55) also remains satisfied. This is shown in figure 3.3 for two arbitrary values of stiffness parameter K_1 greater than K^*. As the load increases, the two roots of C approach each other and become equal when

$$K_1 = \frac{\beta^2}{-\alpha} . \tag{3.61}$$

Substituting equation (3.61) in equation (3.50) yields the value of damping constant for this load as

$$C^2 = -\frac{\beta^4}{\alpha^3} \tag{3.62}$$

The stable regions of figure 3.3 (for typical K_1 values greater than K^*) are shown separately in figures 3.4 and 3.5 for clarity. Figure 3.4 shows the stability boundary for $K_1 = 700$. As the load is increased from $P_f = 20.05$, the roots of C approach each other and become equal when equation (3.61) is satisfied. Beyond this load, the roots diverge and then become equal again when $K_1 = 2\beta$, forming a loop. For higher loads, the roots become complex. Although equation (3.48) is satisfied at every point on the curve shown in figure 3.4, inequality (3.44) is violated for loads higher than that which satisfies equation (3.61). This happens when the two roots of C first become equal. The stability region is shown by the crosshatched area in figure 3.4. All other regions are unstable.

Now consider the behavior for yet higher values of the stiffness parameter K_1 depicted in figure 3.5. The value of K_1 is arbitrarily selected as $K_1 = 1000$. The roots of the damping constant C again approach each other as load increases from P_f and become equal when equation (3.61) is satisfied. For higher loads the roots do not form a loop such as the

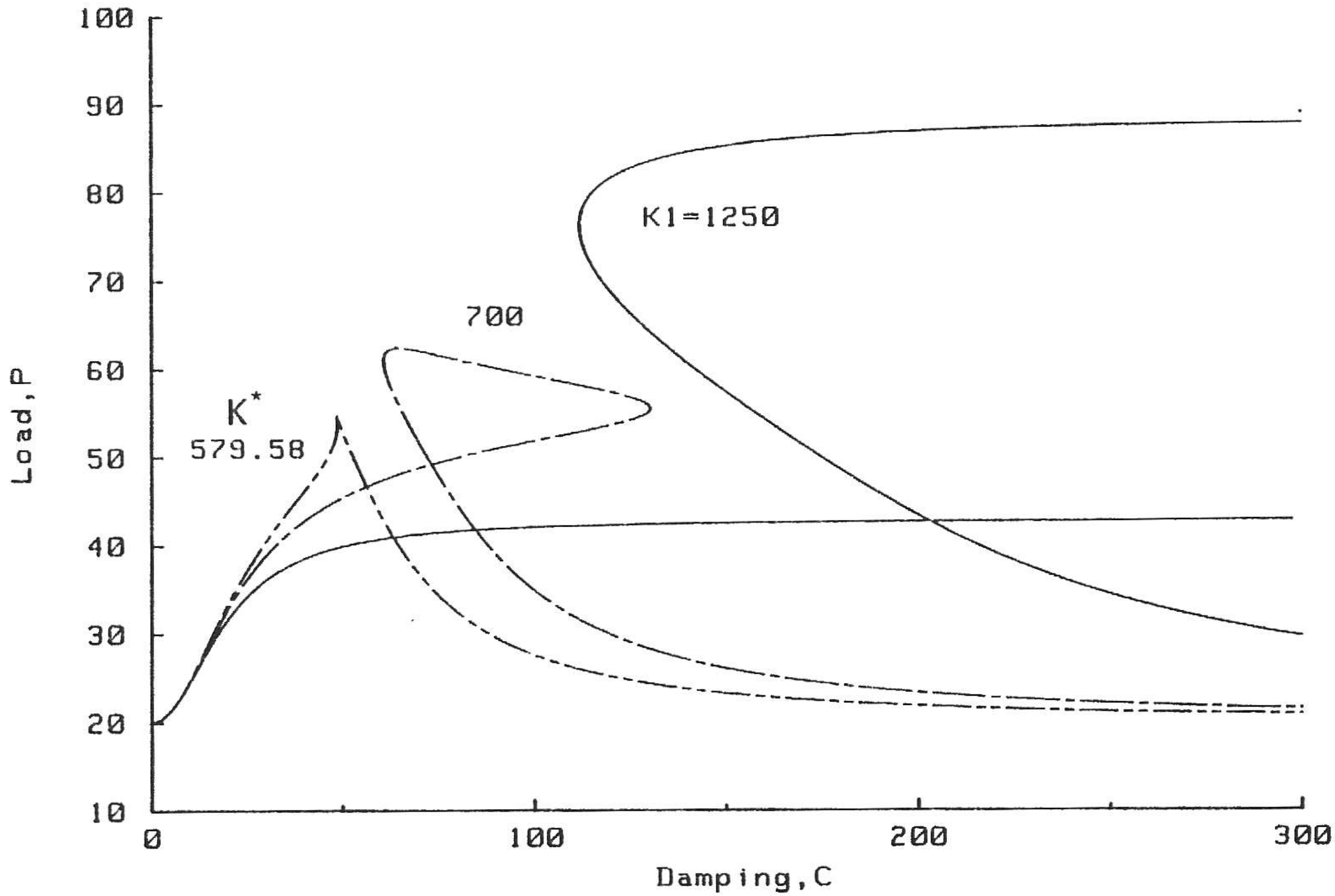


Figure 3.3 Effect of Maxwell Foundation on Flutter Load ($K_1 \geq K^*$)

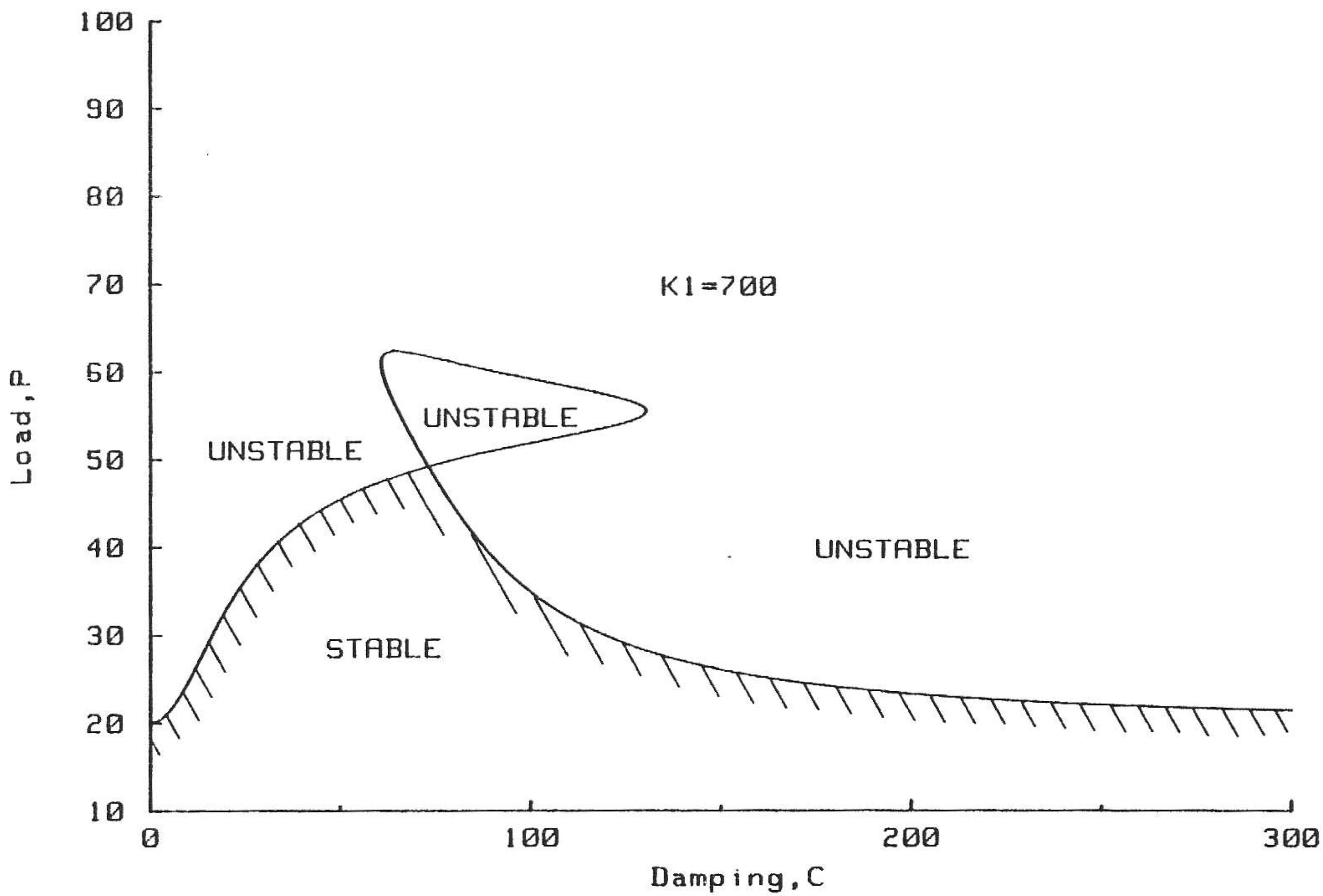


Figure 3.4 Stability Region for Maxwell Foundation (K_1=700)

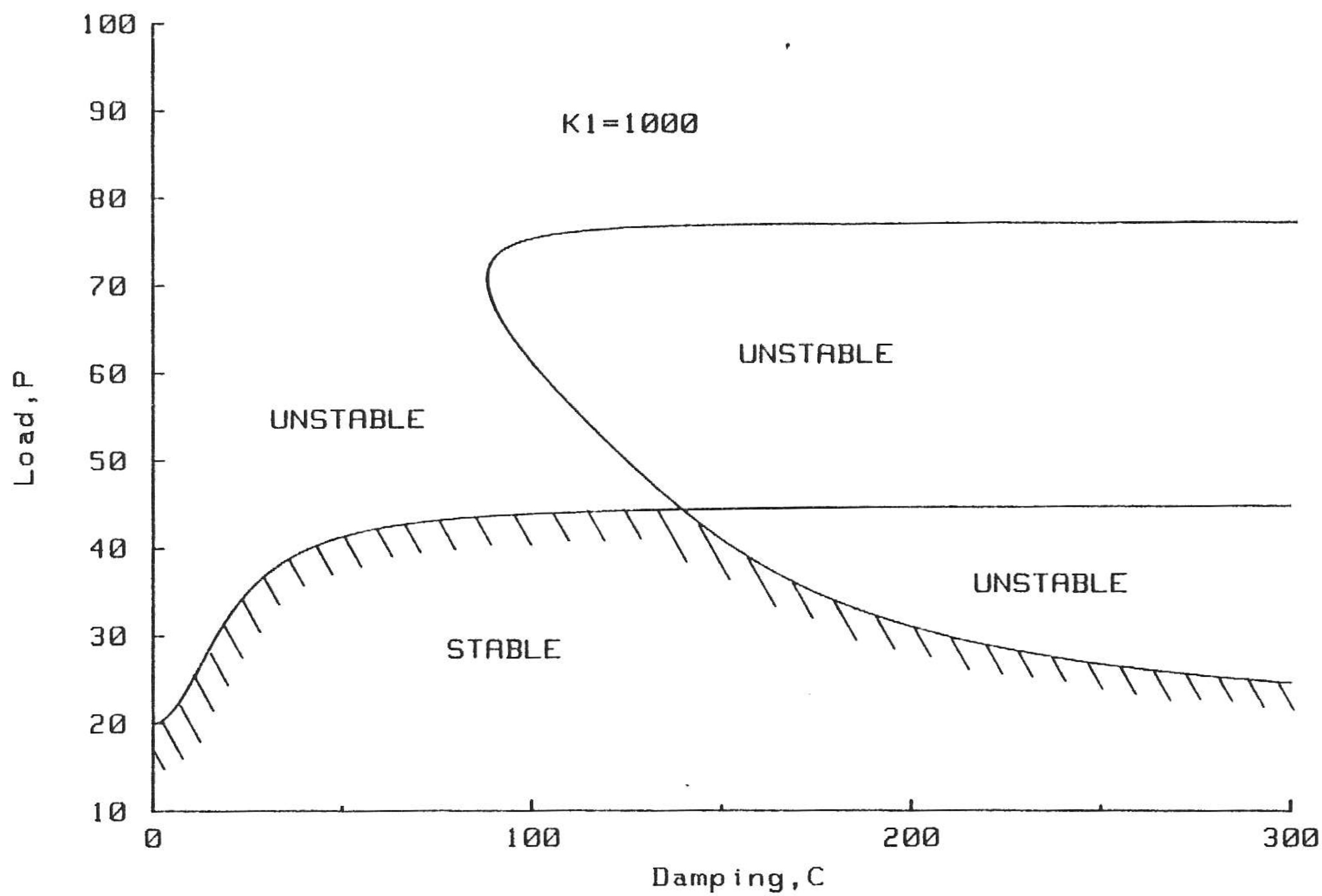


Figure 3.5 Stability Region for Maxwell Foundation (K_1=1000)

one shown in figure 3.4. Instead, the roots form an open curve shown in the figure. For this case, real values of damping are obtained until the condition in (3.56) becomes an equality. Recall that α can be negative or positive and β always remains positive.

Since (3.56) is a quadratic in α and β, there are two values of load P for which (3.56) becomes an equality. The first occurs when the lowest root approaches infinity at P = 42.9 (α = -39.42, β = 218.54). A higher load yields a complex lower root and causes the higher root to first decrease and then increase and finally approach infinity. At this point, P = 88.5 (α = -800.73, β = 599.89) and the equality is again satisfied. Beyond this load, both roots become complex. Similar to the case of K_1 = = 700, inequality (3.44) is violated when the load is increased beyond the point where the roots are equal. As shown in figure (3.5), the stability region is bounded by the load vs. damping curves up to this maximum load. All regions outside the crosshatched area are unstable.

The stability characteristics for $0 < K_1 < \infty$ can be summarized as follows: For $K_1 < K^*$ the peak load occurs when equation (3.57) is satisfied, while for $K_1 > K^*$ the peak load is attained when equation (3.61) is satisfied. At K^* both constraints on the stiffness parameter K_1 are met simultaneously, i.e.,

$$2\beta = \frac{\beta^2}{-\alpha} , \tag{3.63}$$

or

$$\beta = -2\alpha . \tag{3.64}$$

This relationship is satisfied at a unique load value of approximately 54.9 (α = - 145.45, β = 289.79). The value of the stiffness parameter for this load is

$$K^* = 2\beta = 579.58, \tag{3.65}$$

and the required damping constant is given by

$$C^* = 2\beta(\alpha + \beta)^{-\frac{1}{2}} = K^*(\alpha + \beta)^{-\frac{1}{2}} = 48.24. \tag{3.66}$$

The stability boundaries for a wide range of K_1 is represented in figure 3.6. Notice that the combination of K^* and C^* allow the optimum load P^*. Stiffness values other than K^* yield critical loads which are less than P^*. As K_1 is increased from zero to K^*, peak loads increase from $P_f(= 20.05)$ to $P^*(= 54.9)$. For values of $K_1 > K^*$, the peak loads decrease while the corresponding damping values increase. This behavior is expected, since the Maxwell model (figure 2.2(c)) becomes a viscous dashpot as the value of K_1 approaches infinity. Taking the limit of equation (3.48) as K_1 approaches infinity reduces the expression for the stability limit to

$$- C^2\alpha + \beta^2 = 0, \tag{3.67}$$

or

$$\alpha = \frac{\beta^2}{C^2} , \tag{3.68}$$

which is identical to the stability condition obtained from (3.34). As shown in Section 3.3, the critical load for Beck's column in the presence of infinite viscous damping is 37.7. It is seen from figure 3.6, that as K_1 becomes large, the allowable peak load approaches 37.7 and the corresponding damping value approaches infinity. This verifies the results of the viscous damping case.

A plot of the peak loads for $0 \leq K_1 \leq \infty$ is shown in figure 3.7.

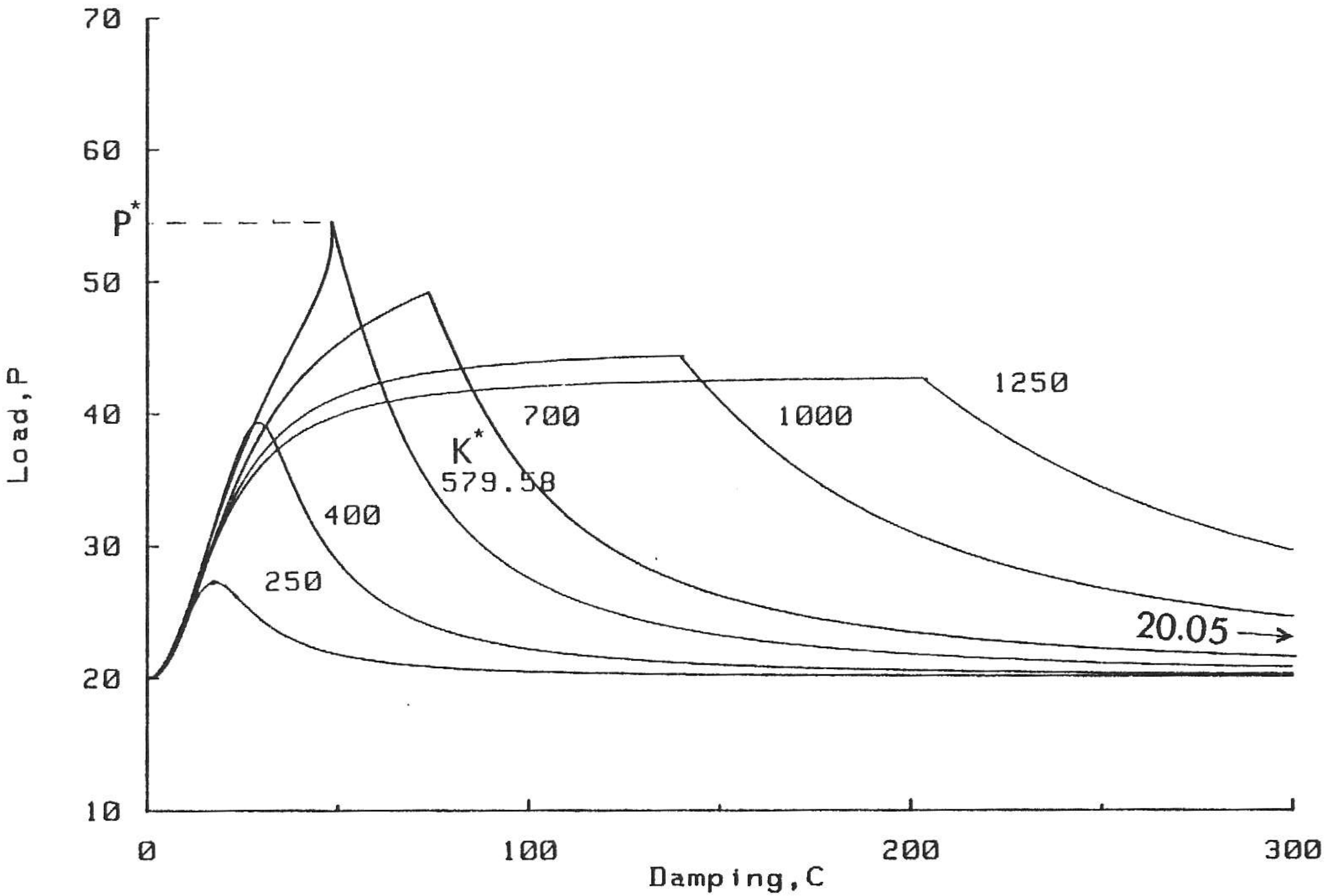


Figure 3.6 Stability Region for Maxwell Foundation ($0 \leq K_1 \leq 1250$)

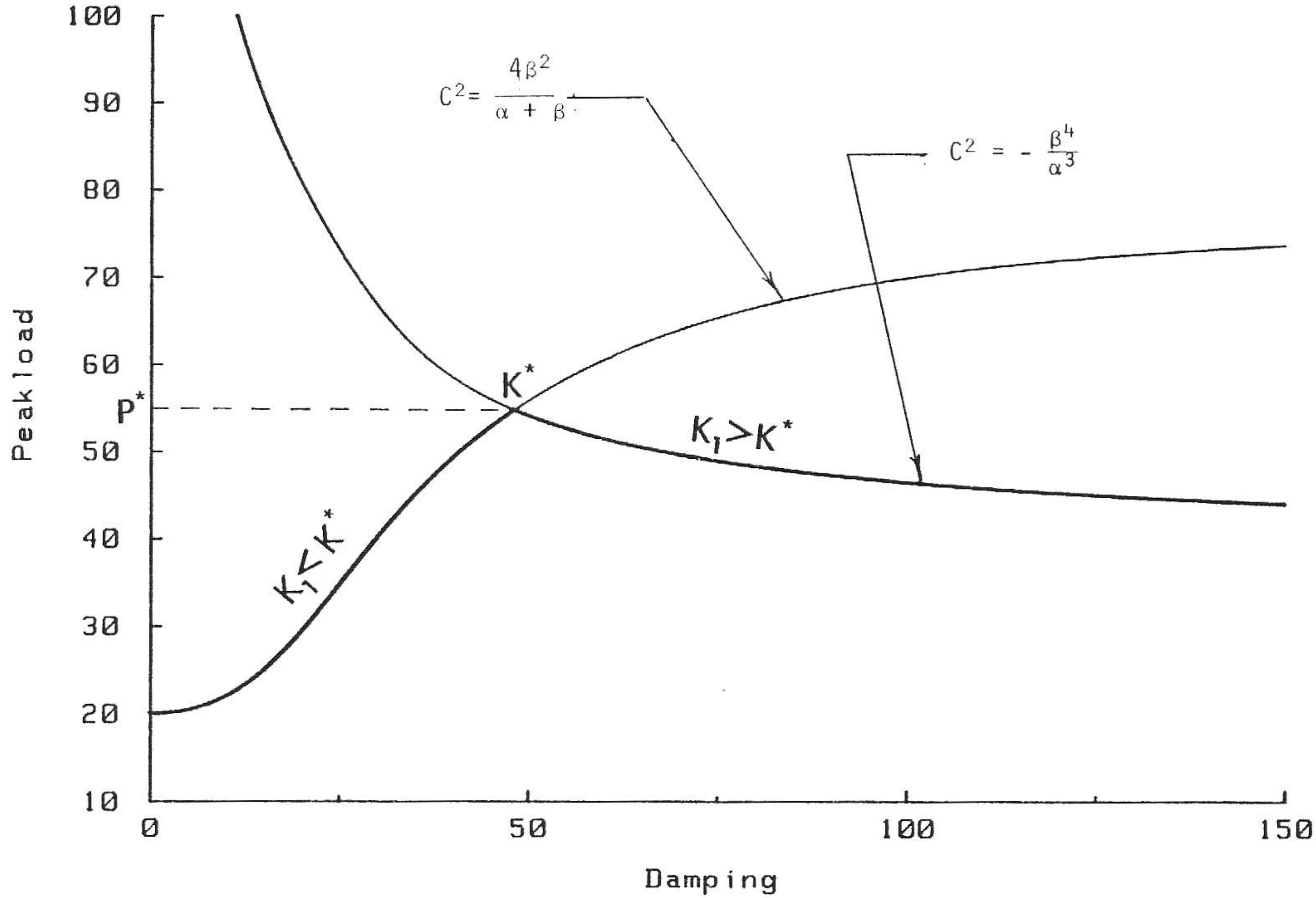


Figure 3.7 Peak Flutter Loads for Maxwell Foundation ($0 \leq K_1 \leq \infty$)

For a stiffness less than K^* the peak load is along the curve described by equation (3.59), while for a stiffness greater than K^* the peak load is along the curve obtained from equation (3.62). The intersection occurs at $K_1 = K^*$. Therefore, the peak loads follow the path represented by the solid line.

In summary, the critical load for Beck's column supported by a Maxwell foundation increases from $P_f(20.05)$ to $P^*(54.9)$ when $K_1 < K^*$ and decreases to $P = 37.7$, as K_1 approaches infinity, when appropriate damping is present.

3.5 Standard Linear Foundation

The motion of Beck's column supported by a Standard Linear foundation (figure 2.2(a)) is modeled by equation (2.17). The temporal part of the solution depends on the equivalent sixth order characteristic equation (3.7), developed in Section 3.1. According to the RHM criteria, introduced in Section 3.2, for this sixth order system to be asymptotically stable, it is necessary and sufficient for all the coefficients, a_0 through a_6, to be positive and the inequalities (3.44) and (3.46) be satisfied simultaneously. In terms of the system parameters, inequality (3.44) becomes

$$K_1(\alpha + K_2) + \beta^2 > 0 \tag{3.69}$$

and inequality (3.46) is

$$d_1 C^4 + d_2 C^2 + d_3 < 0,$$

where

$$d_1 = \beta^2\left[(\alpha + K_2)^2 + K_1(\alpha + K_2) + \beta^2\right],$$

$$d_2 = K_1^2\left[(\alpha + K_2)(2\beta^2 - K_1^2) - K_1\beta^2\right], \tag{3.70}$$

$$d_3 = K_1^4\beta^2.$$

Inequality (3.70) becomes an equality at the stability boundary and can be solved to obtain the damping values. The two roots of C^2 are once again given by equation (3.50), where the d's are now defined by (3.70).

Although this foundation model can be reduced to the Kelvin-Voigt and the Maxwell model as special cases, the analysis is more closely related to that of the Maxwell model. Notice that replacing α in equation (3.38) for the Maxwell foundation by the quantity $\alpha + K_2$ yields the characteristic equation (3.7) for the Standard Linear foundation. As a result, the analysis follows the same pattern as in Section 3.4 with $\alpha + K_2$ replacing α throughout the development.

For real damping constant C in equation (3.70), the inequalities (3.52) and (3.54) become

$$(\alpha + K_2 + \beta^2/K_1)(K_1^2 - 4\beta^2)^{\frac{1}{2}} \geq 0, \tag{3.71}$$

and

$$(\alpha + K_2)^2 + (\alpha + K_2)K_1 + \beta^2 \geq 0, \tag{3.72}$$

respectively. These inequalities place bounds on K_1 as

$$2\beta \leq K_1 \leq \frac{\beta^2}{-(\alpha + K_2)}, \tag{3.73}$$

and

$$K_1 \leq \frac{(\alpha + K_2)^2 + \beta^2}{-(\alpha + K_2)} \quad . \tag{3.74}$$

Analogous to K^* in the Maxwell analysis, the Standard Linear foundation also has an optimal combination of parameters when the upper and lower bounds of (3.73) become equal, i.e., when

$$-2(\alpha + K_2) = \beta. \tag{3.75}$$

For this condition, each value of K_2 has a unique peak load and a critical K_1 value of

$$K_1 = 2\beta. \tag{3.76}$$

If K_1 is less than this critical value, the damping corresponding to the peak load is given from (3.59) as

$$C^2 = \frac{4\beta^2}{\alpha + K_2 + \beta} \, , \tag{3.77}$$

and if K_1 is greater than this critical value, then from (3.62)

$$C^2 = \frac{-\beta^4}{(\alpha + K_2)^3} \quad . \tag{3.78}$$

Although the analysis for this model is similar to that of the Maxwell model, the presence of the additional stiffness parameter K_2 does modify the stability characteristics. The stability boundaries for the Standard Linear foundation model are shown in figures 3.8 and 3.9 for a wide range of parameters K_1 and K_2. First, consider the case when $0 \leq K_1 \leq K^*$ (figure 3.8). Observe that the peak loads for this range of K_1 corresponds to $\beta = K_1/2$, which is independent of K_2. Therefore, the addition of K_2 only shifts the damping values at which peaks occur. This is also indicated by equation (3.77). For loads higher than the peak load, the roots

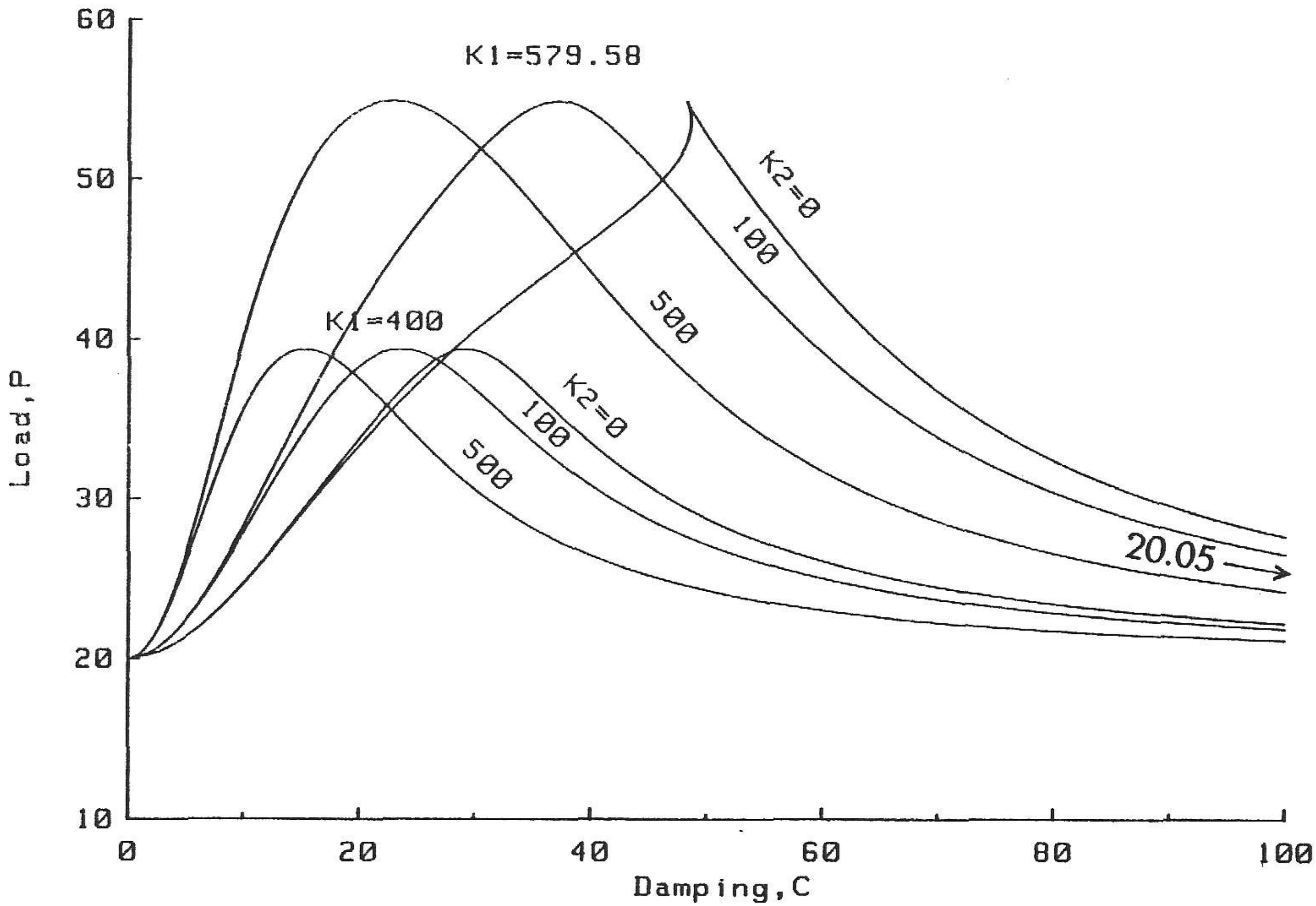


Figure 3.8 Effect of Standard Linear Foundation on Flutter Load ($K_1 \leq K^*$)

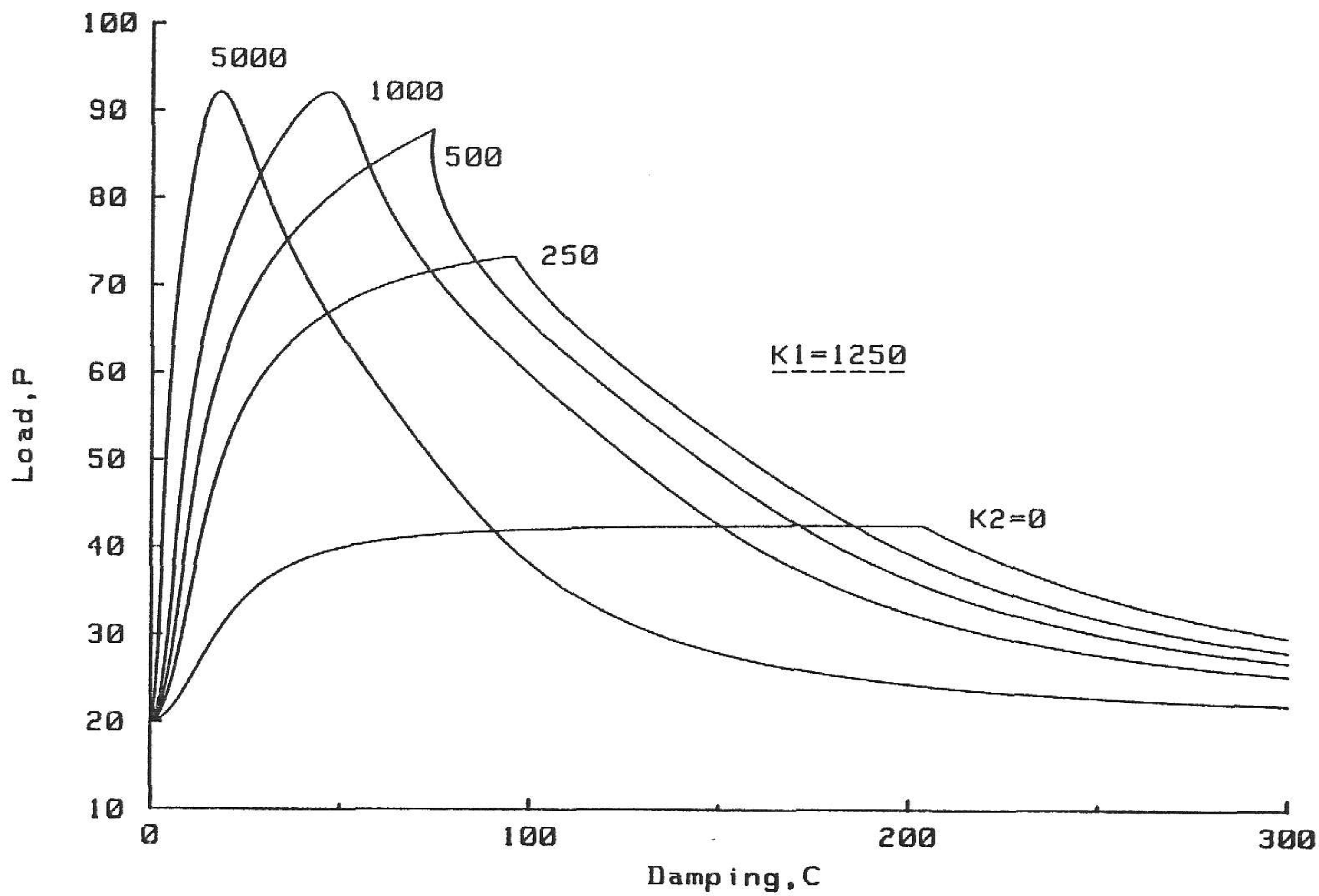


Figure 3.9 Effect of Standard Linear Foundation on Flutter Load ($K_1 > K^*$)

of damping constant C become complex. The results are shown for $K_1 = 400$ and $K_1 = K^* = 579.58$. Increasing K_2 decreases the amount of damping necessary for stability.

Now, consider the case when $K_1 > K^*$ (figure 3.9). As the load is increased from P_f, the two roots of damping approach each other. They become equal at the peak load corresponding to

$$K_1 = \frac{\beta^2}{-(\alpha + K_2)} \; . \tag{3.79}$$

For higher loads, inequality (3.69) is violated. As K_2 is increased the peak load increases to the value where equation (3.79) is satisfied, and the corresponding damping coefficient decreases according to equation (3.78). As K_2 is increased further, the upper and lower bounds on K_1 given by (3.73) become equal at the critical load corresponding to $\beta = K_1/2$. This condition defines the optimal combination of foundation parameters. Still higher values of K_2 only decreases the amoung of damping for the peak load with no increase in the peak load itself.

The effect of an increase in K_2 on the peak loads is shown in figure 3.10. If K_1 is less than the optimal value of 2β , where β is defined by equation (3.75), the peak load is along the curve described by equation (3.77). For K_1 greater than this value, the peak laod is along the curve described by equation (3.78). The shift of the peak laod curve for several values of K_2 is shown in figure 3.11.

Notice, from figure 3.11, that the Standard Linear foundation combines the characteristics of both the Maxwell and Kelvin-Voigt foundations. For a given K_2, there exists an optimal combination of foundation parameter, which is similar to the behavior of the Maxwell model (figure

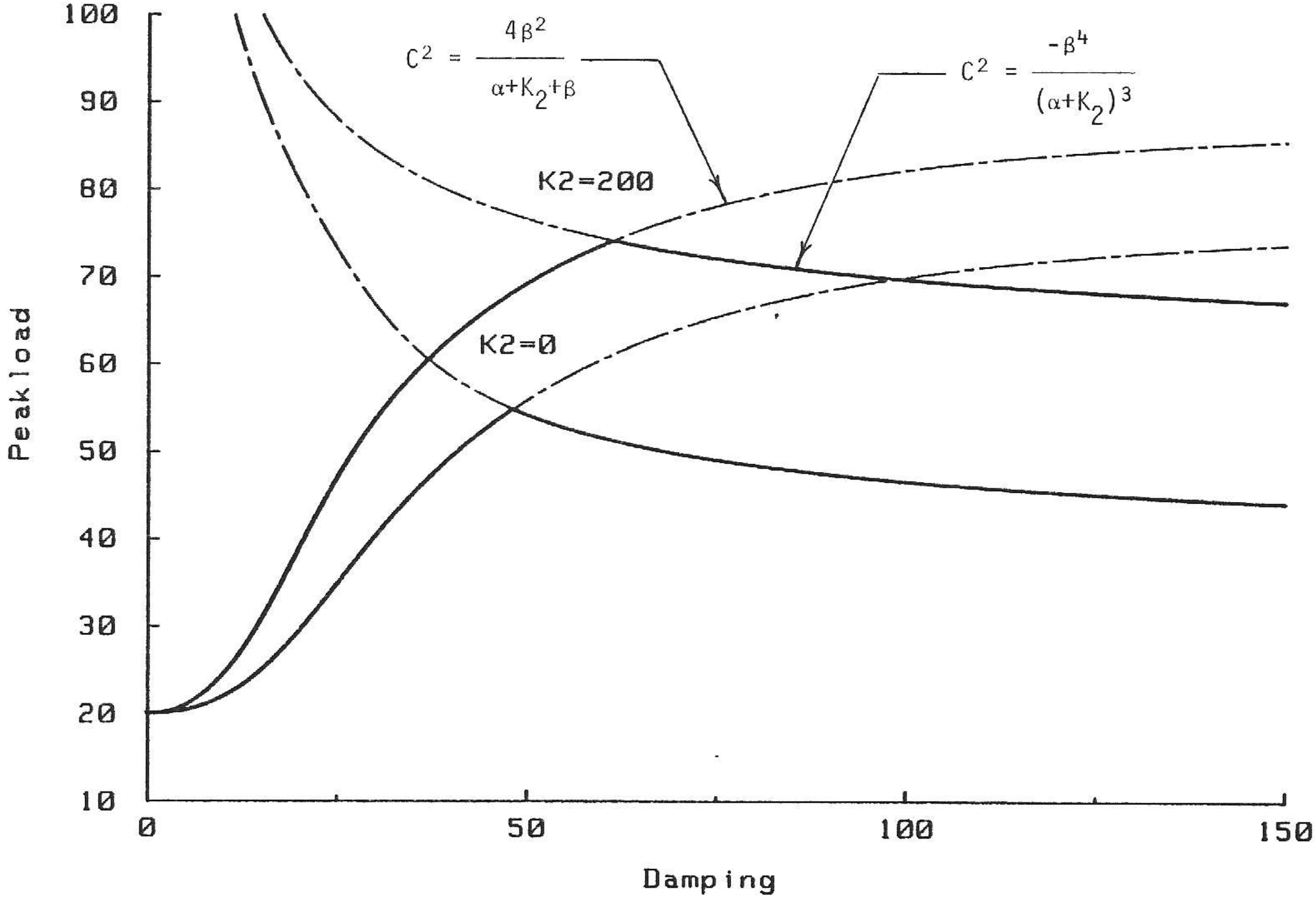
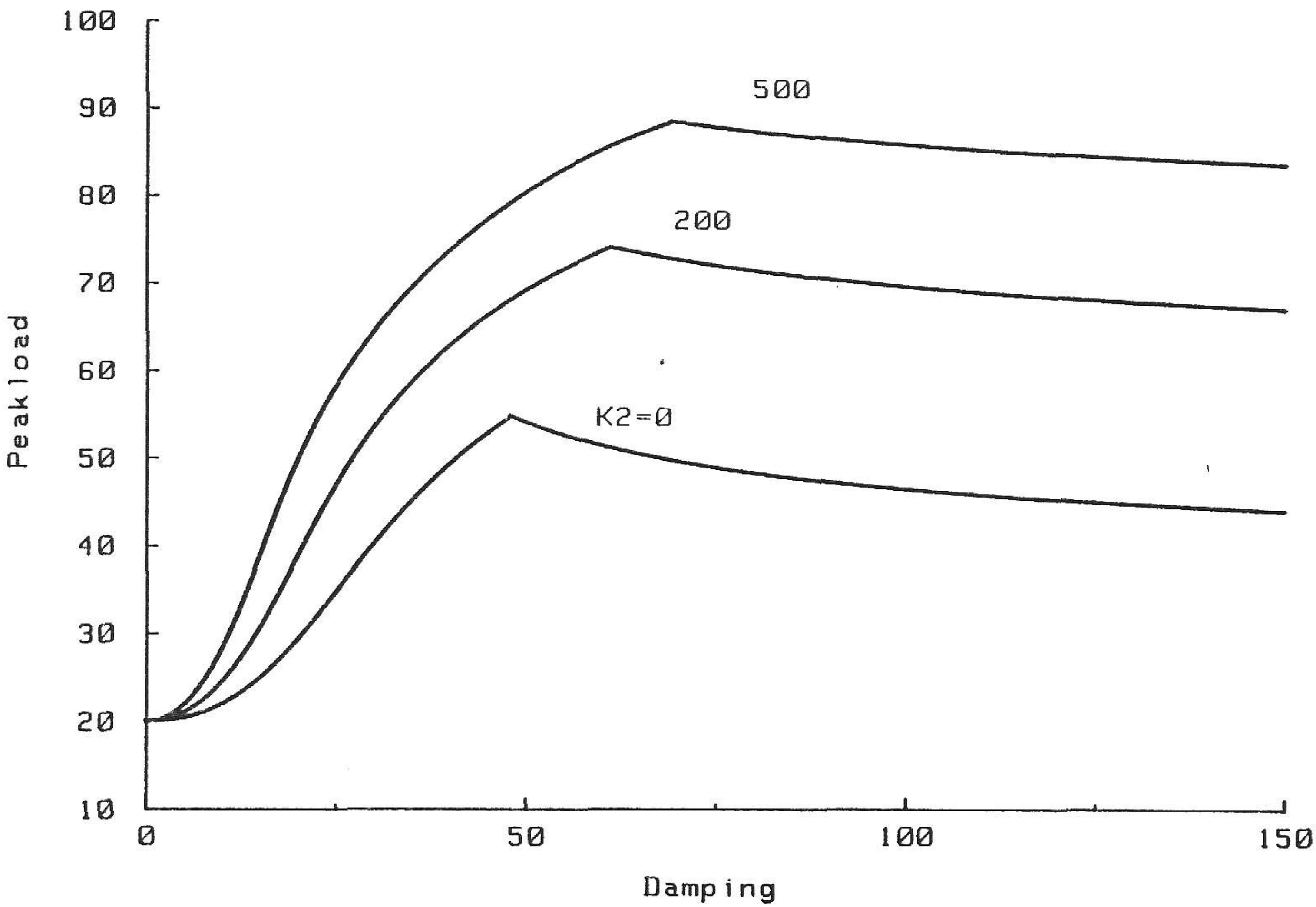


Figure 3.10 Peak Flutter Loads for Standard Linear Foundation (K_2 = 200)



<u>Figure 3.11</u> Peak Flutter Loads for Standard Linear Foundation ($0 \leq K_2 \leq 500$)

3.6). Also, the presence of K_2 increases the flutter load for a given damping, as exhibited by the Kelvin-Voigt model (figure 3.1).

IV. DISCUSSION AND CONCLUSIONS

The present investigation deals with the special class of non-conservative elastic stability problems which contain follower forces. More specifically, the problem considers the effect of various visco-elastic foundations on the dynamic stability of a tangentially loaded column.

The equation of motion for a cantilever column continuously supported by the Standard Linear foundation is derived in Chapter II, including the "in-phase" mass M^* of the foundation in the inertia term. This foundation has as special cases the Kelvin-Voigt and the Maxwell foundations. The equation of motion for the column when supported by a Winkler (elastic) foundation [26,27,28] and when in the presence of external viscous damping [20,21,22] are also degenerate cases of the general equation of motion.

It is shown that a separable solution exists and allows an exact dynamic analysis to be performed. The boundary-value problem thus obtained is the same as the original Beck's column [10]. The resulting transcendental equation is transformed into a complex form and a simple Newton-Raphson iteration scheme is used to solve for the real and complex eigenvalues corresponding to a given load. Since the eigenvalues are complex and appear in the coefficients of the characteristic equation, a general method is given in Chapter III for converting an n^{th} order polynomial with complex coefficients to a polynomial of order 2n with real coefficients. Instead of solving for the roots of the characteristic equation, the Routh-Hurwitz-Mikhailov (RHM) criteria, developed in Section 3.2, is used to yield exact closed form expressions which reveal the effect of the foundation parameters on the stability of the column.

First, the Kelvin-Voigt model is presented. The second order temporal equation transforms into a fourth order characteristic polynomial. The RHM criteria yield stability conditions, involving the system parameters. Unlike the approximate analyses of Wahed [29] and Kar [30], these expressions are not restricted to a small range of parameters. The RHM conditions show the effect of the full range of foundation parameters on the stability of the column. For a given stiffness parameter K_2, the critical flutter load increases with increasing damping to the limiting value where the real part of the corresponding eigenvalue is of the same magnitude as K_2. Since the Kelvin-Voigt model with infinite stiffness reduces to a viscous dashpot, the stability condition for the column in the presence of external viscous damping is recovered.

The analysis also reveals the effect of a Maxwell foundation on elastic systems which fail in flutter. Such studies have not been undertaken in the past, possibly due to the fact that a Maxwell foundation has no stabilizing effect on conservative elastic systems. In contrast to the case of conservative loading, it is found that the Maxwell foundation does have pronounced stabilizing effect on the column under tangential loading. Furthermore, there exists an optimal combination of the stiffness value $K_1 = K^* = 579.58$ and the damping parameter $C^* = 48.29$ which yields the maximum flutter load of $P^* = 54.9$. All other combinations of stiffness K_1 and damping C lead to flutter loads which are less than this optimum value. The flutter load increases from $P_f = 20.05$ to $P^* = 54.9$ when $0 \leq K_1 \leq K^*$ and then decreases to $P = 37.7$ as K_1 approaches infinity when appropriate damping is present.

As expected, the Standard Linear foundation combines the characteristics of both the Kelvin-Voigt and the Maxwell foundations. Each K_2 value has a specific combination of foundation parameters K_1 and C which result in the optimal flutter load, similiar to the behavior of the Maxwell model. Also, the presence of K_2 increases the flutter load for a given damping as shown by the Kelvin-Voigt model.

In summary, an exact analysis has been presented to investigate the stability of a cantilever column supported by a Standard Linear viscoelastic foundation under the action of a constant tangential follower force. This study also resulted in the development of a direct approach to study the stability of systems described by ordinary differential equations with complex coefficients. Instead of employing a numerical procedure , exact closed form stability conditions are derived for the entire range of foundation parameters. From the results, it is found that the Standard Linear foundation has a positive influence on the stability of this special nonconservative problem. Any combination of foundation parameters increases the flutter load beyond that of the unsupported cantilever column subjected to a tangential follower force. Perhaps, the most important contribution of this study is the discovery that a Maxwell support may stabilize some nonconservative systems. It is anticipated that the results of the present investigation should be very helpful and serve as the starting point for understanding the behavior of more general nonconservative problems.

With an understanding of the behavior of the Standard Linear foundation, and in particular the Maxwell foundation, many extensions and applications can be proposed from this problem. A few suggestfollow.

1. Study the effect of viscoelastic foundation on the stability of a column loaded by a distributed tangential follower forces.

2. Investigate the effect of supporting the continuous column at discrete points with a single Standard Linear element.

3. Consider the viscoelastic support of discrete systems.

4. Investigate the effect of a Standard Linear foundation on the dynamic instability of pipes conveying fluid.

5. Apply these results to related dynamic stability problems outside the Solid Mechanics area.

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APPENDIX

APPENDIX A

```
C     FORTRAN PROGRAM IN COMPLEX VARIABLES TO SOLVE
C     TRANSCENDENTAL EQUATION (2.42) USING NEWTON-RAPHSON
C     METHOD DESCRIBED IN SECTION 2.5
C
C
C         INPUT REQUIRED
C
C            ETA= FOLLOWER PARAMETER
C            ALPHA2= STARTING LOAD VALUE
C            STEP= LOAD INCREMENT SIZE
C            L= NUMBER OF LOAD INCREMENTS
C            EPSI= TOLERANCE FOR SUCCESSIVE ROOTS
C            LAMDA= COMPLEX FORM (REAL,IMAG) OF INITIAL GUESS
C
C
C         OUTPUT FOR EACH LOAD INCREMENT
C
C            LOAD, LAMDA SQUARED (ALPHA,BETA), LAMDA (REAL,IMAG)
      REAL*8 ALPHA2,ETA, IGUESS,RGUESS
      COMPLEX*8 LAMDA,COM1,COM2,COM3,COM4
      COMPLEX*16 CDSIN,CDCOS,CDSQRT,DCMPLX,GSQR,GBSQR,G,GBAR
      COMPLEX*16 C1,C2,F1,F2,PARF1,PARF2,D
      COMPLEX*16 PARD,DEL,LAMDA2
   25 FORMAT('1',5X,'ETA=',F5.2,8X,'ALPHA2',28X,'LAMDA2',20X
     2,'LAMDA')
   50 FORMAT('1')
   75 FORMAT(' ',22X,F5.1,10X,'(',E16.7,',',E16.7,')',10X,
     1  '(',E16.7,',',E16.7,')')
      READ,ETA,ALPHA2,STEP,L,EPSI
      PRINT 25,ETA
      READ,LAMDA
      DO 200 I=1,L
      DEL=0
      CALL NEWTON(LAMDA,ALPHA2,ETA,DEL,EPSI)
      LAMDA2=LAMDA**2
      PRINT 75,ALPHA2,LAMDA2,LAMDA
  200 ALPHA2=ALPHA2+STEP
      PRINT 50
      STOP
      END
      SUBROUTINE NEWTON(LAMDA,ALPHA2,ETA,DEL,EPSI)
      REAL*8 ALPHA2,ETA
      COMPLEX*8 LAMDA,COM1,COM2,COM3,COM4
      COMPLEX*16 CDSIN,CDCOS,CDSQRT,DCMPLX,GSQR,GBSQR,G,GBAR
      COMPLEX*16 C1,C2,F1,F2,PARF1,PARF2,D
      COMPLEX*16 PARD,DEL
      ITMAX=200
      DO 100 I=1,ITMAX
      GSQR=DCMPLX(ALPHA2-2*DBLE(AIMAG(LAMDA)),
     22*DBLE(REAL(LAMDA)))
      GBSQR=DCMPLX(ALPHA2+2*DBLE(AIMAG(LAMDA)),
     3-2*DBLE(REAL(LAMDA)))
      G=CDSQRT(GSQR)
      GBAR=CDSQRT(GBSQR)
      F1=1+.5*(CDCOS(G)+CDCOS(GBAR))
      COM2=.5*(CDSIN(G)/G-CDSIN(GBAR)/GBAR)
      PARF1=DCMPLX(DBLE(AIMAG(COM2)),-1*DBLE(REAL(COM2)))
      IF (ALPHA2.EQ.0) GOTO 250
      C1=DCMPLX(2*(DBLE(REAL(LAMDA))**2-DBLE(AIMAG(LAMDA))**2)
     1+(1-ETA)*ALPHA2**2,4*DBLE(REAL(LAMDA))*DBLE(AIMAG(LAMDA)))
```

```
      C2=(2*ETA-1)*ALPHA2
      COM1=.5*LAMDA*(CDCOS(G)-CDCOS(GBAR))
      F2=DCMPLX(ALPHA2-DBLE(AIMAG(COM1)),DBLE(REAL(COM1)))
      COM3=.5*(CDCOS(G)-CDCOS(GBAR))
      COM4=.5*LAMDA*(CDSIN(G)/G+CDSIN(GBAR)/GBAR)
      PARF2=DCMPLX(DBLE(REAL(COM4))-DBLE(AIMAG(COM3)),
     2DBLE(REAL(COM3))+DBLE(AIMAG(COM4)))
      D=C1*F1+C2*F2
      PARD=4*LAMDA*F1+C1*PARF1+C2*PARF2
      DEL=-1*D/PARD
      GO TO 275
  250 DEL=-1*F1/PARF1
  275 IF(CDABS(DEL).LE.EPSI)GO TO 400
  100 LAMDA=LAMDA+DEL
  400 RETURN
      END
```

APPENDIX B

```
10     ! Program file: "Rroots"
20     !
30     ! Language: BASIC
40     !
50     ! Purpose: To solve for first two eigenvalues of eq.(2.42)
60     !          for given follower parameter Eta as load is
70     !          incremented from zero.
80     !
90     ! Method: Muller's method is used to search for solutions of
100    !         the transcendental equation given in the function
110    !         subprogram.
120    !
130    ! Input:
140    !        Eta= follower force parameter
150    !        Input prompts provided as needed during execution
160    !
170    ! Output: First two real eigenvalues for each load increment
180    !         is displayed on CRT. When roots become equal, the
190    !         critical load is displayed. Plotting option given.
200    !
210    !
220   DIM Root(2),Lamda(2,400)
230     INPUT "What value of Eta?",Eta
240     INPUT "Step size for load (Alpha2)?",Step
250     Nroots=2
260    ! PRINTER IS 0
270     PRINT "Root(Nroot,Alpha2,Eta)",LIN(2)
280     Plot=0
290     PRINTER IS 16
300     Alpha2=-Step
310     FOR L=0 TO 400
320     Alpha2=Alpha2+Step
330     IF Alpha2=0 THEN 350
340    GOTO 180
350    Root(1)=5
360    Root(2)=20
370    GOTO 380
380   CALL Muller1(Nroots,Alpha2,Eta,Root(*),Lamda(*),L)
390     IF ABS(Root(2)-Root(1))<.0001 THEN 410
400    GOTO 490
410    Crload=Alpha2-Step
420    Lmax=L-1
430   ! PRINTER IS 0
440    PRINT USING 460;Eta,Crload
450    PRINTER IS 16
460    IMAGE 5X,"CRITICAL LOAD FOR Eta=",D.DD," IS",DDD.DD
470    PRINT LIN(3)
480    GOTO 500
490    NEXT L
500    INPUT "DO YOU WANT A GRAPH? (Y/N)",G$
510    IF (G$="N") OR (G$="n") THEN GOTO 1490
520    IF (G$="Y") OR (G$="y") THEN GOTO 540
530    GOTO 500
540    INPUT "MAXIMUM X AND Y COORDINATES? (X,Y)",Xmax,Ymax
550    INPUT "X-AXIS MINOR TICK?",Xtm,Xet
560    INPUT "Y-AXIS MINOR TICK?",Ytm,Yet
570    INPUT "NUMBER OF DECIMAL PLACES FIXED AFTER DECIMAL POINT?",F
580    PLOTTER IS 13,"GRAPHICS"
590    GRAPHICS
600    LIMIT 0,184.47,0,139
610    LOCATE 15,130,13,88
620    SCALE 0,Xmax,0,Ymax
630    IF Plot>1 THEN GOTO 650
640    AXES Xtm,Ytm,0,0,Xet,Yet
```

```
650    LINE TYPE 1
660    CSIZE 4,.6
670    MOVE Lamda(1,0),0
680    Alpha2=-Step
690    FOR L=0 TO Lmax
700    Alpha2=Alpha2+Step
710    DRAW Lamda(1,L),Alpha2
720    NEXT L
730    Alpha2=Crload+Step
740    FOR L=Lmax TO 0 STEP -1
750    Alpha2=Alpha2-Step
760    DRAW Lamda(2,L),Alpha2
770    NEXT L
780    IF Plot>1 THEN GOTO 1050
790    LINE TYPE 1
800    UNCLIP
810    DEG
820    LDIR 0
830    LORG 6
840    FOR X_label=0 TO Xmax STEP Xet*Xtm
850    FIXED F
860    MOVE X_label,-1
870    LABEL X_label
880    NEXT X_label
890    LDIR 0
900    LORG 8
910    FOR Y_label=0 TO Ymax STEP Yet*Ytm
920    FIXED F
930    MOVE 0,Y_label
940    LABEL Y_label
950    NEXT Y_label
960    SETGU
970    LDIR 90
980    MOVE 1,55
990    LABEL "Load,P"
1000   LDIR 0
1010   MOVE 80,3
1020   LABEL "Lambda"
1030   WAIT 10000
1040   PEN 0
1050   INPUT "DO YOU WANT TO CHANGE DIMENSIONS OF GRAPH? (Y/N)",D$
1060   IF (D$="N") OR (D$="n") THEN 1100
1070   IF (D$="Y") OR (D$="y") THEN 540
1080   GOTO 1050
1090   EXIT GRAPHICS
1100   PRINT "Label Eigencurve, then press CONT to exit LETTER mode"
1110   PEN 1
1120   LETTER
1130   PEN 0
1140   PRINT LIN(1),"DO YOU WANT A COPY ON THERMAL PAPER? (press T)"
1150   PRINT LIN(1),"DO YOU WANT A COPY ON THE PLOTTER? (press P)"
1160   PRINT LIN(1),"IF NEITHER, PRESS N"
1170   INPUT T$
1180   IF (T$="N") OR (T$="n") THEN 1490
1190   IF (T$="T") OR (T$="t") THEN 1220
1200   IF (T$="P") OR (T$="p") THEN 1270
1210   GOTO 1140
1220   PRINTER IS 0
1230   PRINT PAGE;
1240   DUMP GRAPHICS
1250   PRINTER IS 16
1260   GOTO 1140
1270   PLOTTER IS 7,5,"9872A"
1280   Plot=Plot+1
1290   PEN 0
1300   PRINTER IS 7,5
```

```
1310  PRINT "VS1"
1320  PRINTER IS 16
1330  PRINT LIN(1),"GRAPH IS PLACED ON 8-1/2 by 11 PAGE WITH LONGEST EDGE ON THE
 VERTICAL."
1340  PRINT LIN(1),"GRAPH POSITION? UPPER HALF(U),LOWER HALF(L), CENTER(C)"
1350  INPUT U$
1360  IF (U$="U") OR (U$="u") THEN 1400
1370  IF (U$="L") OR (U$="l") THEN 1430
1380  IF (U$="C") OR (U$="c") THEN 1460
1390  GOTO 1340
1400  LIMIT 28,182,140,225
1410  PEN 1
1420  GOTO 610
1430  LIMIT 28,182,25,140
1440  PEN 1
1450  GOTO 610
1460  LIMIT 28,182,82,197
1470  PEN 1
1480  GOTO 610
1490   LINPUT "Do you want to change Eta (Y/N)?",C$
1500   IF (C$="Y") OR (C$="y") THEN 200
1510   IF (C$="N") OR (C$="n") THEN 1520
1520 END
1530 SUB Muller1(Nroots,Alpha2,Eta,Root(*),Lamda(*),L)
1540  Itmax=50               !Maximum number of iterations
1550  Tolf=.0000000001       !Tolerance for function
1560  Eps=.0000001           !Spread tolerance for multiple roots
1570  Et=.00001              !Restart value for multiple roots
1580  Digits=6               !Number of significant digits in roots
1590 CALL Muller(Root(*),Nroots,Itmax,Tolf,Eps,Et,Digits,Alpha2,Eta)
1600  ! PRINTER IS 0
1610 FOR I=1 TO Nroots
1620    PRINT USING 1630;I,Alpha2,Eta,Root(I)
1630    IMAGE "Root(",D,",",DD.DD,",",D.DD,")=",MZ.6DE
1640    Lamda(I,L)=Root(I)
1650 NEXT I
1660 PRINT LIN(2)
1670  PRINTER IS 16
1680 SUBEXIT
1690 SUBEND
1700 SUB Muller(Root(*),Nroots,Itmax,Tolf,Eps,Et,Digits,Alpha2,Eta)
1710 Baddta=(Nroots<=0) OR (Itmax<=0) OR (Tolf<=0) OR (Digits<=0)
1720 IF Baddta=0 THEN 1780
1730 PRINT LIN(2),"ERROR IN SUBPROGRAM Muller."
1740 PRINT "NROOTS=";Nroots;"  Itmax=";Itmax
1750 PRINT "Tolf=";Tolf;"  DIGITS=";Digits,LIN(2)
1760 PAUSE
1770 GOTO 1710
1780 Digits=10^(-Digits)
1790 P=-1
1800 P1=1
1810 P2=0
1820 H=0
1830 FOR I=1 TO Nroots
1840    IF I=1 THEN 1860
1850     IF Root(I-1)=9.999999E99 THEN 2340
1860    J=0
1870    IF Root(I)=0 THEN 1910
1880    P=.9*Root(I)
1890    P1=1.1*Root(I)
1900    P2=Root(I)
1910    Rt=P
1920    GOTO 2320
1930    IF J<>1 THEN 1970
1940    Rt=P1
1950    X0=Fprt
```

```
1960     GOTO 2320
1970     IF J<>2 THEN 2010
1980     Rt=P2
1990     X1=Fprt
2000     GOTO 2320
2010     IF J<>3 THEN 2230
2020     X2=Fprt
2030     D=-.5
2040     IF Root(I)=0 THEN 2070
2050     H=-.1*Root(I)
2060     GOTO 2080
2070     H=-1
2080     Dd=D+1
2090     Bi=X0*D*D-X1*Dd*Dd+X2*(Dd+D)
2100     Den=Bi*Bi-4*X2*D*Dd*(X0*D-X1*Dd+X2)
2110     IF Den>0 THEN Den=SQR(Den)
2120     IF Den<=0 THEN Den=0
2130     Dn=Bi+Den
2140     Dm=Bi-Den
2150     IF ABS(Dn)<=ABS(Dm) THEN Den=Dm
2160     IF ABS(Dn)>ABS(Dm) THEN Den=Dn
2170     IF Den=0 THEN Den=1
2180     Di=-Dd*2*X2/Den
2190     H=Di*H
2200     Rt=Rt+H
2210     IF (ABS(H)<ABS(Rt)*Digits) AND (H<>0) THEN 2520
2220     GOTO 2320
2230     IF ABS(Fprt)>=ABS(X2*10) THEN 2290
2240     X0=X1
2250     X1=X2
2260     X2=Fprt
2270     D=Di
2280     GOTO 2080
2290     Di=Di*.5
2300     H=H*.5
2310     Rt=Rt-H
2320     J=J+1
2330     IF J<Itmax THEN 2390
2340     PRINT LIN(2),"ERROR IN SUBPROGRAM Muller."
2350     PRINT USING 2360;I
2360     IMAGE "MAXIMUM # OF ITERATIONS EXCEEDED ON Root(",DD,")",//
2370     Root(I)=9.999999E99
2380     GOTO 2530
2390     Frt=FNF(Rt,Alpha2,Eta)
2400     Fprt=Frt
2410     IF I<2 THEN 2470
2420     FOR I1=2 TO I
2430        Temp=Rt-Root(I1-1)
2440        IF ABS(Temp)<Eps THEN 2490
2450        Fprt=Fprt/Temp
2460     NEXT I1
2470     IF (ABS(Frt)<Tolf) AND (ABS(Fprt)<Tolf) THEN 2520
2480     GOTO 1930
2490     Rt=Rt+Et
2500     J=J-1
2510     GOTO 2320
2520     Root(I)=Rt
2530 NEXT I
2540 Itmax=J
2550 SUBEXIT
2560 SUBEND
2570 DEF FNCosh(X)
2580 E=EXP(X)
2590 Cosh=(E+1/E)/2
2600 RETURN Cosh
2610 FNEND
```

```
2620 DEF FNSinh(X)
2630 IF X>0 THEN Flg=0
2640 IF X<0 THEN Flg=1
2650 IF X=0 THEN Sinh=0
2660 IF X=0 THEN Out
2670   X=ABS(X)
2680 IF X>10 THEN 2810
2690 IF X>1 THEN 2780
2700 IF X>.5 THEN 2740
2710 X2=X*X
2720 Sinh=X*(((.00019979150018*X2+.00833311842443)*X2+.166666677361)*X2+.9999999
99917)
2730 GOTO Out
2740 E=EXP(X)
2750 D=E-1
2760 Sinh=.5*(D+D/E)
2770 GOTO Out
2780 E=EXP(X)
2790 Sinh=(E-1/E)/2
2800 GOTO Out
2810 Sinh=EXP(X)/2
2820 Out: IF Flg=1 THEN Sinh=-Sinh
2830 IF Flg=1 THEN X=-X
2840 RETURN Sinh
2850 FNEND
2860 DEF FNF(X,Alpha2,Eta)
2870  IF Alpna2=0 THEN 2940
2880  Beta1=SQR(.5*Alpha2+SQR(.25*Alpha2^2+X^2))
2890  Beta2=SQR(-.5*Alpha2+SQR(.25*Alpha2^2+X^2))
2900  F1=Alpha2*X*(2*Eta-1)*SIN(Beta1)*FNSinh(Beta2)+Eta*Alpha2^2
2910  F2=2*X^2+(2*X^2+Alpha2^2*(1-Eta))*COS(Beta1)*FNCosh(Beta2)
2920  F=F1+F2
2930  GOTO 2950
2940  F=1+COS(SQR(SQR(X^2)))*FNCosh(SQR(SQR(X^2)))
2950 RETURN F
2960 FNEND
```

APPENDIX C

LOAD	ALPHA	BETA
20.1	121.0209	10.61037
20.2	120.365	18.4863
20.3	119.709	23.88528
20.4	119.053	28.2641
20.5	118.3965	32.04398
20.6	117.7397	35.41714
20.7	117.0827	38.49072
20.8	116.4253	41.33151
20.9	115.7679	43.98477
21	115.1101	46.4827
21.1	114.452	48.8491
21.2	113.7937	51.10229
21.3	113.1351	53.2565
21.4	112.4763	55.32355
21.5	111.8172	57.31274
21.6	111.1578	59.2319
21.7	110.4981	61.08777
21.8	109.838	62.88589
21.9	109.1779	64.63116
22	108.5174	66.3277
22.1	107.8566	67.97922
22.2	107.1956	69.5889
22.3	106.5342	71.15958
22.4	105.8725	72.69388
22.5	105.2106	74.194
22.6	104.5484	75.66208
22.7	103.8859	77.09982
22.8	103.223	78.50906
22.9	102.56	79.89117
23	101.8966	81.24768
23.1	101.233	82.57977
23.2	100.569	83.8887
23.3	99.90474	85.1755
23.4	99.24017	86.44113
23.5	98.5753	87.6866
23.6	97.91	88.91275
23.7	97.2446	90.1203
23.8	96.5788	91.31
23.9	95.9127	92.4829
24	95.2463	93.6392
24.1	94.5795	94.7797
24.2	93.9125	95.905
24.3	93.2451	97.0155
24.4	92.5775	98.112
24.5	91.9095	99.1946
24.6	91.24124	100.264
24.7	90.57259	101.32
24.8	89.9035	102.365
24.9	89.2343	103.397
25	88.5647	104.417
25.1	87.8947	105.427
25.2	87.22446	106.425
25.3	86.55386	107.413
25.4	85.8829	108.39
25.5	85.2116	109.357
25.6	84.5399	110.315
25.7	83.868	111.263
25.8	83.19568	112.202
25.9	82.5229	113.132
26	81.85	114.053
26.1	81.1766	114.965
26.2	80.5029	115.869

26.3	79.82883	116.7655
26.4	79.1543	117.653
26.5	78.4796	118.533
26.6	77.8044	119.406
26.7	77.12894	120.2719
26.8	76.453	121.13
26.9	75.7767	121.981
27	75.1	122.825
27.1	74.42322	123.6628
27.2	73.7458	124.493
27.3	73.068	125.318
27.4	72.3899	126.136
27.5	71.71144	126.948
27.6	71.0325	127.754
27.7	70.3533	128.554
27.8	69.67363	129.348
27.9	68.9935	130.136
28	68.3131	130.919
28.1	67.6323	131.697
28.2	66.95108	132.469
28.3	66.269	133.236
28.4	65.5874	133.998
28.5	64.9049	134.754
28.6	64.222	135.506
28.7	63.5388	136.253
28.8	62.8552	136.996
28.9	62.171	137.733
29	61.4865	138.466
29.1	60.8016	139.195
29.2	60.11626	139.9193
29.3	59.43047	140.639
29.4	58.74428	141.3546
29.5	58.0576	142.066
29.6	57.3705	142.773
29.7	56.683	143.476
29.8	55.995	144.176
29.9	55.30669	144.871
30	54.6178	145.563
30.1	53.9285	146.251
30.2	53.2387	146.935
30.3	52.5485	147.616
30.4	51.8579	148.293
30.5	51.16682	148.966
30.6	50.47522	149.636
30.7	49.78319	150.303
30.8	49.09067	150.967
30.9	48.3976	151.627
31	47.7042	152.284
31.1	47.0102	152.938
31.2	46.3158	153.589
31.3	45.6209	154.237
31.4	44.92552	154.882
31.5	44.22961	155.5239
31.6	43.53322	156.1629
31.7	42.83633	156.799
31.8	42.1389	157.432
31.9	41.441	158.063
32	40.7426	158.691
32.1	40.0437	159.316
32.2	39.34433	159.939
32.3	38.6443	160.559
32.4	37.9439	161.177
32.5	37.249	161.792
32.6	36.5413	162.405
32.7	35.8393	163.015
32.8	35.1367	163.623

32.9	34.4335	164.2295
33	33.7299	164.833
33.1	33.0256	165.434
33.2	32.3208	166.033
33.3	31.6155	166.63
33.4	30.9097	167.225
33.5	30.2032	167.818
33.6	29.49619	168.408
33.7	28.7885	168.997
33.8	28.0804	169.584
33.9	27.3716	170.169
34	26.6623	170.752
34.1	25.9524	171.333
34.2	25.2419	171.912
34.3	24.5307	172.49
34.4	23.8191	173.066
34.5	23.1067	173.64
34.6	22.3938	174.212
34.7	21.6803	174.783
34.8	20.9662	175.352
34.9	20.2514	175.919
35	19.536	176.485
35.1	18.8199	177.0502
35.2	18.10332	177.613
35.3	17.386	178.174
35.4	16.6681	178.734
35.5	15.9495	179.293
35.6	15.2302	179.85
35.7	14.5103	180.406
35.8	13.7897	180.96
35.9	13.06856	181.514
36	12.3466	182.066
36.1	11.624	182.616
36.2	10.9008	183.166
36.3	10.1768	183.714
36.4	9.452194	184.261
36.5	8.72686	184.808
36.6	8.0008	185.353
36.7	7.27406	185.897
36.8	6.54658	186.439
36.9	5.81842	186.981
37	5.08949	187.522
37.1	4.35983	188.062
37.2	3.62948	188.601
37.3	2.89836	189.139
37.4	2.1665	189.676
37.5	1.43386	190.213
37.6	.7005157	190.7484
37.7	-.0336456	191.283
37.8	-.768524	191.816
37.9	-1.50421	192.35
38	-2.24066	192.882
38.1	-2.97789	193.414
38.2	-3.7159	193.944
38.3	-4.45475	194.475
38.4	-5.19438	195.004
38.5	-5.93483	195.533
38.6	-6.676086	196.062
38.7	-7.41818	196.59
38.8	-8.1611	197.117
38.9	-8.90483	197.644
39	-9.64942	198.17
39.1	-10.3948	198.696
39.2	-11.1411	199.221
39.3	-11.8883	199.746
39.4	-12.6363	200.271

39.5	-13.3852	200.795
39.6	-14.1349	201.319
39.7	-14.8856	201.842
39.8	-15.6372	202.365
39.9	-16.3896	202.888
40	-17.143	203.411
40.1	-17.8973	203.933
40.2	-18.6525	204.455
40.3	-19.4087	204.977
40.4	-20.1658	205.499
40.5	-20.9238	206.02
40.6	-21.6828	206.542
40.7	-22.4427	207.063
40.8	-23.036	207.584
40.9	-23.9655	208.105
41	-24.7284	208.626
41.1	-25.4923	209.147
41.2	-26.2571	209.668
41.3	-27.023	210.189
41.4	-27.7899	210.71
41.5	-28.5578	211.231
41.6	-29.3267	211.753
41.7	-30.0967	212.274
41.8	-30.8677	212.795
41.9	-31.6399	213.317
42	-32.413	213.839
42.1	-33.1872	214.36
42.2	-33.9625	214.883
42.3	-34.7389	215.405
42.4	-35.5164	215.927
42.5	-36.295	216.45
42.6	-37.0748	216.973
42.7	-37.8556	217.497
42.8	-38.6376	218.0211
42.9	-39.4208	218.545
43	-40.2051	219.069
43.1	-40.9906	219.594
43.2	-41.7772	220.119
43.3	-42.565	220.645
43.4	-43.354	221.171
43.5	-44.1443	221.698
43.6	-44.9357	222.225
43.7	-45.7284	222.753
43.8	-46.5222	223.281
43.9	-47.317	223.81
44	-48.113	224.339
44.1	-48.9114	224.869
44.2	-49.7103	225.399
44.3	-50.5105	225.931
44.4	-51.312	226.462
44.5	-52.1149	226.995
44.6	-52.919	227.528
44.7	-53.7245	228.062
44.8	-54.5313	228.597
44.9	-55.3395	229.132
45	-56.149	229.668
45.1	-56.9599	230.205
45.2	-57.,7722	230.743
45.3	-58.5858	231.282
45.4	-59.4009	231.821
45.5	-60.2174	232.2362
45.6	-61.03545	232.903
45.7	-61.8548	233.45
45.8	-62.6756	233.988
45.9	-63.4979	234.532
46	-64.3217	235.077

46.1	-65.147	235.623
46.2	-65.9738	236.17
46.3	-66.8021	236.719
46.4	-67.632	237.268
46.5	-68.4634	237.818
46.6	-69.2964	238.369
46.7	-70.1309	238.922
46.8	-70.967	239.475
46.9	-71.8047	240.03
47	-72.6441	240.586
47.1	-73.485	241.143
47.2	-74.3276	241.701
47.3	-75.1718	242.261
47.4	-76.0178	242.822
47.5	-76.865	243.384
47.6	-77.7146	243.947
47.7	-78.5657	244.512
47.8	-79.4184	245.078
47.9	-80.2729	245.645
48	-81.1291	246.214
48.1	-81.9871	246.784
48.2	-82.8469	247.356
48.3	-83.7085	247.928
48.4	-84.57198	248.503
48.5	-85.43718	249.079
48.6	-86.3042	249.656
48.7	-87.1732	250.235
48.8	-88.044	250.8155
48.9	-88.9168	251.397
49	-89.7914	251.98
49.1	-90.668	252.565
49.2	-91.5465	253.152
49.3	-92.427	253.74
49.4	-93.3094	254.33
49.5	-94.193	254.921
49.6	-95.0803	255.515
49.7	-95.9689	256.109
49.8	-96.8594	256.706
49.9	-97.752	257.304
50	-98.6467	257.904
50.1	-99.5435	258.506
50.2	-100.442	259.109
50.3	-101.343	259.715
50.4	-102.246	260.322
50.5	-103.152	260.931
50.6	-104.059	261.542
50.7	-104.969	262.154
50.8	-105.881	262.769
50.9	-106.795	263.385
51	-107.712	264.004
51.1	-108.631	264.624
51.2	-109.552	265.246
51.3	-110.475	265.871
51.4	-111.401	266.497
51.5	-112.329	267.125
51.6	-113.26	267.755
51.7	-114.193	268.387
51.8	-115.128	269.022
51.9	-116.066	269.658
52	-117.007	270.297
52.1	-117.95	270.937
52.2	-118.895	271.58
52.3	-119.843	272.225
52.4	-120.793	272.872
52.5	-121.746	273.521
52.6	-122.702	274.172

52.7	-123.66	274.826
52.8	-124.621	275.482
52.9	-125.585	276.14
53	-126.551	276.8
53.1	-127.52	277.462
53.2	-128.492	278.127
53.3	-129.466	278.794
53.4	-130.443	279.464
53.5	-131.423	280.135
53.6	-132.406	280.809
53.7	-133.392	281.486
53.8	-134.381	282.164
53.9	-135.372	282.845
54	-136.367	283.529
54.1	-137.364	284.215
54.2	-138.365	284.903
54.3	-139.368	285.594
54.4	-140.374	286.287
54.5	-141.384	286.983
54.6	-142.396	287.681
54.7	-143.412	288.381
54.8	-144.431	289.085
54.9	-145.4531	289.7905
55	-146.478	290.498
55.1	-147.506	291.209
55.2	-148.538	291.922
55.3	-149.572	292.638
55.4	-150.61	293.356
55.5	-151.652	294.077
55.6	-152.697	294.8
55.7	-153.745	295.526
55.8	-154.796	296.255
55.9	-155.851	296.986
56	-156.91	297.72
56.1	-157.971	298.457
56.2	-159.037	299.196
56.3	-160.106	299.938
56.4	-161.178	300.682
56.5	-162.254	301.429
56.6	-163.334	302.179
56.7	-164.417	302.932
56.8	-165.504	303.687
56.9	-166.594	304.448
57	-167.689	305.205
57.1	-168.787	305.969
57.2	-169.889	306.735
57.3	-170.995	307.504
57.4	-172.104	308.275
57.5	-173.218	309.049
57.6	-174.335	309.826
57.7	-175.456	310.606
57.8	-176.581	311.389
57.9	-177.711	312.174
58	-178.844	312.962
58.1	-179.981	313.753
58.2	-181.123	314.547
58.3	-182.268	315.343
58.4	-183.418	316.142
58.5	-184.572	316.944
58.6	-185.73	317.749
58.7	-186.893	318.557
58.8	-188.059	319.367
58.9	-189.23	320.18
59	-190.405	320.996
59.1	-191.585	321.815
59.2	-192.769	322.637

59.3	-193.957	323.461
59.4	-195.15	324.288
59.5	-196.348	325.118
59.6	-197.55	325.951
59.7	-198.756	326.786
59.8	-199.967	327.625
59.9	-201.183	328.466
60	-202.404	329.31
60.1	-203.635	330.154
60.2	-204.864	331.004
60.3	-206.099	331.856
60.4	-207.338	332.711
60.5	-208.582	333.569
60.6	-209.831	334.43
60.7	-211.085	335.293
60.8	-212.344	336.159
60.9	-214.376	337.899
61	-214.876	337.899
61.1	-216.149	338.774
61.2	-217.428	339.651
61.3	-218.712	340.531
61.4	-220	341.413
61.5	-221.295	342.299
61.6	-222.593	343.186
61.7	-223.898	344.077
61.8	-225.207	344.97
61.9	-226.522	345.866
62	-227.842	346.765
62.1	-229.168	347.666
62.2	-230.498	348.57
62.3	-231.834	349.477
62.4	-233.176	350.386
62.5	-234.522	351.298
62.6	-235.875	352.212
62.7	-237.233	353.129
62.8	-238.596	354.048
62.9	-239.965	354.97
63	-241.34	355.895
63.1	-242.72	356.822
63.2	-244.105	357.752
63.3	-245.497	358.684
63.4	-246.894	359.619
63.5	-248.297	360.556
63.6	-249.706	361.495
63.7	-251.12	362.437
63.8	-252.541	363.381
63.9	-253.966	364.328
64	-255.399	365.277
64.1	-256.837	366.299
64.2	-258.281	367.182
64.3	-259.731	368.138
64.4	-261.187	369.097
64.5	-262.648	370.057
64.6	-264.1169	371.02
64.7	-265.59	371.985
64.8	-267.07	372.953
64.9	-268.557	373.922
65	-270.049	374.894
65.1	-271.548	375.867
65.2	-273.052	376.843
65.3	-274.563	377.821
65.4	-276.08	378.801
65.5	-277.604	379.783
65.6	-279.134	380.768
65.7	-280.67	381.754
65.8	-282.212	382.742

65.9	-283.762	383.732
66	-285.317	384.724
66.1	-286.878	385.718
66.2	-288.447	386.713
66.3	-290.021	387.711
66.4	-291.602	388.71
66.5	-293.19	389.711
66.6	-294.784	390.714
66.7	-296.385	391.719
66.8	-297.992	392.725
66.9	-299.606	393.733
67	-301.226	394.743
67.1	-302.853	395.754
67.2	-304.487	396.766
67.3	-306.128	397.781
67.4	-307.774	398.797
67.5	-309.429	399.814
67.6	-311.089	400.833
67.7	-312.756	401.853
67.8	-314.43	402.875
67.9	-316.111	403.898
68	-317.798	404.922
68.1	-319.492	405.948
68.2	-321.194	406.974
68.3	-322.902	408.002
68.4	-324.616	409.032
68.5	-326.338	410.062
68.6	-328.067	411.094
68.7	-329.802	412.127
68.8	-331.544	413.16
68.9	-333.294	414.195
69	-335.05	415.231
69.1	-336.813	416.268
69.2	-338.582	417.305
69.3	-340.359	418.344
69.4	-342.143	419.383
69.5	-343.933	420.423
69.6	-345.732	421.465
69.7	-347.536	422.507
69.8	-349.348	423.549
69.9	-351.167	424.592
70	-352.992	425.636
70.1	-354.825	426.681
70.2	-356.665	427.726
70.3	-358.511	428.772
70.4	-360.365	429.818
70.5	-362.225	430.865
70.6	-364.094	431.912
70.7	-365.968	432.959
70.8	-367.85	434.007
70.9	-369.738	435.055
71	-371.634	436.104
71.1	-373.537	437.153
71.2	-375.447	438.202
71.3	-377.364	439.251
71.4	-379.288	440.301
71.5	-381.219	441.35
71.6	-383.157	442.4
71.7	-385.101	443.45
71.8	-387.054	444.499
71.9	-389.013	445.549
72	-390.979	446.599
72.1	-392.952	447.649
72.2	-394.932	448
72.3	-396.919	449.747
72.4	-398.914	450.797

72.5	-400.915	451.846
72.6	-402.923	452.895
72.7	-404.939	453.943
72.8	-406.96	454.991
72.9	-408.99	456.039
73	-411.026	457.087
73.1	-413.069	458.134
73.2	-415.118	459.18
73.3	-417.176	460.227
73.4	-419.24	461.272
73.5	-421.31	462.317
73.6	-423.388	463.362
73.7	-425.472	464.405
73.8	-427.564	465.449
73.9	-429.663	466.491
74	-431.768	467.533
74.1	-433.88	468.574
74.2	-436	469.615
74.3	-438.125	470.654
74.4	-440.258	471.693
74.5	-442.398	472.731
74.6	-444.544	473.768
74.7	-446.697	474.804
74.8	-448.857	475.839
74.9	-451.024	476.873
75	-453.197	477.906
75.1	-455.378	478.939
75.2	-457.564	479.97
75.3	-459.758	481
75.4	-461.959	482.029
75.5	-464.166	483.056
75.6	-466.379	484.083
75.7	-468.6	485.108
75.8	-470.827	486.132
75.9	-473.061	487.155
76	-475.3	488.176
76.1	-477.547	489.197
76.2	-479.801	490.216
76.3	-482.061	491.233
76.4	-484.327	492.249
76.5	-486.601	493.264
76.6	-488.88	494.277
76.7	-491.166	495.289
76.8	-493.458	496.299
76.9	-495.758	497.308
77	-498.063	498.316
77.1	-500.375	499.321
77.2	-502.692	500.325
77.3	-505.017	501.328
77.4	-507.348	502.329
77.5	-509.686	503.328
77.6	-512.029	504.326
77.7	-514.379	505.322
77.8	-516.735	506.316
77.9	-519.097	507.309
78	-521.466	508.299
78.1	-523.841	509.288
78.2	-526.222	510.275
78.3	-528.609	511.261
78.4	-531.002	512.244
78.5	-533.402	513.226
78.6	-535.807	514.205
78.7	-538.219	515.183
78.8	-540.637	516.159
78.9	-543.061	517.133
79	-545.49	518.105

79.1	-547.926	519.075
79.2	-550.368	520.043
79.3	-552.815	521.009
79.4	-555.269	521.974
79.5	-557.729	522.936
79.6	-560.194	523.896
79.7	-562.666	524.854
79.8	-565.143	525.809
79.9	-567.627	526.763
80	-570.115	527.715
80.1	-572.611	528.665
80.2	-575.111	529.612
80.3	-577.617	530.557
80.4	-580.129	531.5
80.5	-582.647	532.441
80.6	-585.17	533.38
80.7	-587.699	534.316
80.8	-590.234	535.25
80.9	-592.774	536.183
81	-595.32	537.112
81.1	-597.782	538.04
81.2	-600.429	538.965
81.3	-602.991	539.888
81.4	-605.559	540.809
81.5	-608.134	541.727
81.6	-610.712	542.643
81.7	-613.297	543.557
81.8	-615.887	544.468
81.9	-618.483	545.377
82	-621.083	546.284
82.1	-623.69	547.189
82.2	-626.301	548.09
82.3	-628.917	548.99
82.4	-631.54	549.888
82.5	-634.167	550.782
82.6	-636.8	551.675
82.7	-639.438	552.565
82.8	-642.081	553.345
82.9	-644.73	554.338
83	-647.383	555.22
83.1	-650.042	556.101
83.2	-652.705	556.979
83.3	-655.375	557.854
83.4	-658.048	558.727
83.5	-660.727	559.597
83.6	-663.411	560.465
83.7	-666.101	561.331
83.8	-668.795	562.194
83.9	-671.494	563.054
84	-674.199	563.913
84.1	-676.908	564.768
84.2	-679.622	565.621
84.3	-682.341	566.472
84.4	-685.064	567.32
84.5	-687.793	568.166
84.6	-690.527	569.008
84.7	-693.266	569.849
84.8	-696.008	570.687
84.9	-698.757	571.522
85	-701.51	572.356
85.1	-704.268	573.186
85.2	-707.029	574.013
85.3	-709.796	574.838
85.4	-712.569	575.662
85.5	-715.345	576.481
85.6	-718.126	577.299

85.7	-720.913	578.114
85.8	-723.703	578.927
85.9	-726.498	579.737
86	-729.298	580.544
86.1	-732.102	581.349
86.2	-734.911	582.151
86.3	-737.724	582.951
86.4	-740.542	583.749
86.5	-743.365	584.543
86.6	-746.191	585.335
86.7	-749.023	586.125
86.8	-751.858	586.912
86.9	-754.699	587.696
87	-757.544	588.478
87.1	-760.392	589.257
87.2	-763.246	590.034
87.3	-766.104	590.808
87.4	-768.967	591.58
87.5	-771.833	592.349
87.6	-774.704	593.116
87.7	-777.58	593.879
87.8	-780.459	594.641
87.9	-783.343	595.4
88	-786.231	596.156
88.1	-789.123	596.91
88.2	-792.02	597.661
88.3	-794.92	598.41
88.4	-797.825	599.156
88.5	-800.734	599.899
88.6	-803.648	600.641
88.7	-806.565	601.379
88.8	-809.487	602.115
88.9	-812.413	602.848
89	-815.344	603.58
89.1	-818.277	604.308
89.2	-821.215	605.034
89.3	-824.158	605.757
89.4	-827.103	606.478
89.5	-830.054	607.196
89.6	-833.008	607.912
89.7	-835.967	608.626
89.8	-838.929	609.336
89.9	-841.895	610.044
90	-844.866	610.75
90.1	-847.84	611.454
90.2	-850.818	612.154
90.3	-853.801	612.853
90.4	-856.786	613.548
90.5	-859.776	614.242
90.6	-862.771	614.933
90.7	-865.768	615.621
90.8	-868.77	616.307
90.9	-871.775	616.991
91	-874.785	617.672
91.1	-877.798	618.35
91.2	-880.815	619.027
91.3	-883.836	619.7
91.4	-886.861	620.372
91.5	-889.89	621.041
91.6	-892.923	621.707
91.7	-895.959	622.371
91.8	-898.998	623.033
91.9	-902.042	623.692
92	-905.089	624.349
92.1	-908.141	625.003
92.2	-911.196	625.655

92.3	-914.253	626.305
92.4	-917.316	626.952
92.5	-920.382	627.597
92.6	-923.451	628.239
92.7	-926.525	628.879
92.8	-929.602	629.517
92.9	-932.683	630.152
93	-935.766	630.785
93.1	-938.855	631.416
93.2	-941.946	632.044
93.3	-945.041	632.67
93.4	-948.14	633.294
93.5	-951.242	633.916
93.6	-954.348	634.535
93.7	-957.458	635.151
93.8	-960.571	635.766
93.9	-963.688	636.378
94	-966.807	636.988
94.1	-969.931	637.596
94.2	-973.059	638.201
94.3	-976.189	638.804
94.4	-979.324	639.405
94.5	-982.461	640.003
94.6	-985.603	640.6
94.7	-988.748	641.194
94.8	-991.896	641.786
94.9	-995.048	642.376
95	-998.203	642.963
95.1	-1001.36	643.548
95.2	-1004.52	644.131
95.3	-1007.69	644.712
95.4	-1010.85	645.291
95.5	-1014.03	645.867
95.6	-1017.2	646.441
95.7	-1020.38	647.013
95.8	-1023.57	647.583
95.9	-1026.75	648.151
96	-1029.946	648.716
96.1	-1033.13	649.28
96.2	-1036.33	649.841
96.3	-1039.53	650.401
96.4	-1042.73	650.958
96.5	-1045.94	651.513
96.6	-1049.15	652.066
96.7	-1052.36	652.617
96.8	-1055.58	653.166
96.9	-1058.8	653.712
97	-1062.02	654.256
97.1	-1065.25	654.799
97.2	-1068.48	655.339
97.3	-1071.71	655.878
97.4	-1074.95	656.414
97.5	-1078.19	656.949
97.6	-1081.43	657.481
97.7	-1084.68	658.011
97.8	-1087.93	658.54
97.9	-1091.18	659.066
98	-1094.44	659.59
98.1	-1097.7	660.113
98.2	-1100.96	660.633
98.3	-1104.23	661.151
98.4	-1107.5	661.668
98.5	-1110.77	662.183
98.6	-1114.05	662.695
98.7	-1117.33	663.206
98.8	-1120.61	663.714

98.9	-1123.9	664.221
99	-1127.18	664.726
99.1	-1130.48	665.229
99.2	-1133.77	665.731
99.3	-1137.07	666.23
99.4	-1140.37	666.727
99.5	-1143.68	667.223
99.6	-1146.99	667.716
99.7	-1150.3	668.208
99.8	-1153.62	668.698
99.9	-1156.93	669.186
100	-1160.38	669.67

APPENDIX D

```
10    ! Program file: "RHMplt"
20    !
30    ! Language: BASIC
40    !
50    ! Purpose: To determine STABILITY BOUNDARY for
60    !           Standard Linear, Kelvin-Voigt, or
70    !           Maxwell viscoelastic foundation using
80    !           Routh-Hurwitz-Mikhailov (RHM) Criteria
90    !           developed in Sections 3.3 through 3.5
100   !
110   ! Method: Uses Siljak's method to solve stability
120   !          limit condition for real and complex C
130   !          roots for increasing load values.
140   !
150   !          Checks for positive coefficients of
160   !          characteristic equation.
170   !
180   !          Checks additional condition, if any.
190   !
200   ! Foundation Descriptions:
210   !          Standard Linear (SL)- Series combination of
220   !              K1 and C in parallel with K2.
230   !          Kelvin-Voigt (KV)- Parallel combination of
240   !              K2 and C
250   !          Maxwell (MAX)- Series combination of K1 and C
260   !
270   ! Data File Required:  "Etalrt"
280   !          This data file must be supplied by operator
290   !          with indexed pairs of real and imaginary parts
300   !          of eigenvalues of eq.(2.42) corresponding to load
310   !          values greater than P=20.05, following the format
320   !          "J,Load(J),Alpha(J),Beta(J)",where J is the index.
330   !
340   ! Input: Select foundation type, as described above, and the
350   !         appropiate parameters.  Input prompts provided as
360   !         needed during execution.
370   !
380   ! Output: System parameter values of Load, K1, K2 as well as
390   !          damping roots C are displayed on CRT for each Load
400   !          increment.  When peakload is reached, plotting option
410   !          is given.
420   OPTION BASE 1
430   DIM Rcoef(0:2),Icoef(0:2),Rroot(0:2),Iroot(0:2)
440   DIM C1(300),C2(300),Load(300),Alpha(300),Beta(300),C(300)
450   ASSIGN #1 TO "Etalrt"
460   READ #1;J,Load(*),Alpha(*),Beta(*)
470   PRINTER IS 0
480    PRINT "LOAD","ALPHA","BETA"
490    FOR J=1 TO 300
500    PRINT Load(J),Alpha(J),Beta(J)
510    NEXT J
520   INPUT "Kelvin-Voigt(KV), Maxwell(MAX), or Standard Linear(SL) model?",M$
530   IF M$="KV" THEN Kvoigt
540   IF M$="MAX" THEN Mxwell
550   IF M$="SL" THEN Stdlin
560   GOTO 520
570 Kvoigt:  INPUT "Kelvin-Voigt stiffness (K2)?",K2
580   PRINTER IS 16
590   PRINT LIN(2)
600   PRINT "K2",K2
610   PRINT LIN(1)
620   PRINT "N","Load(N)","C(N)"
630   FOR I=1 TO 300
640   IF Alpha(I)+K2>0 THEN GOTO 700
650   PRINTER IS 0
660   PRINT "Alpha+K2=0 at Load of ",Load(I)
```

```
670   PRINTER IS 16
680   Imax=I-1
690   GOTO 1660
700   C(I)=SQR(Beta(I)^2/(Alpha(I)+K2))
710   PRINTER IS 16
720   PRINT I,Load(I),C(I)
730   ! CHECK FOR POSITIVE DEFINITE COEFFICIENTS
740   A0=1
750   A1=2*C(I)
760   A2=C(I)^2+2*(Alpha(I)+K2)
770   A3=2*C(I)*(Alpha(I)+K2)
780   A4=(Alpha(I)+K2)^2+Beta(I)^2
790   PRINTER IS 0
800   IF A0<=0 THEN PRINT "A0 VIOLATED FOR LOAD ",Load(I)
810   IF A1<=0 THEN PRINT "A1 VIOLATED FOR LOAD ",Load(I)
820   IF A2<=0 THEN PRINT "A2 VIOLATED FOR LOAD ",Load(I)
830   IF A3<=0 THEN PRINT "A3 VIOLATED FOR LOAD ",Load(I)
840   IF A4<=0 THEN PRINT "A4 VIOLATED FOR LOAD ",Load(I)
850   PRINTER IS 16
860   NEXT I
870   GOTO 1660
880 Stdlin:     INPUT "Kelvin-Voigt stiffness (K2)?",K2
890   PRINTER IS 16
900   GOTO 920
910 Mxwell:  K2=0
920   INPUT "Maxwell stiffness (K1)?",K1
930   FOR I=1 TO 800
940   A=Beta(I)^2*(Alpha(I)+K2)^2+K1*Beta(I)^2*(Alpha(I)+K2)+Beta(I)^4
950   B=-((Alpha(I)+K2)*(K1^4-2*K1^2*Beta(I)^2)+K1^3*Beta(I)^2)
960   C=K1^4*Beta(I)^2
970   N=2
980   Rcoef(0)=C
990   Icoef(0)=0
1000  Rcoef(1)=B
1010  Icoef(1)=0
1020  Rcoef(2)=A
1030  Icoef(2)=0
1040    Tola=.001
1050    Tolf=.001
1060  Itmax=200
1070  Err=0
1080  CALL Siljak(N,Rcoef(*),Icoef(*),Tola,Tolf,Itmax,Rroot(*),Iroot(*),Err
1090  IF Err=1 THEN 1650
1100   ! PRINTER IS 0
1110    PRINTER IS 16
1120   FIXED 10
1130  PRINT LIN(1)
1140  PRINT "LOAD",Load(I)
1150  PRINT "K1","K2"
1160  PRINT K1,K2
1170  PRINT "Siljak Roots 1 and 2"
1180  PRINT Rroot(1),Iroot(1)
1190  PRINT Rroot(2),Iroot(2)
1200  C1(I)=SQR(ABS(Rroot(2)))
1210  C2(I)=SQR(ABS(Rroot(1)))
1220  PRINT "Damping Constants",C1(I),C2(I)
1230  IF C1(I)-C2(I)=0 THEN GOTO 1650
1240  ! CHECK FOR POSITIVE DEFINITE COEFFICIENTS
1250  A01=C1(I)^2
1260  A02=C2(I)^2
1270  A11=2*C1(I)*K1
1280  A12=2*C2(I)*K1
1290  A21=K1^2+2*C1(I)^2*(Alpha(I)+K2+K1)
1300  A22=K1^2+2*C2(I)^2*(Alpha(I)+K2+K1)
1310  A31=2*C1(I)*K1*(2*Alpha(I)+K2+K1)
1320  A32=2*C2(I)*K1*(2*Alpha(I)+K2+K1)
```

```
1330  A41=C1(I)^2*((Alpha(I)+K1+K2)^2+Beta(I)^2)+2*(Alpha(I)+K2)*K1^2
1340  A42=C2(I)^2*((Alpha(I)+K1+K2)^2+Beta(I)^2)+2*(Alpha(I)+K2)*K1^2
1350  A51=2*C1(I)*K1*((Alpha(I)+K2)*(Alpha(I)+K1+K2)+Beta(I)^2)
1360  A52=2*C2(I)*K1*((Alpha(I)+K2)*(Alpha(I)+K1+K2)+Beta(I)^2)
1370  A6=K1^2*((Alpha(I)+K2)^2+Beta(I)^2)
1380  A71=C1(I)^2*(Alpha(I)+K2)+K1^2
1390  A72=C2(I)^2*(Alpha(I)+K2)+K1^2
1400  IF A01<=0 THEN PRINT "A01>0 VIOLATED"
1410  IF A02<=0 THEN PRINT "A02>0 VIOLATED"
1420  IF A11<=0 THEN PRINT "A11>0 VIOLATED"
1430  IF A12<=0 THEN PRINT "A12>0 VIOLATED"
1440  IF A21<=0 THEN PRINT "A21>0 VIOLATED"
1450  IF A22<=0 THEN PRINT "A22>0 VIOLATED"
1460  IF A31<=0 THEN PRINT "A31>0 VIOLATED"
1470  IF A32<=0 THEN PRINT "A32>0 VIOLATED"
1480  IF A41<=0 THEN PRINT "A41>0 VIOLATED"
1490  IF A42<=0 THEN PRINT "A42>0 VIOLATED"
1500  IF A51<=0 THEN PRINT "A51>0 VIOLATED"
1510  IF A52<=0 THEN GOTO 1560
1520  IF A6<=0 THEN PRINT "A6>0 VIOLATED"
1530  IF A71<=0 THEN PRINT "A71>0 VIOLATED"
1540  IF A72<=0 THEN 1640
1550  GOTO 1590
1560  Imax1=I-1
1570  PRINT "A52>0 VIOLATED"
1580  GOTO 1590
1590  PRINTER IS 16
1600   IF C2(I)-C1(I)<.05 THEN Imax1=I
1610  NEXT I
1620   Imax1=I
1630  GOTO 1660
1640  PRINT "A72>0 VIOLATED"
1650  Imax1=I-1
1660  INPUT "DO YOU WANT A GRAPH ? (Y/N)",G$
1670  IF (G$="N") OR (G$="n") THEN 2620
1680  IF (G$="Y") OR (G$="y") THEN 1700
1690  GOTO 1660
1700  INPUT "Maximum X and Y coordinates? (X,Y)",Xmax,Ymax
1710  INPUT "X-Axis MINOR tick? AND X-Axis MAJOR tick every __ minor ticks?",Xtm
,Xet
1720  INPUT "Y-Axis MINOR tick? AND Y-Axis MAJOR tick every __ minor ticks?",Ytm
,Yet
1730  INPUT "Number of decimal places fixed after decimal point?",F
1740  PLOTTER IS 13,"GRAPHICS"
1750  GRAPHICS
1760  LIMIT 0,184.47,0,139
1770  LOCATE 25,130,25,97
1780  SCALE 0,Xmax,0,Ymax
1790  AXES Xtm,Ytm,0,0,Xet,Yet,2
1800  INPUT "Line type (Integer between 1 and 10)",Line
1810  IF M$="KV" THEN 1930
1820  IF (M$="MAX") OR (M$="SL") THEN 1840
1830  GOTO 1810
1840  LINE TYPE Line
1850  MOVE C1(1),Load(1)-10
1860  FOR I=1 TO Imax1
1870  DRAW C1(I),Load(I)-10
1880  NEXT I
1890  FOR J=Imax1 TO 1 STEP -1
1900  DRAW C2(J),Load(J)-10
1910  NEXT J
1920  GOTO 1980
1930  LINE TYPE Line
1940  MOVE 0,20.05-10
1950  FOR I=1 TO Imax
1960  DRAW C(I),Load(I)-10
```

```
1970  NEXT I
1980  LINE TYPE 1
1990  UNCLIP
2000  DEG
2010  LDIR 0
2020  LORG 5
2030  CSIZE 3,.6
2040  FOR X_label=0 TO Xmax STEP Xet*Xtm
2050  FIXED F
2060   MOVE X_label,-3
2070  LABEL X_label
2080  NEXT X_label
2090  LDIR 0
2100  LORG 5
2110  FOR Y_label=10 TO Ymax+10 STEP Yet*Ytm
2120  FIXED F
2130   MOVE -12,Y_label-10
2140  LABEL Y_label
2150  NEXT Y_label
2160  FIXED 5
2170  SETGU
2180  LDIR 90
2190   MOVE 13,60
2200  LABEL "Load,P"
2210  LDIR 0
2220  MOVE 79,19
2230  LABEL "Damping,C"
2240  PEN 0
2250  WAIT 5000
2260  INPUT "Do you want to change dimensions of graph? (Y/N)",D$
2270  IF (D$="Y") OR (D$="y") THEN GOTO 1700
2280  IF (D$="N") OR (D$="n") THEN GOTO 2300
2290  GOTO 2260
2300  EXIT GRAPHICS
2310  PRINT "Label graph then exit by pressing CONT"
2320  INPUT "Lettering Angle, from positive X-axis in degrees",Ldir
2330  LDIR Ldir
2340  PEN 1
2350  LETTER
2360  PEN 0
2370  PRINT LIN(1),"Do you want a copy on THERMAL PAPER? (key T)"
2380  PRINT LIN(1),"Do you want a copy from the PLOTTER? (key P)"
2390  PRINT LIN(1),"If neither, key N"
2400  INPUT T$
2410  IF (T$="N") OR (T$="n") THEN 2620
2420  IF (T$="T") OR (T$="t") THEN 2450
2430  IF (T$="P") OR (T$="p") THEN 2500
2440  GOTO 2370
2450  PRINTER IS 0
2460  PRINT PAGE;
2470  DUMP GRAPHICS
2480  PRINTER IS 16
2490  GOTO 2370
2500  EXIT GRAPHICS
2510  PRINTER IS 16
2520  PRINT LIN(2),"Graph is placed on 8-1/2 by 11 sheet with longest edge on th
e horizontal."
2530  PRINT LIN(1),"BE SURE PLOTTER IS ON!!!!"
2540  PRINT LIN(1),"Press CONT when sheet is in place and plotter is ON"
2550  PAUSE
2560  PLOTTER IS 7,5,"9872A"
2570  PRINTER IS 7,5
2580    PRINT "VS1"
2590  PRINTER IS 16
2600   LIMIT 0,225,0,165
2610  GOTO 1770
```

```
2620   INPUT "Do you want to change parameters?",C$
2630   IF (C$="Y") OR (C$="y") THEN GOTO 520
2640   IF (C$="N") OR (C$="n") THEN GOTO 2660
2650   GOTO 2620
2660   END
2670 SUB Siljak(N,Rcoef(*),Icoef(*),Tola,Tolf,Itmax,Rroot(*),Iroot(*),Err)
2680 Baddta=(N<=0) OR (Tola<=0) OR (Tolf<=0) OR (Itmax<=0)
2690 IF Baddta=0 THEN 2760
2700 PRINT LIN(2),"ERROR IN SUBPROGRAM Siljak."
2710 PRINT "N=";N,"Tola=";Tola
2720 PRINT "Tolf=";Tolf,"Itmax=";Itmax,LIN(2)
2730 PAUSE
2740 GOTO 2680
2750 DIM Xsiljak(0:N),Ysiljak(0:N)
2760 MAT Rroot=(9.999999E99)
2770 MAT Iroot=(9.999999E99)
2780 Nn=N
2790 IF N=1 THEN 3410
2800 Y=Ysiljak(1)=Xsiljak(0)=1
2810 X=Xsiljak(1)=.1
2820 Ysiljak(0)=L=0
2830 GOSUB Siljak
2840 G=F
2850 M=Q=P=0
2860 L=L+1
2870 FOR K=1 TO N
2880    P=P+K*(Rcoef(K)*Xsiljak(K-1)-Icoef(K)*Ysiljak(K-1))
2890    Q=Q+K*(Rcoef(K)*Ysiljak(K-1)+Icoef(K)*Xsiljak(K-1))
2900 NEXT K
2910 Z=P*P+Q*Q
2920 Deltax=-(U*P+V*Q)/Z
2930 Deltay=(U*Q-V*P)/Z
2940 M=M+1
2950 Xsiljak(1)=X+Deltax
2960 Ysiljak(1)=Y+Deltay
2970 GOSUB Siljak
2980 IF F>=G THEN 3040
2990 IF (ABS(Deltax)<Tola) AND (ABS(Deltay)<Tola) THEN 3220
3000 IF L>Itmax THEN 3160
3010 X=Xsiljak(1)
3020 Y=Ysiljak(1)
3030 GOTO 2840
3040 IF M>20 THEN 3080
3050 Deltax=Deltax/4
3060 Deltay=Deltay/4
3070 GOTO 2940
3080 IF (ABS(U)<=Tolf) AND (ABS(V)<=Tolf) THEN 3220
3090 PRINT LIN(2),"ERROR IN SUBPROGRAM Siljak."
3100 PRINT "THE INTERVAL SIZE HAS BEEN QUARTERED 20 TIMES AND "
3110 PRINT "THE TOLERANCE FOR FUNCTIONAL EVALUATIONS IS STILL NOT MET."
3120 PRINT "Tolf=";Tolf,"U=";U,"V=";V,LIN(2)
3130  Err=1
3140  GOTO 3490
3150 PAUSE
3160 PRINT LIN(2),"ERROR IN SUBROUTINE Siljak."
3170 PRINT "MAXIMUM # OF ITERATIONS HAS BEEN EXCEEDED."
3180 PRINT "L=";L,"Itmax=";Itmax,LIN(2)
3190 PAUSE
3200  Err=1
3210 GOTO 3490
3220 Rroot(N)=Xsiljak(1)
3230 Iroot(N)=Ysiljak(1)
3240 A=Rcoef(N)
3250 B=Icoef(N)
3260 Rcoef(N)=Icoef(N)=0
3270 X=Xsiljak(1)
```

```
3280 Y=Ysiljak(1)
3290 FOR K=N-1 TO 0 STEP -1
3300     C=Rcoef(K)
3310     D=Icoef(K)
3320     U=Rcoef(K+1)
3330     V=Icoef(K+1)
3340     Rcoef(K)=A+X*U-Y*V
3350     Icoef(K)=B+X*V+Y*U
3360     A=C
3370     B=D
3380 NEXT K
3390 N=N-1
3400 IF N<>1 THEN 2800
3410 A=Rcoef(0)
3420 U=Rcoef(1)
3430 B=Icoef(0)
3440 V=Icoef(1)
3450 T=U*U+V*V
3460 Rroot(1)=-(A*U+B*V)/T
3470 Iroot(1)=(A*V-U*B)/T
3480 N=Nn
3490 SUBEXIT
3500 Siljak: Z=Xsiljak(1)*Xsiljak(1)+Ysiljak(1)*Ysiljak(1)
3510 T=2*Xsiljak(1)
3520 FOR K=0 TO N-2
3530     Xsiljak(K+2)=T*Xsiljak(K+1)-Z*Xsiljak(K)
3540     Ysiljak(K+2)=T*Ysiljak(K+1)-Z*Ysiljak(K)
3550 NEXT K
3560 U=V=0
3570 FOR K=0 TO N
3580     U=U+Rcoef(K)*Xsiljak(K)-Icoef(K)*Ysiljak(K)
3590     V=V+Rcoef(K)*Ysiljak(K)+Icoef(K)*Xsiljak(K)
3600 NEXT K
3610 F=U*U+V*V
3620 RETURN
3630 SUBEND
```

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VITA

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EFFECT OF VISCOELASTIC FOUNDATION ON THE STABILITY OF A TANGENTIALLY LOADED CANTILEVER COLUMN

by

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ABSTRACT

The present study investigates the effect of various viscoelastic foundations on the stability of a cantilever column under the action of a constant tangential follower force at the free end. The equation of motion is derived for the column when supported by the Standard Linear foundation, which has as special cases the Kelvin-Voigt and the Maxwell foundations.

It is shown that a separable solution exists and allows an exact dynamic analysis to be performed. The transcendental equation resulting from the boundary-value problem is transformed into a complex form and a simple Newton-Raphson iteration scheme is used to solve for the real and complex eigenvalues. A general procedure is developed for analyzing systems described by ordinary differential equations with complex coefficients.

A stability analysis of each foundation model is presented involving the entire range of foundation parameters. From the results, it is found that the Standard Linear foundation has a positive influence on the stability of this nonconservative problem. Any combination of foundation parameters increases the flutter load beyond that of the unsupported cantilever column. By supporting the column with the Maxwell foundation, there exists an optimum combination of parameters which allows the maximum flutter load.