# A METHOD FOR THE ANALYSIS AND SYNTHESIS OF SECOND-ORDER SYSTEMS WITH CONTINUOUS TIME DELAY

by

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# INTRODUCTION

Few areas of science in recent years have experienced the tremendous development and interest which has been encountered throughout the field of automatic feedback control. An important benefit of the increasing application of automatic controls has been to relieve man of many monotonous activities. Of even greater significance, however, is that modern complex controls can perform functions which were previously beyond the physical abilities of a human operator.

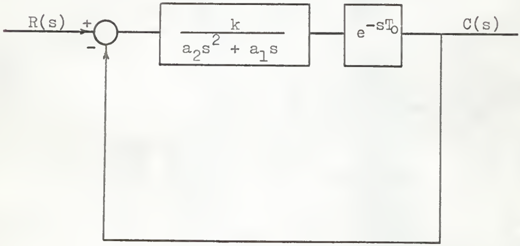
As new and more complex applications are found daily, the demand grows for more complete and accurate methods of analysis. Such a demand has been focused on systems with continuous time delays. This type of situation might be encountered, for instance, in a steel mill where the thickness of a moving sheet is being measured at one point and controlled at another. A time delay might also exist because of the physical properties of a signal-carrying medium. Yet another system containing a time delay is the control loop partially or completely composed of a human operator. Application of closed loop techniques to a manually operated process is relatively new but could prove to be quite valuable.

The purpose of this study is not primarily to derive a simple stability criterion, since satisfactory procedures already exist for determining stability, but rather, to provide a simplified and accurate method of analyzing the transient behavior of a second-order linear control system with time delay when subjected to a unit step input.

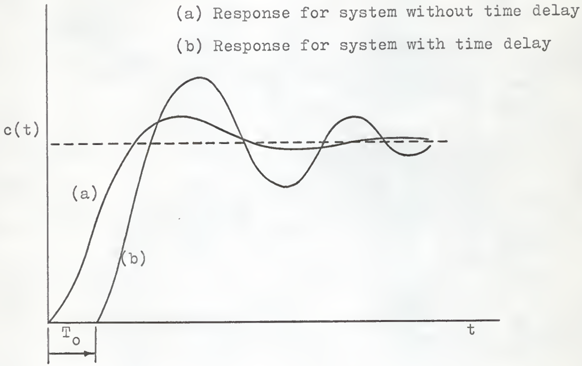
Information concerning the transient response characteristics has been presented in the form of three charts. The stable region has been defined on each of these charts; and for a given time delay, system parameters may be selected for the desired response.

## SYSTEM DESCRIPTION

As has been previously stated, the system under consideration is a linear, second-order control system whose block diagram has the general form shown in the sketch below.




The time delay, $T_o$, is actually a nonlinear characteristic which, fortunately, can be represented in the complex s-domain by the Laplace transform, $e^{-sT_o}$. The addition of even a small delay has a detrimental effect on almost all aspects of transient performance. This fact is illustrated below.




The ideal response should resemble a unit step as closely as possible. Thus, the greater oscillations and longer time required to reach and settle down to the final value are very undesirable features of the time delay system's response.

In addition to the unstabilizing effect on transient behavior, a time delay greatly complicates the analysis of any control system. The reason for this can be traced to the fact that the introduction of time delay creates an infinite number of system poles, the knowledge of which is essential to most analysis procedures.

It has been assumed throughout this text that the time delay term exists in the forward path of the control system. With the time delay in this position, the system's transfer function is

$$\frac{C(s)}{R(s)} = \frac{ke^{-sT_o}}{a_2s^2 + a_1s + a_oe^{-sT_o}}. \quad * \tag{1a}$$

This may be written as

$$\frac{C(s)}{R(s)} = F(s)\ e^{-sT_o}. \tag{1b}$$

The time delay term is not restricted, however, to the forward path. If the delay existed in the feedback loop, the transfer function would be

$$\frac{C(s)}{R(s)} = \frac{k}{a_2s^2 + a_1s + a_oe^{-sT_o}}. \tag{2a}$$

This may be expressed as

$$\frac{C(s)}{R(s)} = F(s). \tag{2b}$$

Since equations (1b) and (2b) differ only by the term, $e^{-sT_o}$, it is obvious from inverse Laplace transform theory that the time response of the first equation has the same form as that of the second but is displaced in time by the amount, $T_o$.

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* For unity feedback, $a_o$=k; while for systems with a gain, $k_1$, in the feedback path, $a_o=kk_1$.

Thus, the theory presented for the system with a time delay in the forward path is applicable, with slight modification, to the system with a time delay in the feedback path.

With a unit step input ($R(s) = \frac{1}{s}$) the steady-state solution becomes

$$c(t)ss = \frac{k}{a_o} \; . \tag{3}$$

In order that the work which follows be applicable to both cases, the ordinate of the response plot has been considered to be "per cent of c(t)ss."

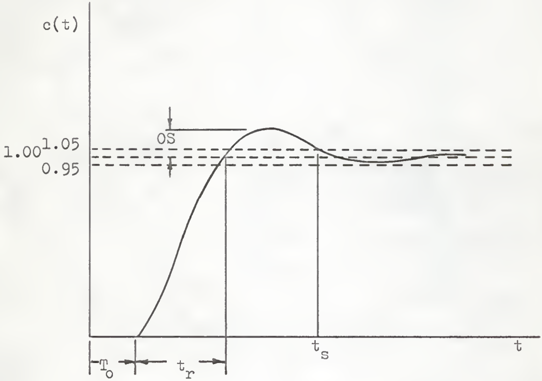
With a unit step input, the following figures of merit are used to evaluate system performance:

Overshoot, os - the difference between the magnitude of the maximum and final values of the response, expressed as a percentage.

Rise Time, $t_r$ - the time required for the response to first reach its final value once it has begun to react to the input signal.

Settling Time, $t_s$ - the time required for the response to reach and thereafter remain within a specified percentage of its final value.

These terms are illustrated below.




So that information could be presented for a general second-order system, it was necessary to transform Eq. (1a) to the new form

$$\frac{C(s)}{R(s)} = \frac{Be^{-sT}}{s^2 + s + Be^{-sT}}. \tag{4}$$

Curves of constant overshoot, rise time, and settling time could then be plotted on charts with coordinates, B and T.

# CONVENTIONAL METHODS OF ANALYSIS

Conventional methods of analyzing systems with time delay may be logically divided into two groups: methods not requiring a determination of system poles and methods requiring this information.

## Methods Not Requiring Pole Determination

The most common means for examining systems with time delay is by Nyquist's frequency response techniques (1).* When discussing the frequency response of a system, the input is considered to be a sine wave of variable frequency. After transient effects have died out, the output is a sine wave of the same frequency as the input but of different magnitude and phase angle. It may be shown that this is done effectively, by substituting $j\omega$ for the Laplace operator, s.

The general system represented by Eq. (1a) might have been represented by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + A(s)} \; . \tag{5}$$

The term, G(s), is defined as the product of all elements in the forward path of the control system, while A(s), the loop gain, is the product of all elements around the loop. With s replaced by $j\omega$ , the loop gain for Eq. (1a) is

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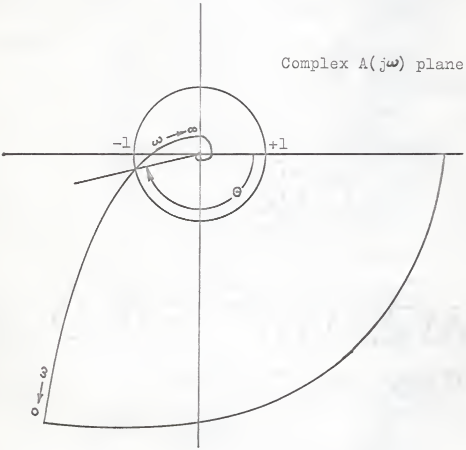
*Numbers in parentheses refer to references.

$$A(j\omega) = \frac{a_o e^{-j\omega T_o}}{j\omega(a_2 j\omega + a_1)} \quad . \tag{6}$$

A "Nyquist diagram" of the complex quantity, $A(j\omega)$, would have the general form shown below.




The Nyquist stability criterion can now be stated as follows:

1. Substitute $j\omega$ for s in the loop gain expression.
2. Plot the polar curve of $A(j\omega)$ as $\omega$ varies from 0 to $\infty$.

3. For stability the curve must cross the unit circle such that $\theta$ is less than $180^\circ$.

It is possible to show that this is entirely equivalent to specifying that the system has no poles in the right half s-plane.

The term, $e^{-j\omega T_o}$, has unity magnitude and a phase angle of $-\omega T_o$. Therefore, the angle, $\theta$, for the second-order system under consideration is

$$\theta = \measuredangle\, j\omega(a_2 j\omega + a_1) + \omega T_o. \qquad (7)$$

It is obvious then that this method presents a relatively fast and easy means of determining stability. This procedure can also permit a design based on a particular degree of stability or phase margin, $\Phi$, given by

$$\Phi = \pi - \theta. \qquad (8)$$

Its usefulness is limited, however, in that only an intuitive knowledge can be obtained regarding the system's response to a sudden distrubance of its input.

Another procedure not requiring a knowledge of the system's poles makes use of a theorem by Pontryagin (2) which is applied to the system's characteristic equation. The theorem may be stated as follows:

Let $h(z,e^z)$ be a polynomial in z and $e^z$ possessing a "principle term" (a term containing the highest power of z multiplied by the highest power of $e^z$). If all the zeroes of

$H(z) \equiv h(z, e^z)$ have negative real parts, then all the zeroes of $F(y)$ and $G(y)$ $(H(jy) \equiv F(y) + jG(y))$ are real alternative and

$$\frac{dG(y)}{dy} F(y) - G(y) \frac{dF(y)}{dy} > 0 \ . \tag{a}$$

Sufficient conditions for $H(z)$ to have all its zeroes in the left half plane are alternatively:

1. All zeroes of $F(y)$ and $G(y)$ are real alternative and Eq. (a) holds for at least one value of $y$.
2. All zeroes of $F(y)$ are real and Eq. (a) holds for every zero, $y_1$, of $F(y)$, i.e.

$$\frac{dF(y_1)}{dy} G(y_1) < 0 \ .$$

3. All zeroes of $G(y)$ are real and Eq. (a) holds for every zero, $y_2$, of $G(y)$, i. e.

$$\frac{dG(y_2)}{dy} F(y_2) > 0 \ .$$

This theorem usually supplies no information other than whether the system is stable or not; thus, its usefulness in any complete design is very limited.

Perhaps of greatest value in any control system analysis is an exact expression for the time response to a unit step input. This is very difficult to obtain since the usual methods of taking the inverse Laplace transform require a knowledge of the system's poles. A little used method of obtaining the inverse transform, however, finds a great deal of application when control systems involve time delay.* The transfer function given by Eq. (1a) when expanded gives the power series

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + A(s)} = G(s) - G(s)A(s) + G(s)A(s)^2 - \dots\ . \qquad (9)$$

The series converges for $A(s) < 1$; therefore, its usefulness is limited to systems whose coefficients lie in certain bounded regions. For the general second-order system this method would yield the response

$$c(t) = \mathcal{L}^{-1}\left[\frac{k}{s(a_2 s^2 + a_1 s)}\right] u(t-T_o) - \mathcal{L}^{-1}\left[\frac{k^2}{s(a_2 s^2 + a_1 s)^2}\right] u(t-2T_o) + \dots\ . \qquad (10)$$

This expression differs from other infinite series representations in that only a finite number of terms need to be evaluated to find the exact response for a particular time. Although the calculations may become quite involved for certain system parameters, the method, in general, is very useful.

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*This method has been presented by Tyner (3).

## Methods Requiring Pole Determination

A certain amount of information about the transient behavior of the control system can be obtained from a knowledge of the pole locations for different values of the system's "static loop sensitivity." For the general system represented by Eq. (1a), the constant, $a_o/a_2$, is defined as the static loop sensitivity; and the path traced on the complex s-plane as $a_o/a_2$ varies is known as the root locus of the system. A technique for finding this locus for systems with time delay has been developed by Chu (4) and is described for the second-order system below.

The characteristic equation,

$$s^2 + \frac{a_1}{a_2} s + \frac{a_o}{a_2} e^{-sT_o} = 0, \tag{11}$$

is actually a vector equation. Thus, for a value of s to be a root of the equation, a magnitude condition,

$$\left|s\right| \cdot \left|s + \frac{a_1}{a_2}\right| = \left|- \frac{a_o}{a_2} e^{-sT_o}\right|, \tag{12}$$

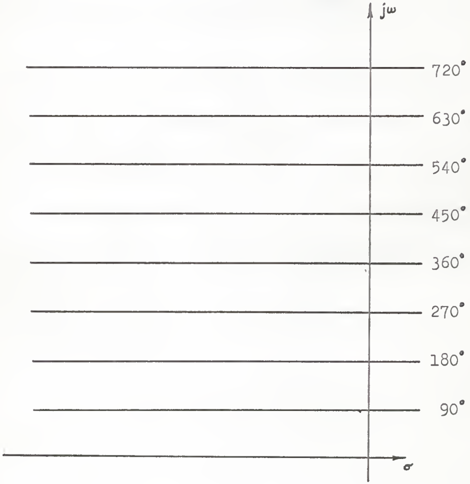
and an angle condition,

$$\measuredangle s + \measuredangle\left(s + \frac{a_1}{a_2}\right) + \measuredangle\left(e^{+sT_o}\right) = \pi \pm k2\pi \tag{13}$$

for k=0,1,2,3,...., must be satisfied. The locus itself is defined by the angle condition while the location of roots on this locus for a given value of $a_0/a_2$ is determined by the magnitude condition. In plotting the root locus it is helpful to consider the time delay term and the remaining part of the system, separately. Letting

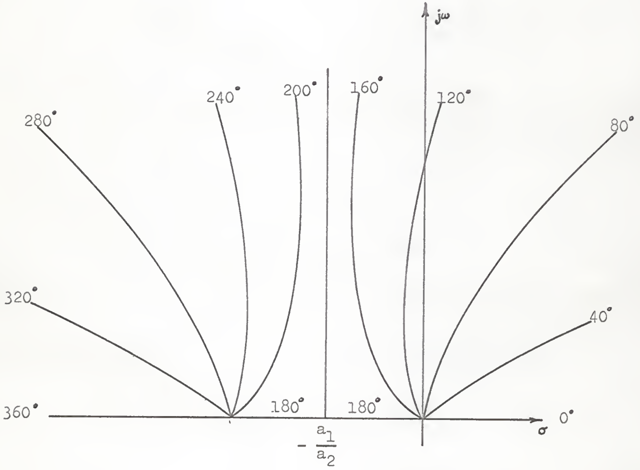
$$\Phi_1 = \measuredangle e^{+sT_0} = wT, \tag{14}$$

a family of loci is obtained as has been shown below.

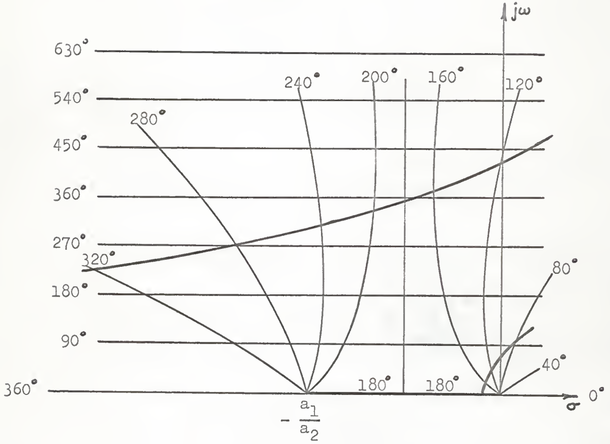


The remaining terms yield another family of curves given by

$$\Phi_2 = \measuredangle s + \measuredangle\left(s + \frac{a_1}{a_2}\right) . \tag{15}$$




By noting that $\Phi_1 + \Phi_2$ must equal $\pi \pm k2\pi$ and superimposing the two sets of curves as has been done on the following page, the complete locus may be easily plotted.




Since the curves are symmetrical about the real axis, only the upper half has been shown.

An important feature of this graph is that since the characteristic equation has an infinite number of roots, the number of branches of the locus is infinite.

Although a knowledge of the system's poles gives the designer a certain amount of qualitative information regarding the transient behavior, an expression for the time response is desirable.

D'Azzo and Houpis (5) have presented an approximate method for finding the time response which is quite useful in many systems with time delay. For the approximation to be sufficiently accurate, two requirements must be satisfied:

1. The system must have a dominant pair of complex poles with all other poles lying far to the left of this pair.
2. Any other pole which is not far to the left of the dominant complex poles must be near a zero so that the magnitude of the transient term due to that pole is small.

When these requirements are met, the time response for a general system,

$$\frac{C(s)}{R(s)} = \frac{P(s)}{Q(s)} = \frac{Kg\prod_{m=1}^{w}(s-z_m)}{\prod_{c=1}^{v}(s-p_c)}\ , \tag{16}$$

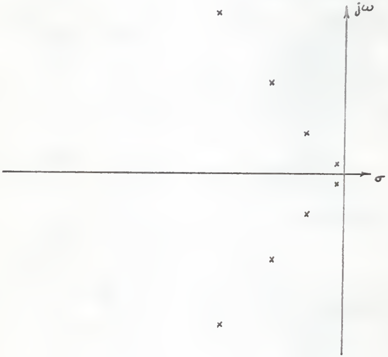
may be approximated by

$$c(t) = \frac{P(0)}{Q(0)} + 2\left|\frac{K_g \prod_{m=1}^{w}(p_1-z_m)}{p_1 \prod_{c=2}^{v}(p_1-p_c)}\right| e^{\sigma t}\cos\left[\omega_d t + \measuredangle P(p_1) - \measuredangle p_1 - \measuredangle Q'(p_1)\right], \tag{17}$$

where $p_1$ = the dominant complex pole = $\sigma + j\omega_d$

$$Q' = \frac{dQ}{ds}\ ,$$

One advantage of this technique is that values of the terms in the time response expression may be found graphically. Also, it is particularly applicable to the type of system under consideration since a plot of the poles of Eq. (1a) has the general form shown below.




Probably the most straight forward technique for computing the time response involves substituting a sufficient number of the system's infinite number of poles into the Heaviside expansion formula as presented by Tyner (3). The time response may be represented by

$$c(t) = \mathcal{L}^{-1}\left[\frac{G(s)}{1+A(s)} \cdot R(s)\right] = \mathcal{L}^{-1}\left[\frac{N(s)}{D(s)}\right] \qquad (18)$$

The Heaviside expansion formula may then be written as

$$c(t) = \sum_{k=1}^{n} \frac{N(s_k)}{D'(s_k)} e^{s_k t} \tag{19}$$

where the $s_k$'s are the poles.

This procedure requires a great amount of tedious calculations, but it has the advantage that a high degree of accuracy may be achieved simply by including more poles.

## METHOD DEVELOPMENT

### System Transformation

All reference to the second-order system under consideration thus far has been made to its most general case given by Eq. (1a). To simplify the discussion in the remaining portion of this text, it has been assumed that the system has unity feedback and is described by

$$\frac{C(s)}{R(s)} = \frac{a_o e^{-sT_o}}{a_2 s^2 + a_1 s + a_o e^{-sT_o}} . \tag{20}$$

No loss of generality is incurred by this since, as was described earlier, the response has been considered to be "per cent of c(t)ss", and the transient characteristic curves which have been constructed are applicable to either case.

By making use of the linear transformation,

$$s = \frac{a_1}{a_2} P, \tag{21}$$

the characteristic equation of Eq. (20) is transformed to

$$P^2 + P + Be^{-PT_o} = 0, \tag{22}$$

where
$$B = \frac{a_o a_2}{a_1^{\,2}} , \tag{23a}$$

and
$$T = \frac{a_1}{a_2} T_o \ . \tag{23b}$$

The clearest way to examine this transformation is to consider the time scale of the response to have been multiplied by the ratio, $a_1/a_2$. The characteristic equation may be written

$$\frac{a_2^{\,2}}{a_1^{\,2}} \frac{d^2c}{dt^2} + \frac{a_2}{a_1} \frac{dc}{dt} + \frac{a_o a_2}{a_1^{\,2}} c(t-T_o) = 0 \ . \tag{24}$$

With
$$t = \frac{a_2}{a_1} \tau \ , \tag{25}$$

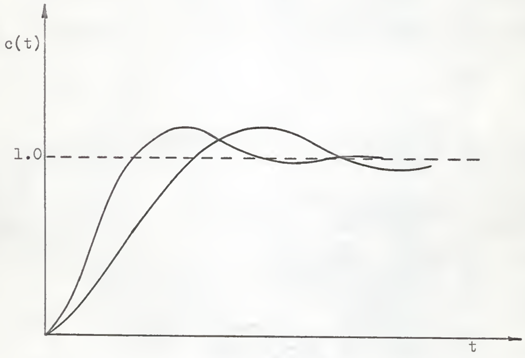
this equation becomes

$$\frac{d^2c}{d\tau^2} + \frac{dc}{d\tau} + \frac{a_o a_2}{a_1^{\,2}} c(\frac{a_2}{a_1}\tau - \frac{a_2}{a_1} T_o) = 0 \ . \tag{26}$$

A Laplace transform would then yield

$$s^2 + s + Be^{-sT} = 0 \; , \tag{27}$$

which is the same form obtained by setting $s = \frac{a_1}{a_2} P$. The time scaling approach, however, attaches a physical significance to the transformation which is illustrated in the figure below.




The overshoot of the system has not been affected. Rise time and settling time before and after the transformation are related as follows:

$$t_{rt} = \frac{a_2}{a_1} t_r \; , \tag{28a}$$

and $t_{st} = \frac{a_2}{a_1} t_r$ , (28b)

## Region of Stability

To define the region of stability on charts with coordinates, B and T, it is necessary to examine the general form of the transformed system's root locus. From the previous discussion of root locus, it should be recalled that a magnitude and a phase angle condition must be satisfied. For the transformed system, these two equations are

$$\left|s\right| \cdot \left|s+1\right| = \left|-Be^{-sT}\right| \tag{29a}$$

and $\measuredangle s + \measuredangle(s+1) + \measuredangle\ e^{+sT}) = \pi \pm k2\pi$ . (29b)

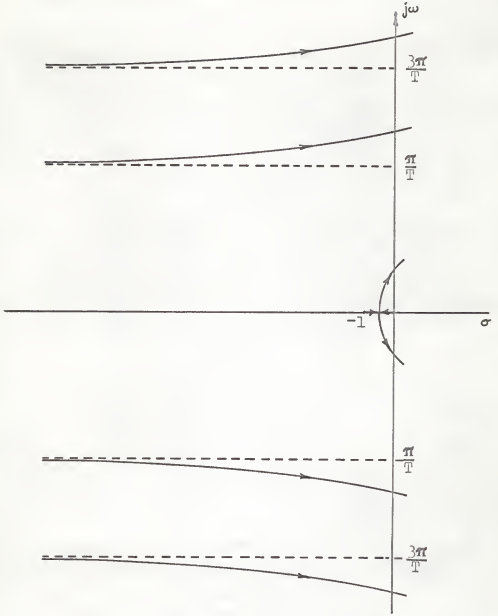
When the poles are complex, these equations become

$$\left[\sigma^{2} + \omega^2\right] \left[(1+\sigma)^2 + \omega^2\right] = B^2 e^{-2T\sigma} \tag{30a}$$

and $\tan^{-1}\left(\frac{\omega}{1+\sigma}\right) + \omega T + \tan^{-1}\left(\frac{\omega}{\sigma}\right) = \pm k2\pi$ for $-1<\sigma<o$ (30b)

$$\pi - \tan^{-1}\left(\frac{\omega}{-1-\sigma}\right) + \omega T + \tan^{-1}\left(\frac{\omega}{\sigma}\right) = \pm k2\pi \quad \text{for } \sigma<-1 \tag{30c}$$

As was explained in a previous section of this text, the root locus defined by the above equations has an infinite number of branches. The fundamental and a few higher branches of this root locus have been given on the following page.




The arrows in the figure indicate the directions in which the poles move as B is increased. A careful inspection of the plot and the magnitude equation reveals that for a given value of B the poles on any particular branch will always be farther to the left than the poles on lower branches.

Therefore, as B is increased, the poles moving along the fundamental branch will reach the imaginary axis first. Hence, only the fundamental branch need be considered in any discussion of stability.

When the poles lying on the fundamental branch reach the imaginary axis, their real part will be zero, and the magnitude and phase angle equations become, respectively,

$$(1 + \omega^2)\ (\omega^2) = B^2 \quad , \tag{31a}$$

and

$$\tan^{-1}\omega + \omega T = \frac{3\pi}{2} \quad . \tag{31b}$$

These two equations define the region of stability shown in Fig. 1.

Another curve on the B-T charts defines a region of special interest. When the value of B is such that the poles defined by the fundamental branch lie on the negative real axis, the response is of an overdamped nature. This is based on the fact that the real poles are much more dominant than the complex poles. For very large values of T, this is not necessarily true; but for the range of T considered in this text, the assumption is a good one.

When the poles of the fundamental branch lie on the real axis between 0 and -1, the magnitude equation becomes

$$(1 + \sigma)\ (-\sigma) = Be^{-\sigma T} \tag{32}$$

The point, $\sigma = \sigma_1$, for which B attains its greatest value, is the "breakaway point" of the system. These maximum values of B for various values of T define the line of critical damping shown in Fig. 1. In the charts for overshoot and rise time it was found that the underdamped region could be ignored. For the settling time charts, however, this region is of interest.

## Time Response

In order to derive relationships for overshoot, rise time, and settling time, a mathematical expression for the time response must be found. The approximation resulting from the substitution of eight poles into the Heaviside expansion formula was found to be best for the purposes of this study. Factors affecting this choice included the accuracy of the approximation and applicability for all values of B and T. This particular method also seemed to lend itself well to computer programing. By substituting the first eight system poles and the pole at s = 0 (resulting from the unit step input, $\frac{1}{s}$) into

$$c(t) = \sum_{k=1}^{9} \frac{Be^{-sT}}{s(s+1)} \cdot e^{st} , \tag{32}$$

the following expression for the time response was obtained:

$$c(t) = 1 + \frac{2B}{\sqrt{x_1^2 + y_1^2}} e^{\sigma_1(t-T)} \sin(\omega_1(t-T) + \alpha_1) + \ldots\ldots$$

$$+ \frac{2B}{\sqrt{x_4^2 + y_4^2}} e^{\sigma_4(t-T)} \sin(\omega_4(t-T) + \alpha_4), \tag{33}$$

where

$$x_n = 3\sigma_n^{\,2} - 3\omega_n^{\,2} + 2\sigma_n + Be^{-T\sigma_n}\cos\omega_n T - TBe^{-T\sigma_n}\sigma_n\cos\omega_n T$$

$$-TBe^{-\sigma_n T}\omega_n\sin\omega_n T \qquad (35)$$

$$y_n = -6\sigma_n\omega_n - 2\omega_n + Be^{-T\sigma_n}\sin(\omega_n T) - TBe^{-T\sigma_n}\sigma_n\sin\omega_n T$$

$$+TBe^{-T\sigma_n}\omega_n\cos\omega_n T \qquad (36)$$

$$\alpha_n = \frac{\pi}{2} + \tan^{-1} y_n/x_n \quad . \qquad (37)$$

Overshoot Chart

The equations defining the system's poles may be written as follows:

$$F(\omega,\sigma,T,k) = \tan^{-1}\left(\frac{\omega}{1+\sigma}\right) + \omega T + \tan^{-1}\left(\frac{\omega}{\sigma}\right) \pm k2\pi = 0$$

$$\text{for } -1 < \sigma < 0 \qquad (38a)$$

$$F(\omega,\sigma,T,k) = \pi + \tan^{-1}\left(\frac{\omega}{1+\sigma}\right) + \omega T + \tan^{-1}\left(\frac{\omega}{\sigma}\right) \pm k2\pi = 0$$

$$\text{for } \sigma < -1 \qquad (38b)$$

$$G(\omega,\sigma,B,T) = \frac{\sigma^4+2\sigma^3+\sigma^2+2\sigma^2\omega^2+2\sigma\omega^2+\omega^2+\omega^4}{e^{-2\sigma T}} - B^2 = 0 \quad . \qquad (38c)$$

For a particular branch of the root locus, k is fixed. Then with B and T given, there remains two equations in two variables. The Newton-Raphson method for two equations was employed to solve for $\omega$ and $\sigma$ .* With the poles determined for particular values of B and T, the system's time response, c(t), is defined. The Newton-Raphson method for one equation was then applied to $\dot{c}(t) = 0$ in order to find $t_{os}$.** With this value determined, the system's overshoot could be evaluated. The general procedure for constructing lines of constant overshoot consisted of selecting a value of T and evaluating the overshoot at several values of B.*** The results, when plotted on a graph with coordinates B and os, yielded a smooth curve from which B could be selected for any given overshoot. An example of one of the B-os curves is given in Fig. 2. From a set of these curves each representing a value of T, the lines of constant overshoot were plotted and are shown in Fig. 3.

## Rise Time Chart

The definition of rise time is often a matter of convenience. The definition employed here is the one given earlier in this text; the time required for the response to first reach its final value once it begins to respond to the input disturbance. The program for rise time remains much the same as the

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* A generalization of the Newton-Raphson method for two equations appears in Appendix A-II.

** The Newton-Raphson method for one equation appears in Appendix A-I.

*** Values were obtained with an IBM 1410 computer. Fortran programs for all charts are shown in Appendix B.

program for overshoot. After solving for the system's poles the Newton-Raphson method for one equation is again used to solve

$$c(t) - 1 = 0 \tag{39}$$

for the time when the response first reaches its final value. The system's rise time is then given by

$$t_{rt} = t_{(\text{final value})} - T \quad . \tag{40}$$

The general procedure again consists of selecting values of T and solving for $t_{rt}$ at various values of B. The results yielded smooth curves for B versus $t_{rt}$ from which the lines of constant rise time could be constructed. A sample B versus $t_{rt}$ curve and the constant rise time chart has been given in Figs. 4 and 5, respectively.

## Settling Time Charts

Settling time is defined as the time for the response to first reach and thereafter remain within a specified percentage of its final value. In this thesis five per cent was selected.

A common practice, when analyzing settling time, is to work with the envelope of the time response. Information obtained in this manner is just as useful, and the mathematics is greatly simplified.

Using this approach, the equation,

$$\text{envelope of } c(t) - .95 = 0 \ , \tag{41}$$

was solved for $t_{st}$ by means of the Newton-Raphson method for one equation.

The procedure again had the same general form; with T held fixed, $t_{st}$ was found for various values of B. Lines of constant settling time were then constructed from graphs of B versus $t_{st}$.

Two programs were actually used to obtain the settling time curves. One program located all values above the critical damping curve, and the second program was applied to the region below this curve. A sample B versus $t_{st}$ curve is given in Fig. 6 while the chart of constant settling time appears in Fig. 7.

## Discussion of Results

It is obvious from the transient response charts that their usefulness is restricted to systems for which B is less than 4 and T is less than 7. In the writer's opinion the majority of physical systems may be analyzed with these charts provided the time delay is not extremely large (i.e., $T_o$ less than 10 seconds). For very large T values of rise time and overshoot may be found without the aid of the charts. Feedback control systems with a transformed time delay, T, will behave as though they are open loop systems for $t_t < 2T$.

Therefore, for sufficiently large T and B, the rise time may be calculated easily from the open loop expression

$$\frac{C(s)}{R(s)} = \frac{Be^{-sT}}{s^2+s} \quad . \tag{42}$$

After construction of the B-os curves, it was noticed that as T became larger these curves became very linear. A careful examination of the curves yielded the empirical equation,

$$os = 1.1T(B - \frac{0.294}{T^{.8}}) \ . \tag{43}$$

No detailed evaluation of this expression for T greater than 7 was undertaken. It is the writer's belief, however, that where extremely accurate results are not required this expression can be of great value.

In discussing the accuracy of the transient response charts the object of greatest concern is the mathematical approximation for time response. When presenting information in a graphical form, the difficulty involved in obtaining exact results is in many cases unwarranted. It was the intention of the author to form an approximation for time response that would produce no readable error in any of the charts presented.

In order to examine the effect of neglected terms, one must study carefully the general form of the time response approximation.

$$c(t) = 1 + \dots + \frac{2B}{\sqrt{x_n^2 + y_n^2}} e^{\sigma_n(t-T)} \sin(\omega_n(t-T) + \alpha_n) + \dots \quad (44)$$

where

$$x_n = 3\sigma_n^2 - 3\omega_n^2 + 2\sigma_n + Be^{-T\sigma_n}\cos\omega_n T - TBe^{-T\sigma_n}\,{}_n\cos\omega_n T$$

$$-TBe^{-\sigma_n T}\omega_n \sin\omega_n T \quad (45)$$

$$y_n = -6\sigma_n\omega_n - 2\omega_n + Be^{-T\sigma_n}\sin\omega_n T - TBe^{-T\sigma_n}\sigma_n \sin\omega_n T$$

$$+TBe^{-T\sigma_n}\omega_n \cos\omega_n T \quad (46)$$

$$\alpha_n = \frac{\pi}{2} + \tan^{-1}\frac{y_n}{x_n} \quad (47)$$

The factor, $\sin(\omega_n(t-T) + \alpha_n)$, continues to oscillate between +1 and -1 and has no convergent effect on the general term. Since $\sigma_n$ is negative, however, $e^{\sigma_n(t-T)}$ begins to approach zero rapidly for t greater than T. In previous discussions of the systems root locus, it was pointed out that $\sigma_n < \sigma_{n-1}$. Thus each successive term is appreciably more damped than the one preceding it. Although not at once apparent the factor, $2B/\sqrt{x_n^2 + y_n^2}$, for a particular pair of complex poles was found to be much smaller than for a previous pair of poles. The net effect of this convergence of successive terms on the accuracy of the time response approximation can best be judged by a comparison of the approximate response with the exact time response found by means of the power series expansion method discussed previously. Comparisons were made for the following three points: (T = 0.50, B = 0.830); (T = 2.00, B = 0.30); (T = 5.50, B = 0.16). At each of these points, the approximate responses were found for fifteen values of time lying in the range between when the system began to respond and about the time overshoot occurred. The error did not exceed 0.005 for any of these values; and because of the convergent nature of the approximation, it may be assumed that the error is much less for larger values of time.

For all of the charts, computer results were found with a high degree of precision; and the only other possible source of error was the actual plotting of the curves. A great deal of effort was made to keep this at a minimum.

To the writer's knowledge no significant error exists in the charts given, and they may be used with confidence for the analysis and synthesis of automatic control systems.*

## CONCLUSION AND RECOMMENDATIONS

The analysis and synthesis of feedback control systems with time delay has been a challenging problem for many years. The recent increase in applications for control systems, particularly in the area of process control, has illustrated the need for additional design and analysis methods to cope with the problems presented by a delay in the control loop.

*An example of the synthesis procedure appears in Appendix C.

This thesis is concerned with the development of a method for the investigation of a control system's transient behavior. Discussion is centered about a second-order, linear system which is transformed such that it may be completely described in terms of a single constant, B, and a constant, T, related to the time delay. Lines of constant overshoot, rise time, and settling time are then presented on charts with coordinates, B and T. Although the ranges of B and T are limited on the graphs, it is felt that the information presented will be a valuable asset in the analysis and synthesis of the majority of second-order control systems.

As an extension of the work initiated in this text several areas could be investigated. In particular, the following recommendations are made:

1. The methods and techniques employed in this thesis be extended to higher order systems and systems with zeroes in their transfer functions.
2. An attempt be made to present transient information for an infinite range of B and T. If this is not possible, it would at least be useful to extend the investigation to larger values of B and T.

# ACKNOWLEDGMENT

The author wishes to express his gratitude to Dr. Kenneth K. Gowdy for his assistance and encouragement in the writing and organization of this thesis.

# REFERENCES


1. Raven, F. H.
   Automatic control systems. New York: McGraw-Hill, 1961.

2. Pontryagin, L. S.
   On the zeroes of some elementary transcendental functions. American Mathematical Society Translations, Series 2, Volume 1, 1955.

3. Tyner, M.
   Computing the time response of a deadtime process. Control Engineering, Volume II, Number 3, 1964.

4. Chu, Y.
   Feedback control systems with deadtime lag or distributed lag by root-locus method. Transactions of the American Institute of Electrical Engineers, Volume 71, Part II, 1951.

5. D'Azzo, J. J. and C. H. Houpis.
   Feedback control system analysis and synthesis. New York: McGraw-Hill, 1960.

6. McCracken, D. D. and W. S. Dorn.
   Numerical methods and fortran programming. New York: John Wiley and Sons, Inc., 1964.

7. Gowdy, K. K.
   A method for the analysis and synthesis of linear third-order systems. Oklahoma State University: Ph.D. Dissertation, 1965.

# NOMENCLATURE

| | |
|---|---|
| $a_n$ | Coefficient of general system |
| B | Coefficient for transformed system |
| c | Time response of system |
| C | Laplace transform of system's output |
| k | Forward gain constant of general system |
| $k_1$ | Feedback gain constant of general system |
| os | Per cent overshoot |
| R | Laplace transform of system's input |
| s | Laplace transform complex variable |
| $t_r$ | Rise time for general system |
| $t_s$ | Settling time for general system |
| $t_{rt}$ | Rise time for transformed system |
| $t_{st}$ | Settling time for transformed system |
| $t_t$ | Transformed time |
| $t_{os}$ | Time at which overshoot occurs |
| | Imaginary part of s |
| | Real part of s |
| $T_o$ | Actual time delay |
| T | Transformed time delay |
| $J_t$ | System's inertia |
| $f_t$ | System's viscous friction |
| $K_t$ | Motor constant |
| $K_a$ | Amplifier gain |
| v | Velocity of material |
| d | Distance between controlling point and measuring point |

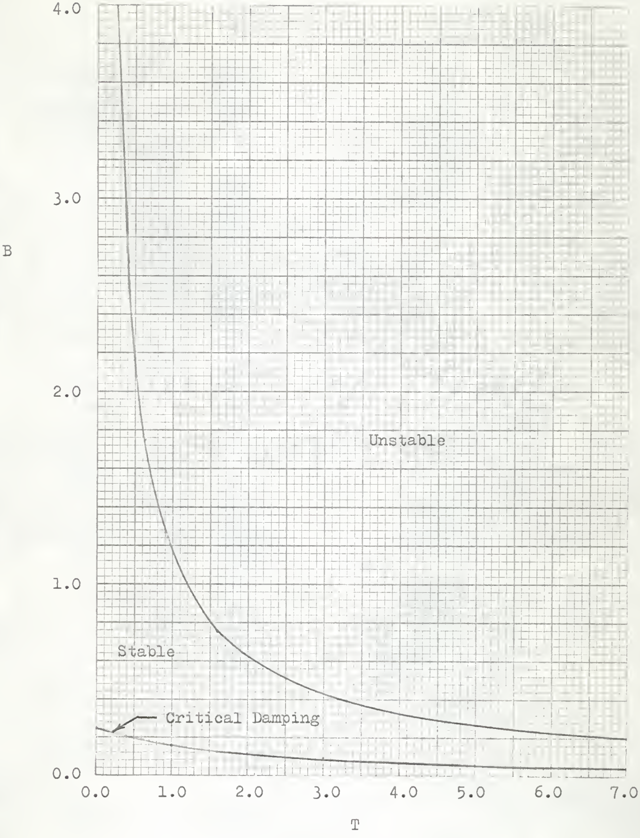



Fig. 1 Region of Stability

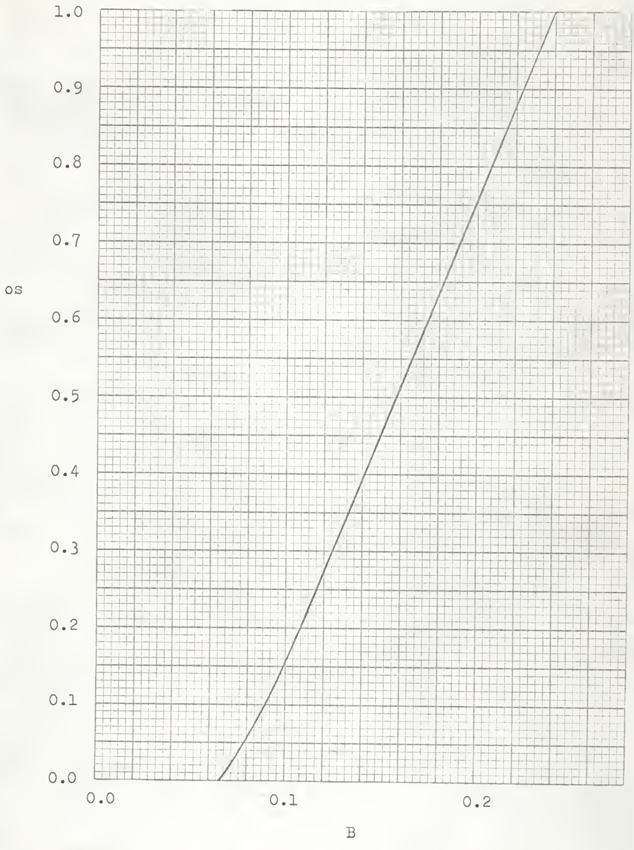



Fig. 2 Overshoot Curve For T = 5.5

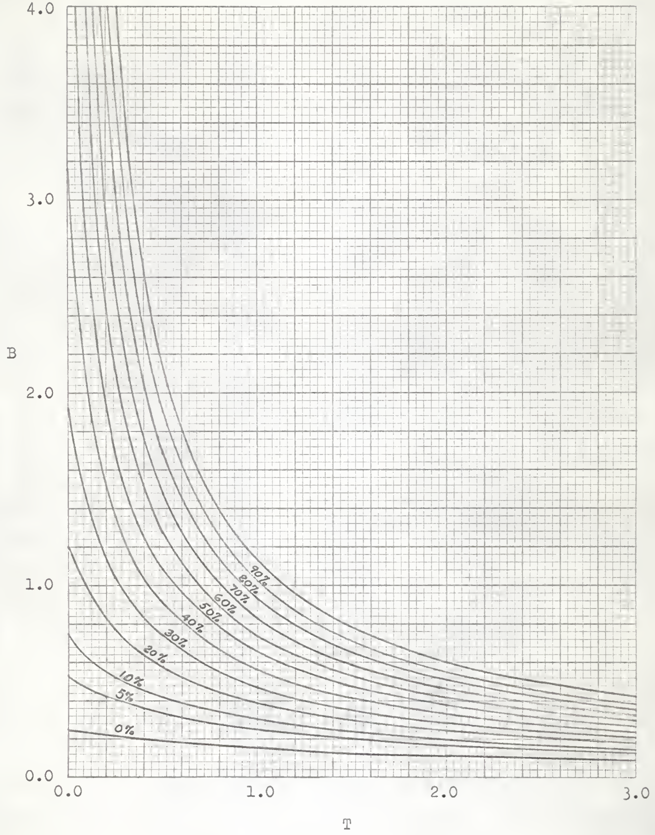



Fig. 3 Lines of Constant Overshoot

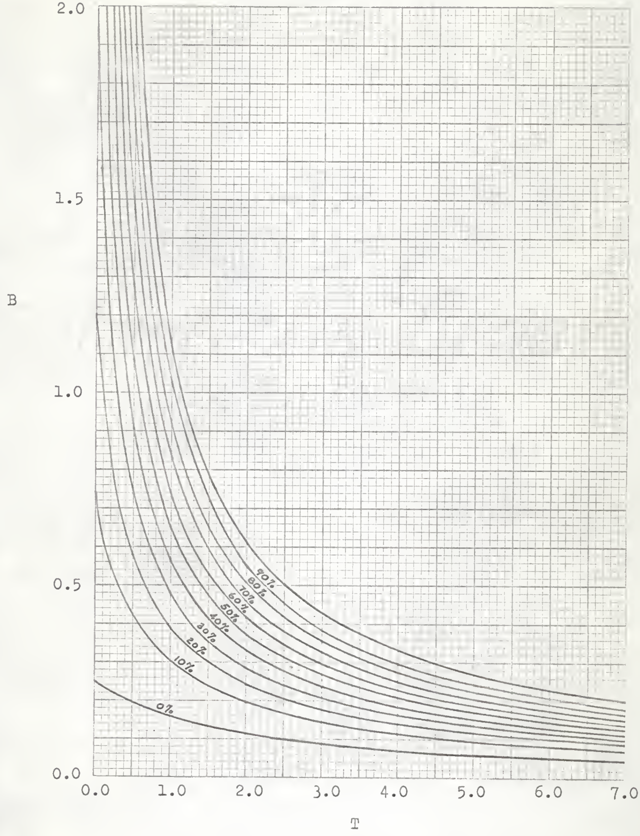



Fig. 3 (Continued)

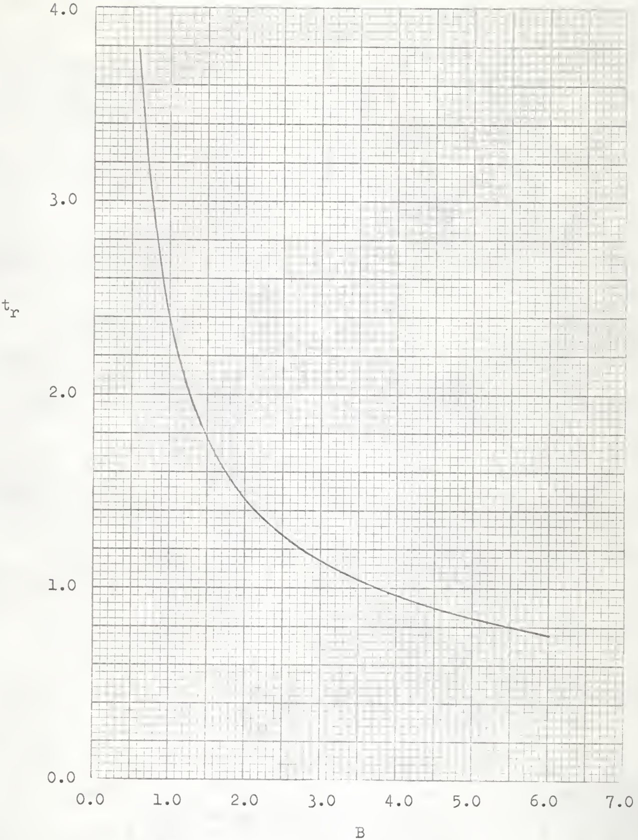



Fig. 4 Rise Time Curve for T = .062

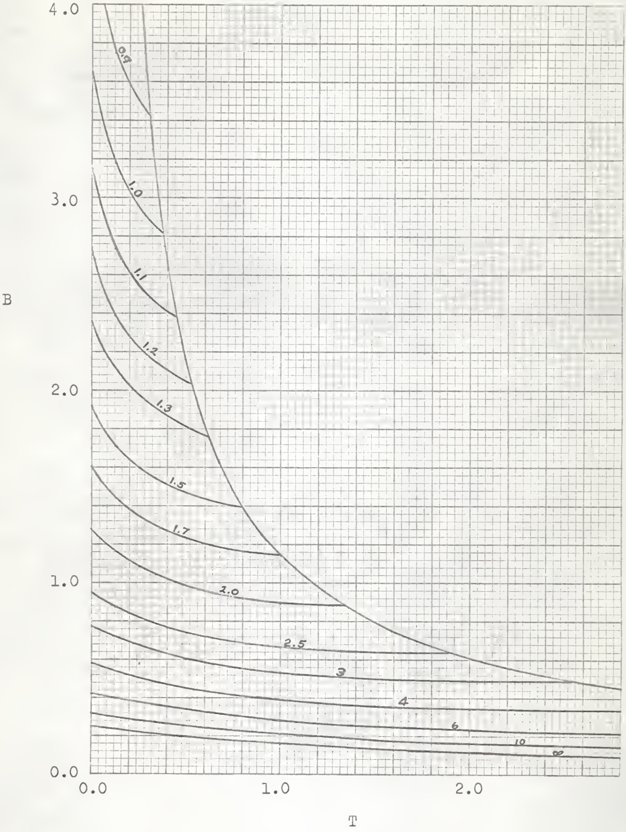



Fig. 5 Lines of Constant Rise Time

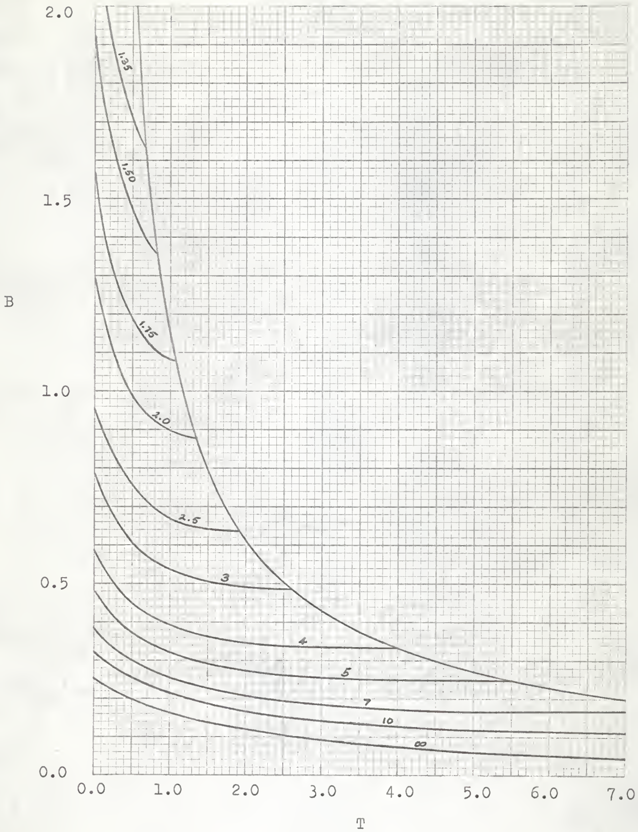



Fig. 5 (Continued)

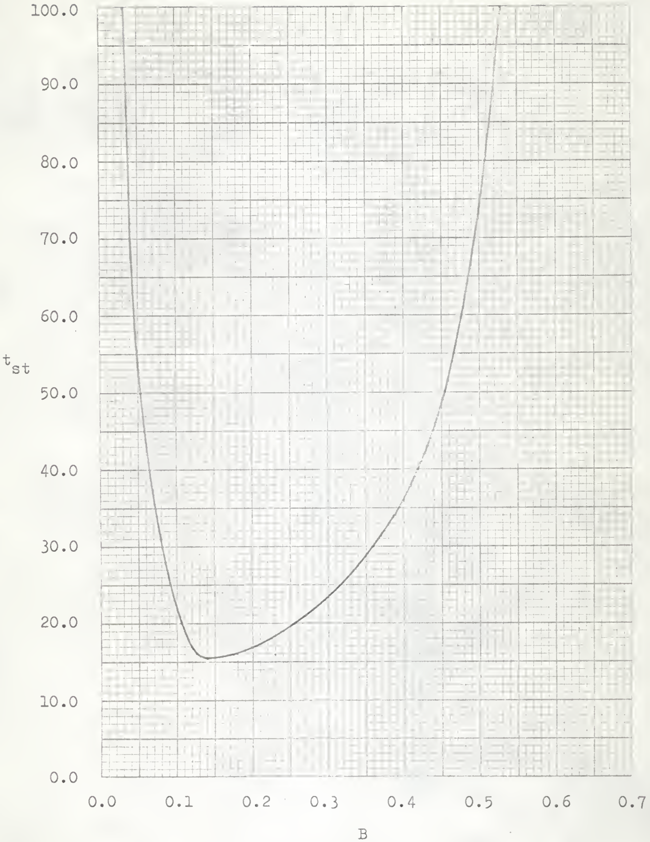



Fig. 6 Settling Time Curve for T = 2.00

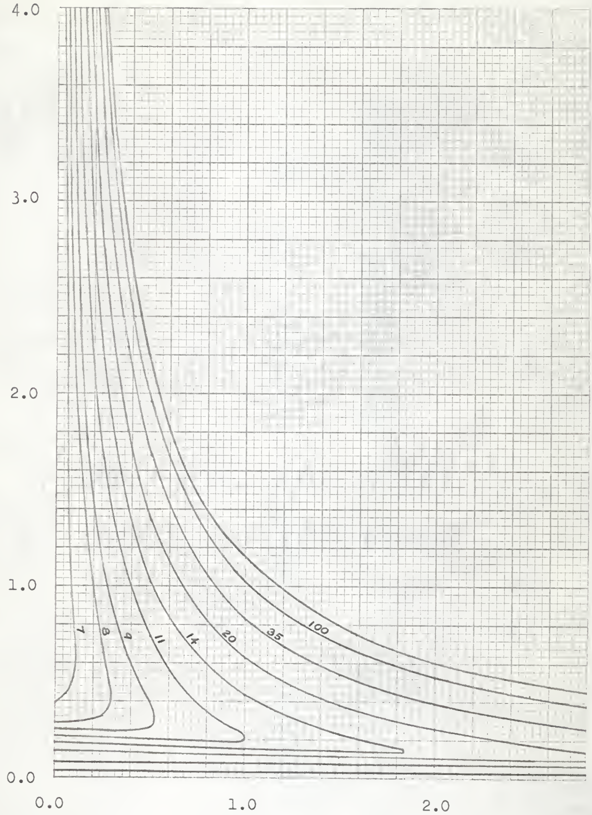



Fig. 7 Lines of Constant Settling Time

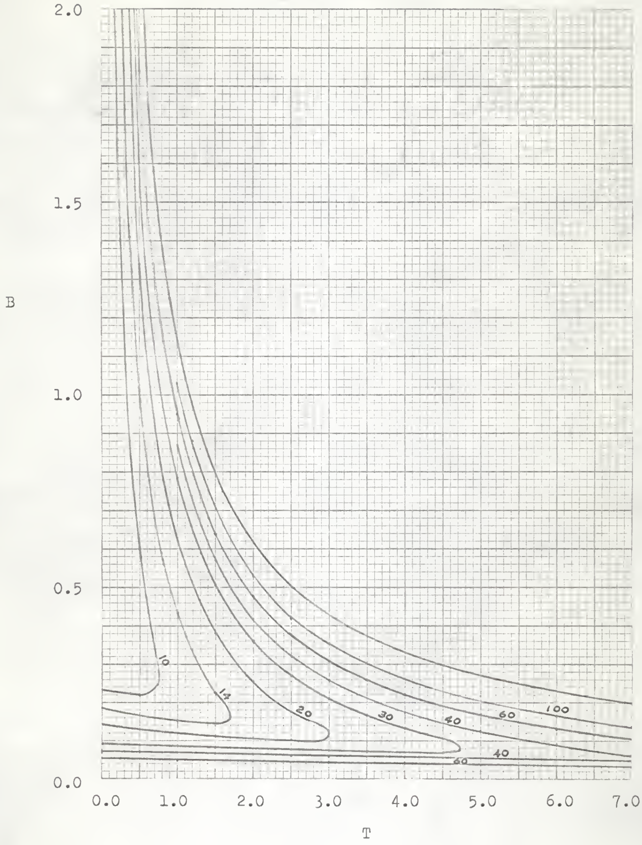



Fig. 7 (Continued)

APPENDIX A-I

## NEWTON-RAPHSON METHOD FOR ONE EQUATION

Given:

$$F(t) = 0$$

Development of method:

1. Expand $F(t)$ in a Taylor series about $t = t_k$ and truncate after two terms.

$$F(t) = F(t_k) + (t-t_k)\,\frac{dF}{dt}\,(t_k)$$

2. Solve for t

$$t = t_k - \frac{F(t_k)}{\dfrac{dF(t_k)}{dt}}$$

Procedure:

1. Let $t_k$ be an initial approximation to the solution of the given equation.
2. As the next approximation, take

$$t_{k+1} = t_k - \frac{F(t_k)}{\dfrac{dF(t_k)}{dt}}$$

3. Continue iterations until the process converges to a solution for t.

Reference: McCracken and Dorn (6), pp. 133, 156.

APPENDIX A-II

# GENERALIZATION OF NEWTON-RAPHSON METHOD FOR TWO EQUATIONS

Given:

$$F(\omega,\sigma) = 0$$

$$G(\omega,\sigma) = 0 .$$

Development of method:

1. Expand both $F(\omega,\sigma)$ and $G(\omega,\sigma)$ in Taylor series about $\omega = \omega_k, \sigma = \sigma_k$ and truncate after two terms. For simplification let

$$F = F(\omega,\sigma) \qquad F_k = F(\omega_k,\sigma_k)$$

$$G = G(\omega,\sigma) \qquad G_k = G(\omega_k,\sigma_k)$$

then

$$F = F_k + (\omega-\omega_k)\frac{\partial F_k}{\partial \omega} + (\sigma-\sigma_k)\frac{\partial F_k}{\partial \sigma} + \ldots = 0$$

$$G = G_k + (\omega-\omega_k)\frac{\partial G_k}{\partial \omega} + (\sigma-\sigma_k)\frac{\partial G_k}{\partial \sigma} + \ldots = 0$$

2. Rearrange with $F_k$ and $G_k$ on the right

$$(\omega-\omega_k)\frac{\partial F_k}{\partial \omega} + (\sigma-\sigma_k)\frac{\partial F_k}{\partial \sigma} = -F_k$$

$$(\omega-\omega_k)\frac{\partial G_k}{\partial \omega} + (\sigma-\sigma_k)\frac{\partial G_k}{\partial \sigma} = -G_k$$

3. Use Cramer's rule to solve for $\omega$ and $\sigma$.

$$\omega = \omega_k - \left[ F_k \frac{\partial G_k}{\partial \sigma} - G_k \frac{\partial F_k}{\partial \sigma} \right] \Big/ J$$

$$\sigma = \sigma_k + \left[ F_k \frac{\partial G_k}{\partial \omega} - G_k \frac{\partial F_k}{\partial \omega} \right] \Big/ J$$

where

$$J = \frac{\partial F_k}{\partial \omega} \frac{\partial G_k}{\partial \sigma} - \frac{\partial F_k}{\partial \sigma} \frac{\partial G_k}{\partial \omega} .$$

Procedure:

1. Let $\omega_k$ and $\sigma_k$ be initial approximations for the solution of the given equations.
2. Use the equations above to solve for the next approximations $\omega_{k+1}$ and $\sigma_{k+1}$.
3. Continue the iterations until two successive approximations are found to be sufficiently close to each other.

Reference: McCracken and Dorn (6), pp. 144-145, 156-157.

APPENDIX B-I

FORTRAN PROGRAM FOR VALUES OF OVERSHOOT

```
      DIMENSION TA(12)
      DIMENSION BBA(12,9),TOS(12,9)
      DIMENSION WTA(12,9,4),STA(12,9,4)
      DIMENSION WA(4),SA(4),X(4),Y(4)
      DIMENSION AL(4),D(4),FA(4)
 1    FORMAT(2XF8.3)
 2    FORMAT(2X,F6.3)
 3    FORMAT(2X,F10.3)
 4    FORMAT(2X,4F7.2)
 5    FORMAT(2X,4F29.6)
 6    FORMAT(2X,3F30.10)
 7    FORMAT(15X,F15.5)
      READ(1,1)TA
      DO8K=1,12
 8    READ(1,2)(BBA(K,J),J=1,9)
      DO9K=1,12
 9    READ(1,3)(TOS(K,J),J=1,9)
      DO10K=1,12
      DO10J=1,9
10    READ(1,4)(WTA(K,J,L),L=1,4)
      DO11K=1,12
      DO11J=1,9
11    READ(1,4)(STA(K,J,L),L=1,4)
      DO28K=1,12
      T=TA(K)
      WRITE(3,1)T
      DO27J=1,9
      B=BBA(K,J)
      WRITE(3,1)B
      DO18L=1,4
      BP=L-1
      WB=WTA(K,J,L)
      SB=-STA(K,J,L)
12    WK=WB
      SK=SB
      G=(SK**4+2.*SK**3+SK**2+2.*SK*SK*WK*WK+2.*SK*WK*WK+WK
        **2+WK**4)/(2.7183**(-2.*SK*T))-B*B
      GS=2.*T*(G+B*B)+4.*SK**3+6.*SK*SK+2.*SK+4.*SK*WK*WK+
         2.*WK*WK)/(2.7183**(-2.*SK*T))
      GW=(4.*SK*SK*WK+4.*SK*WK+2.*WK+4.*WK**3)/(2.7183**(-2.
         *SK*T))
      FS=-WK/((1.+SK)**2+WK*WK)-WK/(SK*SK+WK*WK)
      FW=(1.+SK)/((1.+SK)**2+WK*WK)+T+SK/(SK*SK+WK*WK)
      P=FW*GS-FS*GW
      IF(-SK-1.)13,14,15
13    F=ATAN(WK/(1.+SK))+WK*T-ATAN(-WK/SK)-BP*6.2832
      GOTO16
14    F=1.5708+WK*T-ATAN(WK)-BP*6.2832
      GOTO16
```

```
15  F=3.1416-ATAN(WK/(-1.-SK))+WK*T-ATAN(-WK/SK)-BP*6.2832
16  WB=WK-(F*GS-G*FS)/P
    SB=SK+(F*GW-G*FW)/P
    WRITE(3,5)WB,SB,G,F
    IF(ABS(WB-WK)+ABS(SB-SK)-.001)17,17,12
17  WA(L)=WB
18  SA(L)=SB
    DO21L=1,4
    W=WA(L)
    S=SA(L)
    X(L)=3.*S*S-3.*W*W+2.*S+B*2.7183**(-T*S)*COS(W*T)-T*B*
         2.7183**(-T*S)*S*COS(W*T)-T*B*2.7183**(-T*S)*W*
         SIN(W*T)
    Y(L)=-6.*S*W-2.*W+B*2.7183**(-T*S)*SIN(W*T)-T*B*2.7183
         **(-T*S)*S*SIN(W*T)+T*B*2.7183**(-T*S)*W*COS(W*T)
    IF(X(L)-0.0)20,19,19
19  AL(L)=ATAN(Y(L)/X(L))+1.5708
    GOTO21
20  AL(L)=4.7124-ATAN(Y(L)/(-X(L)))
21  WRITE(3,6)AL(L),X(L),Y(L)
    TO=TOS(K,J)
    IN=1
22  TK=TO
    IN=IN+1
    IF(IN-12)23,23,27
23  DO26I=1,4
    IF(ABS(SA(I)*TK-SA(I)*T)-220.)25,25,24
24  D(I)=0.0
    FA(I)=0.0
    GOTO26
25  D(I)=2.*B/(SQRT(X(I)*X(I)+Y(I)*Y(I)))*2.7183**(SA(I)*
         TK-SA(I)*T)*SIN(WA(I)*(TK-T)+AL(I))
    FA(I)=2.*B/(SQRT(X(I)*X(I)+Y(I)*Y(I)))*2.7183**(SA(I)
          *TK-SA(I)*T)*COS(WA(I)*(TK-T)+AL(I))
26  CONTINUE
    WRITE(3,5)D(1),D(2),D(3),D(4)
    TO=TK-(D(1)*SA(1)+FA(1)*WA(1)+D(2)*SA(2)+FA(2)*WA(2)+
       D(3)*SA(3)+FA(3)*WA(3)+D(4)*SA(4)+FA(4)*WA(4))/(D(
       1)*SA(1)*SA(1)+FA(1)*SA(1)*WA(1)*2.-D(1)*WA(1)*WA(
       1)+D(2)*SA(2)*SA(2)+FA(2)*SA(2)*WA(2)*2.-D(2)*WA(2)
       *WA(2)+D(3)*SA(3)*SA(3)+FA(3)*SA(3)*WA(3)*2.-D(3)*
       WA(3)*WA(3)+D(4)*SA(4)*SA(4)+FA(4)*SA(4)*WA(4)*2.-
       D(4)*WA(4)*WA(4))
    WRITE(3,7)TO
    IF(ABS(TO-TK)-.001)30,30,22
30  OS=D(1)+D(2)+D(3)+D(4)
    WRITE(3,7)OS
27  CONTINUE
28  CONTINUE
    END
```

# APPENDIX B-II

FORTRAN PROGRAM FOR VALUES OF RISE TIME

```
      DIMENSION TA(12)
      DIMENSION BBA(12,9),TRA(12,9)
      DIMENSION WTA(12,9,4),STA(12,9,4)
      DIMENSION WA(4),SA(4),S(4),Y(4)
      DIMENSION AL(4),D(4),FA(4)
    1 FORMAT(2X,F8.3)
    2 FORMAT(2X,F6.3)
    3 FORMAT(2X,F10.3)
    4 FORMAT(2X,4F7.2)
    5 FORMAT(2X,4F29.6)
    6 FORMAT(2X,3F30.10)
    7 FORMAT(15X,F15.5)
      READ(1,1)TA
      DO8K=1,12
    8 READ(1,2)(BBA(K,J),J=1,9)
      DO10K=1,12
    9 READ(1,3)(TRA(K,J),J=1,9)
      DO10K=1,12
      DO10J=1,9
   10 READ(1,4)(WTA(K,J,L),L=1,4)
      DO11K=1,12
      DO11J=1,9
   11 READ(1,4)(STA(K,J,L),L=1,4)
      DO28K=1,12
      T=TA(K)
      WRITE(3,1)T
      DO27J=1,9
      B=BBA(K,J)
      WRITE(3,1)B
      DO18L=1,4
      BP=L-1
      WB=WTA(K,J,L)
      SB=-STA(K,J,L)
   12 WK=WB
      SK=SB
      G=(SK**4+2.*SK**3+SK**2+2.*SK*SK*WK*WK+2.*SK*WK*WK+WK
        **2+WK**4)/(2.7183**(-2.*SK*T))-B*B
      GS=2.*T*(G+B*B)+(4.*SK**3+6.*SK*SK+2.*SK+4.*SK*WK*WK+
         2.*WK*WK)/(2.7183**(-2.*SK*T))
      GW=(4.*SK*SK*WK+4.*SK*WK+2.*WK+4.*WK**3)/(2.7183**(-2.
         *SK*T))
      FS=-WK/((1.+SK)**2+WK*WK-WK/(SK*SK+WK*WK)
      FW=(1.+SK)/((1.+SK)**2+WK*WK)+T+SK/(SK*SK+WK*WK)
      P=FW*GS-FS*GW
      IF(-SK-1.)13,14,15
   13 F=ATAN(WK/(1.+SK))+WK*T-ATAN(-WK/SK)-BP*6.2832
      GOTO16
   14 F=1.5708+WK*T-ATAN(+WK)-BP*6.2832
      GOTO16
   15 F=3.1416-ATAN(WK/(-1.-SK))+WK*T-ATAN(-WK/SK)-BP*6.2832
```

```
16  WB=WK-(F*GS-G*FS)/P
    SB=SK+(F*GW-G*FW)/P
    WRITE(3,5)WB,SB,G,F
    IF(ABS(WB-WK)+ABS(SB-SK)-.001)17,17,12
17  WA(L)=WB
18  SA(L)=SB
    DO21L=1,4
    W=WA(L)
    S=SA(L)
    X(L)=3.*S*S-3.*W*W+2.*S+B*2.7183**(-T*S)*COS(W*T)-T*
         B*2.7183**(-T*S)*S*COS(W*T)-T*B*2.7183**(-T*S)*
         W*SIN(W*T)
    Y(L)=-6.*S*W-2.*W+B*2.7183**(-T*S)*SIN(W*T)-T*B*2.7183
         **(-T*S)*S*SIN(W*T)+T*B*2.7183**(-T*S)*W*COS(W*T)
    IF(X(L)-0.0)20,19,19
19  AL(L)=ATAN(Y(L)/X(L))+1.5708
    GOTO21
20  AL(L)=4.7124-ATAN(Y(L)/(-X(L)))
21  WRITE(3,6)AL(L),X(L),Y(L)
    TR=TRA(K,J)
    IN=1
22  TK=TR
    IN=IN+1
    IF(IN-12)23,23,27
23  DO26I=1,4
    IF(ABS(SA(I)*TK-SA(I)*T)-220.)25,25,24
24  D(I)=0.0
    FA(I)=0.0
    GOTO26
25  D(I)=2.*B/(SQRT(X(I)*X(I)+Y(I)*Y(I)))*2.7183**(SA(I)*
         TK-SA(I)*T)*SIN(WA(I)*(TK-T)+AL(I))
    FA(I)=2.*B/(SQRT(X(I)*X(I)+Y(I)*Y(I)))*2.7183**(SA(I)
          *TK-SA(I)*T)*COS(WA(I)*(TK-T)+AL(I))
26  CONTINUE
    WRITE(3,5)D(1),D(2),D(3),D(4)
    TR=TK-(D(1)+D(2)+D(3)+D(4))/(D(1)*SA(1)+FA(1)*WA(1)+D
       (2)*SA(2)+FA(2)*WA(2)+D(3)*SA(3)+FA(3)*WA(3)+D(4)*
       SA(4)+FA(4)*WA(4))
    WRITE(3,7)TR
    IF(ABS(TR-TK)-.001)27,27,22
27  CONTINUE
28  CONTINUE
    END
```

APPENDIX B-III

FORTRAN PROGRAM FOR VALUES OF UNDERDAMPED SETTLING TIME

```
      DIMENSION TA(12)
      DIMENSION BBA(12,3),TSA(12,3)
      DIMENSION WTA(12,3,4),STA(12,3,4)
      DIMENSION WA(4),SA(4),X(4),Y(4)
      DIMENSION AL(4),D(4)
    1 FORMAT(2X,F8.3)
    2 FORMAT(2X,F6.3)
    3 FORMAT(2X,F10.3)
    4 FORMAT(2X,4F7.2)
    5 FORMAT(2X,4F29.6)
    6 FORMAT(2X,3F30.10)
    7 FORMAT(15X,F15.5)
      READ(1,1)TA
      DO8K=1,12
    8 READ(1,2)(BBA(K,J),J=1,3)
      DO9K=1,12
    9 READ(1,3)(TSA(K,J),J=1,3)
      DO10K=1,12
      DO10J=1,3
   10 READ(1,4)(WTA(K,J,L),L=1,4)
      DO11K=1,12
      DO11J=1,3
   11 READ(1,4)(STA(K,J,L),L=1,4)
      DO28K=1,12
      T=TA(K)
      WRITE(3,1)T
      DO27J=1,3
      B=BBA(K,J)
      WRITE(3,1)B
      DO18L=1,4
      BP=L-1
      WB=WTA(K,J,L)
      SB=-STA(K,J,L)
   12 WK=WB
      SK=SB
      G=(SK**4+2.*SK**3+SK**2+2.*SK*SK*WK*WK+2.*SK*WK*WK+WK
        **2+WK**4)/(2.7183**(-2.*SK*T))-B*B)
      GS=2.*T*(G+B*B)+(4.*SK**3+6.*SK*SK+2.*SK+4.*SK*WK*WK+
         2.*WK*WK(/(2.7183**(-2.*SK*T))
      GW=(4.*SK*SK*WK+4.*SK*WK+2.*WK+4.*WK**3)/(2.7183**(-2.
         *SK*T))
      FS=-WK/((1.+SK)**2+WK*WK)-WK/(SK*SK+WK*WK)
      FW=(1.+SK)/((1.+SK)**2+WK*WK)+T+SK/(SK*SK+WK*WK)
      P=FW*GS-FS*GW
      IF(-SK-1.)13,14,15
   13 F=ATAN(WK/(1.+SK))+WK*T-ATAN(-WK/SK)-BP*6.2832
      GOTO16
   14 F=1.5708+WK*T-ATAN(WK)-BP*6.2832
```

```
      GOTO16
15    F=3.1416-ATAN(WK/(-1.-SK))+WK*T-ATAN(-WK/SK)-BP*6.2832
16    WB=WK-(F*GS-G*FS)/P
      SB=SK+F*GW-G*FW)/P
      WRITE(3,5)WB,FB,G,F
      IF(ABS(WB-WK)+ABS(SB-SK)-.001)17,17,12
17    WA(L)=WB
18    SA(L)=SB
      DO21L=1,4
      W=WA(L)
      S=SA(L)
      X(L)=3.*S*S-3.*W*W+2.*S+B*2.7183**(-T*S(*COS(W*T)-T*B*
           2.7183**(-T*S)*S*COS(W*T)-T*B*2.7183**(-T*S)*W*SIN
           (W*T)
      Y(L)=-6.*S*W-2.*W+B*2.7183**(-T*S)*SIN(W*T)-T*B*2.7183
           **(-T*S)*S*SIN(W*T)+T*B*2.7183**(-T*S)*W*COS(W*T)
      IF(X(L)-0.0)20,19,19
19    AL(L)=ATAN(Y(L)/X(L))+1.5708
      GOTO21
20    AL(L)=4.7124-ATAN(Y(L)/(-X(L)))
21    WRITE(3,6)AL(L),X(L),Y(L)
      TS=TSA(K,J)
      IN=1
22    TK=TS
      IN=IN+1
      IF(IN-12)23,23,27
23    DO26I=1,4
      IF(ABS(SA(I)*TK-SA(I)*T)-220.)25,25,24
24    D(I)=0.0
      GOTO26
25    D(I)=2.*B/(SQRT(X(I)*X(I)+Y(I)*Y(I)))*2.7183**(SA(I)*
           TK-SA(I)*T)
26    CONTINUE
      WRITE(3,5)D(1),D(2),D(3),D(4)
      TS=TK-(-.05+D(1)+D(2)+D(3)+D(4))/(D(1)*SA(1)+FA(1)*WA
         (1)+D(2)*SA(2)+FA(2)*WA(2)+D(3)*SA(3)+FA(3)*WA(3)+
         D(4)*SA(4)+FA(4)*WA(4))
      WRITE(3,7)TS
      IF(ABS(TS-TK)-.001)27,27,22
27    CONTINUE
28    CONTINUE
      END
```

APPENDIX B-IV

FORTRAN PROGRAM FOR VALUES OF OVERDAMPED SETTLING TIME

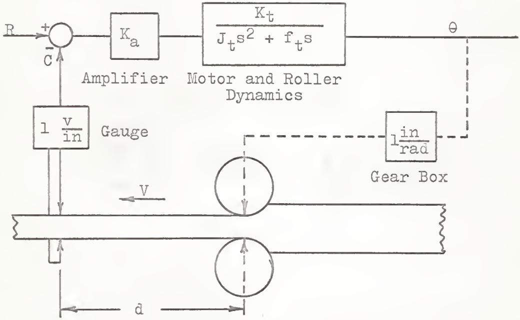
```
      DIMENSION TA(12)
      DIMENSION BBA(12,9),TSA(12,9)
      DIMENSION WTA(12,9,4),STA(12,9,4)
      DIMENSION WA(4),SA(4),X(4),Y(4)
      DIMENSION AL(4),D(4)
    1 FORMAT(2X,F8.3)
    2 FORMAT(2X,4F9.4)
    3 FORMAT(2X,3F30.10)
    4 FORMAT(20X,F10.5)
    5 FORMAT(2X,4F29.6)
      READ(1,1)TA
      DO6K=1,12
    6 READ(1,1)(BBA(K,J),J=1,9)
      DO7K=1,12
    7 READ(1,1)(TSA(K,J),J=1,9)
      DO8K=1,12
      DO8J=1,9
    8 READ(1,2)(WTA(K,J,L),L=1,4)
      DO9K=1,12
      DO9J=1,9
    9 READ(1,2)(STA(K,J,L),L=1,4)
      DO34K=1,12
      T=TA(K)
      WRITE(3,1)T
      DO33L=1,9
      BA=BBA(K,L)
      WRITE(3,1)BA
      DO19I=1,4
      BP=I-1
      IF(I-1)10,10,11
   10 WA(I)=WTA(K,L,I)
      SB(I)=STA(K,L,I)
      GOTO19
   11 WB=WTA(K,L,I)
      SB=-STA(K,L,I)
   12 WK=WB
      SK=SB
      G=(SK**4+2.*SK**3+SK**2+2.*SK*SK*WK*WK+2.*SK*WK*WK+WK
     **2+WK**4)/(2.7183**(-2.*SK*T))-BA*BA
      GS=2.*T*(G+BA*BA)+(4.*SK**3+6.*SK*SK+2.*SK+4.*SK*WK*WK
     +2.*WK*WK)/(2.7183**(-2.*SK*T))
      GW=(4.*SK*SK*WK+4.*SK*WK+2.*WK+4.*WK**3)/(2.7183**(-2.
     *SK*T))
      FS=-WK/((1.+SK)**2+WK*WK)-WK/(SK*SK+WK*WK)
      FW=(1.+SK)/((1.+SK)**2+WK*WK)+T+SK/(SK*SK+WK*WK)
      P=FW*GS-FS*GW
      IF(-SK-1.)13,14,15
   13 F=ATAN(WK/(1.+SK))+WK*T-ATAN(-WK/SK)-BP*6.2832
      GOTO16
   14 F=1.5708+WK*T-ATAN(WK)-BP*6.2832
```

```
      GOTO16
15    F=3.1416-ATAN(WK/(-1.-SK))+WK*T-ATAN(-WK/SK)-BP*6.2832
16    WB=WK-(F*GS-G*FS)/P
      SB=SK+(F*GW-G*FW)/P
      WRITE(3,5)WB,SB,G,F
      IF(ABS(WB-WK)+ABS(SB-SK)-.001)17,17,12
17    WA(I)=WB
18    SA(I)=SB
19    CONTINUE
      DO22I=1,4
      W=WA(I)
      S=SA(I)
      X(I)=3.*S*S-3.*W*W+2.*S+B*2.7183**(-T*S)*COS(W*T)-T*B*
           2.7183**(-T*S)*S*COS(W*T)-T*B*2.7183**(-T*S)*W*SIN
           (W*T)
      Y(I)=-6.*S*W-2.*W+B*2.7183**(-T*S)*SIN(W*T)-T*B*2.7183*
           *(-T*S)*S*SIN(W*T)+T*B*2.7183**(-T*S)*W*COS(W*T)
      IF(X(I)-0.0)21,20,20
20    AL(I)=ATAN(Y(I)/X(I))+1.5708
      GOTO22
21    AL(I)=4.7124-ATAN(Y(I)/(-X(I)))
22    WRITE(3,3)AL(I),X(I),Y(I)
      TS=TSA(K,L)
      IN=1
23    TK=TS
      IN=IN+1
      IF(IN-20)24,24,32
24    DO31I=1,4
      IF(I-1)25,25,28
25    IF(ABS(SA(I)*TK-SA(I)*T)-220.)26,32,32
26    IF(ABS(WA(I)*TK-WA(I)*T)-220.)27,32,32
27    SAM=(B*2.7183**(SA(I)*(TK-T)))/(3.*SA(I)*SA(I)+2.*SA(I)
          +(B-SA(I)*T*B)*2.7183**(-T*SA(I)))
      WAM=(B*2.7183**(WA(I)*(TK-T)))/(3.*WA(I)*WA(I)+2.*WA(I)
          +(B-WA(I)*T*B)*2.7183**(-T*WA(I)))
      D(I)=SAM+WAM
      GOTO31
28    IF(ABS(SA(I)*TK-SA(I)*T)-220.)30,30,29
29    D(I)=0.0
      GOTO31
30    D(I)=2.*B/(SQRT(X(I)*X(I)+Y(I)*Y(I)))*2.7183**(SA(I)*TK
          -SA(I)*T)
31    CONTINUE
      TS=TK-(.05+D(1)-D(2)-D(3)-D(4))/(SAM*SA(1)+WAM*WA(1)-D(
         2)*SA(2)-D(3)*SA(3)-D(4)*SA(4))
      WRITE(3,4)TS
      IF(ABS(TS-TK)-.001)32,32,23
32    CONTINUE
33    CONTINUE
34    CONTINUE
      END
```

# APPENDIX C

THE SYNTHESIS OF A SYSTEM WITH ONE UNKNOWN COEFFICIENT

To illustrate the use of the charts in designing systems, the following example has been selected.




The transfer function for this system is

$$\frac{C(s)}{R(s)} = \frac{K_a K_t}{J_t s^2 + f_t s + K_a K_t}$$

where

$$K_t = 2 \ \frac{\text{in-oz}}{\text{v}}$$

$$J_t = 90 \text{ oz-in-sec}^2$$

$$f_t = 60 \ \frac{\text{oz-in}}{\text{rad/sec}}$$

$$V = 5 \text{ in/sec}$$

$$d = 10 \text{ in}$$

The amplifier gain, $K_a$, is to be found such that the following specifications are met as nearly as possible:

$$os = 10\%$$
$$t_r = 6 \text{ sec}$$
$$t_s = 15 \text{ sec}$$

The time delay is

$$T_o = \frac{d}{V} = \frac{10}{5} = 2.$$

In terms of the general system, the coefficient may be written

$$a_2 = 90$$
$$a_1 = 60$$
$$a_o = 2K_a$$

Therefore, the transformed time delay is

$$T = \frac{90}{60}\, 2 = 3,$$

and the rise time and settling time become

$$t_{rt} = \frac{90}{60}\, 6 = 9$$

$$t_{st} = \frac{90}{60}\, 15 = 22.5 \quad .$$

A careful inspection of the charts indicate that the restrictions may be met very closely when B = 0.15. From the relation

$$B = \frac{a_2 a_o}{a_1^{\,2}}$$

it is found that $K_a$ must equal 3.

# A METHOD FOR THE ANALYSIS AND SYNTHESIS OF SECOND-ORDER SYSTEMS WITH CONTINUOUS TIME DELAY

by

DAVID LEE HEMMEL

B. S., University of Missouri at Rolla, 1964

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AN ABSTRACT OF A MASTER'S THESIS

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MASTER OF SCIENCE

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KANSAS STATE UNIVERSITY
Manhattan, Kansas

1966

A method is presented in this thesis for the analysis and synthesis of second-order control systems with time delay. In almost all cases where time delays occur, the analysis is made much more difficult, and a thorough investigation of the system's transient response is often impossible. This difficulty may be traced to the fact that the introduction of even a small time delay creates an infinite number of poles. In certain cases, information regarding the transient response may be obtained without determining the locations of these poles while in other situations it may be possible to determine the locations of a sufficient number of the system's poles and form an approximation to the time response. These conventional methods of analysis are explained in detail, and it is found that no method exists which is applicable to a large number of systems and also gives accurate results without a great deal of labor.

In the method presented here, it is pointed out that a general second-order system with time delay may be transformed to a system with one coefficient and a new time delay related to the original value. The overshoot, rise time, and settling time for the general system are found to be related to corresponding values for the transformed system. It is then seen that transient response characteristics for a general system may be found easily if corresponding values are known for the transformed system. To solve for the overshoot, rise time, and settling time of the transformed system, the Heaviside expansion formula is employed. Four pairs of complex poles are used, and the resulant approximation is checked and found to be extremely accurate. Transient response information in the form of lines of constant overshoot, rise time, and settling time for the transformed system are found with the aid of an IBM 1410 computer. These curves are plotted on graphs with the transformed coefficient and the transformed time delay as coordinates.

Although this information is presented for limited ranges of the transformed system's coefficient and time delay, it is believed that the method presented in this thesis will prove applicable to a great number of control systems with time delay.