FREE VIBRATION OF
A RECTANGULAR RIGID FRAME

by

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NOTATION

x, y Rectangular coordinates, x in the longitudinal direction, y in the direction of deflection.

$x_1, y_1,$ Rectangular coordinates, for the vertical member of the frame.

$x_2, y_2,$ Rectangular coordinates, for the horizontal member of the frame.

$S_1, S_2,$ Shear force in section 1 and 2.

$M_1, M_2,$ Bending moment in section 1 and 2.

w Lateral distributed load.

A Cross section area.

ρ Mass density.

E Young's modulus.

I Second moment of area of the cross section about the neutral axis through its centroid.

$I_1, I_2,$ I for the vertical and horizontal member of the frame respectively.

ω Natural frequency of vibration of the frame.

X_{1L1} Deflection of the vertical member at $x_1 = L_1$.

k A parameter equal to $\sqrt[4]{\frac{\omega^2 \rho A}{EI}}$

k_1 Corresponding to k for the vertical member, $k_1 = \sqrt[4]{\frac{\omega^2 (\rho A)_1}{EI_1}}$

k_2 Corresponding to k for the horizontal member $k_2 = \sqrt[4]{\frac{\omega^2 (\rho A)_2}{EI_2}}$

ϕ_1 Equal to $k_1 L_1$.

ϕ_2 Equal to $k_2 L_2$.

INTRODUCTION

The problem of computing natural frequency of vibration of structures is of great practical importance in the analysis and design of structures. If the frequency of the disturbing force is sufficiently close to one of the natural frequencies of the structure, the structure will undergo vibrations which, in the absence of appreciable damping effects, may become prohibitively large.

The determination of the natural frequencies of the lateral vibrations of beams has been the subject of numerous papers and books. The method usually used in engineering applications is known as the energy method, or the methods of Rayleigh and Ritz. A method known as Transfer Matrices has grown out of the well-known method of Holzer and Myklestad. As applied to the determination of natural modes and frequencies, it is characterized by the setting up of a frequency determinant that follows as a consequence of satisfying boundary conditions. The computations, which are conveniently carried out in matrix operations, require the insertion of trial values of the frequency into the transfer matrices. Another powerful method of analysis has been applied to frames by Bishop. Described as a method of receptances, it deals with the forced vibrations of systems in terms of the receptances of its component parts. For determining natural modes and frequencies of the structure, those frequencies are sought for which the overall receptance vanishes. Veletsos and Newmark have devised a method described as an extension of Holzer's method for continuous beams; it is also applicable to frames without sideway. For natural frequency determination, this method seeks those frequencies for which the exciting couple applied to the beam vanishes. This, as in the methods already mentioned, requires repeated trial computations.

In contrast with the above methods, the procedure described in this paper leads to the exact solution of the Bernoulli-Euler equation for the vibration of a rectangular fixed end frame. The frequency equation is obtained from direct expansion of the frequency determinant that follows as a consequence of satisfying boundary conditions. In solving for the eigenfrequency from the frequency equations, the 1620 digital computer is used. The computer program and results are shown in the Appendix of this report. It is known that for a structure of infinite degree of freedom, there is an infinite number of vibration modes. In this report, only the natural frequencies and mode shapes of the first ten modes of vibration of a simple rectangular fixed end frame are presented. However, with the same procedure, one can find as many modes as one desires. Rayleigh's energy method is also used to find the frequency of the first vibration mode. The result is compared with that of the exact solution, and the error is found to be 5.6%.

DERIVATION OF BASIC EQUATION OF MOTION

The method of analysis which is developed here is known as the Bernoulli-Euler theory and is based upon the assumption that plane cross-sections of a beam remain plane during flexure and that the radius of curvature of a bent beam is large compared with the beam's depth. Axial change in length, shear deformation and the rotary inertia effect are neglected; only the transverse bending deflections will be considered.

Fig. 1 shows a short element of a beam. It is of length δx and is bounded by plane faces which are perpendicular to the axis; the faces are identified by the numbers 1 and 2. The forces and couples which act upon the element are also shown in the figure; they are the shear forces S_1, S_2, the bending moments M_1, M_2 and the applied lateral load $w\delta x$. If the deflexion of the beam is small, as the theory presupposes, then the inclination of the beam element from the unstrained position is also small. Under these conditions, the equation of motion perpendicular to the axis Ox of the undeflected beam is

$$S_1 - S_2 + w\,\delta x = (\rho A \delta x)\,\frac{\partial^2 y}{\partial t^2} \qquad (1)$$

where A is the area of cross-section, ρ is the mass density and y is the deflexion. If the equation is divided by δx and if the length of the element is then made indefinitely short, the equation of motion is obtained in the form

$$\frac{\partial S}{\partial x} + w = \rho A\,\frac{\partial^2 y}{\partial t^2} \qquad (2)$$

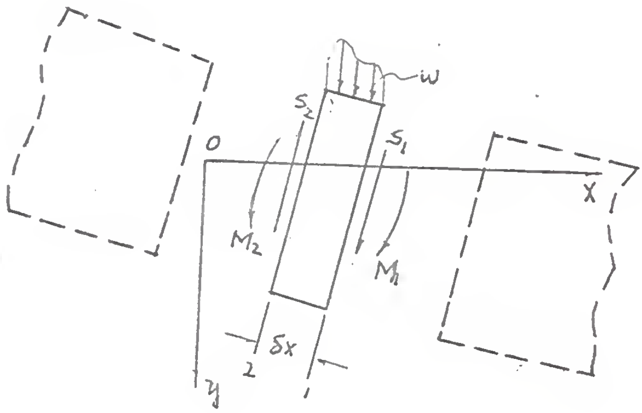


Fig. 1

The equation for the rotational motion of the element about Section 1 may be written

$$\delta_2 \delta x + M_1 - M_2 + kw^* \delta x^2 = 0 \qquad (3)$$

where w* is the mean value of ω distributed on δx and k is the distance from the center of the distributed load ω to section 2. When the length of the element is made small, this equation gives

$$S = - \frac{\partial M}{\partial x} \, . \qquad (4)$$

Now from the simple bending theory we have the relation

$$M = EI \frac{\partial^2 y}{\partial x^2} \qquad (5)$$

where E is Young's modulus of the material and I is the second moment of area of the cross-section about the neutral axis through its centroid. If this is combined with equation (4) it gives

$$S = - EI \frac{\partial^3 y}{\partial x^3} \tag{6}$$

This expression for S may be substituted in equation (2) which then gives the differential equation of motion in the form

$$\frac{\partial^2 y}{\partial t^2} + \frac{EI}{A\rho} \frac{\partial^4 y}{\partial x^4} = \frac{w}{A\rho} \tag{7}$$

When the applied lateral load w is equal to zero, equation (7) becomes

$$\frac{EI}{A\rho} \frac{\partial^4 y}{\partial x^4} = - \frac{\partial^2 y}{\partial t^2} \tag{8}$$

For the normal mode of vibration, it is reasonable to set

$$y(t,x) = T(t)X(x) \tag{9}$$

and

$$\frac{\partial^2 y}{\partial t^2} = X(x) \frac{d^2}{dt^2} T(t) \tag{10}$$

$$\frac{EI}{A\rho} \frac{\partial^4 y}{\partial x^4} = T(t) \frac{EI}{A\rho} \frac{d^4}{dx^4} X(x) \tag{11}$$

substitute equation (10) and equation (11) into equation (8) and rearrange

$$\frac{EI}{A\rho} \frac{\frac{d^4}{dx^4} X(x)}{X(x)} = - \frac{\frac{d^2}{dt^2} T(t)}{T(t)} \tag{12}$$

The separability of x and t in equation (12) proves the assumption of equation (9) is right. The left-hand member is a function of x alone and the righthand member is a function of t alone, the only way for them to be equal is both equal to a constant, say ω^2. Therefore, equation (12) can be reduced to the solution of two equations. Namely

$$\frac{d^2}{dt^2} T(t) + \omega^2 T(t) = 0 \tag{13}$$

and

$$\frac{d^4}{dx^4} X(x) - \frac{\omega^2 \rho A}{EI} X(x) = 0 \tag{14}$$

Let

$$\frac{\omega^2 \rho A}{EI} = k^4 \tag{15}$$

and substitute equation (15) into equation (14) to get

$$\frac{d^4}{dx^4} X(x) - k^4 X(x) = 0 \tag{16}$$

Equation (13) and (16) are the differential equations for normal vibration of beams.

The solution for equation (13) is

$$T(t) = F_1 \sin\omega t + F_2 \cos\omega t \tag{17}$$

or

$$T(t) = F_3 \sin(\omega t + \theta) \tag{18}$$

where F_1, F_2, F_3 and θ are constants to be determined from initial conditions.

The natural frequency is

$$f = \frac{\omega}{2\pi} \tag{19}$$

The solution for equation (16) is

$$X(x) = G_1 \sin kx + G_2 \cos kx + G_3 \sinh kx + G_4 \cosh kx \tag{20}$$

where G_1, G_2, G_3 and G_4 are constants to be determined from boundary conditions.

Equations (17) or (18) and (19) are the basic equations for normal vibration of beams.

DERIVATION OF THE FREQUENCY EQUATION FOR NORMAL VIBRATION OF A RECTANGULAR RIGID FRAME

A rectangular rigid frame is simply three straight beams joined at right angles, therefore, the vibration characteristics of a beam, as has been shown in the preceding paragraphs, can also be applied in frames.

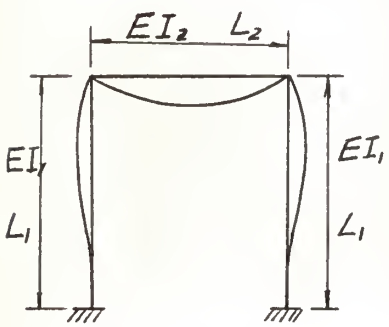


Fig. 2 Symmetrical Mode

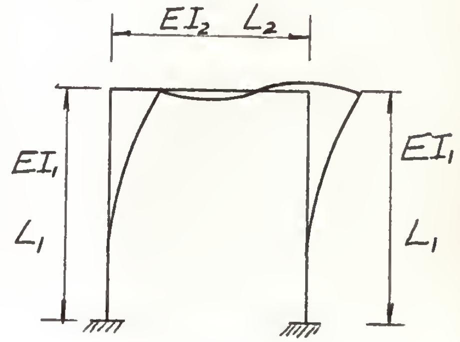


Fig. 3 Anti-symmetrical Mode

Fig. 2 and Fig. 3 show the two different kinds of normal mode vibrations of a rectangular rigid frame, namely, the symmetrical mode and the anti-symmetrical mode. This report will discuss these two systems separately. However, in both cases the method of superposition is freely used. Owing to the symmetry of the frame itself, only one of the two vertical members will be taken into consideration.

1. Symmetrical modes of vibration of a rectangular rigid frame

For normal modes of vibration of beams, the deflection curve is

$$y(t,x) = T(t)\,X(x) \tag{9}$$

where

$$T(t) = F_3 \sin\,(\omega t + \theta) \tag{18}$$

the deflexion curve for vertical bar a-b, can be written as

$$X_1(x) = A_1 \sin k_1 x_1 + A_2 \cos k_1 x_1 + A_3 \sinh k_1 x_1 + A_4 \cosh k_1 x_1 \tag{1-1}$$

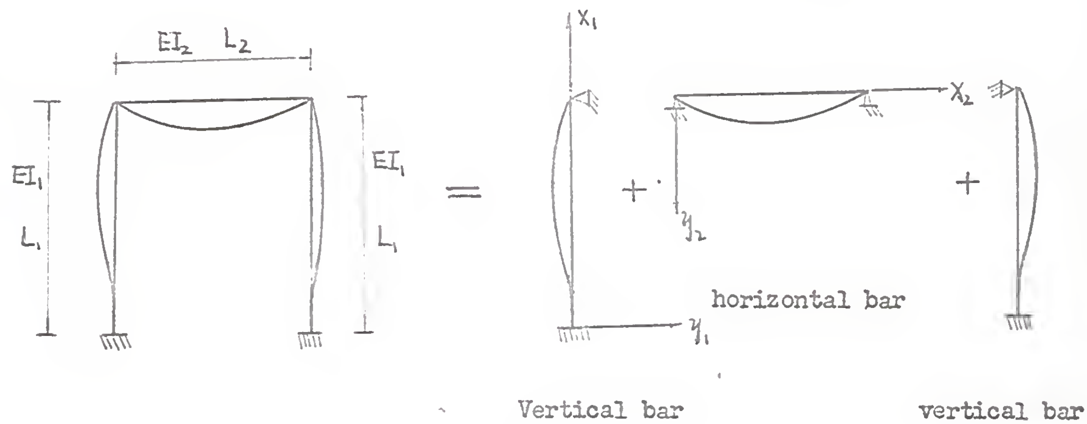


Fig. 4

the deflexion curve for the horizontal bar b-c can be written as

$$X_2(x_2) = B_1 \sin k_2 x_2 + B_2 \cos k_2 x_2 + B_3 \sinh k_2 x_2 + B_4 \cosh k_2 x_2 \tag{1-2}$$

where x_1, x_2, y_1, y_2, are coordinates as shown in Fig. 4, k_1, k_2, are the corresponding to k for bar a-b, b-c, in equation (20). A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4 are constants to be determined by boundary conditions.

Boundary Conditions

(1) For the vertical bar a-b.

(a) $X_1(x_1)\Big|_{x_1=0} = 0 \qquad A_2 = -A_4 \qquad (1\text{-}3)$

(b) $X_1'(x_1)\Big|_{x_1=0} = 0 \qquad A_1 = -A_3 \qquad (1\text{-}4)$

(c) $X_1(x_1)\Big|_{x_1=L_1} = 0$

$$A_3\,(\sinh k_1L_1 - \sin k_1L_1) + A_4\,(\cosh k_1L_1 - \cos k_1L_1) = 0$$

$$A_3 = -\frac{\cosh k_1L_1 - \cos k_1L_1}{\sinh k_1L_1 - \sin k_1L_1}\,A_4 \qquad (1\text{-}5)$$

substitute equations (1-3), (1-4) and (1-5) into equation (1-1) and simplify

$$X_1(x_1) = A_4\left[\cosh k_1x_1 - \cos k_1x_1 + \frac{\cosh k_1L_1 - \cos k_1L_1}{\sinh k_1L_1 - \sin k_1L_1}\,(\sin k_1x_1 - \sinh k_1x_1)\right] \qquad (1\text{-}6)$$

differentiate equation (1-6) and simplify

$$X_1'(x_1) = k_1A_4\left[\sinh k_1x_1 + \sin k_1x_1 + \frac{\cosh k_1L_1 - \cos k_1L_1}{\sinh k_1L_1 - \sin k_1L_1}\,(\cos k_1x_1 - \sinh k_1x_1)\right] \qquad (1\text{-}7)$$

differentiate equation (1-7) and simplify

$$X_1''(x_1) = k_1^2A_4\left[\cosh k_1x_1 + \cos k_1x_1 + \frac{\cosh k_1L_1 - \cos k_1L_1}{\sinh k_1L_1 - \sin k_1L_1}\,(-\sin k_1x_1 - \sinh k_1x_1)\right] \qquad (1\text{-}8)$$

(2) For the horizontal bar b-c

(a) $X_2(x_2)\Big|_{x_2=0} = 0 \qquad B_2 = -B_4 \qquad (1\text{-}9)$

(b) $$X_2(x_2)\Big|_{x_2 = L_2} = 0$$

substitute equation (1-9) into equation (1-2)

$$B_1 \sin k_2L_2 - B_4 \cos k_2L_2 + B_3 \sinh k_2L_2 + B_4 \cosh k_2L_2 = 0 \tag{1-10}$$

(c) $$X_2'(x_2)\Big|_{x_2 = 0} = X_2'(x_2)\Big|_{x_2 = L_2}$$

Combine boundary condition (c) and equation (1-9)

$$-B_1 \sin k_2L_2 + B_4 \cos k_2L_2 + B_3 \sinh k_2L_2 + B_4 \cosh k_2L_2 = 2B_4 \tag{1-11}$$

solve equation (1-10) and equation (1-11) for B_1 and B_3 in terms of B_4,

$$B_1 = \frac{\cos k_2L_2 - 1}{\sin k_2L_2} B_4 \tag{1-12}$$

$$B_3 = \frac{1 - \cosh k_2L_2}{\sinh k_2L_2} B_4 \tag{1-13}$$

substitute equation (1-9), (1-12) and (1-13) into equation (1-2)

$$X_2(x_2) = B_4\left(\frac{\cos k_2L_2 - 1}{\sin k_2L_2} \sin k_2x_2 - \cos k_2x_2 + \frac{1-\cosh k_2L_2}{\sinh k_2L_2}(\sinh k_2x_2 + \cosh k_2x_2)\right) \tag{1-14}$$

Differentiate equation (1-14) and simplify

$$X_2'(x_2) = k_2B_4\left(\frac{\cos k_2L_2 - 1}{\sin k_2L_2} \cos k_2x_2 + \sin k_2x_2 + \frac{1-\cosh k_2L_2}{\sinh k_2L_2} \cosh k_2x_2 + \sinh k_2x_2\right) \tag{1-15}$$

differentiate equation (1-15) and simplify

$$X_2''(x_2) = k_2^2 B_4 \left(-\frac{\cos k_2 L_2 - 1}{\sin k_2 L_2}\sin k_2 x_2 + \cos k_2 x_2 + \frac{1-\cosh k_2 L_2}{\sinh k_2 L_2}\sinh k_2 x_2 + \cosh k_2 x_2\right) \tag{1-16}$$

Compatibility conditions

(1) $X_1(x_1)\Big|_{x_1 = L_1} = X_2(x_2)\Big|_{x_2 = 0}$

equating equation (1-7) and equation (1-15) and simplify

$$k_1 A_4 \frac{2(\cos k_1 L_1 \cosh k_1 L_1 - 1)}{\sinh k_1 L_1 - \sin k_1 L_1} - k_2 B_4 \left(\frac{\cos k_2 L_2 - 1}{\sin k_2 L_2} + \frac{1 - \cosh k_2 L_2}{\sinh k_2 L_2}\right) = 0 \tag{1-17}$$

(2) $EI_1 X_1''(x_1)\Big|_{x_1 = L_1} = EI_2 X_2''(x_2)\Big|_{x_2 = 0}$

substituted from equation (1-8) and equation (1-16) and simplify

$$\frac{2(\cos k_1 L_1 \sinh k_1 L_1 - \sin k_1 L_1 \cosh k_1 L_1)}{\sinh k_1 L_1 - \sin k_1 L_1} k_1 A_4 I_1 - k_2 B_4 I_2 = 0 \tag{1-18}$$

solve A_4, B_4 from equations (1-17) and (1-18). Both equations (1-17) and (1-18) are homogeneous equations. In order to have a non-trivial solution, the determinant of these equations must equal to zero.

$$\Delta = \begin{vmatrix} \dfrac{2(\cos k_1 L_1 \cosh k_1 L_1 - 1)}{\sinh k_1 L_1 - \sin k_1 L_1} k_1 & -2\left(\dfrac{\cos k_2 L_2 - 1}{\sin k_2 L_2} - \dfrac{1 - \cosh k_2 L_2}{\sinh k_2 L_2}\right) \\ \dfrac{2(\cosh_1 L_1 \sinh k_1 L_1 - \sin k_1 L_1 \cosh k_1 L_1)}{\sinh k_1 L_1 - \sin k_1 L_1} k_1^2 I_1 & -2k_2^2 I_2 \end{vmatrix}$$

$$= 0$$

simplify and rearrange

$$\frac{\cos k_2L_2 - 1}{\sin k_2L_2} + \frac{1 - \cosh k_2L_2}{\sinh k_2L_2} = 2\,\frac{k_2I_2}{k_1I_1}\,\frac{\cos k_1L_1 \cosh k_1L_1 - 1}{\cos k_1L_1 \sinh k_1L_1 - \sin k_1L_1 \cosh k_1L_1} \tag{1-19}$$

Let $k_1L_1 = \phi_1$ and $k_2L_2 = \phi_2$, substitute into equation (1-19)

$$\frac{\cos\phi_2 - 1}{\sin\phi_2} + \frac{1 - \cosh\phi_2}{\sinh\phi_2} = 2\,\frac{k_2I_2}{k_1I_1}\,\frac{\cos\phi_1 \cosh\phi_1 - 1}{\cos\phi_1 \sinh\phi_1 - \sin\phi_1 \cosh\phi_1} \tag{1-20}$$

equation (1-19) or equation (1-20) is the frequency equation for symmetrical modes of vibration of rectangular rigid frame.

2. Anti-symmetrical modes of vibration of a rectangular rigid frame

For normal modes of vibration of beams, the deflection curve is

$$y(t,x) = T(t)\,X(x) \tag{9}$$

where

$$T(t) = F\sin(\omega t + \theta) \tag{18}$$

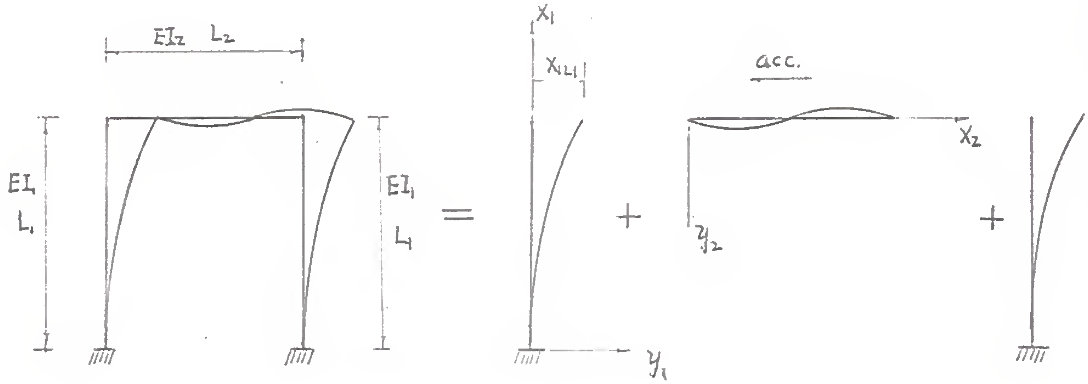


Fig. 5

The deflection shape for the vertical bar a-b can be written as

$$X_1(x_1) = C_1 \sin k_1 x_1 + C_2 \cos k_1 x_1 + C_3 \sinh k_1 x_1 + C_4 \cosh k_1 x_1 \qquad (2\text{-}1)$$

The deflection shape for the horizontal bar b-c can be written as

$$X_2(x_2) = D_1 \sin k_2 x_2 + D_2 \cos k_2 x_2 + D_3 \sinh k_2 x_2 + D_4 \cosh k_2 x_2 \qquad (2\text{-}2)$$

where X_1, X_2, y_1, y_2, are coordinates as shown in Fig. 5, k_1, k_2 are the corresponding to k for bar a-b, b-c in equation (20). C_1, C_2, C_3, C_4, and D_1, D_2, D_3, D_4, are constants to be determined by boundary conditions.

Boundary conditions

(1) For the vertical bar a-b

(a) $X_1(x_1)\Big|_{x_1 = 0} = 0$ $\qquad C_2 = -C_4 \qquad (2\text{-}3)$

(b) $X_1'(x_1)\Big|_{x_1 = 0} = 0$ $\qquad C_1 = -C_3 \qquad (2\text{-}4)$

substitute equation (2-3) and equation (2-4) into equation (2-1) and simplify

$$X_1(x_1) = C_3(\sinh k_1 x_1 - \sin k_1 x_1) + C_4(\cosh k_1 x_1 - \cos k_1 x_1) \qquad (2\text{-}5)$$

differentiate and simplify

$$X_1'(x_1) = k_1 C_3(\cosh k_1 x_1 - \cos k_1 x_1) + k_1 C_4(\sinh k_1 x_1 + \sin k_1 x_1) \qquad (2\text{-}6)$$

differentiate equation (2-6) and simplify

$$X_1''(x_1) = k_1^2 C_3(\sinh k_1 x_1 + \sin k_1 x_1) + k_1^2 C_4(\cosh k_1 x_1 + \cos k_1 x_1) \qquad (2\text{-}7)$$

differentiate equation (2-7) and simplify

$$X_1'''(x_1) = k_1^3 C_3(\cosh k_1 x_1 + \cos k_1 x_1) + k_1^3 C_4(\sinh k_1 x_1 - \sin k_1 x_1) \qquad (2\text{-}8)$$

(2) For the horizontal bar

(a) $$X_2(x_2)\Big|_{x_2 = 0} = 0 \qquad D_2 = -D_4 \tag{2-9}$$

(b) $$X_2(x_2)\Big|_{x_2 = 0} = 0$$

combine equation (2-9) and equation (2-2)

$$-D_1 \sin k_2L_2 - D_4 \cos k_2L_2 + D_3 \sinh k_2L_2 + D_4 \cosh k_2L_2 = 0 \tag{2-10}$$

(c) $$X_2''(x_2)\Big|_{x_2 = 0} = -X_2''(x_2)\Big|_{x_2 = L_2}$$

$$-D_1 \sin k_2L_2 + D_4 \cos k_2L_2 + D_3 \sinh k_2L_2 + D_4 \cosh k_2L_2 = -2D_4 \tag{2-11}$$

solve equation (2-10) and equation (2-11) for D_1, D_3, in terms of D_4

$$D_3 = -\frac{1 + \cosh k_2L_2}{\sinh k_2L_2} D_4 \tag{2-12}$$

$$D_1 = \frac{1 + \cos k_2L_2}{\sin k_2L_2} D_4 \tag{2-13}$$

substituting equations (2-9), (2-12), (2-13) into equation (2-2) and simplify

$$X_2(x_2) = D_4\Big[\Big(\frac{1 + \cos k_2L_2}{\sin k_2L_2}\Big) \sin k_2x_2 - \cos k_2x_2 - \Big(\frac{1 + \cosh k_2L_2}{\sinh k_2L_2}\Big) \sinh k_2x_2 + \cosh k_2x_2\Big] \tag{2-14}$$

differentiate equation (2-14) and simplify

$$X_2'(x_2) = k_2D_4\Big[\Big(\frac{1 + \cosh_2L_2}{\sin k_2L_2}\Big) \cos k_2x_2 + \sin k_2x_2 - \Big(\frac{1 + \cosh k_2L_2}{\sinh k_2L_2}\Big) \cosh k_2x_2 + \sinh k_2x_2\Big] \tag{2-15}$$

differentiate equation (2-15) and simplify

$$X_2''(x_2) = k_2 D_4 \Big[- \left(\frac{1 + \cos k_2 L_2}{\sin k_2 L_2}\right) \sin k_2 x_2 + \cos k_2 x_2$$

$$- \left(\frac{1 + \cosh k_2 L_2}{\sinh k_2 L_2}\right) \sinh k_2 x_2 + \cosh k_2 x_2 \Big] \qquad (2\text{-}16)$$

differentiate equation (2-16) and simplify

$$X_2'''(x_2) = k_2 D_4 \Big[- \left(\frac{1 + \cos k_2 L_2}{\sin k_2 L_2}\right) \cos k_2 x_2 - \sin k_2 x_2$$

$$- \left(\frac{1 + \cosh k_2 L_2}{\sinh k_2 L_2}\right) \cosh k_2 x_2 + \sinh k_2 x_2 \Big] \qquad (2\text{-}17)$$

<u>Compatibility equations</u>

(1) To satisfy the continuity condition at joint b

$$X_1'(x_1)\Big|_{x_1 = L_1} = X_2'(x_2)\Big|_{x_2 = 0}$$

substituted from equations (2-6) and (2-5) and simplify

$$C_3 k_1 (\cosh k_1 L_1 - \cos k_1 L_1) + C_4 k_1 (\sinh k_1 L_1 + \sin k_1 L_1)$$

$$- k_2 D_4 \left(\frac{1 + \cos k_2 L_2}{\sin k_2 L_2} - \frac{1 + \cosh k_2 L_2}{\sinh k_2 L_2}\right) = 0 \qquad (2\text{-}18)$$

(2) To satisfy the static equilibrium condition at joint b

$$EI_1 \left[X_1''(x_1) \;\; _{x_1 = L_1}\right] = EI_2 \left[X_2''(x_2) \;\; _{X_2 = 0}\right]$$

substituted from equation (2-7) and (2-16) and simplify

$$C_3 I_1 k_1^2 (\sinh k_1 L_1 + \sin k_1 L_1) + (\cosh k_1 L_1 + \cos k_1 L_1)\, C_4 I_1 k_1^2 - 2D_4 I_2 k_2^2 = 0 \qquad (2\text{-}19)$$

(3) To satisfy the dynamic equilibrium condition

The force to produce the horizontal motion of the horizontal bar is the shear force at the top ends of the vertical bars.

total shear force = mass of horizontal bar times acc. of horizontal bar.

$$2EI_1 \left[\frac{\partial^3}{\partial x^3} y(t, x_1) \right]_{x_1 = L_1} = - (A\rho)_2 L_2 \frac{\partial^2}{\partial t^2} y(t, x_1) \Big|_{x_1 = L_1}$$

from equation (9)

$$y(t,x_1) = T(t)\, X_1(x_1) \qquad (2\text{-}20)$$

$$\frac{\partial^3}{\partial x^3} y(t,x_1) = T(t)\, X_1'''(x_1) \qquad (2\text{-}21)$$

$$\frac{\partial^2}{\partial t^2} y(t,x_1) = \frac{d^2}{dt^2} T(t)\, X_1(x_1) \qquad (2\text{-}22)$$

since

$$T(t) = F_1 \sin \omega t + F_2 \cos \omega t \qquad (17)$$

$$\frac{d^2}{dt^2} T(t) = - \omega^2 T(t) \qquad (2\text{-}23)$$

substitute equations (2-20), (2-21), 2-22) and (2-23) into the dynamic equilibrium equation and simplify

$$C_3 \left[EI_1k_1^3(\cosh k_1L_1 + \cos k_1L_1) + 1/2\ (A\rho)_2L_2\omega^2(\sinh k_1L_1 - \sin k_1L_1)\right]$$

$$C_4 \left[EI_1k_1^3(\sinh k_1L_1 - \sin k_1L_1) + 1/2\ (A\rho)_2L_2\omega^2(\cosh k_1L_1 - \cos k_1L_1)\right] = 0 \tag{2-24}$$

solve C_3, C_4, D_4 from equations (2-18), (2-19) and (2-24). Both equations (2-18), (2-19), and (2-24) are homogeneous equations. In order to have a non-trivial solution, the determinant of these equations must equal zero.

$$\Delta = \begin{vmatrix} k_1(\cosh k_1L_1 - \cos k_1L_1) & k_1(\sinh k_1L_1 + \sin k_1L_1) & k_2\left(\frac{1\ \ \cos k_2L_2}{\sin k_2L_2} - \frac{1\ \ \cosh k_2L_2}{\sinh k_2L_2}\right) \\ I_1k_1^2(\sinh k_1L_1 + \sin k_1L_1) & I_1k_1(\cosh k_1L_1 + \cos k_1L_1) & 2I_2k_2^2 \\ \begin{array}{l} EI_1k_1^3(\cosh k_1L_1 + \cos k_1L_1) + \\ \frac{(A\rho)_2}{2} L_2\omega^2(\sinh k_1L_1 - \sin k_1L_1) \end{array} & \begin{array}{l} EI_1k_1(\sinh k_1L_1 - \sin k_1L_1) + \\ \frac{(A\rho)_2}{2} L_2\omega^2(\cosh k_1L_1 - \cos k_1L_1) \end{array} & 0 \end{vmatrix}$$

expanding the determinant and simplify

$$\begin{aligned} \Delta = \frac{1}{2} & \left(\frac{1 + \cos k_2L_2}{\sin k_2L_2} - \frac{1 + \cosh k_2L_2}{\sinh k_2L_2}\right) I_1k_1\omega^2(A\rho)_2L_2(2\sin k_1L_1\cosh k_1L_1 \\ & \qquad - \sinh k_1L_1\cos k_1L_1) \\ & + EI_1^2k_1^4\left(\frac{1 + \cos k_2L_2}{\sin k_2L_2} - \frac{1 + \cosh k_2L_2}{\sinh k_2L_2}\right)(-2 - 2\cosh k_1L_1\cos k_1L_1) \\ & + 2EI_2k_2I_1k_1(2\cosh k_1L_1\sin k_1L_1 + 2\cos k_1L_1\sinh k_1L_1) \\ & - I_2k_2(A\rho)_2L_2\omega^2(2 - \cosh k_1L_1\cos k_1L_1) = 0 \end{aligned}$$

rearrange and simplify, and let $\phi_1 = k_1L_1$, $\phi_2 = k_2L_2$

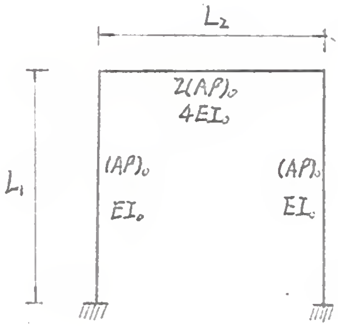
$$\frac{I_1 k_1}{I_2 k_2} \; \frac{\cos\phi_2 \sinh\phi_2 - \sin\phi_2 \cosh\phi_2 + \sinh\phi_2 - \sin\phi_2}{2\sin\phi_2 \sinh\phi_2}$$

$$= \frac{\dfrac{(A\rho)_2 L_2}{(A\rho)_1 L_1} \phi_1 (1 - \cosh\phi_1 \cos\phi_1) - 2(\cosh\phi_1 \sin\phi_1 + \cos\phi_1 \sinh\phi_1)}{\dfrac{(A\rho)_2 L_2}{(A\rho)_1 L_1} \phi_1 (\sin\phi_1 \cosh\phi_1 - \sinh\phi_1 \cos\phi_1) - 2(1 + \cosh\phi_1 \cos\phi_1)} \qquad (2\text{-}25)$$

equation (2-25) is the frequency equation for anti-symmetrical modes of vibration of a rectangular rigid frame.

NUMERICAL ILLUSTRATIVE EXAMPLE

For a rectangular rigid frame as shown in Fig. 6, find the natural frequencies and the shapes of vibration of the first eight normal modes.



$$L_1 = L_2$$

$$\frac{I_1}{I_2} = 0.25$$

$$\frac{(A\rho)_1}{(A\rho)_2} = 0.50$$

Fig. 6

from the data given

$$\frac{I_1 k_1}{I_2 k_2} = \frac{\sqrt[4]{2}}{4} \qquad \text{and} \qquad \frac{(A\rho)_2 I_2}{(A\rho)_1 I_1} = 2.0$$

the frequency equation of anti-symmetrical modes, equation (2-25), takes the form

$$\frac{\sqrt[4]{2}}{8} \; \frac{\cos\phi_2 \sinh\phi_2 - \sin\phi_2 \cosh\phi_2 + \sinh\phi_2 - \sin\phi_2}{2\sin\phi_2 \sinh\phi_2}$$

$$\frac{\phi_1 (1 - \cosh\phi_1 \cos\phi_1) - (\cosh\phi_1 \sin\phi_1 + \cos\phi_1 \sinh\phi_1)}{\phi_1 (\sin\phi_1 \cosh\phi_1 - \sinh\phi_1 \cos\phi_1) - (1 + \cosh\phi_1 \cos\phi_1)} \qquad (3\text{-}1)$$

and the frequency equation of symmetrical modes, equation (1-20) takes the form

$$\frac{\cos\phi_2 - 1}{\sin\phi_2} + \frac{1 - \cosh\phi_2}{\sinh\phi_2} = \frac{8}{\sqrt[4]{2}} \; \frac{\cos\phi_1\cosh\phi_1 - 1}{\cos\phi_1\sinh\phi_1 - \sin\phi_1\cosh\phi_1} \tag{3-2}$$

Now, the problem becomes one of finding the eigenvalue ω to satisfy equation (3-1) or equation (3-2). It can be solved by trial-and-error method with the help of the digital computer.

In this problem, note that

$$\phi_1 = \sqrt[4]{2}\,\phi_2 \tag{3-3}$$

and let

$$X1 = \frac{\phi_1(1 - \cosh\phi_1\cos\phi_1) - (\cosh\phi_1\sin\phi_1 + \cos\phi_1\sinh\phi_1)}{\phi_1(\sin\phi_1\cosh\phi_1 - \sinh\phi_1\cos\phi_1) - (1 + \cosh\phi_1\cos\phi_1)} \tag{3-4}$$

$$X2 = \frac{\sqrt[4]{2}}{8} \; \frac{\cos\phi_2\sinh\phi_2 - \sin\phi_2\cosh\phi_2 + \sinh\phi_2 - \sin\phi_2}{2\sin\phi_2\sinh\phi_2} \tag{3-5}$$

$$Y1 = \frac{\sqrt[4]{2}}{8} \; \frac{\cos\phi_1\cosh\phi_1 - 1}{\cos\phi_1\sinh\phi_1 - \sin\phi_1\cosh\phi_1} \tag{3-6}$$

$$Y2 = \frac{\cos\phi_2 - 1}{\sin\phi_2} + \frac{1 - \cosh\phi_2}{\sinh\phi_2} \tag{3-7}$$

and then, set up Forego program to find the value of X1, X2, Y1 and Y2 for ϕ_1 ranging from 0.0 to 12.0 with each 0.1 increment, the results are shown in Appendix 1. With this results, curves for X1, X2 against ϕ_1 (Fig. 7) and for Y1, Y2 against ϕ_1 (Fig. 8) can be plotted. So the ϕ_1 values for which X1 = X2, and Y1 = Y2 can be located from the figures. It should be noted that the ϕ_1 value thus found will be accurate only for the first decimal place. For example, the ϕ_1 for the first anti-symmetric mode can be between 1.7 and

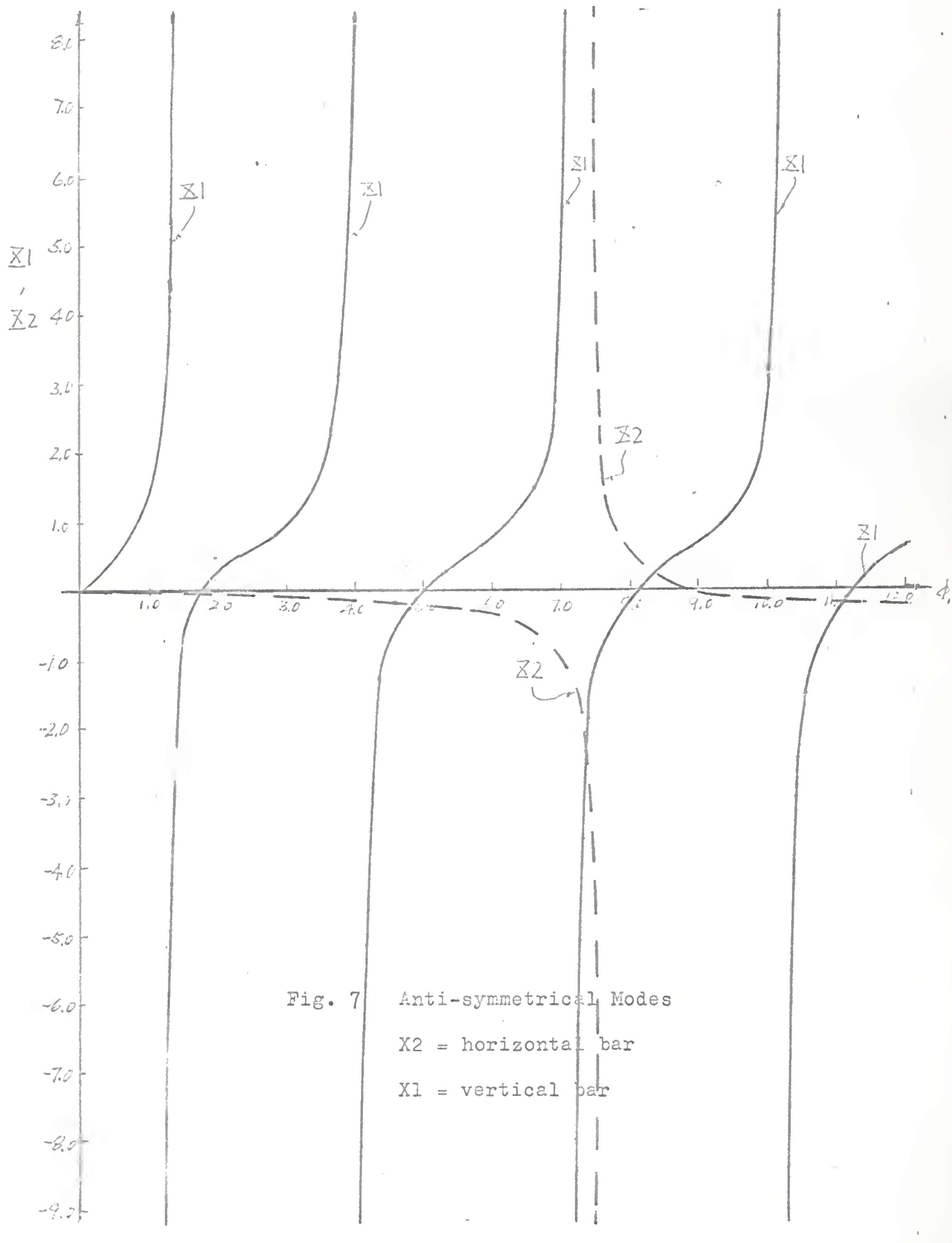


Fig. 7 Anti-symmetrical Modes

X2 = horizontal bar

X1 = vertical bar

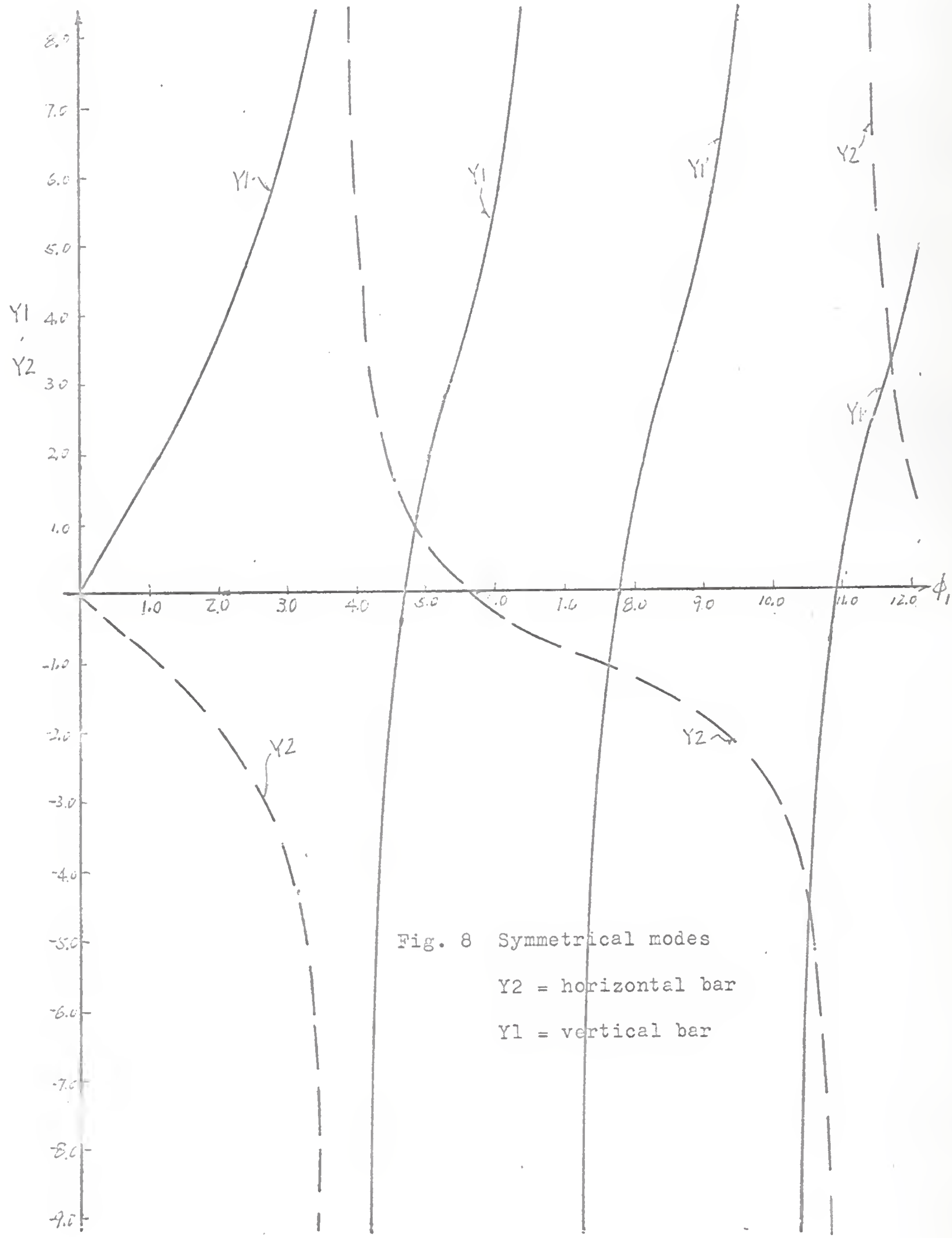


Fig. 8 Symmetrical modes

Y2 = horizontal bar

Y1 = vertical bar

1.8. If two decimal place accuracy is desired, the same program can be used to find the value of X1, X2, Y1, Y2, but for ϕ_1 ranging from 1.71 to 1.80 with each 0.01 increment. Repeating this process, one can find as much accuracy as one wants. This report, however, has reached only four decimal place accuracy. From the relation $\phi_1 = k_1 L$ and

$$\frac{\omega^2(\rho A)_1}{EI_1} = k_1^4 \tag{15}$$

solve for

$$\omega = \phi_1^2 \left(\frac{EI_1}{L^4(\rho A)_1}\right)^{\frac{1}{2}} \tag{3-8}$$

Table 1 shows the ϕ_1 and ϕ_1^2 values for the first ten normal modes of vibration of a rectangular rigid frame as shown in Fig. 6. The value of ϕ_1^2 can determine the natural frequency ω, as expressed in equation (3-8).

Table 1. Values of ϕ_1 and ϕ_1^2 for the first ten normal modes

modes		1	2	3	4	5	6	7	8	9	10
ϕ_1	Anti-symm.	1.6775		4.7187		7.3196		8.2914		10.9617	
	Symm.		3.8063		4.8888		7.7136		10.5998		11.7546
ϕ_1^2		2.814	14.488	22.255	23.900	53.576	59.500	68.747	112.356	120.159	138.171

Now the next step is to find out the deflection shape of the rectangular rigid frame corresponding to each normal mode of vibration. This is equivalent to solving the constants A_1 --- A_4, B_1 --- B_4 in equations (1-1), (1-2),

C_1 --- C_4, D_1 --- D_4 in equations (2-1), (2-2).

For the constants A_1 --- A_4, B_1 --- B_4 in equations (1-1) and (1-2) first solve equation (1-18) for A_4 in terms of B_4

$$A_4 = \frac{2}{4\sqrt{2}} \frac{\sinh\phi_1 - \sin\phi_1}{(\cos\phi_1 \sinh\phi_1 - \sin\phi_1 \cosh\phi_1)} B_4 \tag{3-9}$$

by equation (1-5)

$$A_3 = -\frac{\cosh\phi_1 - \cos\phi_1}{\sinh\phi_1 - \sin\phi_1} A_4 \tag{3-10}$$

by equation (1-3)

$$A_2 = -A_4 \tag{3-11}$$

by equation (1-4)

$$A_1 = -A_3 \tag{3-12}$$

by equation (1-9)

$$B_2 = -B_4 \tag{3-13}$$

by equation (1-12)

$$B_3 = \frac{1 - \cosh\phi_2}{\sinh\phi_2} B_4 \tag{3-14}$$

by equation (1-13)

$$B_1 = \frac{\cos\phi_2 - 1}{\sin\phi_2} B_4 \tag{3-15}$$

Now let

$$B_4 = 1 \tag{3-16}$$

from equation (3-9) to equation (3-16), the constants A_1 --- A_4, B_1 --- B_4

of equations (1-1) and (1-2) can be solved and the characteristic shapes of vibration of the rigid frame for symmetrical modes are known. These values are listed in Table 2.

Table 2. Constants for characteristic shapes of vibration of rectangular rigid frame for symmetrical modes.

Mode	ϕ_1	A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4
2	3.8063	-10.2697	10.1848	10.2697	-10.1848	33.8247	-1	-0.9217	1
4	4.8888	1.4460	-1.4712	-1.4460	1.4712	1.8990	-1	-0.9677	1
6	7.7136	-1.9778	1.9763	1.9778	-1.9763	-0.1019	-1	-0.9970	1
8	10.5998	3.1309	-3.1310	-3.1309	3.1310	-3.8249	-1	-0.9997	1
10	11.7546	1.1896	-1.1896	-1.1896	1.1896	4.2745	-1	-0.9999	1

For the constants C_1 --- C_4, D_1 --- D_4 in equations (2-1) and (2-2) first solve equation (2-18) and (2-19) for C_3 and C_4 in terms of D_4.

$$C_3 = \frac{\frac{1}{4\sqrt{2}}\left(\frac{1+\cos\phi_2}{\sin\phi_2} - \frac{1+\cosh\phi_2}{\sinh\phi_2}\right)(\cos\phi_1 + \cosh\phi_1) - \frac{8}{\sqrt{2}}(\sinh\phi_1 + \sin\phi_1)}{(\cosh^2\phi_1 - \cos^2\phi_1) - (\sinh\phi_1 + \sin\phi_1)^2} \tag{3-17}$$

$$C_4 = \frac{\frac{8}{\sqrt{2}}(\cosh\phi_1 - \cos\phi_1) - \frac{1}{4\sqrt{2}}(\sinh\phi_1 + \sin\phi_1)\left(\frac{1+\cos\phi_2}{\sin\phi_2} - \frac{1+\cosh\phi_2}{\sinh\phi_2}\right)}{(\cosh^2\phi_1 - \cos^2\phi_1) - (\sinh\phi_1 + \sin\phi_1)^2} \tag{3-18}$$

by equation (2-3)

$$C_2 = - C_4 \tag{3-19}$$

by equation (2-4)

$$C_1 = - C_3 \tag{3-20}$$

By equation (2-9)

$$D_2 = - D_4 \tag{3-21}$$

by equation (2-12)

$$D_3 = - \frac{1 + \cosh\phi_2}{\sinh\phi_2} D_4 \tag{3-22}$$

by equation (2-13)

$$D_1 = \frac{1 + \cos\phi_2}{\sin\phi_2} D_4 \tag{3-23}$$

and let

$$D_4 = 1 \tag{3-24}$$

from equations (3-17) to (3-24), the constants C_1 --- C_4, D_1 --- D_4 of equations (2-1) and (2-2) can be solved and the characteristic shapes of vibration of the rectangular rigid frame for anti-symmetrical modes are known. These values are listed in Table 3.

Table 3. Constants for characteristic shapes of vibration of rectangular rigid frame for anti-symmetrical modes

Mode	ϕ_1	A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4
1	1.6775	-4.1451	3.4435	4.1451	-3.4435	1.1745	-1	-1.6455	1
3	3.3992	3.3992	-3.4385	-3.3992	3.4385	-0.4384	-1	-1.0386	1
5	7.3196	-11.3996	11.3974	11.3996	-11.3974	-15.5941	-1	-1.0043	1
7	8.2914	-2.2955	2.2942	2.2955	-2.2942	2.7869	-1	-1.0019	1
9	10.8917	3.2087	-3.2088	-3.2087	3.2088	0.1039	-1	-1.0002	1

with these constants known, the characteristic shapes of the vibrating frame can be computed from equations (1-1), (1-2) for symmetrical modes and from equations (2-1), (2-2) for anti-symmetrical modes as listed in the following tables.

For symmetrical modes, the characteristic shape equations are:

the vertical bar

$$X_1(x_1) = A_1 \sin k_1x_1 + A_2 \cos k_1x_1 + A_3 \sinh k_1x_1 + A_4 \cosh k_1x_1 \qquad (3\text{-}25)$$

the horizontal bar

$$X_2(x_2) = B_1 \sin k_2x_2 + B_2 \cos k_2x_2 + B_3 \sinh k_2x_2 + A_4 \cosh k_2x_2 \qquad (3\text{-}26)$$

For anti-symmetrical modes, the characteristic shape equations are:

the vertical bar

$$X_1(x_1) = C_1 \sin k_1x_1 + C_2 \cos k_1x_1 + C_3 \sinh k_1x_1 + C_4 \cosh k_1x_1 \qquad (3\text{-}27)$$

the horizontal bar

$$X_2(x_2) = D_1 \sin k_2x_2 + D_2 \cos k_2x_2 + D_3 \sinh k_2x_2 + D_4 \cosh k_2x_2 \qquad (3\text{-}28)$$

Table 4 to Table 13 contain the corresponding $X_1(x_1)$ and $X_2(x_2)$ values for equation (3-25) and equation (3-26) in tabular form for the first five symmetrical modes. While Fig. 9 to Fig. 13 are the corresponding characteristic curves. Table 14 to Table 23 contain the corresponding $X_1(x_1)$ and $X_2(x_2)$ values for equation (3-27) and equation (3-28) in tabular form for the first five anti-symmetric modes. While Fig. 14 to Fig. 18 contain the corresponding characteristic curves.

First symmetrical mode -- $\phi_1 = 3.8053$, $\phi_2 = 3.2007$

Table 4. Calculation for equation (3-25), $\phi_1 = 3.8053$

1	2	3	4	5	6	7
k_1x_1	x/L	$A_1 \sin k_1x_1$	$A_2 \cos k_1x_1$	$A_3 \sinh k_1x_1$	$A_4 \cosh k_1x_1$	3+4+5+6
0.3806	0.100	-3.8152	9.4556	4.0042	-10.9313	-1.2867
0.7613	0.200	-7.0861	7.3728	8.5957	-13.2820	-4.3996
1.1419	0.300	-9.3393	4.2359	14.4474	-17.5790	-8.2305
1.5225	0.400	-10.2574	0.4919	22.4167	-24.4527	-11.8015
1.9000	0.500	-9.7182	-3.2927	33.5634	-34.8086	-14.2561
2.3000	0.605	-7.6581	-6.7861	50.7015	-51.3029	-15.0456
2.7000	0.710	-4.3893	-9.2801	76.0605	-76.1161	-13.6530
3.0000	0.798	-1.4491	-10.0830	102.8808	-102.5375	-11.1888
3.4000	0.894	2.6239	-9.8467	153.6902	-152.7588	-6.2914
3.8063	1.000	6.3344	-8.0165	230.8700	-229.1880	-0.0001

Table 5. Calculation for equation (3-26), $\phi_2 = 3.2007$

1	2	3	4	5	6	7
k_2x_2	x/L	$B_1 \sin k_2x_2$	$B_2 \cos k_2x_2$	$B_3 \sinh k_2x_2$	$B_4 \cosh k_2x_2$	3+4+5+6
0.3201	0.100	10.6446	-0.9492	-0.3001	1.0517	10.4470
0.6401	0.200	20.2035	-0.8020	-0.6311	1.2120	19.9824
0.9602	0.300	27.7126	-0.5734	-1.0274	1.4975	27.6093
1.2803	0.400	32.4074	-0.2864	-1.5299	1.9378	32.5288
1.6004	0.500	33.8112	0.0296	-2.1905	2.5784	34.2287
1.9204	0.600	331.7784	0.3425	-3.0772	3.4851	32.5287
2.2000	0.687	27.3473	0.5885	-4.1081	4.5679	28.3954
2.6000	0.813	17.4366	0.8569	-6.1705	6.7690	18.8920
2.9000	0.904	8.0909	0.9710	-8.3502	9.1146	9.8263
3.2007	1.000	-1.9991	0.9983	-11.2947	12.2952	0.0000

Second symmetrical mode -- $\phi_1 = 4.8888$, $\phi_2 = 4.1110$

Table 6. Calculation for equation (3-25), $\phi_1 = 4.8888$

1	2	3	4	5	6	7
k_1x_1	x_1/L	$A_1 \sin k_1x_1$	$A_2 \cos k_1x_1$	$A_3 \sinh k_1x_1$	$A_4 \cosh k_1x_1$	3+4+5+6
0.4889	0.100	0.6792	-1.2989	-0.7354	1.6505	0.2954
0.9778	0.200	1.1992	-0.8221	-1.6502	2.2324	0.9593
1.4667	0.300	1.4382	-0.1529	-2.9673	3.3585	1.6765
1.9556	0.400	1.3403	0.5523	-5.0099	5.3034	2.1861
2.4000	0.491	0.9768	1.0849	-7.9041	8.1753	2.3329
2.9000	0.593	0.3459	1.4285	-13.1002	13.4094	2.0836
3.4000	0.695	-0.3659	0.4224	-21.6400	22.0661	1.4790
4.5000	0.920	-1.4135	0.3101	-65.0743	66.2247	0.0470
4.7000	0.960	-1.4458	0.0182	-79.4852	80.8838	-0.0290
4.8888	1.000	-1.4236	-0.2611	-96.0047	97.6889	-0.0005

Table 7. Calculation for equation (3-26), $\phi_2 = 4.1110$

1	2	3	4	5	6	7
k_2x_2	x_2/L	$B_1 \sin k_2x_2$	$B_2 \cos k_2x_2$	$B_3 \sinh k_2x_2$	$B_4 \cosh k_2x_2$	3+4+5+6
0.4111	0.100	0.7588	-0.9167	-0.4091	1.0857	0.5187
0.8222	0.200	1.3912	-0.6806	-0.8883	1.3575	1.1798
1.2333	0.300	1.7919	-0.3311	-1.5199	1.8619	1.8028
1.6444	0.400	1.8939	0.0735	-2.4119	2.6855	2.2410
2.1000	0.512	1.6202	0.5048	-3.8920	4.1443	2.3773
2.5000	0.608	1.1366	0.8011	-5.8548	6.1323	2.2152
2.9000	0.706	0.4542	0.9710	-8.7670	9.1146	1.7728
3.3000	0.803	-0.2995	0.9875	-13.1006	13.5748	1.1622
3.7000	0.901	-1.0061	0.8481	-19.5585	20.2360	0.5195
4.1110	1.000	-1.5660	0.5658	-29.5107	30.5120	0.0001

Third symmetrical mode -- $\phi_1 = 7.7136$, $\phi_2 = 6.4863$

Table 8. Calculation for equation (3-25), $\phi_1 = 7.7136$

1	2	3	4	5	6	7
k_1x_1	x_1/L	$A_1 \sin k_1x_1$	$A_2 \cos k_1x_1$	$A_3 \sinh k_1x_1$	$A_4 \cosh k_1x_1$	3+4+5+6
0.7714	0.100	-1.3787	1.4168	1.6815	-2.5941	-0.8745
1.5427	0.200	-1.9770	0.0555	4.4139	-4.8330	-2.3406
2.3000	0.298	-1.4748	-1.3168	9.7644	-9.9550	-2.9822
3.1000	0.402	-0.0823	-1.9745	21.9075	-21.9794	-2.1291
3.9000	0.496	1.3603	-1.4346	48.8341	-48.8369	-0.0771
4.6000	0.596	1.9653	-0.2217	98.3700	-98.3154	1.7982
5.4000	0.700	1.5284	1.2544	218.9442	-218.7873	3.2097
6.2000	0.804	0.1642	1.9694	487.2775	-486.9119	2.4992
6.9000	0.895	-1.1440	1.6121	981.2596	-980.5173	1.2104
7.7136	1.000	-1.9584	0.2765	2213.7397	-2212.0588	-0.0010

Table 9. Calculation for equation (3-26), $\phi_2 = 6.4863$

1	2	3	4	5	6	7
k_1x_1	x_1/L	$B_1\sin k_2x_2$	$B_2\cos k_2x_2$	$B_3\sinh k_2x_2$	$B_4\cosh k_2x_2$	3+4+5+6
0.6486	0.100	-0.0616	-0.7969	-0.6929	1.2178	-0.3336
1.2973	0.200	-0.0981	-0.2701	-1.6880	1.9663	-0.0899
1.9459	0.300	-0.0948	0.3664	-3.4182	3.5714	0.4302
2.6000	0.401	-0.0525	0.8569	-6.6746	6.7690	0.8988
3.2000	0.493	0.0060	0.9983	-12.2092	12.2866	1.0817
3.9000	0.601	0.0701	0.7259	-24.6170	24.7113	0.8903
4.5000	0.693	0.0996	0.2108	-44.8680	45.0141	0.4565
5.2000	0.801	0.0900	-0.4685	-90.3615	90.6389	-0.1011
5.8000	0.894	0.0473	-0.8855	-164.6529	165.1513	-0.3398
6.4863	1.000	-0.0206	-0.9794	-327.0608	328.0324	-0.0004

Fourth symmetrical mode -- $\phi_1 = 10.5998$, $\phi_2 = 8.9133$

Table 10. Calculation for equation (3-25), $\phi_1 = 10.5998$

1	2	3	4	5	6	7
k_1x_1	x_1/L	$A_1\sin k_1x_1$	$A_2\cos k_1x_1$	$A_3\sinh k_1x_1$	$A_4\cosh k_1x_1$	3+4+5+6
1.0600	0.100	2.7314	-1.5307	-3.9669	5.0609	2.2945
2.1000	0.198	2.7026	1.5805	-12.5921	12.9758	4.6664
3.2000	0.302	-0.1828	3.1257	-38.3407	38.4693	3.0715
4.2000	0.396	-2.7289	1.5351	-104.3707	104.4210	-1.1435
5.3000	0.499	-2.6058	-1.7358	-313.6094	313.6351	-4.3159
6.4000	0.603	0.3647	-3.1097	-942.1558	942.1909	-2.7099
7.4000	0.678	2.8137	-1.3729	-2561.0508	2561.1345	1.5245
8.5000	0.801	2.5000	1.8849	-7693.8245	7694.0709	4.6313
9.5000	0.896	-0.2354	3.1222	-20913.9843	20914.6526	3.5551
10.5998	1.000	-2.8889	1.2070	-62816.5304	62818.2167	0.0044

Table 11. Calculation for equation (3-26), ϕ_2 = 8.9133

1	2	3	4	5	6	7
k_2x_2	x_2/L	$A_1\sin k_2x_2$	$A_2\cos k_2x_2$	$A_3\sinh k_2x_2$	$A_4\cosh k_2x_2$	3+4+5+6
0.8913	0.100	-2.9754	-0.6284	-1.0138	1.4242	-3.1934
1.7827	0.200	-3.7392	0.2103	2.8880	3.0570	-3.3599
2.7000	0.303	-1.6348	0.9041	-7.4041	7.4735	-0.6613
3.6000	0.404	1.6925	0.8968	-18.2800	18.3128	2.6221
4.5000	0.505	3.7388	0.2108	-44.9895	45.0141	3.9742
5.4000	0.595	2.9559	-0.6347	-110.6677	110.7055	2.3590
6.2000	0.696	2.6128	-0.9965	-246.2996	246.3755	1.6922
7.1000	0.797	-2.7884	-0.6845	-605.8013	605.9839	-3.2905
8.0000	0.898	-3.7844	0.1455	-1490.0371	1490.4791	-3.1915
8.9133	1.000	-1.8723	0.8720	-3713.9995	3715.0001	0.0002

Fifth symmetrical mode -- $\emptyset_1$ 11.7546, $\emptyset_2 = 9.8844$

Table 12. Calculation for equation (3-25), $\emptyset_1 = 11.7546$

1	2	3	4	5	6	7
k_1x_1	x_1/L	$A_1\sin k_1x_1$	$A_2\cos k_1x_1$	$A_3\sinh k_1x_1$	$A_4\cosh k_1x_1$	3+4+5+6
1.1755	0.100	1.0979	-0.4581	-1.7434	2.1106	1.0124
2.4000	0.202	0.8036	0.8772	-6.5026	6.6105	1.7887
3.6000	0.304	-0.5264	1.0668	-21.7524	21.7849	0.5729
4.7000	0.397	-1.1895	0.0148	-65.3914	65.4019	-1.1639
5.9000	0.497	-0.4447	-1.1034	-217.1227	217.1259	-1.5449
7.1000	0.598	0.8671	-0.8143	-720.8775	720.8784	0.0537
8.3000	0.699	1.0731	0.5132	-2393.3992	2393.3994	1.5861
9.5000	0.801	-0.0895	1.1863	-7946.3655	7946.3656	1.0969
10.6000	0.892	-1.0976	0.4586	-23867.4325	23867.4325	-0.6390
11.7546	1.000	-0.8631	-0.8187	-75740.6186	75740.6186	-0.0444

Table 13. Calculation for equation (3-26), $\phi_2 = 9.8844$

1	2	3	4	5	6	7
k_2x_2	x_2/L	$B_1 \sin k_2x_2$	$B_2 \cos k_2x_2$	$B_3 \sinh k_2x_2$	$B_4 \cosh k_2x_2$	3+4+5+6
0.9884	0.100	3.5696	-0.5500	-1.1572	1.5296	3.3920
1.9769	0.200	3.9270	0.3950	-3.5405	3.6794	4.4609
3.0000	0.303	0.6031	0.9900	-10.0169	10.0677	1.6439
4.0000	0.404	-3.2349	0.6536	-27.2872	27.3082	-2.5603
5.0000	0.505	-4.0988	-0.2837	-74.1958	74.2099	-6.9227
5.9000	0.596	-1.5982	-0.9275	-182.4991	182.5201	-2.5047
6.9000	0.697	2.4724	-0.8157	-496.0873	496.1379	1.7073
7.9000	0.798	4.2698	0.0460	-7348.5061	1348.6413	4.4510
8.9000	0.899	2.1415	0.8654	-3666.6201	3665.9868	3.3736,
9.8844	1.000	-1.8962	0.8962	-9809.9350	9810.9350	0.0000

First anti-symmetrical mode -- $\phi_1 = 1.6775$, $\phi_2 = 1.4106$

Table 14. Calculation for equation (3-27), $\phi_1 = 1.6775$.

1	2	3	4	5	6	7
k_1x_1	x_1/L	$C_1 \sin k_1x_1$	$C_2 \cos k_1x_1$	$C_3 \sinh k_1x_1$	$C_4 \cosh k_1x_1$	3+4+5+6
0.1678	0.100	-0.6922	3.3953	0.6989	-3.4921	-0.0901
0.3355	0.200	-1.3646	3.2514	1.4168	-3.6391	-0.3355
0.5033	0.300	-1.9992	3.0162	2.1753	-3.8899	-0.6968
0.6710	0.400	-2.5774	2.6969	2.9948	-4.2482	-1.1339
0.8388	0.500	-3.0831	2.3013	3.8993	-4.7276	-1.6101
1.0065	0.600	-3.5026	1.8416	4.9132	-5.3402	-2.0880
1.1733	0.700	-3.8218	1.3330	6.0585	-6.0984	-2.5287
1.3410	0.800	-4.0364	0.7844	7.3808	-7.0323	-2.9032
1.5108	0.900	-4.1376	0.2066	8.9313	-8.1800	-3.1791
1.6775	1.000	-4.1215	0.3667	10.7055	-9.5368	-2.5861

Table 15. Calculation for equation (3-28), $\phi_2 = 1.4106$

1	2	3	4	5	6	7
k_2x_2	x_2/L	$D_1\sin k_2x_2$	$D_2\cos k_2x_2$	$D_3\sinh k_2x_2$	$D_4\cosh k_2x_2$	3+4+5+6
0.1411	0.100	0.1651	-0.9901	-0.2330	1.0100	-0.0480
0.2821	0.200	0.3269	-0.9605	-0.4704	1.0401	-0.0639
0.4232	0.300	0.4824	-0.9118	-0.7131	1.0909	-0.0558
0.5642	0.400	0.6280	-0.8450	-0.9784	1.1634	-0.0320
0.7053	0.500	0.7614	-0.7614	-1.2591	1.2592	0.0001
0.8464	0.600	0.8796	-0.6627	-1.5650	1.3801	0.0320
0.9874	0.700	0.9802	-0.5509	-1.9020	1.5284	0.0557
1.1285	0.800	1.0615	-0.4280	-2.2769	1.7073	0.0639
1.2695	0.900	1.1216	-0.2928	-2.6971	1.9200	0.0517
1.4106	1.000	1.1595	-0.1597	-3.1712	2.1712	0.0000

Second anti-symmetrical mode -- $\phi_1 = 4.7181$, $\phi_2 = 3.9680$

Table 16. Calculation for equation (3-27), $\phi_1 = 4.7181$

1	2	3	4	5	6	2
k_1x_1	x_1/L	$C_1\sin k_1x_1$	$C_2\cos k_1x_1$	$C_3\sinh k_1x_1$	$C_4\cosh k_1x_1$	3+4+5+6
0.4179	0.100	1.5453	-3.0627	-1.6642	3.8284	0.6468
0.9437	0.200	2.7523	-2.0177	-3.7055	5.0866	2.1157
1.4156	0.300	3.3584	-0.5316	-6.5880	7.4990	3.7378
1.8875	0.400	3.2303	1.0707	-10.9848	11.6122	4.9484
2.4000	0.508	2.6962	2.5355	-18.5807	19.1074	5.7584
2.8000	0.593	1.1387	3.2398	-27.8459	28.3769	4.9095
3.3000	0.698	-0.5361	3.3955	-46.0180	46.6769	3.5183
3.8000	0.804	-2.0800	2.7199	-75.9361	76.8190	1.5948
4.3000	0.910	-3.1143	1.3782	-125.2371	126.7318	-0.2414
4.7187	1.000	-3.3992	-0.0217	-190.3783	192.6100	-1.1892

Table 17. Calculation for equation (3-28), $\phi = 3.9680$

1	2	3	4	5	6	7
k_2x_2	x_2/L	$D_1 \sin k_2x_2$	$D_2 \cos k_2x_2$	$D_3 \sinh k_2x_2$	$D_4 \cosh k_2x_2$	3+4+5+6
0.3968	0.100	-0.1694	-0.9223	-0.4203	1.0798	-0.4349
0.7936	0.200	-0.3125	-0.7013	-0.9136	1.3318	-0.5965
1.1904	0.300	-0.4071	-0.3713	-1.5479	1.7962	-0.5319
1.5872	0.400	-0.4384	0.0164	-2.4332	2.5473	-0.3079
1.9840	0.500	-0.4015	0.4015	-3.6996	3.7046	0.0050
2.4000	0.605	-0.2961	0.7374	-5.6772	5.5569	0.3210
2.8000	0.706	-0.1469	0.9422	-8.5081	8.2527	0.5399
3.2000	0.807	0.0256	0.9983	-12.7183	12.2866	0.5919
3.6000	0.908	0.1940	0.8968	-18.9912	18.3128	0.4124
3.9679	1.000	0.3224	0.6776	-27.4473	26.4462	0.0011

Third anti-symmetrical mode -- $\phi_1 = 7.3169$, $\phi_2 = 6.1550$

Table 18. Calculation for equation (3-27), $\phi_1 = 7.3196$

1	2	3	4	5	6	7
k_1x_1	x_1/L	$C_1\sin k_1x_1$	$C_2\cos k_1x_1$	$C_3\sinh k_1x_1$	$C_4\cosh k_1x_1$	3+4+5+6
0.7320	0.100	-7.6195	8.4774	9.1094	-14.5898	-4.6225
1.4639	0.200	-11.3346	1.2164	23.3202	-25.9530	-12.7513
2.2000	0.301	-9.2166	-6.7074	50.8092	-52.0622	-17.1770
2.9000	0.396	-2.7268	-11.0669	103.2758	-103.8827	-14.4006
3.7000	0.506	6.0395	-9.6661	230.4007	-230.6378	-3.8637
4.4000	0.602	10.8479	-3.5024	464.1837	-464.2343	7.2947
5.1000	0.696	10.5537	4.3082	934.8573	-934.7464	14.9728
5.9000	0.806	4.2623	10.5711	2080.6254	-2080.2546	15.2042
6.6000	0.902	-3.5510	10.8298	4189.8876	-4189.0950	8.0714
7.3196	1.000	-9.8105	5.8047	8604.4147	-8602.7609	-2.3520

Table 19. Calculation for equation (3-28), $\phi_2 = 6.1550$

1	2	3	4	5	6	7
k_2x_2	x_2/L	$D_1 \sin k_2x_2$	$D_2 \cos k_2x_2$	$D_3 \sinh k_2x_2$	$D_4 \cosh k_2x_2$	3+4+5+6
0.6155	0.100	-8.9983	-0.8165	-0.6579	1.1955	-9.2772
1.2310	0.200	-14.6927	-0.3333	-1.5730	1.8583	-14.7407
1.8465	0.300	-14.9950	0.2722	-3.1032	3.2477	-14.5783
2.5000	0.406	-9.3271	0.8011	-6.0762	6.1323	-8.4699
3.1000	0.504	-0.6483	0.9991	-11.1241	11.1215	0.3482
3.7000	0.602	8.2565	0.8481	-20.2982	20.2360	9.0424
4.3000	0.698	14.2782	0.4008	-37.0015	36.8567	14.5342
4.9000	0.796	15.3114	-0.1865	-67.4299	67.1486	14.8346
5.5000	0.894	10.9946	-0.7087	-122.8700	122.3408	9.7639
6.1550	1.000	1.9916	-0.9918	-236.5452	235.5445	0.0009

Fourth anti-symmetrical mode -- $\phi_1 = 8.2914$, $\phi_2 = 6.9722$

Table 20. Calculation for equation (3-27), $\phi_1 = 8.2914$

1	2	3	4	5	6	7
k_1x_1	x_1/L	$C_1\sin k_1x_1$	$C_2\cos k_1x_1$	$C_3\sinh k_1x_1$	$C_4\cosh k_1x_1$	3+4+5+6
0.8291	0.100	-1.6925	1.5479	2.1288	-3.1288	-1.1428
1.6583	0.200	-2.2868	-0.2005	5.8076	-6.2411	-2.9208
2.5000	0.301	-1.3739	-1.8379	13.8882	-14.0687	-3.3923
3.3000	0.398	0.3620	-2.2655	31.0762	-31.1433	-1.9706
4.2000	0.506	2.0008	-1.1248	76.5221	-76.5132	0.8849
5.0000	0.602	2.2012	0.6509	170.3334	-170.2524	2.9331
5.8000	0.698	1.0665	2.0315	379.0979	-378.8901	3.3058
6.7000	0.796	-0.9292	2.0978	932.4374	-931.9121	1.6936
7.5000	0.903	-2.1532	0.7952	2075.1800	-2074.0061	-0.1841
8.2914	1.000	-2.0788	-0.9718	4518.8522	-4576.2591	-0.4575

Table 21. Calculation for equation (3-28), $\phi_2 = 6.9722$

1	2	3	4	5	6	7
k_2x_2	x_2/L	$D_1 \sin k_2x_2$	$D_2 \cos k_2x_2$	$D_3 \sinh k_2x_2$	$D_4 \cosh k_2x_2$	3+4+5+6
0.6972	0.100	1.7895	-0.7666	-0.7565	1.2530	1.5194
1.3944	0.200	2.7437	-0.1755	-1.8959	2.1403	2.8126
2.1000	0.301	2.4057	0.5048	-4.0295	4.1443	3.0253
2.8000	0.402	0.9336	0.9422	-8.2075	8.2527	1.9210
3.5000	0.502	-0.9776	0.9365	-16.5740	16.5728	-0.0423
4.2000	0.602	-2.4291	0.4903	-33.3990	33.3507	-1.9871
4.9000	0.702	-2.7381	-0.1865	-67.2688	67.1486	-3.0448
5.6000	0.802	-1.7594	-0.7756	-135.4683	135.2151	12.7882
6.3000	0.903	0.0466	-0.9999	-272.8023	272.2869	-1.4685
6.9722	1.000	1.7719	-0.7719	-534.2861	533.2838	-0.0023

Fifth anti-symmetrical mode -- $\phi_1 = 10.9617$, $\phi_2 = 9.2177$

Table 22. Calculation for equation (3-27), $\phi_1 = 10.9617$

1	2	3	4	5	6	7
k_1x_1	x_1/L	$C_1\sin k_1x_1$	$C_2\cos k_1x_1$	$C_3\sinh k_1x_1$	$C_4\cosh k_1x_1$	3+4+5+6
1.0962	0.100	2.8541	-1.4664	-4.2653	5.3378	2.4602
2.2000	0.201	2.5942	1.8884	-14.3015	14.4649	4.6460
3.3000	0.301	-0.5060	3.1687	-43.4391	43.5588	2.7804
4.4000	0.401	-3.0534	0.9861	-130.6560	130.6996	-2.0237
5.5000	0.501	-2.2637	-2.2741	-392.5649	392.5903	-4.5124
6.6000	0.601	0.9995	-3.0490	-1179.3890	1179.3890	-2.0082
7.7000	0.707	3.1708	-0.4922	-3542.9625	3543.0742	2.7903
8.8000	0.802	1.8768	2.6027	-10643.6490	10643.9814	4.8119
9.9000	0.902	-1.4680	2.8533	-31975.2898	31976.2863	2.3818
10.9617	1.000	-3.2068	1.0868	-92449.5778	92452.4590	0.7612

Table 23. Calculation for equation (3-28), $\phi_2 = 9.2177$

1	2	3	4	5	6	7
k_2x_2	x_2/L	$D_1 \sin k_2x_2$	$D_2 \cos k_2x_2$	$D_3 \sinh k_2x_2$	$D_4 \cosh k_2x_2$	3+4+5+6
0.9218	0.100	0.0828	-0.6044	-1.0582	1.4558	-0.1240
1.8435	0.200	0.1001	0.2693	-3.0808	3.2384	0.5270
2.8000	0.304	0.0348	0.9422	-8.1935	8.2527	1.0363
3.7000	0.402	-0.0551	0.8461	-20.2153	20.2360	0.8137
4.6000	0.498	-0.1033	0.1122	-49.7470	49.7472	0.0092
5.5000	0.596	-0.0733	-0.7087	-122.3684	122.3480	-0.8024
6.5000	0.704	0.0224	-0.9766	-322.6366	322.5716	-1.0192
7.4000	0.804	0.0934	-0.4385	-818.1555	817.9925	-0.5081
8.3000	0.902	0.0937	0.4314	-2012.3385	2011.9363	0.1232
9.2177	1.000	0.0214	0.9786	-5037.9414	5036.9300	0.0034

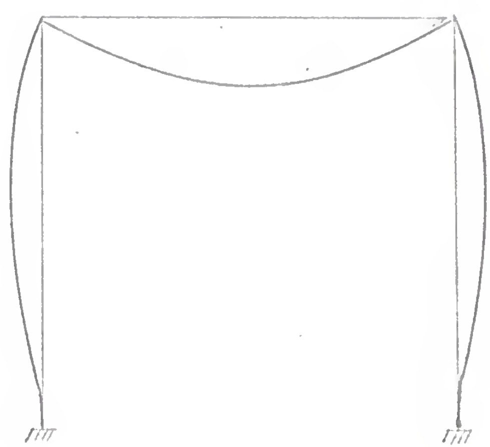


Fig. 9

First symmetric mode

Second normal mode

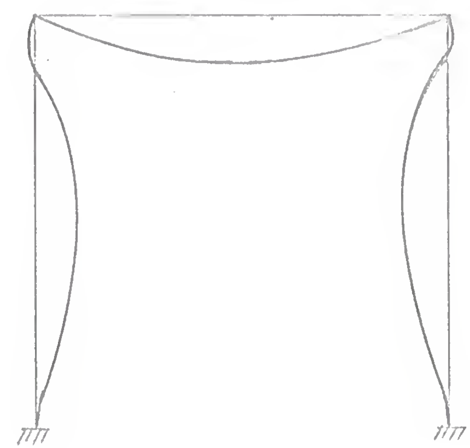


Fig. 10

Second symmetrical mode

Fourth normal mode

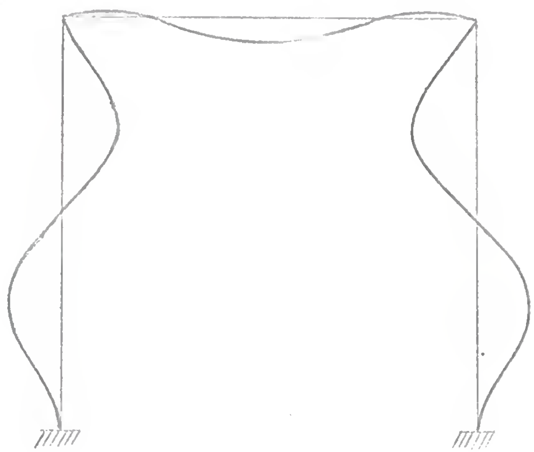


Fig. 11

Third symmetric mode

sixth normal mode

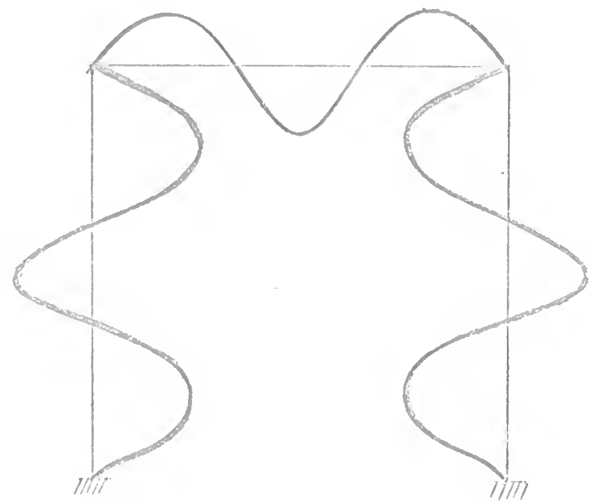


Fig. 12

fourth symmetric mode

eighth normal mode

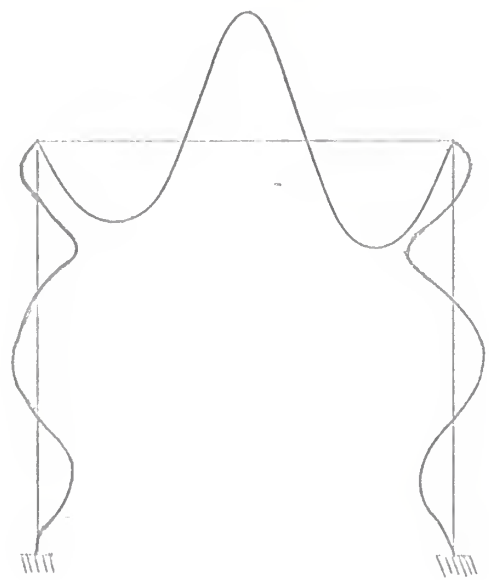


Fig. 13

fifth symmetric mode

tenth normal mode

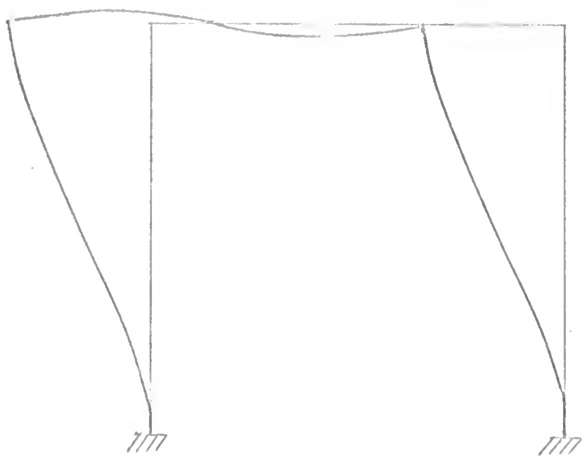


Fig. 14

First anti-symmetric mode

First normal mode

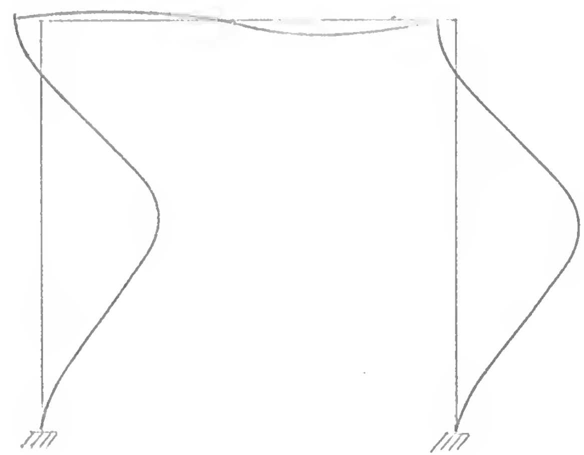


Fig. 15

Second anti-symmetric mode

Third normal mode

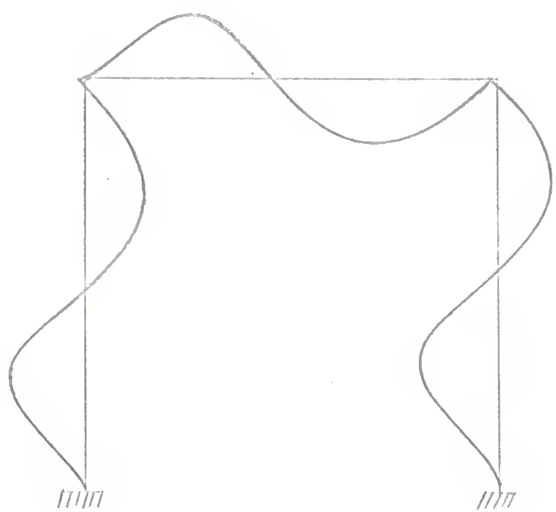


Fig. 16

Third anti-symmetric mode

Fifth normal mode

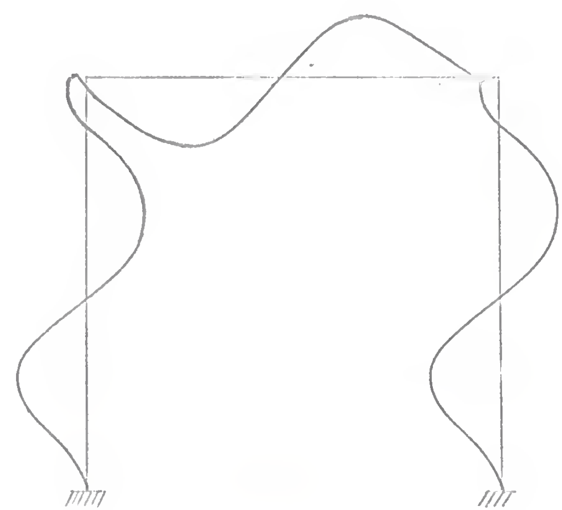


Fig. 17

Fourth anti-symmetric mode

Seventh normal mode

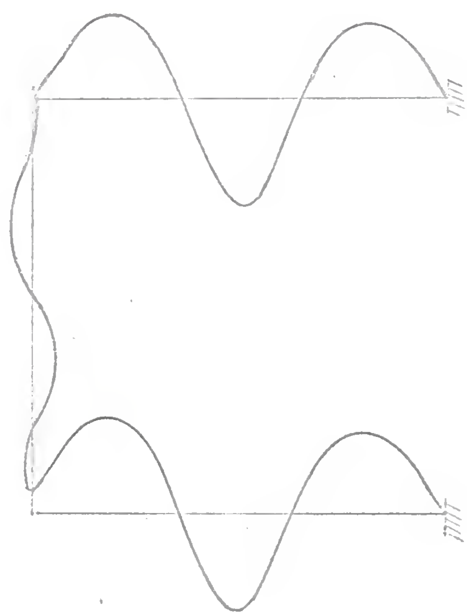


Fig. 18

USE RAYLEIGH'S METHOD TO SOLVE THE NATURAL FREQUENCY OF THE FIRST NORMAL MODE OF VIBRATION OF THE PREVIOUS FRAME

Derivation of formula

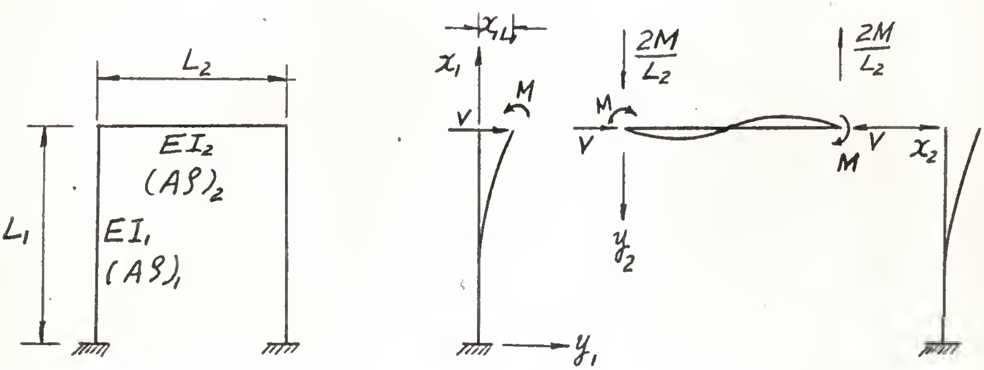


Fig. 17

The above discussion shows that the first normal mode of vibration is the first anti-symmetrical mode as shown in Fig. 17. Because of symmetry, the shear force V and the bending moment M at the end of the vertical bars are the same.

Determination of the deflection equation

For the vertical bar

by equation (5)

$$EI_1 \frac{\partial^2 y}{\partial x_1^2} = EI_1 \frac{d^2}{dx_1^2} X_1(x_1) = V(L_1 - x_1) - M \tag{4-1}$$

integrate equation (4-1)

$$EI_1 \frac{d}{dx_1} X_1(x_1) = V(L_1x_1 - \tfrac{1}{2}x_1^2) - Mx_1 + n_1 \tag{4-2}$$

integrate equation (4-2)

$$EI_1 \left[X_1(x_1)\right] = V(\frac{L_1x_1^2}{2} - \frac{x_1^3}{6}) - \frac{Mx_1^2}{2} + n_1x_1 + n_2 \tag{4-3}$$

boundary condition

(1) $X_1'(x_1)\Big|_{x_1 = 0} = 0$ $\qquad n_1 = 0$

(2) $X_1(x_1)\Big|_{x_1 = 0} = 0$ $\qquad n_2 = 0$

and equation (4-2) takes the form

$$EI_1X_1'(x_1) = V(L_1x_1 - \frac{x_1^2}{2}) - Mx_1 \tag{4-4}$$

equation (4-3) takes the form

$$EI_1X_1(x_1) = V(\frac{L_1x_1^2}{2} - \frac{x_1^3}{6}) - \frac{Mx_1^2}{2} \tag{4-5}$$

For the horizontal bar

by equation (5)

$$EI_2X_2''(x_2) = - M + 2M \frac{x_2}{L_2} \tag{4-6}$$

integrate equation (4-6)

$$EI_2X_2'(x_2) = -Mx_2 + M\frac{x_2^2}{L} + n_3 \tag{4-7}$$

integrate equation (4-7)

$$EI_2X_2(x_2) = -\frac{Mx_2^2}{2} + \frac{Mx_2^3}{3L_2} - n_3x_2 + n_4 \tag{4-8}$$

n_3, n_4, are integrating constants to be determined by boundary conditions

Boundary condition

(1) $X_2(x_2)\Big|_{x_2 = 0} = 0$ $\qquad\qquad n_4 = 0$

(2) $X_2(x_2)\Big|_{x_2 = L/2} = 0$ $\qquad\qquad n_3 = \frac{ML_2}{6}$

with n_3, n_4, thus found, equations (4-7) and (4-8) takes the form

$$EI_2X_2'(x_2) = -Mx_2 + M\frac{x_2}{L_2} + \frac{ML_2}{6} \tag{4-9}$$

$$EI_2X_2(x_2) = -\frac{Mx_2^2}{2} + \frac{Mx_2^3}{3L_2} + \frac{ML_2}{6}x_2 \tag{4-10}$$

compatability equation: the joint at b is rigid

$$X_1'(x_1)\Big|_{x_1 = L_1} = X_2'(x_2)\Big|_{x_2 = 0}$$

$$\frac{1}{EI_1}\left(\frac{VL_1^2}{2} - ML_1\right) = \frac{1}{EI_2}\left(\frac{ML_2}{6}\right)$$

simplify and rearrange

$$M = \frac{1}{\left(2 + \frac{1}{3}\frac{L_2 I_1}{L_1 I_2}\right)} VL_1 = PVL_1 \qquad (4\text{-}11)$$

where

$$\frac{1}{P} = 2 + \frac{1}{3}\frac{L_2 I_1}{L_1 I_2} \qquad (4\text{-}12)$$

Deflection equation for vertical bar

Substitute equation (4-12) into equation (4-5) and rearrange

$$X_1(x_1) = \frac{V}{EI_1}\left[(1 - P)\frac{L_1 x_1^2}{2} - \frac{x_1^3}{6}\right] \qquad (4\text{-}13)$$

Let X_{1L1} be the value of $X_1(x_1)$ at $x_1 = L_1$

$$X_{1L1} = \frac{VL_1^3}{EI_1}\ \frac{2-3P}{6}$$

rearrange

$$\frac{V}{EI_1} = \frac{X_{1L1}}{L_1^3}\ \frac{6}{2-3P} \qquad (4\text{-}14)$$

substitute equation (4-14) into equation (4-13) and simplify

$$X_1(x_1) = \frac{X_{1L1}}{2-3P}\left[3(1 - P)(\frac{x_1}{L_1})^2 - (\frac{x_1}{L_1})^3\right] \tag{4-15}$$

equation (4-15) is the deflection equation for vertical bar

differentiate equation (4-15) twice and simplify

$$X_1''(x_1) = \frac{X_{1L1}}{L_1^2} \frac{6}{2-3P}\left[(1-P) - \frac{x_1}{L_1}\right] \tag{4-16}$$

Deflection equation for horizontal bar

Substitute equation (4-12) into equation (4-10) and rearrange

$$X_2(x_2) = X_{1L1} (\frac{L_2}{L_1})^2 (\frac{P}{2-3P}) \left[(\frac{x_2}{L_2}) - \frac{1}{2}(\frac{x_2}{L_2})^2 + \frac{1}{3}(\frac{x_2}{L_2})^3 (\frac{I_1}{I_2})\right] \tag{4-17}$$

differentiate equation (4-17) twice and simplify

$$X_2''(x_2) = \frac{X_{1L1}}{L_2^2} \frac{L_2^2 I_1}{L_2^2 I_1} \frac{6P}{2-3P} (2 \frac{x_2}{L_2} - 1) \tag{4-18}$$

Determination of the frequency equation

(1) The kinetic energy (K.E.) and the potential energy (P.E.) of the vertical bar

$$X_1(x_1) = \frac{X_{1L1}}{2-3P}\left[3(1-P)(\frac{x_1}{L_1})^2 - (\frac{x_1}{L_1})^3\right] \tag{4-15}$$

$$X_1^2(x_1) = \frac{X_{1L1}^2}{(2-3P)^2} \frac{1}{L_1^6}\left[9(1-P)^2 L_1^2 x_1^4 - 6(1-P)L_1 x_1^5 + x_1^6\right] \tag{4-16'}$$

$$K.E. = \frac{1}{2} (A\rho)_1 \omega^2 \int_0^{L_1} X_1^2(x_1)\, dx_1 \qquad (4\text{-}17)$$

substitute equation (4-16) to equation (4-17) and simplify

$$K.E. = \frac{1}{2} (A\rho)_1 \omega^2 X_{1L1}^2 \frac{L_1}{(2-3P)^2} \left(\frac{33}{15} - \frac{13}{5} P + \frac{9}{5} P^2\right) \qquad (4\text{-}18)$$

square equation (4-16)

$$\left[X_1''(x_1)\right]^2 = \frac{X_{1L1}^2}{L_1^6} \left(\frac{6}{2-3P}\right)^2 \left[L_1^2(1-P)^2 - 2L_1(1-P)x_1 + x_1^2\right] \qquad (4\text{-}20)$$

$$P.E. = \frac{EI_1}{2} \int_0^{L_1} \left[X_1''(x_1)\right]^2 dx_1 \qquad (4\text{-}21)$$

substitute equation (4-20) into equation (4-21) and simplify

$$P.E. = \frac{EI_1}{2} \frac{X_{1L1}^2}{L_1^6} \frac{36}{(2-3P)^2} \left(\frac{1}{3} - P + P^2\right) \qquad (4\text{-}22)$$

(2) The kinetic energy (K.E.) and the potential energy (P.E.) of the horizontal bar

$$K.E._1 = \frac{1}{2} (A\rho)_2 \omega^2 \int_0^{L_2} X_2(x_2)\, dx_2 \qquad (4\text{-}23)$$

substitute equation (4-17) into equation (4-23) and simplify. Note here $K.E._1$ is due to the transverse vibration of the horizontal bar

$$K.E._1 = \frac{1}{2} (A\rho)_2 \omega^2 X^2_{1L1} L_2 \frac{L_2^4 I_1^2}{L_1^4 I_2^2} \frac{P^2}{(2-3P)^2} \left(\frac{1}{210}\right) \tag{4-24}$$

$$K.E._2 = \frac{1}{2} (A\rho)_2 L_2 \omega^2 X^2_{1L1} \tag{4-25}$$

$$K.E. = K.E._1 + K.E._2$$

$$= \frac{1}{2} (A\rho)_2 L_2 \omega^2 X^2_{1L1} \left[1 + \frac{L_2^4 I_1^2}{L_1^4 I_2^2} \frac{P^2}{(2-3P)^2} \frac{1}{210}\right] \tag{4-25}$$

where $K.E._2$ is due to the longitudinal vibration of the horizontal bar

$$P.E. = \frac{1}{2} EI_2 \int_0^{L_2} \left[X_2''(x_2)\right]^2 dx_2 \tag{4-26}$$

substitute equation (4-18) into equa ion (4-26) and simplify

$$P.E. = \frac{1}{2} EI_2 \frac{X^2_{1L1} L_2}{L_1^4} \frac{36P^2}{(2-3P)^2} \frac{1}{3} \tag{4-27}$$

Total K.E. = equation (4-18) + equation (4-25)

$$= \frac{1}{2} \omega^2 X^2_{1L1} L_1 \left[\frac{2(A\rho)_1}{(2-3P)^2} \left(\frac{33}{35} - \frac{13}{5} P + \frac{9}{5} P^2\right) \right.$$

$$\left. + \frac{L_2}{L_1} (A\rho)_2 \left(1 + \frac{L_2^4}{L_1^4} \frac{I_1^2}{I_2^2} \frac{P^2}{(2-3P)^2} \frac{1}{210}\right)\right] \tag{4-28}$$

Total P.E. = equation (4-22) + equation (4-27)

$$= \frac{EI_1}{2} \frac{X_1^2 L_1}{L_1^3} \frac{36}{(2-3P)^2} \left[\frac{2}{3} - 2P + \left(2 + \frac{1}{3} \frac{L_2}{L_1} \frac{I_1}{I_2}\right) P^2 \right] \qquad (4\text{-}29)$$

Rayleigh's equation Total K.E. = Total P.E.

Equating equation (4-28) and equation (4-29) and simplify

$$\omega^2 = \frac{EI_1}{(A\rho)_1 L_1^4} \frac{36}{(2-3P)^2} \frac{\frac{2}{3} - 2P + \left(2 + \frac{1}{3} \frac{L_2 I_1}{L_1 I_1}\right) P^2}{\frac{2}{(2-3P)^2} \left(\frac{33}{35} - \frac{13}{5} P + \frac{9}{5} P^2\right) + \frac{L_2 I_1}{L_1 I_2} \left(1 + \frac{L_2^4 I_1^2}{L_1^4 I_2^2} \frac{P^2}{210(2-3P)^2}\right)} \qquad (4\text{-}30)$$

Numerical illustrative example

This is the same problem as that on page 16. With

$$\frac{I_1}{I_2} = 0.25, \qquad \frac{L_1}{L_2} = 1, \qquad \text{and} \qquad \frac{(A\rho)_1}{(A\rho)_2} = 0.5$$

substitute into equation (4-12) to find P

$$P = 0.48$$

With these values, substitute into equation (4-30), to find ω^2,

$$\omega^2 = 8.8312 \frac{EI_1}{(A\rho)_1 L_1^4}$$

$$\omega = 2.9717 \sqrt{\frac{EI_1}{(A\rho)_1 L_1^4}}$$

From Table 1, for the exact solution, we have

$$\omega = 2.8140 \sqrt{\frac{EI_1}{(A\rho)_1 L_1^4}}$$

Hence, the Rayleigh's method is within an error of 5.6%. For most practical cases, this error is insignificant on account of the safety factor usually used in any structural design and Rayleigh's method is a good approximation.

APPENDIX

```
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    1[illegible] READ 12, PHI[illegible], [illegible], NP
    12 FORMAT (2F20.8, I10)
       DO 21 I=1, NP
       PHIB=(0.5**0.25)*PHIA
       CA=COSF(PHIA)
       SA=SINF(PHI[illegible])
       CB=COSF(PHI[illegible])
       SB=SINF(PHI[illegible])
       HAA=EXPF(P[illegible]I[illegible])
       HBA=EXPF(-[illegible])
       HA[illegible]=EXP[illegible]
       H[illegible]=EXP[illegible](-[illegible])
       SHA=(H[illegible]-[illegible]
       CHA=(H[illegible]+[illegible]
       CHB=(HA[illegible]+[illegible]
       SHB=(HA[illegible]-[illegible].5
       D1=(PHIA*(1.[illegible]-[illegible]*CHA))-(C[illegible]*SA+SA*CHA)
       D2=(PHIA*(SA*C[illegible]-[illegible]*CA))-([illegible].0+CHA*CA)
       X1=D1/D2
       Y1=(8.[illegible]/(2.[illegible].[illegible]))*(CA*C[illegible]A-1.0)/(CA*SHA-SA*CHA)
       X2=(2.0**0.2[illegible]) [illegible]*SHB-S[illegible]*CHB+SHB-SB)/(8.0*SB*SHB)
       Y2=(CB-1.[illegible])/[illegible](1.[illegible]-CHB)/SHB
       PUNCH 2[illegible], PHIA, PHIB, Y1, Y2, X1, X2
    2[illegible] FORMAT (3F2[illegible].8/3F20.8)
    21 PHIA=PHIA+DELP
       STOP
       END
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1.60000000	1.34543420	2.73401340
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1.70000000	1.42952380	2.91791500
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-6.2763942	1.27483370	-.14086132
3.4[illegible]	2.85904760	9.80436530
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3.5[illegible]	2.94313730	11.48287300
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-.92324524	-2.82929090	-2.20114230
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FREE VIBRATION OF
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by

TSUN LEE

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KANSAS STATE UNIVERSITY
Manhattan, Kansas

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ABSTRACT

By applying the Bernoulli-Euler equation, the equation of motion for a vibration elastic beam is derived in the beginning of this report. Since a rectangular rigid frame can be considered as three elastic beams joined rigidly at right angles, the frequency equation can be obtained from direct expansion of the frequency determinant that follows as a consequence of satisfying boundary conditions of the equations of motion for the vibrating elastic beams. In order to show how to use these frequency equations, a numerical example is presented and the natural frequencies and mode shapes of the first ten modes of vibration of a simple rectangular fixed end frame are obtained by the aid of a 1620 digital computer.

Rayleigh's method is also used to find the natural frequency of the first vibration mode for the same numerical example and the error compared to the exact solution is found to be 5.6%.