Classification of 3-component links up to link homotopy

by

Shashika Dilshan Wijerathna

B.S., University of Peradeniya, Sri Lanka, 2020

A REPORT

submitted in partial fulfillment of the
requirements for the degree

MASTER OF SCIENCE

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2024

Approved by:

Major Professor
Victor Turchin

Copyright

Abstract

The concept of link-homotopy was introduced by Milnor's paper in 1954, in which he achieved the classification of 3-component closed links up to link-homotopy. After more than thirty years Levine achieved such a classification for 4-component closed links [3]. Then, in 1990, Habeggerr and Lin provided a new approach to a classification based on the closures of two different string links in the group of link homotopy classes [2]. The closures of two string links are link-homotopic if and only if those two elements are related by a sequence of elements, such that two adjacent elements are either conjugate or partial conjugate, in the link group. We employed Habbeger-Lin's approach to provide a full classification for 3-component string and closed links.

Contents

List of Figures

Acknowledgments

I want to express my deepest gratitude to my advisor Dr. Victor Turchin, for his inestimable guidance, encouragement, and support throughout all these years. His profound expertise, insightful feedback, and unwavering belief in me have been a great resource of motivation for me.

I am also thankful for my master's thesis committee members, Dr. Dave Auckly and Dr. Rina Anno for their support and invaluable contribution. I appreciate the time and effort you invested in providing the support.

I express my heartfelt gratitude to Dr. Ashwini Amarasinghe for his exceptional distance mentorship and constant support throughout my journey in mathematics. Your guidance and motivation through every challenge and your availability at all times have been inestimable.

Dedication

To everyone, who wished for my success.

Chapter 1

Introduction

A Link homotopy is a deformation of one link to another, in which each component is allowed to cross itself, but any two different components are required to stay disjoint. Thus, the classification of links up to link homotopy is coarser than the classification of links up to link isotopy. In this report, we provide a full classification of 3-component links up to link homotopy for both string and closed links based on Habegger-Lin's approach [2]. This classification was first achieved by Milnor [4], where he defines a link group (for links in 3-manifold). The latter is a quotient group of the fundamental group of the link complement, in which two elements are identified if and only if the new links with one more component are link homotopic.

A closed k-component link is a smooth embedding $\sqcup_{i=1}^{k} S_i^1 \hookrightarrow \mathbb{R}^3$. A string link is a smooth embedding $\sqcup_{i=1}^{k} I_i \hookrightarrow D^2 \times I$, where $I = [0,1]$, and such that the boundary points $\{0,1\}$ of the ith interval are sent to $(0, x_i)$ and $(1, x_i)$, respectively, for $i = 1, 2, ..., k$; where $x_1, ..., x_k$ are fixed k distinct points in D^2. The fundamental idea of Habberger-Lin's classification starts with the pure braid group $PB(k)$, which, according to Artin [1], has a semi-direct product decomposition $PB(k) = PB(k-1) \ltimes F(k-1)$; where $F(k-1)$ is a free group with $k-1$ generators. This idea is based on the following short exact sequence, where the map $PB(k) \xrightarrow{p_i} PB(k-1)$ omits the ith strings, with $F(k-1)$ being the kernel of p_i.

$$1 \to F(k-1) \to PB(k) \xrightarrow{p_i} PB(k-1) \to 1$$

It has a splitting given by adding the ith string supported far away from the other strings. It is not hard to prove using induction that a string link is link homotopic to a braid. Thus, the monoid $\mathcal{H}(k)$ of string links up to link homotopy is a quotient of $PB(k)$, and, therefore, is also a group. Habbeger-Lin [2] establish and prove the analogous decomposition $\mathcal{H}(k) = \mathcal{H}(k-1) \ltimes RF(k-1)$, where $RF(k-1)$ is the so called reduced free group of $k-1$ generators. In Chapter 2, we explicitly compute the group $\mathcal{H}(3)$ using the above decomposition. Note that: $\mathcal{H}(1) = PB(1)$ and $\mathcal{H}(2) = PB(2) = \mathbb{Z}$, since the linking number is an invariant of link homotopy.

Any closed link is isotopic to the closure of a string link. However, two disjoint string links may produce the same closed link. By [2, Theorem 2.13] the closures of two string links $\sigma, \sigma' \in \mathcal{H}(k)$ are link homotopic if and only if σ' is obtained from σ by a sequence of conjugations and partial conjugations. (In the case of one and two-component links, the closure is bijective, because any knot is knot-homotopic to the trivial one, while the linking number is still an invariant for closed links as well.) In Chapter 3, we use this result to produce the full classification of 3-components closed links up to link homotopy.

Chapter 2

Classification of String Links up to Link Homotopy

2.1 Reduced free group of 2 generators

The reduced free group $RF(k)$ is a quotient group of the free group $F(k)$ obtained by adding the relations that each generator commutes with all its conjugates [2]. Let T_1 and T_2 be generators of $F(2)$. According to Milnor [4] any element of $RF(2)$ can be uniquely represented as $T_1^i T_2^j T^k$: with $i, j, k \in \mathbb{Z}$ and $T = [T_1, T_2] = T_1 T_2 T_1^{-1} T_2^{-1}$. It is straightforward to verify that T commutes with both T_1 and T_2 since every generator commutes with a conjugate of itself as well as of its inverse.

To compute the group operation we need the following lemma.

Lemma 1. *: $T_1^i T_2^j T_1^{-i} = T_2^j T^{ij}$ for any $i, j \in \mathbb{Z}$.*

Proof: We have $T = T_1 T_2 T_1^{-1} T_2^{-1}$, *i.e.* T is the commutator of T_1 and T_2. It is easy to obtain that $T_1 T_2 T_1^{-1} = T_2 T$. Then for any $j \in \mathbb{Z}$, $T_1 T_2^j T_1^{-1} = T_2^j T^j$. For any $i \in \mathbb{Z}$, consider that $T_1^{(i-1)}(T_1 T_2^j T_1^{-1}) T_1^{-(i-1)} = T_1^{(i-1)}(T_2^j T^j) T_1^{-(i-1)} = T_2^j T^{ij}$. Then, we have $T_1^i T_2^j T_1^{-i} = T_2^j T^{ij}$: for all $i, j \in \mathbb{Z}$.

Then, we compute the group operation,

$$(T_1^{i_1}T_2^{j_1}T^{k_1})(T_1^{i_2}T_2^{j_2}T^{k_2}) = T_1^{i_1}T_2^{j_1}T_1^{i_2}T_2^{j_2}T^{k_1+k_2} \tag{2.1}$$

$$= T_1^{i_1+i_2}(T_1^{-i_2}T_2^{j_1}T_1^{i_2})T_2^{j_2}T^{k_1+k_2} \tag{2.2}$$

$$= T_1^{i_1+i_2}(T_2^{j_1}T^{-j_1 i_2})T_2^{j_2}T^{k_1+k_2} \tag{2.3}$$

$$= T_1^{i_1+i_2}T_2^{j_1+j_2}T^{(k_1+k_2-j_1 i_2)}. \tag{2.4}$$

2.2 The Group $\mathcal{H}(3)$

From Habegger-Lin [2], $\mathcal{H}(3) = \mathcal{H}(2) \ltimes RF(2)$. We will focus on the decomposition omitting the 3^{rd} string with the projection $\mathcal{H}(3) \to \mathcal{H}(2)$. The next step is to provide a geometric interpretation for $RF(2)$ and $\mathcal{H}(2)$.

Let T_{12}, T_{13}, T_{23} be the generators of $PB(3)$ displayed in Fig. 2.1.

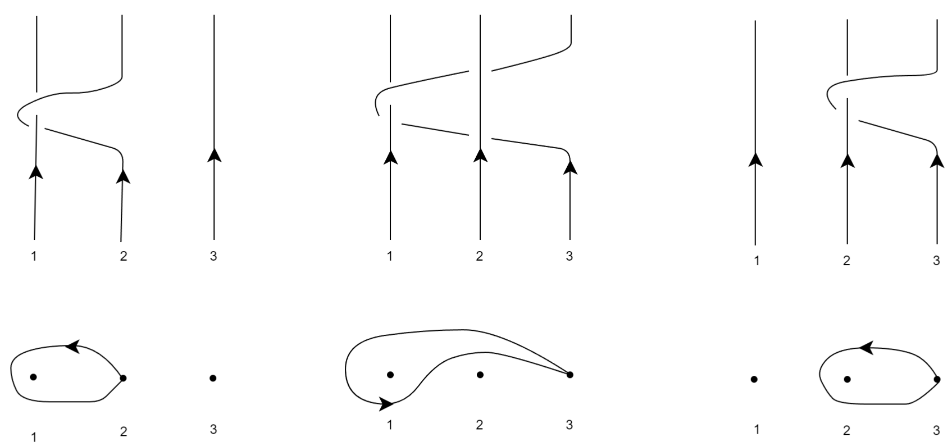

Figure 2.1: Generators of the pure braid group $PB(3)$ denoted by T_{12}, T_{13} and T_{23} respectively.

The group $\mathcal{H}(2) \cong \mathbb{Z}$, is generated by T_{12}, *i.e* $\mathcal{H}(2) \cong \langle T_{12} \rangle$. At the same time, $RF(2)$ is a quotient group of the free group generated by T_{13} and T_{23} subject to the relation that each T_{13} and T_{23} commute with all their respective conjugates. The next step is to understand how $\mathcal{H}(2)$ acts on $RF(2)$. The group action will be explained by the action of T_{12} on T_{13} and T_{23}.

2.2.1 Computation of $T_{12}T_{13}T_{12}^{-1}$

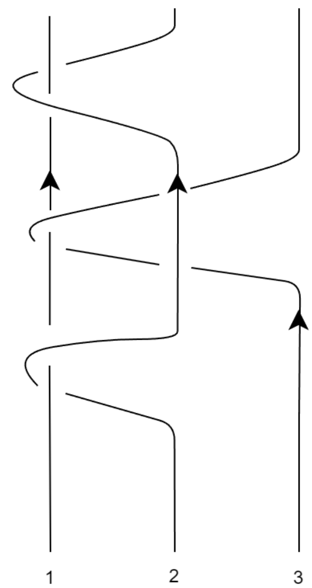

Figure 2.2: The braid $T_{12}T_{13}T_{12}^{-1}$.

The braid $T_{12}T_{13}T_{12}^{-1}$ appears in Fig. 2.2 is isotopic to the one that appears in Fig. 2.3. According to Fig. 2.3,

$$T_{12}T_{13}T_{12}^{-1} = T_{23}^{-1}T_{13}T_{23} = T_{13}T, \tag{2.5}$$

where $T = [T_{13}, T_{23}] = T_{13}T_{23}T_{13}^{-1}T_{23}^{-1}$.

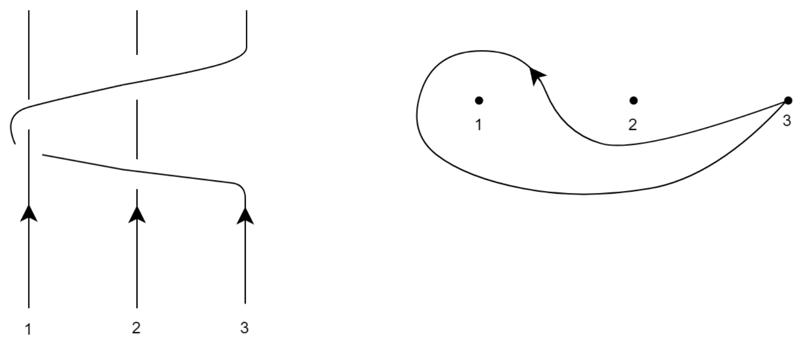

Figure 2.3: A braid isotopic to $T_{12}T_{13}T_{12}^{-1}$.

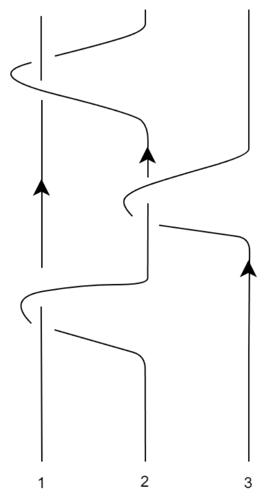

Figure 2.4: The the braid $T_{12}T_{23}T_{12}^{-1}$.

2.2.2 Computation of $T_{12}T_{23}T_{12}^{-1}$

The braid $T_{12}T_{23}T_{12}^{-1}$ that appears in Fig. 2.4 is isotopic to the braid that appears in Fig. 2.5. To proceed the computation of $T_{12}T_{23}T_{12}^{-1}$; From Fig. 2.5,

$$T_{12}T_{23}T_{12}^{-1} = T_{23}^{-1}T_{13}^{-1}T_{23}T_{13}T_{23} \tag{2.6}$$

$$= T_{13}^{-1}T_{23}T_{13} \tag{2.7}$$

$$= T_{23}T^{-1}. \tag{2.8}$$

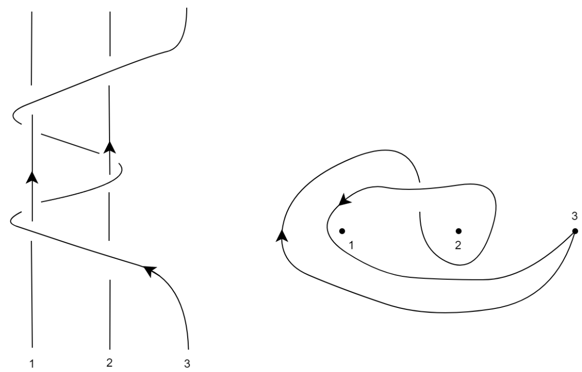

Figure 2.5: A braid homotopic to $T_{12}T_{23}T_{12}^{-1}$.

2.2.3 Computation of $T_{12}T\,T_{12}^{-1}$

The results obtained from (2.5) and (2.8) prove that T_{12} acts on T trivially.

$$T_{12}T\,T_{12}^{-1} = T. \tag{2.9}$$

2.2.4 Corollaries

The results we obtained from (2.5), (2.8) and (2.9) lead us to two interesting corollaries.

Corollary 1. *For any two integers i and j, $T_{12}^{i}T_{13}^{j}T_{12}^{-i} = T_{13}^{j}T^{ij}$ and $T_{12}^{i}T_{23}^{j}T_{12}^{-i} = T_{23}^{i}T^{-ij}$.*

Corollary 2. $T = [T_{13}, T_{23}] = [T_{12}, T_{13}] = [T_{12}, T_{23}]^{-1}$.

2.2.5 Computation of $\mathcal{H}(3)$

Remark 1. *Any element of $\mathcal{H}(3)$ can be uniquely described by $T_{12}^{i}T_{13}^{j}T_{23}^{j}T^{k}$: where i, j, k and $l \in \mathbb{Z}$ and T as described in corollary 2.*

This followed from the decomposition $\mathcal{H}(3) = \mathcal{H}(2) \ltimes RF(2)$ and the descriptions of $H(2)$ and $RF(2)$.

Remark 2. *The aforementioned i, j, k and l are considered to be Milnor invariants of the link denoted by μ and the following interpretation can be provided.*

1. *$i = \mu(12)$: Linking number between the first and the second strings.*

2. *$j = \mu(13)$: Linking number between the first and the third strings.*

3. *$k = \mu(23)$: Linking number between the second and the third strings.*

4. *$l = \mu(123)$: Higher Milnor invariant.*

Let $T_{12}^{i_1}T_{13}^{j_1}T_{23}^{k_1}T^{l_1}, T_{12}^{i_2}T_{13}^{j_2}T_{23}^{k_2}T^{l_2} \in \mathcal{H}(3)$ and we compute the group product bellow $\mathcal{H}(3)$.

$$(T_{12}^{i_1}T_{13}^{j_1}T_{23}^{k_1}T^{l_1})(T_{12}^{i_2}T_{13}^{j_2}T_{23}^{k_2}T^{l_2}) = T_{12}^{i_1+i_2}(T_{12}^{-i_2}T_{13}^{j_1}T_{12}^{i_2})(T_{12}^{-i_2}T_{23}^{k_1}T_{12}^{i_2})T_{13}^{j_2}T_{23}^{k_2}T^{l_1+l_2} \tag{2.10}$$

$$= T_{12}^{i_1+i_2}(T_{13}^{j_1}T^{-i_2j_1})(T_{23}^{k_1}T^{i_2k_1})T_{13}^{j_2}T_{23}^{k_2}T^{l_1+l_2} \tag{2.11}$$

$$= T_{12}^{i_1+i_2}T_{13}^{j_1}T_{23}^{k_1}T_{13}^{j_2}T_{23}^{k_2}T^{(l_1+l_2-i_2j_1+i_2k_1)} \tag{2.12}$$

$$= T_{12}^{i_1+i_2}T_{13}^{j_1+j_2}T_{23}^{k_1+k_2}T^{l_1+l_2-i_2j_1+i_2k_1-k_1j_2}. \tag{2.13}$$

The group operation is now stated.

$$(T_{12}^{i_1}T_{13}^{j_1}T_{23}^{k_1}T^{l_1})(T_{12}^{i_2}T_{13}^{j_2}T_{23}^{k_2}T^{l_2}) = T_{12}^{i_1+i_2}T_{13}^{j_1+j_2}T_{23}^{k_1+k_2}T^{l_1+l_2-i_2j_1+i_2k_1-k_1j_2}.$$

2.2.6 The Center of $\mathcal{H}(3)$

The center of $\mathcal{H}(3)$, is denoted by $\mathcal{ZH}(3)$ is isomorphic to $\mathbb{Z}^2$.

$$\mathcal{ZH}(3) = \{T_{12}^{a}T_{13}^{a}T_{23}^{a}T^{b} | a, b \in \mathbb{Z}\}.$$

The group isomorphism $\mathbb{Z} \times \mathbb{Z} \mapsto \mathcal{ZH}(3)$ is described by

$$(x, y) \mapsto (T_{12}T_{13}T_{23})^x T^y = T_{12}^x T_{13}^x T_{23}^x T^{\left(y - \dfrac{x(x-1)}{2}\right)}.$$

Chapter 3

Classification of 3-Component Closed Links up to Link Homotopy

First, we recall the following theorem.

Theorem 1 ([2], Theorem 2.13)**.** *Let $\sigma, \sigma' \in \mathcal{H}(k)$. The closures of σ and σ' are link homotopic if and only if there is a sequence $\sigma = \sigma_0, \sigma_1, ..., \sigma_n = \sigma'$ of elements of $\mathcal{H}(3)$ such that σ_{j+1} is either a conjugate or a partial conjugate of σ_j, where $j = 0, ..., n-1$.*

3.1 Conjugacy classes

Proposition 1. *An element $\sigma' = T_{12}^{i'} T_{13}^{j'} T_{23}^{k'} T^{l'} \in \mathcal{H}(3)$ is a conjugate of $\sigma = T_{12}^{i} T_{13}^{j} T_{23}^{k} T^{l} \in \mathcal{H}(3)$ if and only if $i = i', j = j', k = k'$ and $l - l'$ is a multiple of $gcd(i-k, j-k)$.*

Thus, conjugacy classes of $\mathcal{H}(3)$ are described as follows:

$$[T_{12}^{i} T_{13}^{j} T_{23}^{k} T^{l}] = \{T_{12}^{i} T_{13}^{j} T_{23}^{k} T^{l+m\alpha} \; : \alpha = gcd(i-j, j-k), m \in \mathbb{Z}\}.$$

Proof. Let any element $\tau = T_{12}^a T_{13}^b T_{23}^c T^d \in \mathcal{H}(3)$ and consider that,

$$\tau\sigma\tau^{-1} = (T_{12}^a T_{13}^b T_{23}^c T^d)(T_{12}^i T_{13}^j T_{23}^k T^l)(T_{12}^{-a} T_{13}^{-b} T_{23}^{-c} T^{-d-ab+ac-bc}).$$

Then we have,

$$\tau\sigma\tau^{-1} = T_{12}^i T_{13}^j T_{23}^k T^{l+a(j-k)+b(k-i)+c(i-j)}.$$

For some $m \in \mathbb{Z}$,

$$\tau\sigma\tau^{-1} = T_{12}^i T_{13}^j T_{23}^k T^{l+a(j-k)+b(k-i)+c(i-j)} = T_{12}^i T_{13}^j T_{23}^k T^{l+gcd(j-k,\, k-i,\, i-j)m}$$

and

$$\tau\sigma\tau^{-1} = T_{12}^i T_{13}^j T_{23}^k T^{l+gcd(i-j,\, j-k)m}.$$

This proves Proposition 1. □

3.2 Partial conjugacy classes

We first recall the notion of partial conjugation due to [2, Definition 2.12].

Definition 1. *Let $\sigma \in \mathcal{H}(k)$ be decomposed as θg corresponding to one of the k decompositions $\mathcal{H}(k) = \mathcal{H}(k-1) \ltimes RF(k-1)$ with $\theta \in \mathcal{H}(k-1)$ and $g \in RF(k-1)$. A partial conjugate of σ is an element σ' such that $\sigma' = \theta h g h^{-1}$ where $h \in RF(k-1)$.*

Proposition 2. *An element $\sigma' = T_{12}^{i'} T_{13}^{j'} T_{23}^{k'} T^{l'} \in \mathcal{H}(3)$ is obtained from $\sigma = T_{12}^i T_{13}^j T_{23}^k T^l \in \mathcal{H}(3)$ by a sequence of partial conjugations if and only if $i = i', j = j', k = k'$ and $l - l'$ is a multiple of $gcd(i, j, k)$.*

Proof. The group $\mathcal{H}(3)$ has three decompositions omitting one of the strings. First, we consider the decomposition associated with omitting the 3^{rd} string. Let $\sigma = T_{12}^i T_{13}^j T_{23}^k T^l$ and we will study how the partial conjugation changes for each generator of $RF(2)$. In this case $\theta = T_{12}^i$, $g = T_{13}^j T_{23}^k T^l$. Let $h = T_{13}$ or T_{23} :

$$\sigma' = T_{12}^i T_{13}(T_{13}^j T_{23}^k T^l) T_{13}^{-1} = T_{12}^i T_{13}^j T_{23}^k T^{l+k} \text{ and } \sigma' = T_{12}^i T_{23}(T_{13}^j T_{23}^k T^l) T_{23}^{-1} = T_{12}^i T_{13}^j T_{23}^j T^{l-j}.$$

Then, consider the decomposition associated with omitting the 2^{nd} string. According to that decomposition, $\mathcal{H}(3) = \mathcal{H}(2) \ltimes RF(2)$: where T_{13} generates $\mathcal{H}(2)$, and T_{12}, T_{23} being generators of $RF(2)$. Then we have $\sigma = T_{13}^j T_{12}^i T_{23}^k T^{l+ij}$, where $\theta = T_{13}^j$, $g = T_{12}^i T_{23}^k T^{l+ij}$. Let $h = T_{12}$ or T_{23} :

$$\sigma' = T_{13}^j T_{12}(T_{12}^i T_{23}^k) T_{12}^{-1} T^{l+ij} = T_{12}^i T_{13}^j T_{23}^k T^{l-k} \text{ and } \sigma' = T_{13}^j T_{23}(T_{12}^i T_{23}^k) T_{23}^{-1} T^{l+ij} = T_{12}^i T_{13}^j T_{23}^k T^{l-i}.$$

The next decomposition is associated with omitting the 1^{st} string. According to that decomposition, $\mathcal{H}(3) = \mathcal{H}(2) \ltimes RF(2)$: where T_{23} generates $\mathcal{H}(2)$, and T_{12}, T_{13} being generators of $RF(2)$. Then we have $\sigma = T_{23}^k T_{12}^i T_{13}^j T^{l+kj-ij}$, where $\theta = T_{23}^k$, $g = T_{12}^i T_{13}^j T^{l+kj-ij}$. Let $h = T_{12}$ and T_{13} :

$$\sigma' = T_{12}^i T_{12}(T_{13}^j T_{23}^k) T_{12}^{-1} T^{l-i} = T_{12}^i T_{13}^j T_{23}^k T^{l-i} \text{ and } \sigma' = T_{12}^i T_{13}(T_{13}^j T_{23}^k) T_{13}^{-1} T^{l+j} = T_{12}^i T_{13}^j T_{23}^k T^{l+j}.$$

Thus, the exponent of T can be changed by any linear combination of i, j and k. That implies Proposition 2. □

3.3 Conclusion

Finally, we conclude that any 3-component closed link can be described by three integers, $i = \mu(12), j = \mu(13), k = \mu(23)$, which are linking numbers between each pair of components, and a remainder modulo $gcd(i, j, k)$. In particular, if any of the linking numbers is non-zero, there are finitely many link homotopy classes with the given linking numbers. The case $i = j = k = 0$ corresponds to almost trivial links. We conclude that there are infinitely many almost trivial closed links parameterized by one integer $\mu(123)$. The closed almost trivial links are in one-to-one correspondence with almost trivial string links.

Bibliography

[1] Emil Artin. Theory of braids. *Annals of Mathematics*, 48(1):101–126, 1947.

[2] Nathan Habegger and Xiao-Song Lin. The classification of links up to link-homotopy. *Journal of the American Mathematical Society*, 3(2):389–419, 1990. doi: 10.1090/S0894-0347-1990-1026062-4.

[3] Jerome P Levine. An approach to homotopy classification of links. *Transactions of the American Mathematical Society*, 306(1):361–387, 1988.

[4] John Willard Milnor. Link groups. *Annals of Mathematics (2)*, 59:177–195, 1954. doi: 10.2307/1969616.