

NON-NEWTONIAN BLOOD FLOW IN TAPERED TUBES

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CHAPTER 1

INTRODUCTION

Only the true capillaries in the living circulation approximate tubes of constant diameter, all others show some taper. The tapered nature of blood vessels has been reported by Knisely and Jeffords [1] as well as by Bloch [2]. In Bloch's article, the mesentric arterioles and venules of rats and frogs were found to exhibit angles of taper varying from $0^{\circ}5'$ to $2^{\circ}12'$. The angle of taper is small, but because of the complexity of the rheological behavior of blood, even a very slight convergence or divergence in the walls of a vessel can have a significant influence on the resistance to flow. Moreover, as the resistance is proportional to the pressure drop, general pressure drops measured in the vascular bed indicate that the resistance of the arterioles constitutes the largest portion of the whole. For example, data presented by Burton [3] on the distribution of intravascular pressures in the systemic circuit shows that the relative resistance of arterioles amounts to 60%. These facts reveal that arterioles must be considered as sections of cones and they offer the major resistance in the vascular bed. The arterioles are believed to be responsible in part for the local regulation of the circulation. Hence an understanding of the flow behavior of blood in tapered vessels is requisite to any serious investigation of regulation in the circulatory system.

In order to investigate the flow behavior of blood in a tapered vessel, it is first necessary to consider the rheological behavior of blood. Blood is a suspension of cells in plasma. Plasma is essentially a Newtonian fluid. Although non-Newtonian behavior of plasma system was reported by several

investigators, their findings have been considered to be an artifact as pointed out by Copley [4]. Red cells compose about 97% of total cell volume in blood. Hence, red cells play an important role in determining the flow properties of blood. The red cell has a biconcave discoid shape when it is not flowing. At low shear rates, human red cells as well as those from some other species aggregate face-to-face to form rouleau which causes the apparent viscosity of blood to increase. The aggregation breaks up when shear is increased, eventually to single cells when shear rates are sufficiently high. This variation of structure with shear rate makes the flow behavior of blood complex. The Casson model has been found to be quite suitable for describing this behavior, although Scott-Blair [5] has cautioned against its usage on physical grounds.

The measurement of blood flow properties in the living body, i.e. in vivo, is more difficult than measurement of the blood sample removed from a living body, i.e. in vitro. The in vitro measurement, however, will encounter the problem of clotting. To prevent the blood for in vitro measurement from clotting, several anticoagulents may be added to the blood sample. The treatment of whole blood with heparin, sequestrene, or oxalate has been shown to have little effect on the flow properties of blood [6].

Besides the properties of blood, the taper of the vessel is another problem to be considered. Newtonian and non-Newtonian fluid flows in a tapered tube have been examined theoretically and to a limited extent experimentally. These works will be discussed in detail in Chapter 5.

The purpose of the present work is to experimentally verify a theoretical pressure-flow relation for blood flow in a tapered tube developed in this work. The pressure flow relation is derived for steady, laminar flow of an

incompressible non-Newtonian fluid in a tapered tube of small angle of taper. In the derivation, a semi-empirical rheological model for blood, the approximate Casson model, is assumed. This model requires the experimental determination of the rheological parameters by independent viscometric measurement.

The thesis presents several chapters concerned with the rheological measurements, the approximate Casson model, and data treatment which are vital inputs to the overall objective. Chapter 2 provides a description of a novel coiled capillary viscometer which was used to gather data to test the approximate Casson model for tube flow presented in Chapter 3. Chapter 4 describes a computer aided data analysis which was used to evaluate the rheological parameters from measurements obtained with a buret viscometer. These parameters were employed in the tapered tube pressure flow relation. The theoretical treatment of the tapered tube flow is presented in Chapter 5 along with experimental comparisons. Experimental pressure flow data for heparinized whole bovine blood in 1° and 2° tapered tubes are compared with the approximate relation. Chapter 6 gives some recommendations for further studies.

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CHAPTER 2

A COILED CAPILLARY VISCOMETER WITH CONTINUOUSLY VARYING PRESSURE HEAD

2.1 Abstract

This chapter provides a description of a coiled capillary viscometer with a continuously varying pressure head and the theory behind its operation. Flow curves are presented for whole bovine blood and aqueous silica suspensions. The data are compared with results from a true bore capillary viscometer as well as with the generalized friction factor correlation of Metzner and Reed. Favorable results were obtained for a shear stress range from 0.1 to 16 dynes/cm².

2.2 Introduction

The determination of the rheological properties or flow curves of homogeneous fluids and suspensions is an integral part of a variety of experimental and analytical biorheological studies. Quite frequently these properties are desired under conditions of low shear and in some cases over a wide range of shear. A wide variety of viscometers are available for this purpose [1], which encompass a spectrum of costs, working range, accuracy and simplicity of operation. Ideally as with any instrumentation it is desirable to have a simple, inexpensive and accurate unit which can be readily adapted to a wide range of working conditions. Unfortunately this is not the case with most if not all of the presently available viscometers. Although a variety of instruments are available to choose from, only two basic types are utilized to any appreciable extent; the capillary viscometers and the rotational viscometers.

Among these, the capillary viscometer is the simplest, least expensive, and most accurate. It has been used extensively in the past for rheological measurements. Over the past ten years however, the rotational viscometers have become increasingly popular and have to a certain extent replaced the capillary instruments. This has come about for a variety of reasons, among which is the interest in the flow properties of non-Newtonian fluids under conditions of low shear and oscillatory flow.

The capillary viscometer is basically not well suited for low shear measurements (unless certain modifications are made). This is largely due to the difficulties in obtaining accurate measurements of very small pressure drops and volumetric flow rates. In addition the data analysis is often quite sensitive. On the other hand, the rotational instruments, with their sophisticated methods for the generation of small angular velocities and the measurement of small torques, are well suited to low shear application. With some, the data analysis is quite simple. Unfortunately, increasing sophistication is accompanied by cost and operational disadvantages. The cost of many rotational viscometers of superior quality may be prohibitive for many researchers. In addition, highly skilled personnel will in general be required for the operation and maintenance of superior instruments. In the case of suspensions, data interpretation is still in question at low shear ($\dot{\gamma} < 1 \text{ sec}^{-1}$).

Several attempts have been made to partially resolve the measurement difficulties associated with the low shear operation of capillary viscometers. Maron and Belner [2] described an absolute viscometer with a continuously varying pressure head. This instrument utilized gravity flow, with pressure drops and volumetric flow rates being related to the height of a fluid

meniscus which could be readily and accurately measured. The viscometer was tested over a shear stress range of $0.06 - 20 \text{ dynes/cm}^2$.

Caraher [3] described a low shear viscometer which employed a 200-foot length of 1/4-inch I.D. Tygon tubing. This instrument employed a constant head gravity flow system with volumetric flow rates being measured with the aid of water and diethyl phthalate manometers. The viscometer was tested over a shear stress range of approximately $2 \times 10^{-3} - 5 \times 10^{-1} \text{ dynes/cm}^2$. This viscometer is not well suited to routine laboratory use due to the large sample required.

Benis [4] described a low shear viscometer which is basically a micro version of Caraher's instrument. It consists of a coiled 4-8 meter length of 1 mm I.D. polyethylene capillary. A large applied pressure is employed to drive the fluid meniscus over a small distance. Benis reported data over the shear stress range $0.1 - 10 \text{ dynes/cm}^2$.

Other modifications such as the use of a differential pressure transducer to measure both pressure difference and volumetric flow rate [5] have been employed with some success. However this type of approach has a high cost associated with it due to the associated supporting instrumentation and will not be considered further in this report.

The purpose of this report is to describe a capillary viscometer which is similar in construction to the viscometer of Benis [4], but is operated with a continuously varying pressure head as the viscometer of Maron and Belner [2].

2.3 Description of the Viscometer

The viscometer is shown schematically in figure 1. It consists of a length of polyethylene capillary tubing A coiled about a retaining spool B,

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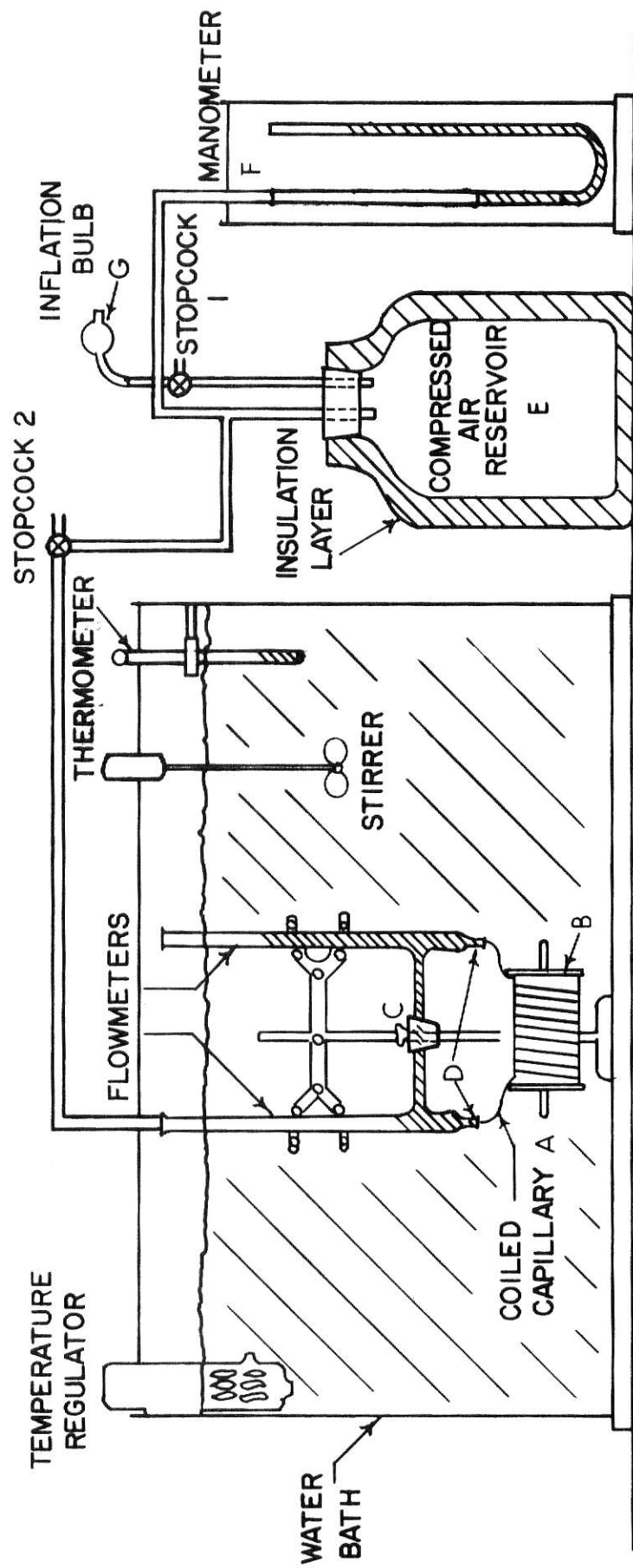


Fig.1 Schematic diagram of the apparatus

approximately 12 cm in diameter. In this study 7 & 8 meter lengths of "Intramedic" tubing (size PE 190, I.D. $\approx$ 1 mm) were used. The viscometer capillary is connected in parallel with a bypass C between two flowmeter tubes of the same diameter. In the present work, flowmeter tubes of 0.940 and 0.798 cm I.D. were utilized to provide measurements over a wide range of flow. Graduated bubble flowmeters with a side arm feature (10 ml and 25 ml) were found suitable as well as precision bore tubing. Luer-Lok connectors D were employed to facilitate filling and cleaning of the capillary. The apparatus was immersed in a temperature bath at $25.00 \pm 0.02^{\circ}\text{C}$, such that the test fluids were under water level during the whole course of a run. The left hand flowmeter was connected through stopcock 2 to an insulated ballast bottle E of 10 liter capacity which permitted an essentially constant driving pressure to be applied to the test fluid. A U-tube manometer F was included in the system for monitoring the driving pressure. In this work two interchangeable manometers were used, one containing mercury and the other water, to permit operation over a wide range of driving pressures.

In operation the capillary A and flowmeter tubes were filled with the test fluid so that on equilibration of the liquid levels the flowmeter tubes were about one half full. In the case of blood, the viscometer was flushed with saline solution before filling with test fluid. The system was disconnected from the pressure source by turning stopcock 2 and the inflation bulb G was used to raise the pressure in the ballast bottle E to a selected level. With bypass C closed and the fluid levels equilibrated, stopcock 2 was opened to apply the driving pressure to the fluid. Simultaneously a stopwatch was started and the rise of the meniscus in the right hand flowmeter was timed at selected height intervals. A stopwatch with a split-hand feature was used to

obtain a set of height versus time data. Heights were measured by one of two methods; with graduated flowmeters the graduations were directly used as reference points, with non-graduated flowmeters a cathetometer was utilized. After a set of height versus time data was taken, the system was again disconnected from the pressure source and the bypass C was opened to speed up the liquid level equilibration. The bypass also provided for the mixing of suspensions in between runs.

Experiments consisted of obtaining a minimum of three sets of height versus time data for a given driving pressure. These experiments were repeated over a range of driving pressures from 10 cm H₂O to 30 cm Hg to provide sufficient data for the establishment of the flow curve for the test fluid.

2.4 Theoretical Development

The analytical approach that was employed is similar to that utilized by Maron and Belner [2]. Because the capillary is coiled, curvative effects should be taken into consideration. However, according to the arguments presented by Benis [4] for Newtonian flow, these effects are not significant. Experimental results also indicate that curvature effects are negligible. The present viscometer is similar to that of Benis with the exception that the radius of the coil is more than twice as large which insures that curvature effects can be neglected for all practical purposes.

Reference is made to figure 2 which is a simplified diagram of the viscometer. The pressure drop across the capillary at any time t is given by

$$\Delta P = \Delta P_m + \rho g (x - y) \quad (1)$$

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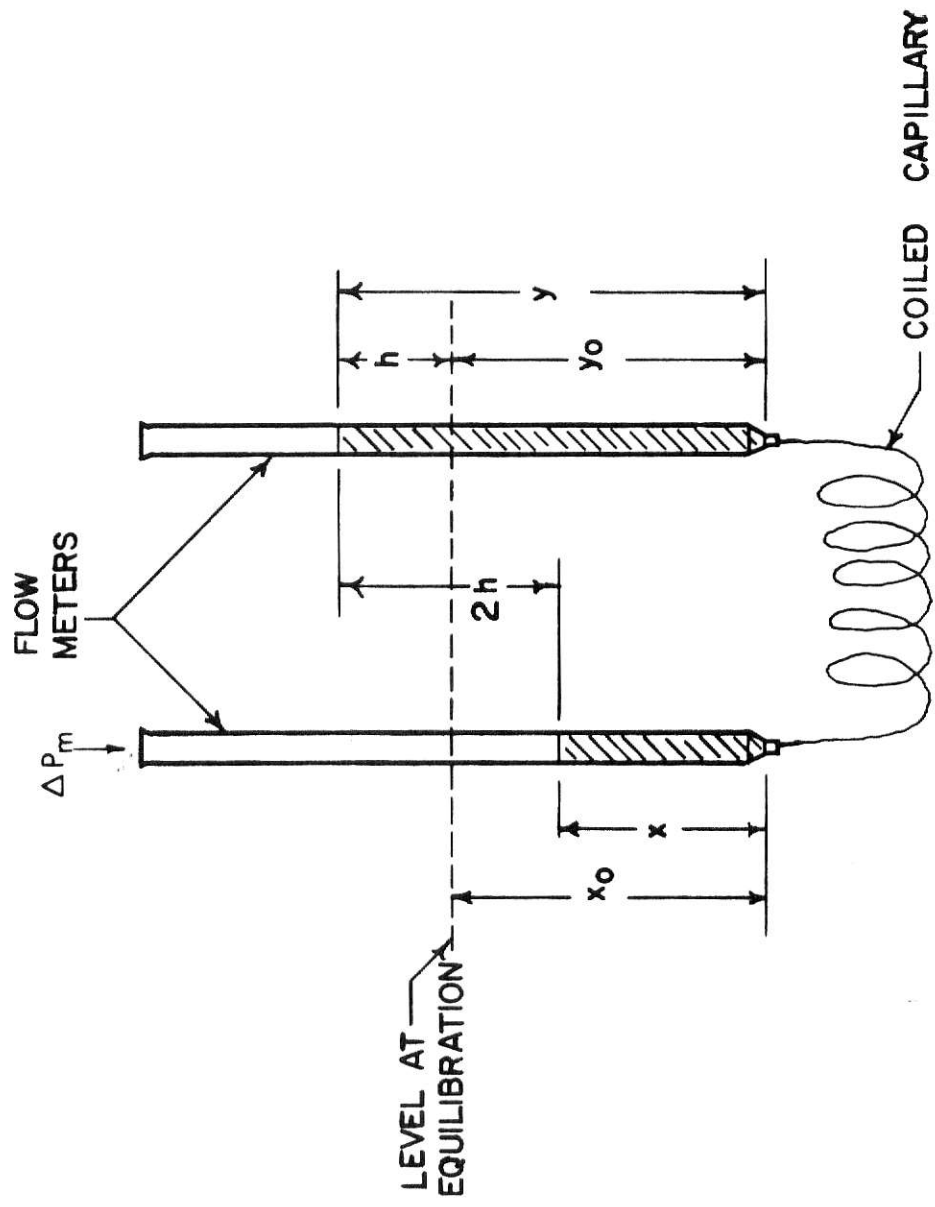


Fig. 2 Simplified diagram of the viscometer

where

ρ = fluid density

g = acceleration of gravity

ΔP_m = the driving pressure

At hydrostatic balance x and y are equal. Noting these values as x_o and y_o respectively equation (1) can be expanded to

$$\Delta P = \Delta P_m + \rho g (x - x_o) - \rho g (y - y_o) \quad (2)$$

Since the two flowmeter tubes are of the same diameter and since there is no loss of fluid in the system it is apparent that

$$x - x_o = -(y - y_o) \quad (3)$$

Substituting equation (3) into equation (2) and noting $y - y_o$ as h gives

$$\Delta P = \Delta P_m - 2 \rho g h \quad (4)$$

The corresponding volumetric flow rate at any time can be expressed as

$$Q = \pi R_F^2 \frac{dh}{dt} = - \frac{\pi R_F^2}{2\rho g} \frac{d\Delta P}{dt} \quad (5)$$

Here R_F is the flowmeter tube radius.

Using a quasi steady-state approach, the pressure drop and flow rate can be related for a Newtonian fluid to give

$$Q = \frac{\pi R_c^4 \Delta P}{8\eta L} \quad (6)$$

where

R_c = capillary radius

L = capillary length

η = coefficient of viscosity

Substitution of equation (5) into equation (6) followed by rearrangement and integration gives

$$\log_{10} \Delta P = \frac{B\rho}{\eta} t + C \quad (7)$$

where B is an instrumental constant defined by

$$B = \frac{g R_c^4}{9.212 L R_F^2} \quad (8)$$

and C is a constant of integration.

From equation (7) it follows that a plot of $\log_{10} (\Delta P)$ versus t should be linear, hence the viscosity can be evaluated from the slope. It should also be noted that the slope should remain constant irrespective of the driving pressure. This is valid only if the fluid is Newtonian and if the flow is slow enough to neglect losses due to the fittings and end effects.

With non-Newtonian fluids the semi-log plot may or may not be linear. In the case of linear plots the slope will change with driving pressure. The resulting data can be utilized to evaluate the wall shear rate ($\dot{\gamma}_w$) as a function of wall shear stress (τ_w), which in turn can be employed to calculate viscosity as a function of shear. A method similar to that developed by Krieger and Maron [6, 7, 8] was used to obtain the necessary relations. This approach is a modification of that used to formulate the Rabinowitsch-Mooney equation [9, 10].

Differentiation of equation (7) with respect to t gives

$$\frac{d \log_{10} (\Delta P)}{dt} = - \frac{B\rho}{\eta_e} = m \quad (9)$$

Here η_e is the effective viscosity and m is the slope of the semi-logarithmic plot at any point. Introduction of the effective fluidity ($\phi_e = 1/\eta_e$) permits rearrangement of equation (9) to define ϕ_e in terms of the slope m .

$$\phi_e = - \frac{m}{B\rho} \quad (10)$$

It can be readily shown [9, 10] for flow in a cylindrical tube that the volumetric flow rate Q and the wall shear stress τ_w are related by the following relation.

$$Q = \frac{R_c^3}{\tau_w^3} \int_0^{\tau_w} \tau^2 f(\tau) d\tau \quad (11)$$

with τ_w defined as

$$\tau_w = \frac{R_c \Delta P}{2L}, \quad (12)$$

$$\tau = \frac{r \Delta P}{2L}, \quad (13)$$

and

$$f(\tau) = \dot{\gamma} \quad (14)$$

where $\dot{\gamma}$ notes the shear rate and τ the shear stress.

This relation is subject to the restrictions of:

- (1) steady laminar flow of an incompressible fluid
- (2) time independent fluid, that is the shear rate at any point is a function only of the shear stress at that point (i.e. $\dot{\gamma} = f(\tau)$)
- (3) no slip at the tube wall.

Multiplying equation (11) by $[4/R_c^3 \tau_w]$ and recognizing that

$$\frac{4Q}{R_c^3 \tau_w} = \phi_e \quad (15)$$

gives the relation

$$\phi_e = \frac{4}{\tau_w^4} \int_0^{\tau_w} \tau^2 \dot{\gamma} d\tau \quad (16)$$

Equation (16) applies in general, no matter how complex the functional relationship between $\dot{\gamma}$ and τ , provided that the stated conditions are satisfied.

Differentiation of equation (16) with respect to τ_w (with the aid of Leibnitz's Rule) gives on simplification the relation for the wall shear rate $\dot{\gamma}_w$.

$$\dot{\gamma}_w = \phi_e \tau_w \left[1 + \frac{1}{4} \frac{d \log \phi_e}{d \log \tau_w} \right] \quad (17)$$

The derivative in equation (17) can be evaluated for the viscometer under consideration with the aid of the following relations.

From equation (10) it can be shown that

$$d \log \phi_e = \frac{1}{2.303} \frac{dm}{m} \quad (18)$$

and

$$\tau_w = \frac{R_c (\Delta P)}{2L} \quad (19)$$

from which it follows

$$d \log \tau_w = d \log (\Delta P) = m d t. \quad (20)$$

The later equality results from equation (9).

Substitution of the results in equations (18) and (20) (along with the definition of ϕ_e provided by equation (10)) into equation (17) gives

$$\dot{\gamma}_w = - \frac{\tau_w m}{B\rho} \left[1 + \frac{1}{9.212 m^2} \frac{dm}{dt} \right] \quad (21)$$

Equation (21) permits calculation of the wall shear rate from the slopes of a semilogarithmic plot of ΔP versus t .

2.5 Data Analysis

A set of h versus t data can be obtained by converting either the reference points on the flowmeter or the readings from the cathetometer into height differences (h 's). For a given driving pressure, three sets of h versus t data were averaged and equation (4) was used to calculate a set of ΔP versus t data. A semilogarithmic plot of ΔP versus time was then made, from which slopes were determined.

In the case of Newtonian behavior m , coupled with equation (9), can be used to calculate the viscosity if the instrumental constant B is known. For calibration purposes m can be used to determine the instrumental constant B when a fluid of known viscosity is employed.

With non-Newtonian fluids two types of behavior are observed in the log ΔP versus time plots. The first type as illustrated by figure 3 shows a near linear plot, but the slope changes with the level of the driving pressure. This is observed when the shear stress range covered by a run is small (high driving pressure). In this case the expression for the wall shear rate, equation (21), simplifies to

$$\dot{\gamma}_w = - \frac{\tau_w m}{B\rho} \quad (22)$$

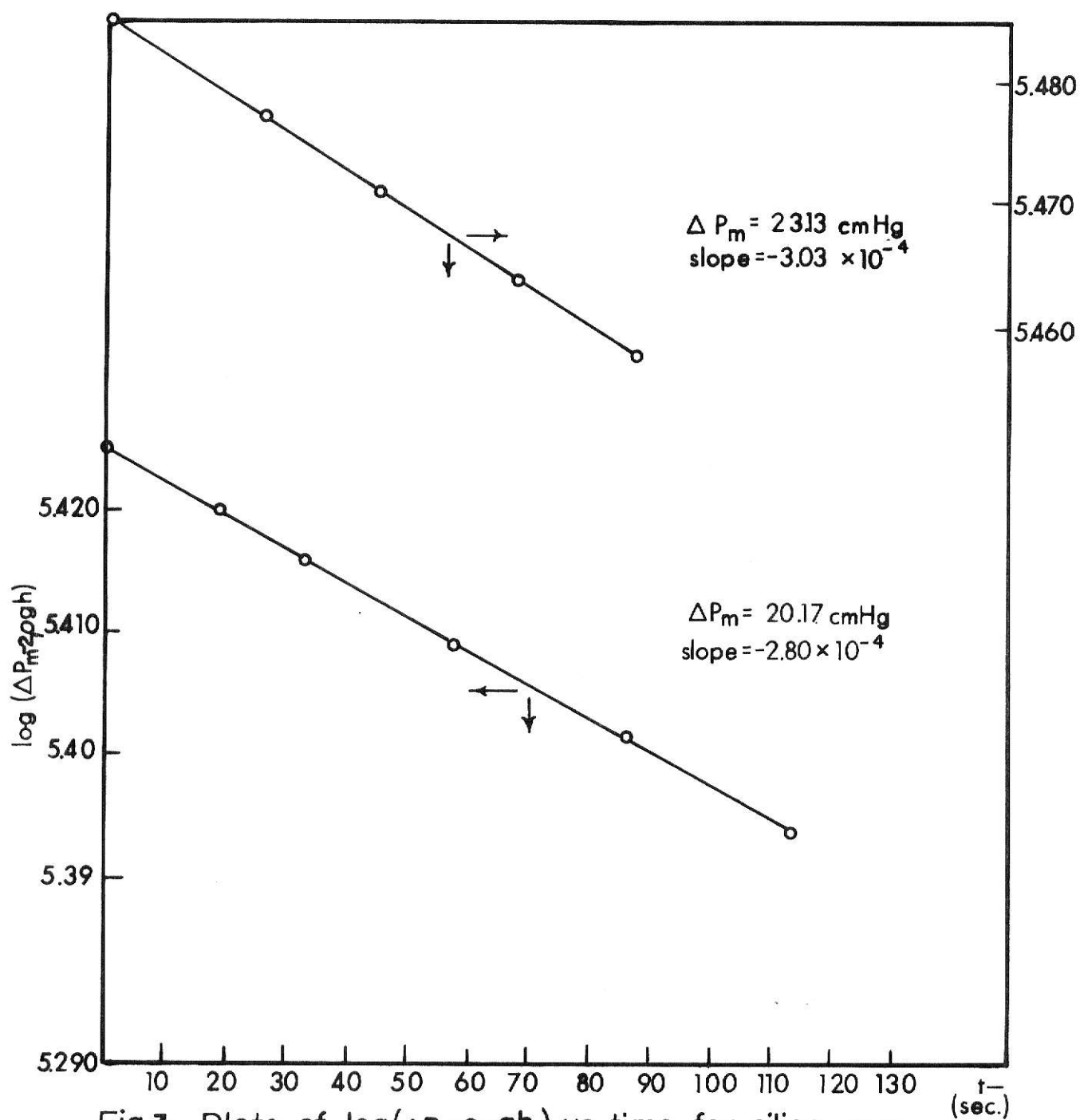


Fig.3 Plots of $\log(\Delta P_m - 2\rho gh)$ vs time for silica suspension concentration of $\phi = 0.20$, using the 25 ml flowmeters (from Cala [13]).

The second type of behavior as illustrated by figure 4, shows notable curvature, and requires the evaluation of the slope from point to point on the curve. This can be accomplished graphically or by curve fitting techniques. Maron and Belner [2] presented a convenient method for improving graphical accuracy. It is also necessary to evaluate dm/dt in this case. This can be determined from a plot of m versus t . The results at each point are then substituted into equation (21) for the calculation of $\dot{\gamma}_w$.

2.6 Viscometer Calibration

Since the polyethylene capillary employed in this study was not a true bore tube and because of its excessive length the average radius (R_c) was determined from flow measurements with distilled water at 25°C. From the slope of the semilogrithmic plot (m), the viscosity and density of water at 25°C, the tube length (L) and the radius of the flowmeter tube (R_F), the radius of the tube can be calculated with the relation

$$R_c^4 = \left[\frac{9.212 \eta L R_F^2}{\rho g} \right] m \quad (23)$$

which is obtained from equations (7) and (8).

Determinations were made over a range of driving pressures (10.0 cm H₂O < ΔP_m < 70.3 cm H₂O) with a reproducibility of 0.3-0.5% on the radius.

The radius of the flowmeter tubes was obtained by measuring the distance between the initial and final graduations, the known volume and the proper geometric relation. In the case of the capillary flowmeter the radius was obtained from the mass of the thread of mercury which filled the tube, the length and the proper geometric relation.

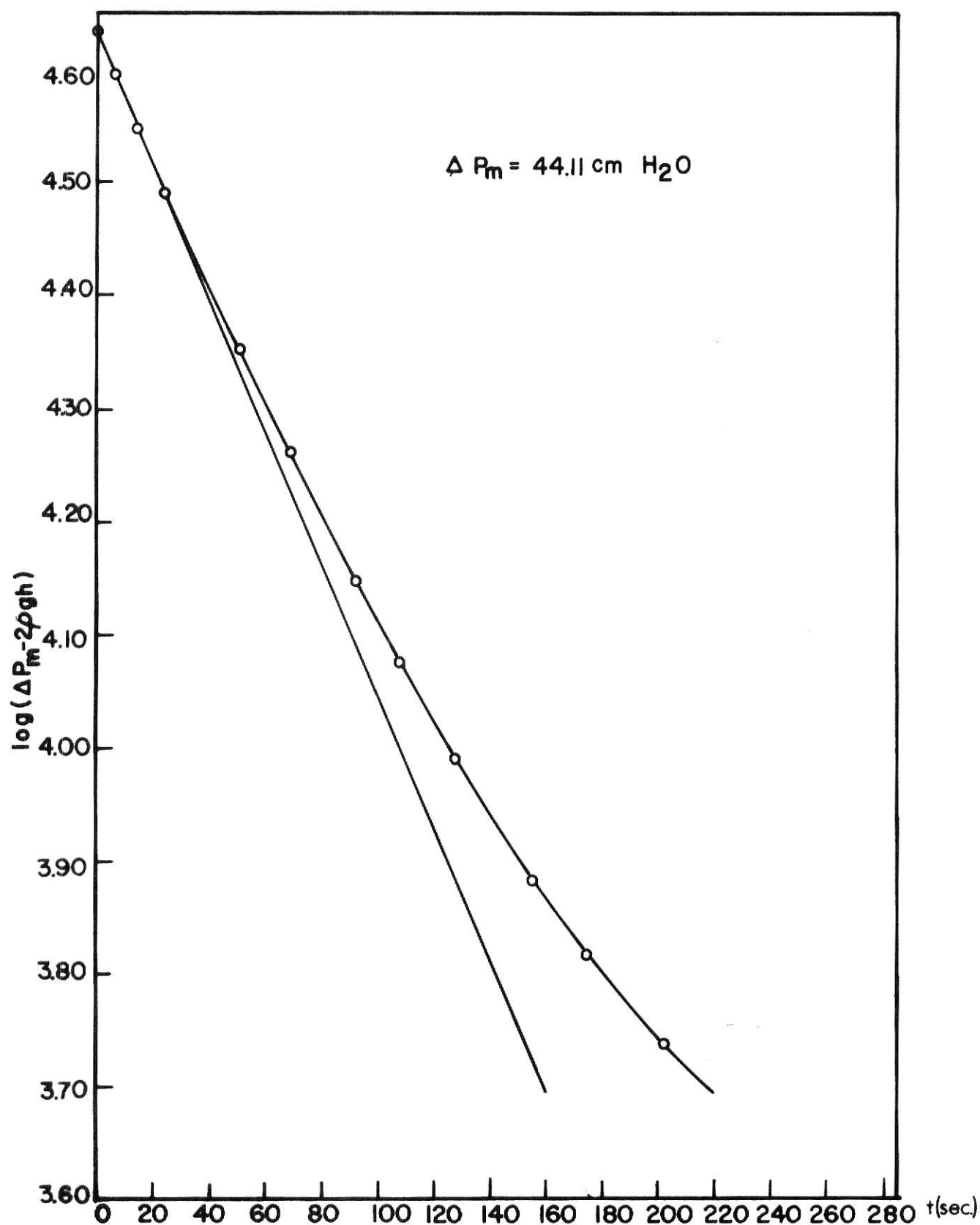


Fig.4 Plot of $\log(\Delta P_m - 2\rho gh)$ vs. time for silica suspension concentration of $\phi = 0.05$ using the capillary flowmeters (from Cala [13]).

The capillary and flowmeter radii are summarized in Table 1. Slope values, m , for distilled water are shown in Table 2 for various driving pressures.

2.7 Test Fluid

Whole bovine blood was employed as test fluid. Whole bovine blood was collected directly into a flask heparinized with 1,000 units of heparin per 200 ml of blood. The blood was stored at $0 \sim 4^{\circ}\text{C}$ prior to use. It was filtered through glass wool and warmed in the constant temperature bath prior to each experiment. Hematocrit values of the blood samples used range from 29 to 38.

2.8 Results

The data from the experimental studies were used to construct flow curves. Typical wall shear stress versus wall shear rate data for whole bovine blood are presented in figure 5. As shown in figure 5 the range of shear covered by the present viscometer is from $.3 \text{ dyne/cm}^2$ to 16 dyne/cm^2 . Since almost the full range of available driving pressures has been employed in this experiment, the extension of the flow curve in figure 5 can only be accomplished by using a capillary of different dimensions. Although hematocrit is not the sole factor determining blood behavior, the two blood samples in figure 5, show very similar flow curves. This flow curve can be approximated by a straight line (slope different from unity) on the log-log plot. Similar behavior was observed in other experiments.

Figure 6 presents the flow curve of silica suspensions covering a shear range of $0.2 \sim 2. \text{ dyne/cm}^2$. From this figure, it appears that the suspension

Table 1 Dimensions of capillaries and flowmeter tubes

	I.D. (cm)	Length (cm)
10 ml flowmeter (graduate)	0.798	-
25 ml flowmeter (graduate)	0.940	-
Capillary flowmeter [*]	0.278	-
P. E. Coiled [*] Capillary I	.1046	800.
P. E. Coiled Capillary II	.1086	692.

^{*} Dimensions employed by Cala [13].

Table 2 Slopes of semi-log plots of distilled water
with different driving pressures, 25°C

driving pressure, ΔP_m (cm H ₂ O)	slope, m
10.0	-6.83×10^{-4}
24.5	-6.86×10^{-4}
46.7	-6.62×10^{-4}
70.3	-6.76×10^{-4}

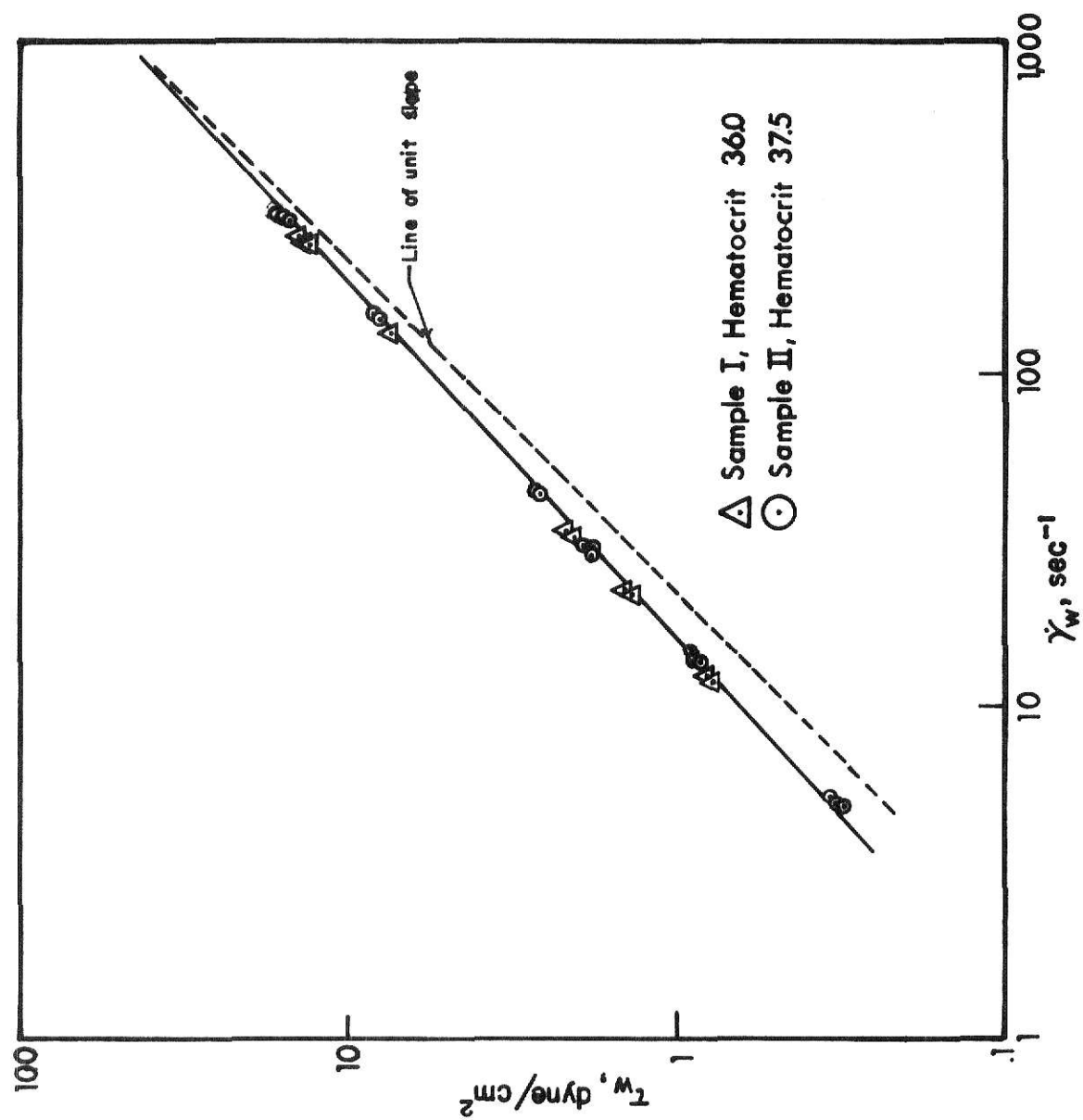


Fig. 5 Wall shear stress vs. wall shear rate plot for blood data.

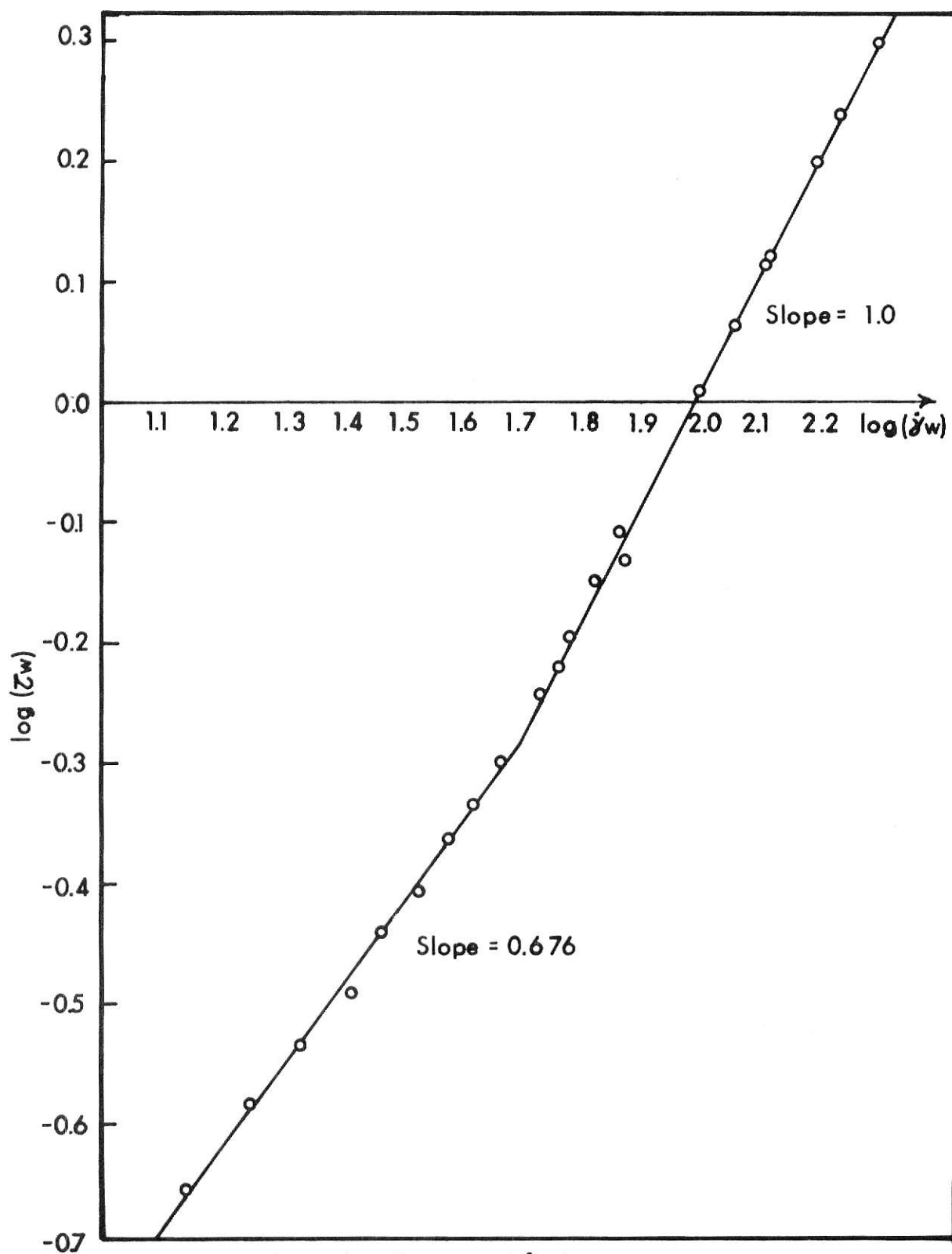


Fig. 6. Plot of $\log(\zeta w)$ vs $\log(\dot{\gamma} w)$ for silica suspension concentration of $\phi = 0.05$ using both, Ten ml graduate and capillary Flowmeters. (from Cala [13])

behaves as a Newtonian fluid at higher shear rates and a non-Newtonian fluid at lower shear rates. Similar results were observed in other experiments but the change to Newtonian behavior could not be observed for $\phi = 0.2$ up to $\tau = 16$.

For the purpose of comparison with an absolute viscometer, the same sample of whole bovine blood was run in the present viscometer and a buret-capillary viscometer which has been described by Cerny [11]. The buret-capillary viscometer for this study consisted primarily of a 50 ml buret and a precision bore capillary tube (19.9 cm in length and 0.102 cm in diameter). The results of the comparison are given in Table 3 and figure 7. The agreement is reasonably good, particularly when one considers the deviations possible in the same instrument. In Table 3, the discrepancy in shear stress at the same shear rate varies from 3% to 10% and it is larger when the rate of shear is lower. This greater discrepancy at lower shear may be the result of the inherent inaccuracies of the buret-capillary viscometer when rate of shear is low. These inaccuracies are associated with both pressure drop and flow rate measurement.

The accuracy of the experimental data was also tested by comparison with the generalized correlation of Metzner and Reed [12]. For this correlation, the friction factor f is defined as

$$f = \frac{2 \tau_w}{\rho (Q/\pi R_c^2)^2} \quad (24)$$

and the generalized Reynolds number as

Table 3 Comparison between shear stress data of this viscometer
and that of an absolute viscometer for heparinized
whole bovine blood

Shear rate, $\dot{\gamma}_w$ sec^{-1}	τ_w of coiled capillary viscometer	τ_w of Buret- capillary viscometer
265	15.0	15.5
170	9.8	10.8
100	6.2	6.9
70	4.6	5.1

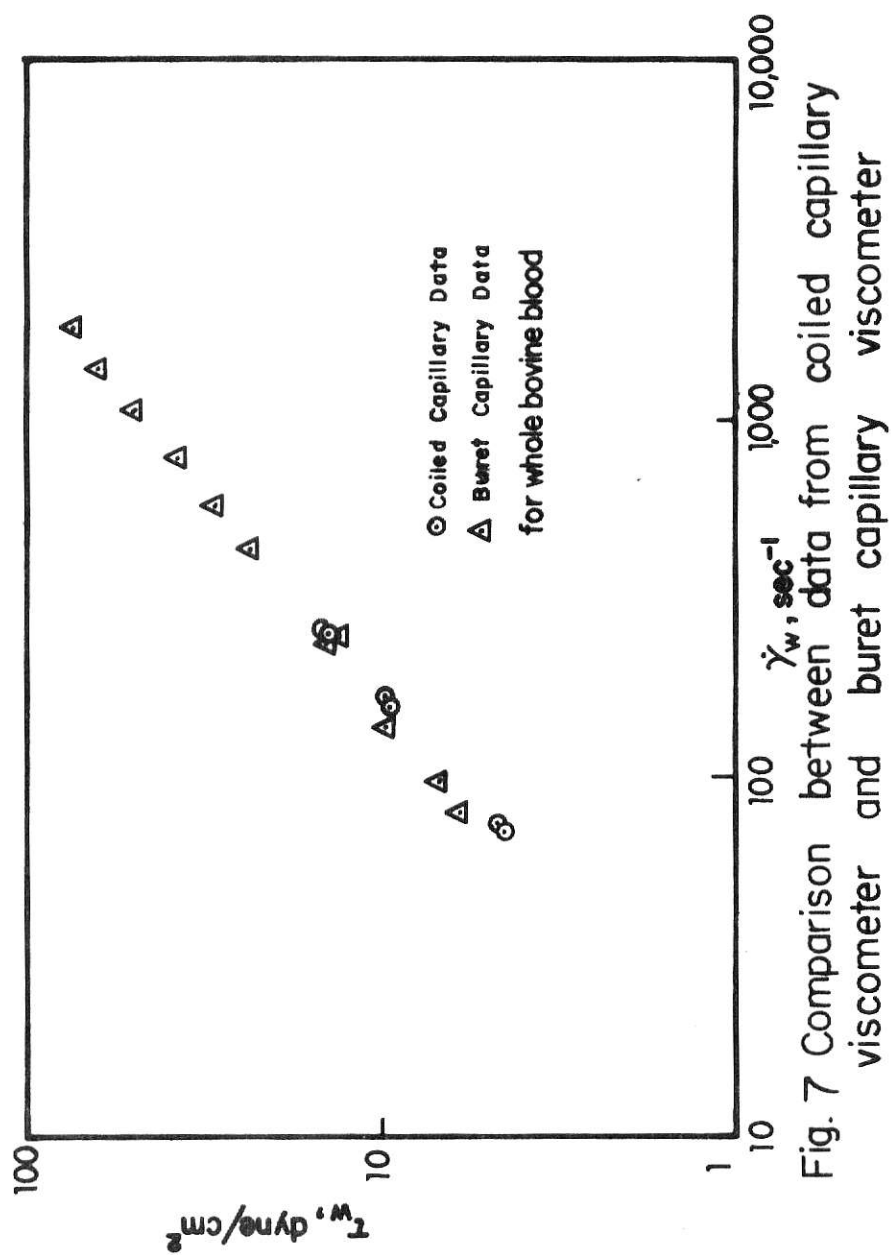


Fig. 7 Comparison between data from coiled capillary viscometer and buret capillary viscometer

$$Re' = \frac{(2R_c)^n \left[\frac{Q}{2}\right]^{2-n} \rho}{8^{n-1} k} \quad (25)$$

where

$$k' = k \left(\frac{3n+1}{4n} \right)^n \quad (26)$$

k is the consistency index and n the flow behavior index of the power law fluid. Here, n is evaluated from the slope of a log-log plot of τ_w versus $\dot{\gamma}_w$ such as figure 5 and k has the value $\log \tau_w$ when $\dot{\gamma}_w = 1$ on $\log \tau_w$ versus $\log \dot{\gamma}_w$ plot.

The theoretical correlation states that all laminar flow data should fall on the line given by

$$f = \frac{16}{Re'} \quad (27)$$

Volumetric flow rates were calculated from slopes determined from the h versus t data and equation (5). The comparison with the theoretical relationship was quite favorable as indicated in figure 8 for both blood and silica suspensions. This result indicates that the data are substantially correct.

2.9 Discussion

The viscometer described in this chapter has a number of desirable features particularly from the point of view of routine laboratory application. The first of these is that the instrument is relatively inexpensive. The components for the most part are readily available in any laboratory or can be purchased at a modest price. In addition the unit can be assembled in the laboratory by the user and does not require the services of any skilled personnel for assembly or repair.

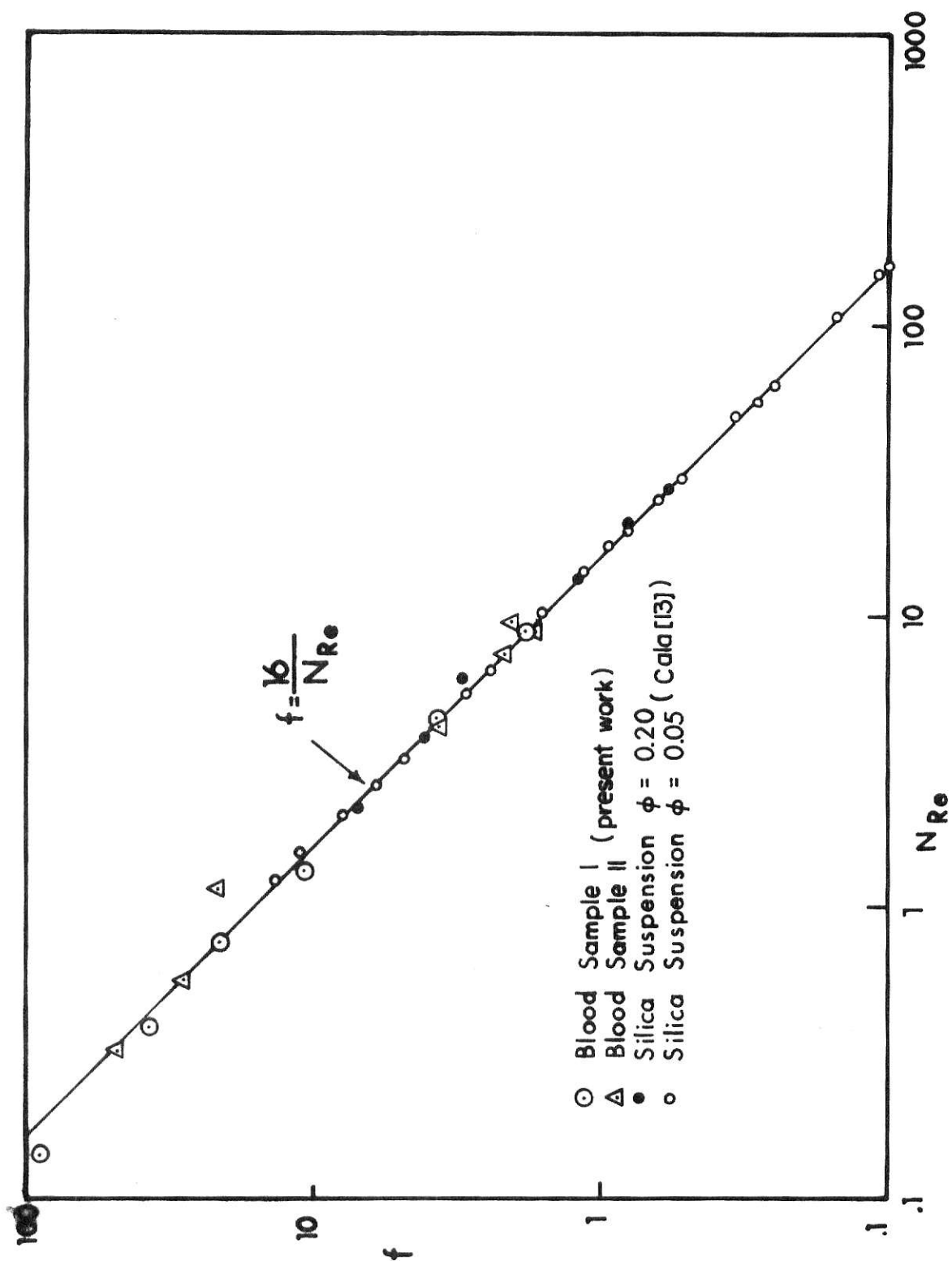


Fig. 8 The friction factor plot for bovine blood and silica suspensions.

The operation is simple and any user can obtain reliable data after a few trial runs. In most cases the resulting data can be analyzed, without difficulty, by graphical methods, as a result of linear behavior for the $\log \Delta P$ versus t plot (i.e. $\dot{\gamma}_w = -\tau_w m/B\rho$).

A comparison with other low shear capillary viscometers is also in order. Although the construction is basically similar to the viscometer of Benis [4] the operation and data interpretation are not. The present method is far superior in accuracy to the steady state method proposed by Benis and in addition permits the working range of the instrument to be extended considerably. As an example, in determining the radius by the method of Benis the results differed by as much as 10% whereas with the present method a deviation of the order of 0.5% was observed. Consideration of equation (4) which gives the pressure drop across the capillary reveals that ΔP can only be considered constant when $\Delta P_m \ll 2\rho gh$. This basically restricts Benis's approach to producing a single data point per experimental run and in addition sets a rather severe lower limit on the applied pressure that can be utilized.

The operation of the viscometer is similar to that proposed by Maron and Belner [2], that is, the rise of a fluid column towards an equilibrium position is measured as a function of time. Data interpretation is also similar. The basic difference between them is the feature that the present instrument is capable of producing low shear data with a relatively high pressure drop across the capillary whereas with the other the pressure drop is quite small. This is a direct consequence of the excessive length of capillary employed in the present unit. In addition, the distance from equilibrium is magnified with the present viscometer, providing for greater accuracy.

The instrument also poses some disadvantages, however in situations involving biological fluids this will be of little consequence. One restriction on the instrument arises from the use of polyethylene for the viscometer capillary. This material should be suitable with most aqueous solutions and suspensions, however a test for swelling of the capillary should be conducted before using organic liquids. In addition it should be mentioned that appreciable pressure losses (associated with the connectors) may be encountered at high flow rates. In the present study, non-linear semi-logarithmic plots were observed in the experiments with water when the driving pressures were of the order of 30 cm Hg and greater. This difficulty could be resolved to a certain extent by proper design of the connectors.

Mention of the working range of the viscometer is also in order at this point. In the present study two lengths and one I.D. were used and a shear stress range of approximately $0.1 - 16 \text{ dynes/cm}^2$ was covered. The shear stress range can be raised or lowered by changing the capillary I.D., length or both although no attempt was made to do so in the present work. Because of the difficulties at high flow rate it is suggested that the instrument be restricted to low shear application.

As a result of the simplicity and low cost of the instrument it should be quite suitable for student laboratory experiments as well as routine laboratory use by the non-specialist.

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CHAPTER 3

THE APPROXIMATE CASSON MODEL FOR TUBE FLOW

3.1 Abstract

An approximate Casson model for tube flow is derived based on order of magnitude considerations. The approximation is expressed in terms of a pseudo shear rate and is principally of value for tube flow measurement in which Q and ΔP are experimentally measured. Data for heparinized bovine blood and silica suspensions are in good agreement with the approximate expression. The model is compared with an approximation proposed earlier by Merrill and co-workers.

3.2 Introduction

A number of empirical models describing the bulk rheological behavior of non-Newtonian fluids are available in the literature [1, 2]. Perhaps one of the lesser known models is the Casson model which is semi-empirical in nature. It is particularly adaptable to dispersed systems composed of interacting particles. Casson [3] originally proposed the model in an effort to describe the rheological behavior of printing inks and lithographic varnishes. The original manuscript was reviewed by Scott-Blair who discovered that the model could describe the rheology of blood [4]. Other workers including Charm and Kurland [5] and Merrill and co-workers [6] verified the findings of Scott-Blair, and the Casson model found widespread use in describing blood rheology. However, Scott-Blair [7] has argued against the use of this model for blood on physical grounds, and Schmidt-Schöbein et al. [8] indicate that the model should only be applied to intermediate shear rates.

In the development of the model, Casson considered a system of interacting particles which would tend to aggregate into rod-like structures. The length of the rod was related to the level of the shear. In an effort to relate the viscosity change of the suspension due to the change in structure, the energy dissipation rate by the rod was considered. The rate of dissipation was related to the rod length and the orientation of the rod with respect to the flow of the medium. The relation between the length-width ratio and level of shear, however, was only given theoretically at extremely high and low shear rates. Hence a linear equation which fitted a limited range of shear rate was arbitrarily selected. The parameters of this linear equation were empirically determined. The resulting model of the suspension was expressed in terms of the parameters of the linear equation, rod volume fraction in the suspension, viscosity of the medium, and a parameter determined by the orientation of the rod.

The resulting semi-empirical model can be expressed as

$$\sqrt{\tau} = \sqrt{\tau_y} + S\sqrt{\dot{\gamma}} \quad (1)$$

In which τ_y is the apparent yield stress and S is the Casson constant. Oka [9] has presented and discussed a more general form of the Casson model.

Merrill and co-workers [6] developed an alternate form of the Casson model which is approximate in nature. Their model is expressed in terms of what we choose to term a pseudo shear rate, $\bar{U}$,

$$\bar{U} = \frac{Q}{2\pi R_c^3} \quad (2)$$

The resulting relation was expressed as

$$\sqrt{\tau} = \sqrt{\tau_y} + 2\sqrt{2} S \sqrt{\bar{U}} \quad (3)$$

This equation is principally of value for tube flow measurements in which Q and ΔP are the measured experimental values. From these measurements and a square root plot of τ versus $\bar{U}$, the apparent yield stress τ_y , and the Casson constant S , can be determined. This considerably simplifies the required data analysis, in comparison to the determination of shear rate.

In the development of equation (3), Merrill and co-workers employed the following treatment which contains assumptions that do not appear to be justified. From the Rabinowitsch equation [1], $\dot{\gamma}_w$ was expressed in terms of $\bar{U}$ as

$$\dot{\gamma}_w = 2\bar{U} (3 + N) \quad (4)$$

where

$$N = \frac{d \ln \bar{U}}{d \ln \tau_w} \quad (5)$$

Substitution of equation (4) into equation (1) gives

$$\sqrt{\tau_w} = \sqrt{\tau_y} + S[2(3 + N)]^{1/2} \sqrt{\bar{U}} \quad (6)$$

Merrill et al. [6] then made the assumption that N can be considered constant over limited range of τ_w ; and in addition assumed $N = 1$ (the case of Newtonian behavior). With these assumptions, equation (3) results from equation (6). At first glance, the assumption of $N = 1$ does not appear realistic, nor does the constant N assumption. Fortunately, for the case of blood the end result is reasonably correct (if the shear rate is not too low), but the assumptions employed give no indication of the conditions under which this approximation is valid.

In this chapter we will present an alternative approach to obtaining an approximate expression of the form of equation (4) on the basis of order of magnitude considerations. The specific limitations follow directly from the analysis. The resulting expression is tested with data for blood and silica suspensions* and compared with Merrill's approximation.

3.3 Analytical Considerations

The classic relation between volume flow and shear stress for flow in a cylindrical tube has been presented in a variety of standard texts [1, 2], and is generally presented as

$$\frac{Q}{\pi R^3} = \frac{1}{\tau_w^3} \int_0^{\tau_w} \tau^2 f(\tau) d\tau \quad (7)$$

where

$$f(\tau) = \text{shear rate.}$$

The result of equation (7) is based on the assumptions of:

1. steady-state, laminar flow
2. incompressible fluid
3. time-independent fluid (i.e., $\dot{\gamma} = f(\tau)$ alone)
4. no slip at the tube wall.

When a constitutive equation is available (i.e., $f(\tau)$ is a known function of τ in terms of rheological constants), equation (7) can be integrated to obtain an expression for the volume flow in terms of τ_w and the rheological constants. For the Casson fluid, $f(\tau)$ is

*Data for silica suspensions are from Cala [11].

$$f(\tau) = \frac{1}{S^2} [\tau - 2\sqrt{\tau_y \tau} + \tau_y] \quad (8)$$

Hence equation (7) can be integrated readily with the aid of equation (8) to yield

$$\frac{Q}{\pi R^3} = \frac{\tau_w}{S^2} \left[\frac{1}{4} + \frac{1}{3} \frac{\tau_y}{\tau_w} - \frac{4}{7} \left(\frac{\tau_y}{\tau_w} \right)^{1/2} - \frac{1}{84} \left(\frac{\tau_y}{\tau_w} \right)^4 \right] \quad (9)$$

Equation (9) can be expressed in terms of the pseudo shear rate $\bar{U}$ by dividing both sides by 2.

$$\bar{U} = \frac{\tau_w}{2S^2} \left[\frac{1}{4} + \frac{1}{3} \frac{\tau_y}{\tau_w} - \frac{4}{7} \left(\frac{\tau_y}{\tau_w} \right)^{1/2} - \frac{1}{84} \left(\frac{\tau_y}{\tau_w} \right)^4 \right] \quad (10)$$

A similar 4th power term in $\frac{\tau_y}{\tau_w}$ results when the Bingham model is employed as $f(\tau)$ to generate Buckingham's equation. For that case a good approximation is obtained by neglecting the 4th power term on the basis of order of magnitude consideration [1]. In the present case this simplification is also justified.

$$\bar{U} = \frac{\tau_w}{2S^2} \left[\frac{1}{4} + \frac{1}{3} \frac{\tau_y}{\tau_w} - \frac{4}{7} \left(\frac{\tau_y}{\tau_w} \right)^{1/2} \right] \quad (11)$$

Equations (10) and (11) agree to within 1.5% for $\frac{\tau_y}{\tau_w} \leq 0.4$ and for practically encountered values of $\frac{\tau_y}{\tau_w}$ the two equations can be considered identical (i.e., $\tau_w > 2.5\tau_y$).

Equation (11) can be solved explicitly for τ_w by noting that the equation can be expressed as a quadratic in terms of $\sqrt{\tau_w}$. The algebraic manipulations result in

$$\tau_w = \frac{188}{147} \tau_y + \frac{32 S^2 \tau_y^{1/2}}{7} \sqrt{\frac{2\bar{U}}{S^2} - \frac{\tau_y}{147 S^2}} + 8 S^2 \bar{U} \quad (12)$$

As a further simplification, it can be noted that for most practical cases

$$\frac{2\bar{U}}{S^2} \gg \frac{\tau_y}{147 S^4} \quad (13)$$

The inequality can alternatively be expressed as

$$\bar{U} \gg .0034 \frac{\tau_y}{S^2} . \quad (14)$$

Since $\frac{\tau_y}{S^2}$ is in general of order of magnitude of 1 or less it appears that

$\frac{\tau_y}{147 S^4}$ will be negligible for $\bar{U}$ greater than about 0.1 - 1. This is about

the lower limit of measurement with capillary tubes. With this simplification introduced equation (12) becomes

$$\tau_w = \frac{188}{147} \tau_y + \frac{32 \sqrt{2} S \tau_y^{1/2}}{7} \bar{U}^{1/2} + 8 S^2 \bar{U} \quad (15)$$

Equation (15) is approximately the square of an expression of the form

$$\sqrt{\tau_w} = X \sqrt{\tau_y} + Y S \sqrt{\bar{U}} \quad (16)$$

A slight modification of the coefficient of the τ_y term in equation (15) will make the expression an exact square of an expression of the form of equation (16). The fractional coefficient of the τ_y term of equation (15) is 188/147. Changing this coefficient to 192/147 gives an exact square. This is about a 2% change in the value of the coefficient, and will be relatively

unimportant since this term contributes only a small part to the sum of the right hand side of equation (15). It will be important, of course, at low values of $\bar{U}$, but here the restriction of $\tau_w > 2.5 \tau_y$ will be in effect and hence equation (15) itself, would not be valid. Within the limits of the values on τ_w and $\bar{U}$ imposed previously, this change in coefficient is quite acceptable. Noting that $\frac{192}{147} = \frac{64}{49}$; equation (15) can be expressed as:

$$\tau_w = \frac{64}{49} \tau_y + \frac{32 \sqrt{2} S}{7} \tau_y^{1/2} \bar{U}^{1/2} + 8 S^2 \bar{U} \quad (17)$$

Taking the square root of equation (17) gives

$$\sqrt{\tau_w} = \frac{8}{7} \sqrt{\tau_y} + 2 \sqrt{2} S \sqrt{\bar{U}} \quad (18)$$

Equation (18) is quite similar to Merrill's approximation equation (3),

$$\sqrt{\tau_w} = \sqrt{\tau_y} + 2S\sqrt{2} \sqrt{\bar{U}} \quad (3)$$

The result of equation (18) has been obtained on the basis of order of magnitude considerations and the limits of application in terms of τ_w and $\bar{U}$ have been established. As with Merrill's approximation, equation (18) predicts a linear $\sqrt{\tau_w}$ versus $\sqrt{\bar{U}}$ plot. However the intercept does not give $\sqrt{\tau_y}$ directly, with $\sqrt{\tau_y}$ being about 14% lower than the intercept. The two equations do however indicate the same value of S from a square root-square root plot.

3.4 Experimental

A coiled capillary viscometer, with a continuously varying pressure head, was used to obtain experimental data. A schematic diagram of the viscometer is shown in figure 1. The capillary was 7 meters long and had a radius

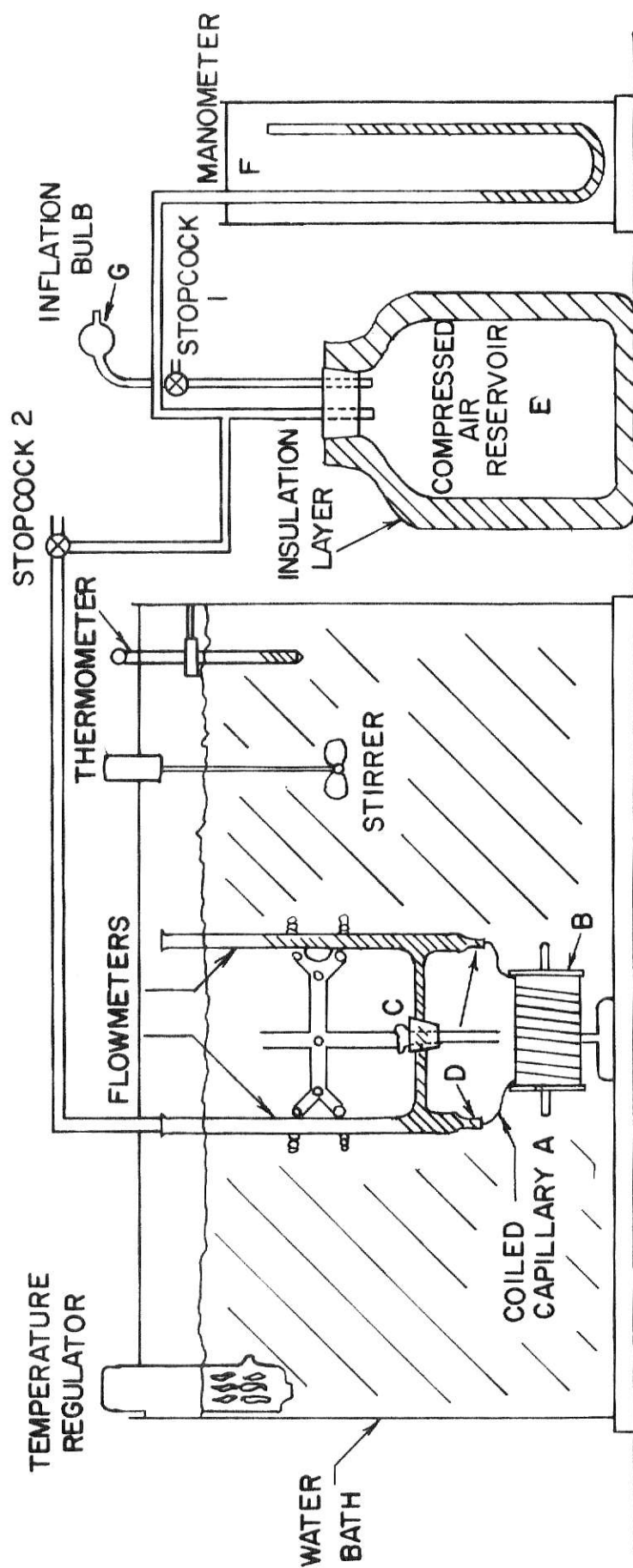


Fig. 1 Schematic diagram of the apparatus

of 0.05 cm. A detailed description of the viscometer and its operation is presented in 2.3. The raw data consisted of a set of fluid level (height) versus time data. For each fluid sample, a minimum of three sets of data were taken while maintaining the same driving pressure. The three sets of data were averaged and were used to calculate shear stress, shear rate and $\bar{U}$ values. The driving pressures ranged from 10 cm H₂O to 30 cm Hg to provide sufficient data for the establishment of a flow curve for the test fluid.

The fluids studied were whole heparinized bovine blood. The blood samples had a hematocrit range of 29-38. Data for silica suspensions by Cala [11] are also presented for comparison.

3.5 Data Analysis

The analysis of the data is similar to the approach used by Maron and Belner [10]. Details of the data treatment for the coiled capillary viscometer have been given in 2.5, however, the principal equations will be briefly outlined. Reference should be made to figure 2 and the nomenclature presented at the end of the text.

The shear stress at tube wall is given by

$$\tau_w = \frac{R_c \Delta P}{2L} \quad (19)$$

where $\Delta P = \Delta P_m - \rho gh$

$$\dot{\gamma}_w = - \frac{\tau_w}{B\rho} \left[1 + \frac{1}{9.212 \text{ m}^2} \frac{dm}{dt} \right] \quad (20)$$

where B is defined as

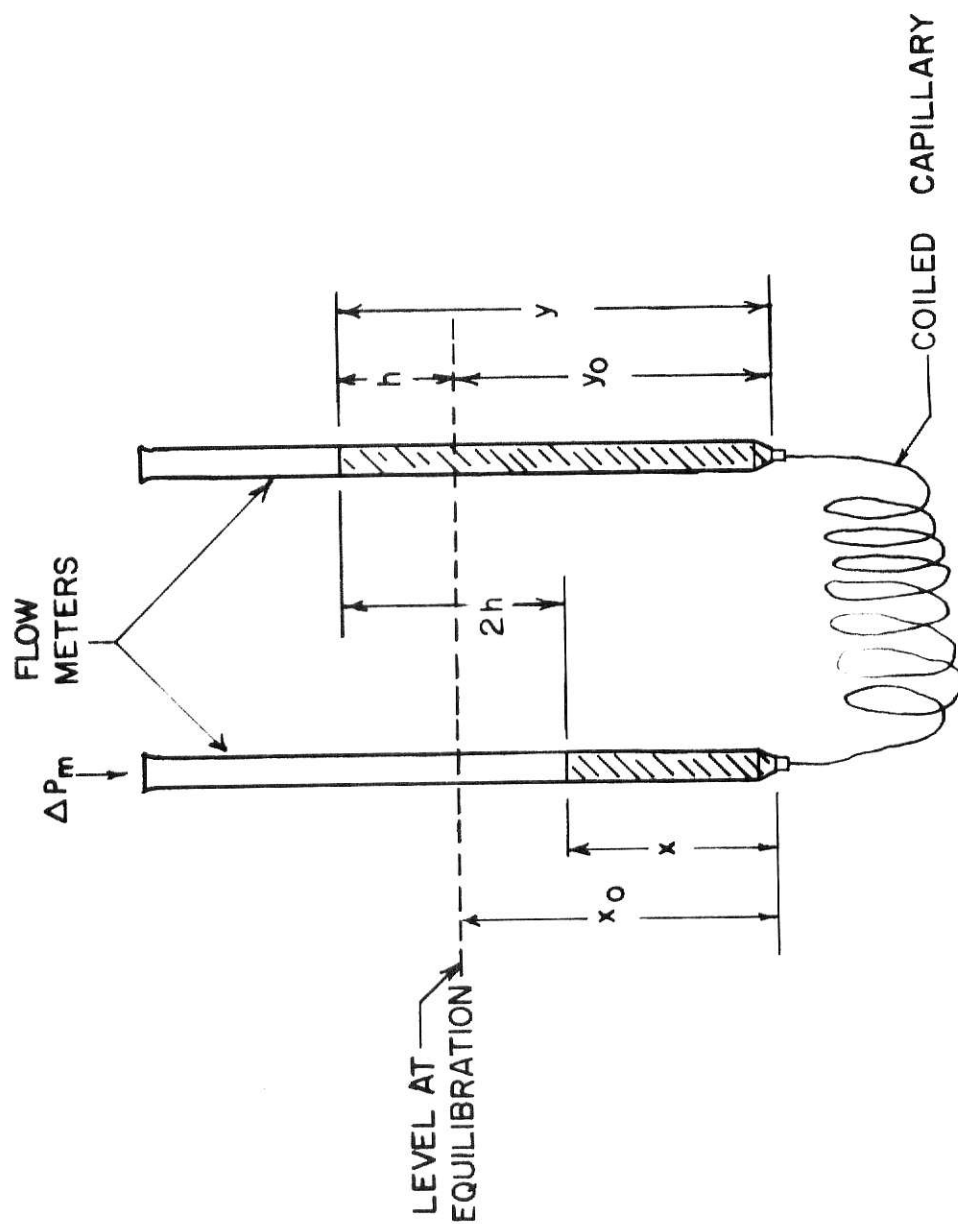


Fig.2. Definition of various terms.

$$B = \frac{g R_c^4}{9.212 L R_f^2} \quad (21)$$

and m is the slope from a $\log_{10} \Delta P$ versus t plot. Most of the semilog plots of ΔP versus t are linear, especially when the driving pressures are high. In this case, equation (20) reduces to

$$\dot{\gamma}_w = - \frac{\tau_w m}{B \rho} \quad (22)$$

For the case of curvature in the $\log_{10} \Delta P$ versus t plot, the term $\frac{dm}{dt}$ in equation (20) must also be evaluated through a plot of m versus t . The pseudo shear rate ($\bar{U}$) is evaluated from the flow rate which can be obtained readily from a known m value and the following relation,

$$Q = A_f \frac{dh}{dt} = - A_f \frac{2.303 m \Delta P}{\rho g} \quad (23)$$

$$\text{Here, note that } m = \frac{d \log_{10} \Delta P}{dt} = - \frac{\rho g}{2.303 \Delta P} \frac{dh}{dt} .$$

3.6 Results and Discussion

The experimental data were plotted as $\sqrt{\tau}$ versus $\sqrt{\gamma}$ and $\sqrt{\tau}$ versus $\sqrt{\bar{U}}$ in accordance with equations (1) and (18). Figures 3 and 4 present the results for bovine blood. Figures 5 and 6 are for silica suspensions. In all cases, the data fit straight lines reasonably well. The lines shown in the figures are the result of least square fitting of the data points. As can be seen from the plots, both test fluids can be described by the Casson fluid models provided that the shear rate is not sufficiently high to give Newtonian behavior as is the case in figure 6. The bovine blood data in figures 3 and 4 do not show a transition to Newtonian behavior up to shear rates of

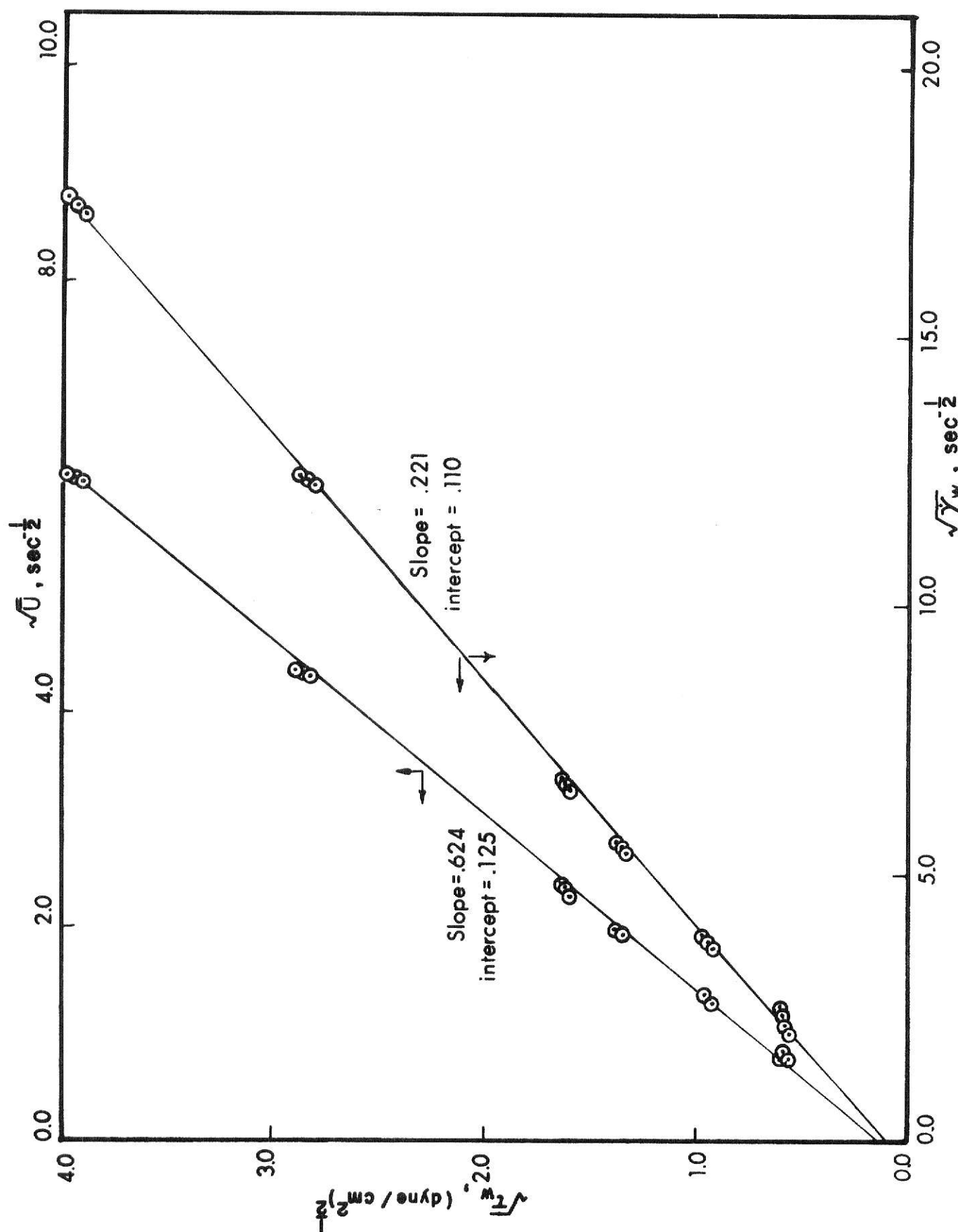


Fig. 3. $\sqrt{\tau_w}$ vs. $\sqrt{\dot{\gamma}_w}$ and $\sqrt{\tau_w}$ vs. $\sqrt{\dot{\gamma}_w}$ plots for blood data. (Sample 1)

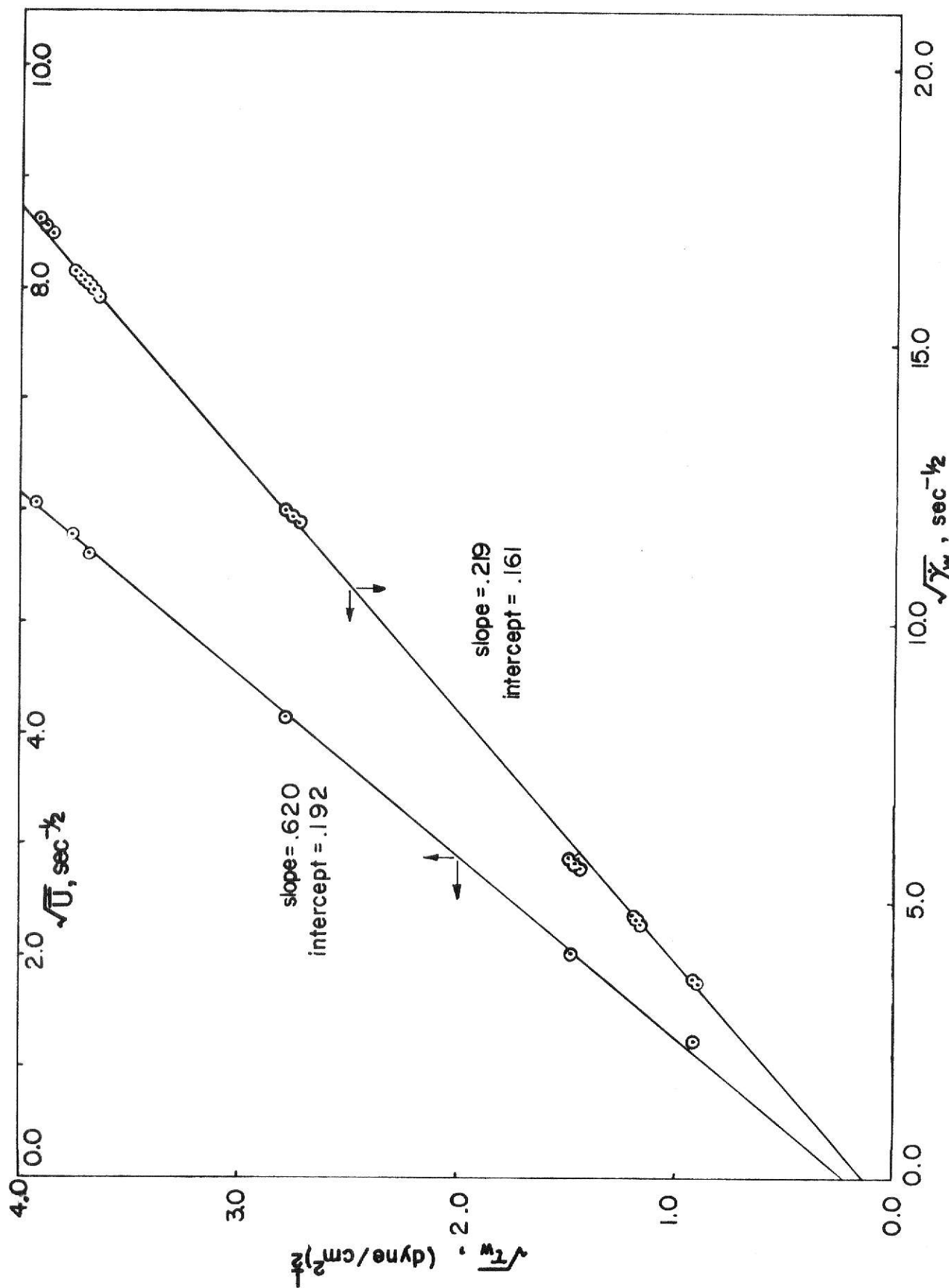


Fig. 4. $\sqrt{\tau_w}$ vs. $\sqrt{\dot{\gamma}_w}$ and $\sqrt{\tau_w}$ vs. $\sqrt{\dot{\gamma}_w}$ plots for blood (sample 2).

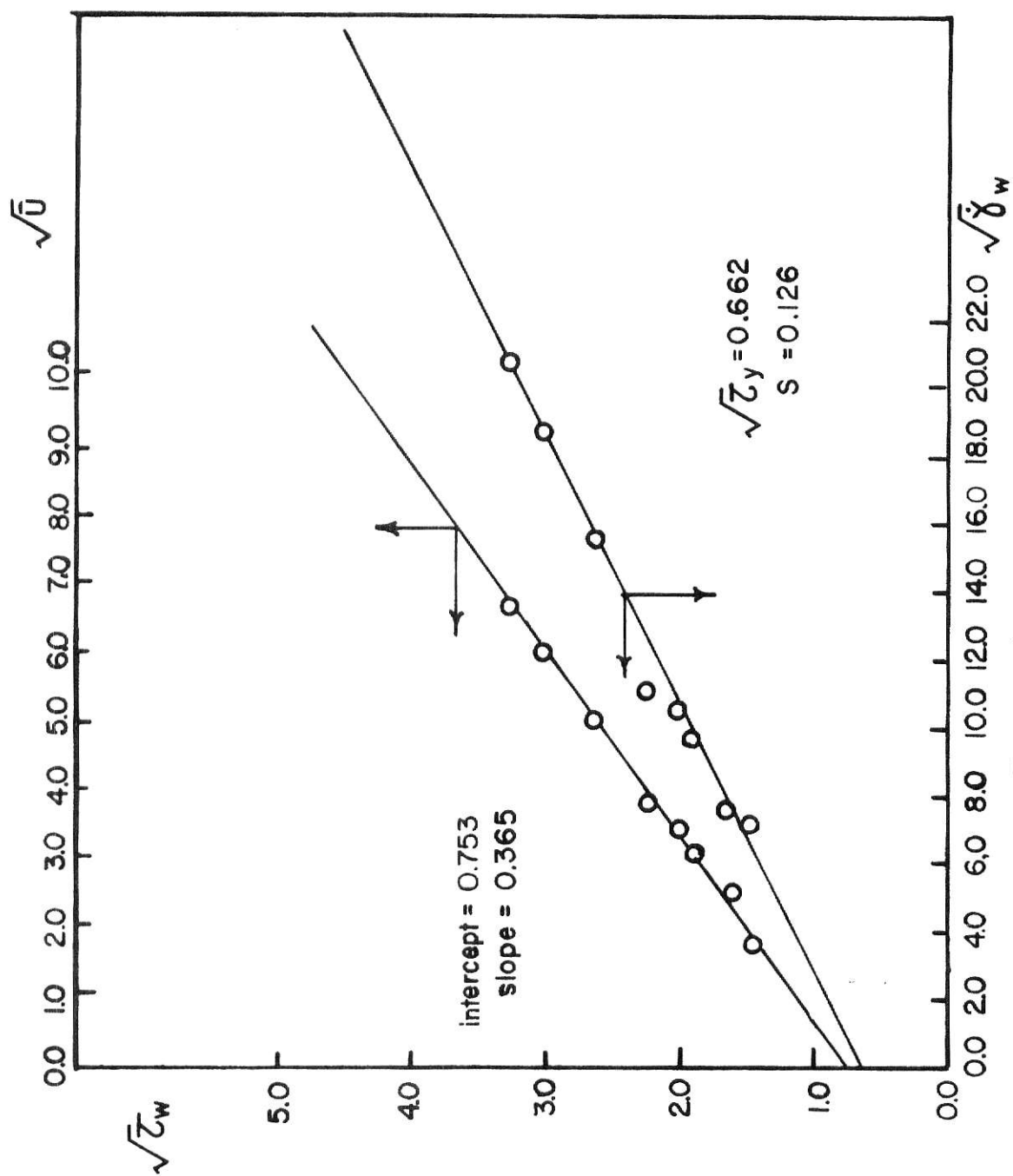


Fig. 5. Plots of $\sqrt{\zeta_w}$ vs. $\sqrt{\gamma_w}$ and $\sqrt{\zeta_w}$ vs. $\sqrt{U}$ silica suspension concentration of $\phi = 0.20$ (from Cata [1]).

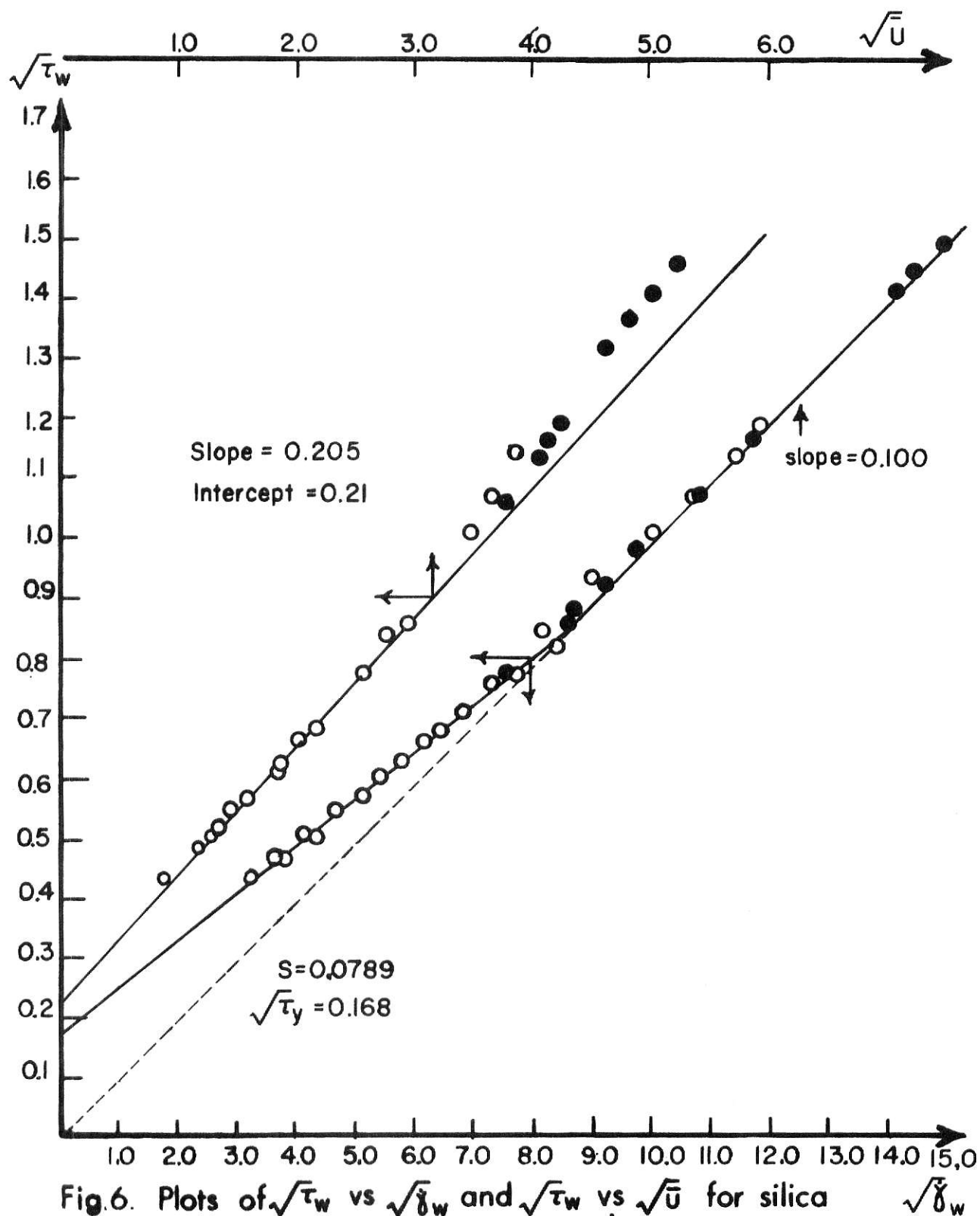


Fig.6. Plots of $\sqrt{\tau_w}$ vs $\sqrt{\delta_w}$ and $\sqrt{\tau_w}$ vs $\sqrt{U}$ for silica suspension concentration of $\phi = 0.05$ (from Cala[11]).

approximately 350 sec^{-1} . Charm and Kurland [5] have reported observing Casson fluid behavior with blood up to 10^5 sec^{-1} . It can also be noted from the figures, that the two methods of plotting do not produce the same intercept. In all cases the $\sqrt{\tau_w}$ versus $\sqrt{\bar{U}}$ plot gives the larger intercept as predicted by equation (18). From the slopes and intercepts of the best fit lines of the square root plots, values of the yield stress τ_y and Casson constant S were calculated according to equations (1) and (18). The results are presented in Table 1 for comparison.

As indicated in Table 1, S values compare remarkably well for the exact and approximate expressions. The values of τ_y show some deviation (up to 18% in one case) but these deviations are both positive and negative. It is most likely that these deviations result from experimental errors. For the most part, agreement is reasonably good.

The approximate Casson fluid model, expressed in terms of the pseudo shear rate ($\bar{U}$), appears to provide a good approximation to the exact model, expressed in terms of shear rate. The approximate form is based only on order of magnitude considerations and specific limits on its application have been established. These limits are

$$(1) \quad \tau_w > 2.5 \tau_y$$

$$(2) \quad \bar{U} > 0.1 - 1 \text{ here the criterion is } \bar{U} \gg .0034 \frac{\tau_y}{S^2}.$$

In the second limitation, $\frac{\tau_y}{S^2}$ generally is of the order of one or less and the inequality provides an approximate guideline. For most tube flow measurements, these limitations on τ_w and $\bar{U}$ will be below the values that can be measured with reasonable accuracy.

Table 1
Experimental results of whole bovine blood

Hematocrit	from $\sqrt{\tau}$ vs. $\sqrt{\dot{\gamma}}$		from $\sqrt{\tau}$ vs. $\sqrt{U}$	
	τ_y	S	τ_y	S
36	.0121	.211	.0122	.221
37.6	.0259	.219	.0289	.219
37	.0899	.218	.0858	.219
37	.0042	.188	.0044	.188
29	.168	.185	.150	.185

Experimental results of silica suspensions*

Solid volume fraction ϕ	from $\sqrt{\tau}$ vs. $\sqrt{\dot{\gamma}}$		from $\sqrt{\tau}$ vs. $\sqrt{U}$	
	τ_y	S	τ_y	S
0.20	0.438	0.126	0.443	0.129
0.05	0.028	0.0789	0.0329	0.0781

*Cala [11].

The approximation is very similar in form to Merrill's approximation with the only difference in the $\sqrt{\tau}$ term (see equations (3) and (18)). Merrill's equation predicts that the $\sqrt{\tau}$ versus $\sqrt{\dot{\gamma}}$ and $\sqrt{\tau}$ versus $\sqrt{\bar{U}}$ plots should give the same intercept. His experimental data seem to verify this prediction [6]. One explanation for this could be due to the fact that the $\bar{U}$ and $\dot{\gamma}$ data are from different instruments. In addition the difference in intercepts is small, about 14% according to equation (18). These two factors could be responsible for Merrill's results.

3.7 Conclusion

The Casson model is widely employed for describing the bulk rheological behavior of blood. In simple tube flow measurements it has been observed that a square root-square root plot of shear stress versus pseudo shear rate is linear. By means of order of magnitude considerations, the functional relation between $\sqrt{\tau}$ and $\sqrt{\bar{U}}$ can be established along with its limits of application. The resulting relation is given by equation (18). The expression permits the estimation of the Casson fluid parameters τ_y and S , from measurements of Q and ΔP for tube flow.

3.8 Nomenclature

A_f : cross-sectional area of the flowmeter, cm^2 .

L : length of the capillary tube, cm.

N : defined by equation (5).

ΔP : pressure drop across the capillary tube, dyne/cm^2 .

ΔP_m : the driving pressure, see figure 2, dyne/cm^2 .

Q : volumetric flow rate, cm^3/sec .

- R: radius of the tube, cm.
- R_c : radius of the capillary tube, cm.
- R_f : radius of the flowmeter, cm.
- S: the Casson constant, defined by equation (1).
- $\bar{U}$: the pseudo shear rate, defined by equation (2), sec^{-1} .
- g: gravitational acceleration, 980.6 cm/sec^2 .
- h: difference between height of fluid column and the equilibration level, see figure 2, cm.
- m: slope from a $\log_{10} \Delta P$ versus t plot.
- $\dot{\gamma}$: shear rate, sec^{-1} .
- ρ : density of the fluid, g/cm^3 .
- τ : shear stress, dyne/cm^2 .
- τ_y : apparent yield stress, dyne/cm^2 .

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CHAPTER 4

COMPUTERIZED DATA TREATMENT FOR A BURET-CAPILLARY VISCOMETER

4.1 Abstract

A computer-aided data treatment is presented for a buret-capillary viscometer. The technique employs a nonlinear parameter estimation method coupled with a well-established flow curve determination procedure to construct a flow curve. Data for a non-Newtonian fluid are used to illustrate the technique. The flow curve for the non-Newtonian fluid can be readily obtained.

4.2 Introduction

Capillary viscometry is a convenient experimental method for rheological measurements on fluids. With a capillary viscometer, the determination of the flow rate at a single pressure drop is sufficient to define the flow behavior of a Newtonian fluid. In the case of a non-Newtonian fluid, a range of shear stress must be covered in order to generate the flow curve and this is usually accomplished by varying the pressure drop. A viscometer operated with continuously varying pressure head was devised by Ostwald and Auerbach [1], who allowed liquid to drain from a buret under gravity through a capillary and recorded the times as the meniscus passed various buret graduations. This viscometer was discussed some years ago by Cerny [2] for a study on the absolute viscosity of water. Other viscometers with varying pressure head were designed by Maron, Krieger and Sisko [3], and Maron and Belner [4]. They used a well-established procedure [5, 6, 7] to obtain flow curve from the raw data. With the varying pressure viscometer, the data from a single

determination can cover a wide shear stress range. However, graphical differentiations are involved in the procedure for flow curve determination. These graphical differentiations, as a difficulty inherent in the procedure, are usually time consuming and inaccurate. Although Maron and Belner [4] presented a method to improve the graphical accuracy, the method is still tedious.

In an attempt to provide an alternative approach to graphical differentiation for obtaining the flow curve, a computer-aided data treatment has been explored. The viscometer is a modification of one described by Cerny [2]. The raw flow data for a non-Newtonian fluid are expressed in terms of an empirical equation. The parameters of the equation are evaluated via the non-linear parameter estimation method by Bard [8]. The empirical equation can then be used for the evaluation of shear stress and shear rate, following the procedure established by Krieger et al. [5, 6, 7], without any graphical differentiation.

4.3 Experimental Apparatus and Procedure

The viscometer is shown schematically in figure 1. It is a modification of one described some years ago by Cerny [2]. The instrument consists primarily of a precision bore capillary A and a 50 c.c. buret B. The capillary A is placed horizontally with one end inserted into a rubber stopper which is sealed to the collecting flask F and the other end connected to the buret B by means of a piece of tygon tubing. The pinch clamp C is a convenience for filling the viscometer. The flask F has a sidearm which is extended with a piece of tubing to the atmosphere. The buret B is jacketed by a 2-inch diameter glass tube. The water bath is kept at a desired temperature by a

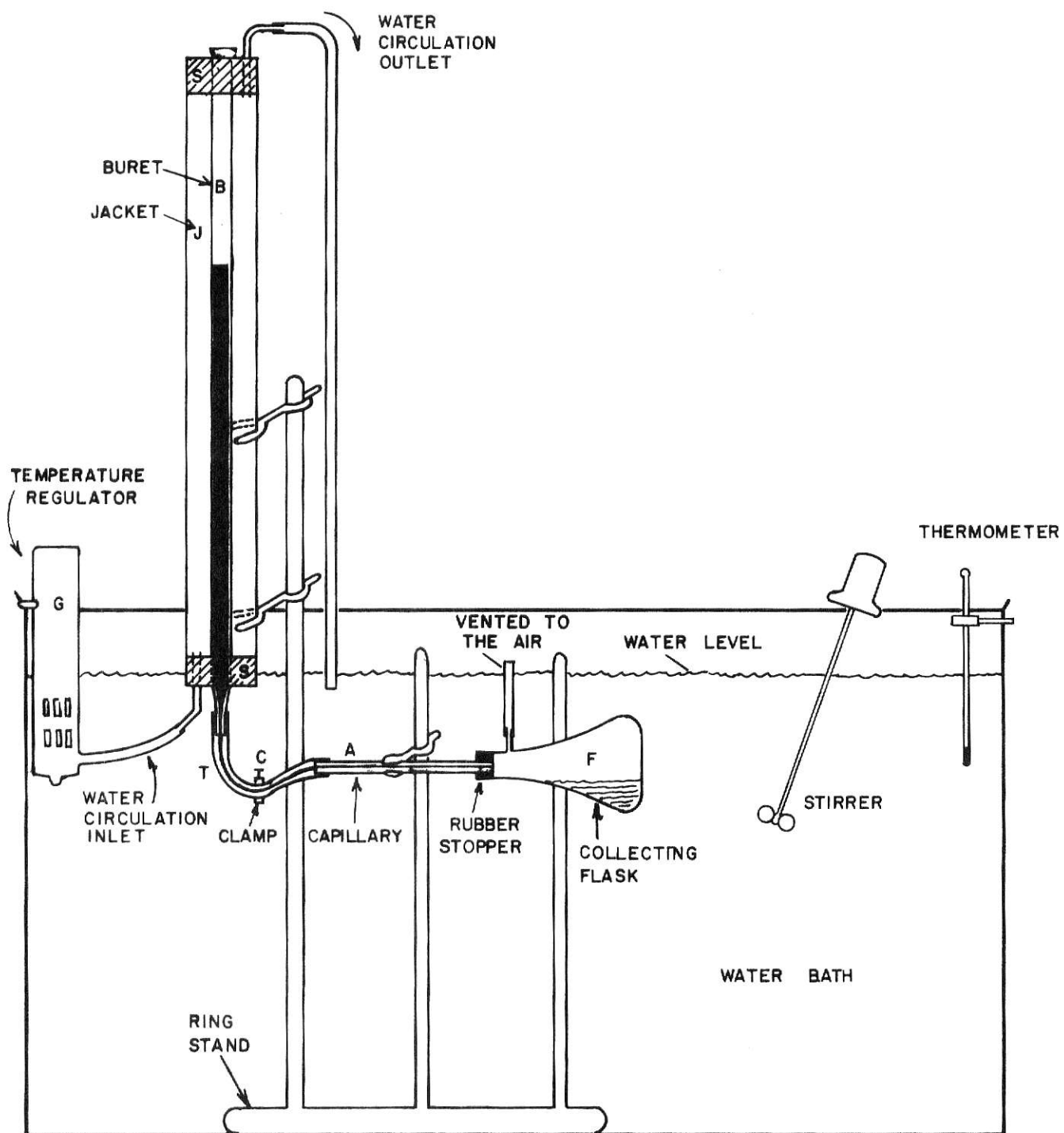


Fig. 1. Schematic diagram of the apparatus.

temperature regulator G. The regulator unit contains a pump which is used for circulation of water through the jacket. This arrangement assures constant temperature for the measurements.

In operation, the buret, connecting tubing, and capillary are filled with the test fluid and the clamp C put in place. Care should be taken to avoid trapping bubbles in the line. Generally the buret is filled well above the top graduation. If the test fluid is not at the bath temperature, about 10 minutes should be allowed to bring it up to bath temperature before starting a run. A run is started by opening clamp C, permitting the fluid in the buret to flow through the capillary. A stopwatch with a split hand feature is used to time the descent of the meniscus in the buret at selected graduations (i.e. 0, 5, 10, 15, ...). The times corresponding to the selected graduations are recorded. Readings can be taken until the meniscus passes the last graduation on the buret or until the descent of the meniscus is too slow to be measurable. A minimum of three sets of graduation (x) versus time data are taken for a given sample. An average of the three sets is used for data analysis.

4.4 Supporting Data

The length of the capillary can be measured directly. The radius is determined by filling the capillary with mercury, weighing the thread of mercury, and calculating the radius from the volume of the mercury thread. The volume is given by $V = (\text{mass of Hg}) \div (\text{density of Hg at measurement temperature})$. The radius then follows from $R = (V/2\pi L)^{1/2}$. A minimum of three determinations are recommended.

The buret cross section is determined by measuring the distance between terminal graduations (i.e. $h_0 - h_{50}$) and dividing the buret volume by this result.

$$A = 50/(h_0 - h_{50})$$

A relation between the buret graduations and the height of the meniscus relative to the capillary outlet is also required for data analysis. Noting the buret graduation as x and the measured distance between the last buret graduation and the capillary outlet as $(h_{50} - h_c)$, the following expression can be written

$$h = \frac{50-x}{A} + (h_{50} - h_c) \quad (1)$$

This gives the height of the meniscus relative to the capillary as a function of the buret graduation reading x .

The test fluid density, if unknown, is determined at the bath temperature with the aid of a pycnometer.

4.5 Theoretical

The theoretical development for Newtonian flow has been discussed by Cerny [2] and is outlined below. This will be followed by the analysis for non-Newtonian flow. The flow of a Newtonian fluid in a capillary tube is described by the Poiseuille equation,

$$\Delta P = \frac{8LQ}{\eta\pi R^4} \quad (2)$$

This expression relates the pressure drop ΔP across the capillary (of radius R and length L) to the volume rate of flow Q and the coefficient of viscosity η . For the viscometer the pressure drop at any moment is also given by

$$\Delta P = \rho gh \quad (3)$$

where h is the height of liquid column in the buret relative to the capillary, ρ the fluid density and g the acceleration of gravity. The volume rate of flow at any moment can be expressed as

$$Q = -A \frac{dh}{dt} \quad (4)$$

where A is the cross sectional area of the buret. From the combination of equations (2), (3) and (4), followed by integration an expression relating h and t results.

$$\ln h = - \left[\frac{\pi R^4 \rho g}{8LA\eta} \right] t + C = - \frac{B\rho}{\eta} t + C = mt + C \quad (5)$$

where

$$B = \frac{\pi R^4 g}{8LA}$$

Thus a plot of $\log_{10} h$ versus t should be linear. The viscosity of a Newtonian fluid can be evaluated from the slope (m) provided that the instrumental dimensions and the fluid density are known.

In the case of a non-Newtonian fluid, the "viscosity" is not constant and varies with the rate of flow or more properly the rate of shear. The $\log h$ versus t plot gives a curve with m varying from point to point. This variation can be utilized to relate the wall shear rate $\dot{\gamma}_w$ to the wall shear stress τ_w from which a flow curve τ_w versus $\dot{\gamma}_w$ can be constructed. An approach similar to that developed by Krieger and Maron [3] is employed. First, an effective fluidity is defined, with reference to equation (2) as

$$\phi_e = \frac{1}{\eta_e} = \frac{\pi R^4 \Delta P}{8LQ} \quad (6)$$

where η_e is the effective viscosity. From the expressions in equation (5) it can be seen that ϕ_e is given by

$$\phi_e = - \frac{m}{B\rho} \quad (7)$$

Under conditions of steady, laminar flow of a time-independent fluid through a cylindrical tube, it can be readily shown [9, 10] that

$$\frac{Q}{\pi R^3} = \frac{1}{\tau_w^3} \int_0^{\tau_w} \tau^2 f(\tau) d\tau \quad (8)$$

where

$$\tau_w = \frac{R\Delta P}{2L} \quad (9)$$

$$f(\tau) = \dot{\gamma} \quad (10)$$

Combination of equations (6), (8), and (9) gives

$$\phi_e = \frac{4Q}{\pi R^3 \tau_w} = \frac{4}{\tau_w^4} \int_0^{\tau_w} \tau^2 f(\tau) d\tau \quad (11)$$

Differentiation of equation (11) with respect to τ_w by Libnitz's rule and rearrangement of the result gives

$$\frac{\dot{\gamma}_w}{\tau_w} = \phi_e \left[1 + \frac{1}{4} \frac{d \ln \phi_e}{d \ln \tau_w} \right] \quad (12)$$

The terms in ϕ_e and τ_w in equation (12) are replaced by equations (7) and (9) so that after some algebraic manipulation equation (12) becomes

$$\frac{\dot{\gamma}_w}{\tau_w} = - \frac{m}{B\rho} \left[1 + \frac{1}{4} \frac{1}{m} \frac{dm}{dt} \right] \quad (13)$$

Equation (13), coupled with equation (9), is used to determine the flow curve of a non-Newtonian fluid.

4.6 Data Analysis

The average of the x versus time data is first converted into h versus t with the aid of equation (1). From equations (3) and (9) we have wall shear stress

$$\tau_w = \frac{R \rho g h}{2L} \quad (14)$$

which can be readily evaluated. To evaluate the wall shear rate from equation (13) we need m and dm/dt. This information can be obtained from the h versus t data with the aid of a non-linear parameter estimation technique (Bard's method [8]). Bard's method is in the form of a computer program provided by IBM. The user must supply the mathematical model, initial guesses and the bounds on the parameters, and the experimental data. The outputs include the estimated parameter values and the deviation of computed values from observed data values. From the deviation we can judge how well the proposed model fits the data points.

The h versus t data obtained by experiment can be described by a function of the form

$$h = h_o \exp \{-kt + (a + b t)^2\} \quad (15)$$

where

h_o = h value when t equals to zero (measured)

k, a, b = parameters to be estimated

The parameters a and b in equation (15) result from the non-linearity of a

ln h versus t plot. Note that when both a and b are zero equation (15) gives a model for Newtonian fluids. For the initial guesses of the parameters, k can be taken as the negative value of the slope of a line fitting the first few points of the ln h versus t plot. The parameters a and b can be taken as the intercept and the slope of the least-square-fit line of a $\sqrt{\delta}$ versus t plot, respectively, with δ defined by^{*}

$$\delta = \ln h - \ln h_0 + kt \quad (16)$$

Only a rough estimation for these parameters is sufficient and this can be easily done on a programmable desk calculator, or available computer program such as the IBM scientific subroutines package.

Both upper and lower bounds must be supplied in the input. The determination of these bounds is somewhat arbitrary. The bounds as suggested from this study are the following,

$$1) \quad k: \text{ initial guess } \times (1.00 \pm 0.30) \quad (17a)$$

$$2) \quad a: 0 < |a| < 0.1 \quad (17b)$$

$$3) \quad b: 0 < b < 0.01 \quad (17c)$$

where the upper bounds of $|a|$ and b are arbitrarily chosen as one order of magnitude greater than the values ordinarily encountered.

The parameters estimated by the computer program can be used to analytically evaluate m and dm/dt. From equation (15),

$$m = \frac{d \ln h}{dt} = -k + 2b(a + bt) . \quad (18)$$

From equation (18), dm/dt results,

^{*} It is apparent that δ represents the deviation of a ln h versus t plot from linearity. This deviation usually is a quadratic function of t. Accordingly, equation (15) was formulated.

$$\frac{dm}{dt} = 2b^2 . \quad (19)$$

Substitution of equations (18) and (19) into equation (13) gives an expression for $\dot{\gamma}_w/\tau_w$ in terms of the parameters. As presented in the program in Appendix, the evaluation of m , dm/dt , τ_w , and $\dot{\gamma}_w/\tau_w$ can be done on the computer with a slight addition to the original Bard's program. In this way, the flow curve information, τ and $\dot{\gamma}$, is obtained directly as computer output.

4.7 An Example of the Method

Pressure flow data (in the form of h versus t) for a non-Newtonian fluid are used to illustrate the procedure. The fitted h versus t curve is then compared with experimental values. The best fit parameters are then employed for the determination of the flow curve for the fluid.

Reference is made to Table 1. Column 2 gives the h values converted from raw data of x versus t by equation (1). Column 3 presents $\ln h$ values which are used for the initial guess of k . A least square fit of these data in the form of $\ln h_{\text{exp},1}$ versus t is made with a programmable Wang calculator. The result for the case gives a slope of -0.0024819 . Hence the initial guess of k is taken as 0.0024819 . Values of δ are then calculated as indicated by equation (16). The values of δ and $\sqrt{\delta}$ are tabulated in columns 4 and 5. The $\sqrt{\delta}$ data are used for the initial guesses of parameters a and b . The $(\sqrt{\delta}$ versus t) data are fitted by least squares with the aid of a programmable calculator. The resulting intercept and slope give the initial guesses of a and b , respectively. These values are presented in Table 2. Bounds for parameters calculated by the program are also shown in Table 2. These parameter values are used in the program to calculate h values by equation (15).

Table 1 Results for the test fluid

Time, t (sec)	$h_{\text{exp}'1}$ (cm)	$\ln h_{\text{exp}'1}$	δ	$\delta^{1/2}$	$h_{\text{cal}'d}$ (cm)	τ (dyne/cm ²)	$\dot{\gamma}$ (sec ⁻¹)
0	56.50	4.0342	0	0	56.54	74.8	1303
7.7	55.44	4.0153			55.45	73.4	1276
23.4	53.31	3.9762			53.31	70.5	1223
39.8	51.19	3.9355			51.18	67.7	1169
56.8	49.07	3.8932	0	0	49.06	64.9	1116
74.8	46.94				46.93	62.1	1063
93.6	44.82				44.80	59.3	1011
113.2	42.69				42.70	56.5	959
134.4	40.57				40.55	53.6	906
156.6	38.45	3.6492	.002	.04	38.42	50.8	854
205.1	34.20				34.19	45.2	751
231.9	32.07				32.08	42.4	700
246.2	31.03				31.01	41.0	675
261.0	29.95	3.3995	.007	.08	29.95	39.6	649
276.5	28.89				28.87	38.2	624
292.4	27.83				27.82	36.8	624
309.2	26.76				26.75	35.4	573
326.3	25.70				25.68	34.0	548
345.1	24.64	3.2044	.017	.13	24.62	32.5	523
364.1	23.58				23.56	31.2	498
384.2	22.52				22.51	29.8	473
427.3	20.39				20.41	27.0	425
450.9	19.33	2.9616	.032	.18	19.35	25.6	400
475.9	18.27				18.30	24.2	376
502.5	17.21				17.25	22.8	352
531.2	16.14				16.20	21.4	328
564.3	15.08				15.07	19.9	303
596.7	14.02	2.6405	.064	.25	14.05	18.6	280
633.9	12.96				12.97	17.2	256
674.5	11.90				11.90	15.8	232
719.2	10.83				10.84	14.3	209
769.9	9.77				9.76	12.9	186
828.4	8.71	2.1645	.143	.38	8.67	11.5	162
896.0	7.65				7.58	10.0	139

Table 2 Values for various parameters

	k	a	b
<u>initial guess</u>	0.0024819	-0.031137	0.00049889
<u>lower bound</u>	0.0017373	-0.1	0
<u>upper bound</u>	0.0032265	0	0.01
<u>estimated parameter</u>	0.0024855	-0.025336	0.00054924

The resulting h values are presented in column 6, Table 1. A comparison between the best fit curve and the experimental data is given in figure 2. As can be seen from the figure, the fitted curve describes the experimental points very well. The error in $h_{cal'd}$ as can be seen in Table 1 never exceeds 1%.

Next τ_w and $\dot{\gamma}_w$, as given by,

$$\tau_w = \frac{R\rho gh}{2L} \quad (15)$$

and

$$\dot{\gamma}_w = \tau_w \cdot \frac{-m}{B\rho} \left[1 + \frac{1}{4m} \frac{dm}{dt} \right] \quad (13)$$

are evaluated, using the estimated parameters. Here h , m and dm/dt are given in terms of the parameters by equations (15), (18) and (19). The results are shown in column 7 and 8 of Table 1 as well as in figure 3. As shown in figure 3 the shear rate range from a single determination covers about one cycle. The flow curve of non-unit slope indicates the non-Newtonian behavior of the test-fluid.

4.8 Conclusion

As is illustrated by the example, a convenient computer-aided technique has been developed for the determination of the flow curve for non-Newtonian fluid from flow data. The height versus time data are fitted to a proposed model equation in terms of several parameters. These parameters are estimated by a computer program for Bard's non-linear parameter estimation method. Height versus time data are input to the program as well as the initial guesses, and lower and upper bounds for the parameters. Only a slight

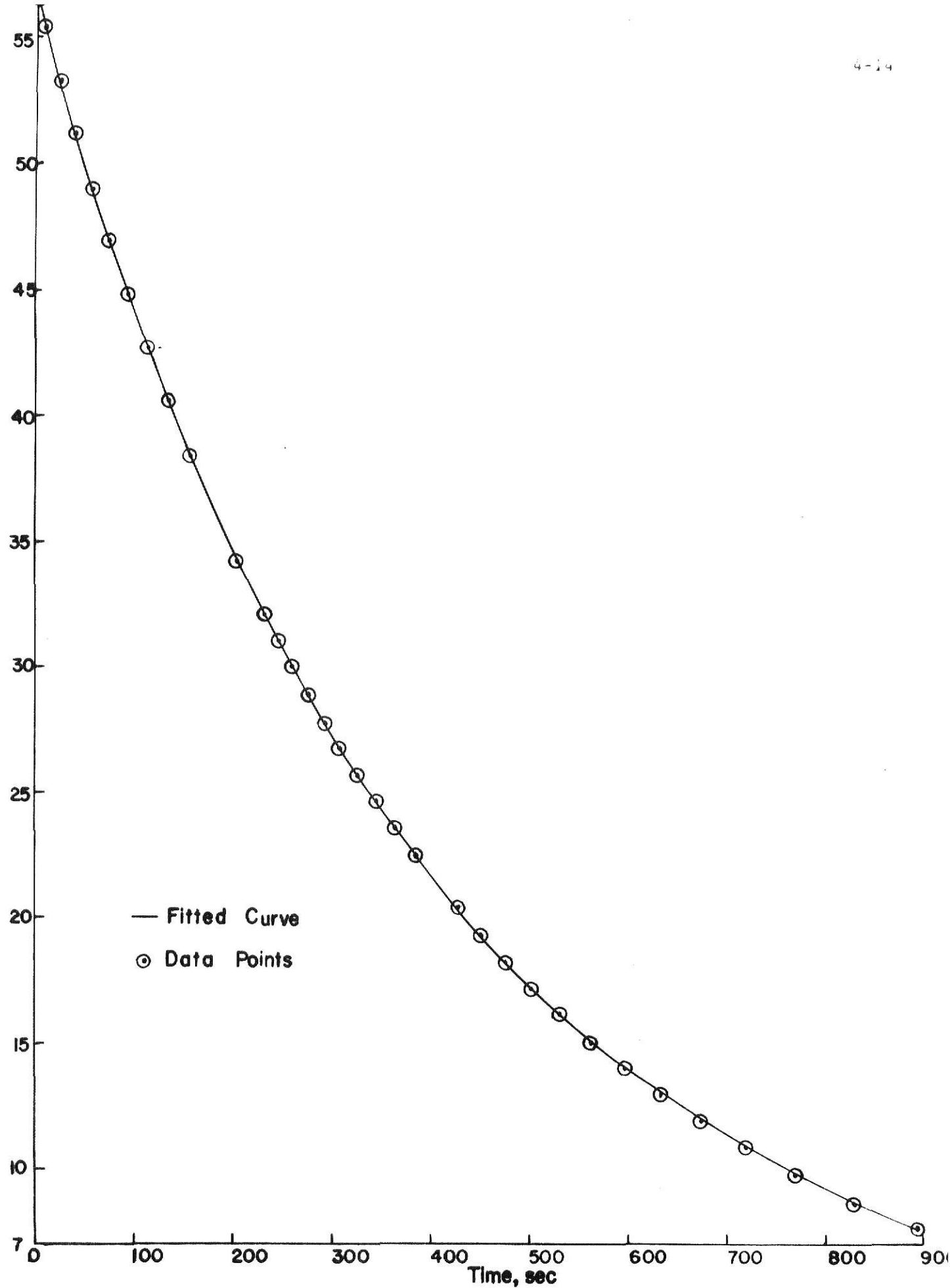


Fig.2. A typical fitted curve for h vs. t data.

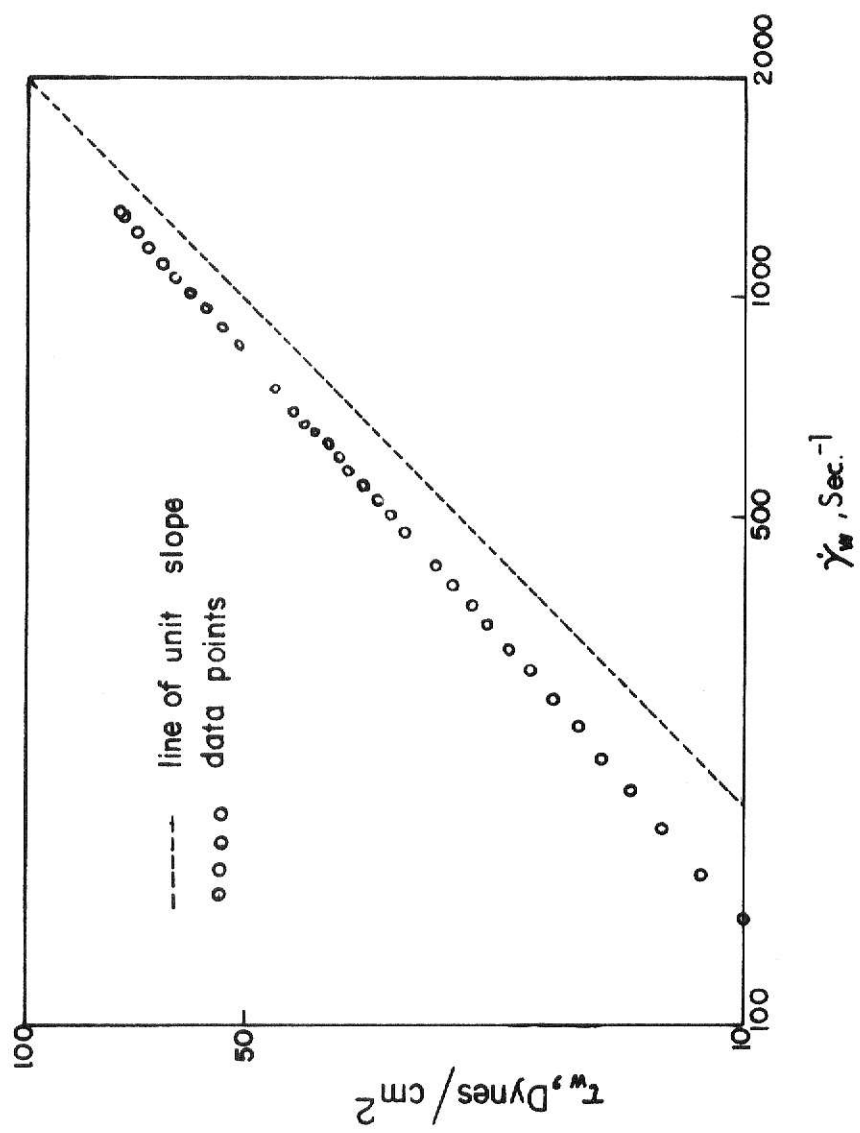


Fig.3 The flow curve.

modification of the original Bard's program provides for a direct print-out of flow curve data. As a result of the computerized data treatment, the capillary viscometer can be a more useful tool for routine laboratory determinations, or student experiments.

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4.10 Appendix

The computer program used consists of seven decks; a main program and six subroutines. In this appendix, details are given only for those which are modified, along with the modifications made. Definitions of the variables added are given in the comment statements. Names of the other subroutines are given to show the entire structure of the program. For details of the entire technique, reference can be made to Bard's original manual [8].

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C DECK 01

C MAIN PROGRAM

COMMON C(20,20),G1(20,20),PSCA,G(20,20),F(20),Y(20),EGV(20),FF(20)
1,TITLE(20),CUR(20),CLB(20),PNL(20),NCON,LOUT,F3,NTH,F6,F7,METH,NPH
2,MD,LS,CI(20)
3,X,XTH(20),A(200,10),M,NA

C THE FOLLOWING COMMON CARD IS ADDED

COMMON HO

1 WRITE(6,2)

2 FORMAT(1H1)

READ(5,2000,END=9999)TITLE

2000 FORMAT(20A4)

C*****

C* THE FOLLOWING TWO CARDS ARE TO READ IN DATA OF :

C* HO ---INITIAL LIQUID COLUMN HEIGHT

C* RC ---CAPILLARY RADIUS

C* RF ---FLOWMETER(BURET) RADIUS

C* GTH---CAPILLARY LENGTH

C* ROF---DENSITY OF LIQUID

READ(5,3099)HO,RC,RF,GTH,ROF

3099 FORMAT(5E10.5)

C*****

CALL NLMAX

C*****

C* THE FOLLOWING CARDS ARE TO CALCULATE:

C* HT---THE CALCULATED HEIGHT

C* SM---M, NEGATIVE OF SLOPE OF LOG H VS. T PLOT

C* DM---DM/DT, SLOPE OF M VS. T PLOT

C* TW---SHEAR STRESS, TAU

C* GM---SHEAR RATE, GAMMA

C* WHERE

C* TM---TIME

C* C(1),C(2),C(3)---PARAMETERS

119 WRITE(6,2099)

2099 FORMAT(1H0,'HEIGHT',7X,'TAU W',8X,'GAMMA W')

DO 98 I=1,M

TM=A(I,1)

HT=HO*EXP(-C1(1)*TM+(C1(2)+C1(3)*TM)**2)

SM=-C1(1)+2.*(C1(2)+C1(3)*TM)*C1(3)

DM=2.*C1(3)**2

TW=HT*ROF*RC/GTH*490.3

GM=-GTH*RF**2/(RC**4*ROF)*TW*SM*(1.+DM/(SM*2.))**2)*0.0081583

WRITE(6,2009) HT,TW,GM

2009 FORMAT(8E13.6)

98 CONTINUE

C*****

GO TO 1

9999 CONTINUE

STOP

END

C DECK 02
C MAXIMIZES FUNCTION USING GAUSS-NEWTON METHOD

SUBROUTINE NLMAX

C DECK 03
C COMPUTES SUM OF SQUARES

SUBROUTINE ACCUM(I1)

C DECK 04
C COMPUTES SCALED EIGENVALUES AND VECTORS

SUBROUTINE FIG(N,I1)

C DECK 05
C OUTPUT FOR LEAST SQUARES PROBLEMS

SUBROUTINE OUT

C LOWER AND UPPER BOUNDS ON PARAMETERS

SUBROUTINE BOUND(I1,H)

C*****
C* MODEL EQUATION IS SUPPLIED INTO SUBROUTINE DLSQ IN THE FORM OF *
C* $X = \text{FUNCTION OF } A(I,J), \text{ AND PARAMETERS } C(I)$ *
C* WHERE *
C* X ---THE DEPENDENT VARIABLE *
C* $A(I,J)$ ---THE INDEPENDENT VARIABLES *
C* AND THE DERIVATIVES OF X WITH RESPECT TO EACH PARAMETER IN THE FORM *
C* OF: *
C* $XTH(I) = \text{FUNCTION OF THE INDEPENDENT VARIABLES AND PARAMETERS}$ *
C*****

SUBROUTINE DLSQ (I1,I)

COMMON ZZ(1364),NTH,F6,F7,METP,OPH,MO,LS,C1(20),X,XTH(20),A(200,10
1),E,QA

COMMON HQ

GO TO (1,1,2),I1

2 RETURN

1 AXP=EXP(-C1(1)*A(I,1)+(C1(2)+C1(3)*A(I,1))**2)

X=HQ*AXP-A(I,2)

GO TO (5,6),I1

6 XTH(1)=-A(I,1)*HQ*AXP

XTH(2)=HQ*AXP*2.*(C1(2)+C1(3)*A(I,1))

XTH(3)=XTH(2)*A(I,1)

5 RETURN

END

CHAPTER 5

NON-NEWTONIAN BLOOD FLOW IN TAPERED TUBES

5.1 Abstract

Steady laminar blood flow in uniformly tapering tubes under non-Newtonian flow condition is examined. The pressure flow relation is studied both theoretically and experimentally. Blood rheology is assumed to be described by the Casson model. Theoretical pressure flow relation for Casson model fluid is derived by a differential-cylinder approach as well as by direct integration. Both methods result in the same expression. Experimental data for bovine blood flow in 1° and 2° tubes agree well with the theoretical predictions.

5.2 Introduction

Almost twenty years ago, Jeffords and Knisely [1] presented evidence to strongly suggest that the segments of blood vessels between branches should be considered as segments of slowly tapering cones rather than cylinders. Data were reported for the mesenteric arterioles of frogs and mice and were later confirmed by the independent work of Bloch [2]. Angles of taper of up to about 2° were observed.

The arterioles are believed to be responsible in part for the local regulation of blood flow. The resistance to flow is related to the rheology of the fluid, the level of shear and the geometry of the vessel. Perhaps the most significant aspect of taper is that it provides a mechanism for increasing shear while the volume flow remains constant. This feature coupled with

non-Newtonian behavior of the blood in the broadest sense could have a considerable influence on the resistance to flow.

There have been numerous theoretical considerations of flow in conical sections with various degrees of simplification involved. Experimental work to confirm theoretical predictions on the other hand is relatively scarce. Some of these efforts will now be reviewed.

A number of theoretical considerations of flow in uniformly tapering vessels have appeared in the literature. The solution for the steady creeping flow of an incompressible Newtonian fluid through conical sections has been presented by several investigators [3, 4, 5]. The pressure development in the vessel was related to the volumetric flow rate and angle of taper. Targ [6] and Oka and Nishimura [7] extended the theoretical solution to the case where the inertial effects cannot be neglected. The numerical solution for both Newtonian and non-Newtonian converging flow in conical sections was presented by Sutterby [8]. Cerny and Walawender [9] presented several solutions to the problem of the viscous flow of an incompressible Newtonian liquid in a converging tube. The solutions include pressure flow relations for the cases of pulsatile and steady flow in a rigid tube and pulsatile flow in an elastic tube. Walawender et al. [10] presented simplified solutions, for the limiting cases of very slow and very fast flow, for a Newtonian fluid in a uniformly tapering tube. By adding a semi-empirical inertial correction term to the solution of very slow case, they presented a solution of practical use for a substantially extended flow range. The steady slow motion of a power-law liquid through a tapered tube was examined by Oka and Takami [11]. They derived a differential equation for velocity distribution. By considering a tapered tube as the integral of a series of differential cylindrical tubes,

Charm and Kurland [12] derived a pressure flow relationship for Newtonian flow in a tapered tube. The result was identical with that presented by Bond [4]. A similar approach was employed by Benis and Lacoste [13] for the flow of a Casson fluid in a tapered vessel. They made the assumption that the local pressure gradient at any point in the tube is the same as that in a uniform tube of the same diameter. Recently, Oka [14, 15] described the pressure development in non-Newtonian flow through a tapered tube. He presented a "general formula" for the volumetric flow rate of non-Newtonian fluids, which verifies the concept that in a slightly tapered tube the volumetric flow rate at any point is the same as that of a cylindrical tube of the same radius. Pressure gradient formulae were obtained based on his "general" flow rate formula for three types of non-Newtonian fluids: power law fluid, Bingham body, and Casson model fluid.

Experimentally, only a limited amount of work on blood flow in tapered tubes has been presented in the literature. Merrill et al. [16] obtained some pressure flow data for suspensions of red cells in plasma in drawn tapered tubes and found that there was no difference between converging and diverging flows. However, the pressure flow data for whole blood in precision tapered tubes obtained by Cerny and Walawender [17] indicate that the flow pattern is different when blood is moving in the convergent or in the divergent direction. These blood data were shown, by means of a generalized flow curve, to be superimposable with the data from cylindrical tubes. Charm and Kurland [12] presented pressure flow data for suspensions of human red cells in plasma. Their data are in agreement with their calculated results based on the consideration of a tapered tube as a series of segments of cylindrical tubes and the existence of a cell free plasma layer at the wall. Whole blood data for

Newtonian flow in 1° and 2° tapered tubes by Walawender et al. [10] agree very well with the simplified theoretical pressure-flow predictions they proposed for Newtonian flow with substantial inertial losses.

The present work is concerned with both theoretical and experimental studies on blood flow in tapered tubes of small angle of taper. Laminar, steady flow of blood in converging tapered tubes under non-Newtonian flow conditions is examined. Blood rheology is assumed to be described by the Casson model. An approximate pressure-flow relationship is proposed to describe the flow behavior of blood in tapered vessels. This relationship is expressed in terms of the rheological parameters of blood and dimensions of the tapered tube. The experimental data are compared with the theoretical predictions.

5.3 Analysis

In the circulatory system, the flow situation is rather complicated both in terms of the geometry of the vessel and the rheology of the fluid as well as the pulsatile nature of the flow. The present analysis considers a highly simplified case however it does include the features of tapered geometry as well as a simplified non-Newtonian rheology. The steady laminar flow of an incompressible non-Newtonian fluid in a uniformly-tapering tube of circular cross-section is considered. The geometry is most conveniently represented by spherical coordinates. A schematic diagram for a section of tapered vessel is shown in figure 1. The flow in the tube is assumed to be axisymmetric hence the ϕ component of the equation of motion is omitted. The angle of taper is small so that the functions $\sin \theta$ and $\cos \theta$ can be approximated as θ and 1 respectively. It is assumed that two dimensional flow is present, but that

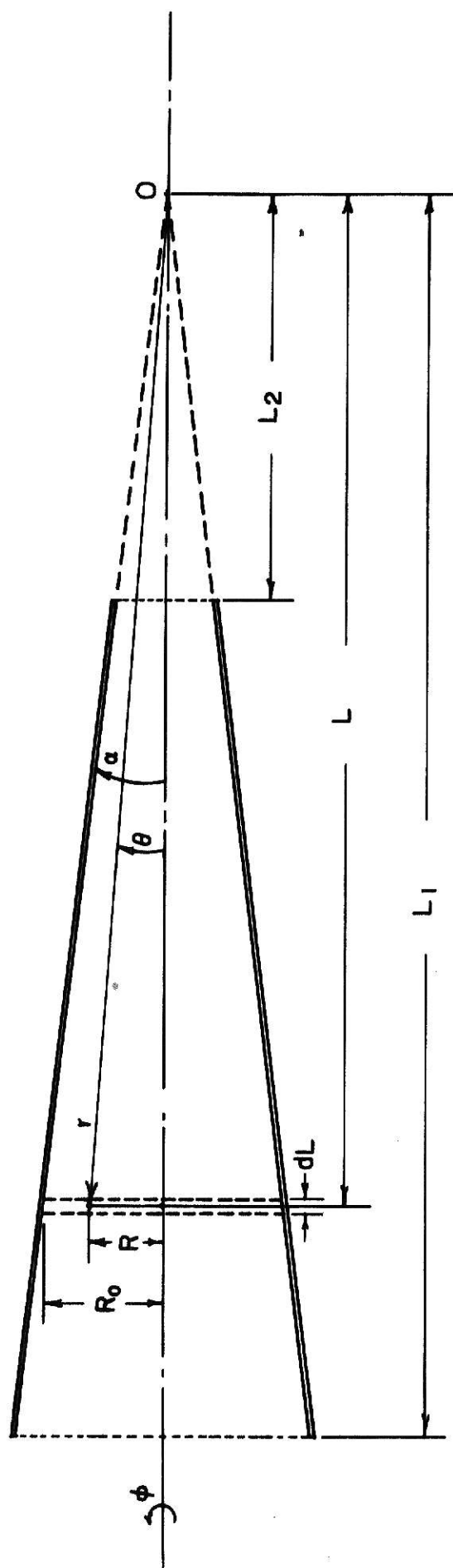


Fig. 1. A tapered tube and the coordinate system.

$V_r \gg V_\theta$. Furthermore it is assumed that V_θ is of the same order of magnitude as α . Order of magnitude considerations coupled with the stated assumptions permit simplification of the equation of motion to

$$\rho V_r \frac{\partial V_r}{\partial r} = - \frac{\partial p}{\partial r} - \frac{1}{r\theta} \frac{\partial}{\partial \theta} (\theta \tau_{\theta r}) \quad (1)$$

Note that only the r component remains. The case of a Newtonian fluid has been discussed by Targ [6] and Sutterby [8]. Sutterby showed through numerical calculations that the θ component of the equation of motion can be neglected. As a further simplification, the case of creeping flow allows equation (1) to be expressed as

$$\frac{dp}{dr} = - \frac{1}{r\theta} \frac{\partial}{\partial \theta} (\theta \tau_{\theta r}) \quad (2)$$

Equation (2) can be integrated at a given r from $\theta = 0$ to $\theta = \theta$, with the boundary condition

$$\tau_{\theta r} = 0 \quad \text{at} \quad \theta = 0$$

to yield an expression for the local shear stress

$$- \frac{\theta r}{2} \frac{dp}{dr} = \tau_{\theta r} \quad (3)$$

Equation (3) gives the local wall shear stress when $\theta = \alpha$,

$$- \frac{\alpha r}{2} \frac{dp}{dr} = \tau_{\theta r} \Big|_{\theta=\alpha} \quad (4)$$

With reference to figure 1, it can be shown for the case of small α that the following approximations can be made.

$$\theta r \doteq R \quad (5)$$

$$\alpha r \doteq R_0 \quad (6)$$

$$r \doteq L \quad (7)$$

As a consequence, equation (4) can be approximated as

$$-\frac{R_o}{2} \frac{dp}{dL} = \tau_{RL} \Big|_{R=R_o} \quad (8)$$

Equation (8) is identical in form to the expression for the wall shear stress, for axial flow in a cylindrical tube of length dL and radius R_o . Recall that in arriving at equation (8) the following assumptions have been made: 1) the angle of taper α is small, 2) the flow is slow, i.e. the inertial effect is negligible, 3) the pressure drop is not dependent on θ , and 4) on the basis of order of magnitude considerations, the normal stress terms of the equation of motion are negligible in comparison with the shear stress.

In Charm and Kurland's work [12], they suggested that a tapered tube be considered as a series of differential cylindrical segments. If an expression for the pressure-flow relation for tube flow is available an expression for dP/dL can be written. The latter can be integrated to yield the pressure flow relation for a tapered tube. For example, the case of uniform taper gives

$$R_o = \alpha L. \text{ For a Newtonian fluid } -\frac{dP}{dL} = \frac{8\eta Q}{\pi R_o^4} = \frac{2}{R_o} \tau_{RL} \Big|_{R=R_o} \text{ or } -\frac{dP}{dL} = \frac{8\eta Q}{\pi \alpha^4 L^4}$$

and the integration is straightforward. The assumptions made in the derivation of equation (8) indicate the limitations on their incremental cylinder approach for the case of non-Newtonian flow in addition to the case of Newtonian flow.

Benis and Lacoste [13] used the method of Charm and Kurland [12] to examine the non-Newtonian flow of blood in a tapering tube. They employed the approximate Casson fluid model for tube flow suggested by Merrill and

coworkers [16]. In this work a slightly modified form of this approximation is employed.

$$\tau_{RL} \Big|_{R=R_0} = \frac{64}{49} \tau_y + \frac{32}{7} S \sqrt{\frac{\tau_y Q}{\pi R_0^3}} + \frac{4S^2 Q}{\pi R_0^3} \quad (9)$$

The conditions under which this approximation is valid have been discussed elsewhere [18]. Substitution of equation (9) into equation (8) (with $R_0 = \alpha L$) followed by integration from $L = L_1$ to $L = L_2$ yields the pressure drop between L_1 and L_2 for uniform taper,

$$\begin{aligned} \Delta P &= P_1 - P_2 \\ &= \frac{128}{49} \frac{\tau_y}{\alpha} \ln \frac{L_1}{L_2} + \frac{128S}{21\alpha} \sqrt{\frac{\tau_y Q}{\pi \alpha^3}} \left(\frac{1}{L_2^{3/2}} - \frac{1}{L_1^{3/2}} \right) \\ &\quad + \frac{8S^2 Q}{3\pi \alpha^4} \left[\frac{1}{L_2^3} - \frac{1}{L_1^3} \right] \end{aligned} \quad (10)$$

where P_1 and P_2 are the pressures at positions L_1 and L_2 respectively. This expression gives the pressure drop for the flow of a Casson fluid in terms of the flow rate, geometric parameters and rheological parameters.

Equation (10) is valid only for slow flow. For larger flow rates, losses due to inertia can be significant. The arithmetic average correction suggested by Walawender et al. [10] is added to equation (10) to allow for inertial losses.

$$\begin{aligned} \Delta P &= \frac{128}{49} \frac{\tau_y}{\alpha} \ln \frac{L_1}{L_2} + \frac{128S}{21\alpha} \sqrt{\frac{\tau_y Q}{\pi \alpha^3}} \left(\frac{1}{L_2^{3/2}} - \frac{1}{L_1^{3/2}} \right) \\ &\quad + \frac{8S^2 Q}{3\pi \alpha^4} \left[\frac{1}{L_2^3} - \frac{1}{L_1^3} \right] + \frac{3\rho Q^2}{4\pi^2 \alpha^4} \left[\frac{1}{L_2^4} - \frac{1}{L_1^4} \right] \end{aligned} \quad (11)$$

An alternative approach to obtain equation (10) is to integrate equation (3) directly in spherical coordinates. This derivation is given in the Appendix.

5.4 Apparatus

The buret-capillary viscometer described some years ago by Cerny [19] also by Cerny and Walawender [17], was modified for this work to indirectly determine pressure-flow data for flow in a tapered tube. In addition the unit was also used for the independent determinations of the rheological parameters for blood using a true bore capillary. A schematic diagram of the apparatus is shown in figure 2. The buret B is jacketed with constant temperature water circulating inside the jacket. The capillary or tapered tube C and collection flask F are immersed in the constant temperature bath during the course of experiment. The temperature regulator G has the functions of controlling the water temperature in the bath and pumping water through the jacket. The buret and the capillary are connected by a piece of tygon tubing T. The compressed air reservoir and manometer are used to apply driving pressure so that high shear data can be obtained for the rheological parameter determinations.

Two tapered tubes were used for the experimental work. A precision bore cylindrical capillary was used for the rheological parameter determinations. The tapered tubes were previously calibrated by Cerny and Walawender [17]. The length of the cylindrical capillary was measured and the radius determined by the conventional method of filling the capillary with mercury and weighing the thread. Table 1 gives the dimensions of the tapered tubes and the capillary tube.

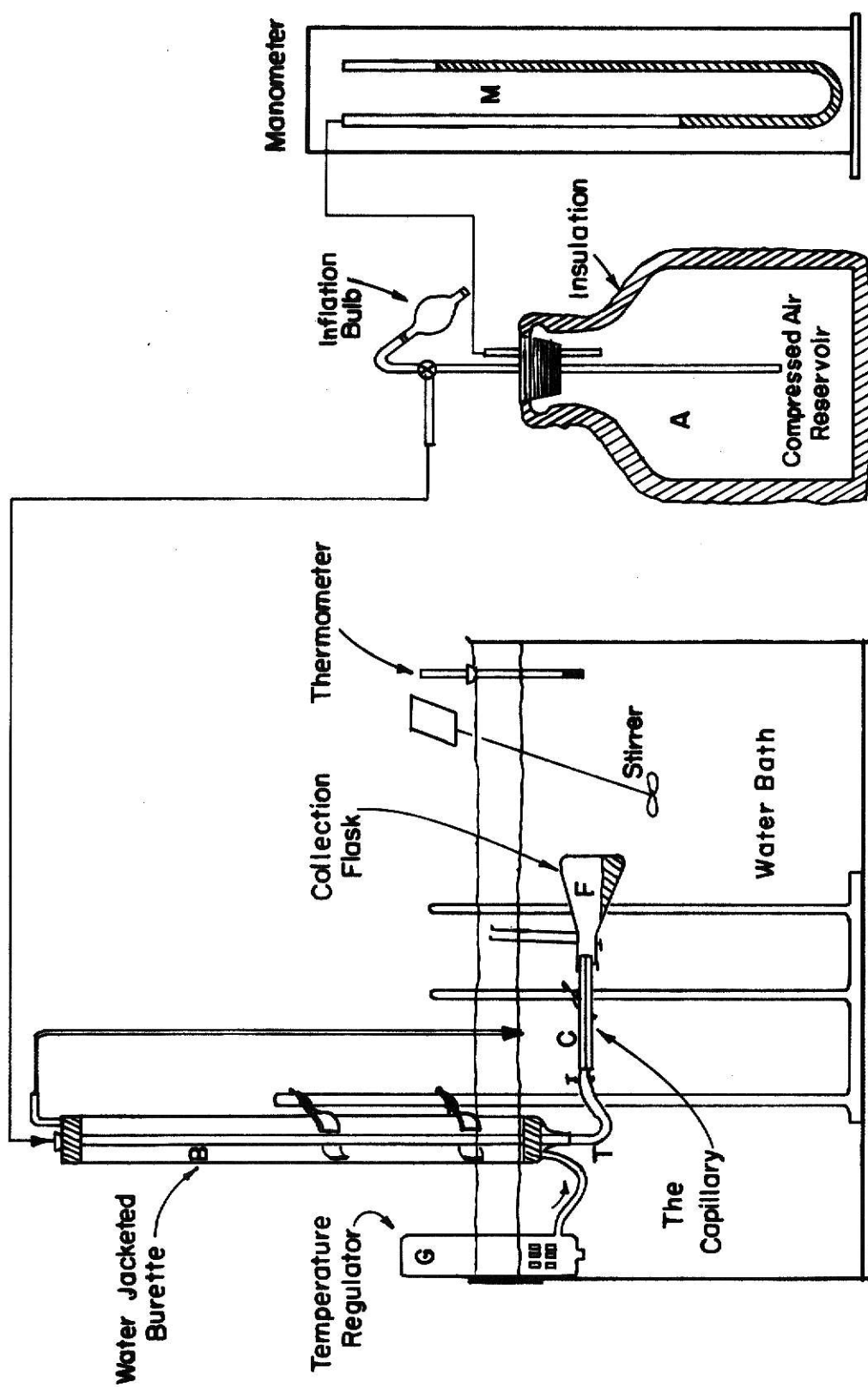


Fig. 2. Schematic diagram of the apparatus.

Table 1 Dimensions of the capillary and tapered tubes

	Length (cm)	I.D. (cm)	
Cylindrical Capillary	$19.88 \pm .01$	.102	
	Large I.D.	Small I.D.	Angle
Taper tube 1	$.2159 \pm .0003$	$.0381 \pm .0003$	$1^\circ \pm 1.5'$
Taper tube 2	$.3886 \pm .0003$	$.0381 \pm .0003$	$1^\circ 58' \pm 1.5'$

5.5 Experimental Procedure

Heparinized whole bovine blood samples were employed in the experimental work. The blood was collected directly into a container and at the same time was heparinized with 1000 units of heparin per 200 ml of blood. It was stored in a refrigerator at $0 - 4^{\circ}\text{C}$ before use. Before each experiment the blood was filtered through a plug of glass wool and warmed to temperature in the constant temperature bath (25°C).

Prior to filling of the apparatus with blood, the unit was filled with physiological saline. Any bubbles trapped in the unit were eliminated at this stage. The saline was then displaced by blood. Measurements consisted of timing the descent of the fluid meniscus at selected graduations on the buret. A stopwatch with a split hand feature was used to obtain a continuous set of time data. Each graduation on the buret can be related to a corresponding height relative to the flow outlet hence the measurements resulted in a set of height versus time data. The set of height versus time data was used to determine pressure drop and flow rate for tapered tube flow as well as the wall shear stress and shear rate for cylindrical capillary flow.

Each experiment for tapered tube flow consisted of at least three sets of height versus time data with the same blood sample. The average value of the sets was used to determine pressure flow data for the blood sample. Similar measurements were made (using the same blood sample) for flow through a cylindrical capillary. These data were used for the independent rheological parameter determinations.

Other supporting data consist of the distance between the capillary and the lowest mark on the buret, density and hematocrit of each blood. The distance between the capillary and the lowest mark on the buret was measured

by means of a cathetometer, the density with a pycnometer, and the hematocrit with a centrifuge.

5.6 Treatment of Data

The measurements of blood flow in a cylindrical capillary were used to provide the supporting rheological parameters. The construction of the flow curve from height versus time data in a cylindrical capillary is similar to the approach developed by Krieger and Maron [20, 21, 22]. Details of the approach have been presented elsewhere [23] so only a brief outline is given below.

As was shown by Krieger and Maron [20, 21, 22], the wall shear rate for non-Newtonian fluids in a capillary is given by

$$\dot{\gamma}_w = \tau_w \phi_e \left[1 + \frac{1}{4} \frac{d \ln \phi_e}{d \ln \tau_w} \right] \quad (12)$$

where τ_w is the wall shear stress given by

$$\tau_w = \frac{\Delta P R_c}{2L} \quad (13)$$

and ϕ_e the effective fluidity (reciprocal of effective viscosity) defined as

$$\phi_e = \frac{8LQ}{\Delta P \pi R_c^4} \quad (14)$$

Here, R_c and L are respectively the radius and length of the capillary. The flow rate Q and pressure drop across the capillary, ΔP , are given for this viscometer as follows:

$$Q = - \pi R_c^2 \frac{dh}{dt} \quad (15)$$

where R_F is the radius of the buret,

$$\Delta P = \begin{cases} \rho_f g h , & \text{without driving pressure} \end{cases} \quad (16a)$$

$$\Delta P = \begin{cases} \rho_f g h + \rho_m g h_m , & \text{with driving pressure} \end{cases} \quad (16b)$$

where g is the acceleration of gravity, ρ_f and ρ_m are densities of blood and manometer fluid respectively. Equations (15) and (16) give Q and ΔP in terms of the height (h) versus time (t) data obtained from the experiments. h_m in equation (16b) is the manometer reading in cm. Substitution of equations (15) and (16) into equation (14) results in

$$\phi_e = - \frac{m}{B\rho_f} \quad (17)$$

where m is the slope of a $\ln \Delta P$ versus t plot, i.e.,

$$m = \frac{d \ln \Delta P}{dt} \quad (18)$$

and B is an instrumental constant given by

$$B = \frac{gR_c^4}{8LR_F^2} . \quad (19)$$

Substitution for ϕ_e in equation (12) followed by some algebraic manipulation gives the wall shear rate as

$$\dot{\gamma}_w = - \tau_w \frac{m}{B\rho_f} \left[1 + \frac{1}{4m^2} \frac{dm}{dt} \right] \quad (20)$$

This equation coupled with equation (13) provides the information for construction of the flow curve from the experimental data.

A method for the evaluation of m and $\frac{dm}{dt}$ using a computer-aided non-linear parameter estimation technique has been discussed in detail elsewhere [23].

This method was employed in the present study and involves curve fitting the h versus t data to a selected function. In the case of no driving pressure, the h versus t data from cylindrical capillary were fitted to a model equation of the form

$$h = h_0 \exp [-kt + (a + bt)^2] \quad (21)$$

where h_0 is the value of h at time zero and k , a , and b are parameters to be determined from the parameter estimation technique. After the determinations of the parameters, m is evaluated from equation (18),

$$m = \frac{d \ln \Delta P}{dt} = \frac{d \ln h}{dt} = -k + 2b(a + bt) \quad (22)$$

and dm/dt by differentiation of equation (22) is

$$\frac{dm}{dt} = 2b^2 \quad (23)$$

Now τ_w can be determined from equations (13), (16), and (21)

$$\tau_w = \frac{\rho_f g h_0 R_c}{2L} \exp [-kt + (a + bt)^2] \quad (24)$$

And $\dot{\gamma}_w$ is given by equation (20) with the terms m , dm/dt , and τ_w replaced by equations (22), (23) and (24) respectively.

For experiments with a driving pressure, the technique is nearly the same, except that h is replaced by h' defined as

$$h' = (\rho_f g h + \rho_m g h_m) / \rho_f g \quad (25)$$

In the calculation of pressure-flow data from height versus time data for a tapered tube, equations (15) and (16a) were employed along with a similar curve fitting technique. Equation (21) was found suitable for describing

h versus t data for the 1° tube. For the 2° tube, the equation is more complicated and is given by

$$h = h_0 \exp [-kt + \{e - (ct + d)^2\}^2] \quad (26)$$

where k, e, c, and d are the parameters to be estimated.

The pressure drop is then obtained from equation (16a) and the flow rate by equation (15), with the appropriate substitutions for h and dh/dt.

The theoretically predicted pressure drop as given by equation (11) was calculated using the independently determined rheological parameters, flow rate data for the tapered tube, and the tapered tube dimensions.

5.7 Results

Since the Casson model was assumed for blood, τ_w and $\dot{\gamma}_w$ data obtained from capillary flow measurements were plotted as $\sqrt{\tau_w}$ versus $\sqrt{\dot{\gamma}_w}$. A typical plot is presented in figure 3. As can be seen from the figure, the flow curve can be well represented by a single straight line. This behavior is similar to that of human blood reported by Charm and Kurland [12]; according to their measurements, human blood can be represented by the Casson model over a shear rate range of $0 \sim 10^5 \text{ sec}^{-1}$. Since there was no obvious break in the data (as suggested by Merrill and coworkers [24]), the rheological parameters τ_y and s were simply evaluated from a least square fit of the $\sqrt{\tau_w}$ versus $\sqrt{\dot{\gamma}_w}$ data. These parameters were used in comparing theory with experiment for tapered tube flow. Recently, Schmid-Schöbein et al. [25] reported that the linear $\sqrt{\tau_w}$ versus $\sqrt{\dot{\gamma}_w}$ plot does not apply for shear rate < 1 for bovine blood, however it does apply for the wall shear conditions of this work.

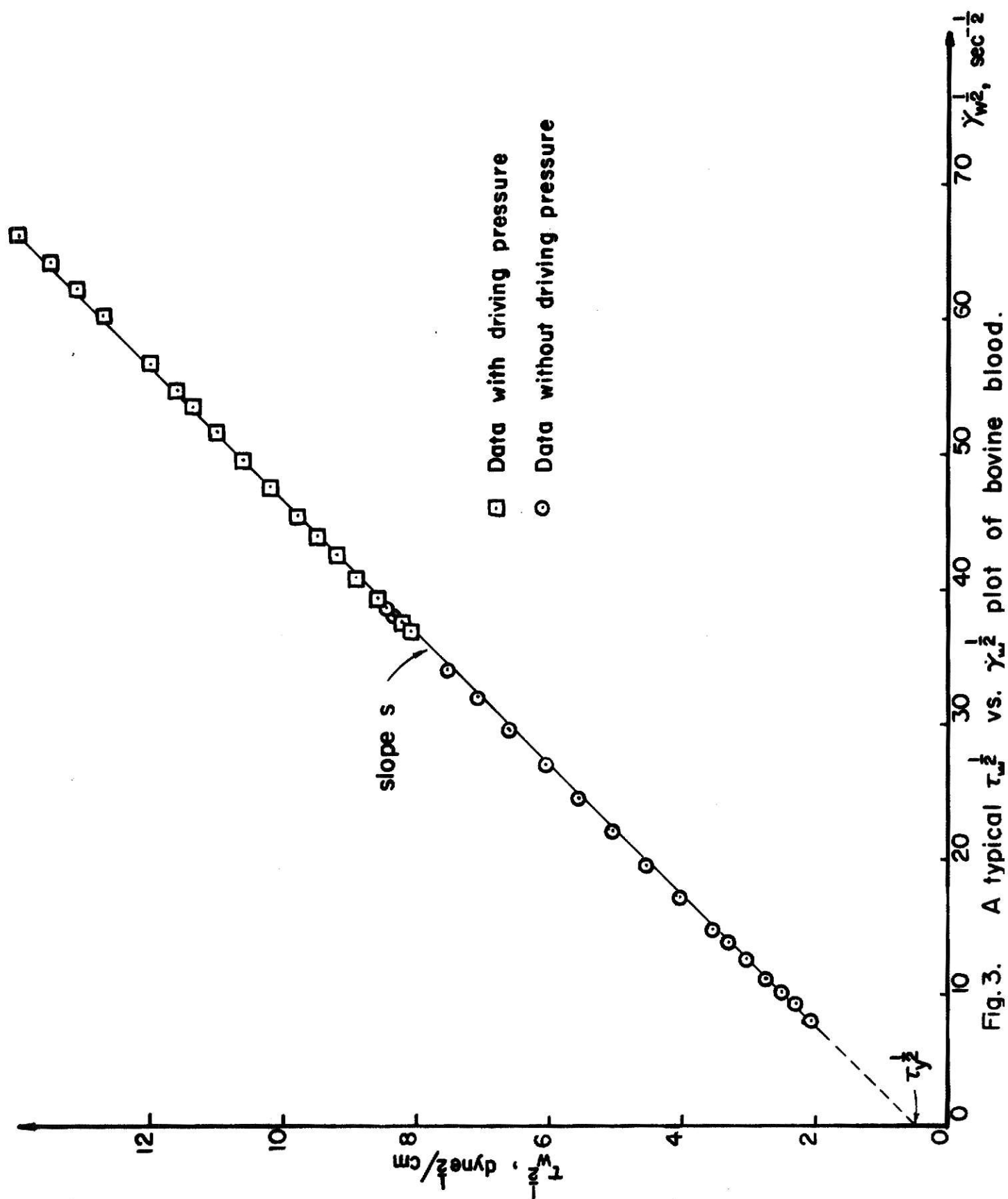


Fig. 3. A typical $\tau_y^{1/2}$ vs. $\gamma_w^{1/2}$ plot of bovine blood.

Typical results for the h versus t curve fit for the blood flow data in tapered tubes are presented in figure 4, for both 1° and 2° tubes. As shown in figure 4, the curve fit is very satisfactory. The difference between the best fit values and the experimental data are in excellent agreement. The largest differences observed did not exceed one per cent. From these curve fittings, values of Q were calculated as indicated by equation (15).

Experimental data for blood flow in tapered tubes are plotted as pressure drop versus flow rate along with the theoretical curve calculated according to equation (11). Four sets of pressure flow data for the 1° tube and five sets for the 2° tube were obtained. Figures 5 and 6 show typical results. As can be seen from the figures, equation (11) provides a very good description of the data.

Comparison of the curves in figure 5 for the 1° tube, and figure 6 for the 2° tube, reveals the different behavior in the two tubes. The curves in figure 5 tend to curve down while the curves in figure 6 tend to curve up. This is due to the effect of the inertial losses. In the 2° tube the inertial loss is significant, accounting for up to 15% of the total pressure drop at the higher flow rates. In the 1° tube however these losses are not very important since the flow rates are in general much lower.

5.8 Discussion

The suggestion that the terminal arteries should be considered as segments of long, slowly tapering cones, rather than cylinders was set forth some time ago as a result of the extensive studies of Knisley and Jeffords [1] as well as those of Bloch [2]. The consideration of flow in tapered vessels has been rather limited and little if anything has been said with respect to the

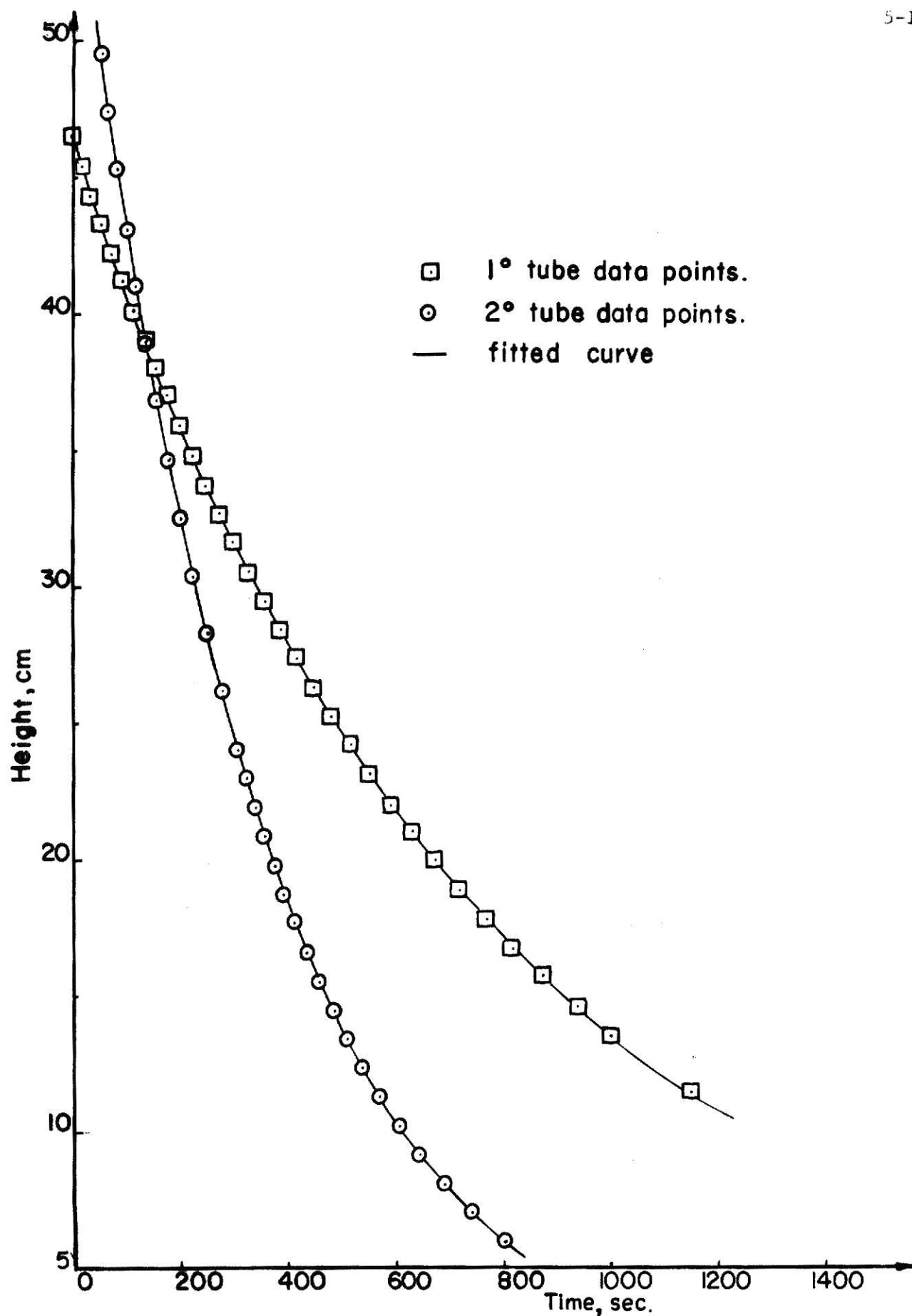


Fig. 4. Typical fitted curves for h vs. t data.

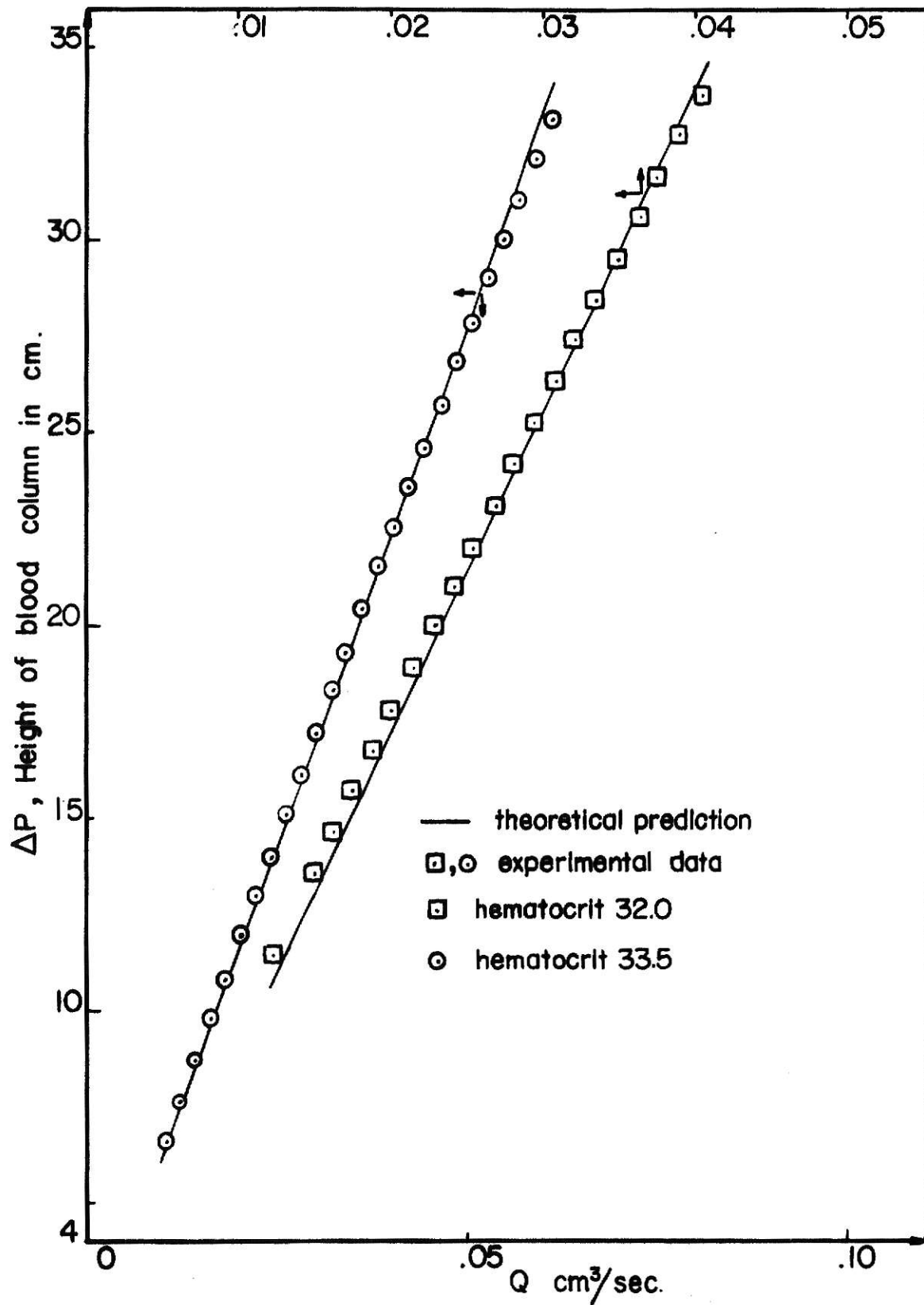


Fig. 5. Pressure drop - flow rate of a 1° tube.

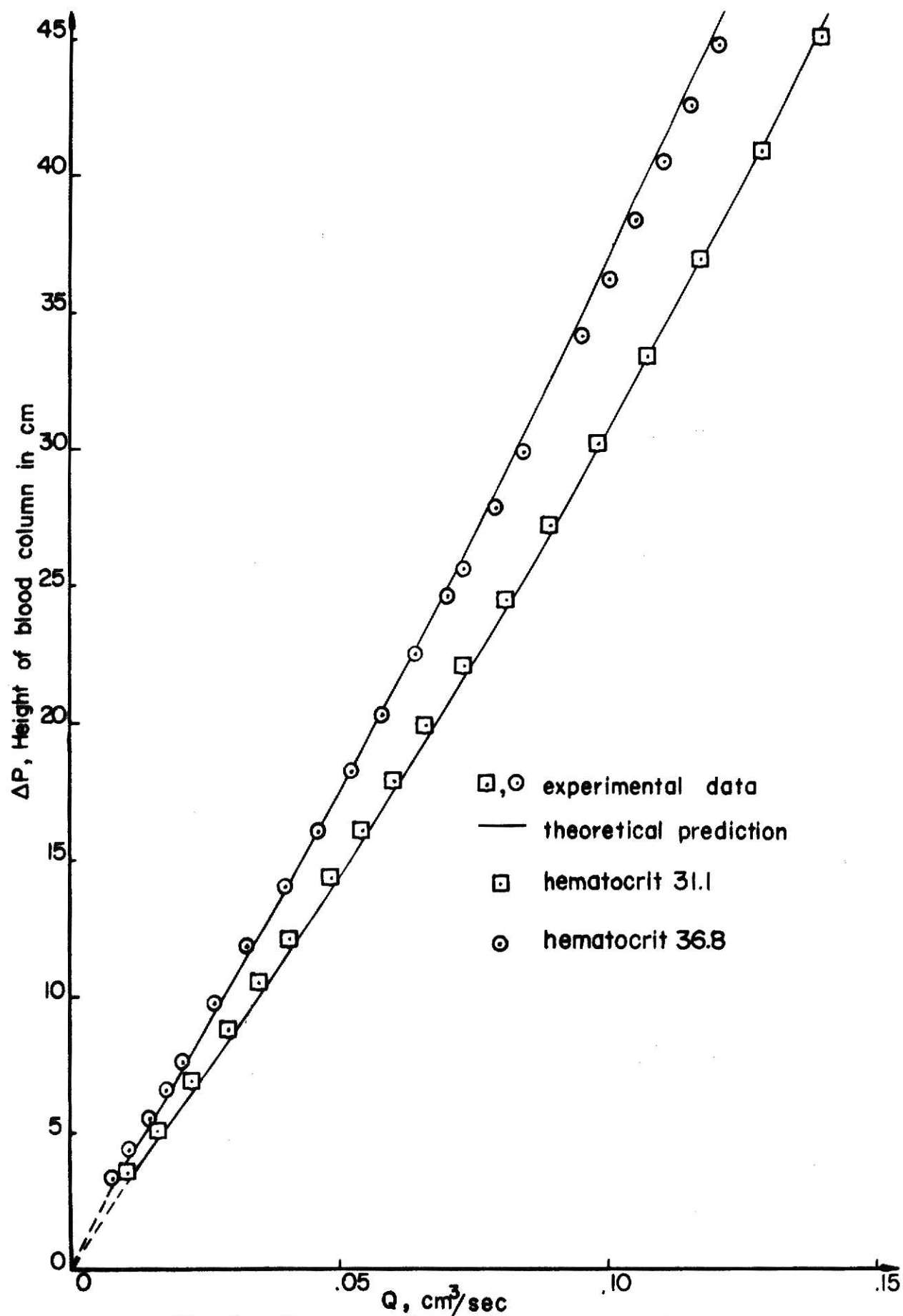


Fig. 6. Pressure drop- flow rate of a 2° tube.

significance of taper and its role in microvascular hemodynamics. The presence of taper provides a mechanism for increasing shear while maintaining volume flow constant. It is well known that increasing shear provides a means for the irreversible breakdown of the structure of blood in the case of in vitro experiments. This breakdown is accompanied by a reduction of the apparent viscosity.

It has long been reported that the resistance under in vivo conditions is substantially lower than predicted on the basis of cylindrical geometry and in vitro experiments. Various explanations have been proposed, but the discrepancy still remains. The Fahraeus effect is most likely responsible for some of the discrepancy but the influence of taper appears to warrant serious consideration.

In the present work a highly simplified case of flow in tapering vessels has been examined. The assumptions of steady-homogeneous flow have been employed and are reasonably valid for the experimental studies. Observations of flow in the living microcirculation indicate a weak pulsatile character as well as two phase slug-type of flow. Nonetheless, the present work may serve as a basis for more detailed considerations.

It is interesting to note that the two analytical approaches presented here as well as the treatment of Oka [15] lead to the same result. For small angles of taper as exist in the case of the arterioles, the analytical treatments indicate that the influence of the normal stress effects is negligible and that the problem can be considered as simple shear flow with a single velocity component.

The experimental results for blood are in good agreement with the predicted pressure-flow relation for the Casson model fluid. In addition data

obtained by Sutterby [8] for the flow of a Natrosol solution in a 7° tapered tube were found to be well described by the present treatment as applied to the case of a power law fluid.

Inertial losses are important only when the flow rate is sufficiently high. This effect was indicated by comparison of figures 5 and 6. Although significant inertial effects were present in the experiments with the 2° tube it is believed that they will be negligible for arteriolar flow. For larger vessel segments, however, they can be quite important.

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5.10 Appendix

It is here assumed that the Casson fluid behavior can be expressed by a simplified constitutive equation of the form

$$\bar{\tau} = -\eta \bar{\Delta} \quad (\text{A1})$$

where η is a function of $\bar{\Delta}$ and depends only on the invariants of $\bar{\Delta}$. For an incompressible fluid, the first invariant is zero. The third invariant vanishes for simple flows, and is normally neglected for more complicated flows. Thus

$$\eta = \eta(I_2) = \eta(\bar{\Delta} : \bar{\Delta}) \quad (\text{A2})$$

The Casson model for simple plan flow is written as

$$\begin{aligned} \tau_{yx}^{1/2} &= \tau_y^{1/2} + S \dot{\gamma}^{1/2} \\ &= \tau_y^{1/2} + S \left(\frac{du_x}{dy} \right)^{1/2} \end{aligned} \quad (\text{A3})$$

In more general form equation (A3) can be expressed as

$$\bar{\tau} = - \left[\frac{\tau_y^{1/2}}{|\sqrt{1/2} \bar{\Delta} : \bar{\Delta}|^{1/2}} + S \right]^2 \bar{\Delta} \quad (\text{A4})$$

For the present flow situation, the rate of deformation tensor can be approximated as

$$\bar{\Delta} = \begin{bmatrix} 0 & \frac{1}{r} \frac{\partial v_r}{\partial \theta} & 0 \\ \frac{1}{r} \frac{\partial v_r}{\partial \theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{A5})$$

The above approximation is based on order of magnitude considerations. Then we have

$$1/2 \bar{\Delta} : \bar{\Delta} \approx \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} \right)^2 \quad (A6)$$

Therefore for flow in a tapering vessel the Casson model can be expressed as

$$\tau_{\theta r} = \left[\tau_y^{1/2} + S \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} \right)^{1/2} \right]^2 \quad (A7)$$

or

$$\tau_{\theta r}^{1/2} = \tau_y^{1/2} + S \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} \right)^{1/2} \quad (A8)$$

Substitution for $\tau_{\theta r}$ from equation (3), followed by rearrangement to give an expression for the shear rate and squaring the result gives an expression for the local velocity gradient

$$\frac{\partial v_r}{\partial \theta} = \frac{r}{S^2} \left[\left(\frac{dp}{dr} r \right) \left(\frac{\theta}{2} \right) - 2 \tau_y^{1/2} \left(\frac{dp}{dr} r \right)^{1/2} \left(\frac{\theta}{2} \right)^{1/2} + \tau_y \right] \quad (A9)$$

Equation (A9) can be integrated at constant r from a point θ to the wall ($\theta = \alpha$) to obtain the local velocity profile, with the aid of the boundary condition

$$v_r = 0 \quad \text{at} \quad \theta = \alpha \quad (A10)$$

The resulting profile is

$$v_r(r, \theta) = \frac{r}{S^2} \left[\left(\frac{dp}{dr} r \right) \frac{\theta^2 - \alpha^2}{4} - \frac{8}{3} \tau_y^{1/2} \left(\frac{dp}{dr} r \right)^{1/2} \left\{ \left(\frac{\theta}{2} \right)^{3/2} - \left(\frac{\alpha}{2} \right)^{3/2} \right\} + \tau_y (\theta - \alpha) \right] \quad (A11)$$

The flow rate of convergent flow in the tube is given by

$$-Q = \int_0^{\alpha} 2\pi r^2 V_r \theta d\theta \quad (A12)$$

Substitution of V_r in equation (A11) into equation (A12) followed by integration gives

$$Q = \frac{2\pi r^3}{S^2} \left[\left(\frac{dp}{dr} r \right) \frac{\alpha^4}{16} - \frac{\sqrt{2}}{7} \left(\frac{dp}{dr} r \right)^{1/2} \tau_y^{1/2} \alpha^{7/2} + \frac{\tau_y \alpha^3}{6} \right] \quad (A13)$$

Equation (A13) is identical to the result obtained by Oka [14] for Casson fluid, if the fourth power yield stress-shear stress ratio term in his result is omitted. Rearrangement of equation (A13) gives a quadratic form in

$$\left(\frac{dp}{dr} r \right)^{1/2},$$

$$\frac{\alpha}{16} \left(\frac{dp}{dr} r \right) - \frac{\sqrt{2}}{7} \tau_y^{1/2} \alpha^{1/2} \left(\frac{dp}{dr} r \right)^{1/2} + \frac{\tau_y}{6} - \frac{S^2 Q}{2\pi r^3 \alpha^3} = 0 \quad (A14)$$

$\left(\frac{dp}{dr} r \right)^{1/2}$ is solved from equation (A14),

$$\left(\frac{dp}{dr} r \right)^{1/2} = \frac{8}{\alpha} \left[\frac{\sqrt{2}}{7} \tau_y^{1/2} \alpha^{1/2} + \left(\frac{S^2 Q}{8\pi r^3} - \frac{\tau_y \alpha}{1176} \right)^{1/2} \right] \quad (A15)$$

Here only the positive root is physically meaningful. The term $\frac{S^2 Q}{8\pi r^3}$ in equation (A15) can be expressed as $\frac{S^2 Q \alpha}{8\pi R_o^3}$. For blood, S usually has a value of 0.2 and τ_y is in the order of 0.1 or less. Hence $\frac{\tau_y \alpha}{1176}$ in equation (A15) can be omitted if $\frac{Q}{2\pi R_o^3} > 0.1$; $\frac{Q}{2\pi R_o^3}$ is a criterion for blood flow study. After the

simplification, equation (A15) is squared and rearranged to yield the pressure gradient,

$$\frac{dp}{dr} = \frac{128}{49} \frac{\tau_y}{\alpha} r^{-1} + \frac{64}{7} S \left[\frac{\tau_y Q}{\pi \alpha^5} \right]^{1/2} r^{-5/2} + \frac{8S^2 Q}{\pi \alpha^4} r^{-4} \quad (A16)$$

The pressure drop is simply obtained by integrating equation (A16)

$$\begin{aligned} \Delta P &= P_1 - P_2 \\ &= \frac{128}{49} \frac{\tau_y}{\alpha} \ln \frac{r_1}{r_2} + \frac{128S}{21} \left[\frac{\tau_y Q}{\pi \alpha^5} \right]^{1/2} [r_2^{-3/2} - r_1^{-3/2}] \\ &\quad + \frac{8}{3} \frac{S^2 Q}{\pi \alpha^4} [r_2^{-3} - r_1^{-3}] \end{aligned} \quad (A17)$$

Equation (A17) is the same as equation (10) if it is noted that $r \doteq L$. This derivation also serves to indicate that the differential cylinder approach can only be valid for the conditions of small angle, creeping flow, $v_\theta \ll v_r$, and no dependence of P on θ .

CHAPTER 6

RECOMMENDATIONS

The analytical treatment of tapered tube flow presented in this work can be extended to the analysis of flow in microcirculation. The present study considered a highly simplified model and hence the limitations should be pointed out:

- 1) Although the pulsatile nature of blood flow in microcirculation is not as important as that in arteries, the flow is still not steady. The present result may be used to describe the mean pressure flow relation but studies on the influence of pulsation are recommended.
- 2) As a result of the present work, the flow behavior of bovine blood is only slightly non-Newtonian though the shear range occurring in the tapered tubes is wide. Studies on other kinds of blood with more pronounced non-Newtonian behavior are recommended.
- 3) The present study is based on homogeneous fluids. Extension to the non-homogeneous case which appears to exist in the microcirculation is recommended.
- 4) In the microcirculation, vessels are not all uniformly tapered.

Extension to the case of non-uniformly tapered is also recommended.

The two viscometers described in the thesis are both inexpensive and easy-to-operate. They may be useful for both routine work as well as instructional purposes for the illustration of non-Newtonian behavior. The data treatment procedures reduce the volume of hard calculation. They also can serve as an instructional aid to illustrate in data analysis and computer usage.

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NON-NEWTONIAN BLOOD FLOW IN TAPERED TUBES

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Blood flow in uniformly tapering tubes of small angle of taper has been examined both theoretically and experimentally. A theoretical pressure flow relation for steady laminar blood flow in a tapered tube under non-Newtonian flow conditions has been derived. Experimental pressure flow data in 1° and 2° tubes for heparinized bovine blood are compared with the theoretical predictions. Topics concerning rheological measurements, a rheological model for blood and data treatment for rheological parameter determination are presented.

In the theoretical derivation, blood is assumed to obey the Casson model. For small angle of taper, theoretical analysis indicates negligible effect of normal stresses. The equations of motion are simplified for slow flow by order of magnitude considerations. The simplified equation is integrated by a differential cylinder approach as well as by direct integration to yield the pressure flow relation. Both approaches lead to the same expression. In addition, an inertia correction term is properly added so that the resulting pressure-flow relation can be applied to an extended range with considerable inertia losses. The pressure flow relation is expressed in terms of rheological parameters of blood and dimensions of the tapered tube. Limits of the application of the theoretical prediction are given along with the derivation.

A novel coiled capillary viscometer was used for rheological measurements. Description of the viscometer and theory behind its operation are presented. Flow curves were obtained from this viscometer for bovine blood and aqueous silica suspensions. The shear stress ranges from $0.1 - 16 \text{ dynes/cm}^2$. The flow curves obtained verify a simple rheological model for blood--the approximate Casson model for tube flow.

The approximation has been derived based on order of magnitude considerations. It is expressed in terms of a pseudo shear rate and is the basis of the differential-cylinder approach in the tapered tube analysis. In addition, rheological parameters are readily evaluated from flow data by the approximation.

A buret capillary viscometer was used for indirect determination of pressure flow data for blood flow in a tapered tube. The same unit was also used to independently determine the rheological parameters for each blood sample using a true bore capillary. In the data treatment for both determinations, a non-linear parameter estimation method was used for curve fitting with the aid of a computer. By the curve fitting technique, pressure flow data for blood flow in a tapered tube resulted readily from the computer. Coupled with a well-established procedure for flow curve determination, the technique provided an easy method for flow curve construction from data of the true bore capillary. Pressure flow data obtained for heparinized bovine blood flow in 1° and 2° tapered tubes agree well with the theoretical predictions.